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Remarks on numerical experiments of Allen–Cahn equations with constraint via Yosida approximation[†]

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Abstract. We consider a one-dimensional Allen–Cahn equation with constraint from the view-point of numerical analysis. Our constraint is the subdifferential of the indicator function on the closed interval, which is the multivalued function. Therefore, it is very difficult to make numerical experiments of our equation. In this paper we approximate our constraint by Yosida approximation. Then, we study the approximating system of our original model numerically. In particular, we give the criteria for the standard forward Euler method to give stable numerical experiments of our approximating equation. Moreover, we give some numerical experiments of approximating equation.

Key Words. Allen–Cahn equation, constraint, subdifferential, Yosida approximation, singular limit, numerical experiments.

1 Introduction

In this paper, for each $\varepsilon \in (0, 1]$ we consider the following Allen–Cahn equation with constraint from the view-point of numerical analysis:

$$u_t^\varepsilon - u_{xx}^\varepsilon + \frac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni \frac{u^\varepsilon}{\varepsilon^2} \quad \text{in } Q := (0, T) \times (0, 1), \quad (1.1)$$

$$u_x^\varepsilon(t, 0) = u_x^\varepsilon(t, 1) = 0, \quad t \in (0, T), \quad (1.2)$$

$$u^\varepsilon(0, x) = u_0^\varepsilon(x), \quad x \in (0, 1), \quad (1.3)$$

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where $0 < T < +\infty$ and u_0^ε is a given initial data. Also, $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1, 1]$ defined by

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z \in [-1, 1], \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.4)$$

More precisely, $\partial I_{[-1,1]}(\cdot)$ is a set-valued mapping defined by

$$\partial I_{[-1,1]}(z) := \begin{cases} \emptyset, & \text{if } z < -1 \text{ or } z > 1, \\ [0, \infty), & \text{if } z = 1, \\ \{0\}, & \text{if } -1 < z < 1 \\ (-\infty, 0], & \text{if } z = -1. \end{cases} \quad (1.5)$$

The Allen–Cahn equation was proposed to describe the macroscopic motion of phase boundaries. In the physical context, the function $u^\varepsilon = u^\varepsilon(t, x)$ in $(P)^\varepsilon := \{(1.1), (1.2), (1.3)\}$ is the nonconserved order parameter that characterizes the physical structure. For instance, let $v = v(t, x)$ be the local ratio of the volume of pure liquid relative to that of pure solid at time t and position $x \in (0, 1)$, defined by

$$v(t, x) := \lim_{r \downarrow 0} \frac{\text{the volume of pure liquid in } B_r(x) \text{ at time } t}{|B_r(x)|},$$

where $B_r(x)$ is the ball in \mathbb{R} with center x and radius r and $|B_r(x)|$ denotes its volume. Put $u^\varepsilon(t, x) := 2v(t, x) - 1$ for any $(t, x) \in Q$. Then, we easily see that $u^\varepsilon(t, x)$ is the nonconserved order parameter that characterizes the physical structure:

$$\begin{cases} u^\varepsilon(t, x) = 1 & \text{on the pure liquid region,} \\ u^\varepsilon(t, x) = -1 & \text{on the pure solid region,} \\ -1 < u^\varepsilon(t, x) < 1 & \text{on the mixture region.} \end{cases}$$

There are vast literatures of Allen–Cahn equation with or without constraint $\partial I_{[-1,1]}(\cdot)$. For such works, we refer to [1, 3, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 22, 23], for instance. In particular, Chen and Elliott [8] considered the singular limit of $(P)^\varepsilon$ as $\varepsilon \rightarrow 0$ in the general bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 1$.

Note that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to make numerical experiments of $(P)^\varepsilon$. Recently, Farshbaf-Shaker et. al [11] gave the results of the limit of a solution u^ε and an element of $\partial I_{[-1,1]}(u^\varepsilon)$, called the Lagrange multiplier, to $(P)^\varepsilon$ as $\varepsilon \rightarrow 0$. Moreover, Farshbaf-Shaker et. al [12] gave numerical experiments to $(P)^\varepsilon$ via the Lagrange multiplier in one dimension of space for sufficient small $\varepsilon \in (0, 1]$. Also, they considered some approximating method. In fact, for $\delta > 0$, they use the following Yosida approximation $(\partial I_{[-1,1]})_\delta(\cdot)$ of $\partial I_{[-1,1]}(\cdot)$ defined by:

$$(\partial I_{[-1,1]})_\delta(z) = \frac{[z - 1]^+ - [-1 - z]^+}{\delta}, \quad \forall z \in \mathbb{R}, \quad (1.6)$$

where $[z]^+$ is the positive part of z . For each $\delta > 0$, they considered the following approximation problem of $(P)^\varepsilon$:

$$(P)_\delta^\varepsilon \begin{cases} (u_\delta^\varepsilon)_t - (u_\delta^\varepsilon)_{xx} + \frac{(\partial I_{[-1,1]})_\delta(u_\delta^\varepsilon)}{\varepsilon^2} = \frac{u_\delta^\varepsilon}{\varepsilon^2} & \text{in } Q := (0, T) \times (0, 1), \\ (u_\delta^\varepsilon)_x(t, 0) = (u_\delta^\varepsilon)_x(t, 1) = 0, & t \in (0, T), \\ u_\delta^\varepsilon(0, x) = u_0^\varepsilon(x), & x \in (0, 1). \end{cases}$$

Then, they gave the following numerical result to $(P)_\delta^\varepsilon$ by the standard explicit finite difference scheme to $(P)_\delta^\varepsilon$ (see [12, Remark 5.3]):

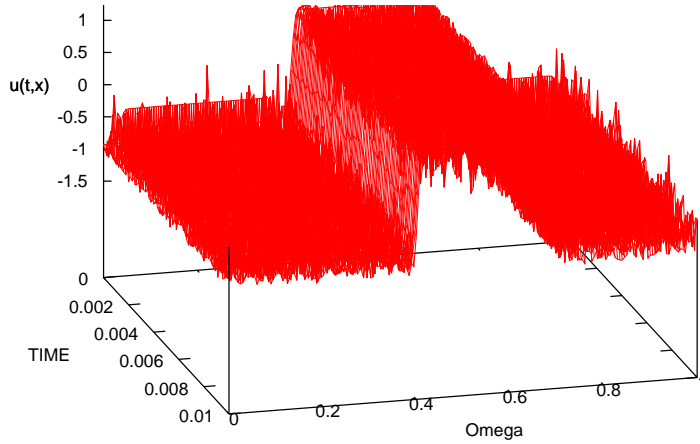


Figure 1: Behaviour of a solution to $(P)_\delta^\varepsilon$ with $\varepsilon = 0.007$ and $\delta = 0.01$.

From Figure 1, we easily see that we have to choose the suitable constants ε , δ and mesh size of time Δt and space Δx in order to get stable numerical results of $(P)_\delta^\varepsilon$. So, in this paper, for each $\varepsilon > 0$ and $\delta > 0$, we give the criteria for the standard explicit finite difference scheme to give stable numerical experiments of $(P)_\delta^\varepsilon$. To this end, we first consider the following ODE problem, denoted by $(E)_\delta^\varepsilon$:

$$(E)_\delta^\varepsilon \begin{cases} (u_\delta^\varepsilon)_t + \frac{(\partial I_{[-1,1]})_\delta(u_\delta^\varepsilon)}{\varepsilon^2} = \frac{u_\delta^\varepsilon}{\varepsilon^2} & \text{in } \mathbb{R}, \text{ for } t \in (0, T), \\ u_\delta^\varepsilon(0) = u_0^\varepsilon & \text{in } \mathbb{R}. \end{cases}$$

Then, we give the criteria to get stable numerical experiments of $(E)_\delta^\varepsilon$. Also, we give some numerical experiments of $(E)_\delta^\varepsilon$. Moreover, we show the criteria to get stable numerical experiments of PDE problem $(P)_\delta^\varepsilon$. Therefore, the main novelties found in this paper are the following:

- (a) We give the criteria to give stable numerical experiments of the ODE problem $(E)_\delta^\varepsilon$. Also, we give numerical experiments to $(E)_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0, 1]$.

