HYDRODYNAMIC LIMIT FOR THE PLANCHEREL ENSEMBLE OF YOUNG DIAGRAMS AND FREE PROBABILITY

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Abstract. Concentration phenomena in statistical ensembles of Young diagrams have been investigated as static models first for the Plancherel ensemble by Vershik–Kerov and Logan–Shepp in 1970s and later for some other group-theoretical ensembles by Biane. On the other hand, a dynamical model of concentration for Young diagrams, which is not directly connected with group representations, was shown by Funaki–Sasada in the framework of hydrodynamic limit. Our aim here is to discuss a dynamical model of concentration for Young diagrams in group-theoretical ensembles. We especially feature Biane's approximate factorization property of ensembles as an origin to give rise to concentration. Starting from an initial ensemble yielding concentration and a microscopic dynamics keeping the Plancherel measure invariant, we derive an evolution of rescaled shapes of Young diagrams through hydrodynamic limit. The resulting evolution along macroscopic time is described in terms of the notions of Voiculescu's free probability theory such as free compression and free convolution of Kerov transition measures.

1. Introduction

The set of Young diagrams of size n, which is denoted by $\mathbb{Y}_n$, parametrizes all the equivalence classes of irreducible representations of the symmetric group $\mathfrak{S}_n$. The Plancherel measure $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ is a fundamental probability measure on $\mathbb{Y}_n$, which is defined by

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda) = \frac{(\dim \lambda)^2}{n!}, \qquad \lambda \in \mathbb{Y}_n, \tag{1.1}$$

where $\dim \lambda$ denotes the dimension of an irreducible representation of $\mathfrak{S}_n$ labeled by $\lambda \in \mathbb{Y}_n$. The probability space $(\mathbb{Y}_n, \mathbb{M}_{\mathrm{Pl}}^{(n)})$ is often called the Plancherel ensemble of Young diagrams of size n. It is known that a Young diagram chosen out of the Plancherel ensemble at random, strictly speaking according to the probability designed by $\mathbb{M}_{\mathrm{Pl}}^{(n)}$, should always look like a special shape. This is the famous concentration phenomenon to the limit

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shape discovered by Logan–Shepp[14] and Vershik–Kerov[17]. A more precise statement is formulated as a weak law of large numbers as follows. Identifying a Young diagram $\lambda \in \mathbb{Y}_n$ with its profile as defined in Subsection2.1(see Figure1), we consider the rescaled function by $1/\sqrt{n}$:

$$\lambda^{\sqrt{n}}(x) = \frac{1}{\sqrt{n}}\lambda(\sqrt{n}x), \qquad x \in \mathbb{R}. \tag{1.2}$$

Set

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x\arcsin\frac{x}{2} + \sqrt{4-x^2}), & |x| \leqq 2, \\ |x| & |x| > 2. \end{cases} \tag{1.3}$$

Then, for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{M}_{\mathrm{Pl}}^{(n)}\big(\{\lambda \in \mathbb{Y}_n \mid \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \Omega(x)| \geqq \epsilon\}\big) = 0. \tag{1.4}$$

The Plancherel measure can be lifted to the probability $\mathbb{M}_{\mathrm{Pl}}$ on the space $\mathfrak{T}$ consisting of the paths of Young diagrams

$$t = \big(t(0) = \varnothing \nearrow t(1) = \square \nearrow t(2) \nearrow \cdots \nearrow t(n) \nearrow \cdots\big), \quad t(n) \in \mathbb{Y}_n \tag{1.5}$$

so that it is the nth marginal distribution as

$$\mathbb{M}_{\mathrm{Pl}}^{(n)}(\lambda) = \mathbb{M}_{\mathrm{Pl}}(\{t \in \mathfrak{T} \mid t(n) = \lambda\}), \qquad \lambda \in \mathbb{Y}_n. \tag{1.6}$$

Then we can show almost sure convergence to the limit shape Ω as a strong law of large numbers. These kinds of concentration are observed in other ensembles than the Plancherel one. In [1] and [2], Biane pointed out the property of "approximate factorization" for states of group algebra $\mathbb{C}[\mathfrak{S}_n]$ induced by probabilities on $\mathbb{Y}_n$ and gave many interesting examples of these concentration phenomena. Approximate factorization property is interpreted as ergodicity in a certain weak sense. The limit shape in the Plancherel ensemble simply means the 1-point function of the system. A thorough treatment including higher correlation functions was developed by Borodin–Okounkov–Olshanski[4].

Let us note that the concentration to the limit shape of Young diagrams is a static result under the micro-macro correspondence by the rescale of (1.2). The purpose of this paper is to capture the evolution of the rescaled shapes of Young diagrams along macroscopic time under the same micro-macro correspondence, in other words, to treat what is called hydrodynamic limit. In [7], Funaki–Sasada showed remarkable results on hydrodynamic limit for the evolution of Young diagrams. In the model they treated, the microscopic transition rule governing the dynamics on the set of Young diagrams is given by creation and annihilation of a box according to the uniform distribution conditioned on their size. Although their model is simple and natural, our aim here is to consider hydrodynamic limit featuring the group-theoretical meaning of Young diagrams (which label the irreducible representations of symmetric groups). The results of Funaki–Sasada in [7] are given in the setting of the grand canonical ensemble under which a Markov chain on the totality of Young diagrams of all sizes allows variation of the number of boxes. The main theorem of this paper is stated in the canonical ensemble setting where we consider a Markov chain on $\mathbb{Y}_n$ and then take a limit of $n \to \infty$. When we discuss the Plancherel ensemble in the grand canonical manner, it seems natural to consider the probability on $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$ through poissonization of $\mathbb{M}_{\mathrm{Pl}}^{(n)}$'s. In Section4, we formulate this model and state some aspects which we meet with in computing its hydrodynamic limit.

The framework of the hydrodynamic limit in this paper is as follows. Recall that, if the microscopic Plancherel ensemble is zoomed out under scaling limit by $1/\sqrt{n}$ as (1.2), one observes Ω of (1.3) macroscopically. We consider a continuous time Markov chain $(X_s^{(n)})_{s \geqq 0}$ on the state space $\mathbb{Y}_n$ which keeps the Plancherel measure $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ invariant. Then, starting from the initial state $\mathbb{M}_{\mathrm{Pl}}^{(n)}$, the macroscopic shape remains Ω as time goes by. As an initial state on $\mathbb{Y}_n$ let us now take a probability $\mathbb{M}_0^{(n)}$ under which the ensemble yields concentration as $n \to \infty$. After $1/\sqrt{n}$-scaling limit of (1.2) with respect to $\mathbb{M}_0^{(n)}$, a certain macroscopic shape ω_0 is observed. When we drive the same Markov chain as above from the initial state $\mathbb{M}_0^{(n)}$, it is expected that the distribution $\mathbb{M}_s^{(n)}$ on $\mathbb{Y}_n$ at time s will tend to $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ as s goes by. Then, seen from the macroscopic point of view, an evolution from ω_0 to Ω should be observed. Speaking the scale more precisely, we assume that the microscopic ensemble $(\mathbb{Y}_n, \mathbb{M}_0^{(n)})$ appears if the macroscopic initial shape ω_0 is zoomed in by rescale of $\sqrt{n}$ multiple. Given a macroscopic time $t > 0$, we consider the situation of the ensemble after microscopically long time $s = tn$ and, observing it by the rescale of $1/\sqrt{n}$, see a macroscopic shape ω_t at time t. Note that the scale of time vs space is the diffusive one. Our aim is to prove this concentration to ω_t at each time t and to describe ω_t as explicitly as possible as a function of t. Since we have convergence to the Plancherel measure as $s \to \infty$ in the microscopic situation, ω_t should converge to the limit shape Ω as $t \to \infty$.

Let us proceed to state our main theorem which realizes the above framework. In order to take a Markov chain on $\mathbb{Y}_n$ which keeps $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ invariant, we consider (a special case of) the up and down transition probabilities used in [5]. For two Young diagrams λ and μ, whose sizes satisfying $|\lambda| + 1 = |\mu|$, we use the notation of $\lambda \nearrow \mu$ if μ is formed by adding one box to λ. For $\lambda \in \mathbb{Y}_n$ ($n \in \mathbb{N}$), set

(1.7)
$$P^{\uparrow}_{\lambda,\mu} = \begin{cases} \frac{\dim \mu}{(n+1)\dim \lambda}, & \lambda \nearrow \mu (\in \mathbb{Y}_{n+1}), \\ 0, & \text{otherwise}, \end{cases} \qquad P^{\downarrow}_{\lambda,\mu} = \begin{cases} \frac{\dim \mu}{\dim \lambda}, & \mu (\in \mathbb{Y}_{n-1}) \nearrow \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

The branching rules of induction and restriction

$$\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} \lambda \cong \bigoplus_{\mu \in \mathbb{Y}_{n+1} \colon \lambda \nearrow \mu} \mu, \qquad \mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \lambda \cong \bigoplus_{\mu \in \mathbb{Y}_{n-1} \colon \mu \nearrow \lambda} \mu$$

yield that $P^{\uparrow}$ and $P^{\downarrow}$ satisfy

$$\sum_{\mu\in\mathbb{Y}_{n+1}} P^{\uparrow}_{\lambda,\mu} = 1, \qquad \sum_{\mu\in\mathbb{Y}_{n-1}} P^{\downarrow}_{\lambda,\mu} = 1.$$

Hence, setting $P^{(n)} = P^{\uparrow}P^{\downarrow}$, namely

$$P^{(n)}_{\lambda,\mu} = \sum_{\nu\in\mathbb{Y}_{n+1}} P^{\uparrow}_{\lambda,\nu}P^{\downarrow}_{\nu,\mu}, \qquad \lambda,\mu\in\mathbb{Y}_n, \tag{1.8}$$

we have $P^{(n)}$ to be a stochastic matrix of degree $|\mathbb{Y}_n|$.

Lemma 1.1. $\mathbb{M}^{(n)}_{\mathrm{Pl}}$ *of* (1.1) *is symmetric with respect to* $P^{(n)}$ *of* (1.8)*, namely we have*

$$\mathbb{M}^{(n)}_{\mathrm{Pl}}(\lambda)P^{(n)}_{\lambda,\mu} = \mathbb{M}^{(n)}_{\mathrm{Pl}}(\mu)P^{(n)}_{\mu,\lambda}, \qquad \lambda,\mu\in\mathbb{Y}_n. \tag{1.9}$$

Hence a Markov chain on $\mathbb{Y}_n$ *with the transition probability matrix* $P^{(n)}$ *keeps* $\mathbb{M}^{(n)}_{\mathrm{Pl}}$ *invariant.*

Proof. We see from (1.8) and (1.7)

$$P^{(n)}_{\lambda,\mu} = \sum_{\nu\in\mathbb{Y}_{n+1}:\lambda\nearrow\nu,\mu\nearrow\nu} \frac{\dim\mu}{(n+1)\dim\lambda}.$$

If $\lambda,\mu\in\mathbb{Y}_n$ are distinct, then Young diagram $\nu\in\mathbb{Y}_{n+1}$ satisfying both $\lambda\nearrow\nu$ and $\mu\nearrow\nu$ can exist at most one, which could be $\nu=\lambda\vee\mu$ (= set-theoretical union of the boxes). The number of Young diagrams $\nu\in\mathbb{Y}_{n+1}$ satisfying $\lambda\nearrow\nu$ agrees with the number of valleys of λ (see (2.1) in Section2). We thus obtain

$$P^{(n)}_{\lambda,\mu} = \begin{cases} \frac{|\{\text{valleys of }\lambda\}|}{n+1}, & \lambda=\mu, \\ \frac{\dim\mu}{(n+1)\dim\lambda}, & \lambda\vee\mu\in\mathbb{Y}_{n+1}, \\ 0, & \text{otherwise} \end{cases} \tag{1.10}$$

for $\lambda,\mu\in\mathbb{Y}_n$. (1.9) follows from (1.1) and (1.10). $\square$

Let us consider a continuous time Markov chain $(X^{(n)}_s)_{s\geqq 0}$ with the transition matrix $P^{(n)}$ on the state space $\mathbb{Y}_n$. $\mathcal{M}^{(n)}$ denotes the induced probability on the set of functions (= paths) from $[0,\infty)$ to $\mathbb{Y}_n$. $\mathbb{M}^{(n)}_0$ being the initial distribution on $\mathbb{Y}_n$, the distribution $\mathcal{M}^{(n)}(X^{(n)}_s = \ \cdot\ )$ at time s is given by

$$\mathcal{M}^{(n)}(X^{(n)}_s=\mu) = \sum_{\lambda\in\mathbb{Y}_n} \mathbb{M}^{(n)}_0(\lambda)(e^{s(P^{(n)}-I)})_{\lambda,\mu}, \qquad \mu\in\mathbb{Y}_n. \tag{1.11}$$

