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# STABLE AND UNSTABLE MANIFOLD ON A FREE BOUNDARY PROBLEM OF THE CURVATURE FLOW WITH DRIVING FORCE

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ABSTRACT. We study a free boundary problem for the curvature flow with driving force for a family of planar curves with two fixed contact angles on the  $x$ -axis. Our aim of this paper is to analyze the dimension of stable and unstable manifolds of traveling wave solution.

## 1. INTRODUCTION

In this paper, we deal with the motion by curvature with driving force of planar curves having two moving end points on the  $x$ -axis with fixed interior contact angles to this axis. The problem is formulated as follows. For any curve  $\gamma$  lying on the upper half-plane having two end points, we shall denote the left and right interior angles by  $Ang_-(\gamma)$  and  $Ang_+(\gamma)$ . For any given such curve  $\Gamma(0)$ , our problem is to look for a family of curves  $\{\Gamma(t)\}_{t \geq 0}$  having two end points on the  $x$ -axis with the same (fixed) contact angles as above evolve by the curvature flow equation with a constant driving force  $c$ , namely,

$$(1.1) \quad V = k + c, \quad Ang_{\pm}(\Gamma(t)) = Ang_{\pm}(\Gamma(0)),$$

where  $V$  is the normal velocity and  $k$  is the (signed) curvature of  $\Gamma(t)$ .

In this paper, we shall always assume that  $\Gamma(0)$  is a graph given by  $y = u^0(x)$ ,  $x \in [l_-^0, l_+^0]$ . Then it can be expected that  $\Gamma(t)$  can be represented as  $y = u(x, t)$ ,  $x \in [l_-(t), l_+(t)]$ , for  $t > 0$  such that  $(u, l_{\pm})$  satisfies the following free boundary problem (P):

$$(1.2) \quad u_t = \frac{u_{xx}}{1 + u_x^2} + c\sqrt{1 + u_x^2}, \quad x \in (l_-(t), l_+(t)), \quad t > 0,$$

$$(1.3) \quad u(l_{\pm}(t), t) = 0, \quad t > 0,$$

$$(1.4) \quad u_x(l_{\pm}(t), t) = \mp \tan \psi_{\pm}, \quad t > 0,$$

$$(1.5) \quad u(x, 0) = u^0(x), \quad l_{\pm}(0) = l_{\pm}^0, \quad x \in (l_-^0, l_+^0)$$

where we assume that  $\psi_{\pm} \in (0, \pi/2)$ ,  $c > 0$  and  $-\infty < l_-^0 < l_+^0 < \infty$ . We also assume that  $u^0$  is a  $C^2$  strict concave function and concentrate the solution that exists globally in time and is uniformly bounded. In fact, this concave assumption is not so essential, since it has already proved in [12] that all bounded solutions become concave eventually. Moreover, any

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concave solution preserve concavity as long as it exists. Here the sign of the curvature is defined so that (1.2) is derived from (1.1). Also, we always assume that

$$(1.6) \quad u^0 > 0 \text{ on } (l_-^0, l_+^0), \quad u^0(l_\pm^0) = 0, \quad u_x^0(l_\pm^0) = \mp \tan \psi_\pm, \quad u^0 \in C^2([l_-^0, l_+^0]), \quad (u^0)_{xx} < 0.$$

For the local existence and uniqueness, including the continuous dependence of the solution to the problem (P), see [3] and [12].

The equation (1.1) comes from various fields. For example, it describes the motion of a superconducting vortex [5]. Also, it appears in the study of the traveling curved fronts (V-shaped waves) [17], the Belousov-Zhabotinsky reaction [2], and the Allen-Cahn model in Chemistry. For the free boundary problem with  $\psi_+ = \psi_-$ , we refer the work of Ei and Yanagida [7, 8], Ei, Sato and Yanagida [6], and Giga and Yamauchi [10]. They analyze the stability of stationary hypersurfaces or curve for more general curvature flow.

The problem (P) with  $c = 0$  appears in the study of evolution of grain domain in polycrystals, see, e.g., [13, 15]. For the mathematical study of the problem (P) with  $c = 0$ , we refer to the work [4] and the references cited therein. If  $c = 0$ , the curve  $\Gamma(t)$  shrinks to a point in finite time in an asymptotically self-similar manner. On the other hand, the problem (P) with  $c > 0$  exhibits much more rich phenomena than that of  $c = 0$ . The asymptotic behavior of the solution depends on the relation between the curvature and the driving force. More precisely, the curvature can dominate the driving force eventually so that the evolution of the curve  $\Gamma(t)$  is just like the case  $c = 0$  such that it shrinks to a point in finite time. On the other hand, the curvature may be dominated by the driving force eventually so that the curve will be expanding for all time. It can also happen that the curvature is balanced with the driving force. In this case, the solution converges either to a stationary solution or to a traveling wave solution of (1.2). The authors in [12] consider the problem (1.1) on a half plane. They proved that all the solutions are classified into three categories:

- (A) (Expanding case)  $T = \infty$ , and both  $L(t)$  and  $A(t)$  tend to  $\infty$  as  $t \rightarrow \infty$ .
- (B) (Bounded case)  $T = \infty$ , and both  $L(t)$  and  $A(t)$  remain bounded from above and below by two positive constants as  $t \rightarrow \infty$ .
- (C) (Shrinking case)  $T < \infty$ , and both  $L(t)$  and  $A(t)$  tend to 0 as  $t \rightarrow T$ .

