FOURTH-ORDER TOTAL VARIATION FLOW WITH DIRICHLET CONDITION : CHARACTERIZATION OF EVOLUTION AND EXTINCTION TIME ESTIMATES

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Abstract

We consider a fourth-order total variation flow, which is studied in image recovery and materials science. In this paper, we characterize a total variation flow in H^{-1}-space with Dirichlet boundary condition. Furthermore, we show an extinction time estimate for the solution of a total variation flow with Dirichlet boundary condition. Giga and Kohn (2011) established the same extinction time estimate in periodic case. Their argument is based on interpolation inequalities. Using extension operators, we derive this type of inequalities, which we apply to the case of Dirichlet boundary condition.

1 Introduction

This is a continuation of [13], where the fourth-order total variation problem under periodic boundary condition was studied with focus on extinction time. The purpose of this paper is to study the following system;

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0, \end{cases} \tag{1}$$

with various boundary conditions. We cannot take this system literally because the right hand side is undefined when $\nabla u = 0$. In this paper we consider the right hand side of system (1) as a subdifferential of a total variation in negative Sobolev space $H^{-1}(\Omega)$. Precisely, we consider the $H^{-1}(\Omega)$ steepest descent with the total variation. The advantage of using this method lies in the fact that the theory of maximal monotone operators initiated by Kōmura [21] guarantees the existence and uniqueness of the solution, which is difficult to prove in many cases of nonlinear partial differential equations. We call system (1) a *fourth-order total variation flow*. In this paper $\int_\Omega |Du|$ denotes the *total variation* of u in Ω, whereas $\int_\Omega |\nabla u|\, dx$ denotes the L^1-norm of ∇u in Ω.

We give a few background of system (1). This system arises in image recovery and materials science.

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First we show how system (1) arises in image recovery. We want to recover the true image from a given image, which is blurred and added a noise. Suppose that a *given gray-scale image* is represented by $u_d : \Omega \to \mathbb{R}$, where Ω is a bounded domain in $\mathbb{R}^2$. We denote a *true image* by $u_{true} : \Omega \to \mathbb{R}$, a *blurred image* by $Ku_{true} : \Omega \to \mathbb{R}$, and a *noise* by $n : \Omega \to \mathbb{R}$. Respectively, we assume that u_d is represented by

$$u_d = Ku_{true} + n.$$

The image recovery problem is to extract u_{true} from u_d. Rudin, Osher, and Fatemi [27] proposed the following minimization problem to consider the image recovery problem

$$u_{true} = \arg\min \left\{ E(u) \mid u \in L^2(\Omega) \right\}, \tag{2}$$

where

$$E(u) := \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega |Ku - u_d|^2 \, dx,$$

for some Lagrange multiplier $\lambda > 0$. The BV-space is suitable for considering the image recovery problem. It is because edges are allowed in $u \in BV(\Omega)$ as jump discontinuities and we can recognize the feature of the picture. We assume for simplicity $Ku = u$ for $u \in L^2(\Omega)$, then the solution u of (2) satisfies Euler-Lagrange equation

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) + \lambda(u - u_d) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where ν is the outward unit normal vector field and $\partial/\partial\nu$ is the directional derivative in the direction of ν. To calculate (3) numerically one possible method is to use 'gradient flow'-the second-order total variation flow with Neumann boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) & \text{in } \Omega \times (0, \infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{4}$$

Note that (3) corresponds to the implicit Euler scheme for this evolution equation by regarding u_d as previous step data and u as next step results, where $1/\lambda$ is the time grid length. For more details we refer the reader to [1, 2]. The right hand side of system (4) is considered as the subdifferential of a total variation in $L^2(\Omega)$. Andreu, Caselles and Mazón [2] considered this system in $\mathbb{R}^n$ and in Ω with Dirichlet and Neumann boundary conditions. Osher, Solé, and Vese [25] introduced another method to recover the image. They considered the minimization problem

$$u_{true} = \arg\min\{E(u) \mid u \in H^{-1}(\Omega)\}, \tag{5}$$

where

$$E(u) := \int_\Omega |Du| + \lambda \|Ku - u_d\|^2_{H^{-1}(\Omega)}.$$

They showed by experiments that we can recover textured details in natural images better than using the method of minimization problem (2). Problem (5) leads to the fourth-order total variation flow with Neumann boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial}{\partial \nu} \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

In materials science, this system is used to describe the relaxation of a crystalline surface below the roughening temperature. In this model, u represents the height of surface. For derivation of this model we refer the reader to [17, 20, 22, 26].

The characterization of subdifferential of convex energy in $H^{-1}(\Omega)$ was done by Kashima [18, 19]. In [19] the subdifferential of

$$F(u) := \int_\Omega \left(|\nabla u(x)| + \frac{\mu}{p} |\nabla u(x)|^p \right) dx, \quad \mu > 0, \quad p > 1, \tag{6}$$

in $H^{-1}(\Omega)$ was characterized as

$$-\Delta \operatorname{div}\left(\frac{\nabla u}{|\nabla u|} + \mu |\nabla u|^{p-2} \nabla u\right),$$

by assuming $u \in W^{1,p}(\Omega)$. The first term of the right hand side in (6) is not a total variation, but an L^1-norm since $\nabla u \in L^1(\Omega)$ by Hölder's inequality. On the other hand, in this paper we allow $u \in BV(\Omega)$, as a result ∇u is just a bounded Radon measure, not necessarily a locally integrable function.

One of the purposes of this paper is to characterize the subdifferential of total variation in H^{-1}-space with Dirichlet boundary condition although the Neumann problem is more popular for image analysis. In [13], the characterization with periodic boundary condition was established. They used the theory in L^2-space, which appears in [5, 7], and extended it to H^{-1}-space. We characterize the subdifferentials of total variations in $\overline{\Omega}$ by using a dual Hilbert space $H^{-1}(\Omega) := (H^1_0(\Omega))^*$. In other words, we consider Dirichlet-Laplacian, so we call this Dirichlet boundary condition. In [13] the way to characterize this subdifferential is indicated but not explicitly given. We cannot apply the method in periodic case directly to the case $\int_{\overline{\Omega}} |Du|$, so we derive an interesting duality representation of $\int_{\overline{\Omega}} |Du|$. By using this representation, we can characterize the subdifferential of $\int_{\overline{\Omega}} |Du|$ in $H^{-1}(\Omega)$.

Another purpose of this paper is to get the extinction time estimate of the solution for system (1) with zero-Dirichlet boundary condition. Let $T(u_0)$ be an extinction time of the solution of system (1) with the initial datum $u|_{t=0} = u_0$. More precisely, we define

$T(u_0)$ by

$$T(u_0) := \inf \left\{ T > 0 \;\middle|\; \begin{array}{l} u(t) = 0 \text{ if } t \geq T \text{ where } u \text{ is the solution} \\ \text{of system (1) with the initial datum } u|_{t=0} = u_0 \end{array} \right\}.$$

We show that $T(u_0)$ is finite and estimated by the form

$$T(u_0) \leq G(u_0)$$

where G is some explicit functional.

Giga and Kohn [13] gives an extinction time estimate for the solution of system (1) with periodic boundary condition. Theorem 3.3 of that paper obtains a scale-invariant result in space dimension 4. However, Theorem 3.8 of that paper (which asserts a scale-invariant result in dimension n, $1 \leq n \leq 4$) is flawed. The proper conclusion based on the argument given there is an extinction time estimate that is only asymptotically scale-invariant. In this paper, we derive a corrected extinction time estimate of system (1) with zero-Dirichlet boundary condition, which should be the same for the periodic case. In [13], they used an energy estimate and an interpolation inequality of the form

$$\|u\|_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} \leq C\|u\|_Y^{1-\theta} \left(\int_{\mathbb{T}^n} |Du| \right)^{\theta}, \tag{7}$$

where $\mathbb{T}^n$ is a torus, Y is a suitable norm, $C > 0$ is a scale-invariant constant and we denote the definition of the functional space $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ in Section 4. We would like to derive this type of inequality, which can be applied to Dirichlet boundary condition. However, in Dirichlet boundary condition the method of the proof of inequality (7) in periodic case breaks down. It is because we use a property of the heat semigroup which is particular to periodic boundary condition in deriving inequality (7). So we take a different method. First, we set a periodic boundary condition, whose periodic cell contains Ω. Then, we define an extension operator $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$, which satisfies some properties. We define this operator differently for the case when Ω is a rectangular domain and the case Ω is a C^∞-domain. Here by using inequality (7) for Eu, we get an interpolation inequality, which can be applied to Dirichlet boundary condition. Finally, by using this interpolation inequality, we get an extinction time estimate of the solution of a total variation flow with Dirichlet boundary condition. We also give an explicit solution when Ω is $\mathbb{R}^n$. The fourth-order total variation flow is less studied compared with the second-order model. We refer a review paper by M-H. Giga and the first author [12] and papers cited there.

Most of the results of this paper are based on the third author's master's thesis [23].

2 Preliminaries

2.1 The total variation in $H^{-1}(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $C^\infty(\Omega)$ denote the space of all smooth functions in Ω. Let $\mathscr{D}(\Omega) = C_0^\infty(\Omega)$ denote the space of all compactly supported functions $\varphi \in C^\infty(\Omega)$ in Ω. We first define the functional spaces as follows:

$$C^\infty_{\mathrm{av}}(\Omega) := \left\{ \varphi \in C^\infty(\Omega) \,\middle|\, \int_\Omega \varphi \, dx = 0 \right\},$$

$$C^\infty_{0,\mathrm{av}}(\Omega) := \left\{ \varphi \in C^\infty_0(\Omega) \,\middle|\, \int_\Omega \varphi \, dx = 0 \right\}$$

i.e. the subset of functions with mean value zero.

Next, we define the total variation for $u \in H^{-1}(\Omega) = (H^1_0(\Omega))^*$, which is the dual of $H^1_0(\Omega)$. Let $\langle \cdot, \cdot \rangle$ denote the canonical pairing of $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Let $\mathscr{D}'(\Omega)$ be the space of Schwartz distributions in Ω, i.e. $\mathscr{D}'(\Omega)$ is the topological dual space of $\mathscr{D}(\Omega) = C^\infty_0(\Omega)$ [24]. By its definition, $H^{-1}(\Omega)$ can be regarded as a subspace of $\mathscr{D}'(\Omega)$.

Lemma 2.1. *Let $u \in H^{-1}(\Omega)$. If*

$$\int_\Omega |Du| := \sup \left\{ \langle u, \operatorname{div} \varphi \rangle \mid \varphi \in C^\infty_0(\Omega; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\Omega)} \le 1 \right\}$$

is finite, then there exists $\tilde{u} \in L^{\frac{n}{n-1}}(\Omega)$ such that $u = \tilde{u}$ in $\mathscr{D}'(\Omega)$.

Proof. First we assume $1 < p < \frac{n}{n-1}$, then the conjugate exponent $p' = \frac{p}{p-1}$ satisfies $p' > n$. For $u \in H^{-1}(\Omega)$, we define a semi-norm $\|u\|_{\tilde{L^p}(\Omega)}$ by

$$\|u\|_{\tilde{L^p}(\Omega)} := \sup \left\{ \langle u, \varphi \rangle \,\middle|\, \varphi \in C^\infty_{0,\mathrm{av}}(\Omega), \ \|\varphi\|_{L^{p'}(\Omega)} \le 1 \right\}.$$

To prove this lemma, we recall the *Bogovskiĭ operator* $B : C^\infty_0(\Omega) \to C^\infty_0(\Omega; \mathbb{R}^n)$ which satisfies the following properties: there exists a constant C such that

$$\operatorname{div}(B\varphi) = \varphi, \qquad \|\nabla(B\varphi)\|_{L^{p'}(\Omega)} \le C \|\varphi\|_{L^{p'}(\Omega)}$$

for all $\varphi \in C^\infty_0(\Omega)$ [11, p.179].