- (b) We give the criteria to give stable numerical experiments of the PDE problem $(P)_\delta^\varepsilon$. Also, we give numerical experiments to $(P)_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0, 1]$.

The plan of this paper is as follows. In Section 2, we recall the solvability and convergence result of $(E)_\delta^\varepsilon$. In Section 3, we consider $(E)_\delta^\varepsilon$ numerically. Then, we prove the main result (Theorem 3.1) corresponding to the item (a) listed above. Also, we give numerical experiments to $(E)_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$. In Section 4, we recall the solvability and convergence result of $(P)_\delta^\varepsilon$. In the final Section 5, we consider $(P)_\delta^\varepsilon$ from the view-point of numerical analysis. Then, we prove the main result (Theorem 5.1) corresponding to the item (b) listed above. Also, we give numerical experiments to $(P)_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$.

Notations and basic assumptions

Throughout this paper, we put $H := L^2(0, 1)$ with usual real Hilbert space structure, and denote by $(\cdot, \cdot)_H$ the inner product in H . Also, we put $V := H^1(0, 1)$ with the usual norm

$$|z|_V := \{|z|_H^2 + |z_x|_H^2\}^{\frac{1}{2}}, \quad z \in V.$$

In Sections 2 and 4, we use some techniques of proper (that is, not identically equal to infinity), l.s.c. (lower semi-continuous), convex functions and their subdifferentials, which are useful in the systematic study of variational inequalities. So, let us outline some notations and definitions. Let W be the real Hilbert space with the inner product $(\cdot, \cdot)_W$. For a proper, l.s.c. and convex function $\psi : W \rightarrow \mathbb{R} \cup \{+\infty\}$, the effective domain $D(\psi)$ is defined by

$$D(\psi) := \{z \in W; \psi(z) < \infty\}.$$

The subdifferential of ψ is a possibly multi-valued operator in W and is defined by $z^* \in \partial\psi(z)$ if and only if

$$z \in D(\psi) \quad \text{and} \quad (z^*, y - z)_W \leq \psi(y) - \psi(z) \quad \text{for all } y \in W.$$

For various properties and related notions of the proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to a monograph by Brézis [4].

Finally, throughout this paper, $C_i = C_i(\cdot)$, $i = 1, 2, 3, \dots$, denotes positive (or nonnegative) constants depending only on its arguments.

2 Solvability and convergence results of $(E)_\delta^\varepsilon$

We begin by giving the rigorous definition of solutions to our problem $(E)_\delta^\varepsilon$ ($\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$).

Definition 2.1. *Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u_0^\varepsilon \in \mathbb{R}$. Then, a function $u_\delta^\varepsilon : [0, T] \rightarrow \mathbb{R}$ is called a solution to $(E)_\delta^\varepsilon$ on $[0, T]$, if the following conditions are satisfied:*

- (i) $u_\delta^\varepsilon \in W^{1,2}(0, T)$.

(ii) The following equation holds:

$$(u_\delta^\varepsilon)_t + \frac{(\partial I_{[-1,1]})_\delta(u_\delta^\varepsilon)}{\varepsilon^2} = \frac{u_\delta^\varepsilon}{\varepsilon^2} \quad \text{in } \mathbb{R}, \text{ for } t \in (0, T).$$

(iii) $u_\delta^\varepsilon(0) = u_0^\varepsilon$ in \mathbb{R} .

Now, let us recall the solvability result of $(E)_\delta^\varepsilon$ on $[0, T]$.

Proposition 2.1. *Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u_0^\varepsilon \in \mathbb{R}$ with $|u_0^\varepsilon| \leq 1$. Then, there exists a unique solution u_δ^ε to $(E)_\delta^\varepsilon$ on $[0, T]$ in the sense of Definition 2.1.*

Proof. We easily prove the uniqueness of solutions to $(E)_\delta^\varepsilon$ on $[0, T]$ by the quite standard arguments: monotonicity and Gronwall's inequality.

Also, we can show the existence of solutions to $(E)_\delta^\varepsilon$ on $[0, T]$ applying the abstract theory of evolution equations governed by subdifferentials. In fact, we define a function $(I_{[-1,1]})_\delta(\cdot)$ on \mathbb{R} by

$$(I_{[-1,1]})_\delta(z) = \frac{|[z-1]^+|^2 + |[-1-z]^+|^2}{2\delta}, \quad \forall z \in \mathbb{R}. \quad (2.1)$$

Clearly, $(I_{[-1,1]})_\delta(\cdot)$ is proper, l.s.c. and convex on \mathbb{R} with $\partial(I_{[-1,1]})_\delta(\cdot) = (\partial I_{[-1,1]})_\delta(\cdot)$ in \mathbb{R} , where $(\partial I_{[-1,1]})_\delta(\cdot)$ is a Yosida approximation of $\partial I_{[-1,1]}(\cdot)$ defined by (1.6).

We easily see that the problem $(E)_\delta^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$(EP)_\delta^\varepsilon \begin{cases} \frac{d}{dt} u_\delta^\varepsilon(t) + \frac{1}{\varepsilon^2} \partial(I_{[-1,1]})_\delta(u_\delta^\varepsilon(t)) - \frac{1}{\varepsilon^2} u_\delta^\varepsilon(t) = 0 & \text{in } \mathbb{R}, \text{ for } t \in (0, T), \\ u_\delta^\varepsilon(0) = u_0^\varepsilon & \text{in } \mathbb{R}. \end{cases} \quad (2.2)$$

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [5, 14, 21]), we can show the existence of a solution u_δ^ε to $(EP)_\delta^\varepsilon$ on $[0, T]$ for each $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$ in the sense of Definition 2.1. Thus, the proof of Proposition 2.1 has been completed. \square

Next, we recall the convergence result of $(E)_\delta^\varepsilon$ as $\delta \rightarrow 0$. To this end, we recall a notion of convergence for convex functions, developed by Mosco [17].

Definition 2.2 (cf. [17]). *Let ψ , ψ_n ($n \in \mathbb{N}$) be proper, l.s.c. and convex functions on a Hilbert space W . Then, we say that ψ_n converges to ψ on W in the sense of Mosco [17] as $n \rightarrow \infty$, if the following two conditions are satisfied:*

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \rightarrow z$ weakly in W as $k \rightarrow \infty$, then

$$\liminf_{k \rightarrow \infty} \psi_{n_k}(z_k) \geq \psi(z);$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in W such that

$$z_n \rightarrow z \text{ in } W \text{ as } n \rightarrow \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \psi_n(z_n) = \psi(z).$$

It is well known that the following lemma holds. Therefore, we omit the detailed proof.

Lemma 2.1 (cf. [2, Section 5], [4, Chapter 2], [15, Section 2]).

$$(I_{[-1,1]})_\delta(\cdot) \longrightarrow I_{[-1,1]}(\cdot) \quad \text{on } \mathbb{R} \text{ in the sense of Mosco [17]} \quad (2.3)$$

as $\delta \rightarrow 0$.

By Lemma 2.1 and the general convergence theory of evolution equations, we easily get the following result.

Proposition 2.2 (cf. [2, Section 5], [15, Section 2]). *Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u_0^\varepsilon \in \mathbb{R}$ with $|u_0^\varepsilon| \leq 1$. Also, let u_δ^ε be the unique solution to $(E)_\delta^\varepsilon$ on $[0, T]$. Then, there exists a unique function $u^\varepsilon \in W^{1,2}(0, T)$ such that*

$$u_\delta^\varepsilon \longrightarrow u^\varepsilon \quad \text{strongly in } C([0, T]) \quad \text{as } \delta \rightarrow 0$$

and u^ε is the unique solution of the following problem $(E)^\varepsilon$ on $[0, T]$:

$$(E)^\varepsilon \begin{cases} u_t^\varepsilon + \frac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni \frac{u^\varepsilon}{\varepsilon^2} & \text{in } \mathbb{R}, \text{ for } t \in (0, T), \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in } \mathbb{R}. \end{cases}$$

3 Stable criteria and numerical experiments for $(E)_\delta^\varepsilon$

In this Section we consider $(E)_\delta^\varepsilon$ from the view-point of numerical analysis.

