Remarks on numerical experiments of Allen–Cahn equations with constraint via Yosida approximation†

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Abstract. We consider a one-dimensional Allen–Cahn equation with constraint from the view-point of numerical analysis. Our constraint is the subdifferential of the indicator function on the closed interval, which is the multivalued function. Therefore, it is very difficult to make numerical experiments of our equation. In this paper we approximate our constraint by Yosida approximation. Then, we study the approximating system of our original model numerically. In particular, we give the criteria for the standard forward Euler method to give stable numerical experiments of our approximating equation. Moreover, we give some numerical experiments of approximating equation.

Key Words. Allen–Cahn equation, constraint, subdifferential, Yosida approximation, singular limit, numerical experiments.

1 Introduction

In this paper, for each $\varepsilon \in (0, 1]$ we consider the following Allen–Cahn equation with constraint from the view-point of numerical analysis:

\begin{align}
  u^\varepsilon_t - u^\varepsilon_{xx} + \frac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni \frac{u^\varepsilon}{\varepsilon^2} & \quad \text{in } Q := (0, T) \times (0, 1), \\
  u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0, & \quad t \in (0, T), \\
  u^\varepsilon(0, x) = u_0^\varepsilon(x), & \quad x \in (0, 1),
\end{align}

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where $0 < T < +\infty$ and $u_0$ is a given initial data. Also, $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1,1]$ defined by

$$I_{[-1,1]}(z) := \begin{cases} 
0, & \text{if } z \in [-1,1], \\
+\infty, & \text{otherwise}.
\end{cases} \quad (1.4)$$

More precisely, $\partial I_{[-1,1]}(\cdot)$ is a set-valued mapping defined by

$$\partial I_{[-1,1]}(z) := \begin{cases} 
\emptyset, & \text{if } z < -1 \text{ or } z > 1, \\
[0, \infty), & \text{if } z = 1, \\
\{0\}, & \text{if } -1 < z < 1 \\
(-\infty, 0], & \text{if } z = -1.
\end{cases} \quad (1.5)$$

The Allen–Cahn equation was proposed to describe the macroscopic motion of phase boundaries. In the physical context, the function $u^\varepsilon = u^\varepsilon(t, x)$ in $(P)^\varepsilon := \{1.1, 1.2, 1.3\}$ is the nonconserved order parameter that characterizes the physical structure. For instance, let $v = v(t, x)$ be the local ratio of the volume of pure liquid relative to that of pure solid at time $t$ and position $x \in (0, 1)$, defined by

$$v(t, x) := \lim_{\epsilon \to 0} \frac{\text{the volume of pure liquid in } B_r(x) \text{ at time } t}{|B_r(x)|},$$

where $B_r(x)$ is the ball in $\mathbb{R}$ with center $x$ and radius $r$ and $|B_r(x)|$ denotes its volume. Put $u^\varepsilon(t, x) := 2v(t, x) - 1$ for any $(t, x) \in Q$. Then, we easily see that $u^\varepsilon(t, x)$ is the nonconserved order parameter that characterizes the physical structure:

$$\begin{cases} 
u(t, x) = 1 & \text{on the pure liquid region}, \\
\nu(t, x) = -1 & \text{on the pure solid region}, \\
-1 < \nu(t, x) < 1 & \text{on the mixture region.}
\end{cases}$$

There are vast literatures of Allen–Cahn equation with or without constraint $\partial I_{[-1,1]}(\cdot)$. For such works, we refer to [1, 3, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 22, 23], for instance. In particular, Chen and Elliott [8] considered the singular limit of $(P)^\varepsilon$ as $\varepsilon \to 0$ in the general bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 1$.

Note that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to make numerical experiments of $(P)^\varepsilon$. Recently, Farshbaf-Shaker et. al [11] gave the results of the limit of a solution $u^\varepsilon$ and an element of $\partial I_{[-1,1]}(u^\varepsilon)$, called the Lagrange multiplier, to $(P)^\varepsilon$ as $\varepsilon \to 0$. Moreover, Farshbaf-Shaker et. al [12] gave numerical experiments to $(P)^\varepsilon$ via the Lagrange multiplier in one dimension of space for sufficient small $\varepsilon \in (0, 1]$. Also, they considered some approximating method. In fact, for $\delta > 0$, they use the following Yosida approximation $(\partial I_{[-1,1]})_\delta(\cdot)$ of $\partial I_{[-1,1]}(\cdot)$ defined by:

$$(\partial I_{[-1,1]})_\delta(\cdot) := \begin{cases} 
\frac{z - 1}{\delta} - \frac{1 - z}{\delta}, & \forall z \in \mathbb{R},
\end{cases} \quad (1.6)$$
where \([z]^+\) is the positive part of \(z\). For each \(\delta > 0\), they considered the following approximation problem of \((P)^\varepsilon\) :

\[
(P)^\varepsilon_{\delta}
\begin{cases}
(u^\varepsilon_0)_t - (u^\varepsilon_0)_{xx} + \frac{1}{\varepsilon^2} \partial I_{[-1,1]}(u^\varepsilon_0) = \frac{u^\varepsilon_0}{\varepsilon^2} & \text{in } Q := (0, T) \times (0, 1), \\
(u^\varepsilon_0)_x(t, 0) = (u^\varepsilon_0)_x(t, 1) = 0, & t \in (0, T), \\
u^\varepsilon_0(0, x) = u^\delta_0(x), & x \in (0, 1).
\end{cases}
\]

Then, they gave the following numerical result to \((P)^\varepsilon_{\delta}\) by the standard explicit finite difference scheme to \((P)^\varepsilon_{\delta}\) (see [12, Remark 5.3]):

Figure 1: Behaviour of a solution to \((P)^\varepsilon_{\delta}\) with \(\varepsilon = 0.007\) and \(\delta = 0.01\).

From Figure 1, we easily see that we have to choose the suitable constants \(\varepsilon\), \(\delta\) and mesh size of time \(\Delta t\) and space \(\Delta x\) in order to get stable numerical results of \((P)^\varepsilon_{\delta}\). So, in this paper, for each \(\varepsilon > 0\) and \(\delta > 0\), we give the criteria for the standard explicit finite difference scheme to give stable numerical experiments of \((P)^\varepsilon_{\delta}\). To this end, we first consider the following ODE problem, denoted by \((E)^\varepsilon_{\delta}\) :

\[
(E)^\varepsilon_{\delta}
\begin{cases}
(u^\varepsilon_0)_t + \frac{1}{\varepsilon^2} \partial I_{[-1,1]}(u^\varepsilon_0) = \frac{u^\delta_0}{\varepsilon^2} & \text{in } \mathbb{R}, \text{ for } t \in (0, T), \\
u^\varepsilon_0(0) = u^\delta_0 & \text{in } \mathbb{R}.
\end{cases}
\]

Then, we give the criteria to get stable numerical experiments of \((E)^\varepsilon_{\delta}\). Also, we give some numerical experiments of \((E)^\varepsilon_{\delta}\). Moreover, we show the criteria to get stable numerical experiments of PDE problem \((P)^\varepsilon_{\delta}\). Therefore, the main novelties found in this paper are the following:

(a) We give the criteria to give stable numerical experiments of the ODE problem \((E)^\varepsilon_{\delta}\). Also, we give numerical experiments to \((E)^\varepsilon_{\delta}\) for sufficient small \(\varepsilon \in (0, 1]\).
We give the criteria to give stable numerical experiments of the PDE problem \((P)_{\delta}\). Also, we give numerical experiments to \((P)_{\delta}\) for sufficient small \(\varepsilon \in (0, 1]\).

The plan of this paper is as follows. In Section 2, we recall the solvability and convergence result of \((E)_{\delta}\). In Section 3, we consider \((E)_{\delta}\) numerically. Then, we prove the main result (Theorem 3.1) corresponding to the item (a) listed above. Also, we give numerical experiments to \((E)_{\delta}\) for sufficient small \(\varepsilon \in (0, 1]\) and \(\delta \in (0, 1]\). In Section 4, we recall the solvability and convergence result of \((P)_{\delta}\). In the final Section 5, we consider \((P)_{\delta}\) from the view-point of numerical analysis. Then, we prove the main result (Theorem 5.1) corresponding to the item (b) listed above. Also, we give numerical experiments to \((P)_{\delta}\) for sufficient small \(\varepsilon \in (0, 1]\) and \(\delta \in (0, 1]\).

