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Author(s)	Katsurada, Hidenori
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# On the special values of certain L-series related to half-integral weight modular forms

Hidenori Katsurada\*

## Abstract

Let  $h$  be a cuspidal Hecke eigenform of half-integral weight, and  $E_{n/2+1/2}$  be Cohen's Eisenstein series of weight  $n/2+1/2$ . For a Dirichlet character  $\chi$  we define a certain linear combination  $R^{(\chi)}(s, h, E_{n/2+1/2})$  of the Rankin-Selberg convolution products of  $h$  and  $E_{n/2+1/2}$  twisted by Dirichlet characters related with  $\chi$ . We then prove a certain algebraicity result for  $R^{(\chi)}(l, h, E_{n/2+1/2})$  with  $l$  integers.

## 0 Introduction

For two modular forms  $h_1(z)$  and  $h_2(z)$  of half-integral weights  $k_1 + 1/2$  and  $k_2 + 1/2$ , respectively, for  $\Gamma_0(4)$ , and a primitive character  $\chi$  we define the Rankin-Selberg convolution product  $\tilde{R}(s, h_1, h_2, \chi)$  twisted by  $\chi$  as

$$\tilde{R}(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi_{-1}^{k_1 - k_2} \chi^2) \sum_{m=1}^{\infty} \frac{c_1(m)c_2(m)\chi(m)}{m^s},$$

where  $c_1(m)$  and  $c_2(m)$  denote the  $m$ -th Fourier coefficients of  $h_1$  and  $h_2$ , respectively, and  $L(s, \chi_{-1}^{k_1 - k_2} \chi^2)$  is the Dirichlet  $L$ -function for  $\chi_{-1}^{k_1 - k_2} \chi^2$  (for the precise definition of  $\chi_{-1}$  see Section 1.)

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The analytic properties of this Dirichlet series were investigated by Shimura [Sh2]. Furthermore the algebraicity of the values of this Dirichlet series at half-integers was deeply investigated by Shimura [Sh2]. However, as far as we know, there is no literature on the algebraicity of its special values at integers except for [K-M]. Therefore we naturally ask the following question:

**Question.** What can one say about the algebraicity of  $\tilde{R}(m, h_1, h_2, \chi)$  with  $m$  an integer?

In [K-M], we gave a partial answer to the above question in the case  $h_1$  is a cuspidal Hecke eigenform in Kohnen's plus subspace for  $\Gamma_0(4)$  and  $h_2$  is Zagier's Eisenstein series of weight  $3/2$ . In this paper, we consider the above question in the case  $h_1$  is a cuspidal Hecke eigenform in Kohnen's plus subspace for  $\Gamma_0(4)$  and  $h_2$  is Cohen's Eisenstein series. This paper is a summary of our paper [Ka], which will be published elsewhere. To state our main result more explicitly, we define another Dirichlet series  $R(s, h_1, h_2, \chi)$  by

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s}.$$

Assume that  $k_1 + k_2$  is even, and that the conductor of  $\chi$  is odd. Then, as will be explained in Section 1, it suffices to consider the above question for  $R(m, h_1, h_2, \chi)$  with integer  $m$ . Now let  $k$  and  $n$  be even integers such that  $n \geq 4$  and  $2k - n \geq 12$ . Let  $h$  be a Hecke eigenform of weight  $k - n/2 + 1/2$  for  $\Gamma_0(4)$  belonging to Kohnen's plus subspace, and  $S(h)$  the normalized Hecke eigenform of weight  $2k - n$  for  $SL_2(\mathbf{Z})$  corresponding to  $h$  under the Shimura correspondence. Moreover let  $E_{n/2+1/2}$  be Cohen's Eisenstein series of weight  $n/2 + 1/2$  (for the precise definition of  $E_{n/2+1/2}$ , see Section 2). Let  $\chi$  be a primitive character of conductor  $N$ . We assume that  $N$  is square free and let  $N = p_1 \cdots p_r$  be the prime decomposition of  $N$ . Put  $l_j = l_{n, p_j} = \text{G.C.D}(n, p_j - 1)$ . For an  $r$ -tuple  $(i_1, i_2, \dots, i_r)$  of integers put

$$\chi_{(i_1, \dots, i_r)} = \chi \prod_{j=1}^r \left( \frac{*}{p_j} \right)_{l_j}^{i_j},$$

where  $\left( \frac{*}{p_j} \right)_{l_j}$  denotes the  $l_j$ -th power residue symbol mod  $p_j$ . For two Dirich-

let characters  $\eta_1$  and  $\eta_2 \pmod N$ , we define  $J_m(\eta_1, \eta_2)$  by

$$J_m(\eta_1, \eta_2) = \sum_Z \eta_1(\det Z) \eta_2(1 - \operatorname{tr}(Z)),$$

where  $Z$  runs over all symmetric matrices of degree  $m$  with entries in  $\mathbf{Z}/N\mathbf{Z}$  and  $\operatorname{tr}(Z)$  denotes the trace of a matrix  $Z$ . We note that  $J_1(\eta_1, \eta_2)$  is the Jacobi sum  $J(\eta_1, \eta_2)$  associated with  $\eta_1$  and  $\eta_2$ . We also put  $J_m(\eta_1) = J_m(\eta_1 \left(\frac{*}{N}\right)^{m-1}, \eta_1)$ , where  $\left(\frac{*}{N}\right)$  is the Jacobi symbol. We then define

$$R^{(\chi)}(s, h, E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1, \dots, i_r)}(2^n)} R(s, h, E_{n/2+1/2}, \chi_{(i_1, \dots, i_r)}) \\ \times \overline{J(\chi_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right))} \overline{J_{n-1}(\chi_{(i_1, \dots, i_r)})} \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi_{(i_1, \dots, i_r)}^2),$$

where  $L(s, S(h), \chi_{(i_1, \dots, i_r)}^2)$  is Hecke's  $L$ -function of  $S(h)$  twisted by  $\chi_{(i_1, \dots, i_r)}^2$ . Then our main result (Theorem 2.1) can be stated as follows:

*There exists a finite dimensional  $\overline{\mathbf{Q}}$ -vector space  $W_{h, E_{n/2+1/2}}$  in  $\mathbf{C}$  such that*

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h, E_{n/2+1/2}}$$

*for any integer  $m$  such that  $n/2 + 1 \leq m \leq k - n/2 - 1$  and a character  $\chi$  of odd square free conductor such that  $\chi^n$  is primitive.*

From the above result we easily obtain the following (cf. Theorem 2.2):

*Let  $r > \dim_{\overline{\mathbf{Q}}} W_{h, E_{n/2+1/2}}$ . Let  $m_1, m_2, \dots, m_r$  be integers such that  $n/2 + 1 \leq m_1, m_2, \dots, m_r \leq k - n/2 - 1$  and  $\chi_1, \chi_2, \dots, \chi_r$  be Dirichlet characters of odd square free conductors  $N_1, N_2, \dots, N_r$ , respectively such that  $\chi_i^n$  is primitive for any  $i = 1, 2, \dots, r$ . Then the values  $\frac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \dots, \frac{R^{(\chi_r)}(m_r, h, E_{n/2+1/2})}{\pi^{m_r n}}$  are linearly dependent over  $\overline{\mathbf{Q}}$ .*

This is a certain generalization of a main result in [K-M] as will be explained later.