Identifying a Young diagram with its profile, put it into the space $\mathbb{D}$ consisting of the continuous diagrams in order to discuss its scaling limit. Here

$\omega \in \mathbb{D}$ is by definition a function $\mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following conditions:

$$|\omega(x) - \omega(y)| \leqq |x - y|, \qquad x, y \in \mathbb{R}, \tag{1.12}$$

$$\omega(x) = |x| \quad \text{if } |x| \text{ is large enough.} \tag{1.13}$$

$\mathfrak{m}_\omega$ denotes the transition measure of $\omega \in \mathbb{D}$ (see (2.2) in Subsection2.1). The transition measure of Ω in (1.3) is the standard semi-circle distribution such that

$$\mathfrak{m}_\Omega(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\, 1_{[-2,2]}(x)dx. \tag{1.14}$$

For a probability on $\mathbb{R}$, its kth moment and free cumulant are denoted by $M_k(\ \cdot\ )$ and $R_k(\ \cdot\ )$ respectively, $k \in \mathbb{N}$ (see Subsection2.2). A decisive fact is that the free cumulant sequence of (1.14) is

$$\begin{aligned} &R_1(\mathfrak{m}_\Omega) = 0, \quad R_2(\mathfrak{m}_\Omega) = 1, \\ &R_3(\mathfrak{m}_\Omega) = R_4(\mathfrak{m}_\Omega) = R_5(\mathfrak{m}_\Omega) = \cdots = 0. \end{aligned} \tag{1.15}$$

The ensembles admitting approximate factorization property cover a wide range of known examples for concentration phenomena ([2]). We formulate the relevant conditions and resulting concentration as follows (partly recalling [8], [9]). Given a probability $\mathbb{M}$ on $\mathbb{Y}_n$, we have a positive-definite normalized central function on $\mathfrak{S}_n$ by

$$f(x) = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{M}(\lambda)\tilde{\chi}^\lambda(x), \qquad x \in \mathfrak{S}_n. \tag{1.16}$$

Here χ^λ is the irreducible character of $\mathfrak{S}_n$ labeled by λ, and we set $\tilde{\chi}^\lambda = \chi^\lambda / \dim\lambda$. The map $\mathbb{M} \mapsto f$ of (1.16) is an affine bijection from the set of probabilities on $\mathbb{Y}_n$ onto

$$\bigl\{ f : \mathfrak{S}_n \longrightarrow \mathbb{C} \,\big|\, \text{positive-definite},\ f(e) = 1,\ f(x^{-1}yx) = f(y)\ (x, y \in \mathfrak{S}_n) \bigr\}. \tag{1.17}$$

Moreover, f in (1.17) is uniquely extended to a tracial state of $\mathbb{C}[\mathfrak{S}_n]$ by linearity. If f is a central function on $\mathfrak{S}_n$ and σ is a Young diagram with $|\sigma| \leqq n$, the value of f at an element of cycle type σ in $\mathfrak{S}_n$ is denoted by $f_{(\sigma, 1^{n-|\sigma|})}$. Note that σ can contain one-box rows in this notation.

Definition 1.2. Given ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)})\}_{n \in \mathbb{N}}$, let $f^{(n)}$ be the corresponding tracial state of $\mathbb{C}[\mathfrak{S}_n]$ determined by (1.16). We refer to the following set of conditions as Condition (AF).

•(approximate factorization property) For any $p \in \mathbb{N}$ and any $k_1, \cdots, k_p \in \{2, 3, \cdots\}$,

$$f^{(n)}_{((k_1)\sqcup\cdots\sqcup(k_p), 1^{n-(k_1+\cdots+k_p)})} - f^{(n)}_{(k_1, 1^{n-k_1})} \cdots f^{(n)}_{(k_p, 1^{n-k_p})} = o\bigl(n^{-\frac{(k_1-1)+\cdots+(k_p-1)}{2}}\bigr) \tag{1.18}$$

as $n \to \infty$.

•(asymptotic cycle mean) There exist $a > 0$ and a real sequence $\{r_3, r_4, \cdots\}$ such that, for any $k \in \{2, 3, \cdots\}$,

$$\lim_{n\to\infty} n^{\frac{k-1}{2}} f^{(n)}_{(k,1^{n-k})} = r_{k+1}, \tag{1.19}$$

$$|r_{k+1}| \leqq a^{k+1}. \tag{1.20}$$

□

The meaning of growth order $n^{\frac{k-1}{2}}$ for the k-cycle value appearing in (1.18) and (1.19) of Condition (AF) may be unclear at a glance. One will be convinced by knowing the form of Kerov polynomials that this growth order fits with our rescale of (1.2). See the approximate computation (2.17) in Subsection2.3.

Theorem 1.3. *Let ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)})\}_{n\in\mathbb{N}}$ satisfy Condition (AF) of Definition1.2. Then, there exists $\omega \in \mathbb{D}$ such that*

$$\lim_{n\to\infty} \mathbb{M}^{(n)}\big(\big\{\lambda \in \mathbb{Y}_n \bigm| \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \omega(x)| \geqq \epsilon\big\}\big) = 0 \tag{1.21}$$

for any $\epsilon > 0$.

We call the continuous diagram ω the macroscopic shape for $\mathbb{M}^{(n)}$.

This concentration result is due to Biane ([1], [2]). Proof of Theorem1.3 is reconstructed in Section3 for the use of proving the main theorem. A stronger fact than (1.21) is shown in (3.8) and (3.10). As typical examples of ensembles satisfying Condition (AF), we mention later in AppendixA balanced irreducible characters and Thoma ensembles. Something like independence is suggested by (1.18) for indicator functions with disjoint supports. It will be also clarified there that approximate factorization property can be regarded as a certain weak ergodicity of the ensemble.

Here is our main theorem. Free convolution of two probabilities μ and ν on $\mathbb{R}$ is denoted by $\mu \boxplus \nu$. For a probability μ on $\mathbb{R}$ and $c \in (0, 1]$, μ_c denotes the probability on $\mathbb{R}$ obtained as free compression by a projection of expectation c (see Subsection2.2).

Theorem 1.4. *For the Markov chain $(X^{(n)}_s)_{s\geqq 0}$ of (1.11), assume that initial ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)}_0)\}_{n\in\mathbb{N}}$ satisfies Condition (AF) of Definition1.2. For any (macroscopic time) $t \geqq 0$, consider the distribution of the Markov chain at $s = tn$:*

$$\mathbb{M}^{(n)}_t(\lambda) = \mathcal{M}^{(n)}(X^{(n)}_{tn} = \lambda), \qquad \lambda \in \mathbb{Y}_n. \tag{1.22}$$

Then, $\{(\mathbb{Y}_n, \mathbb{M}^{(n)}_t)\}_{n\in\mathbb{N}}$ satisfies Condition (AF) for any $t \geqq 0$. There exists a function $t \in [0, \infty) \longmapsto \omega_t \in \mathbb{D}$ such that, for any $t \geqq 0$ and any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{M}^{(n)}_t\big(\big\{\lambda \in \mathbb{Y}_n \bigm| \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \omega_t(x)| \geqq \epsilon\big\}\big) = 0. \tag{1.23}$$

Here the macroscopic shape $\omega_t \in \mathbb{D}$ at time t is characterized through its transition measure $\mathfrak{m}_{\omega_t}$ by the following operations of free compression and *free convolution:*

$$\mathfrak{m}_{\omega_t} = (\mathfrak{m}_{\omega_0})_{e^{-t}} \boxplus (\mathfrak{m}_{\Omega})_{1-e^{-t}}, \qquad t > 0. \tag{1.24}$$

Remark 1.5. In terms of a free cumulant sequence [1], (1.24) is translated into

$$\begin{aligned} &R_1(\mathfrak{m}_{\omega_t}) = 0, \quad R_2(\mathfrak{m}_{\omega_t}) = 1, \\ &R_k(\mathfrak{m}_{\omega_t}) = R_k(\mathfrak{m}_{\omega_0})e^{-(k-1)t}, \qquad k \geqq 3 \end{aligned} \tag{1.25}$$

(see (2.10) and (2.8) in Section2 for the free cumulants obtained by free compression and free convolution). Since we have for any $k \in \mathbb{N}$

$$\lim_{t\to\infty} R_k(\mathfrak{m}_{\omega_t}) = R_k(\mathfrak{m}_{\Omega}) \tag{1.26}$$

by (1.25), ω_t converges to Ω in $\mathbb{D}$ as $t \to \infty$ in a macroscopic point of view.

Remark 1.6. If we adopt $P^{\downarrow}P^{\uparrow}$ instead of $P^{\uparrow}P^{\downarrow}$ as the transition matrix of a Markov chain on $\mathbb{Y}_n$ governing the microscopic dynamics, we still have invariance of $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ and the same result for hydrodynamic limit. See Remark3.5.

In [7] and most results on hydrodynamic limit in other models also, a fundamental task is to describe the evolution along macroscopic time by specifying a (non-linear) partial differential equation for ω_t obtained through the scaling limit. In the contexts of this paper, as may be seen from the strategy of the proof of Theorem1.4 stated in the next paragraph, the shape of an element of $\mathbb{D}$ can be efficiently analyzed by the free cumulant sequence of the transition measure. The correspondence between

$$\omega \in \mathbb{D} \;\longleftrightarrow\; \mathfrak{m}_{\omega} \;\longleftrightarrow\; \{R_k(\mathfrak{m}_{\omega})\}_{k\in\mathbb{N}}$$

will be explained in Subsections 2.1 and 2.2 . As for a partial differential equation for the evolution, we derive the one for the Stieltjes transform of $\mathfrak{m}_{\omega_t}$:

$$G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_{\omega_t}(dx), \qquad z \in \mathbb{C}^+ \tag{1.27}$$

in Proposition3.3.

Let us note brief ideas for the proofs of Theorem1.3 and Theorem1.4, whose details are written in Section3. Since the Plancherel ensemble comes from representations of symmetric groups, it is congenial to group characters in computation. Our basic trick is as follows. In computing expectations, devise to replace free cumulant $R_{k+1}(\mathfrak{m}_\lambda)$ of the transition measure $\mathfrak{m}_\lambda$ of $\lambda \in \mathbb{Y}_n$ by the (suitably normalized) irreducible character value at k-cycle of $\mathfrak{S}_n$ corresponding to λ. These procedures are justified by analysis of the Kerov–Olshanski algebra consisting of polynomial functions of coordinates of Young diagrams as well as a (rough) application of the so-called Kerov polynomial. We review these notions in Subsection2.3. Such replacement being justified, the proof of Theorem1.4 will proceed by using induction/restriction

[1] Note that, for ω_0 also, (1.21) yields $R_1(\mathfrak{m}_{\omega_0}) = 0$, $R_2(\mathfrak{m}_{\omega_0}) = 1$.

of irreducible characters and similar devices for proving concentration to the limit shape Ω.

After Introduction, the subsequent sections are organized as follows. Section2 is devoted to reviewing necessary notions and properties. Introduced are the profile of a Young diagram, its transition measure, several notions in Voiculescu's free probability theory and so on. Then we give explanations on structure of the Kerov–Olshanski algebra and the Kerov polynomials. The proofs of Theorem1.3 and Theorem1.4 are given in Section3 along the strategy stated above. In Section4, we consider the poissonized Plancherel ensemble in which the microscopic dynamics allows variation of sizes of Young diagrams and discuss a setting of the grand canonical ensemble for hydrodynamic limit. Finally, we supplement comments about some results on approximate factorization property for a state of the group algebra of a symmetric group in AppendixA.