Notations and basic assumptions
Throughout this paper, we put \(H := L^2(0, 1)\) with usual real Hilbert space structure, and denote by \((\cdot, \cdot)_H\) the inner product in \(H\). Also, we put \(V := H^1(0, 1)\) with the usual norm 
\[
|z|_V := \left\{ \frac{|z|^2_H + |z_x|^2_H}{2} \right\}^{\frac{1}{2}}, \quad z \in V.
\]

In Sections 2 and 4, we use some techniques of proper (that is, not identically equal to infinity), l.s.c. (lower semi-continuous), convex functions and their subdifferentials, which are useful in the systematic study of variational inequalities. So, let us outline some notations and definitions. Let \(W\) be the real Hilbert space with the inner product \((\cdot, \cdot)_W\). For a proper, l.s.c. and convex function \(\psi : W \to \mathbb{R} \cup \{+\infty\}\), the effective domain \(D(\psi)\) is defined by
\[
D(\psi) := \{ z \in W; \ \psi(z) < \infty \}.
\]
The subdifferential of \(\psi\) is a possibly multi-valued operator in \(W\) and is defined by \(z^* \in \partial \psi(z)\) if and only if
\[
z \in D(\psi) \quad \text{and} \quad (z^*, y - z)_W \leq \psi(y) - \psi(z) \quad \text{for all} \ y \in W.
\]
For various properties and related notions of the proper, l.s.c., convex function \(\psi\) and its subdifferential \(\partial \psi\), we refer to a monograph by Brézis [4].

Finally, throughout this paper, \(C_i = C_i(\cdot), i = 1, 2, 3, \ldots\), denotes positive (or nonnegative) constants depending only on its arguments.

2 Solvability and convergence results of \((E)_{\delta}^{\varepsilon}\)
We begin by giving the rigorous definition of solutions to our problem \((E)_{\delta}^{\varepsilon}\) \((\varepsilon \in (0, 1]\) and \(\delta \in (0, 1]\)).

**Definition 2.1.** Let \(\varepsilon \in (0, 1], \ \delta \in (0, 1] \) and \(u_0^\varepsilon \in \mathbb{R}\). Then, a function \(u_\delta^\varepsilon : [0, T] \to \mathbb{R}\) is called a solution to \((E)_{\delta}^{\varepsilon}\) on \([0, T]\), if the following conditions are satisfied:

1. \(u_\delta^\varepsilon \in W^{1,2}(0, T)\).
The following equation holds:

\[(u_\delta^\varepsilon)_t + \frac{(\partial I_{[-1,1]})_\delta(u_\delta^\varepsilon)}{\varepsilon^2} = \frac{u_\delta^\varepsilon}{\varepsilon^2} \text{ in } \mathbb{R}, \text{ for } t \in (0,T).\]

(iii) \(u_\delta^\varepsilon(0) = u_0^\varepsilon \text{ in } \mathbb{R}.\)

Now, let us recall the solvability result of \((E)_\delta^\varepsilon\) on \([0,T]\).

**Proposition 2.1.** Let \(\varepsilon \in (0,1], \delta \in (0,1]\) and \(u_0^\varepsilon \in \mathbb{R}\) with \(|u_0^\varepsilon| \leq 1\). Then, there exists a unique solution \(u_\delta^\varepsilon\) to \((E)_\delta^\varepsilon\) on \([0,T]\) in the sense of Definition 2.1.

**Proof.** We easily prove the uniqueness of solutions to \((E)_\delta^\varepsilon\) on \([0,T]\) by the quite standard arguments: monotonicity and Gronwall’s inequality.

Also, we can show the existence of solutions to \((E)_\delta^\varepsilon\) on \([0,T]\) applying the abstract theory of evolution equations governed by subdifferentials. In fact, we define a function \((I_{[-1,1]})_\delta(\cdot)\) on \(\mathbb{R}\) by

\[(I_{[-1,1]})_\delta(z) = \frac{|z-1|^2 + |-1-z|^2}{2\delta}, \quad \forall z \in \mathbb{R}.
\]

Clearly, \((I_{[-1,1]})_\delta(\cdot)\) is proper, l.s.c. and convex on \(\mathbb{R}\) with \(\partial(I_{[-1,1]})_\delta(\cdot) = (\partial I_{[-1,1]})_\delta(\cdot)\) in \(\mathbb{R}\), where \((\partial I_{[-1,1]})_\delta(\cdot)\) is a Yosida approximation of \(\partial I_{[-1,1]}(\cdot)\) defined by (1.6).

We easily see that the problem \((E)_\delta^\varepsilon\) can be rewritten as in an abstract framework of the form:

\[
(EP)_\delta^\varepsilon \begin{cases} 
\frac{d}{dt} u_\delta^\varepsilon(t) + \frac{1}{\varepsilon^2} \partial(I_{[-1,1]})_\delta(u_\delta^\varepsilon(t)) - \frac{1}{\varepsilon^2} u_\delta^\varepsilon(t) = 0 \text{ in } \mathbb{R}, \text{ for } t \in (0,T), \\
u_\delta^\varepsilon(0) = u_0^\varepsilon \text{ in } \mathbb{R}. 
\end{cases}
\]

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [5, 14, 21]), we can show the existence of a solution \(u_\delta^\varepsilon\) to \((EP)_\delta^\varepsilon\) on \([0,T]\) for each \(\varepsilon \in (0,1]\) and \(\delta \in (0,1]\) in the sense of Definition 2.1. Thus, the proof of Proposition 2.1 has been completed.

Next, we recall the convergence result of \((E)_\delta^\varepsilon\) as \(\delta \to 0\). To this end, we recall a notion of convergence for convex functions, developed by Mosco [17].

**Definition 2.2** (cf. [17]). Let \(\psi, \psi_n (n \in \mathbb{N})\) be proper, l.s.c. and convex functions on a Hilbert space \(W\). Then, we say that \(\psi_n\) converges to \(\psi\) on \(W\) in the sense of Mosco [17] as \(n \to \infty\), if the following two conditions are satisfied:

(i) for any subsequence \(\{\psi_{n_k}\} \subset \{\psi_n\}\), if \(z_k \to z\) weakly in \(W\) as \(k \to \infty\), then

\[\liminf_{k \to \infty} \psi_{n_k}(z_k) \geq \psi(z);\]

(ii) for any \(z \in D(\psi)\), there is a sequence \(\{z_n\}\) in \(W\) such that

\[z_n \to z \text{ in } W \text{ as } n \to \infty \text{ and } \lim_{n \to \infty} \psi_n(z_n) = \psi(z).\]
It is well known that the following lemma holds. Therefore, we omit the detailed proof.

**Lemma 2.1** (cf. [2, Section 5], [4, Chapter 2], [15, Section 2]).

\[(I_{[-1,1]})_\delta(\cdot) \rightarrow I_{[-1,1]}(\cdot) \text{ on } \mathbb{R} \text{ in the sense of Mosco [17]} \quad (2.3)\]

as \(\delta \rightarrow 0\).

By Lemma 2.1 and the general convergence theory of evolution equations, we easily get the following result.

**Proposition 2.2** (cf. [2, Section 5], [15, Section 2]). Let \(u^\varepsilon \in W^{1,2}(0,T)\) such that

\[u^\varepsilon \rightarrow u^\varepsilon \text{ strongly in } C([0,T]) \text{ as } \delta \rightarrow 0\]

and \(u^\varepsilon\) is the unique solution of the following problem \((E)^\varepsilon\) on \([0,T]\):

\[
(E)^\varepsilon \left\{
\begin{array}{ll}
u_t^\varepsilon + \frac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni \frac{u^\varepsilon}{\varepsilon^2} & \text{in } \mathbb{R}, \text{ for } t \in (0,T), \\

u^\varepsilon(0) = u^\circ_0 & \text{in } \mathbb{R}.
\end{array}
\right.
\]

### 3 Stable criteria and numerical experiments for \((E)^\varepsilon_{\delta}\)

In this Section we consider \((E)^\varepsilon_{\delta}\) from the view-point of numerical analysis.

**Remark 3.1.** Note from Proposition 2.2 that \((E)^\varepsilon_{\delta}\) is the approximating problem of \((E)^\varepsilon\). Also note from (1.5) that the constraint \(\partial I_{[-1,1]}(\cdot)\) is the multivalued function. Therefore, it is very difficult to study \((E)^\varepsilon\) numerically.

In order to make numerical experiments of \((E)^\varepsilon_{\delta}\) via the standard forward Euler method, we consider the following explicit finite difference scheme to \((E)^\varepsilon_{\delta}\), denoted by \((DE)^\varepsilon_{\delta}\):

\[
(DE)^\varepsilon_{\delta} \left\{
\begin{array}{ll}
\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial I_{[-1,1]}(u^n)}{\varepsilon^2} \ni \frac{u^n}{\varepsilon^2} & \text{in } \mathbb{R}, \text{ for } n = 0, 1, 2, \cdots, N_t, \\
u^0 = u^\circ_0 & \text{in } \mathbb{R},
\end{array}
\right.
\]

where \(\Delta t\) is the mesh size of time and \(N_t\) is the integer part of number \(T/\Delta t\). We easily see that \(u^n\) is the approximating solution of \((E)^\varepsilon_{\delta}\) at the time \(t = n\Delta t\).

Clearly, the explicit finite difference scheme \((DE)^\varepsilon_{\delta}\) converges to \((E)^\varepsilon_{\delta}\) as \(\Delta t \rightarrow 0\) since \((DE)^\varepsilon_{\delta}\) is the standard time discretization scheme for \((E)^\varepsilon_{\delta}\).