Here  $A(t)$  denote the area of the domain enclosed by  $\Gamma(t)$  and  $x$ -axis, and  $L(t)$  denote the length of  $\Gamma(t)$ , i.e.,

$$(1.7) \quad A(t) := \int_{l_-(t)}^{l_+(t)} u(x, t) dx, \quad L(t) := \int_{l_-(t)}^{l_+(t)} \sqrt{1 + u_x^2(x, t)} dx,$$

and  $[0, T_{\max})$  is the maximal existence time interval of a classical solution  $u$  to the problem (P) for  $T = T_{\max} \in (0, \infty]$ . Then they provide some concavity properties of the solutions such as eventual concavity for the case (B)-(C) and characterize completely the time asymptotic

behaviors of all solutions. We roughly explain the asymptotic behavior of the solution for each cases. In the case (A), as time passes, the effect of curvature becomes smaller and smaller comparing with the constant forcing term, so it can be expected that the asymptotic behavior of  $\Gamma(t)$  is described by a curve  $\mathcal{C}(t)$  satisfying

$$(1.8) \quad V = c \text{ on } \mathcal{C}(t) \quad \text{and} \quad \text{Ang}_{\pm}(\mathcal{C}(t)) = \psi_{\pm}.$$

This problem has a self-similar structure. For any solution  $\mathcal{C}(t)$ , the similarity transformed curves  $\lambda\mathcal{C}(t/\lambda)$  are also solutions of the problem (1.8), where we choose the center of transformation at the origin. Hence a self-similar solution of (1.8) is the form  $\mathcal{C}_S(t) = t \cdot \mathcal{C}_S(1)$ , which can be seen by substituting  $\lambda = t$ . Next, for the shrinking type (C), the curve  $\Gamma(t)$  shrinks to a point in a self-similar manner as  $t \rightarrow T$  by using a blow-up technique. If we may assume that  $x = x_0$  is the shrinking point, a solution of (P) must converge to the unique self-similar solution of (P) with  $c = 0$  as  $t \rightarrow T_{\max}$ .

$$u(x, t) \approx \sqrt{2(T-t)} \varphi \left( \frac{x - x_0}{\sqrt{2(T-t)}} \right)$$

where  $\varphi$  is the the profile of the self-similar solution to  $(P_0)$ . The existence and uniqueness of the self-similar solution was proved in [4]. Roughly speaking, the solution behaves like the solution of  $V = k$  with the same boundary conditions as  $t \rightarrow T$ . This means that the curvature effect is dominant for the shrinking solutions. Finally, we explain the type (B), which is the most interesting part. For this case, If  $\psi_+ = \psi_-$ ,  $u$  converges to a stationary solution as  $t \rightarrow \infty$  uniformly on  $\mathbb{R}$ . If  $\psi_+ \neq \psi_-$ , then  $u$  converges to a traveling wave  $U(x - \nu t - a)$  with some  $a \in \mathbb{R}$  as  $t \rightarrow \infty$ , where  $U(\xi)$  satisfies

$$(1.9) \quad \begin{cases} \frac{U''}{1+U'^2} + \nu U' + c\sqrt{1+U'^2} = 0 \\ U'(a_{\pm}) = \mp \tan \psi_{\pm} \end{cases} \quad \text{for some } a_{\pm} \in \mathbb{R}.$$

Furthermore, the sign of the speed  $\nu$  coincide with the sign of  $\psi_- - \psi_+$ .

The first main theorem of this paper, which is a different approach of the proof of Theorem 1.6 of [12]. Note that here we also obtain the rate of convergence.

**Theorem 1.1** (Asymptotic shape). *Let  $(u, l_{\pm})$  be a solution of (P) with (1.6) and satisfy (B). Then there exists a constant  $a \in \mathbb{R}$  such that  $l_{\pm}(t) - \nu t \rightarrow a \pm b$  as  $t \rightarrow +\infty$  and  $u(\xi + \nu t + a, t) \rightarrow U(\xi)$  uniformly on  $\mathbb{R}$ , where  $U, \nu, b$  are as in (1.9). Here we regard that  $u \equiv 0$  outside the interval  $[l_-(t), l_+(t)]$  and  $U \equiv 0$  outside the interval  $[-b, b]$ . Furthermore, the Hausdorff distance of the solution curve between the traveling wave converges to zero exponentially fast.*

To prove this theorem, we use a Lyapunov functional borrowed from [9] to prove the convergence of the curvature of  $\Gamma(t)$  to that of a traveling wave. However, we can not conclude

the convergence of the endpoints of the solution only by this argument. The convergence of endpoints and also the convergence rate of the curvature as  $t \rightarrow \infty$  is analyzed. The argument is based on the spectral analysis of the curvature equation.

The second result of this paper is the following:

**Theorem 1.2** (Invariant manifold). *The dimension of unstable manifold of traveling wave is one and the dimension of the center manifold is one. In particular, the codimension of stable manifold of a traveling wave solution is two.*

We shall explain the geometric meaning of Theorem 1.2. Note that the center manifold of the traveling wave solution corresponds the perturbation from traveling wave by a horizontal translation. The unstable manifold must related to the perturbation of the scaling but the center of the dilation is unknown. Other invariants manifolds are all stable. Hence we can find a lot of initial data whose solution converges to one of a traveling wave. In Theorem 5.1 of [12], they construct initial data of the solution that enjoys type (B) behavior by using a one parameter family ordered solutions. On the other hand, from our theorem, if we put the initial data on the stable manifold near the traveling wave, the solution converges to one of a shifted traveling wave, which is on the center manifold. Thus we can find another way of finding initial data whose solution converges to a traveling wave. Moreover, The result of stable and unstable manifold gives us deeper result than that of [12]. In fact, our theorem is that we are able to obtain the convergence rate of the solution, that is essentially impossible to know by intersection number arguments. The main tool used in the proof is zero number theory dealing with the Sturm nodal properties of the solutions and the spectral comparison argument.

The rest of this paper is organized as follows. In Section 2, we provide the curvature equation for a concave solution. In Section 3, we study the asymptotic behaviors of near a traveling wave solution and consider the dimension of invariant manifolds.

## 2. REVIEW OF THE EQUATION FOR THE CURVATURE

We convert the equation (1.2) into an equation for the curvature  $k := u_{xx}/(1 + u_x^2)^{3/2}$ . The independent variables for this equation are  $\theta, t$ , where  $\theta$  is defined by

$$\theta(x, t) = \arctan u_x(x, t).$$

Note that  $\theta$  varies in the interval  $[-\psi_+, \psi_-]$ , if  $u$  is concave. In what follows we assume that  $u$  is strictly concave. This assumption is not essential, since it is proved in [12] that the solution  $u(x, t)$  becomes concave for sufficiently large  $t > 0$  for the case (B)-(C). Hence the correspondence  $x \rightarrow \theta(x, t)$  is one-to-one for each fixed  $t$ . As is shown in [12], the time

evolution problem for the curvature is the following:

$$(2.1) \quad \begin{cases} k_t = k^2(k_{\theta\theta} + k + c), & -\psi_+ < \theta < \psi_+, \quad t > 0, \\ k(\theta, 0) = k_0(\theta), & -\psi_+ < \theta < \psi_+, \\ k_\theta = \cot \theta (k + c), & \theta = \mp\psi_\pm, \quad t > 0, \end{cases}$$

where the initial data  $k_0$  satisfies

$$(2.2) \quad \int_{-\psi_+}^{\psi_-} \frac{\sin \theta}{k_0(\theta)} d\theta = 0.$$