2. Preliminaries

2.1. Young diagram, continuous diagram and transition measure. A Young diagram is presented by $\lambda = (\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{l(\lambda)} > 0)$ where $l(\lambda)$ denotes the number of rows in λ or by $\lambda = (1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots j^{m_j(\lambda)}\cdots)$ where $m_j(\lambda)$ denotes the number of rows of length j in λ. In this paper we mainly display a Young diagram by loading the region $y > |x|$ of the xy-plane with squares (= boxes) of edge length $\sqrt{2}$ as Figure1. The piecewise linear graph presented by bold lines in Figure1 is called the profile of a Young diagram. A Young diagram identified with its profile is regarded as a continuous diagram defined by (1.12) and (1.13). Setting $\mathbb{D}_0 = \{\lambda \in \mathbb{D} \,|\, \lambda \text{ is piecewise linear, } \lambda'(x) = \pm 1\}$, we have $\mathbb{Y} \subset \mathbb{D}_0 \subset \mathbb{D}$. We encode $\lambda \in \mathbb{D}_0$ by using the x-coordinates of the valleys and peaks of λ

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{r-1} < y_{r-1} < x_r \tag{2.1}$$

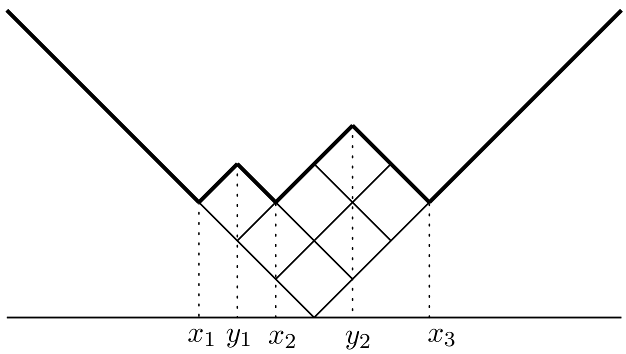

Figure 1. Young diagram, its profile and coordinates

as Figure1. Conversely, interlacing x_i's and y_i's as (2.1) are the coordinates of some $\lambda \in \mathbb{D}_0$ if and only if they satisfy

$$\sum_{i=1}^{r} x_i - \sum_{i=1}^{r-1} y_i = 0.$$

An atomic probability $\mathfrak{m}_\lambda$ on $\mathbb{R}$ is uniquely assigned to any $\lambda = (x_1 < y_1 < \cdots < y_{r-1} < x_r) \in \mathbb{D}_0$ by

$$\frac{(z-y_1)\cdots(z-y_{r-1})}{(z-x_1)\cdots(z-x_r)} = \int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_\lambda(dx), \qquad z \in \mathbb{C}\setminus\mathbb{R}. \tag{2.2}$$

Clearly, $\mathrm{supp}\mathfrak{m}_\lambda$ coincides with $\{x_1, \cdots, x_r\}$. $\mathfrak{m}_\lambda$ is called the (Kerov) transition measure of λ. On the other hand, the Rayleigh measure of $\lambda \in \mathbb{D}_0$ is defined by

$$\tau_\lambda = \sum_{i=1}^{r} \delta_{x_i} - \sum_{i=1}^{r-1} \delta_{y_i} \tag{2.3}$$

as a signed measure on $\mathbb{R}$ with total measure 1. The moment sequences $\{M_n(\mathfrak{m}_\lambda)\}_{n=0,1,2,\cdots}$ and $\{M_n(\tau_\lambda)\}_{n=0,1,2,\cdots}$ for $\lambda \in \mathbb{D}_0$ are related by

$$\sum_{n=0}^{\infty} \frac{M_n(\mathfrak{m}_\lambda)}{z^n} = \exp\Big(\sum_{k=1}^{\infty} \frac{M_k(\tau_\lambda)}{kz^k}\Big), \qquad z \in \mathbb{C},\ |z| \gg 1. \tag{2.4}$$

As a consequence of (2.4), we see that $\{M_n(\mathfrak{m}_\lambda)\}$ and $\{M_n(\tau_\lambda)\}$ are expressed as polynomials of each other. The transition measure $\mathfrak{m}_\omega$ can be defined for any continuous diagram $\omega \in \mathbb{D}$ also through a limiting procedure. A convenient way is that, taking a uniformly approximate sequence $\{\lambda^{(k)}\}_{k\in\mathbb{N}} \subset \mathbb{D}_0$ for a given $\omega \in \mathbb{D}$ and recognizing that $\{M_n(\mathfrak{m}_{\lambda^{(k)}})\}_{k\in\mathbb{N}}$ forms a Cauchy sequence for any $n \in \mathbb{N}$ by the above mentioned polynomial relations, we apply an easy moment problem for a compactly supported probability on $\mathbb{R}$. Then, (2.4) is extended to

$$\int_{\mathbb{R}} \frac{1}{z-x}\mathfrak{m}_\omega(dx) = \frac{1}{z}\exp\Big\{\int_{\mathbb{R}} \frac{1}{x-z}\Big(\frac{\omega(x)-|x|}{2}\Big)' dx\Big\}, \qquad z \in \mathbb{C}\setminus\mathbb{R} \tag{2.5}$$

for $\omega \in \mathbb{D}$. Conversely, given a probability μ on $\mathbb{R}$ with compact support and mean 0, there corresponds a unique $\omega \in \mathbb{D}$ such that $\mu = \mathfrak{m}_\omega$, though Rayleigh measure τ_ω does not necessarily exist. In this case, we mention a Rayleigh function $F : \mathbb{R} \longrightarrow [0,1]$, which satisfies by definition $F(x) = 0$ on $(-\infty, a]$ and $F(x) = 1$ on $[b, \infty)$ for some $a, b \in \mathbb{R}$ and coincides with the distribution function of the Rayleigh measure if it exists. The relation corresponding to (2.5) between μ, F and ω is given by

$$\int_{\mathbb{R}} \frac{1}{z-x}\mu(dx) = \frac{1}{z}\exp\Big\{-\int_{-\infty}^{0} \frac{F(x)}{z-x}dx + \int_{0}^{\infty} \frac{1-F(x)}{z-x}dx\Big\}, \tag{2.6}$$

$$\omega(u) = \int_{-\infty}^{u} F(x)dx + \int_{u}^{\infty} (1-F(x))dx. \tag{2.7}$$

In this paper, we treat only the case where suppμ is compact. Including consideration of probabilities with non-compact supports and certain moment conditions, a thorough treatment of interplay between μ, F and ω is given in [12]. The map $F \mapsto \mu$ is called the Markov transform. Let us refer to the topology on $\mathbb{D}$ defined by the family of semi-distances

$$\left\{ d_k(\omega_1, \omega_2) = |M_k(\mathfrak{m}_{\omega_1}) - M_k(\mathfrak{m}_{\omega_2})| \ (\omega_1, \omega_2 \in \mathbb{D}) \right\}_{k \in \mathbb{N}}.$$

as the moment topology on $\mathbb{D}$. We see the following facts about topologies on $\mathbb{D}$ from the definition of $\mathbb{D}$ and (2.5).

Lemma 2.1. *(1) The pointwise and uniform convergence topologies on $\mathbb{D}$ coincide.*

(2) Restricted on $\{\omega \in \mathbb{D} \,|\, \omega(x) = |x| \text{ for } |x| \geqq a\}$ for $a > 0$, the uniform convergence and moment topologies coincide.

(3) The moment topology is stronger than the uniform convergence one on $\mathbb{D}$.

Proof. (1) directly follows from the conditions (1.12) and (1.13) to define $\mathbb{D}$. (2) and (3) are shown by using (2.5). □

2.2. **Free probability theory.** Let us recall several notions in free probability theory due to Voiculescu. See [15] for details. Here we mainly pick up combinatorial aspects only for one-dimensional distributions. Consider a C^*-probability space $(\mathcal{A}, \phi)$ consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ of $\mathcal{A}$. Let μ, ν be the distributions of self-adjoint $a, b \in \mathcal{A}$ respectively. If a and b are free, the distribution of $a+b$ is denoted by $\mu \boxplus \nu$ and called the (additive) free convolution of μ and ν. Given compactly supported probabilities μ, ν on $\mathbb{R}$, $\mu \boxplus \nu$ is uniquely determined and compactly supported. In terms of free cumulants of probabilities on $\mathbb{R}$, the free convolution is characterized by

$$R_k(\mu \boxplus \nu) = R_k(\mu) + R_k(\nu), \qquad k \in \mathbb{N}. \tag{2.8}$$

Here $R_k(\mu)$ denotes the kth free cumulant of μ, which is defined (from a combinatorial viewpoint) by the following cumulant-moment formula (2.9). $\mathrm{NC}(n)$ denoting the set of non-crossing partitions of $\{1, 2, \cdots, n\}$, extend subscripts of cumulants (and moments also) to partition $\pi = (v_1, \cdots, v_l) \in \mathrm{NC}(n)$ by setting

$$R_\pi(\mu) = R_{|v_1|}(\mu) \cdots R_{|v_l|}(\mu)$$

multiplicatively (where $|v_i|$ denotes the cardinality of block v_i). Then, we have

$$M_n(\mu) = \sum_{\pi \in \mathrm{NC}(n)} R_\pi(\mu), \qquad n \in \mathbb{N}. \tag{2.9}$$

The Möbius function of the poset $\mathrm{NC}(n)$ enables us to write an inversion formula of (2.9), which expresses $R_n(\mu)$ explicitly in terms of $M_k(\mu)$'s.

If $q \in \mathcal{A}$ is a projection ($q^2 = q = q^*$) such that the expectation $\phi(q) \neq 0$, setting $\mathcal{B} = q\mathcal{A}q$ and $c = \phi(q)$, we have a C^*-probability space $(\mathcal{B}, \frac{1}{c}\phi\big|_{\mathcal{B}})$.

Moreover, if self-adjoint $a \in \mathcal{A}$ and q are free, then the distribution of qaq with respect to $\frac{1}{c}\phi\big|_{\mathcal{B}}$ is called the free compression of the distribution μ of a and denoted by μ_c. Given a compactly supported probability μ on $\mathbb{R}$ and $c \in (0,1]$, the free compression μ_c of μ is uniquely determined. In terms of free cumulants, the free compression is characterized by

$$R_k(\mu_c) = c^{k-1}R_k(\mu) \ \big(= \frac{1}{c}R_k(\mu(\frac{1}{c}\,\cdot\,))\big), \qquad k \in \mathbb{N}. \tag{2.10}$$

For a compactly supported probability μ on $\mathbb{R}$, set

$$K_\mu(\zeta) = \frac{1}{\zeta} + \sum_{k=0}^{\infty} R_{k+1}(\mu)\zeta^k, \qquad \zeta \in \mathbb{C}. \tag{2.11}$$

The Taylor series part of (2.11) is Voiculescu's R-transform of μ. K_μ of (2.11) and the Stieltjes transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu(dx), \qquad z \in \mathbb{C}$$

of μ are inverse functions of each other in appropriate domains such that $|\zeta| \ll 1 \longleftrightarrow |z| \gg 1$.

2.3. **Kerov–Olshanski algebra.** An algebra of polynomial functions in the coordinates encoding Young diagrams was introduced in [13] and plays an important role in asymptotic representation theory of symmetric groups. It is essential to compute the transition coefficients between several generating systems of the algebra as explicitly as possible. Substantial contributions in this direction were given in [11]. It is natural to define this algebra by introducing as generators the power sums

$$p_n(\lambda) = \sum_i \big(a_i^n + (-1)^{n-1}b_i^n\big)$$

which are supersymmetric with respect to the Frobenius coordinates (a_i, b_i) of Young diagrams. In this paper, however, since we mainly utilize the x-coordinates of the valleys and peaks (2.1) as coordinates of Young diagrams, we pick up defining generators in terms of these coordinates from the beginning.

For $\lambda = (x_1 < y_1 < \cdots < y_{r-1} < x_r) \in \mathbb{Y}$, consider the nth moment of the Rayleigh measure τ_λ (2.3) of λ:

$$M_n(\tau_\lambda) = \sum_{i=1}^{r} x_i^n - \sum_{i=1}^{r-1} y_i^n, \qquad n \in \mathbb{N}. \tag{2.12}$$

$M_n(\tau_\lambda)$'s for $n \geqq 2$ are algebraically independent (over $\mathbb{R}$). Actually,

$$\big(M_2(\tau_\lambda)\ M_3(\tau_\lambda)\ \cdots\big) = \big(2p_1(\lambda)\ 3p_2(\lambda)\ \cdots\big)A$$

holds for an upper triangular matrix $A = (a_{ij})$ with $a_{ii} = 1$, $a_{ij} \geqq 0$. Let us call the algebra $\mathbb{A}$ generated by $\{1, M_n(\tau_\lambda) \,|\, n \in \{2,3,\cdots\}\}$ (over $\mathbb{R}$) the Kerov–Olshanski algebra after [13]. Since $M_n(\tau_\lambda)$ looks like a power sum, we declare $M_n(\tau_\lambda)$ to be an nth homogeneous element of $\mathbb{A}$. This gives rise to a filtration in $\mathbb{A}$. Then, the top degree of $f \in \mathbb{A}$ is denoted by $\mathrm{wt}(f)$. The symbol wt comes from "weight" (see [11]). We have $\mathrm{wt}(M_n(\tau_\lambda)) = n$ by definition and $\mathrm{wt}(M_n(\mathfrak{m}_\lambda)) = n$ by (2.4). Moreover, (2.9) yields $\mathrm{wt}(R_n(\mathfrak{m}_\lambda)) = n$.

