On the special values of certain L-series related to half-integral weight modular forms

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Abstract

Let h be a cuspidal Hecke eigenform of half-integral weight, and $E_{n/2+1/2}$ be Cohen's Eisenstein series of weight $n/2+1/2$. For a Dirichlet character χ we define a certain linear combination $R^{(\chi)}(s,h,E_{n/+1/2})$ of the Rankin-Selberg convolution products of h and $E_{n/2+1/2}$ twisted by Dirichlet characters related with χ. We then prove a certain algebraicity result for $R^{(\chi)}(l,h,E_{n/2+1/2})$ with l integers.

0 Introduction

For two modular forms $h_1(z)$ and $h_2(z)$ of half-integral weights $k_1+1/2$ and $k_2+1/2$, respectively, for $\Gamma_0(4)$, and a primitive character χ we define the Rankin-Selberg convolution product $\widetilde{R}(s,h_1,h_2,\chi)$ twisted by χ as

$$\widetilde{R}(s,h_1,h_2,\chi)=L(2s-k_1-k_2+1,\chi_{-1}^{k_1-k_2}\chi^2)\sum_{m=1}^{\infty}\frac{c_1(m)c_2(m)\chi(m)}{m^s},$$

where $c_1(m)$ and $c_2(m)$ denote the m-th Fourier coefficients of h_1 and h_2, respectively, and $L(s,\chi_{-1}^{k_1-k_2}\chi^2)$ is the Dirichlet L-function for $\chi_{-1}^{k_1-k_2}\chi^2$ (for the precise definition of χ_{-1} see Section 1.)

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The analytic properties of this Dirichlet series were investigated by Shimura [Sh2]. Furthermore the algebraicity of the values of this Dirichlet series at half-integers was deeply investigated by Shimura [Sh2]. However, as far as we know, there is no literature on the algebraicity of its special values at integers except for [K-M]. Therefore we naturally ask the following question:

Question. What can one say about the algebraicity of $\widetilde{R}(m, h_1, h_2, \chi)$ with m an integer?

In [K-M], we gave a partial answer to the above question in the case h_1 is a cuspidal Hecke eigenform in Kohnen's plus subspace for $\Gamma_0(4)$ and h_2 is Zagier's Eisenstein series of weight 3/2. In this paper, we consider the above question in the case h_1 is a cuspidal Hecke eigenform in Kohnen's plus subspace for $\Gamma_0(4)$ and h_2 is Cohen's Eisenstein series. This paper is a summary of our paper [Ka], which will be published elsewhere. To state our main result more explicitly, we define another Dirichlet series $R(s, h_1, h_2, \chi)$ by

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s}.$$

Assume that $k_1 + k_2$ is even, and that the conductor of χ is odd. Then, as will be explained in Section 1, it suffices to consider the above question for $R(m, h_1, h_2, \chi)$ with integer m. Now let k and n be even integers such that $n \geq 4$ and $2k - n \geq 12$. Let h be a Hecke eigenform of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$ belonging to Kohnen's plus subspace, and $S(h)$ the normalized Hecke eigenform of weight $2k - n$ for $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. Moreover let $E_{n/2+1/2}$ be Cohen's Eisenstein series of weight $n/2 + 1/2$ (for the precise definition of $E_{n/2+1/2}$, see Section 2). Let χ be a primitive character of conductor N. We assume that N is square free and let $N = p_1 \cdots p_r$ be the prime decomposition of N. Put $l_j = l_{n,p_j} =$ G.C.D$(n, p_j - 1)$. For an r-tuple $(i_1, i_2, \cdots, i_r)$ of integers put

$$\chi_{(i_1, \cdots, i_r)} = \chi \prod_{j=1}^{r} \left(\frac{*}{p_j}\right)_{l_j}^{i_j},$$

where $\left(\frac{*}{p_j}\right)_{l_j}$ denotes the l_j-th power residue symbol mod p_j. For two Dirichlet characters η_1 and η_2 mod N, we define $J_m(\eta_1, \eta_2)$ by

$$J_m(\eta_1, \eta_2) = \sum_Z \eta_1(\det Z)\eta_2(1 - \mathrm{tr}(Z)),$$

where Z runs over all symmetric matrices of degree m with entries in $\mathbf{Z}/N\mathbf{Z}$ and $\mathrm{tr}(Z)$ denotes the trace of a matrix Z. We note that $J_1(\eta_1, \eta_2)$ is the Jacobi sum $J(\eta_1, \eta_2)$ associated with η_1 and η_2. We also put $J_m(\eta_1) = J_m(\eta_1 \left(\frac{*}{N}\right)^{m-1}, \eta_1)$, where $\left(\frac{*}{N}\right)$ is the Jacobi symbol. We then define

$$R^{(\chi)}(s, h, E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1,\cdots,i_r)}(2^n)} R(s, h, E_{n/2+1/2}, \chi_{(i_1,\ldots,i_r)})$$

$$\times \overline{J(\chi_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))}\, \overline{J_{n-1}(\chi_{(i_1,\cdots,i_r)})} \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi^2_{(i_1,\cdots,i_r)}),$$

where $L(s, S(h), \chi^2_{(i_1,\cdots,i_r)})$ is Hecke's L-function of $S(h)$ twisted by $\chi^2_{(i_1,\cdots,i_r)}$. Then our main result (Theorem 2.1) can be stated as follows:

There exists a finite dimensional $\overline{\mathbf{Q}}$-vector space $W_{h,E_{n/2+1/2}}$ in $\mathbf{C}$ such that

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h,E_{n/2+1/2}}$$

for any integer m such that $n/2 + 1 \leq m \leq k - n/2 - 1$ and a character χ of odd square free conductor such that χ^n is primitive.