This condition means that  $u(l_\pm(t), t) = 0$ , and we can easily prove that

$$(2.3) \quad \int_{-\psi_+}^{\psi_-} \frac{\sin \theta}{k(\theta, t)} d\theta = 0 \quad \text{for all } t \geq 0$$

for any solution of (2.1) provided that (2.2) holds. The stationary solution of (2.1) is  $-\nu \sin \theta - c$  for any constant  $\nu \in \mathbb{R}$ . If  $\nu = 0$ , the corresponding stationary solution is  $k = -c$ . Now a natural question arises. What is the meaning of the parameter  $\nu$  when it is positive? For the case when  $\psi_- \neq \psi_+$ , let us consider a traveling wave solution, namely,  $u(x, t) = U(y)$ ,  $y := x - \nu t - a$ , where  $a \in \mathbb{R}$  is any constant and  $\nu \neq 0$  is the wave speed. More precisely, the limiting solution is a triple  $(U, \nu, \beta)$  that satisfies

$$(2.4) \quad \begin{cases} 0 = \frac{U''}{1 + U'^2} + c\sqrt{1 + U'^2} + \nu U', \quad U > 0, \quad 0 < y < \beta, \\ U(0) = U(\beta) = 0, \quad U'(0) = \tan \psi_-, \quad U'(\beta) = -\tan \psi_+. \end{cases}$$

Introducing the angle function  $\theta$  as  $\theta := \arcsin\left(\frac{U'}{\sqrt{1 + U'^2}}\right)$ , then the first equation in (2.4) becomes  $k + c + \nu \sin \theta = 0$ , where  $k$  is the curvature function. This is exactly the solution obtained above.

One of the typical questions about the traveling wave is whether there is the uniqueness (up to translations). The following proposition in [12] gives us the answer to this question.

**Proposition 2.1.** *For any given  $\psi_\pm$ . The problem (2.4) has a unique solution. Also*

$$\nu \begin{cases} > \\ = \\ < \end{cases} 0 \quad \text{if and only if} \quad \psi_- \begin{cases} > \\ = \\ < \end{cases} \psi_+.$$

*Proof.* The restriction (2.3) again gives us the following necessary condition:

$$\int_{-\psi_+}^{\psi_-} \frac{\sin \theta}{c + \nu \sin \theta} d\theta = 0 \quad \implies \quad \int_0^{\psi_-} \frac{\sin \theta}{c + \nu \sin \theta} d\theta = \int_0^{\psi_+} \frac{\sin \theta}{c - \nu \sin \theta} d\theta.$$

This implies that  $\nu > 0$  if and only if  $\psi_- > \psi_+$ . Let us define

$$F(\nu) := \int_0^{\psi_-} \frac{\sin \theta}{c + \nu \sin \theta} d\theta, \quad G(\nu) := \int_0^{\psi_+} \frac{\sin \theta}{c - \nu \sin \theta} d\theta$$

for any fixed  $\psi_{\pm}$ . Without loss of generality, we can assume  $\psi_- > \psi_+$ , thus  $\nu > 0$ . We can easily check that  $F(\nu)$  is monotone decreasing and  $G(\nu)$  is monotone increasing for  $\nu > 0$ . Also  $F(0) > G(0)$  by the assumption  $\psi_- > \psi_+$ . By the intermediate theorem, we conclude that there exists a unique  $\alpha$  for the pair  $(\psi_-, \psi_+)$ . Thus we conclude the uniqueness of the profile of the traveling wave.  $\square$

### 3. ASYMPTOTIC BEHAVIORS AND INVARIANT MANIFOLD

We shall start to prove Theorem 1.1. Here we use the Lyapunov function of the curvature equation and spectral analysis instead of intersection argument. One of the advantage of this argument is that we are able to know also the convergence rate to the traveling wave, which is impossible to obtain by the intersection number principle in [12]. Furthermore, since we analyze the spectral of linearized operator around the traveling wave more precise dynamics of is obtained.

In this section, we explain about Lyapunov functional of the curvature equation (2.1). We first associate the problem (2.1) with the following energy functional:

(3.1)  $J[h] = \int_{-\psi_+}^{\psi_-} \left( \frac{1}{2} h_{\theta}^2 - \frac{1}{2} h^2 - ch \right) d\theta - \cot \psi_- \left( ch + \frac{1}{2} h^2 \right) \Big|_{\theta=\psi_-} - \cot \psi_+ \left( ch + \frac{1}{2} h^2 \right) \Big|_{\theta=-\psi_+}$   
 defined on  $\{h \in H^2([-\psi_+, \psi_-]) \mid h_{\theta} = \cot \theta (h + c) \text{ for } \theta = \mp \psi_{\pm}\}$ . By differentiating  $J[k(\cdot, t)]$  by  $t$ , we have

$$\begin{aligned} \frac{d}{dt} J[k(\cdot, t)] &= k_{\theta} k_t \Big|_{\theta=\psi_-} - k_{\theta} k_t \Big|_{\theta=-\psi_+} - \int_{-\psi_+}^{\psi_-} k_t (k_{\theta\theta} + k + c) d\theta \\ &\quad - \cot \psi_- k_t (k + c) \Big|_{\theta=\psi_-} - \cot \psi_+ k_t (k + c) \Big|_{\theta=-\psi_+} = - \int_{-\psi_+}^{\psi_-} \frac{k_t^2(\theta, t)}{k^2(\theta, t)} d\theta. \end{aligned}$$

Hence, for any solution of (2.1), we have

$$(3.2) \quad \frac{d}{dt} J[k(\cdot, t)] = - \int_{-\psi_+}^{\psi_-} \frac{k_t^2}{k^2} d\theta \leq 0.$$

Thus we obtain

$$(3.3) \quad \int_0^{\infty} \int_{-\psi_+}^{\psi_-} \frac{k_t^2}{k^2} d\theta dt < \infty.$$

if  $k$  is a global solution. By the standard omega limiting argument as below, once we have the boundedness of  $-\varepsilon_0 < k < -\varepsilon_0^{-1}$  for some small  $\varepsilon_0 > 0$ , we can show that  $k(\theta, t)$  approaches a set of stationary solution as  $t \rightarrow \infty$ . This gives us accurate interpretation of the asymptotic behavior of the global solution. Let  $k(\theta, t)$  be a solution of (2.1).