A main tool for proving Theorem 2.1 is the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift of  $h$ . To explain this, we define the twisted Koecher-Maaß series of a Siegel modular form in a more general setting. Let  $F(Z)$  be a modular form of weight  $k$  with respect to the symplectic group  $Sp_n(\mathbf{Z})$ . For a positive integer  $N$  let  $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \pmod{N}\}$ , and  $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$ . For a primitive Dirichlet character  $\chi \pmod{N}$  we define the Koecher-Maaß series  $L(s, F, \chi)$  of  $F$  twisted by  $\chi$  as

$$L(s, F, \chi) = \sum_T \frac{\chi(\text{tr}(T))c_F(T)}{e_N(T)(\det T)^s},$$

where  $T$  runs over a complete set of representatives of  $SL_{n,N}(\mathbf{Z})$ -equivalence classes of positive definite half-integral matrices of degree  $n$ , and  $c_F(T)$  denotes the  $T$ -th Fourier coefficient of  $F$ . We note that this Dirichlet series coincides with the Hecke  $L$ -function associated to  $F$  twisted by  $\chi$  in case  $n = 1$ . Though we are mainly concerned with  $L(s, F, \chi)$  in this paper, we also define another type of twisted Koecher-Maaß series  $L^*(s, F, \chi)$  as

$$L^*(s, F, \chi) = \sum_T \frac{\chi(\det(2T))c_F(T)}{e(T)(\det T)^s},$$

where  $T$  runs over a complete set of representatives of  $SL_n(\mathbf{Z})$ -equivalence classes of positive definite half-integral matrices of degree  $n$ , and  $e(T) = e_1(T)$ . These two Dirichlet series  $L(s, F, \chi)$  and  $L^*(s, F, \chi)$  essentially coincide with each other in case  $n = 1$ , but they don't in general. To distinguish these two Dirichlet series, we sometimes call  $L(s, F, \chi)$  and  $L^*(s, F, \chi)$  the twisted Koecher-Maaß series of the first and second kind, respectively. In Section 3, we will discuss a relation between these two Dirichlet series (cf. Theorem 3.5.) Now for the integers  $k$  and  $n$  stated above, let  $h$  a cuspidal Hecke eigenform  $h$  in Kohnen's plus subspace of weight  $k - n/2 + 1/2$  for  $\Gamma_0(4)$ . Let  $I_n(h)$  be the Duke-Imamoglu-Ikeda lift of  $h$  to the space of Siegel cusp forms of degree  $n$ . Then, in Section 4, first we give an explicit formula of  $L^*(s, I_n(h), \eta)$  in terms of the Rankin-Selberg series  $R(s, h, E_{n/2+1/2}, \eta)$  and shifted products of Hecke's  $L$ -functions of  $S(h)$  twisted by  $\eta^2$  in the case  $\eta$  is a primitive character (cf. Theorem 4.1.) Next, by this result combined with Theorem 3.5, we give an explicit formula of  $L(s, I_n(h), \chi^n)$  in terms of  $R^{(\chi)}(s, h, E_{n/2+1/2})$  and a sum of the shifted products  $\prod_{j=1}^{n/2-1} L(2s - 2j + 1, S(h), \chi_{(i_1, \dots, i_r)}^2)$  (cf.

Theorem 4.2 and its corollary.) This implies that  $R^{(\chi)}(s, h, E_{n/2+1/2})$  can be expressed in terms of  $L(s, I_n(h), \chi^n)$  and the sum of the shifted products. Thus we can prove our main result using the algebraicity of Hecke's  $L$ -function of  $S(h)$  (cf. Theorem 1.1) combined with the arithmetic properties of  $L(s, I_n(h), \chi^n)$ , which were investigated by Choie and Kohnen [C-K] in a more general setting (cf. Theorem 3.2). We can also prove a functional equation for  $R^{(\chi)}(s, h, E_{n/2+1/2})$  in case  $n \equiv 2 \pmod{4}$  using the functional equation for  $L(s, F, \chi^n)$  (cf. Theorem 2.3.)

**Notation.** We denote by  $e(x) = \exp(2\pi\sqrt{-1}x)$  for a complex number  $x$ . For a commutative ring  $R$ , we denote by  $M_{mn}(R)$  the set of  $(m, n)$ -matrices with entries in  $R$ . For an  $(m, n)$ -matrix  $X$  and an  $(m, m)$ -matrix  $A$ , we write  $A[X] = {}^tXAX$ , where  ${}^tX$  denotes the transpose of  $X$ . Let  $a$  be an element of  $R$ . Then for an element  $X$  of  $M_{mn}(R)$  we often use the same symbol  $X$  to denote the coset  $X \bmod aM_{mn}(R)$ . Put  $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$ , and  $SL_m(R) = \{A \in M_m(R) \mid \det A = 1\}$ , where  $\det A$  denotes the determinant of a square matrix  $A$  and  $R^*$  is the unit group of  $R$ . We denote by  $S_n(R)$  the set of symmetric matrices of degree  $n$  with entries in  $R$ . In particular, if  $S$  is a subset of  $S_n(\mathbf{R})$  with  $\mathbf{R}$  the field of real numbers, we denote by  $S_{>0}$  (resp.  $S_{\geq 0}$ ) the subset of  $S$  consisting of positive definite (resp. semi-positive definite) matrices. The group  $SL_n(\mathbf{Z})$  acts on the set  $S_n(\mathbf{R})$  in the following way:

$$SL_n(\mathbf{Z}) \times S_n(\mathbf{R}) \ni (g, A) \longrightarrow {}^tgAg \in S_n(\mathbf{R}).$$

Let  $G$  be a subgroup of  $GL_n(\mathbf{Z})$ . For a subset  $\mathcal{B}$  of  $S_n(\mathbf{R})$  stable under the action of  $G$  we denote by  $\mathcal{B}/G$  the set of equivalence classes of  $\mathcal{B}$  with respect to  $G$ . We sometimes identify  $\mathcal{B}/G$  with a complete set of representatives of  $\mathcal{B}/G$ . Two symmetric matrices  $A$  and  $A'$  with entries in  $R$  are said to be equivalent with each other with respect to  $G$  and write  $A \sim_G A'$  if there is an element  $X$  of  $G$  such that  $A' = A[X]$ . Let  $\mathcal{L}_n$  denote the set of half-integral matrices of degree  $n$  over  $\mathbf{Z}$ , that is,  $\mathcal{L}_n$  is the set of symmetric matrices of degree  $n$  whose  $(i, j)$ -component belongs to  $\mathbf{Z}$  or  $\frac{1}{2}\mathbf{Z}$  according as  $i = j$  or not.