From the above result we easily obtain the following (cf. Theorem 2.2):

Let $r > \dim_{\overline{\mathbf{Q}}} W_{h,E_{n/2+1/2}}$. Let $m_1, m_2, \cdots, m_r$ be integers such that $n/2 + 1 \leq m_1, m_2, \cdots, m_r \leq k - n/2 - 1$ and $\chi_1, \chi_2, \cdots, \chi_r$ be Dirichlet characters of odd square free conductors $N_1, N_2, \cdots, N_r$, respectively such that χ_i^n is primitive for any $i = 1, 2, \cdots r$. Then the values
$\frac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \cdots, \frac{R^{(\chi_r)}(m_r, h, E_{n/2+1/2})}{\pi^{m_r n}}$
are linearly dependent over $\overline{\mathbf{Q}}$.

This is a certain generalization of a main result in [K-M] as will be explained later.

A main tool for proving Theorem 2.1 is the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift of h. To explain this, we define the twisted Koecher-Maaß series of a Siegel modular form in a more general setting. Let $F(Z)$ be a modular form of weight k with respect to the symplectic group $Sp_n(\mathbf{Z})$. For a positive integer N let $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \bmod N\}$, and $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$. For a primitive Dirichlet character χ mod N we define the Koecher-Maaß series $L(s, F, \chi)$ of F twisted by χ as

$$L(s, F, \chi) = \sum_T \frac{\chi(\mathrm{tr}(T))c_F(T)}{e_N(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_{n,N}(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $c_F(T)$ denotes the T-th Fourier coefficient of F. We note that this Dirichlet series coincides with the Hecke L-function associated to F twisted by χ in case $n = 1$. Though we are mainly concerned with $L(s, F, \chi)$ in this paper, we also define another type of twisted Koecher-Maaß series $L^*(s, F, \chi)$ as

$$L^*(s, F, \chi) = \sum_T \frac{\chi(\det(2T))c_F(T)}{e(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $e(T) = e_1(T)$. These two Dirichlet series $L(s, F, \chi)$ and $L^*(s, F, \chi)$ essentially coincide with each other in case $n = 1$, but they don't in general. To distinguish these two Dirichlet series, we sometimes call $L(s, F, \chi)$ and $L^*(s, F, \chi)$ the twisted Koecher-Maaß series of the first and second kind, respectively. In Section 3, we will discuss a relation between these two Dirichlet series (cf. Theorem 3.5.) Now for the integers k and n stated above, let h a cuspidal Hecke eigenform h in Kohnen's plus subspace of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$. Let $I_n(h)$ be the Duke-Imamoglu-Ikeda lift of h to the space of Siegel cusp forms of degree n. Then, in Section 4, first we give an explicit formula of $L^*(s, I_n(h), \eta)$ in terms of the Rankin-Selberg series $R(s, h, E_{n/2+1/2}, \eta)$ and shifted products of Hecke's L-functions of $S(h)$ twisted by η^2 in the case η is a primitive character (cf. Theorem 4.1.) Next, by this result combined with Theorem 3.5, we give an explicit formula of $L(s, I_n(h), \chi^n)$ in terms of $R^{(\chi)}(s, h, E_{n/2+1/2})$ and a sum of the shifted products $\prod_{j=1}^{n/2-1} L(2s - 2j + 1, S(h), \chi^2_{(i_1,\cdots,i_r)})$ (cf.

Theorem 4.2 and its corollary.) This implies that $R^{(\chi)}(s, h, E_{n/2+1/2})$ can be expressed in terms of $L(s, I_n(h), \chi^n)$ and the sum of the shifted products. Thus we can prove our main result using the algebraicity of Hecke's L-function of $S(h)$ (cf. Theorem 1.1) combined with the arithmetic properties of $L(s, I_n(h), \chi^n)$, which were investigated by Choie and Kohnen [C-K] in a more general setting (cf. Theorem 3.2). We can also prove a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$ in case $n \equiv\ 2 \bmod 4$ using the functional equation for $L(s, F, \chi^n)$ (cf. Theorem 2.3.)

Notation. We denote by $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for a complex number x. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset $X \bmod aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, and $SL_m(R) = \{A \in M_m(R) \mid \det A = 1\}$, where $\det A$ denotes the determinant of a square matrix A and R^* is the unit group of R. We denote by $S_n(R)$ the set of symmetric matrices of degree n with entries in R. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $SL_n(\mathbf{Z})$ acts on the set $S_n(\mathbf{R})$ in the following way:

$$SL_n(\mathbf{Z}) \times S_n(\mathbf{R}) \ni (g, A) \longrightarrow {}^tgAg \in S_n(\mathbf{R}).$$

Let G be a subgroup of $GL_n(\mathbf{Z})$. For a subset $\mathcal{B}$ of $S_n(\mathbf{R})$ stable under the action of G we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. Two symmetric matrices A and A' with entries in R are said to be equivalent with each other with respect to G and write $A \sim_G A'$ if there is an element X of G such that $A' = A[X]$. Let $\mathcal{L}_n$ denote the set of half-integral matrices of degree n over $\mathbf{Z}$, that is, $\mathcal{L}_n$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to $\mathbf{Z}$ or $\frac{1}{2}\mathbf{Z}$ according as $i = j$ or not.

