On convexity of simple closed frontals

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Abstract

We study convexity of simple closed frontals in the Euclidean plane by using the curvature of Legendre curves. We show that for a Legendre curve, the simple closed frontal is convex if and only if the sign of both functions of the curvature of the Legendre curve does not change. We also give some examples of convex simple closed frontals.

1 Introduction and main result

In the classical differential geometry of regular curves, we can analyze global properties of curves, such as convexity, width and rotation number by using the curvature (cf. [4],[5]). One of well-known result is a characterization of convexity of simple closed regular curves by using the curvature (cf. [5]):

Theorem 1.1 *A simple closed regular curve is convex if and only if its curvature κ has a constant sign; that is, κ is either always non-positive or always non-negative.*

When we consider singular curves, the above theorem does not hold; that is, there is a simple closed singular curve with the curvature is always non-positive except singular points, but the curve is not convex (the curve divided by a tangent line). For example, let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the astroid $\gamma(t) = (\cos^3 t, \sin^3 t)$. The curvature κ of γ is given by $\kappa(t) = -2/(3|\sin 2t|)$ except four singular points and diverges to $-\infty$ at each singular points. Hence, κ has the constant sign. However, this curve is not convex, see Figure 1.

In the present paper, we give a characterization of convexity for a special class of singular curves called frontals by using the curvature of Legendre curves which has introduced in [3].

Let I be an interval. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma(t), \nu(t))^*\theta = 0$ for all $t \in I$, where θ is the canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ and S^1 is the unit sphere (cf. [1],[2]). This condition is equivalent to $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$, where $\cdot$ is the Euclidean inner product on $\mathbb{R}^2$. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. If γ is a regular curve around a point t_0, then we have the Frenet frame along γ. On the other hand, if γ is singular at a point t_0, then

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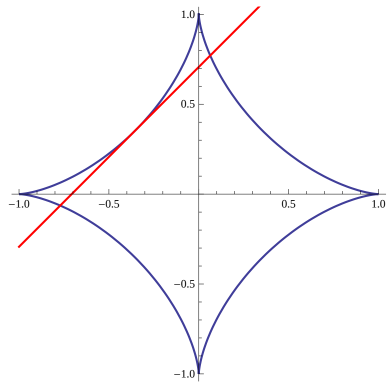

Figure 1: The tangent line at $t = 3\pi/4$ of the astroid $\gamma(t) = (\cos^3 t, \sin^3 t)$ divide the curve.

we can not define the Frenet frame. However, ν is always defined even if t_0 is a singular point of γ. Therefore, we have a frame along a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame along the frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of the frontal (or, the Legendre curve) which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t).$$

The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Remark 1.2 Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ and $(\overline{\gamma}, \overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves are (ℓ, β) and $(\overline{\ell}, \overline{\beta})$ respectively. Suppose that (γ, ν) and $(\overline{\gamma}, \overline{\nu})$ are parametrically equivalent via the change of parameter $t : \overline{I} \to I; u \mapsto t(u)$ with $\dot{t}(u) > 0$, that is, $(\overline{\gamma}(u), \overline{\nu}(u)) = (\gamma(t(u)), \nu(t(u)))$ for all $u \in \overline{I}$. Then, we have

$$\overline{\ell}(u) = \ell(t(u))\dot{t}(u), \ \overline{\beta}(u) = \beta(t(u))\dot{t}(u).$$

Hence the curvature of the Legendre curve is depended on a parametrization.

Definition 1.3 Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t)) = A(\gamma(t)) + \boldsymbol{b}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$, where C is given by the rotation A and the translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

Then we have the following theorems.