Remark 3.1. *Note from Proposition 2.2 that $(E)_\delta^\varepsilon$ is the approximating problem of $(E)^\varepsilon$. Also note from (1.5) that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to study $(E)^\varepsilon$ numerically.*

In order to make numerical experiments of $(E)_\delta^\varepsilon$ via the standard forward Euler method, we consider the following explicit finite difference scheme to $(E)_\delta^\varepsilon$, denoted by of $(DE)_\delta^\varepsilon$:

$$(DE)_\delta^\varepsilon \begin{cases} \frac{u^{n+1} - u^n}{\Delta t} + \frac{(\partial I_{[-1,1]})_\delta(u^n)}{\varepsilon^2} = \frac{u^n}{\varepsilon^2} & \text{in } \mathbb{R}, \text{ for } n = 0, 1, 2, \dots, N_t, \\ u^0 = u_0^\varepsilon & \text{in } \mathbb{R}, \end{cases}$$

where Δt is the mesh size of time and N_t is the integer part of number $T/\Delta t$. We easily see that u^n is the approximating solution of $(E)_\delta^\varepsilon$ at the time $t = n\Delta t$

Clearly, the explicit finite difference scheme $(DE)_\delta^\varepsilon$ converges to $(E)_\delta^\varepsilon$ as $\Delta t \rightarrow 0$ since $(DE)_\delta^\varepsilon$ is the standard time discretization scheme for $(E)_\delta^\varepsilon$.

Here, we give the unstable numerical experiment of $(DE)_\delta^\varepsilon$ in the case when $T = 0.002$, $\varepsilon = 0.003$, $\delta = 0.01$, the initial data $u_0^\varepsilon = 0.1$ and the mesh size of time $\Delta t = 0.000001$:

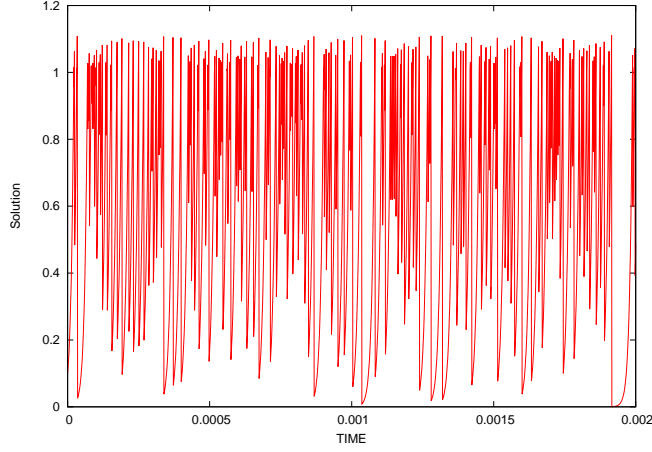


Figure 2: Behaviour of a solution u^n to $(DE)_\delta^\varepsilon$ with $\varepsilon = 0.003$ and $\delta = 0.01$.

From Figure 2, we easily see that we have to choose the suitable constants ε , δ and mesh size of time Δt in order to get stable numerical results of $(DE)_\delta^\varepsilon$.

Now, let us mention our first main result in this paper, which is concerned with the criteria to give stable numerical experiments of $(DE)_\delta^\varepsilon$.

Theorem 3.1. *Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$ and $\Delta t \in (0, 1]$. Assume $u_0^\varepsilon \in (0, 1]$ (resp. $u_0^\varepsilon \in [-1, 0)$) and $T = \infty$. Let $\{u^n; n \geq 0\}$ be the solution to $(DE)_\delta^\varepsilon$. Then, we have:*

- (i) *If $\Delta t \in (0, \delta\varepsilon^2/(1 - \delta))$, u^n converges to $1/(1 - \delta)$ (resp. $-1/(1 - \delta)$) monotonically as $n \rightarrow \infty$.*
- (ii) *If $\Delta t \in (\delta\varepsilon^2/(1 - \delta), 2\delta\varepsilon^2/(1 - \delta))$, u^n oscillates and converges to $1/(1 - \delta)$ (resp. $-1/(1 - \delta)$) as $n \rightarrow \infty$.*

Proof. We give the proof of Theorem 3.1 in the case of the initial value $u_0^\varepsilon \in (0, 1]$.

For simplicity, we set:

$$f_\delta(z) := (\partial I_{[-1,1]})_\delta(z) - z \quad \text{for } z \in \mathbb{R}. \quad (3.1)$$

Then, we easily observe that

$$f_\delta(z) = \begin{cases} \frac{1+z}{\delta} - z, & \text{if } z \leq -1, \\ -z, & \text{if } z \in [-1, 1], \\ \frac{z-1}{\delta} - z, & \text{if } z \geq 1 \end{cases} \quad (3.2)$$

and $z = 0, 1/(1 - \delta), -1/(1 - \delta)$ are zero points of $f_\delta(\cdot)$. Also, we observe that the difference equation of $(DE)_\delta^\varepsilon$ is reformulated in the following form:

$$u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) \quad \text{in } \mathbb{R}, \text{ for } n = 0, 1, 2, \dots. \quad (3.3)$$

Note from (3.2) and (3.3) that if $u^n \in (0, 1]$ we have:

$$u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) = u^n - \frac{\Delta t}{\varepsilon^2} \cdot (-u^n) \geq u^n, \quad (3.4)$$

which implies that u^n is increasing with respect to n until $u^{n+1} \geq 1$.

Now, we prove (i). To this end, we assume that $\Delta t \in (0, \delta\varepsilon^2/(1 - \delta))$. At first, by the mathematical induction, we show:

$$u^i \in \left(0, \frac{1}{1 - \delta}\right) \quad \text{for all } i \geq 0. \quad (3.5)$$

Clearly (3.5) holds for $i = 0$ because of $u^0 = u_0^\varepsilon \in (0, 1]$.

Now, we assume that (3.5) holds for all $i = 0, 1, \dots, n$. Suppose $u^n \in (0, 1]$. Then, we infer from (3.4) that

$$u^{n+1} = \left(1 + \frac{\Delta t}{\varepsilon^2}\right) u^n \leq 1 + \frac{\Delta t}{\varepsilon^2} < 1 + \frac{\delta}{1 - \delta} = \frac{1}{1 - \delta}.$$

Therefore, by (3.4) and the inequality as above, we observe that

$$u^{n+1} \in \left(0, \frac{1}{1 - \delta}\right), \quad \text{if } u^n \in (0, 1]. \quad (3.6)$$

Next, if $u^n \in [1, 1/(1 - \delta))$, we observe from (3.2) and (3.3) that

$$\begin{aligned} u^n \leq u^{n+1} &= u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) \\ &= u^n - \frac{\Delta t}{\varepsilon^2} \cdot \left(\frac{u^n - 1}{\delta} - u^n\right) \\ &= u^n + \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - (1 - \delta)u^n}{\delta} \\ &< u^n + \frac{1 - (1 - \delta)u^n}{1 - \delta} = \frac{1}{1 - \delta}, \end{aligned}$$

which implies that

$$u^{n+1} \in \left[1, \frac{1}{1 - \delta}\right), \quad \text{if } u^n \in \left[1, \frac{1}{1 - \delta}\right). \quad (3.7)$$

From (3.6) and (3.7) we infer that (3.5) holds for $i = n + 1$. Therefore, we conclude from the mathematical induction that (3.5) holds.