# 1 Review on the algebraicity of L-values of elliptic modular forms of integral and half-integral weights

Before stating our main results, we review on the special values of L functions of elliptic modular forms of integral and half-integral weights. Put  $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$ , where  $1_n$  and  $O_n$  denotes the unit matrix and the zero matrix of degree  $n$ , respectively. Furthermore, put

$$Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let  $l$  be an integer or a half-integer, and let  $\Gamma$  be a congruence subgroup of  $Sp_n(\mathbf{Z})$ . We then denote by  $M_l(\Gamma)$  the space of modular forms of weight  $l$  with respect to  $\Gamma$ , and by  $S_l(\Gamma)$  the subspace of  $M_l(\Gamma)$  consisting of cusp forms. We also denote by  $\Gamma_0(4)$  the subgroup of  $SL_2(\mathbf{Z})$  consisting of matrices whose left lower entries are congruent to 0 mod  $N$ . Let

$$f(z) = \sum_{m=1}^{\infty} c_f(m) \mathbf{e}(mz)$$

be a normalized Hecke eigenform in  $S_k(SL_2(\mathbf{Z}))$ , and  $\chi$  be a primitive Dirichlet character. Then let us define Hecke's  $L$ -function  $L(s, f, \chi)$  of  $f$  twisted by  $\chi$  as

$$L(s, f, \chi) = \sum_{m=1}^{\infty} c_f(m) \chi(m) m^{-s}.$$

Then we have the following result (cf. [Sh1]):

**Theorem 1.1** *There exist complex numbers  $u_{\pm}(f)$  uniquely determined up to  $\overline{\mathbf{Q}}^{\times}$  multiple such that*

$$L(m, f, \chi) (\pi^m u_j(f))^{-1} \in \overline{\mathbf{Q}}$$

for any integer  $0 < m \leq k - 1$  and a primitive character  $\chi$ , where  $j = +$  or  $-$  according as  $(-1)^m \chi(-1) = 1$  or  $-1$ .

We remark that we have  $L(m, f, \chi) \neq 0$  if  $m \neq k/2$ , and  $L(k/2, f, \chi) \neq 0$  for infinitely many  $\chi$ .

Next let us consider the half-integral weight case. Let

$$h_1(z) = \sum_{m=1}^{\infty} c_{h_1}(m) \mathbf{e}(mz)$$

be a Hecke eigenform in  $S_{k_1+1/2}(\Gamma_0(4))$ , and

$$h_2(z) = \sum_{m=0}^{\infty} c_{h_2}(m) \mathbf{e}(mz)$$

be an element of  $M_{k_2+1/2}(\Gamma_0(4))$ . For positive integers  $e$  and  $l$ , let  $\chi_{(-1)^le}$  be the Dirichlet character corresponding to the extension  $\mathbf{Q}(\sqrt{(-1)^le}/\mathbf{Q})$ . Let  $\chi$  be a primitive character mod  $N$ . Then we define

$$\tilde{R}(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \omega) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s},$$

where  $\omega(d) = \chi_{-1}^{k_1-k_2} \chi^2(d)$ . Now let  $S(h_1)$  be the normalized Hecke eigenform in  $S_{2k_1}(SL_2(\mathbf{Z}))$  corresponding to  $h_1$  under the Shimura correspondence. Then the following result is due to Shimura [Sh2].

**Theorem 1.2** *Assume that  $k_1 > k_2$ . Under the above notation we have*

$$\tilde{R}(m + 1/2, h_1, h_2, \chi) (u_-(S(h_1))\pi^{-k_2+1+2m})^{-1} \in \overline{\mathbf{Q}}(h_1)\overline{\mathbf{Q}}(h_2)$$

for any integer  $k_2 \leq m \leq k_1 - 1$  and a primitive character  $\chi$ , where  $\overline{\mathbf{Q}}(h_i)$  is the field, generated over  $\overline{\mathbf{Q}}$ , by all the Fourier coefficients of  $h_i$ .

**Corollary** *Let the notation be as above. Assume that  $k_1 > k_2$  and that  $c_{h_1}(n), c_{h_2}(n) \in \overline{\mathbf{Q}}$  for any  $n \in \mathbf{Z}_{\geq 0}$ . Then there exists a one-dimensional  $\overline{\mathbf{Q}}$ -vector space  $U_{h_1, h_2}$  in  $\mathbf{C}$  such that*

$$\tilde{R}(m + 1/2, h_1, h_2, \chi) \pi^{-2m} \in U_{h_1, h_2}$$

for any integer  $k_2 \leq m \leq k_1 - 1$  and a primitive character  $\chi$ .

Now we consider the values of  $\tilde{R}(s, h_1, h_2, \chi)$  at integers. Let

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s}.$$



be the Dirichlet series defined in Section 0. Assume that  $k_1 + k_2$  is even, and that the conductor of  $\chi$  is odd. Then we have

$$R(s, h_1, h_2, \chi) = (1 - 2^{-2s+k_1+k_2-1}\chi^2(2))^{-1}\tilde{R}(s, h_1, h_2, \chi).$$

Hence it suffices to consider the question in Section 0 for  $R(m, h_1, h_2, \chi)$  with integer  $m$ .

