Evolutes and involutes of frontals in the Euclidean plane

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Abstract

We have already defined the evolutes and the involutes of fronts without inflection points. For regular curves or fronts, we can not define the evolutes at inflection points. On the other hand, the involutes can be defined at inflection points. In this case, the involute is not a front but a frontal at inflection points. We define evolutes of frontals under conditions. The definition is a generalisation of both evolutes of regular curves and of fronts. By using relationship between evolutes and involutes of frontals, we give an existence condition of the evolute with inflection points. We also give properties of evolutes and involutes of frontals.

1 Introduction

The notions of evolutes and involutes (also known as evolvents) were studied by C. Huygens in his work [13] and studied in classical analysis, differential geometry and singularity theory of planar curves (cf. [3, 4, 6, 10, 11, 12, 16]). The evolute of a regular curve in the Euclidean plane is given by not only the locus of all its centres of the curvature (the caustics of the regular curve), but also the envelope of normal lines of the regular curve, namely, the locus of singular loci of parallel curves (the wave front of the regular curve). On the other hand, the involute of a regular curve is the trajectory described by the end of stretched string unwinding from a point of the curve. Alternatively, another way to construct the involute of a curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute. The length of the line segment is changed by an amount equal to the arc length traversed by the tangent point as it moves along the curve.

In the previous papers [8, 9], we defined the evolutes and the involutes of fronts without inflection points and gave properties of them. In §2, we review the evolutes and the involutes of regular curves and of fronts. We introduce the moving frame along Legendre curves and the curvature of Legendre curves (cf. [7]). Moreover, we also gave properties of the evolutes and the involutes of fronts, for more detail see [8, 9]. For a Legendre immersion without inflection points, the evolute and the involute of the front are also fronts without inflection points. It follows that we can repeat the evolute and the involute of fronts without inflection points. We gave the n-th form of evolutes and involutes of fronts without inflection points for all $n \in \mathbb{N}$ in

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[8, 9]. The evolute and the involute of the front without inflection points are corresponding to the differential and the integral of the curvatures of the Legendre immersions.

The evolutes of fronts can not be defined at inflection points. On the other hand, the involutes of fronts can be defined at inflection points. In this case, the involute is a frontal at inflection points. In this paper, we consider evolutes and involutes of frontals under conditions. In §3, we define evolutes and involutes of frontals by extending to the evolutes and the involutes of fronts. These definitions are generalisations of evolutes and involutes of regular curves and of fronts. Even if evolutes of frontals exists, we don't know whether evolutes of evolutes exists or not. By using relationship between evolutes and involutes of frontals, we give an existence condition for the n-th evolute. In §4, we give examples of Legendre curves and evolutes of frontals. These examples are useful to understand properties and results.

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

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2 Preliminaries

We quickly review on the theory of evolutes and involutes of regular curves and of fronts (cf. [6, 8, 9, 10, 11, 12, 13, 16]). Moreover, we introduce Legendre curves on the unit tangent bundle and the curvature of the Legendre curve (cf. [7]).

2.1 Regular plane curves

Let I be an interval or $\mathbb{R}$ and let $\mathbb{R}^2$ be the Euclidean plane with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$, where $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular plane curve, that is, $\dot{\gamma}(t) = (d\gamma/dt)(t) \neq 0$ for any $t \in I$. We have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$ and the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$, where $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and J is the anti-clockwise rotation by $\pi/2$ on $\mathbb{R}^2$. Then we have the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where the curvature is given by

$$\kappa(t) = \frac{\dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)}{|\dot{\gamma}(t)|} = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}.$$

Note that the curvature $\kappa(t)$ is independent on the choice of a parametrisation.

In this paper, we consider evolutes and involutes of plane curves.

Definition 2.1 The *evolute* $Ev(\gamma) : I \to \mathbb{R}^2$ *of a regular plane curve* γ is given by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t), \tag{1}$$

away from the point $\kappa(t) = 0$, that is, without inflection points (cf. [6, 10, 11]).

Definition 2.2 The *involute* $Inv(\gamma, t_0) : I \to \mathbb{R}^2$ *of a regular plane curve* γ *at* $t_0 \in I$ is given by

$$Inv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} |\dot\gamma(u)|du\right) \boldsymbol{t}(t). \tag{2}$$

The following properties are also well-known in the classical differential geometry of curves:

Proposition 2.3 *Let* $\gamma : I \to \mathbb{R}^2$ *be a regular curve and* $t_0 \in I$.

(1) *If* t *is a regular point of* $Inv(\gamma, t_0)$*, then* $Ev(Inv(\gamma, t_0))(t) = \gamma(t)$.

(2) *If* t *and* t_0 *are regular points of* $Ev(\gamma)$ *and not inflection points of* γ*, then* $Inv(Ev(\gamma), t_0)(t) = \gamma(t) + (1/\kappa(t_0))\boldsymbol{n}(t)$.

We say that t_0 is *an ordinary inflection point* of γ if $\kappa(t_0) = 0$ and $\dot\kappa(t_0) \neq 0$. By definition of the curvature, $\kappa(t_0) = 0$ and $\dot\kappa(t_0) \neq 0$ are equivalent to the conditions

$$\det(\dot\gamma(t_0), \ddot\gamma(t_0)) = 0, \quad \det(\dot\gamma(t_0), \dddot\gamma(t_0)) \neq 0.$$

2.2 Legendre curves and Legendre immersions

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*(t)\theta = 0$ for all $t \in I$, where θ is the canonical contact 1-form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [2, 4]) and S^1 is the unit circle. This condition is equivalent to $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion*. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet formula of the frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame along the frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of the frontal (or, the Legendre curve) which is given by

$$\begin{pmatrix} \dot\nu(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot\nu(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\beta(t)$ such that

$$\dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t).$$

The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t) (cf. [7]). If $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion, then $(\ell(t), \beta(t)) \neq (0, 0)$ for all $t \in I$.

Proposition 2.4 *Let* $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ *be a Legendre curve with the curvature* (ℓ, β) *and* $f : \widetilde{I} \to I$ *be a smooth function. Then* $(\gamma \circ f, \nu \circ f) : \widetilde{I} \to \mathbb{R}^2 \times S^1$ *is also a Legendre curve with the curvature* $((\ell \circ f)f', (\beta \circ f)f')$.