Also, by (3.2) and (3.5) we observe that $f_\delta(u^n) \leq 0$ for all $n \geq 0$. Therefore, we observe from (3.3) that

$$u^{n+1} = u^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u^n) \geq u^n \quad \text{for all } n \geq 0. \quad (3.8)$$

Therefore, we infer from (3.5) and (3.8) that $\{u^n; n \geq 0\}$ is a bounded and increasing sequence with respect to n . Thus, there exist a subsequence $\{n_k\}$ of $\{n\}$ and a point $u^\infty \in \mathbb{R}$ such that $n_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$u^{n_k} \rightarrow u^\infty \text{ in } \mathbb{R} \text{ as } k \rightarrow \infty. \quad (3.9)$$

By taking the limit in (3.3), we easily observe from the continuity of $f_\delta(\cdot)$ that $u^\infty = 1/(1-\delta)$, which is the zero point of $f_\delta(\cdot)$. Hence, taking into account of the uniqueness of the limit point, the proof of (i) has been completed.

Next, we show (ii). To this end, we assume that $\Delta t \in (\delta\varepsilon^2/(1-\delta), 2\delta\varepsilon^2/(1-\delta))$. Then, we can find the minimal number $n_0 \in \mathbb{N}$ so that

$$u^{n_0} \in \left(1, \frac{1+\delta}{1-\delta}\right) \text{ and } u^i \in (0, 1] \text{ for all } i = 0, 1, \dots, n_0 - 1. \quad (3.10)$$

In fact, if $u^i \in (0, 1]$ for all $i = 0, 1, \dots, k$, we observe from (3.4) that

$$\begin{aligned} u^{k+1} &= u^k - \frac{\Delta t}{\varepsilon^2} f_\delta(u^k) = \left(1 + \frac{\Delta t}{\varepsilon^2}\right) u^k \\ &= \left(1 + \frac{\Delta t}{\varepsilon^2}\right)^2 u^{k-1} \\ &= \dots \\ &= \left(1 + \frac{\Delta t}{\varepsilon^2}\right)^{k+1} u^0. \end{aligned} \quad (3.11)$$

Taking into account of (3.11), $u_0 \in (0, 1]$ and

$$1 + \frac{\Delta t}{\varepsilon^2} > 1 + \frac{\delta}{1-\delta} > 1,$$

we can find the minimal number $n_0 \in \mathbb{N}$ so that

$$u^{n_0} > 1 \text{ and } u^i \in (0, 1] \text{ for all } i = 0, 1, \dots, n_0 - 1.$$

Also, by (3.4) we observe that

$$u^{n_0} = u^{n_0-1} - \frac{\Delta t}{\varepsilon^2} f_\delta(u^{n_0-1}) = \left(1 + \frac{\Delta t}{\varepsilon^2}\right) u^{n_0-1} < \left(1 + \frac{2\delta}{1-\delta}\right) \cdot 1 = \frac{1+\delta}{1-\delta},$$

thus, (3.10) holds.

To show (ii), we put

$$\Delta t := \frac{\delta\varepsilon^2}{1-\delta} \tau \quad \text{for some } \tau \in (1, 2).$$

Then, we observe from (3.2) and (3.3) that

$$\begin{aligned}
u^{n_0+1} &= u^{n_0} - \frac{\Delta t}{\varepsilon^2} f_\delta(u^{n_0}) \\
&= u^{n_0} + \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - (1 - \delta)u^{n_0}}{\delta} \\
&= (1 - \tau)u^{n_0} + \frac{\tau}{1 - \delta}.
\end{aligned} \tag{3.12}$$

From (3.12) it follows that

$$\frac{u^{n_0+1} + (\tau - 1)u^{n_0}}{\tau} = \frac{1}{1 - \delta}. \tag{3.13}$$

Therefore, we observe from (3.13) and $\tau \in (1, 2)$ that the zero point $1/(1 - \delta)$ of $f_\delta(\cdot)$ is in the interval between u^{n_0} and u^{n_0+1} .

Also, by (3.12) we observe that

$$u^{n_0+1} = (1 - \tau)u^{n_0} + \frac{\tau}{1 - \delta} > (1 - \tau)\frac{1 + \delta}{1 - \delta} + \frac{\tau}{1 - \delta} = \frac{1 + \delta - \tau\delta}{1 - \delta} \geq 1$$

and

$$u^{n_0+1} = (1 - \tau)u^{n_0} + \frac{\tau}{1 - \delta} < (1 - \tau) \cdot 1 + \frac{\tau}{1 - \delta} = \frac{1 - \delta + \tau\delta}{1 - \delta} \leq \frac{1 + \delta}{1 - \delta},$$

which implies that

$$u^{n_0+1} \in \left(1, \frac{1 + \delta}{1 - \delta}\right).$$

Therefore, by (3.10), (3.13) and by repeating the procedure as above, we observe that

$$u^n \in \left(1, \frac{1 + \delta}{1 - \delta}\right) \quad \text{for all } n \geq n_0 \tag{3.14}$$

and u^n oscillates around the zero point $1/(1 - \delta)$ for all $n \geq n_0$.

Also, we observe from (3.12) and (3.14) that

$$\left|u^{n+1} - \frac{1}{1 - \delta}\right| = |1 - \tau| \left|u^n - \frac{1}{1 - \delta}\right| \quad \text{for all } n \geq n_0. \tag{3.15}$$

Therefore, by $\tau \in (1, 2)$, (3.14) and (3.15), there exist a subsequence $\{n_k\}$ of $\{n\}$ such that u^{n_k} oscillates and converges to $1/(1 - \delta)$ as $k \rightarrow \infty$. Hence, taking into account of the uniqueness of the limit point, the proof of (ii) has been completed. \square

Remark 3.2. Assume $\Delta t \in [2\delta\varepsilon^2/(1 - \delta), \infty)$ and put $\Delta t := \delta\varepsilon^2\tau/(1 - \delta)$ for some $\tau \geq 2$. Then, we observe that

$$1 + \frac{\Delta t}{\varepsilon^2} > 1 + \frac{2\delta}{1 - \delta} > 1 \quad \text{and} \quad |1 - \tau| \geq 1.$$

Therefore, we infer from the proof of Theorem 3.1 (cf. (3.11), (3.13), (3.15)) that the solution u^n to $(DE)_\delta^\varepsilon$ oscillates as $n \rightarrow \infty$, in general.

Remark 3.3. By (3.3) we easily see that

$$u^n \equiv 0 \text{ for all } n \geq 1, \text{ if } u_0 = 0,$$

$$u^n \equiv \frac{1}{1-\delta} \text{ for all } n \geq 1, \text{ if } u_0 = \frac{1}{1-\delta}$$

and

$$u^n \equiv \frac{-1}{1-\delta} \text{ for all } n \geq 1, \text{ if } u_0 = \frac{-1}{1-\delta}.$$

By (ii) of Theorem 3.1, we observe that u_n oscillates and converges to the zero point of $f_\delta(\cdot)$ in the case when $\Delta t \in (\delta\varepsilon^2/(1-\delta), 2\delta\varepsilon^2/(1-\delta))$. However, in the case when $\Delta t = 2\delta\varepsilon^2/(1-\delta)$, we have the following special case that the solution to $(DE)_\delta^\varepsilon$ does not oscillate and coincides with the zero point of $f_\delta(\cdot)$ after some finite number of iteration.