## 2 Main results

For a non-negative integer  $m$  and a positive integer  $l$ , Cohen's function  $H(l, m)$  is given by  $H(l, m) = L_{-m}(1 - l)$ . Here

$$L_D(s) = \begin{cases} \zeta(2s - 1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a)\chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D \neq 0, D \equiv 0, 1 \pmod{4} \\ 0, & D \equiv 2, 3 \pmod{4}, \end{cases}$$

where the positive integer  $f$  is defined by  $D = D_K f^2$  with the discriminant  $D_K$  of  $K = \mathbf{Q}(\sqrt{D})$ ,  $\chi_{D_K}$  is the Kronecker symbol,  $\mu$  is the Möbius function and  $\sigma_s(n) = \sum_{d|n} d^s$ . Furthermore we define Cohen's Eisenstein series  $E_{l+1/2}(z)$  by

$$E_{l+1/2}(z) = \sum_{m=0}^{\infty} H(l, m)\mathbf{e}(mz).$$

It is known that  $E_{l+1/2}(z)$  is a modular form of weight  $l + 1/2$  belonging to Kohnen's plus space. Let  $k$  and  $n$  be positive even integers such that  $n \geq 4$ ,  $2k - n \geq 12$ . Let  $h(z)$  be a Hecke eigenform in Kohnen's plus subspace  $S_{k-n/2+1/2}^+(\Gamma_0(4))$  (cf. [Ko]), and  $S(h)$  be the normalized Hecke eigenform in  $S_{2k-n}(SL_2(\mathbf{Z}))$  corresponding to  $h$  under the Shimura correspondence. Let  $p$  be a prime number and  $l$  be a positive integer dividing  $p - 1$ . Take an  $l$ -th root of unity  $\zeta_l$  and a prime ideal  $\mathfrak{p}$  of  $\mathbf{Q}(\zeta_l)$  lying above  $p$ . Let  $a$  be an integer prime to  $p$ . Then we have  $a^{(p-1)/l} \equiv \zeta_l^i \pmod{\mathfrak{p}}$  with some  $i \in \mathbf{Z}$ . We then put  $\left(\frac{a}{p}\right)_l = \zeta^i$ . We call  $\left(\frac{*}{p}\right)_l$  the  $l$ -th power residue symbol mod  $p$ . In the case

$l = 2$ , this is the Legendre symbol, and we write it as  $\left(\frac{*}{p}\right)$  as usual. We

note that this definition of the power residue symbol is different from the usual one, and depends on the choice of  $\mathfrak{p}$  and  $\zeta_l$  except the case  $l = 2$ . We denote by  $\left(\frac{*}{N}\right)$  the Jacobi symbol for a positive odd integer  $M$ . Let  $\chi$  be a primitive Dirichlet character of conductor  $N$ . We assume that  $N$  is a square free odd integer, and write  $N = p_1 \cdots p_r$  with  $p_1, \dots, p_r$  prime numbers. Put  $l_j = l_{n, p_j} = \text{G.C.D}(n, p_j - 1)$ . For an  $r$ -tuple  $(i_1, i_2, \dots, i_r)$  of integers put

$$\chi_{(i_1, \dots, i_r)} = \chi \prod_{j=1}^r \left(\frac{*}{p_j}\right)_{l_j}^{i_j}.$$

For two Dirichlet characters  $\eta_1$  and  $\eta_2 \pmod{N}$ , let  $J_m(\eta_1, \eta_2)$  and  $J_m(\eta_1)$  be as those defined in Section 0. By definition,  $J_m(\eta_1, \eta_2)$  is an algebraic number. As in Section 0, we define

$$\begin{aligned} & R^{(\chi)}(s, h, E_{n/2+1/2}) \\ &= \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1, \dots, i_r)}(2^n) J(\chi_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right) J_{n-1}(\chi_{(i_1, \dots, i_r)})} \\ & \quad \times R(s, h, E_{n/2+1/2}, \chi_{(i_1, \dots, i_r)}) \mathbf{L}_n(s, S(h), \chi_{(i_1, \dots, i_r)}), \end{aligned}$$

where

$$\mathbf{L}_n(s, S(h), \eta) = \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \eta^2)$$

for a primitive character  $\eta$ . We note that  $R^{(\chi)}(s, h, E_{n/2+1/2})$  does not depend on the choice of an  $l_i$ -th root of unity  $\zeta_{l_i}$  and an prime ideal  $\mathfrak{p}_i$  of  $\mathbf{Q}(\zeta_{l_i})$  lying above  $p_i$ .

**Remark.** (1) Let  $m$  be an integer s.t.  $n/2 + 1 \leq m \leq k - n/2 - 1$ . Then the value  $\frac{\mathbf{L}_n(m, S(h), \chi_{(i_1, \dots, i_r)}^2)}{\pi^{m(n-2)}}$  belongs to  $\overline{\mathbf{Q}}u_+(S(h))^{n/2-1} \pi^{-n^2/4+n/2}$  for any  $\chi$ .

In particular if  $n \equiv 2 \pmod{4}$ , then it is nonzero for any  $\chi$ , and if  $n \equiv 0 \pmod{4}$ , then it is nonzero for infinitely many  $\chi$ .

(2) As will be stated in Section 3,  $J_{n-1}(\chi_{(i_1, \dots, i_r)})$  is expressed as a product of Jacobi sums, and it is non-zero algebraic number if  $\chi^n$  is rewrote.

**Theorem 2.1** *There exists a finite dimensional  $\overline{\mathbf{Q}}$ -vector space  $W_{h, E_{n/2+1/2}}$  in  $\mathbf{C}$  such that*

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h, E_{n/2+1/2}}$$

for any integer  $n/2 + 1 \leq m \leq k - n/2 - 1$  and a character  $\chi$  of odd square free conductor such that  $\chi^n$  is rewrote.

**Theorem 2.2** Let  $r > \dim_{\overline{\mathbf{Q}}} W_{h, E_{n/2+1/2}}$ . Let  $m_1, m_2, \dots, m_r$  be integers such that  $n/2 + 1 \leq m_1, m_2, \dots, m_r \leq k - n/2 - 1$  and  $\chi_1, \chi_2, \dots, \chi_r$  be Dirichlet characters of odd square free conductors  $N_1, N_2, \dots, N_r$ , respectively such that  $\chi_i^n$  is primitive for any  $i = 1, 2, \dots, r$ . Then the values  $\frac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \dots, \frac{R^{(\chi_r)}(m_r, h, E_{n/2+1/2})}{\pi^{m_r n}}$  are linearly dependent over  $\overline{\mathbf{Q}}$ .