Define a function Σ_ρ on $\mathbb{Y}$ for any $\rho \in \mathbb{Y}$ by

$$\Sigma_\rho(\lambda) = |\lambda|(|\lambda|-1)\cdots(|\lambda|-|\rho|+1)\tilde{\chi}^\lambda_{(\rho,1^{|\lambda|-|\rho|})}, \qquad \lambda \in \mathbb{Y}. \tag{2.13}$$

Here $(\rho, 1^{|\lambda|-|\rho|}) \in \mathbb{Y}_{|\lambda|}$ is the Young diagram having $|\lambda| - |\rho|$ one-box rows besides ρ. Note that $m_1(\rho) > 0$ can occur in (2.13). Furthermore, $\Sigma_\rho(\lambda) = 0$ by definition for $\lambda \in \mathbb{Y}$ such that $|\lambda| < |\rho|$. We can see that

$$\Sigma_\rho \in \mathbb{A}, \qquad \mathrm{wt}(\Sigma_\rho) = |\rho| + l(\rho) \tag{2.14}$$

hold for any $\rho \in \mathbb{Y}$, and

$$\Sigma_\rho \Sigma_\sigma = \Sigma_{\rho\sqcup\sigma} + \sum_{\tau\in\mathbb{Y}:\mathrm{wt}(\Sigma_\tau)<\mathrm{wt}(\Sigma_\rho)+\mathrm{wt}(\Sigma_\sigma)} c_\tau \Sigma_\tau, \qquad c_\tau \in \mathbb{R} \tag{2.15}$$

holds for any $\rho, \sigma \in \mathbb{Y}$. In (2.15), the range of τ is also expressed as

$$|\tau| + l(\tau) \leqq |\rho| + l(\rho) + |\sigma| + l(\sigma) - 1.$$

See [11] for (2.14) and (2.15). In particular, if $\rho = (k)$ is a one-row diagram of length k, $\Sigma_{(k)}$ is abbreviated to Σ_k. We have $\mathrm{wt}(\Sigma_k) = k + 1$ for $k \in \mathbb{N}$. $\Sigma_k(\lambda)$'s and $R_k(\mathfrak{m}_\lambda)$'s are combined by the Kerov polynomial as follows.

Theorem 2.2 (Kerov polynomial)**.** *For $k \in \mathbb{N}$ such that $k \geqq 3$, there exists a polynomial $P_k(x_2, \cdots, x_{k-1})$ in $k-2$ variables satisfying*

$$\Sigma_k(\lambda) = R_{k+1}(\mathfrak{m}_\lambda) + P_k(R_2(\mathfrak{m}_\lambda), \cdots, R_{k-1}(\mathfrak{m}_\lambda)), \qquad \lambda \in \mathbb{Y} \tag{2.16}$$

where each term consisting of the lower part $P_k(R_2(\mathfrak{m}_\lambda), \cdots, R_{k-1}(\mathfrak{m}_\lambda))$ can take $k-1, k-3, \cdots$ only as its weight. The polynomial P_k is unique. □

We see from the definition that $\Sigma_1(\lambda) = R_2(\mathfrak{m}_\lambda) = |\lambda|$, $\Sigma_2(\lambda) = R_3(\mathfrak{m}_\lambda)$ for $k = 1, 2$. $\mathbb{A}$ is generated also by $\{1, \Sigma_k \mid k \in \mathbb{N}\}$. Derivation of the Kerov polynomials based on Okounkov's idea is given in [3]. Kerov's conjecture that all the coefficients of the Kerov polynomials belong to $\mathbb{N}$ was proved by Feray in [6]. In this paper, we need only the top term correspondence in (2.16) and do not use fine properties of the Kerov polynomial. $M_k(\mathfrak{m}_\lambda)$'s and $R_k(\mathfrak{m}_\lambda)$'s are useful for describing the shape of $\lambda \in \mathbb{Y}$ through its transition measure $\mathfrak{m}_\lambda$ while $\Sigma_k(\lambda)$'s are congenial to computation of expectations with respect to the Plancherel measure. The basic trick of replacing $R_{k+1}(\mathfrak{m}_\lambda)$ by $\Sigma_k(\lambda)$, which is anticipated in Introduction, is supported by (2.16). Roughly speaking, under the weight filtration we have

$$\begin{aligned}\tilde{\chi}^\lambda(k\text{-cycle}) &\doteqdot n^{-k}\Sigma_k(\lambda) \doteqdot n^{-k}R_{k+1}(\mathfrak{m}_\lambda) \\ &= n^{-\frac{k-1}{2}} n^{-\frac{k+1}{2}} R_{k+1}(\mathfrak{m}_\lambda) = n^{-\frac{k-1}{2}} R_{k+1}(\mathfrak{m}_{\lambda\sqrt{n}}).\end{aligned} \tag{2.17}$$

Since $\mathfrak{m}_{\lambda\sqrt{n}}$ should have a macroscopic meaning in our rescale of (1.2) for $\lambda \in \mathbb{Y}_n$, (2.17) suggests the growth order in Condition (AF).

3. Concentration and Hydrodynamic Limit

The expectation with respect to a probability $\mathbb{M}$ is denoted by $E_{\mathbb{M}}[\ \cdot\ ]$. Condition (AF) is presented in Definition1.2. Recall the correspondence $\mathbb{M}^{(n)} \leftrightarrow f^{(n)}$ of (1.16).

Lemma 3.1. *Condition (AF) for ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)})\}_{n\in\mathbb{N}}$ yields*

$$f^{(n)}_{(\sigma,1^{n-|\sigma|})} = O\big(n^{-\frac{|\sigma|-l(\sigma)}{2}}\big) \qquad \text{as} \quad n\to\infty \tag{3.1}$$

for any $\sigma \in \mathbb{Y}$. We then have

$$E_{\mathbb{M}^{(n)}}[\Sigma_\rho] = O\big(n^{\frac{|\rho|+l(\rho)}{2}}\big) \qquad \text{as} \quad n\to\infty \tag{3.2}$$

for any $\rho \in \mathbb{Y}$.

Proof. Since $m_1(\sigma) = 0$ is assumed without loss of generality, let $\sigma = (\sigma_1 \geqq \cdots \geqq \sigma_{l(\sigma)} \geqq 2)$. Then, (1.18) and (1.19) immediately give (3.1). $\square$

Our strategy for proving the main theorem is as follows. First we show Theorem1.3. This implies concentration for $\mathbb{M}_0^{(n)}$. Next we show that Condition (AF) is inherited from $\mathbb{M}_0^{(n)}$ to $\mathbb{M}_t^{(n)}$ for any $t>0$. At the same time, we clarify the connection of the asymptotic cycle means between time 0 and t. This gives the desired characterization of ω_t.

Proof. (Proof of Theorem1.3) Keeping (2.16) in mind, let us show the convergence of variance

$$\lim_{n\to\infty} E_{\mathbb{M}^{(n)}}\Big[\Big(\frac{\Sigma_{k_1}}{n^{\frac{k_1+1}{2}}}\cdots\frac{\Sigma_{k_p}}{n^{\frac{k_p+1}{2}}} - E_{\mathbb{M}^{(n)}}\big[\frac{\Sigma_{k_1}}{n^{\frac{k_1+1}{2}}}\big]\cdots E_{\mathbb{M}^{(n)}}\big[\frac{\Sigma_{k_p}}{n^{\frac{k_p+1}{2}}}\big]\Big)^2\Big] = 0 \tag{3.3}$$

for any $p\in\mathbb{N}$ and $k_1,\cdots,k_p\in\mathbb{N}$. Since $\Sigma_1(\lambda) = |\lambda| = n$ holds, we can assume $k_i \geqq 2$ without loss of generality. In

$$\begin{aligned}
&\frac{1}{n^{k_1+\cdots+k_p+p}} E_{\mathbb{M}^{(n)}}\Big[\big(\Sigma_{k_1}\cdots\Sigma_{k_p} - E_{\mathbb{M}^{(n)}}[\Sigma_{k_1}]\cdots E_{\mathbb{M}^{(n)}}[\Sigma_{k_p}]\big)^2\Big] \\
&= \frac{1}{n^{k_1+\cdots+k_p+p}}\Big(E_{\mathbb{M}^{(n)}}\big[\Sigma_{k_1}^2\cdots\Sigma_{k_p}^2\big] \\
&\qquad - 2E_{\mathbb{M}^{(n)}}\big[\Sigma_{k_1}\cdots\Sigma_{k_p}\big]E_{\mathbb{M}^{(n)}}[\Sigma_{k_1}]\cdots E_{\mathbb{M}^{(n)}}[\Sigma_{k_p}] \\
&\qquad + E_{\mathbb{M}^{(n)}}[\Sigma_{k_1}]^2\cdots E_{\mathbb{M}^{(n)}}[\Sigma_{k_p}]^2\Big),
\end{aligned} \tag{3.4}$$

we linearize the product expressions in $\mathbb{A}$ by using (2.15) and then apply Condition (AF). For $j_1,\cdots,j_q \geqq 2$, since

$$\Sigma_{j_1}\cdots\Sigma_{j_q} = \Sigma_{(j_1)\sqcup\cdots\sqcup(j_q)} + \sum_{\sigma\in\mathbb{Y}:|\sigma|+l(\sigma)\leqq j_1+\cdots+j_q+q-1} a_\sigma \Sigma_\sigma$$

holds for some $a_\sigma\in\mathbb{R}$, (3.2) of Lemma3.1 yields

$$E_{\mathbb{M}^{(n)}}\big[\Sigma_{j_1}\cdots\Sigma_{j_q}\big] = E_{\mathbb{M}^{(n)}}\big[\Sigma_{(j_1)\sqcup\cdots\sqcup(j_q)}\big] + O\big(n^{\frac{j_1+\cdots+j_q+q-1}{2}}\big).$$

Applying (1.18), we have

$$
\begin{aligned}
&E_{\mathbb{M}^{(n)}}\big[\Sigma_{(j_1)\sqcup\cdots\sqcup(j_q)}\big]\\
&= n(n-1)\cdots(n-(j_1+\cdots+j_q)+1)f^{(n)}_{((j_1)\sqcup\cdots\sqcup(j_q),1^{n-(j_1+\cdots+j_q)})}\\
&= n(n-1)\cdots(n-(j_1+\cdots+j_q)+1)\Big(f^{(n)}_{(j_1,1^{n-j_1})}\cdots f^{(n)}_{(j_q,1^{n-j_q})}\\
&\qquad + o\big(n^{-\frac{(j_1-1)+\cdots+(j_q-1)}{2}}\big)\Big)\\
&= \big(1+O(n^{-1})\big)E_{\mathbb{M}^{(n)}}[\Sigma_{j_1}]\cdots E_{\mathbb{M}^{(n)}}[\Sigma_{j_q}] + o\big(n^{\frac{j_1+\cdots+j_q+q}{2}}\big)\\
&= E_{\mathbb{M}^{(n)}}[\Sigma_{j_1}]\cdots E_{\mathbb{M}^{(n)}}[\Sigma_{j_q}] + o\big(n^{\frac{j_1+\cdots+j_q+q}{2}}\big)
\end{aligned}
$$

where the last equality follows from (3.2). Hence we get

$$
E_{\mathbb{M}^{(n)}}\big[\Sigma_{j_1}\cdots\Sigma_{j_q}\big] = E_{\mathbb{M}^{(n)}}[\Sigma_{j_1}]\cdots E_{\mathbb{M}^{(n)}}[\Sigma_{j_q}] + o\big(n^{\frac{j_1+\cdots+j_q+q}{2}}\big). \tag{3.5}
$$

Putting (3.5) into (3.4) and noting (3.2) again, we see that (3.4) is $o(1)$ as $n\to\infty$. This implies (3.3).