Corollary 3.1. Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$, $\Delta t = 2\delta\varepsilon^2/(1-\delta)$ and $n \in \mathbb{N}$. Assume $u_0^\varepsilon := (1-\delta)^{n-1}/(1+\delta)^n$. Then, the solution to $(DE)_\delta^\varepsilon$ is given by

$$u^i = \begin{cases} \frac{(1-\delta)^{n-1-i}}{(1+\delta)^{n-i}}, & \text{if } i = 0, 1, \dots, n-1, \\ \frac{1}{1-\delta}, & \text{if } i \geq n. \end{cases} \quad (3.16)$$

Proof. Note that $u_0^\varepsilon := (1-\delta)^{n-1}/(1+\delta)^n \in (0, 1)$. Therefore, by (3.2) and (3.3) we observe that

$$u^1 = u^0 - \frac{\Delta t}{\varepsilon^2} f_\delta(u^0) = u_0^\varepsilon - \frac{2\delta}{1-\delta}(-u_0^\varepsilon) = \frac{1+\delta}{1-\delta}u_0^\varepsilon = \frac{(1-\delta)^{n-2}}{(1+\delta)^{n-1}}.$$

Similarly, we have:

$$u^2 = u^1 - \frac{\Delta t}{\varepsilon^2} f_\delta(u^1) = \frac{1+\delta}{1-\delta}u^1 = \frac{(1-\delta)^{n-3}}{(1+\delta)^{n-2}}.$$

Repeating this procedure, we easily observe that the solution to $(DE)_\delta^\varepsilon$ is given by (3.16). \square

Taking into account of Theorem 3.1, we give numerical experiments of $(DE)_\delta^\varepsilon$ as follows. To this end, we use the following numerical data:

Numerical data of $(DE)_\delta^\varepsilon$.

- $T = 0.002$;
- $\varepsilon = 0.01$;
- $\delta = 0.01$;
- The initial data $u_0^\varepsilon = 0.1$.

Then, we easily observe that:

$$\frac{1}{1-\delta} = \frac{1}{1-0.01} = 1.010101010\dots$$

and

$$\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\dots.$$

3.1 The case when $\Delta t = 0.000001$

Now we consider the case when $\Delta t = 0.000001$. In this case, we have:

$$\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\dots > \Delta t = 0.000001,$$

which implies that (i) of Theorem 3.1 holds. Thus, we have the following stable numerical result of $(DE)_\delta^\varepsilon$. Namely, the solution to $(DE)_\delta^\varepsilon$ converges to the stationary solution $1/(1-\delta) = 1/(1-0.01) = 1.010101010\dots$.

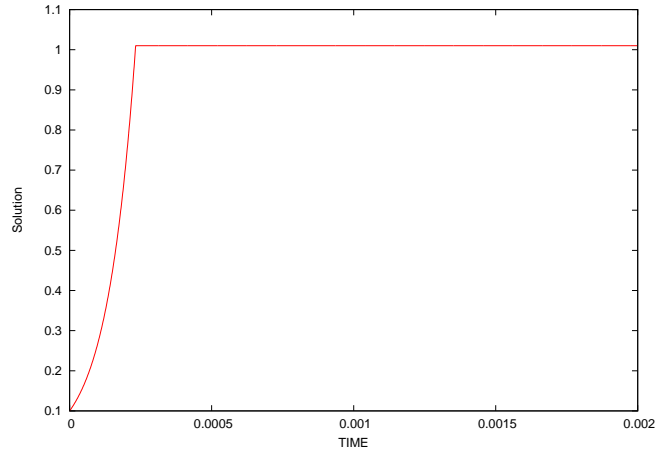


Figure 3: $\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\dots > \Delta t = 0.000001$.

Table 1: Numerical data: $\Delta t = 000001$.

number of iterations i	the value of u^i	number of iterations i	the value of u^i
0	0.100000	227	0.957088
1	0.101000	228	0.966659
2	0.102010	229	0.976325
3	0.103030	230	0.986089
4	0.104060	231	0.995950
5	0.105101	232	1.005909
\vdots	\vdots	233	1.010059
210	0.808144	234	1.010101
211	0.816225	235	1.010101
212	0.824387	236	1.010101
213	0.832631	237	1.010101
214	0.840957	238	1.010101
215	0.849367	239	1.010101
216	0.857861	240	1.010101
217	0.866439	241	1.010101
218	0.875104	242	1.010101
219	0.883855	243	1.010101
220	0.892693	244	1.010101
221	0.901620	245	1.010101
222	0.910636	246	1.010101
223	0.919743	247	1.010101
224	0.928940	248	1.010101
225	0.938230	249	1.010101
226	0.947612	250	1.010101

3.2 The case when $\Delta t = 0.000002$

Now we consider the case when $\Delta t = 0.000002$. In this case, we have:

$$\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010 \dots < \Delta t = 0.000002 < \frac{2\delta\varepsilon^2}{1-\delta},$$

which implies that (ii) of Theorem 3.1 holds. Thus, we have the following numerical result of $(DE)_\delta^\varepsilon$ that the solution to $(DE)_\delta^\varepsilon$ oscillates and converges to the stationary solution $1/(1-\delta) = 1/(1-0.01) = 1.010101010 \dots$.

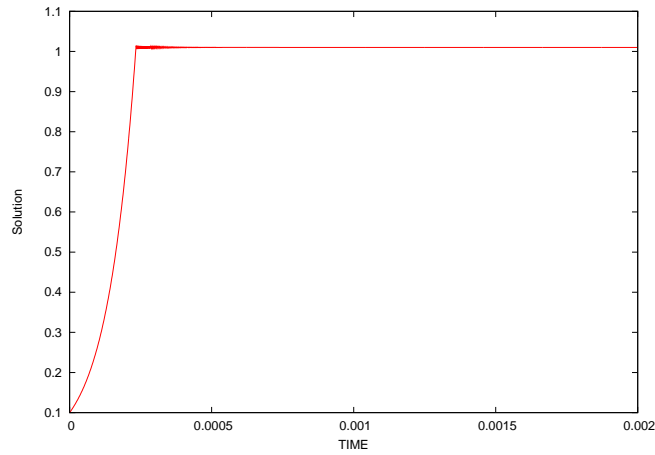


Figure 4: $\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010 \dots < \Delta t = 0.000002 < \frac{2\delta\varepsilon^2}{1-\delta}$.

Table 2: Numerical data: $\Delta t = 0.000002$.

number of iterations i	the value of u^i	number of iterations i	the value of u^i
0	0.100000		
1	0.102000		
2	0.104040		
3	0.106121		
4	0.108243		
5	0.110408		
\vdots	\vdots		
112	0.918796	550	1.010100
113	0.937172	551	1.010102
114	0.955916	552	1.010100
115	0.975034	553	1.010102
116	0.994535	554	1.010100
117	1.014425	555	1.010102
118	1.005863	556	1.010100
119	1.014254	557	1.010102
120	1.006031	558	1.010100
121	1.014090	559	1.010102
122	1.006192	560	1.010100
123	1.013932	561	1.010102
124	1.006347	562	1.010100
125	1.013780	563	1.010102
126	1.006496	564	1.010100
127	1.013634	565	1.010102
128	1.006638	566	1.010101
129	1.013494	567	1.010101
130	1.006775	568	1.010101
131	1.013360	569	1.010101
132	1.006907	570	1.010101
133	1.013231	571	1.010101
134	1.007034	572	1.010101
135	1.013107	573	1.010101
136	1.007155	574	1.010101
137	1.012988	575	1.010101
\vdots	\vdots	576	1.010101
		577	1.010101
		578	1.010101
		579	1.010101
		580	1.010101
		581	1.010101
		582	1.010101

3.3 The case when $\Delta t = 2\frac{\delta\varepsilon^2}{1-\delta}$

Now we consider the case when $\Delta t = 2\delta\varepsilon^2/(1-\delta) = 0.0000020202020\dots$. In this case, we observe Remark 3.2. In fact, we have the following numerical result of $(DE)_\delta^\varepsilon$ that the solution to $(DE)_\delta^\varepsilon$ oscillates.