Proof. Since

$$(\gamma \circ f)'(u) = \dot\gamma(f(u))f'(u) = \beta(f(u))f'(u)\boldsymbol{\mu}(f(u)) = (\beta \circ f(u))f'(u)\boldsymbol{\mu} \circ f(u)$$

and

$$(\nu\circ f)'(u)=\dot\nu(f(u))f'(u)=\ell(f(u))f'(u)\boldsymbol{\mu}(f(u))=(\ell\circ f)(u)f'(u)\boldsymbol{\mu}\circ f(u),$$

it holds that $(\gamma\circ f,\nu\circ f)$ is a Legendre curve with the curvature $((\ell\circ f(u))f'(u),(\beta\circ f(u))f'(u))$. $\Box$

Example 2.5 Let n,m and k be natural numbers with $m=n+k$. Let $(\gamma,\nu):\mathbb{R}\to\mathbb{R}^2\times S^1$ be

$$\gamma(t)=\left(\frac{1}{n}t^n,\frac{1}{m}t^m\right),\ \nu(t)=\frac{1}{\sqrt{t^{2k}+1}}\left(-t^k,1\right).$$

It is easy to see that (γ,ν) is a Legendre curve, and a Legendre immersion when $n=1$ or $k=1$. We call γ is of *type* (n,m). For example, the frontal of type $(2,3)$ has the $3/2$ cusp (A_2 singularity) at $t=0$, of type $(3,4)$ has the $4/3$ cusp (E_6 singularity) at $t=0$, of type $(3,5)$ has the $5/3$ cusp (E_8 singularity) at $t=0$. These types $(2,3)$ and $(3,4)$ are Legendre immersions. The type $(3,5)$ is a Legendre curve but not a Legendre immersion, see Example 4.1. By definition, we have $\boldsymbol{\mu}(t)=(1/\sqrt{t^{2k}+1})(-1,-t^k)$ and

$$\ell(t)=\frac{kt^{k-1}}{t^{2k}+1},\ \beta(t)=-t^{n-1}\sqrt{t^{2k}+1}.$$

Definition 2.6 Let (γ,ν) and $(\widetilde\gamma,\widetilde\nu):I\to\mathbb{R}^2\times S^1$ be Legendre curves. We say that (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde\gamma(t)=C(\gamma(t))=A(\gamma(t))+\boldsymbol{b}$ and $\widetilde\nu(t)=A(\nu(t))$ for all $t\in I$, where C is given by the rotation A and the translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

We have the existence and the uniqueness for Legendre curves in the unit tangent bundle like as regular plane curves, see [7].

Theorem 2.7 (The Existence Theorem) *Let $(\ell,\beta):I\to\mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ whose associated curvature of the Legendre curve is (ℓ,β).*

Theorem 2.8 (The Uniqueness Theorem) *Let (γ,ν) and $(\widetilde\gamma,\widetilde\nu):I\to\mathbb{R}^2\times S^1$ be Legendre curves whose curvatures of Legendre curves (ℓ,β) and $(\widetilde\ell,\widetilde\beta)$ coincide. Then (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ are congruent as Legendre curves.*

In fact, the Legendre curve whose associated curvature of the Legendre curve is (ℓ,β), is given by the form

$$\begin{aligned}\gamma(t)&=\left(-\int\beta(t)\sin\left(\int\ell(t)dt\right)dt,\ \int\beta(t)\cos\left(\int\ell(t)dt\right)dt\right),\\ \nu(t)&=\left(\cos\int\ell(t)dt,\ \sin\int\ell(t)dt\right).\end{aligned}$$

Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ be a Legendre curve with the curvature of the Legendre curve (ℓ,β).

Definition 2.9 We say that a point $t_0\in I$ is an *inflection point* of the frontal γ (or, the Legendre curve (γ,ν)) if $\ell(t_0)=0$.

Remark that the definition of the inflection point of the frontal is a generalisation of the definition of the inflection point of a regular curve (cf. [7]).

We also recall the notion of the contact between Legendre curves (cf. [7]). Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t), \nu(t))$ and $(\widetilde{\gamma}, \widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u), \widetilde{\nu}(u))$ be Legendre curves respectively and let k be a natural number. We say that (γ, ν) *and* $(\widetilde{\gamma}, \widetilde{\nu})$ *have at least k-th order contact at* $t = t_0, u = u_0$ if

$$(\gamma, \nu)(t_0) = (\widetilde{\gamma}, \widetilde{\nu})(u_0),\ \frac{d}{dt}(\gamma, \nu)(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\nu})(u_0), \cdots, \frac{d^{k-1}}{dt^{k-1}}(\gamma, \nu)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma}, \widetilde{\nu})(u_0).$$

In general, we may assume that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ have at least first order contact at any point $t = t_0, u = u_0$, up to congruence as Legendre curves. We denote the curvatures of the Legendre curves $(\ell(t), \beta(t))$ of $(\gamma(t), \nu(t))$ and $(\widetilde{\ell}(u), \widetilde{\beta}(u))$ of $(\widetilde{\gamma}(u), \widetilde{\nu}(u))$, respectively.

Theorem 2.10 ([7, Theorem 3.1]) *If* (γ, ν) *and* $(\widetilde{\gamma}, \widetilde{\nu})$ *have at least* $(k+1)$*-th order contact at* $t = t_0, u = u_0$*, then*

$$(\ell, \beta)(t_0) = (\widetilde{\ell}, \widetilde{\beta})(u_0),\ \frac{d}{dt}(\ell, \beta)(t_0) = \frac{d}{du}(\widetilde{\ell}, \widetilde{\beta})(u_0),\ \cdots,\ \frac{d^{k-1}}{dt^{k-1}}(\ell, \beta)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\ell}, \widetilde{\beta})(u_0). \quad (3)$$

Conversely, if the condition (3) *holds, then* (γ, ν) *and* $(\widetilde{\gamma}, \widetilde{\nu})$ *have at least* $(k+1)$*-th order contact at* $t = t_0, u = u_0$*, up to congruence as Legendre curves.*

Definition 2.11 We say that a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a *part of a circle* if there exist a smooth function $\theta : I \to \mathbb{R}$ and constants $r, a, b \in \mathbb{R}$ such that

$$\gamma(t) = (r \cos\theta(t) + a, r \sin\theta(t) + b),\ \nu(t) = (\cos\theta(t), \sin\theta(t)).$$

Proposition 2.12 *Let* $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ *be a Legendre curve with the curvature* (ℓ, β)*. The Legendre curve* (γ, ν) *is a part of a circle if and only if there exists a constant* $r \in \mathbb{R}$ *such that* $\beta(t) = r\ell(t)$ *for all* $t \in I$*.*

Proof. Assume the Legendre curve (γ, ν) is a part of a circle. There exist a smooth function $\theta : I \to \mathbb{R}$ and constants $r, a, b \in \mathbb{R}$ such that

$$\gamma(t) = (r \cos\theta(t) + a, r \sin\theta(t) + b),\ \nu(t) = (\cos\theta(t), \sin\theta(t)).$$

Since $\boldsymbol{\mu}(t) = (-\sin\theta(t), \cos\theta(t))$, we have $\ell(t) = \dot{\theta}(t)$ and $\beta(t) = r\dot{\theta}(t)$. Thus, $\beta(t) = r\ell(t)$ holds.

By the existence and uniqueness Theorems 2.7 and 2.8, the converse is holded. □

Note that a part of a circle may have singular points and inflection points.

2.3 Evolutes and involutes of fronts

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre curve (ℓ, β). If (γ, ν) dose not have inflection points, namely, $\ell(t) \neq 0$ for all $t \in I$, then (γ, ν) is a Legendre immersion. In this subsection, we assume that (γ, ν) dose not have inflection points.