Theorem 1.4 (The Existence Theorem, [3]) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem 1.5 (The Uniqueness Theorem, [3]) *Let* (γ, ν) *and* $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ *be Legendre curves whose curvatures of Legendre curves* (ℓ, β) *and* $(\widetilde{\ell}, \widetilde{\beta})$ *coincide. Then* (γ, ν) *and* $(\widetilde{\gamma}, \widetilde{\nu})$ *are congruent as Legendre curves.*

For $n \in \mathbb{N} \cup \{0\}$, we say that a Legendre curve $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ is C^n-*closed* if $(\gamma^{(k)}(a), \nu^{(k)}(a)) = (\gamma^{(k)}(b), \nu^{(k)}(b))$ for all $k \in \{0, \cdots, n\}$, where $\gamma^{(k)}(a)$, $\nu^{(k)}(a)$, $\gamma^{(k)}(b)$ and $\nu^{(k)}(b)$ mean one-sided k-th differential. Similarly, we say that a Legendre curve $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ is C^∞-*closed* if $(\gamma^{(k)}(a), \nu^{(k)}(a)) = (\gamma^{(k)}(b), \nu^{(k)}(b))$ for all $k \in \mathbb{N} \cup \{0\}$. In this paper, we say that (γ, ν) is a *closed* Legendre curve, if the curve is at least C^1-closed. Note that if (γ, ν) is a closed Legendre curve, the domain of the curve can be extended from $[a, b]$ to $\mathbb{R}$ so that $(\gamma, \nu)([a, b]) = (\gamma, \nu)(\mathbb{R})$ and the extended map $(\gamma, \nu) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ is at least C^1 map. Moreover, a frontal $\gamma : [a, b] \to \mathbb{R}^2$ is *simple closed* if for $t_1 < t_2$, we have $\gamma(t_1) = \gamma(t_2)$ if and only if $t_1 = a$ and $t_2 = b$

We define a convex frontal in the Euclidean plane. From now on, I is a closed interval. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. We denote the tangent line at t of γ by L_t, that is, $L_t = \{\lambda \boldsymbol{\mu}(t) + \gamma(t) \mid \lambda \in \mathbb{R}\}$. Any tangent line L_t divides $\mathbb{R}^2$ into two half-planes H_+ and H_- such that $H_+ \cup H_- = \mathbb{R}^2$ and $H_+ \cap H_- = L_t$. By using ν, the half-planes H_+ and H_- are presented by $H_+ = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (\boldsymbol{x} - \gamma(t)) \cdot \nu(t) \geq 0\}$ and $H_- = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (\boldsymbol{x} - \gamma(t)) \cdot \nu(t) \leq 0\}$. For a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we say that (γ, ν) is a *convex Legendre curve* (or, γ is a *convex frontal*) if $\gamma(I) \subset H_+$ for any tangent line of γ or $\gamma(I) \subset H_-$ for any tangent line of γ. Note that if γ is a regular curve, then $\boldsymbol{\mu}(t)$ is equal to the unit tangent vector of γ at $\gamma(t)$ up to sign. Therefore, γ is a convex curve as a frontal if and only if γ is a convex curve as the usual mean when γ is a regular curve (cf. [5]).

By definition, convexity of a Legendre curve is preserved under a congruence as Legendre curves. Moreover, if $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a convex Legendre curve, then $(\gamma \circ u, \nu \circ u) : \overline{I} \to \mathbb{R}^2 \times S^1$ is also convex for a change of parameter $u : \overline{I} \to I$ and any smooth function $u : \overline{I} \to I$ as well.

The main result of this paper is stated as follows:

Theorem 1.6 *Let* $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ *be a closed Legendre curve with the curvature* (ℓ, β) *which the frontal* γ *is simple closed. Suppose that zeros of* ℓ *and of* β *are isolated points. Then the frontal* γ *is convex if and only if the curvature satisfy one of the following condition:*
(i) Both of $\ell(t)$ *and* $\beta(t)$ *are always non-negative,*
(ii) $\ell(t)$ *is always non-negative and* $\beta(t)$ *is always non-positive,*
(iii) Both of $\ell(t)$ *and* $\beta(t)$ *are always non-positive,*
(iv) $\ell(t)$ *is always non-positive and* $\beta(t)$ *is always non-negative.*

We prove this theorem in Section 2. Moreover, we give examples of convex simple closed frontals in Section 3.