1 Review on the algebraicity of L-values of elliptic modular forms of integral and half-integral weights

Before stating our main results, we review on the special values of L functions of elliptic modular forms of integral and half-integral weights. Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n and O_n denotes the unit matrix and the zero matrix of degree n, respectively. Furthermore, put

$$Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let l be an integer or a half-integer, and let Γ be a congruence subgroup of $Sp_n(\mathbf{Z})$. We then denote by $M_l(\Gamma)$ the space of modular forms of weight l with respect to Γ, and by $S_l(\Gamma)$ the subspace of $M_l(\Gamma)$ consisting of cusp forms. We also denote by $\Gamma_0(4)$ the subgroup of $SL_2(\mathbf{Z})$ consisting of matrices whose left lower entries are congruent to 0 mod N. Let

$$f(z) = \sum_{m=1}^{\infty} c_f(m)\mathbf{e}(mz)$$

be a normalized Hecke eigenform in $S_k(SL_2(\mathbf{Z}))$, and χ be a primitive Dirichlet character. Then let us define Hecke's L-function $L(s, f, \chi)$ of f twisted by χ as

$$L(s, f, \chi) = \sum_{m=1}^{\infty} c_f(m)\chi(m)m^{-s}.$$

Then we have the following result (cf. [Sh1]):

Theorem 1.1 *There exist complex numbers $u_\pm(f)$ uniquely determined up to $\overline{\mathbf{Q}}^\times$ multiple such that*

$$L(m, f, \chi)(\pi^m u_j(f))^{-1} \in \overline{\mathbf{Q}}$$

for any integer $0 < m \leq k-1$ and a primitive character χ, where $j = +$ or $-$ according as $(-1)^m\chi(-1) = 1$ or -1.

We remark that we have $L(m, f, \chi) \neq 0$ if $m \neq k/2$, and $L(k/2, f, \chi) \neq 0$ for infinitely many χ.

Next let us consider the half-integral weight case. Let

$$h_1(z) = \sum_{m=1}^{\infty} c_{h_1}(m)\mathbf{e}(mz)$$

be a Hecke eigenform in $S_{k_1+1/2}(\varGamma_0(4))$, and

$$h_2(z) = \sum_{m=0}^{\infty} c_{h_2}(m)\mathbf{e}(mz)$$

be an element of $M_{k_2+1/2}(\varGamma_0(4))$. For positive integers e and l, let $\chi_{(-1)^l e}$ be the Dirichlet character corresponding to the extension $\mathbf{Q}(\sqrt{(-1)^l e}/\mathbf{Q})$. Let χ be a primitive character mod N. Then we define

$$\widetilde{R}(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \omega) \sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s},$$

where $\omega(d) = \chi_{-1}^{k_1-k_2}\chi^2(d)$. Now let $S(h_1)$ be the normalized Hecke eigenform in $S_{2k_1}(SL_2(\mathbf{Z}))$ corresponding to h_1 under the Shimura correspondence. Then the following result is due to Shimura [Sh2].

Theorem 1.2 *Assume that $k_1 > k_2$. Under the above notation we have*

$$\widetilde{R}(m + 1/2, h_1, h_2, \chi)(u_-(S(h_1))\pi^{-k_2+1+2m})^{-1} \in \overline{\mathbf{Q}}(h_1)\overline{\mathbf{Q}}(h_2)$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character χ, where $\overline{\mathbf{Q}}(h_i)$ is the field, generated over $\overline{\mathbf{Q}}$, by all the Fourier coefficients of h_i.

Corollary *Let the notation be as above. Assume that $k_1 > k_2$ and that $c_{h_1}(n), c_{h_2}(n) \in \overline{\mathbf{Q}}$ for any $n \in \mathbf{Z}_{\geq 0}$. Then there exists a one-dimensional $\overline{\mathbf{Q}}$-vector space U_{h_1,h_2} in $\mathbf{C}$ such that*

$$\widetilde{R}(m + 1/2, h_1, h_2, \chi)\pi^{-2m} \in U_{h_1,h_2}$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character χ.

Now we consider the values of $\widetilde{R}(s, h_1, h_2, \chi)$ at integers. Let

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s}.$$

be the Dirichlet series defined in Section 0. Assume that k_1+k_2 is even, and that the conductor of χ is odd. Then we have

$$R(s,h_1,h_2,\chi)=(1-2^{-2s+k_1+k_2-1}\chi^2(2))^{-1}\widetilde{R}(s,h_1,h_2,\chi).$$

Hence it suffices to consider the question in Section 0 for $R(m,h_1,h_2,\chi)$ with integer m.

2 Main results

For a non-negative integer m and a positive integer l, Cohen's function $H(l,m)$ is given by $H(l,m)=L_{-m}(1-l)$. Here

$$L_D(s) = \begin{cases} \zeta(2s-1), & D=0 \\ L(s,\chi_{D_K})\sum_{a|f}\mu(a)\chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D\neq 0,\ D\equiv 0,1 \bmod 4 \\ 0, & D\equiv 2,3 \bmod 4, \end{cases}$$

where the positive integer f is defined by $D=D_Kf^2$ with the discriminant D_K of $K=\mathbf{Q}(\sqrt{D})$, χ_{D_K} is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n)=\sum_{d|n}d^s$. Furthermore we define Cohen's Eisenstein series $E_{l+1/2}(z)$ by