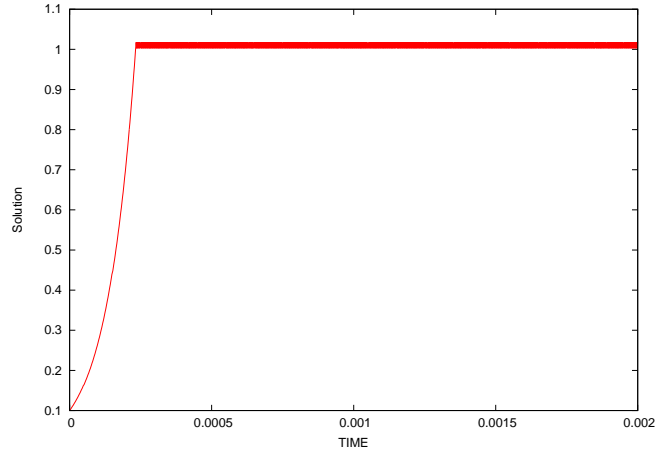


Figure 5: $\Delta t = 2\frac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\dots$.

Table 3: Numerical data: $\Delta t = 2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020 \dots$.

number of iterations i	the value of u^i	number of iterations i	the value of u^i
0	0.100000	121	1.002556
1	0.102020	122	1.017646
2	0.104081	123	1.002556
3	0.106184	124	1.017646
4	0.108329	125	1.002556
5	0.110517	126	1.017646
\vdots	\vdots	127	1.002556
100	0.738955	128	1.017646
101	0.753883	129	1.002556
102	0.769113	130	1.017646
103	0.784651	131	1.002556
104	0.800502	132	1.017646
105	0.816674	133	1.002556
106	0.833173	134	1.017646
107	0.850004	135	1.002556
108	0.867176	136	1.017646
109	0.884695	137	1.002556
110	0.902568	138	1.017646
111	0.920801	139	1.002556
112	0.939403	140	1.017646
113	0.958381	141	1.002556
114	0.977742	142	1.017646
115	0.997495	143	1.002556
116	1.017646	144	1.017646
117	1.002556	145	1.002556
118	1.017646	146	1.017646
119	1.002556	147	1.002556
120	1.017646	148	1.017646

3.4 The case when $\Delta t = 0.000005$

Now we consider the case when $\Delta t = 0.000005$. In this case, we have:

$$2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020 \dots < \Delta t = 0.000005.$$

Therefore, we observe Remark 3.2. In fact, we have the following numerical result of $(DE)_\delta^\varepsilon$ that the solution to $(DE)_\delta^\varepsilon$ oscillates.

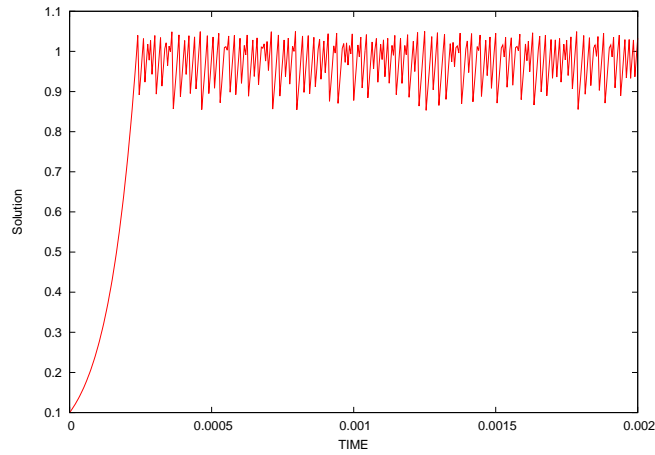


Figure 6: $2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020 \dots < \Delta t = 0.000005$.

Table 4: Numerical data: $\Delta t = 0.000005$.

number of iterations i	the value of u^i	number of iterations i	the value of u^i
0	0.100000	52	1.032021
1	0.105000	53	0.923517
2	0.110250	54	0.969693
3	0.115763	55	1.018178
4	0.121551	56	0.978198
5	0.127628	57	1.027108
\vdots	\vdots	58	0.942925
40	0.703999	59	0.990072
41	0.739199	60	1.039575
42	0.776159	61	0.893678
43	0.814967	62	0.938362
44	0.855715	63	0.985280
45	0.898501	64	1.034544
46	0.943426	65	0.913551
47	0.990597	66	0.959228
48	1.040127	67	1.007189
49	0.891498	68	1.021602
50	0.936073	69	0.964674
51	0.982877	70	1.012907

3.5 The case when $\Delta t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$

Now we consider the case when $\Delta t = 15\delta\varepsilon^2/(1 - \delta)$. In this case, we observe Remark 3.2. In fact, we have the following numerical result of $(DE)_\delta^\varepsilon$ that the solution to $(DE)_\delta^\varepsilon$ oscillates between three zero points of $f_\delta(\cdot)$.

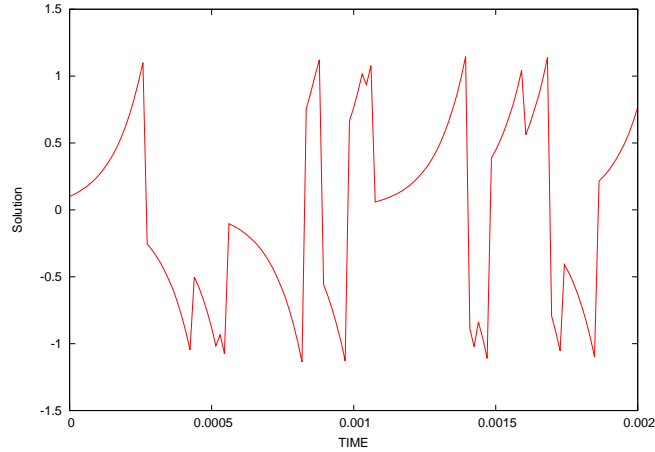


Figure 7: $\Delta t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$.

Table 5: Numerical data: $\Delta t = 15 \frac{\delta \varepsilon^2}{1 - \delta}$.

number of iterations i	the value of u^i	number of iterations i	the value of u^i
0	0.100000	35	-0.933445
1	0.115152	36	-1.074876
2	0.132599	37	-0.103248
3	0.152689	38	-0.118892
4	0.175824	39	-0.136906
5	0.202464	40	-0.157649
\vdots	\vdots	41	-0.181536
6	0.233141	42	-0.209041
7	0.268465	43	-0.240714
8	0.309141	44	-0.277186
9	0.355981	45	-0.319183
10	0.409918	46	-0.367545
11	0.472026	47	-0.423233
12	0.543545	48	-0.487359
13	0.625901	49	-0.561202
14	0.720734	50	-0.646232
15	0.829936	51	-0.744146
16	0.955684	52	-0.856896
17	1.100485	53	-0.986728
18	-0.255276	54	-1.136232
19	-0.293954	55	0.755739
20	-0.338493	56	0.870245
21	-0.389780	57	1.002100
22	-0.448837	58	1.122108
23	-0.516843	59	-0.558000
24	-0.595152	60	-0.642546
25	-0.685327	61	-0.739901
26	-0.789164	62	-0.852007
27	-0.908735	63	-0.981099
28	-1.046422	64	-1.129751
29	-0.501608	65	0.664998
30	-0.577609	66	0.765755
31	-0.665126	67	0.881779
32	-0.765902	68	1.015381
33	-0.881948	69	0.936177
34	-1.015576	70	1.078022

3.6 Numerical result of Corollary 3.1

In this subsection, we consider Corollary 3.1 numerically. To this end, we use the following initial data:

$$u_0 := \frac{(1 - \delta)^5}{(1 + \delta)^6} = \frac{(1 - 0.01)^5}{(1 + 0.01)^6} = 0.8958756 \dots .$$

Then, we have the following numerical experiment of $(DE)_\delta^\varepsilon$ that Corollary 3.1 holds. Namely, we observe that (3.16) holds with $n = 6$:

Table 6: Numerical data: $\Delta t = 2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020 \dots .$

number of iterations i	the value of u^i
0	0.895876
1	0.913974
2	0.932438
3	0.951275
4	0.970493
5	0.990099
6	1.010101
7	1.010101
8	1.010101
9	1.010101
10	1.010101