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2 Proof of the main result

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a closed Legendre curve with the curvature (ℓ, β). In this paper, we assume that zeros of ℓ and β are isolated points. First, we prove that if the sign of ℓ or the sign of β change, then the frontal γ is not convex.

Lemma 2.1 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a closed Legendre curve. If the sign of $\ell(t)$ or the sign of $\beta(t)$ change, then the frontal γ is not convex.*

Proof. Let $t_0 \in I$ be a point such that the sign of ℓ or the sign of β change, that is, locally $\ell(t) > 0$ (respectively, $\ell(t) < 0$) if $t < t_0$ and $\ell(t) < 0$ (respectively, $\ell(t) > 0$) if $t > t_0$, or $\beta(t) > 0$ (respectively, $\beta(t) < 0$) if $t < t_0$ and $\beta(t) < 0$ (respectively, $\beta(t) > 0$) if $t > t_0$. Convexity of the frontal does not change by a congruence of Legendre curves, hence we may assume $\gamma(t_0)$ is the origin of the Euclidean plane without loss of generality.

If the sign of $\gamma(t) \cdot \nu(t_0)$ change around t_0, then L_{t_0} divide the frontal γ. To prove that the frontal γ is not convex, we show that the sign of $\gamma(t) \cdot \nu(t_0)$ changes around t_0.

By the definition of β, we have

$$\frac{d}{dt}(\gamma(t) \cdot \nu(t_0)) = \beta(t)\boldsymbol{\mu}(t) \cdot \nu(t_0).$$

Since $\|\boldsymbol{\mu}(t)\| = \|\nu(t_0)\| = 1$, there is a smooth function φ such that $\boldsymbol{\mu}(t) \cdot \nu(t_0) = \cos\varphi(t)$. Moreover, by the definition of ℓ, we have

$$-\dot{\varphi}(t)\sin\varphi(t) = \frac{d}{dt}(\boldsymbol{\mu}(t) \cdot \nu(t_0)) = -\ell(t)\nu(t) \cdot \nu(t_0). \tag{1}$$

Since $\cos\varphi(t_0) = \boldsymbol{\mu}(t_0) \cdot \nu(t_0) = 0$, we have $\sin\varphi(t_0) = \pm 1$. Substitute t_0 for the equation (1), we obtain

$$\dot{\varphi}(t_0) = \pm\ell(t_0). \tag{2}$$

First, we consider the case of the sign of ℓ changes and the sign of β does not change around t_0. In this case, $\dot{\varphi}(t_0) = 0$ by (2). Since $\nu(t) \cdot \nu(t_0)$ is a continuous function and $\nu(t_0) \cdot \nu(t_0) = 1$, we have $\nu(t) \cdot \nu(t_0) > 0$ around t_0. By the equation (1), the sign of $\dot{\varphi}(t)$ changes around t_0. This conclude that the sign of $\cos\varphi(t)$ does not change around t_0. Hence $\boldsymbol{\mu}(t) \cdot \nu(t_0) \geq 0$ or $\boldsymbol{\mu}(t) \cdot \nu(t_0) \leq 0$ around t_0. Moreover, since the sign of β does not change around t_0, we obtain $(d/dt)(\gamma(t) \cdot \nu(t_0)) \geq 0$ or $(d/dt)(\gamma(t) \cdot \nu(t_0)) \leq 0$, that is, $\gamma(t) \cdot \nu(t_0)$ is a monotone function around t_0. By the assumption $\gamma(t_0) = 0$, we have $\gamma(t_0) \cdot \nu(t_0) = 0$ and the sign of $\gamma(t) \cdot \nu(t_0)$ changes around t_0. Therefore, γ is not convex.

Second, we consider the case of the sign of β changes and the sign of ℓ does not change around t_0. By the equation (1), the sign of $\dot{\varphi}(t)$ does not change around t_0, that is, $\dot{\varphi}(t)$ is a monotone function around t_0. This conclude that the sign of $\cos\varphi(t)$ changes around t_0. Hence the sign of $\boldsymbol{\mu}(t) \cdot \nu(t_0)$ changes around t_0. It follows that the sign of $(d/dt)(\gamma(t) \cdot \nu(t_0))$ does not change around t_0. By the same argument of the first case, γ is not convex.