Since properties of B and Gagliardo-Nirenberg-Sobolev inequality, there exists a constant C such that for any $\varphi \in C^\infty_{0,\mathrm{av}}(\Omega)$ with $\|\varphi\|_{L^{p'}(\Omega)} \le 1$

$$\|B\varphi\|_{L^\infty(\Omega)} \le C \|\nabla(B\varphi)\|_{L^{p'}(\Omega)} \le C \|\varphi\|_{L^{p'}(\Omega)} \le C.$$

Therefore the following inequality

$$\begin{aligned} \langle u, \varphi \rangle &= \langle u, \operatorname{div}(B\varphi) \rangle \\ &\le \sup \left\{ \langle u, \operatorname{div} \psi \rangle \mid \psi \in C^\infty_0(\Omega; \mathbb{R}^n), \|\psi\|_{L^\infty(\Omega)} \le C \right\} = C \int_\Omega |Du| \end{aligned}$$

holds. Hence we have

$$\|u\|_{\tilde{L^p}(\Omega)} \le C \int_\Omega |Du|. \tag{8}$$

Next we fix $\eta \in C^\infty_0(\Omega)$ such that $\int_\Omega \eta \, dx = 1$. And for any $\psi \in C^\infty_0(\Omega)$, we put $C_\psi := \int_\Omega \psi \, dx$ and $\phi := \psi - C_\psi \eta$. Of course the function ϕ satisfies $\int_\Omega \phi \, dx = 0$, then

$\phi \in C_{0,\mathrm{av}}^{\infty}(\Omega)$. Therefore, by the definition of $\|u\|_{\tilde{L^p}(\Omega)}$ and using Hölder's inequality, we find that

$$\begin{aligned}|\langle u, \psi\rangle| &\le |\langle u, \phi\rangle| + |\langle u, C_\psi \eta\rangle| \\ &\le \|u\|_{\tilde{L^p}(\Omega)}\|\phi\|_{L^{p'}(\Omega)} + |C_\psi\langle u, \eta\rangle| \\ &\le \|u\|_{\tilde{L^p}(\Omega)}(\|\psi\|_{L^{p'}(\Omega)} + |C_\psi|\|\eta\|_{L^{p'}(\Omega)}) + |C_\psi\langle u, \eta\rangle| \\ &\le \{\|u\|_{\tilde{L^p}(\Omega)}(1 + |\Omega|^{1/p}\|\eta\|_{L^{p'}(\Omega)}) + |\Omega|^{1/p}|\langle u, \eta\rangle|\}\|\psi\|_{L^{p'}(\Omega)}\end{aligned}$$

for all $\psi \in C_0^\infty(\Omega)$. Therefore as a consequence of Riesz' representation theorem, there exists $\tilde{u} \in L^p(\Omega)$ such that

$$\langle u, \varphi\rangle = \int_\Omega \tilde{u}\varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$, and since (8) we have

$$\begin{aligned}\|\tilde{u}\|_{L^p(\Omega)} &\le \|u\|_{\tilde{L^p}(\Omega)}(1 + |\Omega|^{1/p}\|\eta\|_{L^{p'}(\Omega)}) + |\Omega|^{1/p}|\langle u, \eta\rangle| \\ &\le C\left(\int_\Omega |Du| + \|u\|_{H^{-1}(\Omega)}\right),\end{aligned} \tag{9}$$

where a constant C is independent of u. Moreover, the assumption of lemma leads $\tilde{u} \in BV(\Omega)$ and by using the embedding theorem $BV(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega)$ ([15]), we find $\tilde{u} \in L^{\frac{n}{n-1}}(\Omega)$. □

Remark 2.2. *We can estimate the total variation of $\tilde{u}$ in $\overline{\Omega}$. As is written in [10, p.183], we can decompose $\int_{\overline{\Omega}} |D\tilde{u}|$ into $\int_\Omega |D\tilde{u}|$ and $\int_{\partial\Omega} |T\tilde{u}| \, d\mathcal{H}^{n-1}$, where the trace $T\tilde{u} \in L^1(\partial\Omega)$ satisfies*

$$\|T\tilde{u}\|_{L^1(\partial\Omega)} \le C\left(\|\tilde{u}\|_{L^1(\Omega)} + \int_\Omega |D\tilde{u}|\right).$$

By using (9) *in the proof of Lemma 2.1, we observe that*

$$\|\tilde{u}\|_{L^1(\Omega)} \le |\Omega|^{1/p'}\|\tilde{u}\|_{L^p(\Omega)} \le C\left(\|\tilde{u}\|_{H^{-1}(\Omega)} + \int_\Omega |D\tilde{u}|\right).$$

As a result, we see that

$$\int_{\overline{\Omega}} |D\tilde{u}| \le C\left(\|\tilde{u}\|_{H^{-1}(\Omega)} + \int_\Omega |D\tilde{u}|\right). \tag{10}$$

Remark 2.3. *Let $v \in H_0^1(\Omega)$ and $n \le 4$. There exists $\{\varphi_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that*

$$\varphi_j \to v \quad \text{in } H^1(\Omega) \quad \text{as } j \to \infty.$$

By using Sobolev's embedding theorem, we see that

$$\varphi_j \to v \quad \text{in } L^n(\Omega) \quad \text{as } j \to \infty \quad (\text{weakly in the case } n = 4).$$

Therefore, for any $u \in H^{-1}(\Omega)$ with $\int_\Omega |Du| < +\infty$, we find that

$$\langle u, v\rangle = \lim_{j\to\infty}\langle u, \varphi_j\rangle = \lim_{j\to\infty}\int_\Omega \tilde{u}\varphi_j \, dx = \int_\Omega \tilde{u}v \, dx.$$

2.2 The Dirichlet Laplacian and inner product in $H^{-1}(\Omega)$

We recall the Dirichlet Laplacian $-\Delta_D$ on $H_0^1(\Omega)$.

Definition 2.4. *Let define the operator $-\Delta_D : H_0^1(\Omega) \to H^{-1}(\Omega)$ by*

$$\langle -\Delta_D u, v \rangle := \int_\Omega \nabla u \cdot \nabla v \, dx$$

for all $u, v \in H_0^1(\Omega)$.

We here recall a fundamental property of $-\Delta_D$.

Lemma 2.5. *The operator $-\Delta_D : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometry.*

By using Lemma 2.5, we define the inverse mapping

$$(-\Delta_D)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega),$$

and define $H^{-1}(\Omega)$ as a Hilbert space equipped with the inner product

$$(f, g)_{H^{-1}(\Omega)} := \langle f, (-\Delta_D)^{-1} g \rangle$$

for all $f, g \in H^{-1}(\Omega)$.

In this paper we use the pairings between measures and bounded functions. These pairings are constructed in [3]. Before giving pairings, we recall a fundamental property of BV functions. Assume $u \in BV(\Omega)$, then there exists a Radon measure Du such that

$$\int_\Omega \varphi \, d[Du] = -\int_\Omega u \operatorname{div} \varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^n)$. Next we introduce a trace for $BV(\Omega)$. See [10, 15, 28] for more details about $BV(\Omega)$. Let us denote by $\nu(x)$ the outward unit normal to $\partial\Omega$.

In this subsection we assume that Ω is a bounded domain with Lipschitz boundary.

Lemma 2.6. *There exists a bounded linear operator $T : BV(\Omega) \to L^1(\partial\Omega)$ such that*

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \, d[Du] + \int_{\partial\Omega} (\varphi \cdot \nu) T u \, d\mathcal{H}^{n-1}$$

for all $u \in BV(\Omega)$ and $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

We introduce some function spaces.

Definition 2.7. *Let $1 \le p \le \infty$. Define function spaces $X(\Omega)_p$ and $X(\Omega)$ by*

$$X(\Omega)_p := \{ z \in L^\infty(\Omega; \mathbb{R}^n) \mid \operatorname{div} z \in L^p(\Omega) \},$$

$$X(\Omega) := \{ z \in L^\infty(\Omega; \mathbb{R}^n) \mid \operatorname{div} z \in H_0^1(\Omega) \}.$$

Remark 2.8. *By Sobolev's embedding Theorem, we see that* $H_0^1(\Omega) \subset L^n(\Omega)$ *if* $n \leq 4$. *Therefore, we find that* $X(\Omega) \subset X(\Omega)_n$ *if* $n \leq 4$.

We introduce a Radon measure defined for $u \in BV(\Omega) \cap L^{p'}(\Omega)$ and $z \in X(\Omega)_p$, where $p' = p/(p-1)$ is the conjugate exponent of p.

Definition 2.9. *Let* $1 \leq p \leq n$, $z \in X(\Omega)_p$ *and* $u \in BV(\Omega) \cap L^{p'}(\Omega)$. *Then we define a functional* $(z, Du) : \mathscr{D}(\Omega) \to \mathbb{R}$ *by*

$$\langle (z, Du), \varphi \rangle := -\int_\Omega u\varphi \operatorname{div} z \, dx - \int_\Omega uz \cdot \nabla\varphi \, dx$$

for all $\varphi \in \mathscr{D}(\Omega)$.

We give a key property of the functional (z, Du).

Lemma 2.10. *The functional* (z, Du) *defined in Definition* 2.9 *is identified with a Radon measure. Furthermore,* (z, Du) *satisfies the inequality*

$$\int_\Omega |(z, Du)| \leq \|z\|_{L^\infty(\Omega)} \int_\Omega |Du|$$

where $|(z, Du)|$ *is the total variation measure of* (z, Du).

We next recall *Green's formula* as follows.

Lemma 2.11. *Let* $1 \leq p \leq n$. *Then there exists a linear operator* $\gamma : X(\Omega)_p \to L^\infty(\partial\Omega)$ *such that* γ *satisfies the following properties for all* $z \in X(\Omega)_p$ *and* $u \in BV(\Omega) \cap L^{p'}(\Omega)$*:*

$$\|\gamma(z)\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)},$$

$$\int_\Omega u \operatorname{div} z \, dx + \int_\Omega (z, Du) = \int_{\partial\Omega} \gamma(z) Tu \, d\mathcal{H}^{n-1},$$

$$(\gamma(z))(x) = z(x) \cdot \nu(x) \text{ for all } x \in \partial\Omega \quad \text{if} \quad z \in C^1(\overline{\Omega}; \mathbb{R}^n).$$

Above lemmas are proved in [3, 4].

2.3 The duality functions

We use the duality functions to characterize the subdifferential of the total variation on $H^{-1}(\Omega)$. So we recall the definition of duality functions and their properties.

Let $\mathbf{H}$ be a real Hilbert space and $(\cdot, \cdot)_\mathbf{H}$ be its inner product. For a function $L : \mathbf{H} \to [0, \infty]$, we define the duality function $\tilde{L} : \mathbf{H} \to [0, \infty]$ of L by

$$\tilde{L}(u) := \sup_{v \in \mathbf{H} \setminus \{0\}} \frac{(u, v)_\mathbf{H}}{L(v)}$$

for $u \in \mathbf{H}$. To prove the characterization of the subdifferential, we use the following three facts.

Lemma 2.12. *Let $L_1, L_2 : \mathbf{H} \to [0, \infty]$ be functions. If*

$$L_1(u) \leq L_2(u)$$

for all $u \in \mathbf{H}$, then

$$\tilde{L_2}(u) \leq \tilde{L_1}(u)$$

for all $u \in \mathbf{H}$.

Lemma 2.13. *Let $L : \mathbf{H} \to [0, \infty]$ be a function. If L is convex, lower semi-continuous, and positively homogeneous of degree 1, then*

$$\tilde{\tilde{L}}(u) = L(u)$$

for all $u \in \mathbf{H}$.

Lemma 2.14. *Let $L : \mathbf{H} \to [0, \infty]$ be a function. Assume that L is convex, lower semi-continuous, and positively homogeneous of degree 1. Let $u \in \mathbf{H}$. Then*

$$v \in \partial F(u),$$

if and only if

$$\tilde{F}(v) \leq 1, \qquad (u, v)_{\mathbf{H}} = F(u),$$

where ∂F is the subdifferential of F in $\mathbf{H}$.

These facts are proved in [8], so we omit these proofs.

3 Characterization of The Subdifferential of $\int_{\overline{\Omega}} |Du|$ in $H^{-1}(\Omega)$

In this section we assume that $\Omega \subset \mathbb{R}^n$ is a rectangular domain or a bounded domain with C^∞-boundary.

We consider the energy $\int_{\overline{\Omega}} |Du|$ (defined in Definition 3.1). This energy is suitable to identify the extinction of the solution of a total variation flow because $\int_{\overline{\Omega}} |Du| = 0$ leads to $u = 0$. We characterize the subdifferential of the total variation $\int_{\overline{\Omega}} |Du|$ in $H^{-1}(\Omega)$ and the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta \operatorname{div} \left(\dfrac{\nabla u}{|\nabla u|} \right) & \text{in } \Omega \times (0, \infty), \\ u = \operatorname{div} \left(\dfrac{\nabla u}{|\nabla u|} \right) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{11}$$

First we define $\int_{\overline{\Omega}} |Du|$ for $u \in H^{-1}(\Omega)$. We cannot define $\int_{\overline{\Omega}} |Du|$ directly because $\operatorname{div} \varphi$ does not necessarily belong to $H_0^1(\Omega)$ for $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. To define $\int_{\overline{\Omega}} |Du|$, we use Lemma 2.1.

Definition 3.1. *Let $u \in H^{-1}(\Omega)$. If $\int_\Omega |Du| < +\infty$, then define $\int_{\overline{\Omega}} |Du|$ by*

$$\int_{\overline{\Omega}} |Du| := \sup\left\{ \int_\Omega \tilde{u} \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n)} \le 1 \right\}.$$

If $\int_\Omega |Du| = +\infty$, then we define $\int_{\overline{\Omega}} |Du| = +\infty$.

This definition is reasonable because if $\int_{\overline{\Omega}} |Du| < +\infty$, then $\int_\Omega |Du| < +\infty$ automatically and there exists $\tilde{u} \in L^{\frac{n}{n-1}}(\Omega)$ such that

$$u = \tilde{u} \quad \text{in } \mathscr{D}'(\Omega).$$

It is easy to check the properties of this functional. In fact $\int_{\overline{\Omega}} |Du|$ is convex, lower semi-continuous, proper, and positively homogeneous of degree 1 in $H^{-1}(\Omega)$.