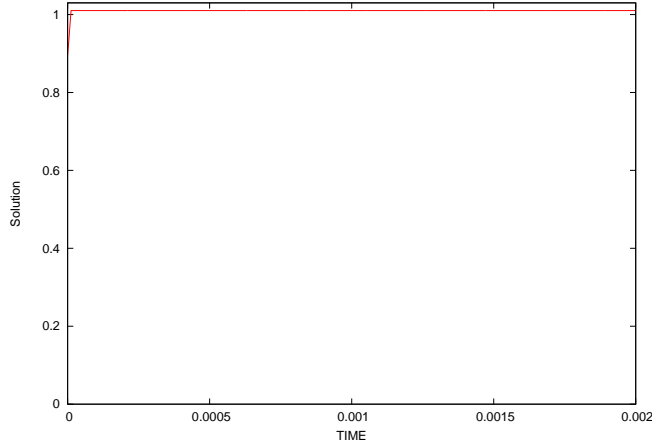


Figure 8: $\Delta t = 2 \frac{\delta \varepsilon^2}{1 - \delta} = 0.0000020202020 \dots$

3.7 Conclusion of ODE problem $(DE)_\delta^\varepsilon$

By Theorem 3.1 and numerical experiments as above, we conclude that

- (i) the mesh size of time Δt must be smaller than $\delta \varepsilon^2 / (1 - \delta)$ in order to get the stable numerical experiments of $(DE)_\delta^\varepsilon$.
- (ii) we have the stable numerical experiments of $(DE)_\delta^\varepsilon$ with the initial data $u_0^\varepsilon := (1 - \delta)^{n-1} / (1 + \delta)^n$, even if the mesh size of time Δt is equal to $2\delta \varepsilon^2 / (1 - \delta)$.

4 Solvability and convergence results for $(P)_\delta^\varepsilon$

We begin by giving the rigorous definition of solutions to our PDE problem $(P)_\delta^\varepsilon$ ($\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$).

Definition 4.1. *Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u_0^\varepsilon \in H$. Then, a function $u_\delta^\varepsilon : [0, T] \rightarrow H$ is called a solution to $(P)_\delta^\varepsilon$ on $[0, T]$, if the following conditions are satisfied:*

- (i) $u_\delta^\varepsilon \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$.
- (ii) *The following variational identity holds:*

$$\left((u_\delta^\varepsilon)_t(t), z \right)_H + \left((u_\delta^\varepsilon)_x(t), z_x \right)_H + \left(\frac{(\partial I_{[-1,1]})_\delta(u_\delta^\varepsilon(t))}{\varepsilon^2}, z \right)_H = \left(\frac{u_\delta^\varepsilon(t)}{\varepsilon^2}, z \right)_H$$

for all $z \in V$ and a.e. $t \in (0, T)$.

(iii) $u_\delta^\varepsilon(0) = u_0^\varepsilon$ in H .

Now, let us recall the solvability result of $(P)_\delta^\varepsilon$ on $[0, T]$.

Proposition 4.1. *Let $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$. Assume the following condition:*

(A) $u_0^\varepsilon \in K := \{z \in V ; -1 \leq z(x) \leq 1 \text{ a.e. } x \in (0, 1)\}$.

Then, for each $u_0^\varepsilon \in K$, there exists a unique solution u_δ^ε to $(P)_\delta^\varepsilon$ on $[0, T]$ in the sense of Definition 4.1.

Proof. By the same argument as in Proposition 2.1, we can show the existence-uniqueness of a solution u_δ^ε to $(P)_\delta^\varepsilon$ on $[0, T]$ for each $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$. In fact, we easily prove the uniqueness of solutions to $(P)_\delta^\varepsilon$ on $[0, T]$ by the quite standard arguments: monotonicity and Gronwall's inequality.

Also, we can show the existence of solutions to $(P)_\delta^\varepsilon$ on $[0, T]$ applying the abstract theory of evolution equations governed by subdifferentials. In fact, we define a functional $\varphi_\delta^\varepsilon$ on H by

$$\varphi_\delta^\varepsilon(z) := \begin{cases} \frac{1}{2} \int_\Omega |z_x|^2 dx + \frac{1}{\varepsilon^2} \int_0^1 (I_{[-1,1]})_\delta(z(x)) dx, & \text{if } z \in V, \\ \infty, & \text{otherwise,} \end{cases} \quad (4.1)$$

where $(I_{[-1,1]})_\delta(\cdot)$ is the function defined in (2.1). Clearly, $\varphi_\delta^\varepsilon$ is proper, l.s.c. and convex on H with the effective domain $D(\varphi) = V$.

We easily see that the problem $(P)_\delta^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$(PP)_\delta^\varepsilon \begin{cases} \frac{d}{dt} u_\delta^\varepsilon(t) + \partial \varphi_\delta^\varepsilon(u_\delta^\varepsilon(t)) - \frac{1}{\varepsilon^2} u_\delta^\varepsilon(t) = 0 & \text{in } H, \text{ for } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in } H. \end{cases}$$

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [5, 14, 21]), we can show the existence of a solution u_δ^ε to $(PP)_\delta^\varepsilon$, hence, $(P)_\delta^\varepsilon$, on $[0, T]$ for each $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$ in the sense of Definition 4.1. Thus, the proof of Proposition 4.1 has been completed. \square

Next, we recall the convergence result of $(P)_\delta^\varepsilon$ as $\delta \rightarrow 0$. Taking account of Lemma 2.1 (cf. (2.3)), we easily observe that the following lemma holds.

Lemma 4.1 (cf. [2, Section 5], [4, Chapter 2], [15, Section 2]). *Let $\varepsilon \in (0, 1]$, and define a functional φ^ε on H by*

$$\varphi^\varepsilon(z) := \begin{cases} \frac{1}{2} \int_\Omega |z_x|^2 dx + \frac{1}{\varepsilon^2} \int_0^1 I_{[-1,1]}(z(x)) dx, & \text{if } z \in V, \\ \infty, & \text{otherwise.} \end{cases}$$

Then,

$$\varphi_\delta^\varepsilon(\cdot) \longrightarrow \varphi^\varepsilon(\cdot) \quad \text{on } H \text{ in the sense of Mosco [17]}$$

as $\delta \rightarrow 0$.

By Lemma 4.1 and the general convergence theory of evolution equations, we easily get the following result.

Proposition 4.2 (cf. [2, Section 5], [15, Section 2]). *Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1]$ and $u_0^\varepsilon \in K$. Also, let u_δ^ε be the unique solution to $(P)_\delta^\varepsilon$ on $[0, T]$. Then, u_δ^ε converges to the unique function u^ε to $(P)^\varepsilon$ on $[0, T]$ in the sense that*

$$u_\delta^\varepsilon \longrightarrow u^\varepsilon \quad \text{strongly in } C([0, T]; H) \quad \text{as } \delta \rightarrow 0. \quad (4.2)$$

Proof. We easily observe that the problem $(P)^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$(PP)^\varepsilon \begin{cases} \frac{d}{dt} u^\varepsilon(t) + \partial\varphi^\varepsilon(u^\varepsilon(t)) - \frac{1}{\varepsilon^2} u^\varepsilon(t) \ni 0 & \text{in } H, \text{ for } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in } H. \end{cases}$$

Therefore, by Lemma 4.1 and the abstract convergence theory of evolution equations (cf. [2, 15]), we observe that the solution u_δ^ε to $(PP)_\delta^\varepsilon$ converges to the unique solution u^ε to $(PP)^\varepsilon$ on $[0, T]$ as $\delta \rightarrow 0$ in the sense of (4.2). Note that u^ε (resp. u_δ^ε) is the unique solution to $(P)^\varepsilon$ (resp. $(P)_\delta^\varepsilon$) on $[0, T]$ (cf. Proposition 4.1). Thus, we conclude that Proposition 4.2 holds. \square

5 Stable criteria and numerical experiments for $(P)_\delta^\varepsilon$

In this Section we consider $(P)_\delta^\varepsilon$ from the view-point of numerical analysis.