Finally, we consider the case of the signs of ℓ and β changes around t_0 simultaneously. We prove by a contradiction. Assume that the frontal γ is a convex curve. Since the signs of ℓ and β change around t_0, similar to the first and second cases, the sign of $(d/dt)(\gamma(t) \cdot \nu(t_0))$ changes around t_0. Moreover,

$$\frac{d}{dt}(\gamma(t) \cdot \boldsymbol{\mu}(t_0)) = \beta(t)\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0).$$

Since the sign of β changes and $\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0) > 0$ around t_0, the sign of $(d/dt)(\gamma(t) \cdot \boldsymbol{\mu}(t_0))$ also changes around t_0. Hence, γ is contained a quadrant in the $\{\nu(t_0), \boldsymbol{\mu}(t_0)\}$-plane around t_0.

By differentiating $\boldsymbol{\mu}(t) \cdot \nu(t_0)$, we have

$$\frac{d}{dt}(\boldsymbol{\mu}(t) \cdot \nu(t_0)) = -\ell(t)\nu(t) \cdot \nu(t_0). \tag{3}$$

By the equation (3), the sign of $(d/dt)(\boldsymbol{\mu}(t)\cdot\nu(t_0))$ changes around t_0 and $(d/dt)(\boldsymbol{\mu}(t_0)\cdot\nu(t_0)) = 0$. Therefore, t_0 is a local maximal or local minimal point of $\boldsymbol{\mu}(t) \cdot \nu(t_0)$. It follows that for any $\varepsilon > 0$, there exist $t_1 \in (t_0 - \varepsilon, t_0)$ and $t_2 \in (t_0, t_0 + \varepsilon)$ such that

$$\boldsymbol{\mu}(t_1) \cdot \nu(t_0) = \boldsymbol{\mu}(t_2) \cdot \nu(t_0) \tag{4}$$

Since $\{\nu(t_0), \boldsymbol{\mu}(t_0)\}$ is a basis on $\mathbb{R}^2$, there exist smooth functions λ and η such that $\boldsymbol{\mu}(t) = \lambda(t)\nu(t_0) + \eta(t)\boldsymbol{\mu}(t_0)$. By the equation (4), we have $\lambda(t_1) = \lambda(t_2)$. Moreover, since $\boldsymbol{\mu}(t_1)$ and $\boldsymbol{\mu}(t_2)$ are unit vectors, we have $\lambda^2(t_1) + \eta^2(t_1) = \lambda^2(t_2) + \eta^2(t_2)$. Hence, $\eta(t_1) = \pm\eta(t_2)$. By the definition of η, we have $\eta(t_0) = 1$. It follows that both $\eta(t_1)$ and $\eta(t_2)$ are positive. Thus $\eta(t_1) = \eta(t_2)$ and hence $\boldsymbol{\mu}(t_1) = \boldsymbol{\mu}(t_2)$. It follows that L_{t_1} is parallel to L_{t_2}. Since zeros of β are isolated points, by change of the choice of t_1 and t_2 if we need, we may assume $\gamma(t_1) \neq \gamma(t_2)$.

Now suppose that $L_{t_1} = L_{t_2}$. Since we assume γ is a convex curve, the curve lies on one-side of $L_{t_1}(= L_{t_2})$ and tangent at t_1 and t_2. Since zeros of ℓ are isolated points, L_{t_1} is a double tangent line of γ. It follows that, there exists a point t_3 around t_1 or t_2 such that L_{t_3} divide $\gamma(I)$. This contradict to convexity of γ. Hence, we have $L_{t_1} \neq L_{t_2}$.