Definition 2.13 The *evolute* $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ of the front γ is given by

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t). \tag{4}$$

Definition 2.14 The *involute* $\mathcal{I}nv(\gamma, t_0) : I \to \mathbb{R}^2$ of the front γ at $t_0 \in I$ is given by

$$\mathcal{I}nv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} \beta(u)du\right) \boldsymbol{\mu}(t). \tag{5}$$

Proposition 2.15 ([9, Proposition 2.14]) *Under the above notations, we have the following.*

(1) *The evolute $\mathcal{E}v(\gamma)$ is also a front. More precisely, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature*

$$\left(\ell(t), \frac{d}{dt}\frac{\beta(t)}{\ell(t)}\right).$$

(2) *The involute $\mathcal{I}nv(\gamma, t_0)$ is also a front for each $t_0 \in I$. More precisely, $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature*

$$\left(\ell(t), \left(\int_{t_0}^{t} \beta(u)du\right) \ell(t)\right).$$

Proposition 2.16 ([9, Proposition 4.1]) *For any $t_0 \in I$, we have the following.*

(1) $\mathcal{E}v(\mathcal{I}nv(\gamma, t_0))(t) = \gamma(t)$.

(2) $\mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) = \gamma(t) - (\beta(t_0)/\ell(t_0))\nu(t)$.

The following results give the relationships between singular points of γ and the properties of the evolutes and involutes.

Proposition 2.17 ([8, Propositions 3.8 and 4.5])

(1) *Suppose that t_0 is a singular point of γ. Then γ is diffeomorphic to the* $3/2$ *cusp at t_0 if and only if t_0 is a regular point of $\mathcal{E}v(\gamma)$.*

(2) *Suppose that t_0 is a singular point of both γ and $\mathcal{E}v(\gamma)$. Then γ is diffeomorphic to the* $4/3$ *cusp at t_0 if and only if $\mathcal{E}v(\gamma)$ is diffeomorphic to the* $3/2$ *cusp at t_0.*

Proposition 2.18 ([9, Proposition 3.9]) (1) *$\mathcal{I}nv(\gamma, t_0)$ is diffeomorphic to the* $3/2$ *cusp at t_0 if and only if t_0 is a regular point of γ.*

(2) *$\mathcal{I}nv(\gamma, t_0)$ is diffeomorphic to the* $4/3$ *cusp at t_0 if and only if γ is diffeomorphic to the* $3/2$ *cusp at t_0.*

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. By Proposition 2.15, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ and $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ are also Legendre immersions without inflection points for any $t_0 \in I$. Therefore, we can repeat the evolute and the involute of the front.

We give the form of the n-th evolute and the n-th involute of the front, where n is a natural number. We denote $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$ and $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$ for convenience. We define $\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t)$ and

$$\beta_0(t) = \beta(t), \quad \beta_n(t) = \frac{d}{dt}\left(\frac{\beta_{n-1}(t)}{\ell(t)}\right),$$

inductively. Moreover, we denote $\mathcal{I}nv^0(\gamma, t_0)(t) = \gamma(t)$ and $\mathcal{I}nv^1(\gamma, t_0)(t) = \mathcal{I}nv(\gamma, t_0)(t)$ for convenience. We define $\mathcal{I}nv^n(\gamma, t_0)(t) = \mathcal{I}nv(\mathcal{I}nv^{n-1}(\gamma, t_0), t_0)(t)$ and

$$\beta_{-1}(t) = \left(\int_{t_0}^{t} \beta(u)du\right) \ell(t), \quad \beta_{-n}(t) = \left(\int_{t_0}^{t} \beta_{-n+1}(u)du\right) \ell(t)$$

inductively.

Theorem 2.19 ([8, 9]) (1) *$(\mathcal{E}v^n(\gamma), J^n(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature (ℓ, β_n), where the n-th evolute of the front is given by*

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v^{n-1}(\gamma)(t) - \frac{\beta_{n-1}(t)}{\ell(t)} J^{n-1}(\nu(t))$$

and J^n is n-times operation of J.

(2) *$(\mathcal{I}nv^n(\gamma, t_0), J^{-n}(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature (ℓ, β_{-n}), where the n-th involute of the front γ at t_0 is given by*

$$\mathcal{I}nv^n(\gamma, t_0)(t) = \mathcal{I}nv^{n-1}(\gamma, t_0)(t) + \frac{\beta_{-n}(t)}{\ell(t)} J^{-n}(\nu(t))$$

and J^{-n} is n-times operation of J^{-1}.

3 Evolutes and involutes of frontals

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β). We can define the involute of the frontal as the same form of the involute of the front.

Definition 3.1 The *involute $\mathcal{I}nv(\gamma, t_0) : I \to \mathbb{R}^2$ of the frontal γ at $t_0 \in I$* is given by

$$\mathcal{I}nv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} \beta(u)du\right) \boldsymbol{\mu}(t). \tag{6}$$

On the other hand, Proposition 2.16 suggests that we may define an evolute of the frontal under existence and uniqueness conditions.

Definition 3.2 The *evolute $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ of the frontal γ* is given by

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \alpha(t)\nu(t), \tag{7}$$

if there exists a unique smooth function $\alpha : I \to \mathbb{R}$ such that $\beta(t) = \alpha(t)\ell(t)$. In this case, we say that *the evolute $\mathcal{E}v(\gamma)$ exists.*

The uniqueness condition is well-known as a topological condition.

Lemma 3.3 *Suppose that there exists a continuous function $\alpha : I \to \mathbb{R}$ such that $\alpha(t) = \beta(t)/\ell(t)$ on $L = \{t \in I \mid \ell(t) \neq 0\}$. Then the function α is a unique if and only if L is a dense subset of I, namely, $\overline{L} = I$.*

Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ be a Legendre curve with the curvature (ℓ,β). In this paper, we assume that $L=\{t\in I \mid \ell(t)\neq 0\}$ is a dense subset of I. This condition follows that if such a smooth function α exists, then the uniqueness condition is satisfied by Lemma 3.3.

Remark 3.4 If the inflection points $\ell(t)=0$ are isolated, then the condition that L is a dense subset of I is satisfied.

If t_0 is an inflection point of (γ,ν) and the evolute $\mathcal{E}v(\gamma)$ exists, then the inflection point must be a singular point of γ. It follows that t_0 is a singular point of Legendre curve (γ,ν), that is, $(\ell(t_0),\beta(t_0))=(0,0)$.