(1.19) yields

$$
E_{\mathbb{M}^{(n)}}\Big[\frac{\Sigma_k}{n^{\frac{k+1}{2}}}\Big] = \frac{n(n-1)\cdots(n-k+1)}{n^{\frac{k+1}{2}}}f^{(n)}_{(k,1^{n-k})} \xrightarrow{n\to\infty} r_{k+1} \tag{3.6}
$$

for $k \geqq 2$. (2.16) and (2.15) enable us to write

$$
\Sigma_k(\lambda) = R_{k+1}(\mathfrak{m}_\lambda) + \sum_{\tau\in\mathbb{Y}:|\tau|+l(\tau)\leqq k-1} b_\tau\Sigma_\tau \tag{3.7}
$$

for some $b_\tau\in\mathbb{R}$. From (3.6), (3.7) and (3.3) with (3.2), we get

$$
\lim_{n\to\infty} E_{\mathbb{M}^{(n)}}\Big[\Big(\frac{R_{k_1+1}(\mathfrak{m}_\lambda)}{n^{\frac{k_1+1}{2}}}\cdots\frac{R_{k_p+1}(\mathfrak{m}_\lambda)}{n^{\frac{k_p+1}{2}}} - r_{k_1+1}\cdots r_{k_p+1}\Big)^2\Big] = 0.
$$

Since

$$
n^{-\frac{k+1}{2}}R_{k+1}(\mathfrak{m}_\lambda) = R_{k+1}(\mathfrak{m}_{\lambda^{\sqrt{n}}}),\quad R_1(\mathfrak{m}_{\lambda^{\sqrt{n}}})=0,\quad R_2(\mathfrak{m}_{\lambda^{\sqrt{n}}})=1
$$

hold, we have shown

$$
\lim_{n\to\infty} E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}}) - r_{k_1}\cdots r_{k_p}\big)^2\big] = 0 \tag{3.8}
$$

for any $p\in\mathbb{N}$ and $k_1,\cdots,k_p\in\mathbb{N}$, where we set $r_1=0$ and $r_2=1$, in particular

$$
\lim_{n\to\infty} E_{\mathbb{M}^{(n)}}\big[R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big] = r_{k_1}\cdots r_{k_p}. \tag{3.9}
$$

Let us verify that $r_1, r_2, r_3, \cdots$ is a free cumulant sequence for a probability on $\mathbb{R}$ with compact support. Set

$$
m_l = \sum_{\pi\in\mathrm{NC}(l)} r_\pi,\qquad r_\pi = r_{|v_1|}\cdots r_{|v_p|}\quad (\pi=\{v_1,\cdots,v_p\}).
$$

Then, (3.9) implies

$$m_l = \lim_{n\to\infty} E_{\mathbb{M}^{(n)}}\big[M_l(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big] = \lim_{n\to\infty} M_l\big(E_{\mathbb{M}^{(n)}}[\mathfrak{m}_{\lambda^{\sqrt{n}}}]\big)$$

for any $l \in \mathbb{N}$. Here $M_l(\,\cdot\,)$ denotes the lth moment of a probability. (1.20) gives m_l a similar bound. Indeed, a rough estimate

$$|m_l| \leqq a^l |\mathrm{NC}(l)| \leqq (4a)^l$$

holds. Hence there exists a unique probability μ on $\mathbb{R}$ with compact support such that $M_l(\mu) = m_l$, which implies that $r_1, r_2, r_3, \cdots$ is the free cumulant sequence of μ. The Rayleigh function obtained from μ (through the inverse Markov transform) gives $\omega \in \mathbb{D}$ such that $\mathfrak{m}_\omega = \mu$ by virtue of (2.6) and (2.7). In particular,

$$R_k(\mathfrak{m}_\omega) = r_k, \qquad k \in \mathbb{N} \tag{3.10}$$

holds. Since (3.8) yields weak law of large numbers with respect to the moment topology on $\mathbb{D}$, we get the consequence of Theorem1.3 by taking Lemma2.1 into account. □

In order to see the situation at time t for the proof of Theorem1.4, we need the following.

Proposition 3.2. *For any $\rho \in \mathbb{Y}$ and $n \in \mathbb{N}$, considering a column vector $(\tilde{\chi}^{\lambda}_{(\rho,1^{n-|\rho|})})_{\lambda\in\mathbb{Y}_n}$, we have*

$$P^{(n)}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}) = \Big(1 - \frac{|\rho| - m_1(\rho)}{n+1}\Big)(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}), \tag{3.11}$$

hence also

$$e^{tn(P^{(n)}-I)}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}) = e^{-\frac{tn(|\rho|-m_1(\rho))}{n+1}}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}). \tag{3.12}$$

For the proof we recall the induced character formula for a finite group. Let G be a finite group and H a subgroup of G. For $\mu \in \widehat{H}$, irreducible character χ^μ is extended to the whole G to be 0 outside H. Then, the formula assures

$$\frac{1}{\dim \mathrm{Ind}_H^G \mu}\chi^{\mathrm{Ind}_H^G \mu}(x) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\dim\mu}\chi^\mu(g^{-1}xg), \qquad x \in G. \tag{3.13}$$

Proof. (Proof of Proposition3.2) From the definition (1.8) of $P^{(n)} = P^{\uparrow}P^{\downarrow}$, the μ-entry of (3.11) for $\mu \in \mathbb{Y}_n$ is computed as

(3.14)

$$\begin{aligned}
&P^{(n)}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})})(\mu) = \sum_{\nu\in\mathbb{Y}_n} P^{(n)}_{\mu,\nu}\tilde{\chi}^{\nu}_{(\rho,1^{n-|\rho|})} \\
&= \sum_{\nu\in\mathbb{Y}_n}\sum_{\xi\in\mathbb{Y}_{n+1}} P^{\uparrow}_{\mu,\xi}P^{\downarrow}_{\xi,\nu}\frac{1}{\dim\nu}\chi^{\nu}_{(\rho,1^{n-|\rho|})} \\
&= \sum_{\xi\in\mathbb{Y}_{n+1}:\mu\nearrow\xi}\frac{\dim\xi}{(n+1)\dim\mu}\sum_{\nu\in\mathbb{Y}_n:\nu\nearrow\xi}\frac{\dim\nu}{\dim\xi}\frac{\chi^{\nu}_{(\rho,1^{n-|\rho|})}}{\dim\nu} \\
&= \frac{1}{(n+1)\dim\mu}\sum_{\xi\in\mathbb{Y}_{n+1}:\mu\nearrow\xi}\chi^{\xi}_{(\rho,1^{n+1-|\rho|})} = \frac{1}{(n+1)\dim\mu}\chi^{\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}\mu}_{(\rho,1^{n+1-|\rho|})}.
\end{aligned}$$

Put $\mathfrak{S}_{n+1} = G$, $\mathfrak{S}_n = H$, $\mu \in \mathbb{Y}_n$, and furthermore $x \in \mathfrak{S}_n$ into (3.13), where the cycle type of x in $\mathfrak{S}_n$ is $(\rho, 1^{n-|\rho|})$ under $|\rho| \leqq n$. $m_1(\rho)$ may not be 0. The cycle type of x in $\mathfrak{S}_{n+1}$ is then $(\rho, 1^{n+1-|\rho|})$. For $g \in \mathfrak{S}_{n+1}$, $g^{-1}xg \in \mathfrak{S}_n$ holds if and only if $g(n+1)$ is a fixed point of x. The number of such $g \in \mathfrak{S}_{n+1}$ equals $(n+1-|\rho|+m_1(\rho))n!$. Then, (3.13) yields

$$\frac{1}{(n+1)\dim\mu}\chi^{\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}\mu}_{(\rho,1^{n+1-|\rho|})} = \frac{(n+1-|\rho|+m_1(\rho))n!}{(n+1)!\dim\mu}\chi^{\mu}_{(\rho,1^{n-|\rho|})}.$$

Putting this into the last expression of (3.14), we have

$$P^{(n)}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})})(\mu) = \frac{n+1-|\rho|+m_1(\rho)}{n+1}\tilde{\chi}^{\mu}_{(\rho,1^{n-|\rho|})}.$$

(3.12) follows directly from (3.11). $\square$

Proof. (Proof of Theorem1.4) Let us show that the ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)}_t)\}_{n\in\mathbb{N}}$ at $t > 0$ admits Condition (AF) of Definition1.2. Set

$$f^{(n,t)}(x) = \sum_{\mu\in\mathbb{Y}_n}\mathbb{M}^{(n)}_t(\mu)\tilde{\chi}^{\mu}(x), \qquad x \in \mathfrak{S}_n \tag{3.15}$$

for $t \geqq 0$ as a state of $\mathbb{C}[\mathfrak{S}_n]$. (1.22) and (1.11) yield

$$\begin{aligned}
f^{(n,t)}_{(\rho,1^{n-|\rho|})} &= \sum_{\mu\in\mathbb{Y}_n}\Big(\sum_{\lambda\in\mathbb{Y}_n}\mathbb{M}^{(n)}_0(\lambda)e^{tn(P^{(n)}-I)}_{\lambda,\mu}\Big)\tilde{\chi}^{\mu}_{(\rho,1^{n-|\rho|})} \\
&= \sum_{\lambda\in\mathbb{Y}_n}\mathbb{M}^{(n)}_0(\lambda)\sum_{\mu\in\mathbb{Y}_n}e^{tn(P^{(n)}-I)}_{\lambda,\mu}\tilde{\chi}^{\mu}_{(\rho,1^{n-|\rho|})}
\end{aligned}$$

for any $\rho \in \mathbb{Y}$. Using (3.12), we have

$$f^{(n,t)}_{(\rho,1^{n-|\rho|})} = \sum_{\lambda\in\mathbb{Y}_n} \mathbb{M}^{(n)}_0(\lambda) e^{-\frac{tn(|\rho|-m_1(\rho))}{n+1}} \tilde{\chi}^\lambda_{(\rho,1^{n-|\rho|})} = e^{-\frac{tn(|\rho|-m_1(\rho))}{n+1}} f^{(n,0)}_{(\rho,1^{n-|\rho|})}. \tag{3.16}$$

Combining (3.16) with (1.18) for the initial $f^{(n,0)}$, we get for any $p \in \mathbb{N}$ and $k_1, \cdots, k_p \geqq 2$

$$\begin{aligned} f^{(n,t)}_{((k_1)\sqcup\cdots\sqcup(k_p),1^{n-(k_1+\cdots+k_p)})} &= e^{-\frac{tn(k_1+\cdots+k_p)}{n+1}} f^{(n,0)}_{((k_1)\sqcup\cdots\sqcup(k_p),1^{n-(k_1+\cdots+k_p)})} \\ &= e^{-\frac{tn(k_1+\cdots+k_p)}{n+1}} \Big(f^{(n,0)}_{(k_1,1^{n-k_1})} \cdots f^{(n,0)}_{(k_p,1^{n-k_p})} + o\big(n^{-\frac{(k_1-1)+\cdots+(k_p-1)}{2}}\big)\Big) \\ &= f^{(n,t)}_{(k_1,1^{n-k_1})} \cdots f^{(n,t)}_{(k_p,1^{n-k_p})} + o\big(n^{-\frac{(k_1-1)+\cdots+(k_p-1)}{2}}\big), \end{aligned}$$

which implies (1.18) holds for $t > 0$ also. Furthermore, if $k \geqq 2$, we have

$$n^{\frac{k-1}{2}} f^{(n,t)}_{(k,1^{n-k})} = n^{\frac{k-1}{2}} e^{-\frac{tnk}{n+1}} f^{(n,0)}_{(k,1^{n-k})} \xrightarrow{n\to\infty} r_{k+1}e^{-kt} \tag{3.17}$$

where r_{k+1} comes from (1.19) for the initial ensemble $\mathbb{M}^{(n)}_0$. As is seen from the last paragraph of the proof of Theorem1.3, r_{k+1} is then the free cumulant $R_{k+1}(\mathfrak{m}_\omega)$ for $\omega = \omega_0$, the macroscopic shape for the initial $\mathbb{M}^{(n)}_0$. Obviously, (1.20) holds by replacing r_{k+1} by $r_{k+1}e^{-kt}$. We have thus shown the ensemble $\mathbb{M}^{(n)}_t$ satisfies Condition (AF). Then, Theorem1.3 assures that $\mathbb{M}^{(n)}_t$ admits concentration so that it satisfies (1.23), where ω_t is the macroscopic shape for $\mathbb{M}^{(n)}_t$.