**Proposition 3.1.** *Let the case (B) hold. The function  $k(\theta, t)$  converges in the  $C^{\infty}$ -topology to the stationary solution of the problem (2.1) as  $t \rightarrow \infty$ .*

**Remark 3.1.** This proposition only implies that the shape of the solution  $\Gamma(t)$  converges to that of traveling wave. This does not mean that the endpoints of  $\Gamma(t)$  also converge to those of a traveling wave.

*Proof.* By the assumption, the solution is global and its all derivatives of the curvature  $k(\theta, t)$  are uniformly bounded by positive constants from above ([3, 12]). We also have  $-\varepsilon_0 < k(\theta, t) < -\varepsilon_0^{-1}$  for some  $\varepsilon_0 > 0$ , because of

$$\left| \log \left\{ \frac{k(\theta_2, t)}{k(\theta_1, t)} \right\} \right| = \left| \int_{\theta_1}^{\theta_2} \frac{k_\theta(\theta, t)}{k(\theta, t)} d\theta \right| \leq \max_{\theta \in [-\psi_+, \psi_-]} |k_\theta| L(t) \leq C \max_{\theta \in [-\psi_+, \psi_-]} |k_\theta|.$$

Therefore,  $J(k(\cdot, t))$  remain uniformly bounded and

$$(3.4) \quad \int_0^\infty \int_{-\psi_+}^{\psi_-} \left( \frac{k_t}{k} \right)^2 d\theta dt \leq \infty.$$

Let  $\{t_j\}$  be any sequence such that  $\lim_{j \rightarrow \infty} t_j = \infty$  and we prove that there exists a subsequence still denoted by  $\{t_j\}$  such that  $k(t_j) \rightarrow k^*$  in  $C^\infty([-\psi_+, \psi_-])$ , where  $k^*$  is the stationary solution of (2.1). To prove this, we define  $k_j(\theta, t) = k(\theta, t + t_j)$  on  $[-\psi_+, \psi_-] \times [0, 1]$ . Then  $\{k_j\}$  is bounded in  $C^{2,1}([-\psi_+, \psi_-] \times [0, 1])$  from the above derivative bounds. Then there exists a subsequence  $\{t_j\}$  (still denoted by the same notation) and  $\zeta \in C^{2,1}([-\psi_+, \psi_-] \times [0, 1])$  such that  $k_j \rightarrow \zeta$  in  $C^{2,1}([-\psi_+, \psi_-] \times [0, 1])$  satisfying

$$(3.5) \quad \begin{cases} \zeta_t = \zeta^2(\zeta_{\theta\theta} + \zeta + c), & -\psi_+ < \theta < \psi_+, \quad 0 \leq t \leq 1, \\ \zeta_\theta = \cot \theta (\zeta + c), & \theta = \mp \psi_\pm, \quad 0 \leq t \leq 1, \end{cases}$$

Combining this, (3.4) and the above lower bound of curvature, we conclude

$$\lim_{j \rightarrow \infty} \int_0^1 \int_{-\psi_+}^{\psi_-} (\partial_t k_j)^2 d\theta dt \leq \lim_{j \rightarrow \infty} \int_{t_j}^\infty \int_{-\psi_+}^{\psi_-} k_t^2 d\theta dt = 0.$$

By Fatou's lemma, we conclude  $\zeta_t \equiv 0$  on  $[-\psi_+, \psi_-] \times [0, 1]$ . Hence the function  $\zeta$  must satisfy

$$\begin{cases} 0 = \zeta^2(\zeta_{\theta\theta} + \zeta + c), & -\psi_+ < \theta < \psi_+, \quad 0 \leq t \leq 1, \\ \zeta_\theta = \cot \theta (\zeta + c), & \theta = \mp \psi_\pm, \quad 0 \leq t \leq 1, \end{cases}$$

Since the stationary solution  $k^*$  of (2.1) is unique except for  $k \equiv 0$ , we conclude that  $\zeta \equiv k^*$  from the classification theorem. Furthermore,  $k(\cdot, t)$  itself must converge to the corresponding stationary solution  $k^*$  as  $t \rightarrow \infty$ , since  $J$  is a strict Lyapunov function (see also Theorem 53.5 of [18]). The derivative bounds then guarantee that  $k(\cdot, t) \rightarrow k^*$  in  $C^\infty([-\psi_+, \psi_-])$ .  $\square$

Our second aim of this section is to prove the exponential stability with shift to a traveling wave solution for the problem (P). By only Proposition 3.1, we can not conclude that any (concave) solution  $\Gamma(t)$  converges to one of the traveling wave as  $t \rightarrow \infty$ . In order to prove the convergence of two endpoints, we analyze the convergence rate of the curvature. This leads



us to consider the spectral analysis of the linearized operator around  $k^*$  for the curvature equation (2.1). We first introduce the weighted  $L^2$  space:

$$L_*^2 := \{v \in L^2([- \psi_+, \psi_-]) \mid \|v\|_* < \infty\}$$

equipped with the inner product

$$(f, g)_* = \int_{-\psi_+}^{\psi_-} fg \frac{d\theta}{(k^*)^2}.$$

By a simple calculation the linearized operator of (2.1) at  $k = k^*$  becomes the following:

$$\mathcal{L}(v) := (k^*)^2(v_{\theta\theta} + v).$$

The domain of the operator is

$$\mathcal{D}(\mathcal{L}) := \left\{ v \in H_*^2([- \psi_+, \psi_-]) \mid v_\theta = \cot \theta v \text{ at } \theta = \mp \psi_\pm \right\},$$

where  $H_*^2([- \psi_+, \psi_-])$  is the associated Sobolev space equipped with the same weighted  $L^2$  norm.

We shall mention that the operator  $\mathcal{L}$  is self-adjoint and its eigenvalues are simple.