Proposition 3.5 *Under the above notations, we have the following.*

(1) *If the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\beta(t)=\alpha(t)\ell(t)$, then the evolute $\mathcal{E}v(\gamma)$ is also a frontal. More precisely, $(\mathcal{E}v(\gamma),J(\nu)):I\to\mathbb{R}^2\times S^1$ is a Legendre curve with the curvature*

$$(\ell(t),\dot{\alpha}(t))\,.$$

(2) *The involute of the frontal $\mathcal{I}nv(\gamma,t_0)$ is also a frontal for each $t_0\in I$. More precisely, $(\mathcal{I}nv(\gamma,t_0),J^{-1}(\nu)):I\to\mathbb{R}^2\times S^1$ is a Legendre curve with the curvature*

$$\left(\ell(t),\left(\int_{t_0}^{t}\beta(u)du\right)\ell(t)\right).$$

Proof. (1) By using the Frenet formula of the Legendre curve, we have

$$\begin{aligned}\dot{\mathcal{E}v}(\gamma)(t) &= \dot{\gamma}(t)-\dot{\alpha}(t)\nu(t)-\alpha(t)\dot{\nu}(t)\\ &= \beta(t)\boldsymbol{\mu}(t)-\dot{\alpha}(t)\nu(t)-\alpha(t)\ell(t)\boldsymbol{\mu}(t)\\ &= \dot{\alpha}(t)J(J(\nu(t))).\end{aligned}$$

It follows that $\dot{\mathcal{E}v}(\gamma)(t)\cdot J(\nu(t))=0$. Since $(d/dt)J(\nu(t))=\ell(t)J(J(\nu(t)))$, the curvature of the Legendre curve $(\mathcal{E}v(\gamma),J(\nu))$ is given by $(\ell(t),\dot{\alpha}(t))$.

(2) By using the Frenet formula of the Legendre curve, we have

$$\dot{\mathcal{I}nv}(\gamma,t_0)(t)=\dot{\gamma}(t)-\beta(t)\boldsymbol{\mu}(t)-\left(\int_{t_0}^{t}\beta(u)du\right)\dot{\boldsymbol{\mu}}(t)=\left(\int_{t_0}^{t}\beta(u)du\right)\ell(t)J(J^{-1}(\nu(t))).$$

It follows that $\dot{\mathcal{I}nv}(\gamma,t_0)(t)\cdot J^{-1}(\nu(t))=0$. Since $(d/dt)J^{-1}(\nu(t))=\ell(t)J(J^{-1}(\nu(t)))$, the curvature of the Legendre curve $(\mathcal{I}nv(\gamma,t_0),J^{-1}(\nu))$ is given by

$$\left(\ell(t),\left(\int_{t_0}^{t}\beta(u)du\right)\ell(t)\right).$$

□

By Proposition 3.5, if t_0 is an inflection point of a Legendre curve (γ,ν), then t_0 is also an inflection point of both the evolute if exists, and the involute of the frontal. Moreover, t_0 is a singular point of the involute of the frontal. The important difference between the evolute and the involute of the frontal is that we can always repeat the involute of the frontal but can not repeat the evolute of the frontal in general.

Proposition 3.6 *Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre curve (ℓ,β).*

(1) *The evolute of the involute of the frontal always exists and $\mathcal{E}v(\mathcal{I}nv(\gamma,t_0))(t) = \gamma(t)$ for any $t_0 \in I$.*

(2) *If the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\beta(t) = \alpha(t)\ell(t)$, then $\mathcal{I}nv(\mathcal{E}v(\gamma),t_0)(t) = \gamma(t) - \alpha(t_0)\nu(t)$ for any $t_0 \in I$.*

Proof. (1) We denote the curvature of the involute of the frontal by $(\ell_{-1}(t),\beta_{-1}(t))$. Since the form of the curvature of the involute of the frontal in Proposition 3.5 (2), $\ell_{-1}(t) = \ell(t)$ and $\beta_{-1}(t) = \alpha(t)\ell(t)$, where

$$\alpha(t) = \int_{t_0}^{t} \beta(u)du.$$

By definition of the evolute of the frontal, it holds that

$$\begin{aligned}
\mathcal{E}v(\mathcal{I}nv(\gamma,t_0))(t) &= \mathcal{I}nv(\gamma,t_0)(t) - \left(\int_{t_0}^{t}\beta(u)du\right)J^{-1}(\nu(t)) \\
&= \gamma(t) - \left(\int_{t_0}^{t}\beta(u)du\right)\boldsymbol{\mu}(t) + \left(\int_{t_0}^{t}\beta(u)du\right)\boldsymbol{\mu}(t) \\
&= \gamma(t).
\end{aligned}$$

(2) By definition of the involute of the frontal, it holds that

$$\begin{aligned}
\mathcal{I}nv(\mathcal{E}v(\gamma),t_0)(t) &= \mathcal{E}v(\gamma)(t) - \left(\int_{t_0}^{t}\dot{\alpha}(u)du\right)J(\boldsymbol{\mu}(t)) \\
&= \gamma(t) - \alpha(t)\nu(t) + \alpha(t)\nu(t) - \alpha(t_0)\nu(t) \\
&= \gamma(t) - \alpha(t_0)\nu(t).
\end{aligned}$$

□

By a direct calculation, we have the following Lemma.

Lemma 3.7 *Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre curve (ℓ,β).*

(1) $\det(\dot\gamma(t),\ddot\gamma(t)) = 0$ *and* $\det(\dot\gamma(t),\dddot\gamma(t)) \neq 0$ *if and only if* $\beta(t) \neq 0, \ell(t) = 0$ *and* $\dot\ell(t) \neq 0$.

(2) $\dot\gamma(t) = 0, \ddot\gamma(t) = 0$, $\det(\gamma^{(3)}(t),\gamma^{(4)}(t)) = 0$ *and* $\det(\gamma^{(3)}(t),\gamma^{(5)}(t)) \neq 0$ *if and only if* $\beta(t) = \dot\beta(t) = 0, \ddot\beta(t) \neq 0$, $\ell(t) = 0$ *and* $\dot\ell(t) \neq 0$.

Proposition 3.8 *Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre curve (ℓ,β).*

(1) *γ is diffeomorphic to the $5/3$ cusp at t_0 if and only if the evolute $\mathcal{E}v(\gamma)$ of the frontal exists, $\beta(t) = \alpha(t)\ell(t)$ around t_0 with $\alpha(t_0) = 0$ and t_0 is an ordinary inflection point of $\mathcal{E}v(\gamma)$.*

(2) *$\mathcal{I}nv(\gamma,t_0)(t)$ is diffeomorphic to the $5/3$ cusp at t_0 if and only if t_0 is an ordinary inflection point of γ.*

Proof. (1) By Lemma 3.7, we have $\beta(t_0) = \dot\beta(t_0) = 0, \ddot\beta(t_0) \neq 0$, $\ell(t_0) = 0$ and $\dot\ell(t_0) \neq 0$. Since t_0 is an isolated singular point of (γ,ν) and

$$\lim_{t\to t_0}\frac{\beta(t)}{\ell(t)} = \lim_{t\to t_0}\frac{\dot\beta(t)}{\dot\ell(t)} = 0,$$

there exists a unique smooth function germ $\alpha : (I, t_0) \to \mathbb{R}$ such that $\beta(t) = \alpha(t)\ell(t)$ with $\alpha(t_0) = 0$. It follows that $\mathcal{E}v(\gamma)$ exists around t_0. Note that $\mathcal{E}v(\gamma)(t_0) = \gamma(t_0)$. Since $\ddot{\beta}(t) = \ddot{\alpha}(t)\ell(t) + 2\dot{\alpha}(t)\dot{\ell}(t) + \alpha(t)\ddot{\ell}(t)$, we have $\dot{\alpha}(t_0) = \ddot{\beta}(t_0)/(2\dot{\ell}(t_0)) \neq 0$. It follows from $(d/dt)(\mathcal{E}v(\gamma)(t)) = \dot{\alpha}(t)J^2(\nu(t))$ that t_0 is a regular point of $\mathcal{E}v(\gamma)$. By Proposition 3.5, the curvature of the Legendre curve of the evolute is $(\ell(t), \dot{\alpha}(t))$. Hence, t_0 is also an ordinary inflection point of $\mathcal{E}v(\gamma)$ by Lemma 3.7.