On the other hand, when $L_{t_1} \neq L_{t_2}$, we obtain

$$\gamma(I) \subset \{\boldsymbol{x} \in \mathbb{R}^2 | (\boldsymbol{x} - \gamma(t_1)) \cdot \nu(t_1) \geq 0\} \cap \{\boldsymbol{x} \in \mathbb{R}^2 | (\boldsymbol{x} - \gamma(t_2)) \cdot \nu(t_2) \geq 0\}$$

or

$$\gamma(I) \subset \{\boldsymbol{x} \in \mathbb{R}^2 | (\boldsymbol{x} - \gamma(t_1)) \cdot \nu(t_1) \leq 0\} \cap \{\boldsymbol{x} \in \mathbb{R}^2 | (\boldsymbol{x} - \gamma(t_2)) \cdot \nu(t_2) \leq 0\}$$

by the definition of the convex. The tangent line L_{t_1} or L_{t_2} divide $\gamma(I)$, since γ lies on a same side of half-planes which are divided by L_{t_1} and L_{t_2}. This contradict to convexity of γ. Therefore, γ is not convex. □

In the rest of this section we prove that if the signs of ℓ and β does not change, then the simple closed frontal γ is convex. In order to prove this claim, we prepare some lemmas and notations.

Lemma 2.2 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β). If $\beta(t) \geq 0$ for all $t \in I$ or $\beta(t) \leq 0$ for all $t \in I$, then there is the smooth map $\Phi(\gamma, \nu) : I \to S^1$ such that $\Phi(\gamma, \nu) = \dot{\gamma}/\|\dot{\gamma}\|$ on $I \setminus \mathcal{Z}_\beta$, where $\mathcal{Z}_\beta = \{t \in I \mid \beta(t) = 0\}$.*

Proof. By the definition of β, we obtain

$$\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} = \frac{\beta(t)}{|\beta(t)|}\frac{\boldsymbol{\mu}(t)}{\|\boldsymbol{\mu}(t)\|} = \frac{\beta(t)}{|\beta(t)|}\boldsymbol{\mu}(t) = \mathrm{sign}(\beta(t))\boldsymbol{\mu}(t)$$

on $I \setminus \mathcal{Z}_\beta$, where $\mathrm{sign}(\beta(t))$ is the sign of $\beta(t)$. Therefore, we can extend this function to I if and only if $\mathrm{sign}(\beta(t))$ is a constant map on $I \setminus \mathcal{Z}_\beta$. By the assumption, we may define $\Phi(\gamma, \nu)(t) = \mathrm{sign}(\beta(t))\boldsymbol{\mu}(t)$. Here we also denote $\mathrm{sign}(\beta(t))$ by 1 if $\beta(t) \geq 0$ for all $t \in I$ and -1 if $\beta(t) \leq 0$ for all $t \in I$. □

We denote the set of closed Legendre curves (γ, ν) with $\beta(t) \geq 0$ (respectively, $\beta(t) \leq 0$) for all $t \in I$ by $\mathcal{R}_+$ (respectively, $\mathcal{R}_-$), and $\mathcal{R}_+ \cup \mathcal{R}_-$ by $\mathcal{R}$. For a Legendre curve $(\gamma, \nu) \in \mathcal{R}$, we define a smooth function $\theta : I \to \mathbb{R}$ such that $\Phi(\gamma, \nu)(t) = (\cos\theta(t), \sin\theta(t))$ by Lemma 2.2. Same as the case of regular curves, for a Legendre curve $(\gamma, \nu) \in \mathcal{R}$, we call the degree of $\Phi(\gamma, \nu)$ the *rotation index* of the Legendre curve (γ, ν).