Next, we define functionals $F, G : H^{-1}(\Omega) \to [0, \infty]$ by

$$F(u) := \sup\left\{ \int_\Omega \tilde{u} \operatorname{div} \varphi \, dx \;\middle|\; \begin{array}{l} \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \\ \|\varphi\|_{L^\infty(\mathbb{R}^n)} \le 1,\ \operatorname{div} \varphi = 0 \text{ on } \partial\Omega \end{array} \right\}, \tag{12}$$

and

$$G(u) := \inf\{ \|z\|_{L^\infty(\Omega)} \mid z \in X(\Omega),\ u = -(-\Delta_D) \operatorname{div} z \}. \tag{13}$$

Then the functionals F and G are convex, lower semi-continuous, proper, and positively homogeneous of degree 1.

The following lemmas are key steps to characterize the subdifferential of $\int_{\overline{\Omega}} |Du|$ in $H^{-1}(\Omega)$. For the moment we admit the following duality representation lemmas and will prove main results.

Lemma 3.2. *Let $F : H^{-1}(\Omega) \to [0, \infty]$ be the functional defined by* (12)*. Then*

$$F(u) = \int_{\overline{\Omega}} |Du|$$

for $u \in H^{-1}(\Omega)$.

Lemma 3.3. *Let $F, G : H^{-1}(\Omega) \to [0, \infty]$ be functionals defined by* (12) *and* (13)*. Then*

$$G(u) = \tilde{F}(u)$$

for $u \in H^{-1}(\Omega)$, where $\tilde{F}$ is the duality functional of F.

We are ready to characterize the subdifferential ∂F of $F(u) = \int_{\overline{\Omega}} |Du|$ in $H^{-1}(\Omega)$. We postpone to prove Lemma 3.2 and Lemma 3.3 after stating Theorem 3.4.

Theorem 3.4. *Let Ω be a rectangular domain or a bounded domain with C^∞-boundary. Assume that $u \in H^{-1}(\Omega)$ satisfies $F(u) = \int_{\overline{\Omega}} |Du| < +\infty$. Then $v \in \partial F(u)$ if and only if there exists $z \in X(\Omega)$ with $\|z\|_{L^\infty(\Omega)} \le 1$ such that*

$$v = -(-\Delta_D) \operatorname{div} z, \quad (u, (-\Delta_D) \operatorname{div} z)_{H^{-1}(\Omega)} = -\int_{\overline{\Omega}} |Du|.$$

Proof of Theorem 3.4. By Lemma 2.14 we see that

$$v \in \partial F(u)$$

if and only if

$$\tilde{F}(v) \leq 1, \quad (u,v)_{H^{-1}(\Omega)} = F(u).$$

Moreover, $G = \tilde{F}$ holds by Lemma 3.3, these conditions are equivalent

$$G(v) \leq 1, \quad (u,v)_{H^{-1}(\Omega)} = \int_{\overline{\Omega}} |Du|.$$

This is the claim of our theorem, so the proof is complete. □

Theorem 3.4 gives a natural and intrinsic characterization of system (11) as

$$\begin{cases} \dfrac{du}{dt}(t) \in -\partial F(u(t)) & \text{in } H^{-1}(\Omega) \text{ for a.e. } t \in (0,\infty), \\ u(0) = u_0 & \text{in } H^{-1}(\Omega). \end{cases} \tag{14}$$

We have the boundary condition

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0,$$

since $\operatorname{div} z \in H_0^1(\Omega)$. Furthermore, we note that $u \in D(\partial F)$ (i.e. $\partial F(u) \neq \emptyset$) formally requires the boundary condition

$$u = 0,$$

since the solution of the total variation flow deceases $\int_{\partial\Omega} |Tu| \, d\mathcal{H}^{n-1}$. So we give the definition of solutions to our problem (11).

Definition 3.5. *Let $u_0 \in H^{-1}(\Omega)$. A function $u \in C([0,\infty); H^{-1}(\Omega))$ is a solution to the fourth order singular diffusion problem* (11) *with the initial datum u_0, if the following conditions are satisfied:*

1. *u is absolutely continuous on any compact subset of* $(0,\infty)$*.*
2. *u is a solution of nonlinear evolution problem* (14)*.*

We obtain the existence and uniqueness of the global-in-time solution for system (11) by applying the nonlinear semigroup theory [6, 21].

Proposition 3.6. *Assume that an initial datum $u_0 \in H^{-1}(\Omega)$ satisfies $\int_{\overline{\Omega}} |Du_0| < +\infty$. Then there exists a unique solution u to the problem* (11) *in the sense of Definition 3.5.*

We conclude this section by proving Lemma 3.2 and Lemma 3.3.

Proof of Lemma 3.2. Let $u \in H^{-1}(\Omega)$. By the definition (12) of F, it is obvious that $F(u) \le \int_{\overline{\Omega}} |Du|$, so we only prove $\int_{\overline{\Omega}} |Du| \le F(u)$. If $F(u) = +\infty$, then the inequality is trivial. So we assume that $F(u) < +\infty$, then $\int_\Omega |Du| < +\infty$ and

$$\int_{\overline{\Omega}} |Du| \le C \left(\int_\Omega |Du| + \|u\|_{H^{-1}(\Omega)} \right) < +\infty,$$

by (10) in Remark 2.2.

First we consider the case that Ω is a rectangular domain represented as $\Omega := \prod_{i=1}^n (0, b_i)$. We give the main idea of the proof. We approximate a test function $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n)} \le 1$ by a following test function

$$\psi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \quad \|\psi\|_{L^\infty(\mathbb{R}^n)} \le 1, \quad \operatorname{div} \psi = 0 \ \text{ on } \partial\Omega.$$

We prepare tools for approximation. Fix any $\varepsilon > 0$. Since Du is a Radon measure in Ω, there exists $\delta > 0$ such that the shaved domain $\Omega^\delta := \prod_{i=1}^n (\delta, b_i - \delta)$ satisfies

$$\int_{\Omega \setminus \Omega^\delta} |Du| < \varepsilon. \tag{15}$$

Next we consider estimates on boundary $\partial\Omega$. To do so, we prepare some notations. We put the index set as follows:

$$I_n := \{\alpha = (\alpha_1, \dots, \alpha_n) \mid \alpha_i = 0 \ \text{ or } \ \alpha_i = b_i \ \text{ for } i = 1, 2, \dots, n\}.$$

Since $T\tilde{u} \in L^1(\partial\Omega)$ where $\tilde{u}$ is defined in Lemma 2.1, there exists $0 < \tau < \delta$ such that the small cube $\Omega_\alpha^\tau := \prod_{i=1}^n (\alpha_i - \tau, \alpha_i + \tau)$, which includes only one vertex of Ω, satisfies

$$\sum_{\alpha \in I_n} \int_{\Omega_\alpha^\tau \cap \partial\Omega} |T\tilde{u}| \, d\mathcal{H}^{n-1} < \varepsilon. \tag{16}$$

Take cut-off functions $\zeta \in C_0^\infty(\Omega)$ such that

$$\begin{cases} \zeta \equiv 1 \ \text{ in } \Omega^\delta, \\ 0 \le \zeta \le 1 \ \text{ in } \Omega, \\ \operatorname{supp}(\zeta) \subset \displaystyle\prod_{i=1}^n (\tau, b_i - \tau), \end{cases}$$

and $\eta_0, \eta_i \in C_0^\infty(\mathbb{R})$ for $i = 1, \dots, n$ such that

$$\begin{cases} \eta_0 \equiv 1 \ \text{ in } \left[-\dfrac{\tau}{8}, \dfrac{\tau}{8}\right], \\ 0 \le \eta_0 \le 1 \ \text{ in } \mathbb{R}, \\ \operatorname{supp}(\eta_0) \subset \left(-\dfrac{\tau}{4}, \dfrac{\tau}{4}\right), \end{cases} \qquad \begin{cases} \eta_i \equiv 1 \ \text{ in } [\tau, b_i - \tau], \\ 0 \le \eta_i \le 1 \ \text{ in } \mathbb{R}, \\ \operatorname{supp}(\eta_i) \subset \left(-\dfrac{3\tau}{4}, b_i - \dfrac{3\tau}{4}\right). \end{cases}$$

For any $\varphi = (\varphi_1, \dots, \varphi_n) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R})} \le 1$, we define an approximating function $\psi = (\psi_1, \dots, \psi_n) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ by

$$\psi_i(x) := \zeta(x)\varphi_i(x) + \tilde{\psi}_i(x) \prod_{j \ne i} \eta_j(x_j)$$

where

$$\tilde{\psi}_i(x) := \eta_0(x_i)\varphi_i(\dots, x_{i-1}, 0, x_{i+1}, \dots) + \eta_0(x_i - b_i)\varphi_i(\dots, x_{i-1}, b_i, x_{i+1}, \dots)$$

for $i = 1, \dots, n$. It is easily to check that $\|\psi\|_{L^\infty(\mathbb{R}^n)} \le 1$ and $\operatorname{div} \psi = 0$ on $\partial\Omega$ because of $\frac{\partial \tilde{\psi}_i}{\partial x_i} = 0$ on $\partial\Omega$. Moreover we find that $\|\varphi - \psi\|_{L^\infty(\mathbb{R})} \le 1$.

Therefore we observe that

$$\begin{aligned} \left| \int_\Omega \tilde{u} \operatorname{div}(\varphi - \psi)\, dx \right| &\le \left| \int_\Omega (\varphi - \psi, D\tilde{u}) \right| + \left| \int_{\partial\Omega} T\tilde{u}(\varphi - \psi) \cdot \nu \, d\mathcal{H}^{n-1} \right| \\ &\le \int_{\Omega \setminus \Omega^\delta} |Du| + \sum_{\alpha \in I_n} \int_{\Omega_\alpha^\tau \cap \partial\Omega} |T\tilde{u}| \, d\mathcal{H}^{n-1} \\ &< 2\varepsilon \end{aligned}$$

by conditions (15) and (16). Taking supremum, we find that

$$\int_{\overline{\Omega}} |Du| \le F(u) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\int_{\overline{\Omega}} |Du| = F(u),$$

which completes the proof for the case when Ω is a rectangular domain.

Next we consider the case that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^∞-boundary. Fix any $\varepsilon > 0$. Since Du is a Radon measure in Ω, there exists $\delta > 0$ such that the shaved domain $\Omega^\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$ satisfies

$$\int_{\Omega \setminus \Omega^\delta} |Du| < \varepsilon. \tag{17}$$

Since $\partial\Omega$ is compact and C^∞, we can take $\{U_i\}_{1 \le i \le m}$, $\gamma_i : \mathbb{R}^{n-1} \to \mathbb{R}$, and $\{\ell_i\}_{1 \le i \le m} \subset \mathbb{N}$ such that

$$U_i \cap \Omega = \{x \in U_i \mid x_{\ell_i} > \gamma_i(x_1, \dots, x_{\ell_i - 1}, x_{\ell_i + 1}, \dots, x_n)\}$$

or

$$U_i \cap \Omega = \{x \in U_i \mid x_{\ell_i} < \gamma_i(x_1, \dots, x_{\ell_i - 1}, x_{\ell_i + 1}, \dots, x_n)\}.$$

Write

$$x'^\ell := (x_1, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_n), \qquad x' := x'^n = (x_1, \dots, x_{n-1}).$$

Denote the set

$$\{x'^{\ell_i} \in \mathbb{R}^{n-1} \mid (x_1, \ldots, x_{\ell_i - 1}, \gamma_i(x'^{\ell_i}), x_{\ell_i + 1}, \ldots, x_n) \in U_i \cap \partial\Omega\}$$

by $\tilde{U}_i$. In the same manner to the case Ω being a rectangular domain, we take $\tilde{W}_i$, $\tilde{V}_i$, which are compactly contained in $\tilde{V}_i$, $\tilde{U}_i$ respectively, and $0 < \tau < \delta$ such that

$$\left(\bigcup_{\tilde{x} \in \tilde{V}_i} \{\tilde{x}\} \times (\gamma_i(\tilde{x}) - \tau, \gamma_i(\tilde{x}) + \tau)\right) \cap \left(\bigcup_{\tilde{x} \in \tilde{V}_j} \{\tilde{x}\} \times (\gamma_j(\tilde{x}) - \tau, \gamma_j(\tilde{x}) + \tau)\right) = \emptyset$$

if $i \neq j$, and

$$\int_{\partial\Omega \setminus \bigcup_{i=1}^m \{[\tilde{x}, \gamma_i(\tilde{x})] \mid \tilde{x} \in \tilde{W}_i\}} |T\tilde{u}| \, d\mathcal{H}^{n-1} < \varepsilon. \tag{18}$$

Next we consider cut-off functions. Set $\xi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \xi \equiv 1 \ \text{ in } \Omega^\delta, \\ 0 \leq \xi \leq 1 \ \text{ in } \mathbb{R}^n, \\ \operatorname{supp}(\xi) \subset \Omega^\delta, \end{cases}$$

and $\eta \in C_0^\infty(\mathbb{R}), \zeta_i \in C_0^\infty(\mathbb{R}^{n-1})$ for $i = 1, \ldots, n$ such that

$$\begin{cases} \eta \equiv 1 \ \text{ in } \left[-\dfrac{\tau}{2}, \dfrac{\tau}{2}\right], \\ 0 \leq \eta \leq 1 \ \text{ in } \mathbb{R}^{n-1}, \\ \operatorname{supp}(\eta) \subset (-\tau, \tau), \end{cases} \qquad \begin{cases} \zeta_i \equiv 1 \ \text{ in } \tilde{W}_i, \\ 0 \leq \zeta_i \leq 1 \ \text{ in } \mathbb{R}^{n-1}, \\ \operatorname{supp}(\zeta_i) \subset \tilde{V}_i. \end{cases}$$