$$E_{l+1/2}(z)=\sum_{m=0}^{\infty}H(l,m)\mathbf{e}(mz).$$

It is known that $E_{l+1/2}(z)$ is a modular form of weight $l+1/2$ belonging to Kohnen's plus space. Let k and n be positive even integers such that $n\geq 4$, $2k-n\geq 12$. Let $h(z)$ be a Hecke eigenform in Kohnen's plus subspace $S^+_{k-n/2+1/2}(\Gamma_0(4))$ (cf. [Ko]), and $S(h)$ be the normalized Hecke eigenform in $S_{2k-n}(SL_2(\mathbf{Z}))$ corresponding to h under the Shimura correspondence. Let p be a prime number and l be a positive integer dividing $p-1$. Take an l-th root of unity ζ_l and a prime ideal $\mathfrak{p}$ of $\mathbf{Q}(\zeta_l)$ lying above p. Let a be an integer prime to p. Then we have $a^{(p-1)/l}\equiv\zeta_l^i \bmod \mathfrak{p}$ with some $i\in\mathbf{Z}$. We then put $\left(\frac{a}{p}\right)_l=\zeta^i$. We call $\left(\frac{*}{p}\right)_l$ the l-th power residue symbol mod p. In the case $l=2$, this is the Legendre symbol, and we write it as $\left(\frac{*}{p}\right)$ as usual. We note that this definition of the power residue symbol is different from the usual one, and depends on the choice of $\mathfrak{p}$ and ζ_l except the case $l = 2$. We denote by $\left(\frac{*}{N}\right)$ the Jacobi symbol for a positive odd integer M. Let χ be a primitive Dirichlet character of conductor N. We assume that N is a square free odd integer, and write $N = p_1 \cdots p_r$ with $p_1, \cdots, p_r$ prime numbers. Put $l_j = l_{n,p_j} = \mathrm{G.C.D}(n, p_j - 1)$. For an r-tuple $(i_1, i_2, \cdots, i_r)$ of integers put

$$\chi_{(i_1,\cdots,i_r)} = \chi \prod_{j=1}^{r} \left(\frac{*}{p_j}\right)_{l_j}^{i_j}.$$

For two Dirichlet characters η_1 and η_2 mod N, let $J_m(\eta_1, \eta_2)$ and $J_m(\eta_1)$ be as those defined in Section 0. By definition, $J_m(\eta_1, \eta_2)$ is an algebraic number. As in Section 0, we define

$$\begin{aligned} & R^{(\chi)}(s, h, E_{n/2+1/2}) \\ = & \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1,\cdots,i_r)}(2^n)}\, \overline{J(\chi_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))}\, \overline{J_{n-1}(\chi_{(i_1,\cdots,i_r)})} \\ & \times R(s, h, E_{n/2+1/2}, \chi_{(i_1,\ldots,i_r)}) \mathbf{L}_n(s, S(h), \chi_{(i_1,\cdots,i_r)}), \end{aligned}$$

where

$$\mathbf{L}_n(s, S(h), \eta) = \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \eta^2)$$

for a primitive character η. We note that $R^{(\chi)}(s, h, E_{n/2+1/2})$ does not depend on the choice of an l_i-th root of unity ζ_{l_i} and an prime ideal $\mathfrak{p}_i$ of $\mathbf{Q}(\zeta_{l_i})$ lying above p_i.

Remark. (1) Let m be an integer s.t. $n/2+1 \le m \le k - n/2 - 1$. Then the value $\dfrac{\mathbf{L}_n(m, S(h), \chi^2_{(i_1,\cdots,i_r)})}{\pi^{m(n-2)}}$ belongs to $\overline{\mathbf{Q}} u_+(S(h))^{n/2-1} \pi^{-n^2/4+n/2}$ for any χ. In particular if $n \equiv 2$ mod 4, then it is nonzero for any χ, and if $n \equiv 0$ mod 4, then it is nonzero for infinitely many χ.
(2) As will be stated in Section 3, $J_{n-1}(\chi_{(i_1,\cdots,i_r)})$ is expressed as a product of Jacobi sums, and it is non-zero algebraic number if χ^n is rewrote.

Theorem 2.1 *There exists a finite dimensional $\overline{\mathbf{Q}}$-vector space $W_{h,E_{n/2+1/2}}$ in $\mathbf{C}$ such that*

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h,E_{n/2+1/2}}$$

for any integer $n/2+1 \leq m \leq k-n/2-1$ and a character χ of odd square free conductor such that χ^n is rewrote.

Theorem 2.2 *Let $r > \dim_{\overline{\mathbf{Q}}} W_{h,E_{n/2+1/2}}$. Let $m_1, m_2, \cdots, m_r$ be integers such that $n/2+1 \leq m_1, m_2, \cdots, m_r \leq k-n/2-1$ and $\chi_1, \chi_2, \cdots, \chi_r$ be Dirichlet characters of odd square free conductors $N_1, N_2, \cdots, N_r$, respectively such that χ_i^n is primitive for any $i = 1, 2, \cdots r$. Then the values $\dfrac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \cdots, \dfrac{R^{(\chi_r)}(m_r, h, E_{n/2+1/2})}{\pi^{m_r n}}$ are linearly dependent over $\overline{\mathbf{Q}}$.*

Corollary *Assume that $n \equiv 2 \bmod 4$. Let r and $m_1, m_2, \cdots, m_r$ be as above. Let $\chi_1, \chi_2, \cdots, \chi_r$ be Dirichlet characters of odd prime conductors $p_1, p_2, \cdots, p_r$, respectively such that χ_i^n is non-trivial for any $i = 1, 2, \cdots r$. Put $l_i = \mathrm{GCD}(n, p_i - 1)$. Then the values $\left\{ \dfrac{R(m_i, h, E_{n/2+1/2}, \chi_{i(j)})}{\pi^{2m_i}} \right\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$ are linearly dependent over $\overline{\mathbf{Q}}$.*

We also have a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$:

Theorem 2.3 *Let h be as above. Let χ be a primitive character of odd square free conductor N. Assume that $n \equiv 2 \bmod 4$, and that χ^n is primitive. Put*

$$\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^n)^{-1} \gamma_n(s) R^{(\chi)}(s, h, E_{n/2+1/2}),$$

where $\tau(\chi^n)$ is the Gauss sum of χ^n, and

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2} \Gamma(s - (i-1)/2).$$

Then $\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2})$ has an analytic continuation to the whole s-plane, and has the following functional equation:

$$\mathcal{R}^{(\chi)}(k - s, h, E_{n/2+1/2}) = \mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}).$$

Remark. (1) The series $\{R(s, h, E_{n/2+1/2}, \chi_{i(j)})\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$ are linearly independent over $\mathbf{C}$ as functions of s.
(2) In the case of $n = 2$, this type of result was given for $R(m, h, E_{3/2})$ with $E_{3/2}$ Zagier's Eisenstein series of weight 3/2 by [K-M]. Cohen's Eisenstein

series is a holomorphic modular form, where as Zagier's Eisenstein series is not. Nevertheless, the former can be regarded as a generalization of the latter. Therefore, our present result can be regarded as a generalization of [K-M]. (3) The meromorphy of this type of series was derived in [Sh2] by using so called the Rankin-Selberg integral expression in a more general setting, but we don't know whether the functional equation of the above type can be directly proved without using the above method.

3 Twisted Koecher-Maaß series

To prove the main results, in this section and the next, we consider the twisted Koecher-Maaß series of a Siegel modular form. Let $F(Z) \in M_k(Sp_n(\mathbf{Z}))$. Then $F(Z)$ has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{L}_{n \geq 0}} c_F(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where $\mathrm{tr}(X)$ denotes the trace of a matrix X. For $N \in \mathbf{Z}_{>0}$, put $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \mod N\}$, and for $T \in \mathcal{L}_{n>0}$ put $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$. For a primitive Dirichlet character χ mod N Let

$$L(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_{n,N}(\mathbf{Z})} \frac{\chi(\mathrm{tr}(T))c_F(T)}{e_N(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of F of the first kind as in Section 0. The following two theorems are due to Choie and Kohnen [C-K].

Theorem 3.1 *Let $F \in S_k(Sp_n(\mathbf{Z}))$, and χ a primitive character of conductor N. Put*

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2}\Gamma(s-(i-1)/2),$$

and

$$\Lambda(s, F, \chi) = N^{2s}\tau(\chi)^{-1}\gamma_n(s)L(s, F, \chi) \quad (\mathrm{Re}(s) >> 0),$$

where $\tau(\chi)$ is the Gauss sum of χ. Then $\Lambda(s, F, \chi)$ has an analytic continuation to the whole s-plane and has the following functional equation:

$$\Lambda(k-s, F, \chi) = (-1)^{nk/2}\chi(-1)\Lambda(s, F, \overline{\chi}).$$

Theorem 3.2 *Let F and χ be as above. Then there exists a finite dimensional $\overline{\mathbf{Q}}$-vector space V_F in $\mathbf{C}$ such that*

$$L(m, F, \chi)\pi^{-nm} \in V_F$$

for any primitive character χ and any integer m such that $(n+1)/2 \leq m \leq k-(n+1)/2$.

Example. Let $n = 1$. Take a basis $\{f_1, \cdots, f_d\}$ of $S_k(SL_2(\mathbf{Z}))$ consisting of normalized Hecke eigenforms. Write $f \in S_k(SL_2(\mathbf{Z}))$ as

$$f = a_1 f_1 + \cdots + a_d f_d$$

with $a_1, \cdots, a_d \in \mathbf{C}$. Then put $w_i = a_i u_+(f_i), w_{d+i} = a_i u_-(f_i)$ $(i = 1, \cdots, d)$ and $V_f = \displaystyle\sum_{i=1}^{2d} \overline{\mathbf{Q}} w_i$. Then V_f satisfies the required property for f.

Now let

$$L^*(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{\chi(\det(2T))c_F(T)}{e(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of F of the second kind as in Section 0. We will discuss a relation between these two Dirichlet series. Let N be a positive integer. Let g be a periodic function on $\mathbf{Z}$ with a period N and ϕ a polynomial in $t_1, ..., t_r$. Then for an element $u = (a_1 \bmod N, ..., a_r \bmod N) \in (\mathbf{Z}/N\mathbf{Z})^r$, the value $g(\phi(a_1, ..., a_r))$ does not depend on the choice of the representative u. Therefore we denote this value by $g(\phi(u))$. Now let χ be a primitive character mod N. For $A \in \mathcal{L}_{n>0}$, put

$$h(A, \chi) = \sum_{U \in SL_n(\mathbf{Z}/N\mathbf{Z})} \chi(\mathrm{tr}(A[U])).$$

The following proposition is due to [[K-M], Proposition 3.1].

Proposition 3.3 *Let*

$$F(Z) = \sum_{A \in \mathcal{L}_{n\geq 0}} c_F(A)\mathbf{e}(\mathrm{tr}(AZ))$$

be an element of $M_k(Sp_n(\mathbf{Z}))$. Let χ be a Dirichlet character mod N. Assume $N \neq 2$. Then we have

$$L(s, F, \chi) = \sum_{A \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{c_F(A)h(A, \chi)}{e(A)(\det A)^s}.$$

For a Dirichlet character χ mod N, let $\chi^{(p)}$ be the p-factor of χ so that $\chi = \prod_{p|N} \chi^{(p)}$. For a prime number p put

$$\gamma_{n,p} = p^{n^2-n(n+1)/2}(1-p^{-n/2}) \prod_{e=1}^{(n-2)/2} (1-p^{-2e})$$

or

$$\gamma_{n,p} = p^{n^2-n(n+1)/2} \prod_{e=1}^{(n-1)/2} (1-p^{-2e})$$

according as n is even or odd. The following result is a technical tool for proving our main result.