Conversely, if t_0 is an ordinary inflection point of $\mathcal{E}v(\gamma)$, we have $\ell(t_0) = 0$ and $\dot{\ell}(t_0) \neq 0$. By $\alpha(t_0) = 0$, we have $\beta(t_0) = 0$ and $\dot{\beta}(t_0) = 0$. Moreover, since t_0 is a regular point of $\mathcal{E}v(\gamma)$, namely $\dot{\alpha}(t_0) \neq 0$, we have $\ddot{\beta}(t_0) \neq 0$. It follows from Lemma 3.7 that γ is diffeomorphic to the 5/3 cusp at t_0 (cf. [5, 14, 15, 17]).

(2) Suppose that $\mathcal{I}nv(\gamma, t_0)(t)$ is diffeomorphic to the 5/3 cusp at t_0. By Proposition 3.6 and (1), it holds that t_0 is an ordinary inflection point of γ.

Conversely, if t_0 is an ordinary inflection point of γ, then $\beta(t_0) \neq 0, \ell(t_0) = 0$ and $\dot{\ell}(t_0) \neq 0$ by Lemma 3.7. By Proposition 3.5, the curvature of the Legendre curve of the involute is given by

$$(\ell_{-1}(t), \beta_{-1}(t)) = \left(\ell(t), \left(\int_{t_0}^{t} \beta(u)du\right)\ell(t)\right).$$

By a direct calculation, we have

$$\beta_{-1}(t_0) = 0,\ \dot{\beta}_{-1}(t_0) = 0,\ \ddot{\beta}_{-1}(t_0) \neq 0.$$

It follows from Lemma 3.7 that $\mathcal{I}nv(\gamma, t_0)$ is diffeomorphic to the 5/3 cusp at t_0. $\Box$

Remark 3.9 If t_0 is an inflection point and a singular point of the frontal γ, then γ is degenerate more than 3/2 cusp at t_0. Indeed, we have $\dot{\gamma}(t_0) = \beta(t_0)\boldsymbol{\mu}(t_0) = 0$ and $\det(\ddot{\gamma}(t_0), \dddot{\gamma}(t_0)) = 2\dot{\beta}(t_0)^2\ell(t_0) = 0$ (cf. Propositions 2.17 and 2.18).

Proposition 3.10 *If $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a part of a circle, then the evolute of the circle exists and is given by a point.*

Proof. There exist a smooth function $\theta : I \to \mathbb{R}$ and constants $r, a, b \in \mathbb{R}$ such that

$$\gamma(t) = (r\cos\theta(t) + a, r\sin\theta(t) + b),\ \nu(t) = (\cos\theta(t), \sin\theta(t)).$$

By Proposition 2.12, $\beta(t) = r\ell(t)$. It follows that $\mathcal{E}v(\gamma)(t) = \gamma(t) - r\nu(t) = (a, b)$. $\Box$

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β).

Definition 3.11 We say that t_0 is a *vertex of the frontal* γ (or, *of the Legendre curve* (γ, ν)) if the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\dot{\mathcal{E}v}(\gamma)(t_0) = 0$.

Suppose that the evolute of the frontal $\mathcal{E}v(\gamma)$ exists and $\beta(t) = \alpha(t)\ell(t)$. Then t_0 is a vertex of the frontal γ if and only if $\dot{\alpha}(t_0) = 0$.

Proposition 3.12 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β) and $t_0 \in I$. Suppose that the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\beta(t) = \alpha(t)\ell(t)$. We denote $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ by a part of a circle with the curvature $(\ell(t), \alpha(t_0)\ell(t))$.*

(1) (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ have at least second order contact at $t = t_0$, up to congruence as Legendre curves.

(2) *If t_0 is a singular point of (γ,ν), then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least third order contact at $t=t_0$, up to congruence as Legendre curves.*

(3) *If t_0 is a singular point and a vertex of (γ,ν), then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least 4-th order contact at $t=t_0$, up to congruence as Legendre curves.*

Proof. (1) Since $(\ell(t_0),\beta(t_0))=(\ell(t_0),\alpha(t_0)\ell(t_0))$, it holds that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least second order contact at $t=t_0$, up to congruence as Legendre curves by Theorem 2.10.

(2) By differentiating $\beta(t)=\alpha(t)\ell(t)$, we have $\dot\beta(t)=\dot\alpha(t)\ell(t)+\alpha(t)\dot\ell(t)$. By the assumption, $\ell(t_0)=\beta(t_0)=0$ holds. It follows that $(\dot\ell(t_0),\dot\beta(t_0))=(\dot\ell(t_0),\alpha(t_0)\dot\ell(t_0))$. By Theorem 2.10, (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least third order contact at $t=t_0$, up to congruence as Legendre curves.

(3) Also we have $\ddot\beta(t)=\ddot\alpha(t)\ell(t)+2\dot\alpha(t)\dot\ell(t)+\alpha(t)\ddot\ell(t)$. By the assumption, $\ell(t_0)=\beta(t_0)=\dot\alpha(t_0)=0$ holds. It follows that $(\ddot\ell(t_0),\ddot\beta(t_0))=(\ddot\ell(t_0),\alpha(t_0)\ddot\ell(t_0))$. By Theorem 2.10, (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least 4-th order contact at $t=t_0$, up to congruence as Legendre curves. $\Box$

Remark 3.13 Suppose that t_0 is a regular point and not an inflection point of γ. It is well-known that if t_0 is a vertex of γ, then γ and the osculating circle have at least third order contact at t_0.

Proposition 3.14 *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}):I\to\mathbb{R}^2\times S^1$ be Legendre curves with the curvatures of the Legendre curves are $(\ell(t),\beta(t))$ and $(\widetilde{\ell}(t),\widetilde{\beta}(t))$ respectively.*

(1) *Suppose that the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\beta(t)=\alpha(t)\ell(t)$. If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves, then the evolute $\mathcal{E}v(\widetilde{\gamma})$ of the frontal exists. Moreover, $(\mathcal{E}v(\gamma),J(\nu))$ and $(\mathcal{E}v(\widetilde{\gamma}),J(\widetilde{\nu}))$ are congruent as Legendre curves.*

(2) *Let $t_0\in I$. (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves if and only if $(\mathcal{I}nv(\gamma,t_0),J^{-1}(\nu))$ and $(\mathcal{I}nv(\widetilde{\gamma},t_0),J^{-1}(\widetilde{\nu}))$ are congruent as Legendre curves.*

Proof. (1) Since $(\ell(t),\beta(t))=(\widetilde{\ell}(t),\widetilde{\beta}(t))$, we have $\widetilde{\beta}(t)=\alpha(t)\widetilde{\ell}(t)$. Thus, the evolute $\mathcal{E}v(\widetilde{\gamma})$ exists. Moreover, the curvatures of the evolutes are the same $(\ell(t),\dot\alpha(t))$. By Theorem 2.8, $(\mathcal{E}v(\gamma),J(\nu))$ and $(\mathcal{E}v(\widetilde{\gamma}),J(\widetilde{\nu}))$ are congruent as Legendre curves.