**Lemma 3.2.**  *$\mathcal{L}$  is self-adjoint with respect to the inner product  $(\cdot, \cdot)_*$  in  $L_*^2([- \psi_+, \psi_-])$ .*

From the standard Sturm-Liouville theory, we have the following lemma.

**Lemma 3.3.** *All eigenvalues of the operator  $\mathcal{L}$  are simple.*

It is easy to check that  $k^* + c = -c \sin \theta$  belongs to  $\mathcal{D}(\mathcal{L})$ . By a simple calculation, we have the following:

**Lemma 3.4.** *0 is an eigenvalue of  $\mathcal{L}$  and the kernel of  $\mathcal{L}$  is spanned by  $k^* + c = \sin \theta$ .*

**Proposition 3.5.** *A set of eigenvalues of the operator  $\mathcal{L}$  consists of one positive number, 0 and countable infinite negative real numbers.*

*Proof.* We are able to apply the standard Sturm-Liouville theorem and to get its spectral property. From Lemma 3.4,  $\mathcal{L}$  has a simple eigenvalue 0, where the corresponding eigenfunction  $\sin \theta$  has only one zero point on  $[- \psi_+, \psi_-]$ . By the standard Sturm theory, 0 is the second largest eigenvalue of the operator  $\mathcal{L}$  (see Theorem 4.1 of [14]). Therefore,  $\mathcal{L}$  has only one positive eigenvalue and its eigenfunction is positive on  $[- \psi_+, \psi_-]$ .  $\square$

Now, from Lemma 3.3 and Lemma 3.4 hold, we can decompose  $L_*^2([- \psi_+, \psi_-])$  into the direct sum  $L_*^2([- \psi_+, \psi_-]) = X_0 \oplus X_\pm$ , where

$$X_0 := \{\mu \sin \theta \in L_*^2([- \psi_+, \psi_-]) \mid \mu \in \mathbb{R}\}, \quad X_\pm := X_0^\perp = \{v \in L_*^2([- \psi_+, \psi_-]) \mid (v, \sin \theta)_* = 0\}.$$

By using Lemma 3.2, we can also decompose the operator  $\mathcal{L}$  with respect to  $L_*^2([-\psi_+, \psi_-]) = X_0 \oplus X_\pm$ , i.e. we set

$$\begin{cases} \mathcal{L}_0 : \mathcal{D}(\mathcal{L}_0) := \mathcal{D}(\mathcal{L}) \cap X_0 \rightarrow X_0, & \mathcal{L}_0(v) := \mathcal{L}(v) = 0 \quad (v \in \mathcal{D}(\mathcal{L}_0)) \\ \mathcal{L}_\pm : \mathcal{D}(\mathcal{L}_\pm) := \mathcal{D}(\mathcal{L}) \cap X_\pm \rightarrow X_\pm, & \mathcal{L}_\pm(v) := \mathcal{L}(v) \quad (v \in \mathcal{D}(\mathcal{L}_\pm)). \end{cases}$$

We also decompose  $X_\pm = X_0^\perp$  into the direct sum  $X_\pm = X_+ \oplus X_-$  and the operator  $\mathcal{L}_\pm$  with respect to  $X_0^\perp = X_+ \oplus X_-$ , where

$$X_+ := \{\mu v_1 \in X_\pm \mid \mu \in \mathbb{R}\}, \quad X_- := \{v \in X_\pm \mid (v, v_1)_* = 0\},$$

where  $v_1$  is the first eigenfunction of  $\mathcal{L}$  and its eigenvalue is positive. Then, the above decomposition and Proposition 3.5 imply the following lemma:

**Lemma 3.6.**  *$\mathcal{L}_\pm$  and  $\mathcal{L}_-$  are self-adjoint with respect to the inner product  $(\cdot, \cdot)_*$  in  $X_\pm$  and  $X_-$ , respectively. Furthermore, all eigenvalues of the operator  $\mathcal{L}_\pm$  and  $\mathcal{L}_-$  are simple, respectively. A set of eigenvalues of the operator  $\mathcal{L}_\pm$  consists of one positive number and countable infinite negative real numbers, and a set of eigenvalues of the operator  $\mathcal{L}_-$  consists of only countable infinite negative real numbers.*

**Remark 3.2.** *Now we consider the geometric meaning of the above decomposition. For any solution  $(u, l_-, l_+)$ , we define the curvature of the solution as  $\kappa(\theta, t)$ . Under the conditions (2.3), all possible perturbation  $v \in \mathcal{D}(\mathcal{L})$  at  $k^*$  must satisfy*

$$(v(\cdot), \sin(\cdot))_* = \int_{-\psi_+}^{\psi_-} v(\theta) \sin \theta \frac{d\theta}{(k^*)^2} = 0.$$

*Thus we need to analyze the spectral properties for the operator  $\mathcal{L}_\pm$  instead of  $\mathcal{L}$  for the problem (P) to know the stability of  $k^*$ .*

By using these properties, we have the following theorem:

**Theorem 3.1.** *Assume the case (B) holds. Let the curvature  $k(\theta, 0)$  sufficiently close to the curvature  $k^*(\theta)$  in uniformly on  $[-\psi_+, \psi_-]$ , then  $k(\theta, t)$  converges to  $k^*$  exponentially on  $[-\psi_+, \psi_-]$  uniformly as  $t \rightarrow \infty$ .*

*Proof.* From Theorems 7.1-7.2 of [12], we have also obtained the well-posedness property for the problem (2.1). Here we measure the distance of two functions in  $C([0, 1])$  for the problem (2.1). Furthermore, if  $k(\theta, t)$  and  $k^*(\theta)$  are uniformly close on  $[-\psi_+, \psi_-]$ , then they become close in  $C^\infty$  for a sufficiently small  $t_0 > 0$ . Now we regard  $k(\theta, t_0)$  as the initial value and apply the general theory of [16]. From Lemma 3.6, the set of eigenvalues of the operator  $\mathcal{L}_\pm$  consists of one positive number and real negative numbers, thus  $\mathcal{L}_\pm$  is sectorial operator and we can apply the instability theory for the equation (2.1) (see Section 9.1.1 of [16]). By the assumption (B) and Theorem 9.1.8 of [16],  $k(\theta, t) - k^*(\theta)$  must be on the stable

manifold of the stationary solution  $k^*$ , whose existence is approved by Lemma 3.6. Under this restriction, we have the exponential convergence of the curvature in  $H^2$ -topology, which comes from Theorem 9.1.2 of [16] and the statement for  $\mathcal{L}_-$  in Lemma 3.6. By the Sobolev inequality, we have the exponential convergence also in  $C^1$  topology. This, in particular gives us the desired result.  $\square$