Theorem 3.4 *Let $A \in \mathcal{L}_{n>0}$. Let N be a square free odd integer, and let $N = \prod_{i=1}^{r} p_i$ be the prime decomposition of N. Let χ be a primitive Dirichlet character mod N. For each positive integer $i \leq r$, put $l_i = \mathrm{G.C.D}(n, p_i - 1)$ and let $u_{0,i}$ be a primitive l_i-th root of unity mod p_i.*
(1). If $\chi^{(p_i)}(u_{0,i}) \neq 1$ for some i. Then we have $h(A,\chi) = 0$.
(2). Assume that $\chi^{(p_i)}(u_{0,i}) = 1$ for any i. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^n = \chi$.
(2.1) Let n be even. Then we have

$$h(A,\chi) = \prod_{i=1}^{r} (-1)^{n(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}}_{(i_1,\cdots,i_r)}(2^n) \widetilde{\chi}_{(i_1,\cdots,i_r)}(\det(2A)) \overline{J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))} \overline{J_{n-1}(\widetilde{\chi}_{(i_1,\cdots,i_r)})}.$$

(2.2) Let n be odd, and assume that χ^2 is primitive. Then we have

$$h(A,\chi) = \prod_{i=1}^{r} (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}}_{(i_1,\cdots,i_r)}(2^n) \widetilde{\chi}_{(i_1,\cdots,i_r)}(\det(2A)) \overline{J_{n-1}(\widetilde{\chi}_{(i_1,\cdots,i_r)})}.$$

The proof of the above theorem is elementary but is rather lengthy. The details will be given in [Ka].

Remark. Let η be a primitive Dirichlet character of odd prime conductor p. Assume that $\eta^2 \neq 1$. Then we can prove that we have

$$J(\eta, \left(\frac{*}{p}\right))J(\eta \left(\frac{*}{p}\right), \eta \left(\frac{*}{p}\right)) = \left(\frac{-1}{p}\right) \bar{\eta}(4)p.$$

(This is not so trivial. For the details, see [Ka].) Hence for $A \in \mathcal{L}_{2>0}$ and a primitive character χ of odd square free conductor N such that $\chi^{(p)}(-1) = 1$ for any prime divisor p of N, we have

$$h(A, \chi) = \prod_{p|N} \left\{ \left(1 + \left(\frac{4 \det A}{p}\right)\right) \left(1 - \left(\frac{-1}{p}\right) p^{-1}\right) \right\} N^2 \left(\frac{-1}{N}\right) \tilde{\chi}(4 \det A)),$$

where $\tilde{\chi}$ is a character such that $\tilde{\chi}^2 = \chi$. This coincides with (2) of Theorem 3.8 in [K-M].

By Theorem 3.4 and Proposition 3.3 we easily obtain:

Theorem 3.5 *Let $N, p_i, l_i, u_{0,i}$ $(i = 1, \cdots, r)$ and χ be as in Theorem 3.4, and let F be an element of $M_k(Sp_n(\mathbf{Z}))$.*
(1). *If $\chi^{(p_i)}(u_{0,i}) \neq 1$ for some i. Then we have $L(s, F, \chi) = 0$.*
(2). *Assume that $\chi^{(p_i)}(u_{0,i}) = 1$ for any i. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^n = \chi$.*
(2.1) *Let n be even. Then we have*

$$L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{n(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}}_{(i_1,\cdots,i_r)}(2^n) \overline{J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))} \overline{J_{n-1}(\widetilde{\chi}_{(i_1,\cdots,i_r)})} L^*(s, F, \widetilde{\chi}_{(i_1,i_2,\cdots,i_r)}).$$

(2.2) *Let n be odd, and assume that χ^2 is primitive. Then we have*

$$L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}}_{(i_1,\cdots,i_r)}(2^n) \overline{J_{n-1}(\widetilde{\chi}_{(i_1,i_2,\cdots,i_r)})} L^*(s, F, \widetilde{\chi}_{(i_1,i_2,\cdots,i_r)}).$$

To give an explicit formula of $J_m(\chi,\eta)$ for primitive characters χ,η mod N, we define $I_m(\chi,\eta)$ as

$$I_m(\chi,\eta)=\sum_{Z\in S_m(\mathbf{Z}/N\mathbf{Z})}\chi(\det Z)\eta(\mathrm{tr}(Z)).$$

Then we have the following two propositions, whose proof will be given precisely in [Ka].

Proposition 3.6 *Let χ and η be primitive character mod an odd prime number p. Assume that $\chi^2\neq 1$ and that η is non-trivial. Put $c_m(\chi,\eta)=1$ or 0 according as $\chi^{m-1}\eta=1$ or not.*
(1) *Assume that m is odd. Then*

$$I_m(\chi,\eta)=c_m(\chi,\eta)\left(\frac{-1}{p}\right)^{(m-1)/2}p^{(m-1)/2}(p-1)J_{m-1}(\chi\left(\frac{*}{p}\right),\eta).$$

(2) *Assume that m is even. Then*

$$I_m(\chi,\eta)=c_m(\chi,\eta)\left(\frac{-1}{p}\right)^{m/2}p^{(m-2)/2}(p-1)\chi(-1)J(\chi,\left(\frac{*}{p}\right))J_{m-1}(\chi\left(\frac{*}{p}\right),\eta).$$