Here, we give the unstable numerical experiment of \((DE)^\varepsilon_{\delta}\) in the case when \(T = 0.002, \varepsilon = 0.003, \delta = 0.01\), the initial data \(u^\circ_0 = 0.1\) and the mesh size of time \(\Delta t = 0.000001\):
From Figure 2, we easily see that we have to choose the suitable constants \( \varepsilon, \delta \) and mesh size of time \( \Delta t \) in order to get stable numerical results of \((\text{DE})_\delta^\varepsilon\).

Now, let us mention our first main result in this paper, which is concerned with the criteria to give stable numerical experiments of \((\text{DE})_\delta^\varepsilon\).

**Theorem 3.1.** Let \( \varepsilon \in (0, 1], \delta \in (0, 1) \) and \( \Delta t \in (0, 1] \). Assume \( u_0^\varepsilon \in (0, 1] \) (resp. \( u_0^\varepsilon \in [-1, 0) \)) and \( T = \infty \). Let \( \{u^n; n \geq 0\} \) be the solution to \((\text{DE})_\delta^\varepsilon\). Then, we have:

(i) If \( \Delta t \in (0, \delta^2/(1 - \delta)) \), \( u^n \) converges to \( 1/(1 - \delta) \) (resp. \( -1/(1 - \delta) \)) monotonically as \( n \to \infty \).

(ii) If \( \Delta t \in (\delta^2/(1 - \delta), 2\delta^2/(1 - \delta)) \), \( u^n \) oscillates and converges to \( 1/(1 - \delta) \) (resp. \( -1/(1 - \delta) \)) as \( n \to \infty \).

**Proof.** We give the proof of Theorem 3.1 in the case of the initial value \( u_0^\varepsilon \in (0, 1] \).

For simplicity, we set:

\[
 f_\delta(z) := (\partial I_{[-1, 1]}^\varepsilon)_\delta(z) - z \quad \text{for } z \in \mathbb{R}. \tag{3.1}
\]

Then, we easily observe that

\[
 f_\delta(z) = \begin{cases} 
 \frac{1+z}{\delta} - z, & \text{if } z \leq -1, \\
 -z, & \text{if } z \in [-1, 1], \\
 \frac{z-1}{\delta} - z, & \text{if } z \geq 1
\end{cases} \tag{3.2}
\]
and \( z = 0, 1/(1 - \delta), -1/(1 - \delta) \) are zero points of \( f_\delta(\cdot) \). Also, we observe that the difference equation of \((\text{DE})^*_\delta \) is reformulated in the following form:

\[
    u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) \quad \text{in } \mathbb{R}, \text{ for } n = 0, 1, 2, \cdots. \tag{3.3}
\]

Note from (3.2) and (3.3) that if \( u^n \in (0, 1] \) we have:

\[
    u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) = u^n - \frac{\Delta t}{\varepsilon^2} \cdot (-u^n) \geq u^n, \tag{3.4}
\]

which implies that \( u^n \) is increasing with respect to \( n \) until \( u^{n+1} \geq 1 \).

Now, we prove (i). To this end, we assume that \( \Delta t \in (0, \delta \varepsilon^2/(1 - \delta)) \). At first, by the mathematical induction, we show:

\[
    u^i \leq \left(0, \frac{1}{1 - \delta}\right) \quad \text{for all } i \geq 0. \tag{3.5}
\]

Clearly (3.5) holds for \( i = 0 \) because of \( u^0 = u_0^* \in (0, 1] \).

Now, we assume that (3.5) holds for all \( i = 0, 1, \cdots, n \). Suppose \( u^n \in (0, 1] \). Then, we infer from (3.4) that

\[
    u^{n+1} = \left(1 + \frac{\Delta t}{\varepsilon^2}\right) u^n \leq 1 + \frac{\Delta t}{\varepsilon^2} < 1 + \frac{\delta}{1 - \delta} = \frac{1}{1 - \delta}.
\]

Therefore, by (3.4) and the inequality as above, we observe that

\[
    u^{n+1} \in \left(0, \frac{1}{1 - \delta}\right), \quad \text{if } u^n \in (0, 1]. \tag{3.6}
\]

Next, if \( u^n \in [1, 1/(1 - \delta)) \), we observe from (3.2) and (3.3) that

\[
    u^n \leq u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) \\
    = u^n - \frac{\Delta t}{\varepsilon^2} \cdot \left( \frac{u^n - 1}{\delta} - u^n \right) \\
    = u^n + \frac{\Delta t}{\varepsilon^2} \cdot \frac{1}{\delta} - \frac{1}{\delta} u^n \\
    < u^n + \frac{1 - (1 - \delta) u^n}{1 - \delta} = \frac{1}{1 - \delta},
\]

which implies that

\[
    u^{n+1} \in \left[1, \frac{1}{1 - \delta}\right], \quad \text{if } u^n \in \left[1, \frac{1}{1 - \delta}\right]. \tag{3.7}
\]

From (3.6) and (3.7) we infer that (3.5) holds for \( i = n + 1 \). Therefore, we conclude from the mathematical induction that (3.5) holds.
Also, by (3.2) and (3.5) we observe that \( f_\delta(u^n) \leq 0 \) for all \( n \geq 0 \). Therefore, we observe from (3.3) that

\[
u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) \geq u^n \quad \text{for all } n \geq 0.
\]

Therefore, we infer from (3.5) and (3.8) that \( \{u^n; n \geq 0\} \) is a bounded and increasing sequence with respect to \( n \). Thus, there exist a subsequence \( \{n_k\} \) of \( \{n\} \) and a point \( u^\infty \in \mathbb{R} \) such that \( n_k \to \infty \) as \( k \to \infty \) and

\[
u^{n_k} \to u^\infty \text{ in } \mathbb{R} \text{ as } k \to \infty.
\]

By taking the limit in (3.3), we easily observe from the continuity of \( f_\delta(\cdot) \) that \( u^\infty = 1/(1-\delta) \), which is the zero point of \( f_\delta(\cdot) \). Hence, taking into account of the uniqueness of the limit point, the proof of (i) has been completed.

Next, we show (ii). To this end, we assume that \( \Delta t \in (\delta \varepsilon^2/(1-\delta), 2\delta \varepsilon^2/(1-\delta)) \).

Then, we can find the minimal number \( n_0 \in \mathbb{N} \) so that

\[
u^{n_0} \in \left(1, \frac{1+\delta}{1-\delta}\right) \quad \text{and} \quad u^i \in (0, 1] \text{ for all } i = 0, 1, \cdots, n_0 - 1.
\]

In fact, if \( u^i \in (0, 1] \) for all \( i = 0, 1, \cdots, k \), we observe from (3.4) that

\[
u^{k+1} = u^k - \frac{\Delta t}{\varepsilon^2} f_\delta(u^k) = \left(1 + \frac{\Delta t}{\varepsilon^2}\right) u^k
= \left(1 + \frac{\Delta t}{\varepsilon^2}\right)^2 u^{k-1}
= \cdots
= \left(1 + \frac{\Delta t}{\varepsilon^2}\right)^{k+1} u^0.
\]

Taking into account of (3.11), \( u_0 \in (0, 1] \) and

\[1 + \frac{\Delta t}{\varepsilon^2} > 1 + \frac{\delta}{1-\delta} > 1,
\]

we can find the minimal number \( n_0 \in \mathbb{N} \) so that

\[u^{n_0} > 1 \text{ and } u^i \in (0, 1] \text{ for all } i = 0, 1, \cdots, n_0 - 1.
\]

Also, by (3.4) we observe that

\[
u^{n_0} = u^{n_0-1} - \frac{\Delta t}{\varepsilon^2} f_\delta(u^{n_0-1}) = \left(1 + \frac{\Delta t}{\varepsilon^2}\right) u^{n_0-1} < \left(1 + \frac{2\delta}{1-\delta}\right) \cdot 1 = \frac{1+\delta}{1-\delta},
\]

thus, (3.10) holds.

To show (ii), we put

\[\Delta t := \frac{\delta \varepsilon^2}{1-\delta} \tau \quad \text{for some } \tau \in (1, 2).
\]
Then, we observe from (3.2) and (3.3) that
\[
u^{n_0+1} = u^{n_0} - \frac{\Delta t}{\varepsilon^2} f_\delta(u^{n_0}) = u^{n_0} + \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - (1 - \delta)u^{n_0}}{\delta} = (1 - \tau)u^{n_0} + \frac{\tau}{1 - \delta}.
\] (3.12)

From (3.12) it follows that
\[
u^{n_0+1} + (\tau - 1)u^{n_0} = \frac{1}{\tau - 1 - \delta},
\] (3.13)

Therefore, we observe from (3.13) and \(\tau \in (1, 2)\) that the zero point \(1/(1 - \delta)\) of \(f_\delta(\cdot)\) is in the interval between \(u^{n_0}\) and \(u^{n_0+1}\).