(2) If $(\ell(t),\beta(t))=(\widetilde{\ell}(t),\widetilde{\beta}(t))$, then

$$\left(\ell(t),\left(\int_{t_0}^t\beta(u)du\right)\ell(t)\right)=\left(\widetilde{\ell}(t),\left(\int_{t_0}^t\widetilde{\beta}(u)du\right)\widetilde{\ell}(t)\right)$$

holds. It follows that $(\mathcal{I}nv(\gamma,t_0),J^{-1}(\nu))$ and $(\mathcal{I}nv(\widetilde{\gamma},t_0),J^{-1}(\widetilde{\nu}))$ are congruent as Legendre curves. The converse is a direct corollary of (1) and Proposition 3.6. $\Box$

Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ be a Legendre curve with the curvature (ℓ,β) and $f:\widetilde{I}\to I$ be a smooth function, where $\widetilde{I}$ is an interval or $\mathbb{R}$. By Proposition 2.4, $(\gamma\circ f,\nu\circ f):\widetilde{I}\to\mathbb{R}^2\times S^1$ is also a Legendre curve with the curvature $((\ell\circ f)f',(\beta\circ f)f')$.

Proposition 3.15 *Under the above notations, we have the following. Suppose that $\widetilde{L}=\{u\in\widetilde{I}|\ell(f(u))f'(u)\neq 0\}$ is a dense subset in $\widetilde{I}$.*

(1) *If $\mathcal{E}v(\gamma)$ exists, then $\mathcal{E}v(\gamma\circ f)$ exists and $\mathcal{E}v(\gamma\circ f)(u)=\mathcal{E}v(\gamma)(f(u))\in\mathcal{E}v(\gamma)(I)$.*

(2) *Let $u_0\in\widetilde{I}$. $\mathcal{I}nv(\gamma\circ f,u_0)(u)=\mathcal{I}nv(\gamma,f(u_0))(f(u))\in\mathcal{I}nv(\gamma,f(u_0))(I)$.*

Proof. (1) There exists a unique smooth function $\alpha : I \to \mathbb{R}$ such that $\beta(t) = \alpha(t)\ell(t)$. It follows that $(\beta \circ f(u))f'(u) = (\alpha \circ f(u))(\ell \circ f(u))f'(u)$. By definition of the evolute of the frontal, we have

$$\mathcal{E}v(\gamma \circ f)(u) = \gamma \circ f(u) - (\alpha \circ f(u))(\nu \circ f(u)) = \mathcal{E}v(\gamma)(f(u)) \in \mathcal{E}v(\gamma)(I).$$

(2) By definition of the involute of the frontal, we have

$$\begin{aligned} \mathcal{I}nv(\gamma \circ f, u_0)(u) &= \gamma \circ f(u) - \left(\int_{u_0}^{u} (\beta \circ f(s))f'(s)ds \right) \boldsymbol{\mu} \circ f(u) \\ &= \gamma \circ f(u) - \left(\int_{f(u_0)}^{f(u)} \beta(t)dt \right) \boldsymbol{\mu} \circ f(u) \\ &= \mathcal{I}nv(\gamma, f(u_0))(f(u)) \in \mathcal{I}nv(\gamma, f(u_0))(I). \end{aligned}$$

$\square$

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. We define a *parallel curve* $\gamma_\lambda : I \to \mathbb{R}^2$ of the frontal γ (or, Legendre curve (γ, ν)) by

$$\gamma_\lambda(t) = \gamma(t) + \lambda\nu(t),$$

where $\lambda \in \mathbb{R}$.

Proposition 3.16 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β).*

(1) *The parallel curve $\gamma_\lambda : I \to \mathbb{R}^2$ is also a frontal for any $\lambda \in \mathbb{R}$. More precisely, $(\gamma_\lambda, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve with the curvature*

$$(\ell(t), \beta(t) + \lambda\ell(t)).$$

(2) *If the evolute $\mathcal{E}v(\gamma)$ exists, then the evolute of a parallel curve of γ exists. Moreover, the evolute $\mathcal{E}v(\gamma_\lambda)$ coincides with the evolute $\mathcal{E}v(\gamma)$.*

Proof. (1) Let $\gamma_\lambda(t) = \gamma(t) + \lambda\nu(t)$ where $\lambda \in \mathbb{R}$. By the Frenet formula, we have

$$\dot{\gamma_\lambda}(t) = \dot{\gamma}(t) + \lambda\dot{\nu}(t) = (\beta(t) + \lambda\ell(t))\boldsymbol{\mu}(t)$$

holds. It follows that $\dot{\gamma_\lambda}(t) \cdot \nu(t) = 0$ and hence (γ_λ, ν) is a Legendre curve. Then the curvature of the Legendre curve (γ_λ, ν) is given by $(\ell(t), \beta(t) + \lambda\ell(t))$.

(2) There exists a unique smooth function α such that $\beta(t) = \alpha(t)\ell(t)$. By the form of the curvature of the parallel curve, we have

$$\beta(t) + \lambda\ell(t) = (\alpha(t) + \lambda)\ell(t).$$

Therefore, the evolute of γ_λ exists again. Moreover,

$$\mathcal{E}v(\gamma_\lambda)(t) = \gamma_\lambda(t) - (\alpha(t) + \lambda)\nu(t) = \gamma(t) - \alpha(t)\nu(t) = \mathcal{E}v(\gamma).$$

$\square$

Proposition 3.17 *For any points $t_0, t_1 \in I$, the involute $\mathcal{I}nv(\gamma, t_1)(t)$ is a parallel curve of $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$.*

Proof. By definition of the involute of the frontal, we have

$$\begin{aligned}
\mathcal{I}nv(\gamma, t_1)(t) &= \gamma(t) - \left(\int_{t_1}^{t} \beta(u)du\right) \boldsymbol{\mu}(t) \\
&= \gamma(t) - \left(\int_{t_0}^{t} \beta(u)du\right) \boldsymbol{\mu}(t) - \left(\int_{t_1}^{t_0} \beta(u)du\right) \boldsymbol{\mu}(t) \\
&= \mathcal{I}nv(\gamma, t_0)(t) + \left(\int_{t_1}^{t_0} \beta(u)du\right) J^{-1}(\nu(t)).
\end{aligned}$$

It follows that $\mathcal{I}nv(\gamma, t_1)(t)$ is a parallel curve of $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$. $\Box$

For a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we denote the set of Legendre curves of the parallel curves of the involute $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ by $\mathcal{PI}(\gamma, \nu)$, that is,

$$\mathcal{PI}(\gamma, \nu) = \{(\mathcal{I}nv(\gamma, t_0) + \lambda J^{-1}(\nu), J^{-1}(\nu)) \mid \lambda \in \mathbb{R}\}.$$

Note that by Proposition 3.17, for any $t_1 \in I$, $(\mathcal{I}nv(\gamma, t_1), J^{-1}(\nu)) \in \mathcal{PI}(\gamma, \nu)$. Hence $\mathcal{PI}(\gamma, \nu)$ is independent on the choice of the initial point t_0.