**Corollary** Assume that  $n \equiv 2 \pmod{4}$ . Let  $r$  and  $m_1, m_2, \dots, m_r$  be as above. Let  $\chi_1, \chi_2, \dots, \chi_r$  be Dirichlet characters of odd prime conductors  $p_1, p_2, \dots, p_r$ , respectively such that  $\chi_i^n$  is non-trivial for any  $i = 1, 2, \dots, r$ . Put  $l_i = \text{GCD}(n, p_i - 1)$ . Then the values  $\left\{ \frac{R(m_i, h, E_{n/2+1/2}, \chi_{i(j)})}{\pi^{2m_i}} \right\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$  are linearly dependent over  $\overline{\mathbf{Q}}$ .

We also have a functional equation for  $R^{(\chi)}(s, h, E_{n/2+1/2})$  :

**Theorem 2.3** Let  $h$  be as above. Let  $\chi$  be a primitive character of odd square free conductor  $N$ . Assume that  $n \equiv 2 \pmod{4}$ , and that  $\chi^n$  is primitive. Put

$$\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^n)^{-1} \gamma_n(s) R^{(\chi)}(s, h, E_{n/2+1/2}),$$

where  $\tau(\chi^n)$  is the Gauss sum of  $\chi^n$ , and

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^n \pi^{(i-1)/2} \Gamma(s - (i-1)/2).$$

Then  $\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2})$  has an analytic continuation to the whole  $s$ -plane, and has the following functional equation:

$$\mathcal{R}^{(\chi)}(k - s, h, E_{n/2+1/2}) = \mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}).$$

**Remark.** (1) The series  $\{R(s, h, E_{n/2+1/2}, \chi_{i(j)})\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$  are linearly independent over  $\mathbf{C}$  as functions of  $s$ .

(2) In the case of  $n = 2$ , this type of result was given for  $R(m, h, E_{3/2})$  with  $E_{3/2}$  Zagier's Eisenstein series of weight  $3/2$  by [K-M]. Cohen's Eisenstein

series is a holomorphic modular form, where as Zagier's Eisenstein series is not. Nevertheless, the former can be regarded as a generalization of the latter. Therefore, our present result can be regarded as a generalization of [K-M]. (3) The meromorphy of this type of series was derived in [Sh2] by using so called the Rankin-Selberg integral expression in a more general setting, but we don't know whether the functional equation of the above type can be directly proved without using the above method.

### 3 Twisted Koecher-Maaß series

To prove the main results, in this section and the next, we consider the twisted Koecher-Maaß series of a Siegel modular form. Let  $F(Z) \in M_k(Sp_n(\mathbf{Z}))$ . Then  $F(Z)$  has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{L}_n \geq 0} c_F(T) \mathbf{e}(\text{tr}(TZ)),$$

where  $\text{tr}(X)$  denotes the trace of a matrix  $X$ . For  $N \in \mathbf{Z}_{>0}$ , put  $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \pmod{N}\}$ , and for  $T \in \mathcal{L}_n \geq 0$  put  $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$ . For a primitive Dirichlet character  $\chi \pmod{N}$  Let

$$L(s, F, \chi) = \sum_{T \in \mathcal{L}_n \geq 0 / SL_{n,N}(\mathbf{Z})} \frac{\chi(\text{tr}(T)) c_F(T)}{e_N(T) (\det T)^s}$$

be the twisted Koecher-Maaß series of  $F$  of the first kind as in Section 0. The following two theorems are due to Choie and Kohnen [C-K].

**Theorem 3.1** *Let  $F \in S_k(Sp_n(\mathbf{Z}))$ , and  $\chi$  a primitive character of conductor  $N$ . Put*

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^n \pi^{(i-1)/2} \Gamma(s - (i-1)/2),$$

and

$$\Lambda(s, F, \chi) = N^{2s} \tau(\chi)^{-1} \gamma_n(s) L(s, F, \chi) \quad (\text{Re}(s) \gg 0),$$

where  $\tau(\chi)$  is the Gauss sum of  $\chi$ . Then  $\Lambda(s, F, \chi)$  has an analytic continuation to the whole  $s$ -plane and has the following functional equation:

$$\Lambda(k - s, F, \chi) = (-1)^{nk/2} \chi(-1) \Lambda(s, F, \bar{\chi}).$$

**Theorem 3.2** *Let  $F$  and  $\chi$  be as above. Then there exists a finite dimensional  $\overline{\mathbf{Q}}$ -vector space  $V_F$  in  $\mathbf{C}$  such that*

$$L(m, F, \chi)\pi^{-nm} \in V_F$$

for any primitive character  $\chi$  and any integer  $m$  such that  $(n+1)/2 \leq m \leq k - (n+1)/2$ .

**Example.** Let  $n = 1$ . Take a basis  $\{f_1, \dots, f_d\}$  of  $S_k(SL_2(\mathbf{Z}))$  consisting of normalized Hecke eigenforms. Write  $f \in S_k(SL_2(\mathbf{Z}))$  as

$$f = a_1 f_1 + \dots + a_d f_d$$

with  $a_1, \dots, a_d \in \mathbf{C}$ . Then put  $w_i = a_i u_+(f_i)$ ,  $w_{d+i} = a_i u_-(f_i)$  ( $i = 1, \dots, d$ )

and  $V_f = \sum_{i=1}^{2d} \overline{\mathbf{Q}} w_i$ . Then  $V_f$  satisfies the required property for  $f$ .

Now let

$$L^*(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{\chi(\det(2T))c_F(T)}{e(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of  $F$  of the second kind as in Section 0. We will discuss a relation between these two Dirichlet series. Let  $N$  be a positive integer. Let  $g$  be a periodic function on  $\mathbf{Z}$  with a period  $N$  and  $\phi$  a polynomial in  $t_1, \dots, t_r$ . Then for an element  $u = (a_1 \bmod N, \dots, a_r \bmod N) \in (\mathbf{Z}/N\mathbf{Z})^r$ , the value  $g(\phi(a_1, \dots, a_r))$  does not depend on the choice of the representative  $u$ . Therefore we denote this value by  $g(\phi(u))$ . Now let  $\chi$  be a primitive character mod  $N$ . For  $A \in \mathcal{L}_{n>0}$ , put

$$h(A, \chi) = \sum_{U \in SL_n(\mathbf{Z}/N\mathbf{Z})} \chi(\text{tr}(A[U])).$$

The following proposition is due to [[K-M], Proposition 3.1].