Also, by (3.12) we observe that
\[
u^{n_0+1} = (1 - \tau)u^{n_0} + \frac{\tau}{1 - \delta} > (1 - \tau) \cdot \frac{1 + \delta}{1 - \delta} + \frac{\tau}{1 - \delta} = \frac{1 + \delta - \tau \delta}{1 - \delta} \geq 1
\]
and
\[
u^{n_0+1} = (1 - \tau)u^{n_0} + \frac{\tau}{1 - \delta} < (1 - \tau) \cdot (1 + \frac{\tau}{1 - \delta}) = \frac{1 - \delta + \tau \delta}{1 - \delta} \leq \frac{1 + \delta}{1 - \delta},
\]
which implies that
\[
u^{n_0+1} \in \left(1, \frac{1 + \delta}{1 - \delta}\right).
\]

Therefore, by (3.10), (3.13) and by repeating the procedure as above, we observe that
\[
u^n \in \left(1, \frac{1 + \delta}{1 - \delta}\right) \quad \text{for all } n \geq n_0
\] (3.14)
and \(\nu^n\) oscillates around the zero point \(1/(1 - \delta)\) for all \(n \geq n_0\).

Also, we observe from (3.12) and (3.14) that
\[
\left|\nu^{n+1} - \frac{1}{1 - \delta}\right| = |1 - \tau| \left|\nu^n - \frac{1}{1 - \delta}\right| \quad \text{for all } n \geq n_0.
\] (3.15)

Therefore, by \(\tau \in (1, 2)\), (3.14) and (3.15), there exist a subsequence \(\{n_k\}\) of \(\{n\}\) such that \(\nu^{n_k}\) oscillates and converges to \(1/(1 - \delta)\) as \(k \to \infty\). Hence, taking into account of the uniqueness of the limit point, the proof of (ii) has been completed.

\begin{remark}
Assume \(\Delta t \in [2\delta \varepsilon^2/(1 - \delta), \infty)\) and put \(\Delta t := \delta \varepsilon^2 \tau/(1 - \delta)\) for some \(\tau \geq 2\). Then, we observe that
\[1 + \frac{\Delta t}{\varepsilon^2} > 1 + \frac{2\delta}{1 - \delta} > 1 \quad \text{and} \quad |1 - \tau| \geq 1.
\]

Therefore, we infer from the proof of Theorem 3.1 (cf. (3.11), (3.13), (3.15)) that the solution \(\nu^n\) to (DE)\(\delta\) oscillates as \(n \to \infty\), in general.
\end{remark}
Remark 3.3. By (3.3) we easily see that

\[ u^n \equiv 0 \quad \text{for all } n \geq 1, \quad \text{if } u_0 = 0, \]
\[ u^n = \frac{1}{1 - \delta} \quad \text{for all } n \geq 1, \quad \text{if } u_0 = \frac{1}{1 - \delta} \]

and
\[ u^n = -\frac{1}{1 - \delta} \quad \text{for all } n \geq 1, \quad \text{if } u_0 = -\frac{1}{1 - \delta}. \]

By (ii) of Theorem 3.1, we observe that \( u_n \) oscillates and converges to the zero point of \( f_\delta(\cdot) \) in the case when \( \Delta t \in (\delta \varepsilon^2/(1 - \delta), 2\delta \varepsilon^2/(1 - \delta)) \). However, in the case when \( \Delta t = 2\delta \varepsilon^2/(1 - \delta) \), we have the following special case that the solution to \( (DE)_\delta \) does not oscillate and coincides with the zero point of \( f_\delta(\cdot) \) after some finite number of iteration.

Corollary 3.1. Let \( \varepsilon \in (0, 1], \delta \in (0, 1), \Delta t = 2\delta \varepsilon^2/(1 - \delta) \) and \( n \in \mathbb{N} \). Assume \( u_0 := (1 - \delta)^{n-1}/(1 + \delta)^n \). Then, the solution to \( (DE)_\delta \) is given by

\[
\begin{align*}
   u^i = & \begin{cases}
       (1 - \delta)^{n-1-i} \big/ (1 + \delta)^n, & \text{if } i = 0, 1, \ldots, n - 1, \\
       \frac{1}{1 - \delta}, & \text{if } i \geq n.
   \end{cases}
\end{align*}
\] (3.16)

Proof. Note that \( u_0 := (1 - \delta)^{n-1}/(1 + \delta)^n \in (0, 1) \). Therefore, by (3.2) and (3.3) we observe that

\[ u^1 = u^0 - \frac{\Delta t}{\varepsilon^2} f_\delta(u^0) = u_0 - \frac{2\delta}{1 - \delta} (-u_0^\varepsilon) = \frac{1 + \delta}{1 - \delta} u_0^\varepsilon = \frac{(1 - \delta)^{n-2}}{(1 + \delta)^{n-1}}. \]

Similarly, we have:

\[ u^2 = u^1 - \frac{\Delta t}{\varepsilon^2} f_\delta(u^1) = \frac{1 + \delta}{1 - \delta} u^1 = \frac{(1 - \delta)^{n-3}}{(1 + \delta)^{n-2}}. \]

Repeating this procedure, we easily observe that the solution to \( (DE)_\delta \) is given by (3.16). \( \square \)

Taking into account of Theorem 3.1, we give numerical experiments of \( (DE)_\delta \) as follows. To this end, we use the following numerical data:

**Numerical data of \( (DE)_\delta \):**

- \( T = 0.002; \)
- \( \varepsilon = 0.01; \)
- \( \delta = 0.01; \)
- The initial data \( u_0^\varepsilon = 0.1. \)
Then, we easily observe that:
\[
\frac{1}{1 - \delta} = \frac{1}{1 - 0.01} = 1.010101010\ldots
\]
and
\[
\frac{\delta \varepsilon^2}{1 - \delta} = 0.0000010101010\ldots.
\]

3.1 The case when $\Delta t = 0.000001$

Now we consider the case when $\Delta t = 0.000001$. In this case, we have:
\[
\frac{\delta \varepsilon^2}{1 - \delta} = 0.0000010101010\ldots > \Delta t = 0.000001,
\]
which implies that (i) of Theorem 3.1 holds. Thus, we have the following stable numerical result of (DE)$_{\delta}$. Namely, the solution to (DE)$_{\delta}$ converges to the stationary solution $1/(1 - \delta) = 1/(1 - 0.01) = 1.010101010\ldots$.
Table 1: Numerical data: $\Delta t = 0.00001$.

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3.2 The case when $\Delta t = 0.000002$

Now we consider the case when $\Delta t = 0.000002$. In this case, we have:

$$\frac{\delta \varepsilon^2}{1 - \delta} = 0.0000010101010 \cdots < \Delta t = 0.000002 < \frac{2\delta \varepsilon^2}{1 - \delta},$$

which implies that (ii) of Theorem 3.1 holds. Thus, we have the following numerical result of (DE)$_\delta$ that the solution to (DE)$_\delta$ oscillates and converges to the stationary solution $1/(1 - \delta) = 1/(1 - 0.01) = 1.010101010 \cdots$.

![Figure 4](attachment:image.png)

Figure 4: $\frac{\delta \varepsilon^2}{1 - \delta} = 0.0000010101010 \cdots < \Delta t = 0.000002 < \frac{2\delta \varepsilon^2}{1 - \delta}.$
Table 2: Numerical data: $\Delta t = 0.000002$.

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3.3 The case when $\Delta t = 2 \frac{\delta \varepsilon^2}{1 - \delta}$

Now we consider the case when $\Delta t = 2\delta \varepsilon^2/(1 - \delta) = 0.0000020202020\cdots$. In this case, we observe Remark 3.2. In fact, we have the following numerical result of $(\text{DE})_\delta^\varepsilon$ that the solution to $(\text{DE})_\delta^\varepsilon$ oscillates.

![Graph](image)

Figure 5: $\Delta t = 2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020\cdots$. 
Table 3: Numerical data: $\Delta t = 2 \frac{\delta x^2}{1 - \delta} = 0.000020202020 \ldots$

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3.4 The case when $\Delta t = 0.000005$

Now we consider the case when $\Delta t = 0.000005$. In this case, we have:

$$2 \frac{\delta e^2}{1 - \delta} = 0.0000020202020\cdots < \Delta t = 0.000005.$$ 

Therefore, we observe Remark 3.2. In fact, we have the following numerical result of (DE)$_{\delta}$ that the solution to (DE)$_{\delta}$ oscillates.

![Figure 6: $2 \frac{\delta e^2}{1 - \delta} = 0.0000020202020\cdots < \Delta t = 0.000005.$](image)
Table 4: Numerical data: $\Delta t = 0.000005$.

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<td>51</td>
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3.5 The case when $\triangle t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$

Now we consider the case when $\triangle t = 15 \delta \varepsilon^2 / (1 - \delta)$. In this case, we observe Remark 3.2. In fact, we have the following numerical result of (DE)$_\delta^\varepsilon$ that the solution to (DE)$_\delta^\varepsilon$ oscillates between three zero points of $f_\delta(\cdot)$.

Figure 7: $\triangle t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$. 

---

This figure illustrates the oscillatory behavior of the solution for $\triangle t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$, highlighting the periodic nature of the solution as it fluctuates between zero points of $f_\delta(\cdot)$. The graph demonstrates the impact of the time step on the solution's dynamics, showcasing the interplay between the step size and the system's oscillatory behavior.
Table 5: Numerical data: $\Delta t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$.