*Proof of Theorem 1.1.* By differentiating (1.3) with respect to  $t$ , and then using (1.2) and (1.4), we get

$$l'_\pm(t) - \nu = \frac{\pm 1}{\sin \psi_\pm} (k(\theta, t) - k^*(\theta))$$

Theorem 3.1 implies that  $k$  converges to  $k^*$  exponentially fast. Thus  $|l'_\pm(t) - \nu| = O(e^{-\lambda t})$  for some  $\lambda > 0$  as  $t \rightarrow \infty$ . By integrating  $|l'_\pm(t) - \nu| = O(e^{-\lambda t})$ , we obtain the convergence of  $l_\pm(t) - l_\pm(0) - \nu t$  as  $t \rightarrow \infty$ . Hence, we can define  $\mu^\pm := \lim_{t \rightarrow \infty} (l_\pm(t) - \nu t) \in \mathbb{R}$ . Let us reconstruct the solution curve  $\Gamma(t)$  from the curvature  $k(\theta, t)$  by the relation:

$$x(\theta, t) = l_-(t) + \int_\theta^{\psi_-} \frac{\cos \theta}{-k(\theta, t)} d\theta, \quad y(\theta, t) = u(x(\theta, t), t) = \int_{-\psi_+}^\theta \frac{\sin \theta}{k(\theta, t)} d\theta.$$

By a similar way, we again construct the curve for the graph of  $u^*$  from the curvature  $k^*$ , whose shape represents a traveling wave solution:

$$x^*(\theta, t) = \mu^- + \nu t + \int_\theta^{\psi_-} \frac{\cos \theta}{-k^*(\theta)} d\theta, \quad y^*(\theta, t) = u^*(x^*(\theta, t), t) = \int_{-\psi_+}^\theta \frac{\sin \theta}{k^*(\theta)} d\theta.$$

Therefore, as  $t \rightarrow \infty$ , we conclude

$$(3.6) \quad x(\theta, t) - x^*(\theta, t) = l_-(t) - (\mu^- + \nu t) + \int_\theta^{\psi_-} \left( \frac{\cos \theta}{-k(\theta, t)} - \frac{\cos \theta}{-k^*(\theta)} \right) d\theta \rightarrow 0$$

$$(3.7) \quad y(\theta, t) - y^*(\theta, t) = \int_{-\psi_+}^\theta \left( \frac{\sin \theta}{k(\theta, t)} - \frac{\sin \theta}{k^*(\theta)} \right) d\theta \rightarrow 0.$$

uniformly for  $\theta \in [-\psi_+, \psi_-]$ . Thus, boundedness of  $u_x^*$  and the mean value theorem imply  $|u(x(\theta, t), t) - u^*(x(\theta, t), t)| \leq |u(x(\theta, t), t) - u^*(x^*(\theta, t), t)| + |u^*(x^*(\theta, t), t) - u^*(x(\theta, t), t)| \rightarrow 0$  as  $t \rightarrow \infty$  as long as  $x(\theta, t) \in [l_-(t), l_+(t)] \cap [\mu^- + \nu t, \mu^+ + \nu t]$ . This and (3.6)-(3.7) give us pointwise convergence  $u(x, t) \rightarrow u^*(x, t)$  on  $\mathbb{R}$  as  $t \rightarrow \infty$ , where we consider zero extension as in the statement. Combining these with space derivative bounds, we conclude the uniform convergence.  $\square$

Next we shall prove Theorem 1.2. We first need to introduce another linearized operator, which can be deduced by linearizing (1.1) around a traveling wave and compare spectral of the operator  $\mathcal{L}$  that was discussed above. In order to explain the second operator, we

quickly sketch the formal derivation of the linear operator deduced from (1.1). This operator is similar to that of [6], but it is deduced by linearizing around a traveling wave.

Let  $\mathcal{W}(t) = \{(x, U(x - \nu t + \bar{a})) \mid x - \nu t + \bar{a} \in (0, \beta)\}$  for  $\bar{a} \in \mathbb{R}$  be a traveling wave solution (or stationary solution) and let

$$\Gamma_\epsilon(t) = \{\vec{Y}(s, t) = \vec{W}(s, t) + \epsilon \Phi(s, t) \vec{n}(\vec{W}(s, t)) \mid \vec{W}(s, t) \in \mathcal{W}(t) + O(\epsilon^2)\}$$

be a perturbation of  $\mathcal{W}(t)$ , where  $\vec{n}$  is the outward unit normal vector of  $\mathcal{W}(t)$  on the transformations of coordinates in Section 6 of [6]. Here the variable  $s$  represents a arclength parameter of  $\mathcal{W}(t)$  from the left endpoints, which is not the arclength parameter of  $\Gamma_\epsilon(t)$ . In fact,  $\vec{W}$  and  $\Phi$  depends on also  $\epsilon$ , but we omit it to make the notation simpler (see Section 6 of [6] for the detail). By  $\vec{n}_t(W(s, t)) = 0$ , we have

$$\begin{aligned} (3.8) \quad & V(\vec{Y}(s, t)) = V(\vec{W}(s, t)) + \epsilon \Phi_t(\vec{W}(s, t)) + O(\epsilon^2), \\ & k(\vec{Y}(s, t)) = k(\vec{W}(s, t)) + \epsilon \left\{ \Phi_{ss}(\vec{W}(s, t)) + k^2 \Phi(\vec{W}(s, t)) \right\} + O(\epsilon^2), \\ & \vec{n}(\vec{Y}(s, t)) = \vec{n}(\vec{W}(s, t)) - \epsilon \nabla_w \Phi + O(\epsilon^2), \end{aligned}$$

where  $\nabla_w$  is the gradient on  $\mathcal{W}(t)$ . By substituting these into (1.1), we obtain

$$\Phi_t = \Phi_{ss} + (k^*)^2 \Phi + O(\epsilon).$$

Motivated from this calculation, we introduce the following linear operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2([0, L^*]) \rightarrow L^2([0, L^*])$ :

$$\mathcal{A}\phi(s) := \phi_{ss}(s) + (k^*(s))^2 \phi(s),$$

where  $k^*$  is the corresponding curvature and  $L^*$  is the length of the traveling wave solution. The domain of the operator is

$$\mathcal{D}(\mathcal{A}) := \{\phi \in H^2([0, L^*]) \mid \phi_s(0) = \cot \psi_- k^*(0) \phi(0), \phi_s(L^*) = -\cot \psi_+ k^*(L^*) \phi(L^*)\}.$$

Hereafter, we write the standard inner product in  $L^2([0, L^*])$  as  $(\cdot, \cdot)_{L^2}$ . This operator is considered, for more generalized curvature flows, by Ei and Yanagida [7, 8], Ei, Sato and Yanagida [6], and Giga and Yamauchi [10]. They analyze the stability of stationary hypersurfaces, but for the stability of traveling waves.