Proposition 3.7 *Let χ,η and p be as in Proposition 3.6.*
(1) *Assume that m is odd. Then*

$$J_m(\chi,\eta)=\left(\frac{-1}{p}\right)^{(m-1)/2}p^{(m-1)/2}$$

$$\times\{J(\chi,\chi^{m-1}\eta)J_{m-1}(\chi\left(\frac{*}{p}\right),\eta)+\eta(-1)I_{m-1}(\chi\left(\frac{*}{p}\right),\eta)\}.$$

(2) *Assume that m is even. Then*

$$J_m(\chi,\eta)=\left(\frac{-1}{p}\right)^{m/2}p^{(m-2)/2}J(\chi,\left(\frac{*}{p}\right))$$

$$\times\{J(\chi,\chi^{m-1}\left(\frac{*}{p}\right)\eta)J_{m-1}(\chi\left(\frac{*}{p}\right),\eta)+\eta(-1)I_{m-1}(\chi\left(\frac{*}{p}\right),\eta)\}.$$

From the above two propositions we have the following:

Theorem 3.8 *Let χ be a primitive character with a prime conductor p such that $\chi^2 \neq 1$.*
(1) *Let m be odd.*
(1.1) *Assume that $\chi^m \neq 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2} J(\chi \left(\frac{*}{p}\right)^i, \chi^m) J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi).$$

(1.2) *Assume that $\chi^m = 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = p^{m-1} \left(\frac{-1}{p}\right)^{i+1} J(\chi \left(\frac{*}{p}\right)^{i+1}, \left(\frac{*}{p}\right)) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

(2) *Let m be even.*
(2.1) *Assume that $\chi^m \left(\frac{*}{p}\right)^{i+1} \neq 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi)$$

$$= \left(\frac{-1}{p}\right)^{m/2-1} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J(\chi \left(\frac{*}{p}\right)^{i+1}, \chi^m \left(\frac{*}{p}\right)^{i+1}) J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi).$$

(2.2) *Assume that $\chi^m \left(\frac{*}{p}\right)^{i+1} = 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \chi(-1) p^{m-1} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

Corollary *Let χ be a primitive character with an odd square free conductor N. Assume that χ^2 is primitive. Then the value $J_m(\chi)$ is nonzero.*

4 An explicit formula for the twisted Koecher-Maaß series of the D-I-I lift

Throughout this section and the next, we assume that n and k are even positive integers. Let h be a Hecke eigenform of weight $k-n/2+1/2$ belonging to Kohnen's plus space. Then h has the following Fourier expansion:

$$h(z) = \sum_e c_h(e) \mathbf{e}(ez),$$

where e runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \bmod 4$. Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m)\mathbf{e}(mz)$$

be the normalized Hecke eigenform of weight $2k-n$ with respect to $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. For a prime number p let β_p be a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}c_{S(h)}(p)$. For a prime number p, let $\mathbf{Q}_p$, and $\mathbf{Z}_p$ be the field of p-adic numbers, and the ring of p-adic integers, respectively. We denote by ν_p the additive valuation on $\mathbf{Q}_p$ normalized so that $\nu_p(p) = 1$, and by $\mathbf{e}_p$ the continuous homomorphism from the additive group $\mathbf{Q}_p$ to $\mathbf{C}^\times$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbf{Z}[p^{-1}]$. For a positive definite half integral matrix T of degree n write $(-1)^{n/2}\det(2T)$ as $(-1)^{n/2}\det(2T) = \mathfrak{d}_T\mathfrak{f}_T^2$ with $\mathfrak{d}_T$ a fundamental discriminant and $\mathfrak{f}_T$ a positive integer. We then define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR))p^{-\nu_p(\mu_p(R))s} \ (s \in \mathbf{C})$$

for each prime number p, where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. Then there exists a polynomial $F_p(T, X)$ in X such that

$$b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \left(\frac{\mathfrak{d}_T}{p}\right)p^{n/2-s})^{-1}\prod_{i=1}^{n/2}(1 - p^{2i-2s})$$

(cf. [Ki].) We then put

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T|)\prod_p (p^{k-n/2-1/2}\beta_p)^{\nu_p(\mathfrak{f}_T)}F_p(T, p^{-(n+1)/2}\beta_p^{-1}).$$

We note that $c_{I_n(h)}(T)$ does not depend on the choice of β_p. Define a Fourier series $I_n(h)(Z)$ by

$$I_n(h)(Z) = \sum_{T \in \mathcal{L}_{n>0}} c_{I_n(h)}(T)\mathbf{e}(\mathrm{tr}(TZ)).$$

In [I] Ikeda showed that $I_n(h)(Z)$ is a cuspidal Hecke eigenform in $S_k(Sp_n(\mathbf{Z}))$ and its standard L-function $L(s, I_n(h), \mathrm{St})$ is given by

$$L(s, I_n(h), \mathrm{St}) = \zeta(s)\prod_{i=1}^{n} L(s+k-i, S(h)).$$

We call $I_n(h)$ the Duke-Imamoglu-Ikeda lift (D-I-I lift) of h. Now using the same argument as in the proof of Theorem 1 of [I-K] we obtain the following. For the details see [Ka].

Theorem 4.1 *Let χ be a primitive Dirichlet character mod N. Then we have*

$$L^*(s,F,\chi) = 2^{ns}\{c_n R(s,h,E_{n/2+1/2},\chi) \prod_{j=1}^{n/2-1} L(2s-2j,S(h),\chi^2)$$

$$+d_n c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1,S(h),\chi^2)\},$$

where c_n and d_n are non-zero rational numbers depending only on n.