<table>
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<tr>
<th>number of iterations $i$</th>
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<th>number of iterations $i$</th>
<th>the value of $u^i$</th>
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</table>
3.6 Numerical result of Corollary 3.1

In this subsection, we consider Corollary 3.1 numerically. To this end, we use the following initial data:

\[ u_0 := \frac{(1 - \delta)^5}{(1 + \delta)^6} = \frac{(1 - 0.01)^5}{(1 + 0.01)^6} = 0.8958756 \ldots \]

Then, we have the following numerical experiment of (DE) that Corollary 3.1 holds. Namely, we observe that (3.16) holds with \( n = 6 \):

Table 6: Numerical data: \( \Delta t = 2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020 \ldots \)

<table>
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<th>number of iterations ( i )</th>
<th>the value of ( u^i )</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>10</td>
<td>1.010101</td>
</tr>
</tbody>
</table>
3.7 Conclusion of ODE problem (DE)$_\delta$

By Theorem 3.1 and numerical experiments as above, we conclude that

(i) the mesh size of time $\Delta t$ must be smaller than $\delta\varepsilon^2/(1 - \delta)$ in order to get the
stable numerical experiments of (DE)$_\delta$.

(ii) we have the stable numerical experiments of (DE)$_\delta$ with the initial data $u_0^\delta := (1 - \delta)^{n-1}/(1 + \delta)^n$, even if the mesh size of time $\Delta t$ is equal to $2\delta\varepsilon^2/(1 - \delta)$.

4 Solvability and convergence results for (P)$_\delta^\varepsilon$

We begin by giving the rigorous definition of solutions to our PDE problem (P)$_\delta^\varepsilon$
($\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$).

Definition 4.1. Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u_0^\delta \in H$. Then, a function $u_\delta^\varepsilon : [0, T] \rightarrow H$ is called a solution to (P)$_\delta^\varepsilon$ on $[0, T]$, if the following conditions are satisfied:

(i) $u_\delta^\varepsilon \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$.

(ii) The following variational identity holds:

$$((u_\delta^\varepsilon)_t(t), z)_H + ((u_\delta^\varepsilon)_x(t), z_x)_H + \left(\frac{(\partial I_{[-1,1]})(u_\delta^\varepsilon(t))}{\varepsilon^2}, z\right)_H = \left(\frac{u_\delta^\varepsilon(t)}{\varepsilon^2}, z\right)_H$$

for all $z \in V$ and a.e. $t \in (0, T)$. 

Figure 8: $\Delta t = 2\frac{\delta\varepsilon^2}{1 - \delta} = 0.0000020202020 \cdots$. 

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Proposition 4.1. Let \( \varepsilon \in (0, 1) \) and \( \delta \in (0, 1) \). Assume the following condition:

\( (A) \) \( u_0^\varepsilon \in K := \{ z \in V : -1 \leq z(x) \leq 1 \text{ a.e. } x \in (0, 1) \} \).

Then, for each \( u_0^\varepsilon \in K \), there exists a unique solution \( u_0^\delta \) to \((P)^\varepsilon_\delta\) on \([0, T]\) in the sense of Definition 4.1.

Proof. By the same argument as in Proposition 2.1, we can show the existence-uniqueness of a solution \( u_0^\delta \) to \((P)^\varepsilon_\delta\) on \([0, T]\) for each \( \varepsilon \in (0, 1) \) and \( \delta \in (0, 1) \). In fact, we easily prove the uniqueness of solutions to \((P)^\varepsilon_\delta\) on \([0, T]\) by the quite standard arguments: monotonicity and Gronwall’s inequality.

Also, we can show the existence of solutions to \((P)^\varepsilon_\delta\) on \([0, T]\) applying the abstract theory of evolution equations governed by subdifferentials. In fact, we define a functional \( \varphi_\delta^\varepsilon \) on \( H \) by

\[
\varphi_\delta^\varepsilon(z) := \begin{cases} 
\frac{1}{2} \int_\Omega |z_x|^2 \, dx + \frac{1}{\varepsilon^2} \int_0^1 (I_{[-1,1]})(z(x)) \, dx, & \text{if } z \in V, \\
\infty, & \text{otherwise},
\end{cases}
\]

where \( (I_{[-1,1]})(\cdot) \) is the function defined in (2.1). Clearly, \( \varphi_\delta^\varepsilon \) is proper, l.s.c. and convex on \( H \) with the effective domain \( D(\varphi) = V \).

We easily see that the problem \((P)^\varepsilon_\delta\) can be rewritten as in an abstract framework of the form:

\[
(PP)^\varepsilon_\delta \quad \begin{cases} 
\frac{d}{dt} u_\delta^\varepsilon(t) + \partial \varphi_\delta^\varepsilon(u_\delta^\varepsilon(t)) - \frac{1}{\varepsilon^2} u_\delta^\varepsilon(t) = 0 \text{ in } H, & \text{for } t > 0, \\
u_\delta^\varepsilon(0) = u_0^\varepsilon \text{ in } H.
\end{cases}
\]

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [5, 14, 21]), we can show the existence of a solution \( u_\delta^\varepsilon \) to \((PP)^\varepsilon_\delta\), hence, \((P)^\varepsilon_\delta\), on \([0, T]\) for each \( \varepsilon \in (0, 1) \) and \( \delta \in (0, 1) \) in the sense of Definition 4.1. Thus, the proof of Proposition 4.1 has been completed.

Next, we recall the convergence result of \((P)^\varepsilon_\delta\) as \( \delta \to 0 \). Taking account of Lemma 2.1 (cf. (2.3)), we easily observe that the following lemma holds.

Lemma 4.1 (cf. [2, Section 5], [4, Chapter 2], [15, Section 2]). Let \( \varepsilon \in (0, 1) \), and define a functional \( \varphi^\varepsilon \) on \( H \) by

\[
\varphi^\varepsilon(z) := \begin{cases} 
\frac{1}{2} \int_\Omega |z_x|^2 \, dx + \frac{1}{\varepsilon^2} \int_0^1 I_{[-1,1]}(z(x)) \, dx, & \text{if } z \in V, \\
\infty, & \text{otherwise}.
\end{cases}
\]

Then, \( \varphi_\delta^\varepsilon(\cdot) \to \varphi^\varepsilon(\cdot) \) on \( H \) in the sense of Mosco [17] as \( \delta \to 0 \).
By Lemma 4.1 and the general convergence theory of evolution equations, we easily get the following result.

**Proposition 4.2** (cf. [2, Section 5], [15, Section 2]). Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u^0_0 \in K$. Also, let $u^\delta_0$ be the unique solution to $(P)^\delta_0$ on $[0, T]$. Then, $u^\delta_0$ converges to the unique function $u^\varepsilon$ to $(P)^\varepsilon$ on $[0, T]$ in the sense that

$$u^\delta_0 \rightharpoonup u^\varepsilon \quad \text{strongly in } C([0, T]; H) \quad \text{as} \quad \delta \to 0. \quad (4.2)$$

**Proof.** We easily observe that the problem $(P)^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$(PP)^\varepsilon \left\{ \begin{array}{ll}
\frac{d}{dt} u^\varepsilon(t) + \partial \varphi^\varepsilon(u^\varepsilon(t)) - \frac{1}{\varepsilon^2} u^\varepsilon(t) \ni 0 & \text{in } H, \quad \text{for } t > 0, \\
u^\varepsilon(0) = u^0_0 & \text{in } H.
\end{array} \right.$$ 

Therefore, by Lemma 4.1 and the abstract convergence theory of evolution equations (cf. [2, 15]), we observe that the solution $u^\delta_0$ to $(PP)^\delta_0$ converges to the unique solution $u^\varepsilon$ to $(PP)^\varepsilon$ on $[0, T]$ as $\delta \to 0$ in the sense of $(4.2)$. Note that $u^\varepsilon$ (resp. $u^\delta_0$) is the unique solution to $(P)^\varepsilon$ (resp. $(P)^\delta_0$) on $[0, T]$ (cf. Proposition 4.1). Thus, we conclude that Proposition 4.2 holds. \( \square \)

## 5 Stable criteria and numerical experiments for $(P)^\varepsilon$

In this Section we consider $(P)^\varepsilon$ from the view-point of numerical analysis.

**Remark 5.1.** Note from Proposition 4.2 that $(P)^\varepsilon$ is the approximating problem of $(P)^\varepsilon$. Also note from (1.5) that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to study $(P)^\varepsilon$ numerically.

In order to make numerical experiments of $(P)^\varepsilon$, we consider the following explicit finite difference scheme to $(PP)^\varepsilon$, denoted by of $(DP)^\varepsilon$:

$$(DP)^\varepsilon_0 \left\{ \begin{array}{l}
\Delta t u^n_k - u^n_{k-1} - 2u^n_k + u^n_{k+1} = \frac{(\partial I_{[-1,1]}(u^n_k))}{\varepsilon^2} \quad \text{for } n = 0, 1, 2, \ldots, N_t \text{ and } k = 1, 2, \ldots, N_x - 1,
\\
u^n_0 = u^n_0, \quad u^n_{N_x} = u^n_{N_x-1} \quad \text{for } n = 1, 2, \ldots, N_t,
\\
u^0_k = u^0_k(x_k) \quad \text{for } k = 0, 1, 2, \ldots, N_x,
\end{array} \right.$$ 

where $\Delta t$ is the mesh size of time, $\Delta x$ is the mesh size of space, $N_t$ is the integer part of number $T/\Delta t$, $N_x$ is the integer part of number $1/\Delta x$ and $x_k := k\Delta x$. We easily see that $u^n_k$ is the approximating solution of $(P)^\varepsilon_0$ at the time $t_n := n\Delta t$ and the position $x_k := k\Delta x$.