**Proposition 3.3** *Let*

$$F(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} c_F(A) \mathbf{e}(\text{tr}(AZ))$$

be an element of  $M_k(Sp_n(\mathbf{Z}))$ . Let  $\chi$  be a Dirichlet character mod  $N$ . Assume  $N \neq 2$ . Then we have

$$L(s, F, \chi) = \sum_{A \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{c_F(A)h(A, \chi)}{e(A)(\det A)^s}.$$

For a Dirichlet character  $\chi \pmod N$ , let  $\chi^{(p)}$  be the  $p$ -factor of  $\chi$  so that  $\chi = \prod_{p|N} \chi^{(p)}$ . For a prime number  $p$  put

$$\gamma_{n,p} = p^{n^2-n(n+1)/2} (1 - p^{-n/2}) \prod_{e=1}^{(n-2)/2} (1 - p^{-2e})$$

or

$$\gamma_{n,p} = p^{n^2-n(n+1)/2} \prod_{e=1}^{(n-1)/2} (1 - p^{-2e})$$

according as  $n$  is even or odd. The following result is a technical tool for proving our main result.

**Theorem 3.4** *Let  $A \in \mathcal{L}_{n>0}$ . Let  $N$  be a square free odd integer, and let  $N = \prod_{i=1}^r p_i$  be the prime decomposition of  $N$ . Let  $\chi$  be a primitive Dirichlet character mod  $N$ . For each positive integer  $i \leq r$ , put  $l_i = \text{G.C.D}(n, p_i - 1)$  and let  $u_{0,i}$  be a primitive  $l_i$ -th root of unity mod  $p_i$ .*

- (1). *If  $\chi^{(p_i)}(u_{0,i}) \neq 1$  for some  $i$ . Then we have  $h(A, \chi) = 0$ .*
  - (2). *Assume that  $\chi^{(p_i)}(u_{0,i}) = 1$  for any  $i$ . Fix a character  $\tilde{\chi}$  such that  $\tilde{\chi}^n = \chi$ .*
- (2.1) *Let  $n$  be even. Then we have*

$$h(A, \chi) = \prod_{i=1}^r (-1)^{n(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) \tilde{\chi}_{(i_1, \dots, i_r)}(\det(2A)) J(\tilde{\chi}_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)})}.$$

- (2.2) *Let  $n$  be odd, and assume that  $\chi^2$  is primitive. Then we have*

$$h(A, \chi) = \prod_{i=1}^r (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}_{(i_1, \dots, i_r)}(2^n) \tilde{\chi}_{(i_1, \dots, i_r)}(\det(2A)) \overline{J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)})}.$$

The proof of the above theorem is elementary but is rather lengthy. The details will be given in [Ka].

**Remark.** Let  $\eta$  be a primitive Dirichlet character of odd prime conductor  $p$ . Assume that  $\eta^2 \neq 1$ . Then we can prove that we have

$$J\left(\eta, \left(\frac{*}{p}\right)\right) J\left(\eta \left(\frac{*}{p}\right), \eta \left(\frac{*}{p}\right)\right) = \left(\frac{-1}{p}\right) \bar{\eta}(4)p.$$

(This is not so trivial. For the details, see [Ka].) Hence for  $A \in \mathcal{L}_{2>0}$  and a primitive character  $\chi$  of odd square free conductor  $N$  such that  $\chi^{(p)}(-1) = 1$  for any prime divisor  $p$  of  $N$ , we have

$$h(A, \chi) = \prod_{p|N} \left\{ \left(1 + \left(\frac{4 \det A}{p}\right)\right) \left(1 - \left(\frac{-1}{p}\right) p^{-1}\right) \right\} N^2 \left(\frac{-1}{N}\right) \tilde{\chi}(4 \det A),$$

where  $\tilde{\chi}$  is a character such that  $\tilde{\chi}^2 = \chi$ . This coincides with (2) of Theorem 3.8 in [K-M].

By Theorem 3.4 and Proposition 3.3 we easily obtain:

**Theorem 3.5** *Let  $N, p_i, l_i, u_{0,i}$  ( $i = 1, \dots, r$ ) and  $\chi$  be as in Theorem 3.4, and let  $F$  be an element of  $M_k(\mathrm{Sp}_n(\mathbf{Z}))$ .*

- (1). *If  $\chi^{(p_i)}(u_{0,i}) \neq 1$  for some  $i$ . Then we have  $L(s, F, \chi) = 0$ .*
  - (2). *Assume that  $\chi^{(p_i)}(u_{0,i}) = 1$  for any  $i$ . Fix a character  $\tilde{\chi}$  such that  $\tilde{\chi}^n = \chi$ .*
- (2.1) *Let  $n$  be even. Then we have*

$$L(s, F, \chi) = \prod_{i=1}^r (-1)^{n(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) J(\tilde{\chi}_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right) J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)}) L^*(s, F, \tilde{\chi}_{(i_1, i_2, \dots, i_r)})}.$$

(2.2) *Let  $n$  be odd, and assume that  $\chi^2$  is primitive. Then we have*

$$L(s, F, \chi) = \prod_{i=1}^r (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) J_{n-1}(\tilde{\chi}_{(i_1, i_2, \dots, i_r)}) L^*(s, F, \tilde{\chi}_{(i_1, i_2, \dots, i_r)})}.$$

To give an explicit formula of  $J_m(\chi, \eta)$  for primitive characters  $\chi, \eta$  mod  $N$ , we define  $I_m(\chi, \eta)$  as

$$I_m(\chi, \eta) = \sum_{Z \in S_m(\mathbf{Z}/N\mathbf{Z})} \chi(\det Z) \eta(\operatorname{tr}(Z)).$$

Then we have the following two propositions, whose proof will be given precisely in [Ka].

**Proposition 3.6** *Let  $\chi$  and  $\eta$  be primitive character mod an odd prime number  $p$ . Assume that  $\chi^2 \neq 1$  and that  $\eta$  is non-trivial. Put  $c_m(\chi, \eta) = 1$  or  $0$  according as  $\chi^{m-1}\eta = 1$  or not.*

(1) *Assume that  $m$  is odd. Then*

$$I_m(\chi, \eta) = c_m(\chi, \eta) \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2} (p-1) J_{m-1}\left(\chi \left(\frac{*}{p}\right), \eta\right).$$

(2) *Assume that  $m$  is even. Then*

$$I_m(\chi, \eta) = c_m(\chi, \eta) \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2} (p-1) \chi(-1) J\left(\chi, \left(\frac{*}{p}\right)\right) J_{m-1}\left(\chi \left(\frac{*}{p}\right), \eta\right).$$

**Proposition 3.7** *Let  $\chi, \eta$  and  $p$  be as in Proposition 3.6.*

(1) *Assume that  $m$  is odd. Then*

$$J_m(\chi, \eta) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2} \\ \times \{J(\chi, \chi^{m-1}\eta) J_{m-1}\left(\chi \left(\frac{*}{p}\right), \eta\right) + \eta(-1) I_{m-1}\left(\chi \left(\frac{*}{p}\right), \eta\right)\}.$$