Lemma 2.3 *If $(\gamma, \nu) \in \mathcal{R}$, then $\ell(t) = -\operatorname{sign}(\beta(t))\dot{\theta}(t)$.*

Proof. Suppose $(\gamma, \nu) \in \mathcal{R}$, we have

$$\boldsymbol{\mu}(t) = \operatorname{sign}(\beta(t))\Phi(\gamma, \nu)(t) = \operatorname{sign}(\beta(t))(\cos\theta(t), \sin\theta(t))$$

and

$$\nu(t) = J^{-1}(\boldsymbol{\mu}(t)) = \operatorname{sign}(\beta(t))(\sin\theta(t), -\cos\theta(t)).$$

Then $\dot{\boldsymbol{\mu}}(t) = -\operatorname{sign}(\beta(t))\dot{\theta}(t)\nu(t)$. By the definition of ℓ, we obtain $\ell(t) = -\operatorname{sign}(\beta(t))\dot{\theta}(t)$. $\square$

Lemma 2.4 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with $(\gamma, \nu) \in \mathcal{R}$. Then there is a point $t_0 \in I$ with the property that $\gamma(I)$ lies entirely to one side of L_{t_0}.*

Proof. Let (x, y) be a coordinate on $\mathbb{R}^2$ and let $\gamma(t_0) = p$ be a point which has maximum height, that is, y-coordinate is maximum in $\gamma(I)$. Without loss of generality, we may assume p is the origin of the Euclidean plane and $(\gamma, \nu) \in \mathcal{R}_+$. Since the sign of β does not change and $\boldsymbol{\mu}(t_0) \cdot \boldsymbol{\mu}(t_0) = 1$, we have

$$\frac{d}{dt}(\gamma(t) \cdot \boldsymbol{\mu}(t_0)) = \beta(t)\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0) \geq 0$$

around t_0. If the sign of $(d/dt)(\gamma(t) \cdot \nu(t_0))$ does not change around t_0, then γ through the origin from the second quadrant to the fourth quadrant or the third quadrant to the first quadrant of the $\{\nu(t_0), \boldsymbol{\mu}(t_0)\}$-plane. This contradict to p has maximum height. Hence, the sign of $(d/dt)(\gamma(t) \cdot \nu(t_0))$ change around t_0. This means γ lies on the under half-plane divided by L_{t_0}. Moreover, if L_{t_0} does not coincides with the x-axis, this contradict to p has maximum height. Therefore, L_{t_0} coincides with the x-axis. By the above, L_{t_0} coincides with the x-axis and $\gamma(I)$ lies on under the x-axis. Therefore, t_0 is a required point. $\square$

The rest of the proof is similar to the case of regular curves (see [5]).

Lemma 2.5 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve which the frontal γ is simple closed and $(\gamma, \nu) \in \mathcal{R}$. Then the rotation index of the Legendre curve (γ, ν) is equal to ± 1.*

Proof. Let $p = \gamma(t_0)$ be a point on $\gamma(I)$ with the property that $\gamma(I)$ lies entirely to one side of L_{t_0}. Such a point t_0 is always exists by Lemma 2.4.

Now we set $I = [0, L]$. Consider the triangular region $\mathcal{T} := \{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leq t_1 \leq t_2 \leq L\}$. By a reparametrization of (γ, ν), we may assume $t_0 = 0$, that is, $\gamma(0) = p$. We define a map $\Sigma : \mathcal{T} \to S^1$ by

$$\Sigma(t_1, t_2) = \begin{cases} \Phi(\gamma, \nu)(t) & (\text{if } t_1 = t_2 = t), \\ -\Phi(\gamma, \nu)(0) & (\text{if } t_1 = 0 \text{ and } t_2 = L), \\ \dfrac{\gamma(t_2) - \gamma(t_1)}{\|\gamma(t_2) - \gamma(t_1)\|} & (\text{otherwise}). \end{cases}$$

By the assumptions and Lemma 2.2, the map Σ is continuous. Let $A = (0, 0)$, $B = (0, L)$ and $C = (L, L)$. Since the restriction of Σ to the segment AC is equal to $\Phi(\gamma, \nu)$, the degree of this map is equal to the rotation index of (γ, ν).