Clearly, the explicit finite difference scheme $(DP)^\varepsilon_0$ converges to $(P)^\varepsilon_0$ as $\Delta t \to 0$ and $\Delta x \to 0$.

From Figure 1, we easily see that we have to choose the suitable constants $\varepsilon$, $\delta$, the mesh size of time $\Delta t$ and the mesh size of space $\Delta x$ in order to get stable numerical results of $(DP)^\varepsilon_0$. Now, let us mention our second main result in this paper, which is concerned with the stability of $(DE)^\varepsilon_0$. 

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Theorem 5.1. Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$, $\Delta t \in (0, 1]$, $\Delta x \in (0, 1]$, $T > 0$ and $u_0^\varepsilon \in K$, where $K$ is the set of initial value defined in Proposition 4.1 (cf. (A)). Let $N_\varepsilon$ be the integer part of number $1/\Delta x$, and let $\{u_n^\varepsilon; n \geq 0, k = 0, 1, \cdots, N_\varepsilon\}$ be the solution to $(DP)^\varepsilon_\delta$. Also, let $c_0 \in (0, 1)$ and assume that

$$0 < \Delta t \leq \frac{c_0 \delta \varepsilon^2}{1 - \delta} \quad \text{and} \quad 0 \leq \frac{\Delta t}{(\Delta x)^2} \leq \frac{1 - c_0}{2}. \quad (5.1)$$

Then, we have:

(i) the solution to $(DP)^\varepsilon_\delta$ is bounded in the following sense:

$$\max_{0 \leq k \leq N_\varepsilon} |u_k^n| \leq \frac{1}{1 - \delta} \quad \text{for all } n \geq 0. \quad (5.2)$$

(ii) $\{u_n^\varepsilon; k = 0, 1, \cdots, N_\varepsilon\}$ does not oscillate with respect to $n \geq 0$.

Proof. We first show (i), i.e., (5.2) by the mathematical induction.

Clearly (5.2) holds for $n = 0$ because of $u_0^\varepsilon \in K$.

Now, we assume that

$$\max_{0 \leq k \leq N_\varepsilon} |u_k^n| \leq \frac{1}{1 - \delta} \quad \text{for all } i = 0, 1, \cdots, n. \quad (5.3)$$

We easily observe that the explicit finite difference problem $(DP)^\varepsilon_\delta$ can be reformulated as in the following form:

$$u_k^{n+1} = ru_{k-1}^n + ru_{k+1}^n + (1 - 2r)u_k^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n)$$

for all $n = 0, 1, 2, \cdots, N_\varepsilon$ and $k = 1, 2, \cdots, N_\varepsilon - 1$, \quad (5.4)

where we put $r := \Delta t/(\Delta x)^2$ and $f_\delta(\cdot)$ is the function defined by (3.2).

We easily observe from (5.1), (5.3) and (5.4) that

$$\frac{1}{1 - \delta} - u_k^{n+1} = r \left( \frac{1}{1 - \delta} - u_{k-1}^n \right) + r \left( \frac{1}{1 - \delta} - u_{k+1}^n \right) + (1 - 2r) \left( \frac{1}{1 - \delta} - u_k^n \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n)$$

$$\geq (1 - 2r) \left( \frac{1}{1 - \delta} - u_k^n \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n)$$

for all $k = 1, 2, \cdots, N_\varepsilon - 1$. \quad (5.5)

From (3.2), (5.1) and (5.3) we infer that the function $[-1/(1 - \delta), 1/(1 - \delta)] \ni z \to (1 - 2r) (1/(1 - \delta) - z) + \Delta t/\varepsilon^2 f_\delta(z)$ is non-negative and continuous. In fact, it follows from (3.2) that the function $[-1/(1 - \delta), 1] \ni z \to (1 - 2r) (1/(1 - \delta) - z) + \Delta t/\varepsilon^2 f_\delta(z)$
attains a minimum value at $z = 1$. Therefore, we observe from (3.2) and (5.1) that

$$(1 - 2r) \left( \frac{1}{1 - \delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z)$$

$$\geq (1 - 2r) \left( \frac{1}{1 - \delta} - 1 \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(1)$$

$$= (1 - 2r) \cdot \frac{\delta}{1 - \delta} - \frac{\Delta t}{\varepsilon^2}$$

$$\geq \frac{c_0 \delta}{1 - \delta} - \frac{\Delta t}{\varepsilon^2}$$

$$\geq 0 \quad \text{for all } z \in \left[ -\frac{1}{1 - \delta}, 1 \right]. \quad (5.6)$$

Also, for any $z \in [1, 1/(1 - \delta)]$, we observe from (3.2) that

$$(1 - 2r) \left( \frac{1}{1 - \delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z)$$

$$= (1 - 2r) \left( \frac{1}{1 - \delta} - z \right) + \frac{\Delta t}{\varepsilon^2} \cdot \left( \frac{z - 1}{\delta} - z \right)$$

$$= \left[ \frac{1 - \delta}{\delta \varepsilon^2} \Delta t - (1 - 2r) \right] z + (1 - 2r) \frac{\delta}{1 - \delta} - \frac{\Delta t}{\delta \varepsilon^2}. \quad (5.7)$$

Here we note from (5.1) that

$$\frac{1 - \delta}{\delta \varepsilon^2} \Delta t - (1 - 2r) \leq \frac{1 - \delta}{\delta \varepsilon^2} \Delta t - c_0 \leq 0.$$

Therefore, we infer from (5.7) that the function $z \mapsto (1 - 2r) \left( 1/(1 - \delta) - z \right) + \Delta t/\varepsilon^2 f_\delta(z)$ is non-increasing and attains a minimum value at $z = 1/(1 - \delta)$. Hence, we have:

$$(1 - 2r) \left( \frac{1}{1 - \delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z) \geq \frac{\Delta t}{\varepsilon^2} f_\delta \left( \frac{1}{1 - \delta} \right) = 0$$

for all $z \in \left[ 1, \frac{1}{1 - \delta} \right]$. \quad (5.8)

Thus, we observe from (5.6) and (5.8) that

$$(1 - 2r) \left( \frac{1}{1 - \delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z) \geq 0, \quad \forall z \in \left[ -\frac{1}{1 - \delta}, \frac{1}{1 - \delta} \right],$$

which implies from (5.3) and (5.5) that

$$\frac{1}{1 - \delta} - u^n_k \geq 0 \quad \text{for all } k = 1, 2, \ldots, N_x - 1. \quad (5.9)$$
Similarly, we observe from (5.1), (5.3) and (5.4) that

\[
u_{k+1} + \frac{1}{1-\delta} = r \left( u_{k-1} + \frac{1}{1-\delta} \right) + r \left( u_k + \frac{1}{1-\delta} \right) + (1-2r) \left( u_k + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(u_k) \\
\geq (1-2r) \left( u_k + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(u_k) \quad (5.10)
\]

for all \( k = 1, 2, \cdots, N_\delta - 1. \)

By the similar arguments as above, we observe that the function \([-1/(1-\delta), 1/(1-\delta)] \ni z \to (1-2r)(z + 1/(1-\delta)) - \Delta t/\varepsilon^2 f_\delta(z)\) is non-negative and continuous. In fact, it follows from (3.2) that the function \([-1, 1/(1-\delta)] \ni z \to (1-2r)(z + 1/(1-\delta)) - \Delta t/\varepsilon^2 f_\delta(z)\) attains a minimum value at \( z = -1. \) Therefore, we observe from (3.2) and (5.1) that

\[
(1-2r) \left( z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) \\
\geq (1-2r) \left( -1 + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(-1) \\
=(1-2r) \cdot \frac{\delta}{1-\delta} - \frac{\Delta t}{\varepsilon^2} \\
\geq \frac{c_0\delta}{1-\delta} - \frac{\Delta t}{\varepsilon^2} \\
\geq 0 \quad \text{for all } z \in \left[ -1, \frac{1}{1-\delta} \right]. \quad (5.11)
\]

Also, for any \( z \in [-1/(1-\delta), -1], \) we observe from (3.2) that

\[
(1-2r) \left( z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) \\
=(1-2r) \left( z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} \cdot \left( \frac{1+z}{\delta} - z \right) \\
=[(1-2r) - \frac{1-\delta}{\delta^2} \frac{\Delta t}{\varepsilon^2}] z + (1-2r) \frac{1}{1-\delta} - \frac{\Delta t}{\varepsilon^2}. \quad (5.12)
\]

Here we note from (5.1) that

\[
(1-2r) - \frac{1-\delta}{\delta^2} \frac{\Delta t}{\varepsilon^2} \geq c_0 - \frac{1-\delta}{\delta^2} \frac{\Delta t}{\varepsilon^2} \geq 0.
\]