Assume $\varphi = (\varphi_1, \ldots, \varphi_n) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfies $\|\varphi\|_{L^\infty(\mathbb{R}^n)} \leq 1$. We suppose for simplicity $\ell_j = n$, then define $g_j : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_j(x') := -\frac{1}{\sqrt{1 + |\nabla_{x'}\gamma_j|^2}} \begin{pmatrix} \nabla_{x'}\gamma_j \\ -1 \end{pmatrix} \cdot \varphi(x', \gamma_j(x'))$$

on $\tilde{U}_j$. In the similar way we define g_i for all $1 \leq i \leq m$. Then g_i is independent of x_{ℓ_i}. In order to use this independence, take $\psi = (\psi_1, \ldots, \psi_n) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ as

$$\psi_k(x) := \xi(x)\varphi_k(x) + \sum_{1 \leq i \leq m, \ell_i = k} g_i(x'^{\ell_i})\zeta_i(x'^{\ell_i})\eta(x_{\ell_i} - \gamma(x'^{\ell_i}))$$

for $1 \leq k \leq n$. Then, we immediately have

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leq 1, \qquad \operatorname{div}\psi = 0 \ \text{ on } \partial\Omega.$$

In addition, by conditions (17) and (18), we see that

$$\begin{aligned} \left| \int_\Omega \tilde{u} \operatorname{div}(\varphi - \psi) \, dx \right| &\leq \left| \int_\Omega (\varphi - \psi, D\tilde{u}) \right| + \left| \int_{\partial\Omega} T\tilde{u}(\varphi - \psi) \cdot \nu \, d\mathcal{H}^{n-1} \right| \\ &\leq \int_{\Omega \setminus \Omega^\delta} |Du| + \sum_{i=1}^m \int_{\partial\Omega \setminus \bigcup_{i=1}^m \{[\tilde{x}, \gamma_i(\tilde{x})] \mid \tilde{x} \in \tilde{W}_i\}} |T\tilde{u}| \, d\mathcal{H}^{n-1} \\ &< 2\varepsilon. \end{aligned}$$

Taking supremum, we find that

$$\int_{\overline{\Omega}} |Du| \le F(u) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\int_{\overline{\Omega}} |Du| = F(u),$$

which completes the proof. □

Proof of Lemma 3.3. First we prove $\tilde{F} \le G$. Let $u \in H^{-1}(\Omega)$. If $G(u) = +\infty$, then this inequality is trivial. So we assume that $G(u) < +\infty$, then there exists $z \in X(\Omega)$ such that $u = -(-\Delta_D)\operatorname{div} z$ by the definition of the functional G. For $v \in H^{-1}(\Omega)$ with $F(v) < +\infty$, it follows that

$$\begin{aligned}
(u, v)_{H^{-1}(\Omega)} &= -\int_{\Omega} \tilde{v} \operatorname{div} z \, dx \\
&= \int_{\Omega} (z, D\tilde{v}) - \int_{\partial\Omega} \gamma(z) T\tilde{v} \, d\mathcal{H}^{n-1} \\
&\le \|z\|_{L^\infty(\Omega)} \left(\int_{\Omega} |D\tilde{v}| + \int_{\partial\Omega} |T\tilde{v}| \, d\mathcal{H}^{n-1} \right) = \|z\|_{L^\infty(\Omega)} F(v),
\end{aligned}$$

where we use Lemma 3.2 in the last equations. Hence we have

$$\tilde{F}(u) = \sup_{v \in H^{-1}(\Omega)\setminus\{0\}} \frac{(u, v)_{H^{-1}(\Omega)}}{F(v)} \le \|z\|_{L^\infty(\Omega)}.$$

Therefore, we see that $\tilde{F}(u) \le G(u)$ by taking infimum of the right hand side.

Next we prove $G \le \tilde{F}$. First we outline the proof. If we prove $\tilde{G} \ge F$, then we see that $\tilde{\tilde{G}} \le \tilde{F}$ by Lemma 2.12. Furthermore, since G is convex, lower semi-continuous, and positively homogeneous of degree 1, by using Lemma 2.13 we show that $\tilde{\tilde{G}} = G$ and $G \le \tilde{F}$. Therefore, we only need to show that $\tilde{G} \ge F$ to prove $G \le \tilde{F}$.

Let $u \in H^{-1}(\Omega)$. By the definition of G, we see that

$$\begin{aligned}
\tilde{G}(u) &= \sup_{v \in H^{-1}(\Omega)\setminus\{0\}} \frac{(u, v)_{H^{-1}(\Omega)}}{G(v)} \\
&\ge \sup_{z \in C^\infty(\mathbb{R}^n;\mathbb{R}^n),\, \operatorname{div} z = 0 \text{ on } \partial\Omega} \frac{(u, -(-\Delta_D)\operatorname{div} z)_{H^{-1}(\Omega)}}{G(-(-\Delta_D)\operatorname{div} z)} \\
&\ge \sup_{z \in C^\infty(\mathbb{R}^n;\mathbb{R}^n),\, \operatorname{div} z = 0 \text{ on } \partial\Omega} \frac{\langle u, -\operatorname{div} z \rangle}{\|z\|_{L^\infty(\Omega)}} \\
&= F(u).
\end{aligned}$$

Therefore, $\tilde{F} = G$ holds on $H^{-1}(\Omega)$. □

4 Main Theorems on Extinction Time Estimates

In this section, we give our main results on the extinction time estimates for the solution of the fourth-order total variation flow with Dirichlet boundary condition. Let Ω be a bounded domain in $\mathbb{R}^n$. Before stating main results, we recall some notations and negative norms.

Let $\mathbb{T}^n$ be a n-dimensional torus defined by

$$\mathbb{T}^n := \prod_{i=1}^{n} \mathbb{R}/(\omega_i \mathbb{Z})$$

with constants $\omega_i > 0$, $i = 1, \dots, n$. Here, we take sufficiently large constants $\omega_i > 0$ such that the periodic cell

$$\Omega_{\mathrm{per}} := \prod_{i=1}^{n} (-\omega_i/2, \omega_i/2)$$

contains the closure $\overline{\Omega}$ of Ω. Next, we define the function spaces as follows:

$$\begin{aligned} C^\infty_{\mathrm{av}}(\mathbb{T}^n) &:= \left\{ \varphi \in C^\infty(\mathbb{T}^n) \;\middle|\; \int_{\mathbb{T}^n} \varphi \, dx = 0 \right\}, \\ H^1_{\mathrm{av}}(\mathbb{T}^n) &:= \left\{ \varphi \in H^1(\mathbb{T}^n) \;\middle|\; \int_{\mathbb{T}^n} \varphi \, dx = 0 \right\}, \\ H^{-1}_{\mathrm{av}}(\mathbb{T}^n) &:= (H^1_{\mathrm{av}}(\mathbb{T}^n))^*. \end{aligned}$$

Definition 4.1. *Let $u \in H^1_{\mathrm{av}}(\mathbb{T}^n)$. Assume $1 \le p \le \infty$ and p' is the conjugate exponent of p. Define $\|u\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)}$ by*

$$\|u\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)} := \sup \left\{ \int_{\mathbb{T}^n} u\varphi \, dx \;\middle|\; \varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n),\ \|\nabla \varphi\|_{L^{p'}(\mathbb{T}^n)} \le 1 \right\}.$$

Since $\overline{\Omega}$ is compact and contained in Ω_{per}, there exists a constant $C_p > 0$ such that

$$\|\varphi\|_{L^p(\partial\Omega)} \le C_p (\operatorname{diam} \Omega)^{\frac{1}{p'}} \|\nabla \varphi\|_{L^p(\Omega_{\mathrm{per}})}$$

for all $\varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n)$. Take $\lambda \Omega_{\mathrm{per}}$ for $\lambda \Omega$, where $\lambda > 0$. Then, we see that C_p is independent of the size of Ω.

Definition 4.2. *For $v \in H^1_0(\Omega)$ we define $\|v\|_{\dot{W}^{-1,p}_0(\Omega)}$ by*

$$\|v\|_{\dot{W}^{-1,p}_0(\Omega)} := \sup \left\{ \int_\Omega v\varphi \, dx \;\middle|\; \begin{array}{l} \varphi \in C^\infty(\overline{\Omega}), \|\nabla\varphi\|_{L^{p'}(\Omega)} \le 1, \\ \|\varphi\|_{L^{p'}(\partial\Omega)} \le C_{p'} (\operatorname{diam}\Omega)^{\frac{1}{p}} \end{array} \right\}.$$

Remark 4.3. *Let $v \in H^1_0(\Omega)$ and $\lambda > 0$. Define $v^\lambda \in H^1_0(\lambda^{-1}\Omega)$ by*

$$v^\lambda(x) := v(\lambda x).$$

We see that

$$\|v^\lambda\|_{\dot{W}^{-1,p}(\lambda^{-1}\Omega)} = \lambda^{-1-\frac{n}{p}} \|v\|_{\dot{W}^{-1,p}(\Omega)}.$$

In this section we show the main theorems of this paper. Let $T(u_0)$ be the extinction time of the solution of system (14) with the initial datum $u|_{t=0} = u_0$. Similar estimates are known for periodic case by Giga and Kohn [13].

Let $1 \le n \le 4$. We consider the next condition:

$$1 + \frac{n}{2} = \theta(n-1) + (1-\theta)\left(3 + \frac{n}{p}\right) \tag{19}$$

for $1 \le p \le \infty$, $\frac{1}{2} < \theta \le 1$.

Theorem 4.4. *Let $1 \le n \le 4$ and $\frac{1}{2} < \theta \le 1$. We assume that the following condition 1 or 2 is satisfied.*

1. *Ω is a rectangular domain, $1 \le p \le \infty$ and the condition (19) holds.*
2. *Ω is a bounded domain with C^∞-boundary, $1 < p < \infty$ and the condition (19) holds.*

And suppose that u is the solution of system (11) with initial datum $u_0 \in H^{-1}(\Omega)$. Then there is a scale-invariant constant $C_* > 0$ such that

$$\|u\|_{H^{-1}(\Omega)}(t)^{2-(1/\theta)} \le \|u_0\|_{H^{-1}(\Omega)}^{2-(1/\theta)} - \left(2 - \frac{1}{\theta}\right) C_*^{-1/\theta} \int_0^t A(s)^{1-(1/\theta)}\, ds$$

for all $t < T(u_0)$, with

$$A(t) = (|\Omega|^{\frac{1}{p}} + C_{p'}(\operatorname{diam}\Omega)^{\frac{1}{p}}|\partial\Omega|^{\frac{1}{p}})t + \|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}.$$

Set

$$a := |\Omega|^{\frac{1}{p}} + C_{p'}|\partial\Omega|^{\frac{1}{p}}(\operatorname{diam}\Omega)^{\frac{1}{p}}, \qquad \gamma := 2 - \frac{1}{\theta}.$$

Then, the extinction time $T(u_0)$ is estimated by

$$T(u_0) \le \frac{\|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}}{a}\left(\left(1 + \frac{aC_*^{1/\theta}\|u_0\|_{H^{-1}(\Omega)}^\gamma}{\|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}^\gamma}\right)^{1/\gamma} - 1\right).$$

Remark 4.5. *If*

$$\frac{aC_*^{1/\theta}\|u_0\|_{H^{-1}(\Omega)}^\gamma}{\|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}^\gamma}$$

is small, then there exists a constant $c > 0$ such that

$$\left(1 + \frac{aC_*^{1/\theta}\|u_0\|_{H^{-1}(\Omega)}^\gamma}{\|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}^\gamma}\right)^{1/\gamma} \le 1 + c\frac{aC_*^{1/\theta}\|u_0\|_{H^{-1}(\Omega)}^\gamma}{\|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}^\gamma}.$$

By using this inequality, we see that

$$T(u_0) \le C_*'\|u_0\|_{H^{-1}(\Omega)}^\gamma \|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}^{1-\gamma}$$

with a scale-invariant constant $C'_* := cC_*^{1/\theta}$.

In the case $n = 4$ and $\theta = 1$, we find that the extinction time estimate obtained in Theorem 4.4 leads to

$$T(u_0) \leq C_* \|u_0\|_{H^{-1}(\Omega)}.$$

These are same to the estimate which is pointed out in [13].

Here we recall the extinction time estimate with periodic boundary condition, which was proved by Giga and Kohn [13]. We review the definition of $-\Delta_{\text{av}}$ and its properties in Section 8.