We verify that its transition measure $\mathfrak{m}_{\omega_t}$ is expressed as (1.24). The free cumulants of $\mathfrak{m}_{\omega_t}$ are given by

$$R_1(\mathfrak{m}_{\omega_t}) = 0, \quad R_2(\mathfrak{m}_{\omega_t}) = 1, \quad R_k(\mathfrak{m}_{\omega_t}) = R_k(\mathfrak{m}_{\omega_0})e^{-(k-1)t} \ (k \geqq 3)$$

by virtue of (3.17). On the other hand, the formula (2.10) for free compression with (1.15) yields

$$\begin{aligned} &R_1\big((\mathfrak{m}_{\omega_0})_{e^{-t}}\big) = 0, && R_1\big((\mathfrak{m}_\Omega)_{1-e^{-t}}\big) = 0, \\ &R_2\big((\mathfrak{m}_{\omega_0})_{e^{-t}}\big) = e^{-t}, && R_2\big((\mathfrak{m}_\Omega)_{1-e^{-t}}\big) = 1 - e^{-t}, \\ &R_k\big((\mathfrak{m}_{\omega_0})_{e^{-t}}\big) = R_k(\mathfrak{m}_{\omega_0})e^{-(k-1)t}, && R_k\big((\mathfrak{m}_\Omega)_{1-e^{-t}}\big) = 0 \quad (k \geqq 3). \end{aligned}$$

Since free convolution is characterized by (2.8), we obtain (1.24). This completes the proof of Theorem1.4. □

Proposition 3.3. *Let $G(t, z)$ be the Stieltjes transform of $\mathfrak{m}_{\omega_t}$ as defined by (1.27) where ω_t is the macroscopic shape at t. Then, $G(t, z)$ satisfies the following partial differential equation:*

$$\frac{\partial G(t,z)}{\partial t} = \frac{1}{G(t,z)}\frac{\partial G(t,z)}{\partial z} + G(t,z) - \frac{1}{2}\frac{\partial}{\partial z}G(t,z)^2. \tag{3.18}$$

Proof. We just verify (3.18) from (1.24). From the expressions (2.11) for $\mu = \mathfrak{m}_{\omega_0}$ and $\mu = \mathfrak{m}_{\omega_t}$, we have

$$K_0(\zeta) = K_{\mathfrak{m}_{\omega_0}}(\zeta) = \frac{1}{\zeta} + \zeta + \sum_{k=2}^{\infty} R_{k+1}(\mathfrak{m}_{\omega_0})\zeta^k,$$

$$K(t,\zeta) = K_{\mathfrak{m}_{\omega_t}}(\zeta) = \frac{1}{\zeta} + \zeta + \sum_{k=2}^{\infty} R_{k+1}(\mathfrak{m}_{\omega_0})e^{-kt}\zeta^k.$$

These yield

$$K_0(\zeta e^{-t}) = \frac{e^t}{\zeta} + \frac{\zeta}{e^t} - \frac{1}{\zeta} - \zeta + K(t,\zeta). \tag{3.19}$$

Differentiating (3.19) by t and ζ respectively and eliminating K_0'-terms , we have

$$\frac{\partial K(t,\zeta)}{\partial t} + \zeta\frac{\partial K(t,\zeta)}{\partial \zeta} + \frac{1}{\zeta} - \zeta = 0. \tag{3.20}$$

Since we have

$$K\big(t, G(t,z)\big) = z, \tag{3.21}$$

differentiating (3.21) by t and ζ respectively and then putting them into (3.20), we get the desired equation (3.18). □

Remark 3.4. Since $(\mathfrak{m}_\Omega)_t$ is the semi-circle distribution with mean 0 and variance t on $\mathbb{R}$, its Stieltjes transform $g(t,z)$ satisfies complex Burgers equation

$$\frac{\partial g(t,z)}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial z} g(t,z)^2$$

as is well-known and easily derived. For very small $t > 0$, the right hand side of (3.18) may be considered as a simple sum of contributions of the initial part $(\mathfrak{m}_{\omega_0})_{e^{-t}}$ and the stationary part $(\mathfrak{m}_\Omega)_{1-e^{-t}}$ under the approximation $1 - e^{-t} \doteqdot t$.

Remark 3.5. If we take $P^{(n)} = P^{\downarrow}P^{\uparrow}$ instead of $P^{\uparrow}P^{\downarrow}$ as a microscopic dynamics, symmetry (1.9) holds and Theorem1.4 remains valid without any modification. In fact, we have only to recognize that (1.10), (3.11) and (3.12) would be the following:

$$P^{(n)}_{\lambda,\mu} = \begin{cases} \frac{|\{\text{peaks of } \lambda\}|}{n}, & \lambda = \mu, \\ \frac{\dim \mu}{n \dim \lambda}, & \lambda \wedge \mu \in \mathbb{Y}_{n-1}, \\ 0, & \text{otherwise} \end{cases}$$

for $\lambda, \mu \in \mathbb{Y}_n$, where $\lambda \wedge \mu$ means set-theoretic intersection of the boxes,

$$P^{(n)}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}) = \Big(1 - \frac{|\rho| - m_1(\rho)}{n}\Big)(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}),$$

$$e^{tn(P^{(n)}-I)}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})}) = e^{-t(|\rho|-m_1(\rho))}(\tilde{\chi}^{\cdot}_{(\rho,1^{n-|\rho|})})$$

for $\rho \in \mathbb{Y}$ and $n \in \mathbb{N}$.

4. Grand Canonical Ensemble

Given an ensemble, namely a sequence of probability space $\{(\mathbb{Y}_n, \mathbb{M}^{(n)})\}_{n\in\mathbb{N}}$, we set

$$\mathbb{M}^{(\xi)} = \sum_{n=0}^{\infty} p_\xi(n)\mathbb{M}^{(n)} \tag{4.1}$$

where $\xi > 0$ is a parameter of the system and p_ξ is a probability on $\mathbb{N} \sqcup \{0\}$. $\mathbb{M}^{(0)}$ is a unique probability on $\mathbb{Y}_0 = \{\varnothing\}$. Then, $\mathbb{M}^{(\xi)}$ is a probability on $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$. This gives us the formalism of grand canonical ensembles. In what follows, we consider poissonization of the ensemble by taking a Poisson distribution as p_ξ:

$$p_\xi = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!}\delta_n. \tag{4.2}$$

Since p_ξ in (4.2) has mean ξ and standard deviation $\sqrt{\xi}$, it tends to concentrate near $n = \xi$ as $\xi \to \infty$.

Proposition 4.1. *Assume that ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)})\}_{n\in\mathbb{N}}$ satisfies Condition (AF) of Definition1.2 with setting $r_1 = 0$ and $r_2 = 1$. Then, we have*

$$\lim_{\xi\to\infty} E_{\mathbb{M}^{(\xi)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}}) \cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}}) - r_{k_1}\cdots r_{k_p}\big)^2\big] = 0 \tag{4.3}$$

for any $p \in \mathbb{N}$ and $k_1, \cdots, k_p \in \mathbb{N}$ with respect to the poissonization $\mathbb{M}^{(\xi)}$ of $\mathbb{M}^{(n)}$. In particular, there exists $\omega \in \mathbb{D}$ such that $R_k(\mathfrak{m}_\omega) = r_k$ ($k \in \mathbb{N}$) and

$$\lim_{\xi\to\infty} \mathbb{M}^{(\xi)}\big(\{\lambda \in \mathbb{Y} \,\big|\, \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{\xi}}(x) - \omega(x)| \geqq \epsilon\}\big) = 0 \tag{4.4}$$

for any $\epsilon > 0$.

Proof. Putting (4.1) and (4.2) into (4.3), we estimate

$$\sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}}) \cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}}) - r_{k_1}\cdots r_{k_p}\big)^2\big] \tag{4.5}$$

by dividing it into two sums:

$$\text{(I)} \sum_{n\,:\,|n-\xi|\leqq\xi^{3/4}} \qquad \text{and} \qquad \text{(II)} \sum_{n\,:\,|n-\xi|>\xi^{3/4}} .$$

For sum (I), we begin with

(4.6)

$$(\mathrm{I}) \leqq 2\Big\{ \sum_{|n-\xi|\leqq \xi^{3/4}} \frac{e^{-\xi}\xi^n}{n!} \times E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}}) - R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big)^2\big] + \sum_{|n-\xi|\leqq \xi^{3/4}} \frac{e^{-\xi}\xi^n}{n!} E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}}) - r_{k_1}\cdots r_{k_p}\big)^2\big]\Big\}$$

The first sum in (4.6) is expressed as

(4.7)

$$\sum_{|n-\xi|\leqq \xi^{3/4}} \frac{e^{-\xi}\xi^n}{n!} \Big\{ \Big(\frac{n}{\xi}\Big)^{\frac{k_1+\cdots+k_p}{2}} - 1\Big\}^2 E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big)^2\big].$$

Since Condition (AF) yields (3.9) for any p and k_i,

$$E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big)^2\big]$$

in (4.7) is convergent and hence bounded. (4.7) is thus bounded as

$$\leqq \text{const.} \sup_{n:\,|n-\xi|\leqq \xi^{3/4}} \Big\{ \Big(\frac{n}{\xi}\Big)^{\frac{k_1+\cdots+k_p}{2}} - 1\Big\}^2 \xrightarrow{\xi\to\infty} 0.$$

The second sum in (4.6) is bounded as

$$\leqq \sup_{n:\,|n-\xi|\leqq \xi^{3/4}} E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}}) - r_{k_1}\cdots r_{k_p}\big)^2\big],$$

which is arbitrarily small for sufficiently large ξ by virtue of (3.8). We have thus shown (I) tends to 0 as $\xi \to \infty$.

For sum (II), we have

$$(4.8)\quad (\mathrm{II}) \leqq 2\Big\{ \sum_{|n-\xi|>\xi^{3/4}} \frac{e^{-\xi}\xi^n}{n!} E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{\xi}}})\big)^2\big] + \sum_{|n-\xi|>\xi^{3/4}} \frac{e^{-\xi}\xi^n}{n!} (r_{k_1}\cdots r_{k_p})^2\Big\}.$$

The first sum in (4.8) is, with setting $k = k_1 + \cdots + k_p$,

$$\sum_{|n-\xi|>\xi^{3/4}} \frac{e^{-\xi}\xi^n}{n!} \Big(\frac{n}{\xi}\Big)^k E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big)^2\big]$$

$$= \sum_{|n-\xi|>\xi^{3/4}} \frac{e^{-\xi}\xi^{n-k}}{(n-k)!} \Big\{ \frac{n^k}{n\cdots(n-k+1)} E_{\mathbb{M}^{(n)}}\big[\big(R_{k_1}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\cdots R_{k_p}(\mathfrak{m}_{\lambda^{\sqrt{n}}})\big)^2\big]\Big\}.$$

Since the inside of $\{\ \}$ is bounded (by (3.9) again), we continue as

$$\leqq \text{const.} \sum_{j:\,|j-\xi|>\xi^{3/4}-k} \frac{e^{-\xi}\xi^j}{j!} \xrightarrow{\xi\to\infty} 0.$$

It is obvious that the second sum in (4.8) tends to 0. We have thus shown that (II) tends to 0 as $\xi \to \infty$, and hence (4.5) also. This completes the proof. $\square$

Taking the Plancherel measure $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ as $\mathbb{M}^{(n)}$ in (4.1), set

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)} = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} \mathbb{M}_{\mathrm{Pl}}^{(n)}, \tag{4.9}$$

namely

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)}(\lambda) = \frac{e^{-\xi}\xi^{|\lambda|}(\dim\lambda)^2}{(|\lambda|!)^2}, \qquad \lambda \in \mathbb{Y}. \tag{4.10}$$

$\{(\mathbb{Y}, \mathbb{M}_{\mathrm{PP}}^{(\xi)})\}_{\xi>0}$ is called the poissonized Plancherel ensemble. Since the Plancherel ensemble $\mathbb{M}_{\mathrm{Pl}}^{(n)}$ satisfies Condition (AF) and the macroscopic shape is Ω of (1.3), Proposition4.1 yields that the poissonized Plancherel ensemble satisfies

$$\lim_{\xi\to\infty} \mathbb{M}_{\mathrm{PP}}^{(\xi)}\big(\{\lambda \in \mathbb{Y} \,\big|\, \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{\xi}}(x) - \Omega(x)| \geqq \epsilon\}\big) = 0 \tag{4.11}$$

for any $\epsilon > 0$. In other words, the limit shape Ω is observed macroscopically in the poissonized Plancherel ensemble also.