Therefore, we infer from (5.12) that the function \([-1/(1-\delta), -1] \ni z \to (1-2r)(z + 1/(1-\delta)) - \Delta t/\varepsilon^2 f_\delta(z)\) is non-decreasing and attains a minimum value at \( z = -1/(1-\delta). \) Hence, we have:

\[
(1-2r) \left( z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) \geq - \frac{\Delta t}{\varepsilon^2} f_\delta \left( \frac{1}{1-\delta} \right) = 0 \quad \text{for all } z \in \left[ -1/(1-\delta), -1 \right]. \quad (5.13)
\]
Thus, we observe from (5.11) and (5.13) that

\[(1 - 2r) \left( z + \frac{1}{1 - \delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) \geq 0, \quad \forall z \in \left[ -\frac{1}{1 - \delta}, \frac{1}{1 - \delta} \right],\]

which implies from (5.3) and (5.10) that

\[u^{n+1}_k + \frac{1}{1 - \delta} \geq 0 \quad \text{for all } k = 1, 2, \ldots, N_x - 1. \quad (5.14)\]

Taking into account of Neumann boundary condition, namely, by \(u^{n+1}_0 = u^{n+1}_1\) and \(u^{n+1}_{N_x} = u^{n+1}_{N_x-1}\), we observe from (5.9) and (5.14) that

\[\max_{0 \leq k \leq N_x} |u^{n+1}_k| \leq \frac{1}{1 - \delta},\]

which implies that (5.3) holds for \(i = n + 1\). Therefore, we conclude from the mathematical induction that (5.2) holds. Hence, the proof of (i) of Theorem 5.1 has been completed.

Next, we show (ii) by the standard arguments. Namely, we reformulate (DP)\(\delta\) as in the following form:

\[
\begin{pmatrix}
    u^{n+1}_1 \\
    u^{n+1}_2 \\
    \vdots \\
    u^{n+1}_{N_x-1}
\end{pmatrix} = \begin{pmatrix}
    1 - 2r & r & r & r & \cdots & r & r \\
    r & 1 - 2r & r & r & \cdots & r & r \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    r & r & 1 - 2r & r & 1 - 2r \\
\end{pmatrix} \begin{pmatrix}
    u^n_1 \\
    u^n_2 \\
    \vdots \\
    u^n_{N_x-1}
\end{pmatrix} + r \begin{pmatrix}
    u^n_0 \\
    0 \\
    \vdots \\
    0 \\
    u^n_{N_x}
\end{pmatrix} + \begin{pmatrix}
    -\frac{\Delta t}{\varepsilon^2} f_\delta(u^n_1) \\
    -\frac{\Delta t}{\varepsilon^2} f_\delta(u^n_2) \\
    \vdots \\
    -\frac{\Delta t}{\varepsilon^2} f_\delta(u^n_{N_x-1})
\end{pmatrix}. \quad (5.15)
\]

Here by taking into account of Neumann boundary condition and initial condition, namely, by \(u^n_0 = u^n_1\) and \(u^n_{N_x} = u^n_{N_x-1}\) for all \(n \geq 0\), we observe that (5.15) is reformulated as in the following form:

\[
\mathbf{u}^{(n+1)} = A\mathbf{u}^{(n)} + \begin{pmatrix}
    ru^n_1 - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n_1) \\
    -\frac{\Delta t}{\varepsilon^2} f_\delta(u^n_2) \\
    \vdots \\
    r u^n_{N_x-1} - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n_{N_x-1})
\end{pmatrix}. \quad (5.16)
\]
where we put  

\[ u^{(n)} := \begin{pmatrix} u_1^{(n)} \\ u_2^{(n)} \\ \vdots \\ u_{N_x-1}^{(n)} \end{pmatrix}, \quad A = \begin{pmatrix} 1 - 2r & r & r \\ r & 1 - 2r & r \\ & \ddots & \ddots & \ddots \\ & & r & 1 - 2r & r \\ & & & r & 1 - 2r \end{pmatrix}. \]

Noting from (3.2) that

\[ f_{\delta}(u_k^n) = \begin{cases} \frac{1 - \delta}{\delta} u_k^n + \frac{1}{\delta} & \text{if } u_k^n \leq -1, \\
-\frac{u_k^n}{\delta} & \text{if } u_k^n \in [-1,1], \\
\frac{1 - \delta}{\delta} u_k^n - \frac{1}{\delta} & \text{if } u_k^n \geq 1. \end{cases} \]  

(5.17)

By putting

\[ b_k^n := \begin{cases} \frac{-1}{\delta} & \text{if } u_k^n \in [-1,1], \\
\frac{1 - \delta}{\delta} u_k^n + \frac{1}{\delta} & \text{if } u_k^n \leq -1, \\
\frac{1 - \delta}{\delta} u_k^n - \frac{1}{\delta} & \text{if } u_k^n \geq 1, \end{cases} \]

(5.18)

we observe from (5.17) and (5.18) that:

\[ \begin{pmatrix} ru_1^n - \frac{\Delta t}{\varepsilon^2} f_{\delta}(u_1^n) \\ -\frac{\Delta t}{\varepsilon^2} f_{\delta}(u_2^n) \\ \vdots \\ ru_{N_x-1}^n - \frac{\Delta t}{\varepsilon^2} f_{\delta}(u_{N_x-1}^n) \end{pmatrix} = \begin{pmatrix} r - \frac{\Delta t}{\varepsilon^2} b_1^n \\ -\frac{\Delta t}{\varepsilon^2} b_2^n \\ \vdots \\ r - \frac{\Delta t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N_x-1}^n \end{pmatrix} + \begin{pmatrix} -\frac{\Delta t}{\varepsilon^2} b_1^n \\ -\frac{\Delta t}{\varepsilon^2} b_2^n \\ \vdots \\ -\frac{\Delta t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix}. \]

By using the matrix as above, we observe that (5.15) can be rewritten as in the following form:

\[ u^{(n+1)} = Au^{(n)} + Bu^{(n)} + \tilde{b}^{(n)}, \quad (n \geq 0), \]  

(5.19)

where we put

\[ B := \begin{pmatrix} r - \frac{\Delta t}{\varepsilon^2} b_1^n \\ -\frac{\Delta t}{\varepsilon^2} b_2^n \\ \vdots \\ r - \frac{\Delta t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix}, \quad \tilde{b}^{(n)} := \begin{pmatrix} -\frac{\Delta t}{\varepsilon^2} b_1^n \\ -\frac{\Delta t}{\varepsilon^2} b_2^n \\ \vdots \\ -\frac{\Delta t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix}. \]
By the general theory, we observe that the eigenvalue \( \lambda_j \) of matrix \( A \) is given by:

\[
\lambda_j := 1 - 4r \sin^2 \left( \frac{j \pi}{2N_x} \right) \quad \text{for} \quad j = 1, 2, \cdots, N_x - 1, \tag{5.20}
\]

which implies that \( \lambda_1 \) (rest. \( \lambda_{N_x-1} \)) is the maximum (rest. minimum) eigenvalue of \( A \).

Now let \( \{ \bar{\lambda}_j; j = 1, 2, \cdots, N_x - 1 \} \) be the set of all eigenvalues of matrix \( \bar{A} := A + B \) such that

\[
\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{N_x-1}.
\]

Also, let \( \{ \lambda_j^B; j = 1, 2, \cdots, N_x - 1 \} \) be the set of all eigenvalues of \( B \) such that

\[
\lambda_1^B \geq \lambda_2^B \geq \cdots \geq \lambda_{N_x-1}^B.
\]

Then, by the abstract perturbation theory of matrix, we observe that:

\[
\lambda_{N_x-1} + \lambda_j^B \leq \bar{\lambda}_j \leq \lambda_1 + \lambda_j^B, \quad \forall j = 1, 2, \cdots, N_x - 1. \tag{5.21}
\]

Since \( B \) is a symmetric matrix, it follows from (5.1), (5.18) and (5.20) that

\[
\lambda_{N_x-1} + \lambda_j^B \geq 1 - 4r \sin^2 \left( \frac{(N_x - 1) \pi}{2N_x} \right) - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta \varepsilon^2}
\geq 1 - 4r - \frac{\Delta t(1 - \delta)}{\delta \varepsilon^2}
\geq 1 - 4 \cdot \frac{1 - c_0}{2} - c_0
= -1 + c_0
> -1 \quad \text{for all} \quad j = 1, 2, \cdots, N_x - 1.
\]

Thus, we conclude from (5.21) and the above inequality that

\[
\bar{\lambda}_j > -1 \quad \text{for all} \quad j = 1, 2, \cdots, N_x - 1. \tag{5.22}
\]

Next, we now assume \( \max_{1 \leq k \leq N_x - 1} |u_k^n| \leq 1 \). Then, we observe from (5.1) and (5.18) that

\[
1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n = 1 - 2 - \frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{\varepsilon^2}
\geq 1 - 2 \cdot \frac{1 - c_0}{2} + \frac{\Delta t}{\varepsilon^2}
> 0 \quad \text{for all} \quad k = 1, 2, \cdots, N_x - 1.
\]

Therefore, all components of \( A + B \) are positive. Hence, the sum of all components in \( k \)-th row of \( A + B \) is the following:

\[
\left| 1 - r - \frac{\Delta t}{\varepsilon^2} b_k^n \right| + |r| = 1 - r - \frac{\Delta t}{\varepsilon^2} b_k^n + r = 1 + \frac{\Delta t}{\varepsilon^2} > 1 \quad \text{for} \quad k = 1 \text{ or } N_x - 1
\]
and
\[ |r| + \left| 1 - 2r - \frac{\Delta t}{\varepsilon^2} b^n_k \right| + |r| = r + 1 - 2r - \frac{\Delta t}{\varepsilon^2} b^n_k + r = 1 + \frac{\Delta t}{\varepsilon^2} > 1 \tag{5.24} \]
for all \( k = 2, 3, \ldots, N_x - 2 \).