Theorem 4.6. *[Theorem 3.8, 13] Suppose that $1 \leq n \leq 4$, $1 \leq p \leq \infty$, and $\frac{1}{2} < \theta \leq 1$ satisfy the equation (19). Let u be the solution of system (1) with periodic boundary condition with initial datum $u_0 \in H^{-1}_{\text{av}}(\mathbb{T}^n)$. Then there is a scale-invariant constant $C_* > 0$ such that*

$$\|u\|_{H^{-1}(\mathbb{T}^n)}(t)^{2-(1/\theta)} \leq \|u_0\|^{2-(1/\theta)}_{H^{-1}(\mathbb{T}^n)} - \left(2 - \frac{1}{\theta}\right) C_*^{-1/\theta} \int_0^t A(s)^{1-(1/\theta)}\, ds$$

for all $t < T(u_0)$, with

$$A(t) := |\Omega_{\text{per}}|^{\frac{1}{p}} t + \|(-\Delta_{\text{av}})^{-1} u_0\|_{\dot{W}^{-1,p}(\mathbb{T}^n)}.$$

Set

$$a := |\Omega_{\text{per}}|^{\frac{1}{p}}, \qquad \gamma := 2 - \frac{1}{\theta}.$$

Then, as a consequence we have

$$T(u_0) \leq \frac{\|(-\Delta_{\text{av}})^{-1} u_0\|_{\dot{W}^{-1,p}(\mathbb{T}^n)}}{a} \left(\left(1 + \frac{a C_*^{1/\theta} \|u_0\|^{\gamma}_{H^{-1}(\mathbb{T}^n)}}{\|(-\Delta_{\text{av}})^{-1} u_0\|^{\gamma}_{\dot{W}^{-1,p}(\mathbb{T}^n)}} \right)^{1/\gamma} - 1 \right).$$

The following interpolation inequality is used in the proof of Theorem 4.6 in [13].

Lemma 4.7. *[Lemma 3.4, 13] Suppose that $1 \leq n \leq 4$, $1 \leq p \leq \infty$, and $\frac{1}{2} < \theta \leq 1$ satisfy the equation (19). Then, there exists a positive constant C_* such that*

$$\|u\|_{H^{-1}(\mathbb{T}^n)} \leq C_* \|(-\Delta_{\text{av}})^{-1} u\|^{1-\theta}_{\dot{W}^{-1.p}(\mathbb{T}^n)} \left(\int_{\mathbb{T}^n} |Du| \right)^{\theta}$$

for all $u \in H^{-1}_{\text{av}}(\mathbb{T}^n) \cap BV(\mathbb{T}^n)$. Moreover, the constant C_* is scale-invariant.

Next we show the Dirichlet version of this interpolation inequality.

Lemma 4.8. *Let $1 \leq n \leq 4$ and $\frac{1}{2} < \theta \leq 1$. We assume that the following condition 1 or 2 is satisfied.*

1. *Ω is a rectangular domain, $1 \leq p \leq \infty$ and the equation (19) holds.*
2. *Ω is a bounded domain with C^∞-boundary, $1 < p < \infty$ and the equation (19) holds.*

Then, there exists a positive constant C_* such that

$$\|u\|_{H^{-1}(\Omega)} \le C_* \|(-\Delta_D)^{-1}u\|^{1-\theta}_{\dot{W}_0^{-1,p}(\Omega)} \left(\int_{\overline{\Omega}} |Du| \right)^{\theta}$$

for all $u \in H^{-1}(\Omega) \cap BV(\Omega)$. Moreover, the constant C_* is scale-invariant.

We prove Lemma 4.8 in the next section, and in Section 6 we prove Theorem 4.4.

5 Extension Operators and Interpolation Inequalities

Here we prove Lemma 4.8. The main idea of the proof is to take an extension operator $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ and use Lemma 4.7 established by Giga and Kohn for periodic problems. First, we define an extension operator for the case that Ω is a rectangular domain. Without loss of generality, we may assume that Ω is represented as

$$\Omega := \prod_{i=1}^{n} (0, b_i)$$

with constants $b_i > 0, i = 1, \dots, n$. Define an extension operator $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ as follows.

Definition 5.1. *Let $\Omega = \prod_{i=1}^{n}(0, b_i)$ be a rectangular domain in $\mathbb{R}^n$. Consider the periodic boundary condition with a periodic cell $\Omega_{\mathrm{per}} = \prod_{i=1}^{n}(-b_i, b_i)$. We set an index set as follows:*

$$J_n := \{\alpha = (\alpha_1, \dots, \alpha_n) \mid \alpha_i = \pm 1 \quad \text{for} \quad i = 1, \dots, n\}.$$

Moreover for $\alpha = (\alpha_1, \dots, \alpha_n) \in J_n$, we denote its signature

$$\operatorname{sgn} \alpha := \sharp\{i \mid \alpha_i = -1\},$$

and a part of domain of Ω_{per}

$$\Omega_\alpha := \{(x_1, \dots, x_n) \in \Omega_{\mathrm{per}} \mid (\alpha_1 x_1, \dots, \alpha_n x_n) \in \Omega\}.$$

Let $u \in H^{-1}(\Omega)$ and $v = (-\Delta_D)^{-1}u \in H^1_0(\Omega)$. Then define the function $\tilde{v}$ on Ω_{per} by

$$\tilde{v}(x) := (-1)^{\operatorname{sgn}\alpha} v(\alpha_1 x_1, \dots, \alpha_n x_n) \quad \text{if} \quad x \in \Omega_\alpha.$$

It follows easily that $\tilde{v} \in H^1_{\mathrm{av}}(\mathbb{T}^n)$. Define an extension operator $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ by

$$Eu := -\Delta_{\mathrm{av}} \tilde{v}.$$

For example, $\tilde{v}$ is represented by

$$\tilde{v}(x,y) := \begin{cases} v(x,y) & \text{in } (0,b_1) \times (0,b_2), \\ -v(x,-y) & \text{in } (0,b_1) \times (-b_2,0), \\ -v(-x,y) & \text{in } (-b_1,0) \times (0,b_2), \\ v(-x,-y) & \text{in } (-b_1,0) \times (-b_2,0) \end{cases}$$

if $n = 2$.

We show the property of this extension operator E, which is important in considering the interpolation inequality.

Lemma 5.2. *Let $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ be the extension operator defined in Definition 5.1. Suppose that $1 \le p \le \infty$ and $u \in H^{-1}(\Omega)$ with $\int_{\overline{\Omega}} |Du| < +\infty$. Then we have*

$$\int_{\mathbb{T}^n} |D(Eu)| \le 2^n \int_{\overline{\Omega}} |Du|,$$

and

$$\|(-\Delta_{\mathrm{av}})^{-1}(Eu)\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)} \le 2^n \|(-\Delta_D)^{-1}u\|_{\dot{W}^{-1,p}_0(\Omega)}.$$

To prove Lemma 5.2, we introduce the following lemma.

Lemma 5.3. *Let $u \in H^{-1}(\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$ and $v = (-\Delta_D)^{-1}u \in H^1_0(\Omega)$. Let $\tilde{v}$ and $\tilde{u}$ be, respectively, the odd extension of v, and the extension $\tilde{u} = Eu := -\Delta_{\mathrm{av}}\tilde{v}$, as in Definition 5.1. Then we see that*

$$\langle Eu, \varphi \rangle = \sum_{\alpha \in J_n} \int_{\Omega_\alpha} \tilde{u}\varphi \, dx$$

for $\varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n)$.

Proof of Lemma 5.3. Assume that $u \in H^{-1}(\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$ and $u = -\Delta_D v$ in Ω. Then by the regularity argument in [16] (see also [14] if Ω is smooth), we conclude that $v \in H^1_0(\Omega) \cap W^{2,\frac{n}{n-1}}(\Omega)$. Therefore, we see that

$$\nabla \tilde{v}|_{\Omega_\alpha} \in W^{1,\frac{n}{n-1}}(\Omega_\alpha), \qquad \left.\frac{\partial \tilde{v}}{\partial \nu}\right|_{\partial\Omega_\alpha} \in L^{\frac{n}{n-1}}(\partial\Omega_\alpha).$$

For $\varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n)$, we find that

$$\begin{aligned}\langle Eu, \varphi \rangle &= \sum_{\alpha \in J_n} \int_{\Omega_\alpha} \nabla \tilde{v} \cdot \nabla \varphi \, dx \\ &= \sum_{\alpha \in J_n} \left(\int_{\partial\Omega_\alpha} \frac{\partial \tilde{v}}{\partial \nu} \varphi \, d\mathcal{H}^{n-1} - \int_{\Omega_\alpha} \Delta \tilde{v} \, \varphi \, dx \right).\end{aligned}$$

Now, we consider the first integration terms on boundary. For example, we take $\alpha = (1, 1, \ldots, 1), \alpha' = (-1, 1, \ldots, 1) \in J_n$. Then $S := \{0\} \times \prod_{i=2}^n (0, b_i)$ is the part of boundary of Ω_α and $\Omega_{\alpha'}$. Now, we see that

$$\int_{S \cap \partial\Omega_\alpha} \frac{\partial \tilde{v}}{\partial \nu} \varphi \, d\mathcal{H}^{n-1} = \int_0^{b_n} \cdots \int_0^{b_2} \left(-\frac{\partial v}{\partial x_1} \right) \varphi \, dx_2 \cdots dx_n,$$

and

$$\int_{S \cap \partial\Omega_{\alpha'}} \frac{\partial \tilde{v}}{\partial \nu} \varphi \, d\mathcal{H}^{n-1} = \int_0^{b_n} \cdots \int_0^{b_2} \frac{\partial v}{\partial x_1} \varphi \, dx_2 \cdots dx_n.$$

Thus, the sum of the integrals on S is 0. By the similar calculation, we have

$$\sum_{\alpha \in J_n} \int_{\partial\Omega_\alpha} \frac{\partial \tilde{v}}{\partial \nu} \varphi \, d\mathcal{H}^{n-1} = 0.$$

Therefore

$$\langle Eu, \varphi \rangle = -\sum_{\alpha \in J_n} \int_{\Omega_\alpha} \Delta \tilde{v} \, \varphi \, dx = \sum_{\alpha \in J_n} \int_{\Omega_\alpha} \tilde{u}\varphi \, dx$$

and the proof of lemma is now complete. $\square$

Proof of Lemma 5.2. First we consider $\int_{\mathbb{T}^n} |D(Eu)|$. This is easily proved. Indeed,

$$\begin{aligned}
\int_{\mathbb{T}^n} |D(Eu)| &:= \sup\left\{\langle Eu, \operatorname{div}\varphi\rangle \mid \varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{T}^n)} \le 1\right\} \\
&= \sup\left\{ \sum_{\alpha\in J_n} \int_{\Omega_\alpha} \tilde{u}\operatorname{div}\varphi\,dx \;\middle|\; \varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{T}^n)} \le 1\right\} \\
&\le \sum_{\alpha\in J_n} \sup\left\{ \int_{\Omega_\alpha} \tilde{u}\operatorname{div}\varphi\,dx \;\middle|\; \varphi \in C^\infty(\overline{\Omega_\alpha};\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\overline{\Omega_\alpha})} \le 1\right\} \\
&= 2^n \int_{\overline{\Omega}} |Du|.
\end{aligned}$$

Next we consider $\|(-\Delta_{\mathrm{av}})^{-1}(Eu)\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)}$. If $\varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n)$ satisfies $\|\nabla\varphi\|_{L^{p'}(\mathbb{T}^n)} \le 1$, then by Poincaré's inequality [10, Chapter 5.8.1] and a trace property, we find that

$$\|\varphi\|_{L^{p'}(\partial\Omega_\alpha)} \le C_{p'}(\operatorname{diam}\Omega)^{\frac{1}{p}} \|\nabla\varphi\|_{L^{p'}(\mathbb{T}^n)}$$

for $\alpha \in J_n$. Therefore, we check that

$$\begin{aligned}
&\|(-\Delta_{\mathrm{av}})^{-1}(Eu)\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)} \\
&\quad= \sup\left\{ \int_{\mathbb{T}^n} (-\Delta_{\mathrm{av}})^{-1}(Eu)\varphi\,dx \;\middle|\; \varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n),\ \|\nabla\varphi\|_{L^{p'}(\mathbb{T}^n)} \le 1\right\} \\
&\quad= \sup\left\{ \sum_{\alpha\in J_n} \int_{\Omega_\alpha} \tilde{v}\varphi\,dx \;\middle|\; \varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n),\ \|\nabla\varphi\|_{L^{p'}(\mathbb{T}^n)} \le 1\right\} \\
&\quad\le \sum_{\alpha\in J_n} \sup\left\{ \int_{\Omega_\alpha} \tilde{v}\varphi\,dx \;\middle|\; \begin{array}{l} \varphi \in C^\infty(\overline{\Omega_\alpha}), \|\nabla\varphi\|_{L^{p'}(\Omega_\alpha)} \le 1, \\ \|\varphi\|_{L^{p'}(\partial\Omega_\alpha)} \le C_{p'}(\operatorname{diam}\Omega)^{\frac{1}{p}} \end{array}\right\} \\
&\quad= 2^n \|(-\Delta_D)^{-1}u\|_{\dot{W}^{-1,p}_0(\Omega)},
\end{aligned}$$

which is the desired conclusion. □

Next we define an extension operator E for the case that Ω is a C^∞-domain.