Our aim of this section is to prove the exponential stability with shift to a traveling wave solution for the problem (P). In order to prove this, we prove the exponential convergence of the corresponding curvature to that of traveling wave, which gives us the convergence of  $\Gamma(t)$  to one of the traveling wave for all solutions of type (B). We will discuss it at the end of this section.

As in the first operator  $\mathcal{L}$ , the operator  $\mathcal{A}$  is self-adjoint with respect to the inner product  $(\cdot, \cdot)_{L^2}$  in  $L^2([0, L^*])$  and its eigenvalues are simple. In particular, all eigenvalues are real number. Furthermore, 0 is a simple eigenvalue of  $\mathcal{A}$  and the kernel of  $\mathcal{A}$  is spanned by  $k^* + c$ .

It is easy to check but we write the proof for the reader's convenience. First,  $k^* + c = -\nu \sin \theta$  belongs to  $\mathcal{D}(\mathcal{A})$ . By the boundary condition of (2.1), we have

$$k_s^* = \mp \cot \psi_{\pm} k^*(k^* + c), \quad \theta = \mp \psi_{\pm}.$$

This is the desired boundary conditions. Next, by a simple calculation, we have

$$\begin{aligned} \mathcal{A}(\nu \sin \Theta^*) &= (\nu \sin \Theta^*)_{ss} + (k^*(s))^2 \nu \sin \Theta^* \\ &= \nu (k^*(s))^2 ((\sin \theta)_{\theta\theta} + \sin \theta) = 0. \end{aligned}$$

The zero eigenvalue of the operator  $\mathcal{A}$  corresponds to the group of translations parallel to the horizontal line. Thus, we may neglect this one dimensional subspace of the kernel corresponding to translations. By neglecting translation, the eigenvalue problem  $\mathcal{A}\phi = \lambda\phi$  and the Sturm-Liouville like problem for  $\mathcal{L}_{\pm}$  have both the same number of negative eigenvalues and the same multiplicity. In order to see this, we introduce another operator  $\mathcal{P} : \mathcal{D}(\mathcal{A}) \ni \phi \mapsto v \in \mathcal{D}(\mathcal{L}_{\pm})$  as

$$(3.9) \quad v(\theta) = \mathcal{P}(\phi) := \phi_{ss}(s) + (k^*(s))^2 \phi(s), \quad \text{where } \theta = \psi_- + \int_0^s k^* ds$$

We first check that this operator is well-defined. Since the operator  $\mathcal{A}$  is self-adjoint and its eigenvalues are simple, for any  $\phi \in \mathcal{D}(\mathcal{A})$ , there exists a sequence  $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$  such that

$$\phi = \sum_{n=1}^{\infty} a_n \phi_n,$$

where  $\phi_n$  is the eigenfunction of  $\mathcal{A}$  corresponding to the  $n$ -th eigenvalue  $\lambda_n$ , which means  $\mathcal{A}(\phi_n) = \lambda_n \phi_n$ . Then, we can see

$$\mathcal{P}(\phi) = \sum a_n \mathcal{P}(\phi_n) = \sum a_n \mathcal{A}(\phi_n) = \sum a_n \lambda_n \phi_n.$$

$\phi_n$  satisfy the Robin boundary condition of  $\mathcal{D}(\mathcal{A})$ , thus

$$(\phi_n)_{\theta}(\theta) = (\kappa^*(s))^{-1} (\phi_n)_s(s) = \cot \theta \phi(\theta),$$

where  $(\theta, s) = (-\psi_+, 0)$  and  $(\psi_-, L^*)$ , hence  $\mathcal{P}(\phi)$  satisfies the Robin boundary condition of  $\mathcal{D}(\mathcal{L}_{\pm})$ . The Robin boundary condition in  $\mathcal{D}(\mathcal{A})$  also follows from

$$(\mathcal{P}(\phi)(\cdot), \sin(\cdot))_* = \int_{-\psi_+}^{\psi_-} (\kappa^*)^2 (\phi_{\theta\theta} + \phi) \sin \theta \frac{d\theta}{(\kappa^*)^2} = \{\phi_{\theta} \sin \theta - \phi \cos \theta\}|_{-\psi_+}^{\psi_-} = 0.$$

Thus,  $\mathcal{P}(\phi) \in \mathcal{D}(\mathcal{L}_{\pm})$  for  $\phi \in \mathcal{D}(\mathcal{A})$ .

The transformation (3.9) gives us the correspondence of the eigenvalues and eigenfunctions between two operators  $\mathcal{L}_{\pm}$  and  $\mathcal{A}$ . More precisely, the following proposition holds:

**Proposition 3.7.** *Let  $\sigma(\mathcal{L}_{\pm})$  and  $\sigma(\mathcal{A})$  be spectral sets of each operator. Then  $\sigma(\mathcal{A}) \setminus \{0\} = \sigma(\mathcal{L}_{\pm})$ .*