Therefore, we observe from (5.23)–(5.24) that \( \max_{1 \leq k \leq N_x - 1} |u^n_k| \) is increasing with respect to \( n \) in the case when \( \max_{1 \leq k \leq N_x - 1} |u^n_k| \leq 1 \).

However, if \( u^n_k \notin [-1, 1] \) for some \( k = 1, 2, \ldots, N_x - 1 \), it follows from (5.1) and (5.18) that
\[ 1 - 2r - \frac{\Delta t}{\varepsilon^2} b^n_k = 1 - 2 - \frac{\Delta t}{(\Delta x)^2} - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} \geq 1 - 2 \cdot \frac{1 - c_0}{2} - c_0 \]
\[ = 0. \]

Therefore, the sum of all components in \( k \)-th row of \( A + B \) is the following:
\[ \left| 1 - r - \frac{\Delta t}{\varepsilon^2} b^n_k \right| + |r| = 1 - r - \frac{\Delta t}{\varepsilon^2} b^n_k + r = 1 - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} < 1 \quad \text{if} \quad u^n_k \notin [-1, 1] \text{ for some } k = 1 \text{ or } N_x - 1 \] 
(5.25)

and
\[ |r| + \left| 1 - 2r - \frac{\Delta t}{\varepsilon^2} b^n_k \right| + |r| = r + 1 - 2r - \frac{\Delta t}{\varepsilon^2} b^n_k + r = 1 - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} < 1 \quad \text{if} \quad u^n_k \notin [-1, 1] \text{ for some } k = 2, 3, \ldots, N_x - 2. \] 
(5.26)

Although \( \max_{1 \leq k \leq N_x - 1} |u^n_k| \) is increasing with respect to \( n \) in the case when \( \max_{1 \leq k \leq N_x - 1} |u^n_k| \leq 1 \) (cf. (5.23)–(5.24)), we conclude from (5.22) and (5.25)–(5.26) that (ii) of Theorem 5.1 holds. Thus, the proof of Theorem 5.1 has been completed.

\[ \square \]

**Remark 5.2.** By (5.1) we get the suitable mesh size of space \( \Delta x \). In fact, for each \( \varepsilon \in (0, 1], \delta \in (0, 1) \) we take the constant \( \tilde{c}_0 \in (0, 1), \Delta t \in (0, 1], \Delta x \in (0, 1] \) such that
\[ \Delta t \leq \frac{\tilde{c}_0 \delta \varepsilon^2}{1 - \delta} \quad \text{and} \quad \frac{\Delta t}{(\Delta x)^2} = \frac{1 - \tilde{c}_0}{2}. \]

Then, we have:
\[ \Delta t = \frac{1 - \tilde{c}_0}{2} \cdot (\Delta x)^2 \leq \frac{\tilde{c}_0 \delta \varepsilon^2}{1 - \delta}. \]

Thus, we have the following condition of \( \Delta x \):
\[ 0 < \Delta x < \varepsilon \sqrt{\frac{2\tilde{c}_0 \delta}{(1 - \tilde{c}_0)(1 - \delta)}}. \]
Remark 5.3. We can take $c_0 = 0$ in (5.1) for the explicit finite difference scheme to the following usual heat equation with Neumann boundary condition:

\[
\begin{align*}
&u_t^\tau - u_{xx}^\tau = 0 \text{ in } Q := (0,T) \times (0,1), \\
&u_x^\tau(t,0) = u_x^\tau(t,1) = 0, \quad t \in (0,T), \\
&w^\tau(0,x) = u_0^\tau(x), \quad x \in (0,1).
\end{align*}
\]

Taking into account of Theorem 5.1, we give numerical experiments of (DP)$^\tau$ as follows. To this end, we use the following numerical data:

\begin{itemize}
  \item $T = 0.01$;
  \item $\delta = 0.01$;
  \item $\Delta x = 0.005$;
  \item $c_0 := 0.6$.
\end{itemize}

Also, we consider the following initial data $u_0^\tau(x)$ defined by

\[
u_0^\tau(x) = \begin{cases} 
-0.5, & \text{if } x \in [0.00, 0.25], \\
-0.5 \sin(2\pi x), & \text{if } x \in [0.25, 0.75], \\
0.5, & \text{if } x \in [0.75, 1.00].
\end{cases}
\]

(5.27)

Figure 9: The graph of initial data $u_0^\tau(x)$ defined by (5.27).

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5.1 The case when $\varepsilon = 0.05$ and $\Delta t = 0.000005$

Now, we consider the case when $\varepsilon = 0.05$ and $\Delta t = 0.000005$. In this case, we easily observe that:

$$\frac{\Delta t}{(\Delta x)^2} = \frac{0.000005}{(0.005)^2} = 0.2 = \frac{1-c_0}{2},$$

and

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} = \frac{0.6 \times 0.01 \times (0.05)^2}{1-0.01} = 0.000015151515\ldots.$$

Therefore, we have

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} > \Delta t,$$

which implies that the criteria condition (5.1) holds. Thus, we have the following stable numerical experiment of (DP)$\delta$:

![Figure 10: $\varepsilon = 0.05$, $\Delta t = 0.000005$, $\Delta x = 0.005$, $\delta = 0.01$.](image)

5.2 The case when $\varepsilon = 0.007$ and $\Delta t = 0.000005$

Now, we consider the case when $\varepsilon = 0.007$ and $\Delta t = 0.000005$. In this case, we easily observe that:

$$\frac{\Delta t}{(\Delta x)^2} = \frac{0.000005}{(0.005)^2} = 0.2 = \frac{1-c_0}{2},$$

and

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} = \frac{0.6 \times 0.01 \times (0.007)^2}{1-0.01} = 0.00000029696969\ldots.$$

Therefore, we have

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} < \Delta t,$$

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which implies that the criteria condition (5.1) does not hold. Therefore, we have the following unstable numerical experiment of (DP)\(^\varepsilon\):

![Figure 11](image-url)

Figure 11: \(\varepsilon = 0.007, \ \triangle t = 0.000005, \ \triangle x = 0.005, \ \delta = 0.01\).

### 5.3 The case when \(\varepsilon = 0.007\) and \(\triangle t = 0.000002\)

Now, we consider the case when \(\varepsilon = 0.007\) and \(\triangle t = 0.000002\). In this case, we have

\[
\frac{\triangle t}{(\triangle x)^2} = \frac{0.0000002}{(0.005)^2} = \frac{1}{125} \leq 0.2 = \frac{1 - c_0}{2}
\]

and

\[
\frac{c_0 \delta \varepsilon^2}{1 - \delta} = \frac{0.0000029696969 \cdots}{\delta} > \triangle t,
\]

which implies that the criteria condition (5.1) holds. Therefore, we have the following stable numerical experiment of (DP)\(^\varepsilon\):
Figure 12: \( \varepsilon = 0.007, \ \Delta t = 0.0000002, \ \Delta x = 0.005, \ \delta = 0.01 \).

Remark 5.4. We observe from Theorem 5.1 that in order to get stable numerical results of (DP)\( \delta \), we have to choose the suitable constants \( \varepsilon, \delta \) and the mesh size of time \( \Delta t \) and space \( \Delta x \). Therefore, if we make a numerical experiment of (P)\( \varepsilon \) for sufficient small \( \varepsilon \), we had better consider the original problem (P)\( \varepsilon \) by using a primal-dual active set method in [3], a Lagrange multiplier method in [12] and so on.

5.4 Conclusion of PDE problem (DP)\( \delta \)

By Theorem 5.1 and numerical experiments as above, we conclude that the mesh size of time \( \Delta t \) and space \( \Delta x \) must be satisfied

\[
0 < \Delta t \leq \frac{c_0 \delta^2}{1 - \delta}, \quad 0 \leq \frac{\Delta t}{(\Delta x)^2} \leq \frac{1 - c_0}{2}
\]

for some constant \( c_0 \in (0, 1) \),

in order to get the stable numerical experiments of (DP)\( \delta \). Also, by Theorems 3.1 and 5.1, we conclude that the value \( \delta \varepsilon^2/(1 - \delta) \) is very important to make numerical experiments of (DE)\( \delta \) and (DP)\( \delta \).

References