(2) *Assume that  $m$  is even. Then*

$$J_m(\chi, \eta) = \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2} J\left(\chi, \left(\frac{*}{p}\right)\right) \\ \times \{J\left(\chi, \chi^{m-1} \left(\frac{*}{p}\right) \eta\right) J_{m-1}\left(\chi \left(\frac{*}{p}\right), \eta\right) + \eta(-1) I_{m-1}\left(\chi \left(\frac{*}{p}\right), \eta\right)\}.$$

From the above two propositions we have the following:



**Theorem 3.8** *Let  $\chi$  be a primitive character with a prime conductor  $p$  such that  $\chi^2 \neq 1$ .*

(1) *Let  $m$  be odd.*

(1.1) *Assume that  $\chi^m \neq 1$ . Then*

$$J_m\left(\chi\left(\frac{*}{p}\right)^i, \chi\right) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2} J\left(\chi\left(\frac{*}{p}\right)^i, \chi^m\right) J_{m-1}\left(\chi\left(\frac{*}{p}\right)^{i+1}, \chi\right).$$

(1.2) *Assume that  $\chi^m = 1$ . Then*

$$J_m\left(\chi\left(\frac{*}{p}\right)^i, \chi\right) = p^{m-1} \left(\frac{-1}{p}\right)^{i+1} J\left(\chi\left(\frac{*}{p}\right)^{i+1}, \left(\frac{*}{p}\right)\right) J_{m-2}\left(\chi\left(\frac{*}{p}\right)^i, \chi\right).$$

(2) *Let  $m$  be even.*

(2.1) *Assume that  $\chi^m \left(\frac{*}{p}\right)^{i+1} \neq 1$ . Then*

$$\begin{aligned} & J_m\left(\chi\left(\frac{*}{p}\right)^i, \chi\right) \\ &= \left(\frac{-1}{p}\right)^{m/2-1} J\left(\chi\left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)\right) J\left(\chi\left(\frac{*}{p}\right)^{i+1}, \chi^m\left(\frac{*}{p}\right)^{i+1}\right) J_{m-1}\left(\chi\left(\frac{*}{p}\right)^{i+1}, \chi\right). \end{aligned}$$

(2.2) *Assume that  $\chi^m \left(\frac{*}{p}\right)^{i+1} = 1$ . Then*

$$J_m\left(\chi\left(\frac{*}{p}\right)^i, \chi\right) = \chi(-1) p^{m-1} J\left(\chi\left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)\right) J_{m-2}\left(\chi\left(\frac{*}{p}\right)^i, \chi\right).$$

**Corollary** *Let  $\chi$  be a primitive character with an odd square free conductor  $N$ . Assume that  $\chi^2$  is primitive. Then the value  $J_m(\chi)$  is nonzero.*

## 4 An explicit formula for the twisted Koecher-Maaß series of the D-I-I lift

Throughout this section and the next, we assume that  $n$  and  $k$  are even positive integers. Let  $h$  be a Hecke eigenform of weight  $k-n/2+1/2$  belonging to Kohnen's plus space. Then  $h$  has the following Fourier expansion:

$$h(z) = \sum_e c_h(e) e(ez),$$

where  $e$  runs over all positive integers such that  $(-1)^{k-n/2}e \equiv 0, 1 \pmod{4}$ . Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m) \mathbf{e}(mz)$$

be the normalized Hecke eigenform of weight  $2k - n$  with respect to  $SL_2(\mathbf{Z})$  corresponding to  $h$  under the Shimura correspondence. For a prime number  $p$  let  $\beta_p$  be a non-zero complex number such that  $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2} c_{S(h)}(p)$ . For a prime number  $p$ , let  $\mathbf{Q}_p$ , and  $\mathbf{Z}_p$  be the field of  $p$ -adic numbers, and the ring of  $p$ -adic integers, respectively. We denote by  $\nu_p$  the additive valuation on  $\mathbf{Q}_p$  normalized so that  $\nu_p(p) = 1$ , and by  $\mathbf{e}_p$  the continuous homomorphism from the additive group  $\mathbf{Q}_p$  to  $\mathbf{C}^\times$  such that  $\mathbf{e}_p(x) = \mathbf{e}(x)$  for  $x \in \mathbf{Z}[p^{-1}]$ . For a positive definite half integral matrix  $T$  of degree  $n$  write  $(-1)^{n/2} \det(2T)$  as  $(-1)^{n/2} \det(2T) = \mathfrak{d}_T \mathfrak{f}_T^2$  with  $\mathfrak{d}_T$  a fundamental discriminant and  $\mathfrak{f}_T$  a positive integer. We then define the local Siegel series  $b_p(T, s)$  by

$$b_p(T, s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR)) p^{-\nu_p(\mu_p(R))s} \quad (s \in \mathbf{C})$$

for each prime number  $p$ , where  $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$ . Then there exists a polynomial  $F_p(T, X)$  in  $X$  such that

$$b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \left(\frac{\mathfrak{d}_T}{p}\right) p^{n/2-s})^{-1} \prod_{i=1}^{n/2} (1 - p^{2i-2s})$$

(cf. [Ki].) We then put

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T|) \prod_p (p^{k-n/2-1/2} \beta_p)^{\nu_p(\mathfrak{f}_T)} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).$$

We note that  $c_{I_n(h)}(T)$  does not depend on the choice of  $\beta_p$ . Define a Fourier series  $I_n(h)(Z)$  by

$$I_n(h)(Z) = \sum_{T \in \mathcal{L}_{n>0}} c_{I_n(h)}(T) \mathbf{e}(\mathrm{tr}(TZ)).$$

In [I] Ikeda showed that  $I_n(h)(Z)$  is a cuspidal Hecke eigenform in  $S_k(Sp_n(\mathbf{Z}))$  and its standard  $L$ -function  $L(s, I_n(h), \mathrm{St})$  is given by

$$L(s, I_n(h), \mathrm{St}) = \zeta(s) \prod_{i=1}^n L(s + k - i, S(h)).$$

We call  $I_n(h)$  the Duke-Imamoglu-Ikeda lift (D-I-I lift) of  $h$ . Now using the same argument as in the proof of Theorem 1 of [I-K] we obtain the following. For the details see [Ka].