In order to discuss hydrodynamic limit, we consider a Markov chain on $\mathbb{Y}$ which is symmetric with respect to $\mathbb{M}_{\mathrm{PP}}^{(\xi)}$. Recall the up and down transition probabilities $P_{\lambda,\mu}^{\uparrow(n)}$ and $P_{\lambda,\mu}^{\downarrow(n)}$ defined in (1.7). Here the superscript $^{(n)}$ is put to make dependence on n explicit. We seek a transition probability $P_{\lambda,\mu}^{(\xi)}$ which has the following form: for some $\alpha_\xi(n) \in (0,1)$,

$$\begin{aligned} P_{\lambda,\mu}^{(\xi)} &= \alpha_\xi(n) P_{\lambda,\mu}^{\uparrow(n)} + (1-\alpha_\xi(n)) P_{\lambda,\mu}^{\downarrow(n)}, \qquad \lambda \in \mathbb{Y}_n,\ n \in \mathbb{N}, \\ P_{\varnothing,\square}^{(\xi)} &= \alpha_\xi(0), \qquad P_{\varnothing,\varnothing}^{(\xi)} = 1-\alpha_\xi(0). \end{aligned} \tag{4.12}$$

We ask whether it is possible to determine $\{\alpha_\xi(n)\}_{n=0,1,2,\cdots}$ such that $0 < \alpha_\xi(n) < 1$ and

$$\mathbb{M}_{\mathrm{PP}}^{(\xi)}(\lambda) P_{\lambda,\mu}^{(\xi)} = \mathbb{M}_{\mathrm{PP}}^{(\xi)}(\mu) P_{\mu,\lambda}^{(\xi)}, \qquad \lambda,\ \mu \in \mathbb{Y} \tag{4.13}$$

hold. It suffices to verify (4.13) for $\lambda \in \mathbb{Y}_n$ and $\mu \in \mathbb{Y}_{n+1}$ such that $\lambda \nearrow \mu$. Combining (4.12) and (4.10), we easily see that (4.13) holds if and only if

$$\alpha_\xi(n+1) = 1 - \frac{n+1}{\xi}\alpha_\xi(n), \qquad n \in \mathbb{N} \sqcup \{0\}. \tag{4.14}$$

We can observe that a solution of (4.14) is given by

$$\alpha_\xi(n) = \sum_{l=0}^{\infty} \frac{(-1)^l \xi^{l+1}}{(n+1)\cdots(n+l+1)}, \qquad n \in \mathbb{N} \sqcup \{0\} \tag{4.15}$$

(where the series expression is essentially a special case of a Kummer function). Successive integration by parts yields

$$\alpha_\xi(n) = \int_0^1 \xi e^{-\xi x}(1-x)^n dx, \qquad n \in \mathbb{N} \sqcup \{0\}. \tag{4.16}$$

For any $\xi > 0$ and $n \in \mathbb{N}$, we see from (4.16)

$$0 < \alpha_\xi(n) < \alpha_\xi(n-1) < \alpha_\xi(0) = 1 - e^{-\xi} < 1.$$

We have thus obtained the following.

Proposition 4.2. *For $\xi > 0$ define $P^{(\xi)} = (P^{(\xi)}_{\lambda,\mu})_{\lambda,\mu \in \mathbb{Y}}$ by* (4.12) *and* (4.16) *(or* (4.15)*). Then $P^{(\xi)}$ gives a transition matrix. A Markov chain on $\mathbb{Y}$ with $P^{(\xi)}$ as its transition matrix is symmetric with respect to $\mathbb{M}^{(\xi)}_{\mathrm{PP}}$, namely it satisfies* (4.13)*.*

Remark 4.3. In the poissonized Plancherel ensemble, the typical size of a Young diagram is ξ, the mean of p_ξ. For $n = \xi$, (4.16) yields

$$\alpha_n(n) = \int_0^1 n e^{-nx}(1-x)^n dx = \int_0^\infty e^{-x}\big(1 - \frac{x}{n}\big)^n 1_{[0,n]}(x) dx \xrightarrow{n\to\infty} \frac{1}{2}.$$

In other words, increase and decrease of the number of boxes are asymptotically balanced near the typical size.

Finally, we give an expression making the structure of the transition matrix $P^{(\xi)}$ more transparent. Set a total order in $\mathbb{Y} = \bigsqcup_{n=0}^\infty \mathbb{Y}_n$ where the number of boxes is non-decreasing and it coincides with the lexicographic order, for example, in each $\mathbb{Y}_n$. Set

$$\varDelta^{(\xi)} = \mathrm{diag}(\varDelta^{(\xi)}_\lambda)_{\lambda \in \mathbb{Y}}, \quad \varDelta^{(\xi)}_\lambda = \sqrt{\mathbb{M}^{(\xi)}_{\mathrm{PP}}(\lambda)} = \frac{e^{-\xi/2}\xi^{|\lambda|/2} \dim \lambda}{|\lambda|!}, \tag{4.17}$$

$$A^{(\xi)} = \mathrm{diag}(\alpha_\xi(|\lambda|))_{\lambda \in \mathbb{Y}}. \tag{4.18}$$

Define $T = (T_{\lambda,\mu})_{\lambda,\mu \in \mathbb{Y}}$ by

$$T_{\lambda,\mu} = \begin{cases} 1, & \lambda \nearrow \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{4.19}$$

Then, $T + T^*$ is the adjacency matrix of the Young graph. As is well-known and easily verified, it satisfies CCR: $TT^* - T^*T = I$. Furthermore, set

$$(E_\varnothing)_{\lambda,\mu} = \begin{cases} 1, & \lambda = \mu = \varnothing, \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

The following fact is directly verified.

Proposition 4.4. *Under the notations of* (4.17) – (4.20)*, the transition matrix* $P^{(\xi)}$ *is expressed as*

$$P^{(\xi)} = \Delta^{(\xi)-1}\Big\{\frac{1}{\sqrt{\xi}}(A^{(\xi)}T + T^* A^{(\xi)}) + e^{-\xi}E_{\varnothing}\Big\}\Delta^{(\xi)}.$$

We note that $\frac{1}{\sqrt{\xi}}(A^{(\xi)}T+T^*A^{(\xi)})$ is a far-reaching analogue of a tridiagonal (infinite) Jacobi matrix

$$\begin{pmatrix} 0 & a_1 & & & \\ a_1 & 0 & a_2 & & \\ & a_2 & 0 & a_3 & \\ & & a_3 & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

and its asymptotic spectral analysis seems to be of independent interest. We refer to [10] for such asymptotic problems about Jacobi matrices and adjacency matrices of several sorts of graphs.

Remark 4.5. In order to consider hydrodynamic limit as in Theorem1.4 in the grand canonical ensemble, we have to treat

$$\mathbb{M}_t^{(\xi)} = \sum_{\lambda\in\mathbb{Y}} \mathbb{M}_0^{(\xi)}(\lambda)\,(e^{t\xi(P^{(\xi)}-I)})_{\lambda,\cdot}$$

as a distribution on $\mathbb{Y}$ at macroscopic time $t > 0$ and asymptotic behavior of such an quantity as (4.3) rescaled by $1/\sqrt{\xi}$ with respect to $\mathbb{M}_t^{(\xi)}$ as $\xi \to \infty$. Microscopic time $t\xi$ is of the same order with the typical size ξ of Young diagrams as $\xi \to \infty$. This situation gives rise to difficulty because we should take into account a wide range of the Young graph which the microscopic chain driven by $P^{(\xi)}$ can spread.

Appendix A. Approximate Factorization Property

In order to observe the evolution of the macroscopic shape by following the procedure of hydrodynamic limit, we assumed approximate factorization property for an initial ensemble in Theorem1.4. This is interpreted as presuming the initial condition to have a certain regularity. In this appendix, we supply some descriptions about related backgrounds. Concretely, the following two items are commented.
(a) In what sense, is approximate factorization property connected to certain weak ergodicity?
(b) What are typical examples admitting approximate factorization property?
Although these are known facts ((a) due to Thoma[16] and Vershik–Kerov[18], and (b) due to Biane[1], [2]), we hope they will serve readers in clarifying the substance of approximate factorization property for ensembles.

(a) Let us consider a special situation that ensemble $\{(\mathbb{Y}_n, \mathbb{M}^{(n)})\}_{n\in\mathbb{N}}$ is a sequence of the marginal distributions of a probability on the path space $\mathfrak{T}$ of (1.5). Namely, consider a probability $\mathbb{M}$ on $\mathfrak{T}$ such that

$$\mathbb{M}^{(n)}(\lambda) = \mathbb{M}\big(\{t \in \mathfrak{T} \,|\, t(n) = \lambda\}\big), \qquad \lambda \in \mathbb{Y}_n$$

for any $n \in \mathbb{N}$. Here $\mathfrak{T}$ is equipped with the canonical Borel structure generated by cylindrical sets. The Plancherel ensemble is exactly the case as is seen in (1.6). A probability $\mathbb{M}$ on $\mathfrak{T}$ is said to be central if it assigns a common value $\varphi(\lambda)$ depending only on λ (and independent of the history before) to any cylindrical set

$$\big\{(\varnothing \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n-1)} \nearrow \lambda \nearrow \mu^{(n+1)} \nearrow \cdots) \,|\, \mu^{(n+i)} \in \mathbb{Y}_{n+i},\ i \in \mathbb{N}\big\}.$$

Then, we have

$$\mathbb{M}^{(n)}(\lambda) = \varphi(\lambda) \dim \lambda, \qquad \lambda \in \mathbb{Y}_n$$

since there are $\dim \lambda$ paths in all connecting $\varnothing$ to λ. $\mathcal{M}(\mathfrak{T})$ denotes the set of central probabilities on $\mathfrak{T}$.

Let $\mathcal{K}(\mathfrak{S}_\infty)$ be the set of $\mathbb{C}$-valued positive-definite, central and normalized functions on the infinite symmetric group $\mathfrak{S}_\infty$. Extending the correspondence given by (1.16), we have an affine homeomorphism between $\mathcal{M}(\mathfrak{T})$ and $\mathcal{K}(\mathfrak{S}_\infty)$. Here $\mathcal{M}(\mathfrak{T})$ is equipped with the weak convergence topology and $\mathcal{K}(\mathfrak{S}_\infty)$ with the pointwise convergence (equivalently, compact-open) topology. Both sets are convex and compact with respect to their topologies. In other words, $\mathbb{M} \in \mathcal{M}(\mathfrak{T})$ and $f \in \mathcal{K}(\mathfrak{S}_\infty)$ are connected by

$$f\big|_{\mathfrak{S}_n} = \sum_{\lambda\in\mathbb{Y}_n} \varphi(\lambda)\chi^\lambda = \sum_{\lambda\in\mathbb{Y}_n} \mathbb{M}^{(n)}(\lambda)\tilde{\chi}^\lambda, \qquad n \in \mathbb{N}. \tag{A.1}$$

Extremal elements in $\mathcal{M}(\mathfrak{T})$ and $\mathcal{K}(\mathfrak{S}_\infty)$ are called ergodic probabilities on $\mathfrak{T}$ and (indecomposable) characters of $\mathfrak{S}_\infty$ respectively. As is known as Thoma's criterion, $f \in \mathcal{K}(\mathfrak{S}_\infty)$ is a character of $\mathfrak{S}_\infty$ if and only if it is factorizable: $f(xy) = f(x)f(y)$ holds for any disjoint cycles x and y in $\mathfrak{S}_\infty$. Since the tracial state corresponding to the ensemble $\mathbb{M}^{(n)}$ is given by $f^{(n)} = f\big|_{\mathbb{C}[\mathfrak{S}_n]}$ from (A.1), approximate factorization property for $f^{(n)}$ generalizes to some extent the factorizability or ergodicity by taking an appropriate decay order into account. Translated into probabilities or ensembles, it therefore suggests ergodicity weakened to some extent.