Definition 5.4. *Let $u \in L^{\frac{n}{n-1}}(\Omega)$, $v \in L^{\frac{n}{n-1}}_{\mathrm{av}}(\mathbb{T}^n)$, $w \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and $1 < p < \infty$. Then we define $\|u\|_{Y_p(\Omega)}$, $\|v\|_{Y_p(\mathbb{T}^n)}$ and $\|w\|_{Y_p(\mathbb{R}^n)}$ by*

$$\begin{aligned}
\|u\|_{Y_p(\Omega)} &:= \sup\left\{ \int_\Omega u\varphi\,dx \;\middle|\; \varphi \in C^\infty(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega,\ \|\varphi\|_{W^{3,p'}(\Omega)} \le 1\right\}, \\
\|v\|_{Y_p(\mathbb{T}^n)} &:= \sup\left\{ \int_{\mathbb{T}^n} v\varphi\,dx \;\middle|\; \varphi \in C^\infty_{\mathrm{av}}(\mathbb{T}^n),\ \|\varphi\|_{W^{3,p'}(\mathbb{T}^n)} \le 1\right\}, \\
\|w\|_{Y_p(\mathbb{R}^n)} &:= \sup\left\{ \int_{\mathbb{R}^n} w\varphi\,dx \;\middle|\; \varphi \in C^\infty(\mathbb{R}^n),\ \|\varphi\|_{W^{3,p'}(\mathbb{R}^n)} \le 1\right\}.
\end{aligned}$$

Here we show the properties of the above norms.

Lemma 5.5. *There exist constants $C_1, C_2 > 0$ such that*

$$\|u\|_{Y_p(\Omega)} \leq C_1 \|(-\Delta_D)^{-1} u\|_{\dot{W}_0^{-1,p}(\Omega)} \leq C_2 \|u\|_{Y_p(\Omega)}$$

for $u \in H^{-1}(\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$.

Lemma 5.6. *There exist constants $C_1, C_2 > 0$ such that*

$$\|u\|_{Y_p(\mathbb{T}^n)} \leq C_1 \|(-\Delta_{\mathrm{av}})^{-1} u\|_{\dot{W}^{-1,p}(\mathbb{T}^n)} \leq C_2 \|u\|_{Y_p(\mathbb{T}^n)}$$

for $u \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \cap L^{\frac{n}{n-1}}_{\mathrm{av}}(\mathbb{T}^n)$.

These lemmas can be easily proved by a duality argument and equivalence of $W^{3,p'}$ norm of φ and $W^{1,p'}$ norm of $(-\Delta)\varphi$, which follows from L^p elliptic higher regularity theorem [14]. The next lemma is concerned with the extension operator $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ for smooth cases of $\partial\Omega$.

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-boundary. Then we can define an extension operator $E : H^{-1}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$, which there exists a constant C such that*

$$\int_{\mathbb{T}^n} |D(Eu)| \leq C \int_{\overline{\Omega}} |Du|,$$

$$\|(-\Delta_{\mathrm{av}})^{-1}(Eu)\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)} \leq C \|(-\Delta_D)^{-1} u\|_{\dot{W}_0^{-1,p}(\Omega)}$$

for each $u \in H^{-1}(\Omega)$, that satisfies the condition $\int_{\overline{\Omega}} |Du| < +\infty$.

Proof of Lemma 5.7. Let $u \in H^{-1}(\Omega)$ with $\int_{\overline{\Omega}} |Du| < +\infty$. Since $\int_{\Omega} |Du| < +\infty$, by using Lemma 2.1 we find that there exists $\tilde{u} \in L^{\frac{n}{n-1}}(\Omega)$ such that

$$u = \tilde{u} \ \text{ in } \mathscr{D}'(\Omega).$$

First we investigate the special case that Ω is a half-ball

$$\Omega := B_1(0) \cap \mathbb{R}^n_+,$$

where

$$B_r(x) := \{y \in \mathbb{R}^n \mid |y - x| < r\}$$

is an open ball, and put

$$\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\},$$
$$\mathbb{R}^n_- := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n < 0\}.$$

Assume that $\zeta \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\begin{cases} \zeta \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1(0), \\ 0 \leq \zeta \leq 1 & \text{in } \mathbb{R}^n. \end{cases}$$

We see at once that ζ vanishes near the curved part of $\partial\Omega$. Here we write the coordinate by $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and define an operator $A : L^{\frac{n}{n-1}}(\Omega) \to L^{\frac{n}{n-1}}(\mathbb{R}^n)$ by

$$A(v)(x) := \begin{cases} v(x', x_n), & \text{if } (x', x_n) \in \Omega, \\ -v(x', -x_n), & \text{if } (x', -x_n) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

This is an odd extension of v in x_n direction. We first claim

$$\int_{\overline{\mathbb{R}^n_-}} |D(A(\zeta\tilde{u}))| = \int_{\overline{\mathbb{R}^n_+}} |D(\zeta\tilde{u})|.$$

This is easy to prove, so we omit the proof of above equation. Furthermore, we show that there exists a constant $C > 0$ such that

$$\int_{\overline{\mathbb{R}^n_+}} |D(\zeta\tilde{u})| = \int_{\overline{\Omega}} |D(\zeta\tilde{u})| \leq C \int_{\overline{\Omega}} |D\tilde{u}|. \tag{20}$$

This inequality is proved as follows. Since $\tilde{u} \in BV(\Omega)$, there exists $\{u^i\}_{i\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that

$$\begin{cases} \|u^i - \tilde{u}\|_{L^1(\Omega)} \to 0, \\ \|u^i - T\tilde{u}\|_{L^1(\{x_n=0\}\cap B_1(0))} \to 0, \\ \displaystyle\int_\Omega |\nabla u^i|\, dx \to \int_\Omega |D\tilde{u}| \end{cases}$$

as $i \to \infty$, where T is the trace operator. This property appears in [3]. These implies

$$\lim_{i\to\infty} \int_{\overline{\Omega}} |Du^i| = \int_{\overline{\Omega}} |Du|.$$

Since $u^i \in C^\infty(\overline{\Omega})$, the fundamental theorem of calculus shows

$$u^i(x', x_n) = u^i(x', 0) + \int_0^{x_n} \frac{\partial u^i}{\partial x_n}(x', s)\, ds$$

for all $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$ satisfying $|x'| \leq 1$ and $0 < x_n < \sqrt{1 - |x'|^2}$. Thus, we find that

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla(\zeta u^i)|\, dx &= \int_{\mathbb{R}^n_+} (|u^i \nabla\zeta| + |\zeta\nabla u^i|)\, dx \\ &\leq C \int_{|x'|<1} \left(|u^i(x', 0)| + \int_0^{\sqrt{1-|x'|^2}} \left| \frac{\partial u^i}{\partial x_n}(x', s) \right| ds \right) dx' + \int_\Omega |\nabla u^i|\, dx \\ &\leq C \int_{\{x_n=0\}\cap B_1(0)} |u^i|\, d\mathcal{H}^{n-1} + C \int_\Omega |\nabla u^i|\, dx \\ &\leq C \int_{\overline{\Omega}} |Du^i|. \end{aligned}$$

Since $\zeta u^i \to \zeta \tilde{u}$ in $L^1(\Omega)$, we see that

$$\int_\Omega |D(\zeta\tilde{u})| \le \liminf_{i\to\infty} \int_\Omega |D(\zeta u^i)| \le \lim_{i\to\infty} C\int_{\overline{\Omega}} |Du^i| = C\int_{\overline{\Omega}} |D\tilde{u}|,$$

and

$$\int_{\overline{\Omega}} |D(\zeta\tilde{u})| = \int_\Omega |D(\zeta\tilde{u})| + \int_{\partial\Omega} |\zeta\tilde{u}|\, d\mathcal{H}^{n-1} \le C\int_{\overline{\Omega}} |D\tilde{u}|.$$

Hence the inequality (20) holds for the case of $\Omega = B_1(0) \cap \mathbb{R}^n_+$.

Next we show

$$\|A(\zeta u)\|_{Y_p(\mathbb{R}^n)} \le 2\|\zeta u\|_{Y_p(\Omega)}. \tag{21}$$

Since $A(\zeta u)$ is an odd function in x_n direction, we find that

$$\int_{\mathbb{R}^n} A(\zeta u)\varphi\, dx = \int_{\mathbb{R}^n} A(\zeta u)\frac{\varphi(x', x_n) - \varphi(x', -x_n)}{2}\, dx$$

for all $\varphi \in C^\infty(\mathbb{R}^n)$. This equation gives

$$\begin{aligned}
&\|A(\zeta u)\|_{Y_p(\mathbb{R}^n)} \\
&\quad = \sup\left\{ \int_{\mathbb{R}^n} A(\zeta u)\frac{\varphi(x', x_n) - \varphi(x', -x_n)}{2}\, dx \;\middle|\; \varphi \in C^\infty(\mathbb{R}^n),\ \|\varphi\|_{W^{3,p'}(\mathbb{R}^n)} \le 1 \right\} \\
&\quad \le 2\sup\left\{ \int_{\mathbb{R}^n_+} \zeta u\psi\, dx \;\middle|\; \psi \in C^\infty(\overline{\Omega}),\ \psi = 0 \text{ on } \{x_n = 0\},\ \|\psi\|_{W^{3,p'}(\Omega)} \le 1 \right\} \\
&\quad = 2\|\zeta u\|_{Y_p(\Omega)}.
\end{aligned}$$

So the inequality (21) holds for the case of $\Omega = B_1(0) \cap \mathbb{R}^n_+$.

We drop the assumption that Ω is a half-ball. Let $x^0 \in \partial\Omega$. Since $\partial\Omega$ is C^∞, by relabeling and reorienting the coordinates axes if necessary, we can assume that

$$\Omega \cap B_r(x^0) = \{x = (x', x_n) \in B_r(x^0) \mid x_n > h(x')\}$$

for some $r > 0$ and some C^∞-function $h : \mathbb{R}^{n-1} \to \mathbb{R}$. We change variables as follows:

$$y = \Phi(x) := (x', x_n - h(x')),$$

and put the inverse transformation $\Psi := \Phi^{-1}$ and a fixed point $y^0 := \Phi(x^0)$.

Take a small radius $s > 0$ such that $U := B_s(y^0) \cap \{y_n > 0\}$ lies in $\Phi(\Omega \cap B_r(x^0))$. Suppose that $\zeta \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\begin{cases} \zeta \equiv 0 & \text{in } \mathbb{R}^n \setminus \Psi(U), \\ 0 \le \zeta \le 1 & \text{in } \mathbb{R}^n. \end{cases}$$

Set

$$u^\sharp(y) := (\zeta\tilde{u})(\Psi(y)), \qquad Bu(x) := Au^\sharp(\Phi(x)),$$

then by using changing variables and (20), (21), we see that

$$\int_{\mathbb{R}^n} |D(Bu)| \leq C \int_{\Omega} |D(\zeta\tilde{u})| \leq C \int_{\Omega} |D\tilde{u}| \tag{22}$$

and

$$\|Bu\|_{Y_p(\mathbb{R}^n)} \leq C\|u\|_{Y_p(\Omega)}, \tag{23}$$

where $C > 0$ is independent of u.

The compactness of $\partial\Omega$ enables us to cover $\partial\Omega$ with finitely many sets $\{U_i\}_{1\leq i\leq m}$, which have the property of $\Psi(U)$ above. Take an open set U_0, which is compactly contained in Ω, such that $\{U_i\}_{0\leq i\leq m}$ is an open covering of Ω. Then, there exists a partition of unity $\{\zeta_i\}_{0\leq i\leq m} \subset C_0^\infty(\mathbb{R}^n)$ associated with $\{U_i\}_{0\leq i\leq m}$. Define an extension operator F by

$$F\tilde{u} = \begin{cases} \tilde{u} & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^m B^i u & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{24}$$

where B^i is an operator which is defined for U_i by the same method as in defining the operator B for $\Psi(U)$ above with $\zeta = \zeta_i$. Since (22) and (23), we have

$$\int_{\mathbb{R}^n} |D(F\tilde{u})| \leq \int_{\Omega} |Du| + \sum_{1\leq i\leq m} \int_{\mathbb{R}^n} |D(B^i u)| \leq C \int_{\Omega} |Du|, \tag{25}$$

and

$$\|F\tilde{u}\|_{Y_p(\mathbb{R}^n)} \leq \|\tilde{u}\|_{Y_p(\Omega)} + \sum_{i=1}^m \|B^i\tilde{u}\|_{Y_p(\mathbb{R}^n)} \leq C\|\tilde{u}\|_{Y_p(\Omega)}. \tag{26}$$