We also define $\mathcal{PI}^n(\gamma, \nu)$ by $\mathcal{PI}^n(\gamma, \nu) = \mathcal{PI}(\mathcal{PI}^{n-1}(\gamma, \nu))$ inductively, that is,

$$\mathcal{PI}^n(\gamma, \nu) = \{(\mathcal{I}nv(\widetilde{\gamma}, t_0) + \lambda J^{-1}(\widetilde{\nu}), J^{-1}(\widetilde{\nu})) \mid (\widetilde{\gamma}, \widetilde{\nu}) \in \mathcal{PI}^{n-1}(\gamma, \nu), \lambda \in \mathbb{R}\}.$$

We denote $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$ and $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$ if the evolute exists. We also define $\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t)$ inductively, if the evolute exists. We give an existence condition of the n-th evolute of the frontal.

Theorem 3.18 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β). Then the following are equivalent.*

(1) *The n-th evolute $\mathcal{E}v^n(\gamma)$ exists and $(\mathcal{E}v^n(\gamma), J^n(\nu))$ is a Legendre curve.*

(2) *There exists a Legendre curve $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ such that $(\gamma, \nu) \in \mathcal{PI}^n(\widetilde{\gamma}, \widetilde{\nu})$.*

Proof. We give the proof by the induction on $n \in \mathbb{N}$. First we consider the case of $n = 1$. If the evolute of the frontal $\mathcal{E}v(\gamma)$ exists and $\beta(t) = \alpha(t)\ell(t)$, then $\mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) = \gamma(t) - \alpha(t_0)\nu(t)$ by Proposition 3.6. Therefore,

$$\gamma(t) = \mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) + \alpha(t_0)J^{-1}(J(\nu(t)))$$

holds. It follows that $(\gamma, \nu) \in \mathcal{PI}(\mathcal{E}v(\gamma), J(\nu))$. Conversely, suppose that there exists a Legendre curve $(\widetilde{\gamma}, \widetilde{\nu})$ such that $(\gamma, \nu) \in \mathcal{PI}(\widetilde{\gamma}, \widetilde{\nu})$. Then $\mathcal{E}v(\gamma)(t) = \widetilde{\gamma}(t)$ by Propositions 3.6 and 3.16. Therefore, the evolute $\mathcal{E}v(\gamma)$ exists and $(\mathcal{E}v(\gamma), J(\nu))$ is a Legendre curve by Propositions 3.5 and 3.6.

Next suppose that the case of $n-1$ is holded. If $\mathcal{E}v^n(\gamma)$ exists, then $(\mathcal{E}v^{n-1}(\mathcal{E}v(\gamma)), J^{n-1}(J(\nu)))$ exists. By the assumption of the induction, there exists a Legendre curve $(\widetilde{\gamma}, \widetilde{\nu})$ such that $(\mathcal{E}v(\gamma), J(\nu)) \in \mathcal{PI}^{n-1}(\widetilde{\gamma}, \widetilde{\nu})$. Moreover, we have $(\gamma, \nu) \in \mathcal{PI}(\mathcal{E}v(\gamma), J(\nu))$. It follows that $(\gamma, \nu) \in \mathcal{PI}^n(\widetilde{\gamma}, \widetilde{\nu})$. Conversely, suppose that there exists a Legendre curve $(\widetilde{\gamma}, \widetilde{\nu})$ such that $(\gamma, \nu) \in \mathcal{PI}^n(\widetilde{\gamma}, \widetilde{\nu})$. Then there exists $(\overline{\gamma}, \overline{\nu}) \in \mathcal{PI}^{n-1}(\widetilde{\gamma}, \widetilde{\nu})$ such that $(\gamma, \nu) \in \mathcal{PI}(\overline{\gamma}, \overline{\nu})$. It follows that $(\mathcal{E}v(\gamma), J(\nu)) = (\overline{\gamma}, \overline{\nu})$. By the assumption of the induction, $\mathcal{E}v^{n-1}(\overline{\gamma})$ exists and $(\mathcal{E}v^{n-1}(\overline{\gamma}), J^{n-1}(\overline{\nu}))$ is a Legendre curve. Therefore, $\mathcal{E}v^{n-1}(\mathcal{E}v(\gamma)) = \mathcal{E}v^n(\gamma)$ exists and $(\mathcal{E}v^n(\gamma), J^n(\nu))$ is a Legendre curve. This completes the proof of Theorem. $\Box$

4 Example

We give examples of evolutes of frontals. These are useful to understand the phenomena and results.

Example 4.1 Let n, m and k be natural numbers with $m = n + k$ and $n \geq k$. Let $(\gamma, \nu) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ be of type (n, m),

$$\gamma(t) = \left(\frac{1}{n}t^n, \frac{1}{m}t^m\right), \ \nu(t) = \frac{1}{\sqrt{t^{2k}+1}}(-t^k, 1),$$

see Example 2.5. Then we have

$$\ell(t) = \frac{kt^{k-1}}{t^{2k}+1}, \ \beta(t) = -t^{n-1}\sqrt{t^{2k}+1},$$

see Example 2.5. Under the condition $n \geq k$, there exists a unique smooth function $\alpha(t) = -t^{n-k}(t^{2k}+1)^{\frac{3}{2}}/k$ such that $\beta(t) = \alpha(t)\ell(t)$. Hence the evolute of γ exists and is given by

$$\mathcal{E}v(\gamma)(t) = \left(-\frac{n-k}{nk}t^n - \frac{1}{k}t^{n+2k}, \frac{n+2k}{(n+k)k}t^{n+k} + \frac{1}{k}t^{n-k}\right).$$

As a concrete example, we take $n = 3, m = 5$ and $k = 2$. That is,

$$\gamma(t) = \left(\frac{1}{3}t^3, \frac{1}{5}t^5\right), \ \nu(t) = \frac{1}{\sqrt{t^4+1}}(-t^2, 1),$$

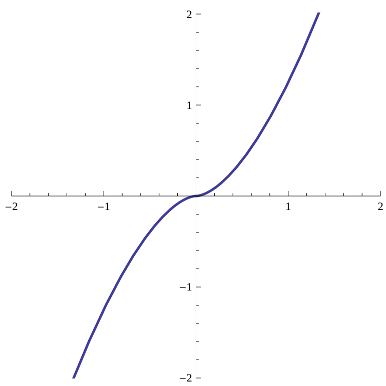
see Figure 1 left. Then the evolute of the 5/3 cusp is given by

$$\mathcal{E}v(\gamma)(t) = \left(-\frac{1}{6}t^3 - \frac{1}{2}t^7, \frac{1}{2}t + \frac{7}{10}t^5\right),$$