Moreover, consider the restriction of Σ to the segment $AB \cup BC$. Set the angle from $\Phi(\gamma, \nu)(0)$ to $-\Phi(\gamma, \nu)(0)$ is equal to π. Since

$$\Sigma|_{AB}(t_1, t_2) = \Sigma(0, t) = (\gamma(t) - \gamma(0))/\|\gamma(t) - \gamma(0)\|,$$

$\Sigma|_{AB}$ covers one half of S^1. Similarly,

$$\Sigma|_{BC}(t_1, t_2) = \Sigma(t, L) = (\gamma(L) - \gamma(t))/\|\gamma(L) - \gamma(t)\|,$$

$\Sigma|_{BC}$ covers other half of S^1. Hence, the degree of the restriction of Σ to the segment $AB \cup BC$ is equal to ± 1 (the sign depends on an orientation of γ).

Note that the restriction of Σ to the segment $AB \cup BC$ is homotopic to the restriction of Σ to the segment AC, that is, $\Phi(\gamma, \nu)$. Because the rotation index is preserved under homotopy, the rotation index of the frontal γ is equal to ± 1. □

The following lemma is the sufficient part of the main theorem.

Lemma 2.6 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a closed Legendre curve which the frontal γ is simple closed. If the signs of ℓ and β does not change, then the frontal γ is convex.*

Proof. By Lemma 2.3, the sign of $\ell(t)$ does not change if and only if θ is a monotone function on I. Suppose that γ is not convex. There is a point $\gamma(s_0) \in \gamma(I)$ such that L_{s_0} divide γ to $\gamma_1 \subset H_+$ and $\gamma_2 \subset H_-$. By the mean value theorem, there are two points $\gamma(s_1) \in \gamma_1$ and $\gamma(s_2) \in \gamma_2$ such that $\theta(s_1) = \theta(s_0)$ up to $\pm n\pi$, $L_{s_1} \neq L_{s_2}$, $\theta(s_2) = \theta(s_0)$ up to $\pm n\pi$ and $L_{s_2} \neq L_{s_0}, L_{s_1}$ for some integer n.

Two of the three points $\gamma(s_0)$, $\gamma(s_1)$ and $\gamma(s_2)$ must have tangents point in the same direction. Thus, there are two points $s_i < s_j$ such that $\Phi(\gamma, \nu)(s_i) = \Phi(\gamma, \nu)(s_j)$ and $\theta(s_i) = \theta(s_j) \pm 2n\pi$ for some integer n. Since θ is monotone and γ is simple closed, $n \in \{0, 1, -1\}$ by Lemma 2.5. If $n = 0$, then $\theta(s_i) = \theta(s_j)$, that is, θ is constant on the closed interval $[s_i, s_j]$. This contradict to $L_{s_i} \neq L_{s_j}$. If $n = \pm 1$, then θ is constant on the set $I \setminus [s_i, s_j]$. This contradict to $L_{s_i} \neq L_{s_j}$. Therefore, the frontal γ is convex. □

We prove the main theorem as follows.

Proof of Theorem 1.6 By combining Lemmas 2.1 and 2.6, we obtain Theorem 1.6. □

3 Examples

We show some examples of simple closed frontals and discuss on convexity of them.

Example 3.1 Consider a closed Legendre curve $(\gamma, \nu) : [0, 2\pi] \to \mathbb{R}^2 \times S^1$ defined by

$$\gamma(t) = (\cos^3 t, \sin^3 t), \nu(t) = (\sin t, \cos t).$$

Then the frontal γ is simple closed. Since $\boldsymbol{\mu}(t) = (-\cos t, \sin t)$, the curvature of the Legendre curve is given by $(\ell(t), \beta(t)) = (-1, 3\cos t \sin t)$. The sign of β change around $t = 0, \pi/2, \pi, 3\pi/2$ and 2π. By Theorem 1.6, γ is not a convex frontal, see Figure 1.

Example 3.2 Consider a closed Legendre curve $(\gamma, \nu) : [0, 2\pi] \to \mathbb{R}^2 \times S^1$ defined by

$$\gamma(t) = \left(\frac{1}{3}\cos^3 t, \sin t - \frac{1}{3}\sin^3 t\right), \nu(t) = (\cos t, \sin t).$$

Then the frontal γ is simple closed. Since $\boldsymbol{\mu}(t) = (-\sin t, \cos t)$, the curvature of the Legendre curve is given by $(\ell(t), \beta(t)) = (1, \cos^2 t)$. By Theorem 1.6, γ is a convex frontal, see Figure 2 left.