Finally we define an extension operator $E : H^{-1}(\Omega) \cap L^{\frac{n}{n-1}}(\Omega) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$. We take sufficiently large constants $\omega_i > 0$ for $1 \leq i \leq n$ such that

$$\mathrm{supp}(F\tilde{u}) \subset (-\omega_1/4, \omega_1/4) \times \prod_{i=2}^n (-\omega_i/2, \omega_i/2)$$

for all $\tilde{u} \in H^{-1}(\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$. Here we remark that the constants ω_i depend only on Ω. We consider a periodic cell

$$\Omega_{\mathrm{per}} := \prod_{i=1}^n (-\omega_i/2, \omega_i/2).$$

Define $\tilde{Eu} \in L^{\frac{n}{n-1}}_{\mathrm{av}}(\mathbb{T}^n)$ by

$$\tilde{Eu}(x) := \begin{cases} F\tilde{u}(x) & \text{in } (-\omega_1/4, \omega_1/4) \times \prod_{i=2}^n (-\omega_i/2, \omega_i/2), \\ -F\tilde{u}(\omega_1/2 - x_1, x_2, \ldots, x_n) & \text{in } (\omega_1/4, \omega_1/2) \times \prod_{i=2}^n (-\omega_i/2, \omega_i/2), \\ -F\tilde{u}(-\omega_1/2 + x_1, x_2, \ldots, x_n) & \text{in } (-\omega_1/2, -\omega_1/4) \times \prod_{i=2}^n (-\omega_i/2, \omega_i/2), \end{cases}$$

where F is the extension operator defined by (24), and define $Eu \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ by

$$\langle Eu, \varphi\rangle = \int_{\mathbb{T}^n} (\tilde{Eu})\varphi \, dx$$

for $\varphi \in H^1_{\mathrm{av}}(\mathbb{T}^n)$. Then we see at once that

$$\int_{\mathbb{T}^n} |D(Eu)| = 2\int_{\mathbb{R}^n} |D(F\tilde{u})|$$

and

$$\|Eu\|_{Y_p(\mathbb{T}^n)} = 2\|F\tilde{u}\|_{Y_p(\mathbb{R}^n)}.$$

By these properties, (25), (26) and Lemma 5.5-5.6, we find that

$$\int_{\mathbb{T}^n} |D(Eu)| = 2\int_{\mathbb{R}^n} |D(F\tilde{u})| \le C\int_{\overline{\Omega}} |Du|,$$

and

$$\begin{aligned}\|(-\Delta_{\mathrm{av}})(Eu)\|_{\dot{W}^{-1,p}_{\mathrm{av}}(\mathbb{T}^n)} \le C\|Eu\|_{Y_p(\mathbb{T}^n)} &= 2C\|F\tilde{u}\|_{Y_p(\mathbb{R}^n)} \\ &\le C\|\tilde{u}\|_{Y_p(\Omega)} \le C\|(-\Delta_D)^{-1}u\|_{\dot{W}^{-1,p}_0(\Omega)},\end{aligned}$$

which completes the proof. □

Finally we prove the interpolation inequality (Lemma 4.8).

Proof of Lemma 4.8. First, we prove the case that Ω is a rectangular domain. Let $u \in H^{-1}(\Omega)$ and $Eu \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ defined in Definition 5.1. Then by Lemma 4.7 and Lemma 5.2 we see that

$$\begin{aligned}\|u\|_{H^{-1}(\Omega)} &\le \|Eu\|_{H^{-1}(\mathbb{T}^n)} \\ &\le C\|(-\Delta_{\mathrm{av}})^{-1}(Eu)\|^{1-\theta}_{\dot{W}^{-1,p}(\mathbb{T}^n)}\left(\int_{\mathbb{T}^n}|D(Eu)|\right)^{\theta} \\ &\le 2^n C\|(-\Delta_D)^{-1}u\|^{1-\theta}_{\dot{W}^{-1,p}_0(\Omega)}\left(\int_{\overline{\Omega}}|Du|\right)^{\theta},\end{aligned}$$

which is our claim.

Next we prove the case that Ω is a bounded domain with C^∞-boundary. In the similar way to the above proof, we can show that there exists a constant $C > 0$ such that

$$\|u\|_{H^{-1}(\Omega)} \le C\|(-\Delta_D)^{-1}u\|^{1-\theta}_{\dot{W}^{-1,p}_0(\Omega)}\left(\int_{\overline{\Omega}}|Du|\right)^{\theta} \tag{27}$$

for $u \in H^{-1}(\Omega) \cap BV(\Omega)$.

So we only consider the scale-invariance of a constant C. For $u \in H^{-1}(\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$ and $\lambda > 0$, set $u^\lambda \in H^{-1}(\lambda\Omega) \cap L^{\frac{n}{n-1}}(\lambda\Omega)$ by putting

$$u^\lambda(x) := u(\lambda^{-1}x).$$

Then we see at once

$$\begin{cases} \|u^\lambda\|_{H^{-1}(\lambda\Omega)} = \lambda^{1+\frac{n}{2}} \|u\|_{H^{-1}(\Omega)}, \\ \|(-\Delta_D)^{-1}u^\lambda\|_{\dot{W}_0^{-1,p}(\lambda\Omega)} = \lambda^{3+\frac{n}{p}} \|(-\Delta_D)^{-1}u\|_{\dot{W}_0^{-1,p}(\Omega)}, \\ \displaystyle\int_{\lambda\Omega} |Du^\lambda| = \lambda^{n-1} \int_\Omega |Du|. \end{cases}$$

Thus, we find that if u satisfies (27), then

$$\|u^\lambda\|_{H^{-1}(\lambda\Omega)} \leq C \|(-\Delta_D)^{-1}u^\lambda\|_{\dot{W}_0^{-1,p}(\lambda\Omega)}^{1-\theta} \left(\int_{\lambda\Omega} |Du^\lambda| \right)^\theta$$

for all $\lambda > 0$ because of (19). Therefore, the constant C is scale-invariance, and the proof is complete. □

6 Extinction Time Estimates

Here we prove Theorem 4.4 by using Lemma 4.8. These proofs are similar to what was done in [13] for periodic boundary condition. There seems to be a flaw in the proof of [13, Theorem 3.8]. We remark that the last inequality (two lines from the end of the proof of [13, Theorem 3.8]) may not be true unless $aT^*A_0^{-1}$ is small and the scale-invariant estimate does not follow from their argument except the case $n = 4$.

Proof of Theorem 4.4. Let u be a solution of the fourth-order total variation flow (11) in the sense of Definition 3.5. Then, for a.e. $t > 0$, there exists $z \in X(\Omega)$ with $\|z\|_{L^\infty(\Omega)} \leq 1$ such that

$$u_t(t) = -\Delta_D \operatorname{div} z, \qquad (u(t), -\Delta_D \operatorname{div} z)_{H^{-1}(\Omega)} = -\int_\Omega |Du(t)|.$$

First we consider $\|(-\Delta_D)^{-1}u(t)\|_{\dot{W}_0^{-1,p}(\Omega)}$. For any $\varphi \in C^\infty(\overline{\Omega})$ satisfying

$$\|\nabla\varphi\|_{L^{p'}(\Omega)} \leq 1, \qquad \|\varphi\|_{L^{p'}(\partial\Omega)} \leq C_{p'} (\operatorname{diam}\Omega)^{\frac{1}{p}},$$

where $C_{p'} > 0$ is independent of the scale of Ω, we see that

$$\begin{aligned} \int_\Omega \varphi \, (-\Delta_D)^{-1} u_t(t) \, dx &= \int_\Omega \varphi \operatorname{div} z \, dx \\ &= -\int_\Omega z \cdot \nabla\varphi \, dx + \int_{\partial\Omega} \gamma(z)\varphi \, d\mathcal{H}^{n-1} \\ &\leq \|z\|_{L^p(\Omega)} + C_{p'} (\operatorname{diam}\Omega)^{\frac{1}{p}} \|\gamma(z)\|_{L^p(\partial\Omega)} \\ &\leq |\Omega|^{\frac{1}{p}} + C_{p'} |\partial\Omega|^{\frac{1}{p}} (\operatorname{diam}\Omega)^{\frac{1}{p}} = a. \end{aligned}$$

Taking supremum, we find that

$$\|(-\Delta_D)^{-1}u_t(t)\|_{\dot{W}_0^{-1,p}(\Omega)} \leq a,$$

therefore the following estimate

$$\|(-\Delta_D)^{-1}u(t)\|_{\dot{W}_0^{-1,p}(\Omega)} \le \|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)} + at =: A(t) \tag{28}$$

holds.

Next we consider $\|u(t)\|_{H^{-1}(\Omega)}$. Write $y(t) := \|u(t)\|_{H^{-1}(\Omega)}$. We apply the theory introduced in Kōmura [21] and find that

$$y(t)\frac{d}{dt}y(t) = \frac{1}{2}\frac{d}{dt}(y(t)^2) = (u(t), u'(t))_{H^{-1}(\Omega)}.$$

By using interpolation inequality Lemma 4.8, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(y(t)^2) &= (u(t), (-\Delta_D)\operatorname{div} z)_{H^{-1}(\Omega)} \\ &= -\int_\Omega |Du(t)| \\ &\le -C_*^{-1/\theta} y(t)^{1/\theta}\|(-\Delta_D)^{-1}u(t)\|^{1-1/\theta}_{\dot{W}_0^{-1,p}(\Omega)}.\end{aligned}$$

where a constant C_* is scale-invariant. Therefore, $y(t)$ is nonincreasing. If $y(t) \neq 0$, then

$$y(t)^{1-1/\theta}\frac{d}{dt}y(t) \le -C_*^{-1/\theta}\|(-\Delta_D)^{-1}u(t)\|^{1-1/\theta}_{\dot{W}_0^{-1,p}(\Omega)} \le -C_*^{-1/\theta}A(t)^{1-1/\theta}.$$

holds because of (28). Integrating over $(0,t)$, we see that

$$\frac{1}{2-1/\theta}\left(y(t)^{2-1/\theta} - y(0)^{2-1/\theta}\right) \le -C_*^{-1/\theta}\int_0^t A(s)^{1-1/\theta}\,ds.$$

Hence we get the following inequality

$$\|u(t)\|^{2-(1/\theta)}_{H^{-1}(\mathbb{T}^n)} \le \|u_0\|^{2-(1/\theta)}_{H^{-1}(\mathbb{T}^n)} - \left(2-\frac{1}{\theta}\right)C_*^{-1/\theta}\int_0^t A(s)^{1-(1/\theta)}\,ds.$$

Moreover we put

$$A_0 := A(0) = \|(-\Delta_D)^{-1}u_0\|_{\dot{W}_0^{-1,p}(\Omega)}, \qquad \gamma := 2 - \frac{1}{\theta}$$

for simplicity. Then

$$\|u(t)\|^\gamma_{H^{-1}(\mathbb{T}^n)} \le \|u_0\|^\gamma_{H^{-1}(\mathbb{T}^n)} - \frac{C_*^{-1/\theta}}{a}\left((at+A_0)^\gamma - A_0^\gamma\right).$$

holds. By using this inequality, we see that the extinction time $T(u_0)$ is estimated by

$$T(u_0) \le \frac{A_0}{a}\left(\left(1 + \frac{aC_*^{1/\theta}\|u_0\|^\gamma_{H^{-1}(\Omega)}}{A_0^\gamma}\right)^{1/\gamma} - 1\right).$$

Since C_* is independent of u_0 and the scale of Ω, this proves the theorem. $\square$

7 Examples

In this section we give some examples of solutions of the system (1). M-H. Giga and Y. Giga studied examples of solutions in the case of $\Omega = \mathbb{T}^1$ [12]. Here we show that the similar argument also holds for radial symmetric solutions in the case of $\Omega = \mathbb{R}^n$. Our method in this paper does not give a characterization of the subdifferential of total variation energy when the domain Ω is $\mathbb{R}^n$, since we cannot use Poincaré's inequality. Instead, we assume that the equation

$$u_t = -\Delta \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

is characterized as for a.e $t > 0$,

$$u_t(t) = -\Delta \operatorname{div} z \quad \text{in } H^{-1}(\mathbb{R}^n) \tag{29}$$

where $z \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|z\|_{L^\infty(\mathbb{R}^n)} \leq 1$ satisfies the conditions

$$\int_{\mathbb{R}^n} (z, Du(t)) = \int_{\mathbb{R}^n} |Du|(t). \tag{30}$$

If such u and z exists, it gives indeed a solution of (1).

Now, we take the initial data $u_0 \in C(\mathbb{R}^n)$ defined by

$$u_0(x) := \begin{cases} h_0 & \text{if } |x| < r_0, \\ v_0(r) & \text{if } |x| \geq r_0 \end{cases} \tag{31}$$

for $r_0 > 0$, where we put $r = |x|$, and assume that the function $v_0(r)$ is positive, continuous and monotone decreasing on $[0, \infty)$ and $v_0(r_0) = h_0$. We hope that the solutions to the system (1) with the initial datum u_0 defined by (31) are also radial symmetry. By formal calculations, if $u(x) = f(r)$ and $f(r)$ is monotone decreasing, then

$$\frac{\nabla u(x)}{|\nabla u(x)|} = \frac{f'(r)\nabla r}{|f'(r)\nabla r|} = -\frac{x}{r}$$

and

$$-\Delta\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) = 0.$$

This formal calculations does not hold around the origin $x = 0$, because the initial data u_0 makes a facet in $|x| \leq r_0$. So we assume that the solution $u(x,t)$ and a function $z(x)$ are represented as

$$u(x,t) := \begin{cases} h(t) & \text{if } |x| < \alpha(t), \\ v_0(r) & \text{if } |x| \geq \alpha(t) \end{cases}$$

and

$$z(x) = \eta(r)\frac{x}{r}$$

for some functions $h(t), \alpha(t), \eta(r)$ with $h(0) = h_0, \alpha(0) = r_0, \eta(0) = 0$ and $\eta(r) = -1$ for $r > \alpha(t)$. In this section we show that it is possible to take these functions satisfying our characterization of the system (1).