(b) First let us consider ensemble $\{(\mathbb{Y}_n, \delta_{\lambda^{(n)}})\}_{n\in\mathbb{N}}$ where $\lambda^{(n)} = (\lambda_1^{(n)} \geqq \lambda_2^{(n)} \geqq \cdots) \in \mathbb{Y}_n$ is prescribed to fulfill the following conditions:

$$\exists c > 0 \text{ such that } \lambda_1^{(n)} \leqq c\sqrt{n},\ \lambda_1^{(n)\prime} \leqq c\sqrt{n} \quad (\forall n \in \mathbb{N}), \tag{A.2}$$

$$\exists \omega_0 \in \mathbb{D} \text{ such that } \lambda^{(n)\sqrt{n}} \text{ converges to } \omega_0 \text{ in } \mathbb{D} \text{ as } n \to \infty. \tag{A.3}$$

Since $\lambda_1^{(n)}\lambda_1^{(n)\prime} \geqq n$ necessarily holds, (A.2) yields lower bounds $\lambda_1^{(n)} \geqq \sqrt{n}/c$, $\lambda_1^{(n)\prime} \geqq \sqrt{n}/c$ also. Young diagrams forming a sequence which satisfies (A.2) are said to be c-balanced. Note that ω_0 in (A.3) satisfies $\mathrm{supp}\mathfrak{m}_{\omega_0} \subset [-c, c]$ because of (A.2). If we take this ensemble as the initial one in Theorem1.4, our Markov chain starts from a single point. For any $\omega_0 \in \mathbb{D}$ there exist some $c > 0$ and c-balanced Young diagrams $\lambda^{(n)} \in \mathbb{Y}_n$ such that $\lambda^{(n)\sqrt{n}}$ converges to ω_0 in $\mathbb{D}$. Let us verify Condition (AF) for $\delta_{\lambda^{(n)}}$. The assumption implies

$$\lim_{n\to\infty} n^{-\frac{k+1}{2}} R_{k+1}(\mathfrak{m}_{\lambda^{(n)}}) = R_{k+1}(\mathfrak{m}_{\omega_0}), \qquad k \in \mathbb{N} \sqcup \{0\}.$$

Using (2.16), we have for $k \geqq 2$

(A.4)

$$\begin{aligned} n^{\frac{k-1}{2}} \tilde{\chi}^{\lambda^{(n)}}_{(k,1^{n-k})} &= \frac{n^{(k-1)/2}}{n(n-1)\cdots(n-k+1)} \big\{ R_{k+1}(\mathfrak{m}_{\lambda^{(n)}}) + (\text{weight} \leqq k-1) \big\} \\ &= \frac{n^k}{n(n-1)\cdots(n-k+1)} \big\{ n^{-\frac{k+1}{2}} R_{k+1}(\mathfrak{m}_{\lambda^{(n)}}) + n^{-\frac{k+1}{2}} (\text{weight} \leqq k-1) \big\} \\ &\xrightarrow{n\to\infty} R_{k+1}(\mathfrak{m}_{\omega_0}) \end{aligned}$$

since each term of lower weight is expressed as a constant multiple of

$$n^{-\frac{k+1}{2}} R_{j_1}(\mathfrak{m}_{\lambda^{(n)}}) \cdots R_{j_l}(\mathfrak{m}_{\lambda^{(n)}}) \qquad \text{where} \quad j_1 + \cdots + j_l \leqq k-1.$$

Together with $\mathrm{supp}\mathfrak{m}_{\omega_0} \subset [-c, c]$, (A.4) implies the asymptotic cycle mean condition (1.19) and (1.20). (1.18) for $\tilde{\chi}^{\lambda^{(n)}}$ is verified by using (2.14)–(2.16) and (A.4) as follows:

$$\begin{aligned} &\tilde{\chi}^{\lambda^{(n)}}_{((k_1)\sqcup\cdots\sqcup(k_p),1^{n-(k_1+\cdots+k_p)})} \\ &= \frac{n\cdots(n-k_1+1)\,\cdots\, n\cdots(n-k_p+1)}{n\cdots(n-(k_1+\cdots+k_p)+1)} \tilde{\chi}^{\lambda^{(n)}}_{(k_1,1^{n-k_1})} \cdots \tilde{\chi}^{\lambda^{(n)}}_{(k_p,1^{n-k_p})} \\ &\quad + \sum_{j_1+\cdots+j_l+l \leqq k_1+\cdots+k_p+p-1} \frac{n\cdots(n-j_1+1)\,\cdots\, n\cdots(n-j_l+1)}{n\cdots(n-(k_1+\cdots+k_p)+1)} \\ &\qquad\qquad \times c_{j_1\cdots j_l}\, \tilde{\chi}^{\lambda^{(n)}}_{(j_1,1^{n-j_1})} \cdots \tilde{\chi}^{\lambda^{(n)}}_{(j_l,1^{n-j_l})} \\ &= \tilde{\chi}^{\lambda^{(n)}}_{(k_1,1^{n-k_1})} \cdots \tilde{\chi}^{\lambda^{(n)}}_{(k_p,1^{n-k_p})} + O\big(n^{-\frac{(k_1-1)+\cdots+(k_p-1)}{2}-1}\big) \\ &\quad + O\big(n^{(j_1+\cdots+j_l)-(k_1+\cdots+k_p)-\frac{(j_1-1)+\cdots+(j_l-1)}{2}}\big), \end{aligned}$$

where $(j_1+\cdots+j_l)-(k_1+\cdots+k_p)-\frac{(j_1-1)+\cdots+(j_l-1)}{2} \leqq -\frac{(k_1-1)+\cdots+(k_p-1)}{2}-\frac{1}{2}$.

Next we mention ensembles closely related to (a). As was shown by Thoma in [16], the extremal points of $\mathcal{K}(\mathfrak{S}_\infty)$ (and hence of $\mathcal{M}(\mathfrak{T})$) are parametrized by

$$\text{(A.5)} \quad \begin{aligned} \Delta = \Big\{ (\alpha,\beta) \,\Big|\, \alpha = (\alpha_i)_{i\in\mathbb{N}},\ \beta = (\beta_i)_{i\in\mathbb{N}},\ \alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0,& \\ \beta_1 \geqq \beta_2 \geqq \cdots \geqq 0,\ \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leqq 1 \Big\},& \end{aligned}$$

which is called the Thoma simplex. To $(\alpha,\beta)\in\Delta$ the corresponding character of $\mathfrak{S}_\infty$ is given by

$$f_{\alpha,\beta}(k\text{-cycle})=\sum_{i=1}^{\infty}(\alpha_i^k+(-1)^{k-1}\beta_i^k),\qquad k\geqq 2. \tag{A.6}$$

Since $f_{\alpha,\beta}$ is factorizable, (A.6) completely determines the values of $f_{\alpha,\beta}$ on $\mathfrak{S}_\infty$. By putting (A.6) into (A.1), the corresponding ergodic probability $\mathbb{M}_{\alpha,\beta}$ is expressed by a (supersymmetric) analogue of Schur function. For a fixed $(\alpha,\beta)\in\Delta$, Vershik–Kerov's theorem assures that

$$\lim_{n\to\infty}\frac{t(n)_i}{n}=\alpha_i,\qquad \lim_{n\to\infty}\frac{t(n)'_i}{n}=\beta_i$$

hold for $\mathbb{M}_{\alpha,\beta}$-a.s. $t\in\mathfrak{T}$. Hence, unless $(\alpha,\beta)=(0,0)$, it is meaningless to discuss a macroscopic shape in $(\mathbb{Y}_n,\mathbb{M}^{(n)}_{\alpha,\beta})$ under our rescale of (1.2) as $n\to\infty$. In order for $t(n)_1$ and $t(n)'_1$ to be of order $\sqrt{n}$, we adjust the Thoma parameter $(\alpha^{(n)},\beta^{(n)})$ to the order $1/\sqrt{n}$. Hence the situation here is not precisely the one treated in (a) except the case of $(\alpha,\beta)=(0,0)$, namely the Plancherel ensemble. Setting $\gamma=1-\sum_{i=1}^{\infty}(\alpha_i+\beta_i)\in[0,1]$ for $(\alpha,\beta)\in\Delta$, we consider a probability

$$\nu_{\alpha,\beta}=\sum_{i=1}^{\infty}(\alpha\delta_{\alpha_i}+\beta_i\delta_{-\beta_i})+\gamma\delta_0 \tag{A.7}$$

on $\mathbb{R}$, which is often called a Thoma measure. For a probability ν on $\mathbb{R}$ and $a>0$, we set

$$\nu^a(B)=\nu(a^{-1}B),\qquad B\ :\ \text{Borel set}\ \subset\mathbb{R}.$$

If a sequence of Thoma parameters $\{(\alpha^{(n)},\beta^{(n)})\}_{n\in\mathbb{N}}\subset\Delta$ satisfies:

$$\alpha_1^{(n)}=O\big(\frac{1}{\sqrt{n}}\big),\qquad \beta_1^{(n)}=O\big(\frac{1}{\sqrt{n}}\big), \tag{A.8}$$

(A.9) $\quad \nu^{\sqrt{n}}_{\alpha^{(n)},\beta^{(n)}}$ weakly converges to a probability ν on $\mathbb{R}$ as $n\to\infty$,

we call $\big\{(\mathbb{Y}_n,\mathbb{M}^{\sqrt{n}}_{\alpha^{(n)},\beta^{(n)}})\big\}_{n\in\mathbb{N}}$ a Thoma ensemble. Note that (A.8) and (A.9) ensure uniform (w.r.t. n) boundedness of the supports of $\nu^{\sqrt{n}}_{\alpha^{(n)},\beta^{(n)}}$. Let us verify that a Thoma ensemble satisfies Condition (AF). Since (A.7) and (A.6) yield

$$\begin{aligned}M_k(\nu^{\sqrt{n}}_{\alpha^{(n)},\beta^{(n)}})&=\int_{\mathbb{R}}x^k\nu^{\sqrt{n}}_{\alpha^{(n)},\beta^{(n)}}(dx)\\&=n^{k/2}\sum_{i=1}^{\infty}\big(\alpha_i^{(n)k+1}+(-1)^k\beta_i^{(n)k+1}\big)=n^{k/2}f_{\alpha^{(n)},\beta^{(n)}}\big((k+1)\text{-cycle}\big)\end{aligned}$$

for $k\in\mathbb{N}$, noting $f^{(n)}_{\alpha^{(n)},\beta^{(n)}}=f_{\alpha^{(n)},\beta^{(n)}}\big|_{\mathfrak{S}_n}$, we have

$$\lim_{n\to\infty}n^{k/2}(f^{(n)}_{\alpha^{(n)},\beta^{(n)}})_{(k+1,1^{n-(k+1)})}=M_k(\nu). \tag{A.10}$$

Hence (1.19) and (1.20) hold by setting $r_{k+1} = M_{k-1}(\nu)$. Note that (A.10) is valid for $k = 0$ also. Approximate factorization property (1.18) obviously holds without error terms because of factorizability of the character $f_{\alpha^{(n)},\beta^{(n)}}$. The macroscopic shape $\omega = \omega_\nu$ for Thoma ensemble $\mathbb{M}^{(n)}_{\alpha^{(n)},\beta^{(n)}}$ is described in terms of ν as follows. Since (3.10) yields $R_k(\mathfrak{m}_{\omega_\nu}) = r_k = M_{k-2}(\nu)$ for $k \geqq 2$, we have

$$\begin{aligned} R_{\mathfrak{m}_{\omega_\nu}}(\zeta) &= \sum_{k=2}^{\infty} R_k(\mathfrak{m}_{\omega_\nu})\zeta^{k-1} = \sum_{k=2}^{\infty} M_{k-2}(\nu)\zeta^{k-1} \\ &= \sum_{k=2}^{\infty} \int_{\mathbb{R}} \zeta x^{k-2}\zeta^{k-2}\nu(dx) = \int_{\mathbb{R}} \frac{\zeta}{1-\zeta x}\nu(dx). \end{aligned}$$

Rigorous procedures exist for obtaining

$$R_{\mathfrak{m}_{\omega_\nu}}(\zeta) + \frac{1}{\zeta} = K_{\mathfrak{m}_{\omega_\nu}}(\zeta) \longrightarrow G_{\mathfrak{m}_{\omega_\nu}}(z) \longrightarrow \mathfrak{m}_{\omega_\nu} \longrightarrow \omega_\nu$$

though explicit computation may be case by case difficult.

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