Example 3.3 Consider a closed Legendre curve $(\gamma, \nu) : [0, 2\pi] \to \mathbb{R}^2 \times S^1$ defined by

$$\gamma(t) = \left(\frac{1}{3}\cos^3 t - \frac{1}{5}\cos^5 t, \frac{1}{3}\sin^3 t - \frac{1}{5}\sin^5 t\right), \nu(t) = (\cos t, \sin t).$$

Then the frontal γ is simple closed. Since $\boldsymbol{\mu}(t) = (-\sin t, \cos t)$, the curvature of the Legendre curve is given by $(\ell(t), \beta(t)) = (1, \cos^2 t \sin^2 t)$. By Theorem 1.6, γ is a convex frontal, see Figure 2 center.

Example 3.4 Consider a closed Legendre curve $(\gamma, \nu) : [0, \pi] \to \mathbb{R}^2 \times S^1$ defined by

$$\begin{aligned}\gamma(t) &= \left(\cos t \sin^3 t, \sin^8 t\right), \\ \nu(t) &= \frac{1}{\sqrt{(8\sin^5 t\cos t)^2 + (3 - 4\sin^2 t)^2}}(8\sin^5 t \cos t, 4\sin^2 t - 3).\end{aligned}$$

Then the frontal γ is simple closed. Since

$$\boldsymbol{\mu}(t) = \frac{1}{\sqrt{(8\sin^5 t\cos t)^2 + (3 - 4\sin^2 t)^2}}(3 - 4\sin^2 t, 8\sin^5 t\cos t),$$

the curvature of the Legendre curve is given by

$$\begin{aligned}\ell(t) &= \frac{8\sin^4 t(16\sin^4 t - 30\sin^2 t + 15)}{(8\sin^5 t\cos t)^2 + (3 - 4\sin^2 t)^2}, \\ \beta(t) &= -\sin^2 t\sqrt{(8\sin^5 t\cos t)^2 + (3 - 4\sin^2 t)^2}.\end{aligned}$$

Note that $16\sin^4 t - 30\sin^2 t + 15 = (4\sin^2 t - 15/4)^2 + 15/16 > 0$. By Theorem 1.6, γ is a convex frontal, see Figure 2 right.

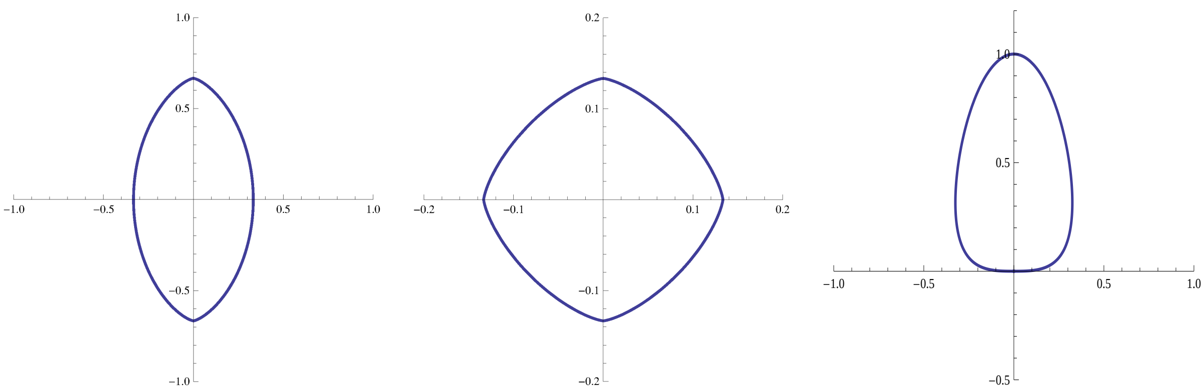

Figure 2: Convex simple closed frontals.

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