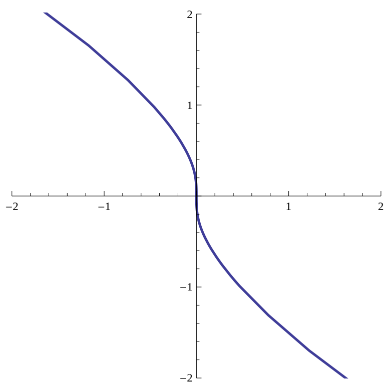
see Proposition 3.8 and Figure 1 centre. Moreover, the parallel curve for each $\lambda \in \mathbb{R}$ is given by

$$\gamma_\lambda(t) = \left(\frac{1}{3}t^3 - \frac{\lambda t^2}{\sqrt{t^4+1}}, \frac{1}{5}t^5 + \frac{\lambda}{\sqrt{t^4+1}}\right),$$

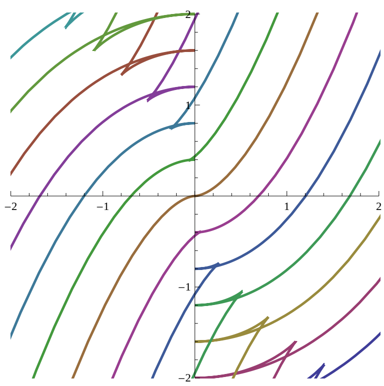
see Figure 1 right (cf. [1, 17]).



the 5/3 cusp



the evolute of the 5/3 cusp



parallel curves of the 5/3 cusp

Figure 1.

Example 4.2 Let n, m and k be natural numbers with $m = n + k$ and $n \geq k$. Let $(\gamma, \nu) : [0, 2\pi) \to \mathbb{R}^2 \times S^1$ be given by

$$\begin{aligned}
\gamma(t) &= (\sin^n t \cos^m t, \sin^m t \cos^n t), \\
\nu(t) &= \frac{1}{\sqrt{((2n+k)\cos^2 t - n - k)^2 \cos^{2k} t + ((2n+k)\sin^2 t - n - k)^2 \sin^{2k} t}} \\
&\quad \left(((2n+k)\sin^2 t - n - k)\sin^k t, ((2n+k)\cos^2 t - n - k)\cos^k t\right).
\end{aligned}$$

By a direct calculation, (γ, ν) is a Legendre curve. By definition, we have

$$\begin{aligned}
\boldsymbol{\mu}(t) &= \frac{1}{\sqrt{((2n+k)\cos^2 t - n - k)^2 \cos^{2k} t + ((2n+k)\sin^2 t - n - k)^2 \sin^{2k} t}} \\
&\quad \left(-((2n+k)\cos^2 t - n - k)\cos^k t, ((2n+k)\sin^2 t - n - k)\sin^k t\right), \\
\ell(t) &= \frac{k \sin^{k-1} t \cos^{k-1} t F(t; n, k)}{8(((2n+k)\cos^2 t - n - k)^2 \cos^{2k} t + ((2n+k)\sin^2 t - n - k)^2 \sin^{2k} t)}, \\
\beta(t) &= -\sin^{n-1} t \cos^{n-1} t \sqrt{((2n+k)\cos^2 t - n - k)^2 \cos^{2k} t + ((2n+k)\sin^2 t - n - k)^2 \sin^{2k} t},
\end{aligned}$$

where

$$F(t; n, k) = -k^2 + 4n(1+n) + 2k(1+2n) + (k^2 + 4n(-1+n) + 2k(-1+2n))\cos 4t.$$

Under the conditions $F(t; n, k) \neq 0$ for all $t \in [0, 2\pi)$ and $n \geq k$, there exists a unique smooth function

$$\alpha(t) = -\frac{8 \sin^{n-k} t \cos^{n-k} t(((2n+k)\cos^2 t - n - k)^2 \cos^{2k} t + ((2n+k)\sin^2 t - n - k)^2 \sin^{2k} t)^{\frac{3}{2}}}{kF(t; n, k)}.$$

Then the evolute of the frontal γ exists and is given by $\mathcal{E}v(\gamma)(t) = \gamma(t) - \alpha(t)\nu(t)$. Note that if $k = 1, 2, 3, 4, 5$ and $n \geq k$, then $F(t; n, k) \neq 0$ for all $t \in [0, 2\pi)$.

As a concrete example, we take $n = 3, m = 5$ and $k = 2$. That is,

$$\begin{aligned}
\gamma(t) &= \left(\sin^3 t \cos^5 t, \sin^5 t \cos^3 t\right), \\
\nu(t) &= \frac{2}{\sqrt{15 + 17\cos 4t + 4\cos 8t}} \left(3\sin^4 t - 5\sin^2 t \cos^2 t, 3\cos^4 t - 5\sin^2 t \cos^2 t\right),
\end{aligned}$$

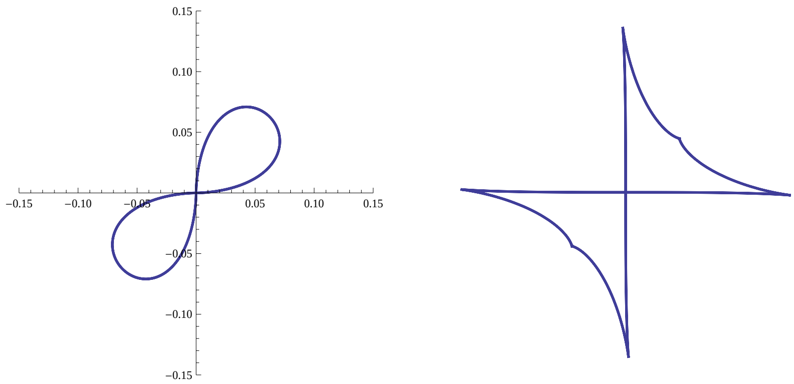
see Figure 2 left. Then we have

$$\begin{aligned}
\boldsymbol{\mu}(t) &= \frac{2}{\sqrt{15 + 17\cos 4t + 4\cos 8t}} \left(-3\cos^4 t + 5\sin^2 t \cos^2 t, 3\sin^4 t - 5\sin^2 t \cos^2 t\right), \\
\ell(t) &= \frac{12(3 + 2\cos 4t)\sin 2t}{15 + 17\cos 4t + 4\cos 8t}, \\
\beta(t) &= -\frac{1}{8}\sqrt{15 + 17\cos 4t + 4\cos 8t}\,\sin^2 2t.
\end{aligned}$$

It follows that there exists a unique smooth function $\alpha : [0, 2\pi) \to \mathbb{R}$,

$$\alpha(t) = -\frac{(15 + 17\cos 4t + 4\cos 8t)^{\frac{3}{2}}}{96(3 + 2\cos 4t)} \sin 2t.$$

The evolute $\mathcal{E}v(\gamma)(t) = \gamma(t) - \alpha(t)\nu(t)$ of the frontal γ see Figure 2 right.



$\gamma(t) = (\sin^3 t \cos^5 t, \sin^5 t \cos^3 t)$ the evolute $\mathcal{E}v(\gamma)(t)$

Figure 2.

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