*Proof.* We investigate the kernel of the operator  $\mathcal{P}$ . If

$$\mathcal{P}(\phi) = (\phi)_{ss} + (k^*)^2\phi = (k^*)^2(\phi_{\theta\theta} + \phi) = 0,$$

then

$$\phi_{\theta\theta} + \phi = 0.$$

Here we used  $k^* \neq 0$ . Moreover, by  $\phi \in \mathcal{D}(\mathcal{A})$ ,  $\phi(s) = \mu \sin \Theta(s)$  for any  $\mu \in \mathbb{R}$  and  $\text{Ker}(\mathcal{P}) = \{\mu \sin \Theta(s) \mid \mu \in \mathbb{R}\}$ . It is easy to check that the operator  $\mathcal{P}^* : \mathcal{D}(\mathcal{L}_{\pm}) \rightarrow \mathcal{D}(\mathcal{A})$  is given by  $\mathcal{P}^*v = v_{\theta\theta} + v$  and  $\text{Ker}(\mathcal{P}^*) = \{\mu \sin \theta \mid \mu \in \mathbb{R}\} \cap \mathcal{D}(\mathcal{L}_{\pm}) = \{0\}$ . By the Fredholm alternative theorem  $\mathcal{P}(\mathcal{D}(\mathcal{A})) = (\text{Ker}(\mathcal{P}^*))^{\perp} = \mathcal{D}(\mathcal{L}_{\pm})$ . Thus  $(\text{Ker}(\mathcal{P}))^{\perp} = \mathcal{D}(\mathcal{A})$ . Therefore, we can define the inverse of the operator  $\mathcal{P}_0 := \mathcal{P}|_{(\text{Ker}(\mathcal{P}))^{\perp}} : (\text{Ker}(\mathcal{P}))^{\perp} \rightarrow \mathcal{D}(\mathcal{L}_{\pm})$  by  $\mathcal{P}_0^{-1} : \mathcal{D}(\mathcal{L}_{\pm}) \rightarrow (\text{Ker}(\mathcal{P}))^{\perp} \subset \mathcal{D}(\mathcal{A})$ .

In order to prove  $\sigma(\mathcal{A}) \setminus \{0\} \subset \sigma(\mathcal{L}_{\pm})$ , we prove  $\mathcal{P}(\mathcal{A}(\phi)) = \mathcal{L}_{\pm}(\mathcal{P}(\phi))$  for any  $\phi \in \mathcal{D}(\mathcal{A})$ . The operator  $\mathcal{A}$  is self-adjoint and all eigenvalues are simple. Therefore,  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$  and  $\mathcal{P}(\mathcal{A}(\phi))$  is well-defined. By using this equation, we have

$$\begin{aligned} \mathcal{P}(\mathcal{A}(\phi)) &= \mathcal{P}\{(\partial_{ss} + (k^*)^2 I)\phi\} = (\partial_{ss} + (k^*)^2 I)(\partial_{ss} + (k^*)^2 I)\phi \\ &= ((k^*)^2 \partial_{\theta\theta} + (k^*)^2 I)\mathcal{P}(\phi) = \mathcal{L}(\mathcal{P}(\phi)), \end{aligned}$$

where  $\partial_{ss}, \partial_{\theta\theta}$  are second derivatives for  $s$  and  $\theta$  and  $I$  is identity mapping in  $L^2$ . Thus  $\mathcal{P}\mathcal{A} = \mathcal{L}_{\pm}\mathcal{P}$  has proved.

Let us substitute  $\phi$  for eigenfunction  $\phi_n \in \mathcal{D}(\mathcal{A})$  whose eigenvalue is  $\lambda_n$  and define  $v_n = \mathcal{P}(\phi_n)$ . Then, we obtain

$$\mathcal{L}_{\pm}(v_n) = \mathcal{L}_{\pm}(\mathcal{P}(\phi_n)) = \mathcal{P}(\mathcal{A}(\phi_n)) = \mathcal{P}(\lambda\phi_n) = \lambda\mathcal{P}(\phi_n) = \lambda v_n.$$

$\text{Ker}(\mathcal{P})$  is one of the eigenspace of  $\mathcal{A}$  whose basis is  $\sin \Theta$  and hence  $\sigma(\mathcal{A}) \setminus \{0\} \subset \sigma(\mathcal{L}_{\pm})$  holds.

To prove  $\sigma(\mathcal{A}) \setminus \{0\} \supset \sigma(\mathcal{L}_{\pm})$ , we prove  $\mathcal{P}_0^{-1}(\mathcal{L}_{\pm}(v)) = \mathcal{A}(\mathcal{P}_0^{-1}(v))$  for any  $v \in \mathcal{D}(\mathcal{L}_{\pm})$ . Now, we note that  $\mathcal{A}((\text{Ker}(\mathcal{P}))^{\perp}) = (\text{Ker}(\mathcal{P}))^{\perp}$ . By using  $\mathcal{P}_0^{-1}(v) \in (\text{Ker}(\mathcal{P}))^{\perp}$ , we have

$$\begin{aligned} \mathcal{P}_0^{-1}(\mathcal{L}_{\pm}(v)) &= \mathcal{P}_0^{-1}[\mathcal{L}_{\pm}\{\mathcal{P}(\mathcal{P}_0^{-1}(v))\}] \\ &= \mathcal{P}_0^{-1}[\mathcal{P}\{\mathcal{A}(\mathcal{P}_0^{-1}(v))\}] \\ &= \mathcal{A}(\mathcal{P}_0^{-1}(v)). \end{aligned}$$

We substitute  $v$  for eigenfunction  $\tilde{v}_n \in \mathcal{D}(\mathcal{L}_{\pm})$  whose eigenvalue is  $\tilde{\lambda}_n$  and define  $\tilde{\phi}_n = \mathcal{P}_0^{-1}(\tilde{v}_n)$ . By a similar argument as above, we obtain  $\mathcal{A}(\tilde{\phi}_n) = \tilde{\lambda}_n \tilde{\phi}_n$  and  $\sigma(\mathcal{A}) \setminus \{0\} \supset \sigma(\mathcal{L}_{\pm})$ .  $\square$

**Remark 3.3.** Since  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2([0, L^*]) \rightarrow L^2([0, L^*])$  is the standard Sturm Liouville operator, all the information of the eigenvalues and eigenfunctions can be obtained as in the argument of the operator  $\mathcal{L}$ . Here we would like to know the correspondence of each

*eigenvalue and eigenfunctions to know what each eigenfunctions means more clearly, and construct the operator  $\mathcal{P}$ .*

By using this correspondence of the eigenvalues with respect to  $\mathcal{A}$  and  $\mathcal{L}_\pm$ , the linear operator  $\mathcal{A}$  has one positive eigenvalue, zero and all other eigenvalues are negative and Theorem 1.2 has proved as in the proof of Theorem 3.1.

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