Now by the above theorem combined with Theorem 3.5 we obtain:

Theorem 4.2 *Let N be a square free odd integer, and $N = p_1 \cdots p_r$ be the prime decomposition of N. For each $i = 1, \cdots, r$ let $l_i =$ G.C.D$(n, p_i - 1)$ and $u_0 \in \mathbf{Z}$ be a primitive l_i-th root of unity mod p_i.*
(1) Assume $\chi^{(p_i)}(u_i) \neq 1$ for some i. Then $L(s, I_n(h), \chi) = 0$.
(2) Assume $\chi^{(p_i)}(u_i) = 1$ for any i. Then

$$L(s,I_n(h),\chi) = 2^{ns} \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}_{(i_1,\cdots,i_r)}(2^n)}\, \overline{J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))}\, \overline{J_{n-1}(\widetilde{\chi}_{(i_1,\cdots,i_r)})}$$

$$\times\{c_{n,N} R(s,h,E_{n/2+1/2},\widetilde{\chi}_{(i_1,\cdots,i_r)}) \prod_{j=1}^{n/2-1} L(2s-2j,S(h),\widetilde{\chi}^2_{(i_1,\cdots,i_r)})$$

$$+d_{n,N} c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1,S(h),\widetilde{\chi}^2_{(i_1,\cdots,i_r)})\},$$

where $c_{n,N}$ and $d_{n,N}$ are non-zero rational numbers depending only on n and N, and $\widetilde{\chi}$ is a character s.t. $\widetilde{\chi}^n = \chi$.

Remark. In the case $n = 2$, an explicit formula for $L(s, I_2(h), \chi)$ was given by Katsurada-Mizuno [K-M].

Corollary *Let χ be a Dirichlet character of odd square free conductor N such that χ^n is primitive. Then for any integer $n/2+1 \le m \le k-n/2-1$*

$$\frac{L(m, I_n(h), \chi^n)}{\pi^{mn}}$$

$$= \{\gamma_{n,N} \frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} + \delta_{n,N} c_h(1) \frac{\mathbf{M}^{(\chi)}(m, S(h))}{\pi^{mn}}\},$$

where $\gamma_{n,N}$ and $\delta_{n,N}$ are non-zero numbers, and

$$\mathbf{M}^{(\chi)}(m, S(h)) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1,\cdots,i_r)}(2^n)} \overline{J(\chi_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))} \overline{J_{n-1}(\chi_{(i_1,\cdots,i_r)})}$$

$$\times \prod_{j=1}^{n/2} L(2m-2j+1, S(h), (\chi_{(i_1,\cdots,i_r)})^2).$$

5 Proof of main results and some comments

We prove the results in Section 2.

Proof of Theorem 2.1. Assume that $n \equiv 2 \bmod 4$. Then we have $c_h(1) = 0$, and by Theorem 3.1 and Corollary to Theorem 4.2, we have

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)}$$

with some complex number u_1, where $V_{I_n(h)}$ is the $\overline{\mathbf{Q}}$-vector space associated with $I_n(h)$ in Theorem 3.1. Assume that $n \equiv 0 \bmod 4$. By Theorem 1.1 we have

$$\frac{\mathbf{M}^{(\chi)}(m, S(h))}{\pi^{mn}} \in \overline{\mathbf{Q}} u_-(S(h))^{n/2} \pi^{-n^2/4}.$$

Hence, again by Theorem 3.1 and Corollary to Theorem 4.2,

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)} + \overline{\mathbf{Q}} u_2$$

with complex numbers u_1 and u_2. This proves the assertion.

Proof of Theorem 2.2 and its corollary. Theorem 2.2 follows directly from Theorem 2.1. We note that $J_{n-1}(\chi_{(i_1,\cdots,i_r)})$ is a non-zero algebraic number by virtue of Corollary to Proposition 3.8. We also note that $\dfrac{\mathbf{L}_n(m,S(h),\eta)}{\pi^{m(n-2)}}$ belongs to $\overline{\mathbf{Q}}u_+(S(h))^{n/2-1}\pi^{-n^2/4+n/2}$, and nonzero for any integer $n/2+1 \le m \le k-n/2-1$ and primitive character η. This proves the corollary.

Proof of Theorem 2.3. The assertion follows from Theorem 3.2.

Now we give some comments. First we are interested in the dimension of $W_{h,E_{n/2+1/2}}$ over $\overline{\mathbf{Q}}$. Therefore we propose the following problem.

Problem 1. *Give* $\dim_{\overline{\mathbf{Q}}} W_{h,E_{n/2+1/2}}$ *explicitly or estimate it.*

This problem is reduced to the following problem:

Problem 2. *Give* $\dim_{\overline{\mathbf{Q}}} V_{I_n(h)}$ *explicitly or estimate it.*

Next we consider a generalization or a refinement of Theorem 2.1. Namely we propose the following conjecture.

Conjecture. *Let* $h_1(z)$ *be a Hecke eigenform in* $S^+_{k_1+1/2}(\Gamma_0(4))$ *and* $h_2(z) \in M_{k_2+1/2}(\Gamma_0(4))$ *with* $k_1 \ge k_2+2$. *Assume that* $c_{h_2}(m) \in \overline{\mathbf{Q}}$ *for any* $m \in \mathbf{Z}_{\ge 0}$. *Then there exists a finite dimensional* $\overline{\mathbf{Q}}$*-vector space* $W_{h_1,h_2} \subset \mathbf{C}$ *such that*

$$R(m,h_1,h_2,\chi)\pi^{-2m} \in W_{h_1,h_2}$$

for any $k_2+1 \le m \le k_1-1$ *and any primitive character* χ.

Problem 3. Prove Theorem 2.1 without using the relation between the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and the twisted Rankin-Selberg series of modular forms of half-integral weight.

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