**Theorem 4.1** *Let  $\chi$  be a primitive Dirichlet character mod  $N$ . Then we have*

$$L^*(s, F, \chi) = 2^{ns} \{c_n R(s, h, E_{n/2+1/2}, \chi) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \chi^2) \\ + d_n c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1, S(h), \chi^2)\},$$

where  $c_n$  and  $d_n$  are non-zero rational numbers depending only on  $n$ .

Now by the above theorem combined with Theorem 3.5 we obtain:

**Theorem 4.2** *Let  $N$  be a square free odd integer, and  $N = p_1 \cdots p_r$  be the prime decomposition of  $N$ . For each  $i = 1, \dots, r$  let  $l_i = \text{G.C.D}(n, p_i - 1)$  and  $u_0 \in \mathbf{Z}$  be a primitive  $l_i$ -th root of unity mod  $p_i$ .*

- (1) *Assume  $\chi^{(p_i)}(u_i) \neq 1$  for some  $i$ . Then  $L(s, I_n(h), \chi) = 0$ .*  
(2) *Assume  $\chi^{(p_i)}(u_i) = 1$  for any  $i$ . Then*

$$L(s, I_n(h), \chi) = 2^{ns} \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) J(\tilde{\chi}_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) \overline{J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)})}} \\ \times \{c_{n,N} R(s, h, E_{n/2+1/2}, \tilde{\chi}_{(i_1, \dots, i_r)}) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \tilde{\chi}_{(i_1, \dots, i_r)}^2) \\ + d_{n,N} c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1, S(h), \tilde{\chi}_{(i_1, \dots, i_r)}^2)\},$$

where  $c_{n,N}$  and  $d_{n,N}$  are non-zero rational numbers depending only on  $n$  and  $N$ , and  $\tilde{\chi}$  is a character s.t.  $\tilde{\chi}^n = \chi$ .

**Remark.** In the case  $n = 2$ , an explicit formula for  $L(s, I_2(h), \chi)$  was given by Katsurada-Mizuno [K-M].

**Corollary** *Let  $\chi$  be a Dirichlet character of odd square free conductor  $N$  such that  $\chi^n$  is primitive. Then for any integer  $n/2 + 1 \leq m \leq k - n/2 - 1$*

$$\begin{aligned} & \frac{L(m, I_n(h), \chi^n)}{\pi^{mn}} \\ &= \left\{ \gamma_{n,N} \frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} + \delta_{n,N} c_h(1) \frac{\mathbf{M}^{(\chi)}(m, S(h))}{\pi^{mn}} \right\}, \end{aligned}$$

where  $\gamma_{n,N}$  and  $\delta_{n,N}$  are non-zero numbers, and

$$\begin{aligned} \mathbf{M}^{(\chi)}(m, S(h)) &= \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \frac{\chi_{(i_1, \dots, i_r)}(2^n) J(\chi_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) J_{n-1}(\chi_{(i_1, \dots, i_r)})}{\pi^{mn}} \\ &\quad \times \prod_{j=1}^{n/2} L(2m - 2j + 1, S(h), (\chi_{(i_1, \dots, i_r)})^2). \end{aligned}$$

## 5 Proof of main results and some comments

We prove the results in Section 2.

**Proof of Theorem 2.1.** Assume that  $n \equiv 2 \pmod{4}$ . Then we have  $c_h(1) = 0$ , and by Theorem 3.1 and Corollary to Theorem 4.2, we have

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)}$$

with some complex number  $u_1$ , where  $V_{I_n(h)}$  is the  $\overline{\mathbf{Q}}$ -vector space associated with  $I_n(h)$  in Theorem 3.1. Assume that  $n \equiv 0 \pmod{4}$ . By Theorem 1.1 we have

$$\frac{\mathbf{M}^{(\chi)}(m, S(h))}{\pi^{mn}} \in \overline{\mathbf{Q}} u_-(S(h))^{n/2} \pi^{-n^2/4}.$$

Hence, again by Theorem 3.1 and Corollary to Theorem 4.2,

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)} + \overline{\mathbf{Q}} u_2$$

with complex numbers  $u_1$  and  $u_2$ . This proves the assertion.

**Proof of Theorem 2.2 and its corollary.** Theorem 2.2 follows directly from Theorem 2.1. We note that  $J_{n-1}(\chi_{(i_1, \dots, i_r)})$  is a non-zero algebraic number by virtue of Corollary to Proposition 3.8. We also note that  $\frac{\mathbf{L}_n(m, S(h), \eta)}{\pi^{m(n-2)}}$  belongs to  $\overline{\mathbf{Q}}u_+(S(h))^{n/2-1}\pi^{-n^2/4+n/2}$ , and nonzero for any integer  $n/2 + 1 \leq m \leq k - n/2 - 1$  and primitive character  $\eta$ . This proves the corollary.

**Proof of Theorem 2.3.** The assertion follows from Theorem 3.2.

Now we give some comments. First we are interested in the dimension of  $W_{h, E_{n/2+1/2}}$  over  $\overline{\mathbf{Q}}$ . Therefore we propose the following problem.

**Problem 1.** Give  $\dim_{\overline{\mathbf{Q}}} W_{h, E_{n/2+1/2}}$  explicitly or estimate it.

This problem is reduced to the following problem:

**Problem 2.** Give  $\dim_{\overline{\mathbf{Q}}} V_{I_n(h)}$  explicitly or estimate it.

Next we consider a generalization or a refinement of Theorem 2.1. Namely we propose the following conjecture.

**Conjecture.** Let  $h_1(z)$  be a Hecke eigenform in  $S_{k_1+1/2}^+(\Gamma_0(4))$  and  $h_2(z) \in M_{k_2+1/2}(\Gamma_0(4))$  with  $k_1 \geq k_2 + 2$ . Assume that  $c_{h_2}(m) \in \overline{\mathbf{Q}}$  for any  $m \in \mathbf{Z}_{\geq 0}$ . Then there exists a finite dimensional  $\overline{\mathbf{Q}}$ -vector space  $W_{h_1, h_2} \subset \mathbf{C}$  such that

$$R(m, h_1, h_2, \chi)\pi^{-2m} \in W_{h_1, h_2}$$

for any  $k_2 + 1 \leq m \leq k_1 - 1$  and any primitive character  $\chi$ .

**Problem 3.** Prove Theorem 2.1 without using the relation between the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and the twisted Rankin-Selberg series of modular forms of half-integral weight.

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Hidenori KATSURADA

Muroran Institute of Technology

27-1 Mizumoto, Muroran, 050-8585, Japan

E-mail: [hidenori@mmm.muroran-it.ac.jp](mailto:hidenori@mmm.muroran-it.ac.jp)