Fix any $t > 0$. The function $\eta(r)$ should be continuous on $[0, \infty)$ and $\eta'(\alpha(t)) = 0$ since $\Delta \operatorname{div} z \in H^{-1}(\mathbb{R}^n)$ and $\eta(r) = -1$ for $r > \alpha(t)$. Moreover, the solution $u(x,t)$ also makes a facet, therefore $\Delta \operatorname{div} z$ have to be constant on $r < \alpha(t)$. Here

$$\begin{aligned}\Delta \operatorname{div} z &= \Delta \operatorname{div}\left(\eta(r)\frac{x}{r}\right)\\ &= \eta'''(r) + \frac{2(n-1)}{r}\eta''(r) + \frac{n^2-4n+3}{r^2}\eta'(r) - \frac{n^2-4n+3}{r^3}\eta(r)\end{aligned}$$

holds for $r < \alpha(t)$. Since $-\Delta \operatorname{div} z$ is constant C_n on $r < \alpha(t)$, we get an ODE

$$-\eta'''(r) - \frac{2(n-1)}{r}\eta''(r) - \frac{n^2-4n+3}{r^2}\eta'(r) + \frac{n^2-4n+3}{r^3}\eta(r) = C_n. \tag{32}$$

First, we consider the case of $\Omega = \mathbb{R}^3$. Then, the ODE (32) becomes a simple equation as follows:

$$-\eta'''(r) - \frac{4}{r}\eta''(r) = C_3. \tag{33}$$

Because $\eta(r)$ have to be continuous on $[0, \infty)$, we have

$$\eta''(r) = -\frac{C_3}{5}r.$$

So the solution $\eta(r)$ to (33) is a polynomial of degree 3. We solve (33) under the conditions $\eta(0) = 0, \eta(\alpha(t)) = -1$ and $\eta'(\alpha(t)) = 0$, then

$$\eta(r) = \frac{1}{2\alpha(t)^3}r^3 - \frac{3}{2\alpha(t)}r \tag{34}$$

and

$$C_3 = -\Delta \operatorname{div} z = -\eta'''(r) - \frac{4}{r}\eta''(r) = -\frac{15}{\alpha(t)^3}$$

hold for $|x| < \alpha(t)$. In general dimensional case, the function $\eta(r)$ defined by (34) is a solution to the ODE (32) with

$$C_n = -\Delta \operatorname{div} z = -\frac{n^2+2n}{\alpha(t)^3}$$

by a similar calculation. Hence the following formula

$$-\Delta \operatorname{div} z = -\Delta \operatorname{div}\left(\eta(r)\frac{x}{r}\right) = -\frac{n^2+2n}{\alpha(t)^3}H(\alpha(t)-r) + \frac{3}{\alpha(t)^2}\delta(\alpha(t)-r) \tag{35}$$

holds in the distribution sense $\mathscr{D}'(\mathbb{R}^n)$, where H is the Heaviside function and δ is the dirac measure.

Next, we can rewrite the solution $u(x,t)$ as

$$u(x,t) = h(t)H(\alpha(t) - r) + v_0(r)H(r - \alpha(t))$$

by using the Heaviside function H. So

$$\begin{aligned} u_t(x,t) &= h'(t)H(\alpha(t) - r) + h(t)\delta(\alpha(t) - r)\alpha'(t) - v_0(r)\delta(\alpha(t) - r)\alpha'(t) \\ &= h'(t)H(\alpha(t) - r) + (h(t) - v_0(\alpha(t)))\alpha'(t)\delta(\alpha(t) - r) \end{aligned} \tag{36}$$

holds in the distribution sense.

Since (35) and (36), we can find the solution to (29) if we solve the ODE system

$$\begin{cases} h'(t) = -\dfrac{n^2 + 2n}{\alpha(t)^3}, \\ (h(t) - v_0(\alpha(t)))\alpha'(t) = \dfrac{3}{\alpha(t)^2}. \end{cases} \tag{37}$$

Hereafter the same arguments hold in [12]. Indeed, we can prove $h(t) - v_0(\alpha(t)) > 0$ for $t > 0$ and the solvability of the system (37). Moreover, it is easy to check the condition (30). Hence we find the solution to the system (1).

8 Characterization of The Subdifferential of $\int_{\mathbb{T}^n} |Du|$ and $\int_\Omega |Du|$ in H^{-1} type space

Here we show the characterization of the steepest descent of $\int_{\mathbb{T}^n} |Du|$ and $\int_\Omega |Du|$ in $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ and $H^{-1}(\Omega)$ respectively. First we give the characterization of the subdifferential of $\int_{\mathbb{T}^n} |Du|$ in $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$. This was proved in [13] by using L^2-subdifferential. We give a different proof without using L^2-subdifferential.

We introduce some function spaces. We already defined $H^1_{\mathrm{av}}(\mathbb{T}^n)$ and $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ in section 4. Now, define a function space $X(\mathbb{T}^n)$ by

$$X(\mathbb{T}^n) := \{z \in L^\infty(\mathbb{T}^n; \mathbb{R}^n) \mid \operatorname{div} z \in H^1_{\mathrm{av}}(\mathbb{T}^n)\}.$$

These spaces construct the framework of periodic boundary condition.

In a similar way to the Dirichlet case, we introduce an operator $-\Delta_{\mathrm{av}} : H^1_{\mathrm{av}}(\mathbb{T}^n) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ and the inner product in $H^1_{\mathrm{av}}(\mathbb{T}^n)$.

Definition 8.1. *Let define an operator $-\Delta_{\mathrm{av}} : H^1_{\mathrm{av}}(\mathbb{T}^n) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ by*

$$\langle -\Delta_{\mathrm{av}} u, v \rangle := \int_{\mathbb{T}^n} \nabla u \cdot \nabla v \, dx$$

for all $u, v \in H^1_{\mathrm{av}}(\mathbb{T}^n)$.

Here we show the property of $-\Delta_{\mathrm{av}}$.

Lemma 8.2. *The operator $-\Delta_{\mathrm{av}} : H^1_{\mathrm{av}}(\mathbb{T}^n) \to H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ is an isometry.*

By using Lemma 8.2, we can define the inverse mapping

$$(-\Delta_{\mathrm{av}})^{-1} : H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \to H^1_{\mathrm{av}}(\mathbb{T}^n),$$

and define $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ as a Hilbert space equipped with the inner product

$$(f, g)_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} := \langle f, (-\Delta_{\mathrm{av}})^{-1} g \rangle$$

for all $f, g \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$.

Now we introduce the approximation of the function in $X(\mathbb{T}^n)$. This approximation is in particular to periodic boundary condition. For any $\varepsilon > 0$, let $\rho^\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be the standard mollifier. Then we obtain the following approximation. For the proof we refer the reader to [19, Lemma 2.7].

Lemma 8.3. *Let $z \in X(\mathbb{T}^n)$ and $z^\varepsilon := z * \rho^\varepsilon$ be the convolution of z and ρ^ε. Then, z^ε satisfies*

$$\|z^\varepsilon\|_{L^\infty(\mathbb{T}^n)} \le \|z\|_{L^\infty(\mathbb{T}^n)}$$

and

$$\operatorname{div} z^\varepsilon \to \operatorname{div} z \quad \text{in } H^1_{\mathrm{av}}(\mathbb{T}^n) \quad \text{as } \varepsilon \searrow 0.$$

Next we consider the characterization of the subdifferential of a total variation with periodic boundary condition. We start with giving the definition of a total variation.

Definition 8.4. *Let define a functional $F : H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \to [0, \infty]$ by*

$$F(u) := \sup\{\langle u, \operatorname{div} \varphi \rangle \mid \varphi \in C^\infty(\mathbb{T}^n; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{T}^n)} \le 1\}.$$

We will denote the value $F(u)$ by $\int_{\mathbb{T}^n} |Du|$. It is easily seen that the functional F defined in Definition 8.4 is convex, lower semi-continuous, proper, and positively homogeneous of degree 1 in $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$. So, we consider the characterization of the subdifferential of F.

Next, we define a functional $G : H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \to [0, \infty]$ by

$$G(u) := \inf\{\|z\|_{L^\infty(\mathbb{T}^n)} \mid z \in X(\mathbb{T}^n),\ u = -(-\Delta_{\mathrm{av}}) \operatorname{div} z\}.$$

It is easy to check that G is convex, lower semi-continuous, and positively homogeneous of degree 1. Moreover there exists a minimal element in the definition of G.

Now we show that $\tilde{F}(u) = G(u)$. This property enables us to characterize the subdifferential of a total variation F.

Lemma 8.5. *Let $F, G : H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \to [0, \infty]$ be the functionals defined in this section. Then*

$$\tilde{F}(u) = G(u)$$

for all $u \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$.

Proof of Lemma 8.5. Let $u \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$. First we show $\tilde{F}(u) \le G(u)$. If $G(u)$ is infinite, then this inequality is obvious. So we assume that $G(u) < +\infty$, then there exists $z \in X(\mathbb{T}^n)$ such that

$$u = -(-\Delta_{\mathrm{av}}) \operatorname{div} z, \qquad \|z\|_{L^\infty(\mathbb{T}^n)} = G(u).$$

Put $z^\varepsilon = z * \rho^\varepsilon$ where ρ is a mollifier. Then, by Lemma 8.3 we show that

$$\begin{aligned}
(u, v)_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} &= \langle v, -\operatorname{div} z \rangle \\
&= \lim_{\varepsilon \to 0} \langle v, -\operatorname{div} z^\varepsilon \rangle \\
&\le \limsup_{\varepsilon \to 0} \|z^\varepsilon\|_{L^\infty(\mathbb{T}^n)} F(v) \\
&\le \|z\|_{L^\infty(\mathbb{T}^n)} F(v).
\end{aligned}$$

for all $v \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$. Therefore, we find that

$$\tilde{F}(u) = \sup_{v \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \setminus \{0\}} \frac{(u, v)_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)}}{F(v)} \le \|z\|_{L^\infty(\mathbb{T}^n)} = G(u),$$

which is the desired inequality.

Next we prove $G(u) \le \tilde{F}(u)$. By the similar argument to the proof of Lemma 3.3, we prove $\tilde{G}(u) \ge F(u)$. We see that

$$\begin{aligned}
\tilde{G}(u) &= \sup_{v \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \setminus \{0\}} \frac{(u, v)_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)}}{G(v)} \\
&\ge \sup_{z \in C^\infty(\mathbb{T}^n; \mathbb{R}^n) \setminus \{0\}} \frac{\langle u, \operatorname{div} z \rangle}{\|z\|_{L^\infty(\mathbb{T}^n)}} = F(u).
\end{aligned}$$

Therefore, the proof is complete. □

Proposition 8.6. *Let $F : H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \to [0, \infty]$ be the functional defined by Definition 8.4. Assume $F(u) < +\infty$. Then $v \in \partial_{H^{-1}} F(u)$ if and only if there exists $z \in X(\mathbb{T}^n)$ such that*

$$\begin{cases}
v = -(-\Delta_{\mathrm{av}}) \operatorname{div} z, \\
\|z\|_{L^\infty(\mathbb{T}^n)} \le 1, \\
(u, (-\Delta_{\mathrm{av}}) \operatorname{div} z)_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} = -F(u),
\end{cases}$$

where $\partial_{H^{-1}} F$ is the subdifferential of F with respect to $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$-topology.

We can prove Proposition 8.6 in a similar way to the proof of Theorem 3.4. The following proposition guarantees the existence and uniqueness of the solution for system (1) with periodic boundary condition.

Proposition 8.7. *Let $u_0 \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ satisfy $\int_{\mathbb{T}^n} |Du_0| < +\infty$. Then there exists a unique solution $u \in C([0, \infty); H^{-1}_{\mathrm{av}}(\mathbb{T}^n))$, which is absolutely continuous on any compact set of $(0, \infty)$, of the system*

$$\begin{cases}
\dfrac{du}{dt}(t) \in -\partial_{H^{-1}} F(u(t)) & \text{in } H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \text{ for a.e. } t \in (0, \infty), \\
u(0) = u_0 & \text{in } H^{-1}_{\mathrm{av}}(\mathbb{T}^n).
\end{cases}$$

The proof of Proposition 8.7 is based on nonlinear semigroup theory.

Next we characterize the steepest-descent of $\int_\Omega |Du|$ in $H^{-1}(\Omega)$. We characterize in a similar way to the characterization of $\int_{\bar\Omega} |Du|$ and we only state result without detailed proof.

Theorem 8.8. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Assume that $u \in H^{-1}(\Omega)$ satisfies $F(u) := \int_\Omega |Du| < +\infty$. Then $v \in \partial_{H^{-1}} F(u)$ if and only if there exists $z \in X(\Omega)$ (defined in Definition 2.7) with $\|z\|_{L^\infty(\Omega)} \le 1$ such that*

$$\begin{cases} v = -(-\Delta_D)\operatorname{div} z, \\ (u, -(-\Delta_D)\operatorname{div} z)_{H^{-1}(\Omega)} = -F(u), \\ \gamma(z) = 0 \ \text{on } \partial\Omega, \end{cases}$$

where $\gamma(z)$ denotes the trace of $z \cdot \nu$ as in Lemma 2.11, and $\partial_{H^{-1}} F$ is the subdifferential of F with respect to $H^{-1}(\Omega)$-topology.

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