Remark 5.1. *Note from Proposition 4.2 that $(P)_\delta^\varepsilon$ is the approximating problem of $(P)^\varepsilon$. Also note from (1.5) that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to study $(P)^\varepsilon$ numerically.*

In order to make numerical experiments of $(P)_\delta^\varepsilon$, we consider the following explicit finite difference scheme to $(P)_\delta^\varepsilon$, denoted by of $(DP)_\delta^\varepsilon$:

$$(DP)_\delta^\varepsilon \begin{cases} \frac{u_k^{n+1} - u_k^n}{\Delta t} - \frac{u_{k-1}^n - 2u_k^n + u_{k+1}^n}{(\Delta x)^2} + \frac{(\partial I_{[-1,1]})_\delta(u_k^n)}{\varepsilon^2} = \frac{u_k^n}{\varepsilon^2} \\ \text{for } n = 0, 1, 2, \dots, N_t \text{ and } k = 1, 2, \dots, N_x - 1, \\ u_0^n = u_1^n, \quad u_{N_x}^n = u_{N_x-1}^n \quad \text{for } n = 1, 2, \dots, N_t, \\ u_k^0 = u_0^\varepsilon(x_k) \quad \text{for } k = 0, 1, 2, \dots, N_x, \end{cases}$$

where Δt is the mesh size of time, Δx is the mesh size of space, N_t is the integer part of number $T/\Delta t$, N_x is the integer part of number $1/\Delta x$ and $x_k := k\Delta x$. We easily see that u_k^n is the approximating solution of $(P)_\delta^\varepsilon$ at the time $t_n := n\Delta t$ and the position $x_k := k\Delta x$.

Clearly, the explicit finite difference scheme $(DP)_\delta^\varepsilon$ converges to $(P)_\delta^\varepsilon$ as $\Delta t \rightarrow 0$ and $\Delta x \rightarrow 0$.

From Figure 1, we easily see that we have to choose the suitable constants ε , δ , the mesh size of time Δt and the mesh size of space Δx in order to get stable numerical results of $(DP)_\delta^\varepsilon$. Now, let us mention our second main result in this paper, which is concerned with the stability of $(DE)_\delta^\varepsilon$.

Theorem 5.1. Let $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$, $\Delta t \in (0, 1]$, $\Delta x \in (0, 1]$, $T > 0$ and $u_0^\varepsilon \in K$, where K is the set of initial value defined in Proposition 4.1 (cf. (A)). Let N_x be the integer part of number $1/\Delta x$, and let $\{u_k^n; n \geq 0, k = 0, 1, \dots, N_x\}$ be the solution to $(\text{DP})_\delta^\varepsilon$. Also, let $c_0 \in (0, 1)$ and assume that

$$0 < \Delta t \leq \frac{c_0 \delta \varepsilon^2}{1 - \delta} \quad \text{and} \quad 0 \leq \frac{\Delta t}{(\Delta x)^2} \leq \frac{1 - c_0}{2}. \quad (5.1)$$

Then, we have:

(i) the solution to $(\text{DP})_\delta^\varepsilon$ is bounded in the following sense:

$$\max_{0 \leq k \leq N_x} |u_k^n| \leq \frac{1}{1 - \delta} \quad \text{for all } n \geq 0. \quad (5.2)$$

(ii) $\{u_k^n; k = 0, 1, \dots, N_x\}$ does not oscillate with respect to $n \geq 0$.

Proof. We first show (i), i.e., (5.2) by the mathematical induction.

Clearly (5.2) holds for $n = 0$ because of $u_0^\varepsilon \in K$.

Now, we assume that

$$\max_{0 \leq k \leq N_x} |u_k^i| \leq \frac{1}{1 - \delta} \quad \text{for all } i = 0, 1, \dots, n. \quad (5.3)$$

We easily observe that the explicit finite difference problem $(\text{DP})_\delta^\varepsilon$ can be reformulated as in the following form:

$$u_k^{n+1} = r u_{k-1}^n + r u_{k+1}^n + (1 - 2r) u_k^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n) \quad (5.4)$$

for all $n = 0, 1, 2, \dots, N_t$ and $k = 1, 2, \dots, N_x - 1$,

where we put $r := \Delta t / (\Delta x)^2$ and $f_\delta(\cdot)$ is the function defined by (3.2).

We easily observe from (5.1), (5.3) and (5.4) that

$$\begin{aligned} \frac{1}{1 - \delta} - u_k^{n+1} &= r \left(\frac{1}{1 - \delta} - u_{k-1}^n \right) + r \left(\frac{1}{1 - \delta} - u_{k+1}^n \right) \\ &\quad + (1 - 2r) \left(\frac{1}{1 - \delta} - u_k^n \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n) \\ &\geq (1 - 2r) \left(\frac{1}{1 - \delta} - u_k^n \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n) \end{aligned} \quad (5.5)$$

for all $k = 1, 2, \dots, N_x - 1$.

From (3.2), (5.1) and (5.3) we infer that the function $[-1/(1 - \delta), 1/(1 - \delta)] \ni z \rightarrow (1 - 2r)(1/(1 - \delta) - z) + \Delta t / \varepsilon^2 f_\delta(z)$ is non-negative and continuous. In fact, it follows from (3.2) that the function $[-1/(1 - \delta), 1] \ni z \rightarrow (1 - 2r)(1/(1 - \delta) - z) + \Delta t / \varepsilon^2 f_\delta(z)$

attains a minimum value at $z = 1$. Therefore, we observe from (3.2) and (5.1) that

$$\begin{aligned}
& (1-2r) \left(\frac{1}{1-\delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z) \\
& \geq (1-2r) \left(\frac{1}{1-\delta} - 1 \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(1) \\
& = (1-2r) \cdot \frac{\delta}{1-\delta} - \frac{\Delta t}{\varepsilon^2} \\
& \geq \frac{c_0 \delta}{1-\delta} - \frac{\Delta t}{\varepsilon^2} \\
& \geq 0 \quad \text{for all } z \in \left[-\frac{1}{1-\delta}, 1 \right].
\end{aligned} \tag{5.6}$$

Also, for any $z \in [1, 1/(1-\delta)]$, we observe from (3.2) that

$$\begin{aligned}
& (1-2r) \left(\frac{1}{1-\delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z) \\
& = (1-2r) \left(\frac{1}{1-\delta} - z \right) + \frac{\Delta t}{\varepsilon^2} \cdot \left(\frac{z-1}{\delta} - z \right) \\
& = \left[\frac{1-\delta}{\delta \varepsilon^2} \Delta t - (1-2r) \right] z + (1-2r) \frac{1}{1-\delta} - \frac{\Delta t}{\delta \varepsilon^2}.
\end{aligned} \tag{5.7}$$

Here we note from (5.1) that

$$\frac{1-\delta}{\delta \varepsilon^2} \Delta t - (1-2r) \leq \frac{1-\delta}{\delta \varepsilon^2} \Delta t - c_0 \leq 0.$$

Therefore, we infer from (5.7) that the function $[1, 1/(1-\delta)] \ni z \rightarrow (1-2r)(1/(1-\delta) - z) + \Delta t/\varepsilon^2 f_\delta(z)$ is non-increasing and attains a minimum value at $z = 1/(1-\delta)$. Hence, we have:

$$\begin{aligned}
(1-2r) \left(\frac{1}{1-\delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z) & \geq \frac{\Delta t}{\varepsilon^2} f_\delta \left(\frac{1}{1-\delta} \right) = 0 \\
& \text{for all } z \in \left[1, \frac{1}{1-\delta} \right].
\end{aligned} \tag{5.8}$$

Thus, we observe from (5.6) and (5.8) that

$$(1-2r) \left(\frac{1}{1-\delta} - z \right) + \frac{\Delta t}{\varepsilon^2} f_\delta(z) \geq 0, \quad \forall z \in \left[-\frac{1}{1-\delta}, \frac{1}{1-\delta} \right],$$

which implies from (5.3) and (5.5) that

$$\frac{1}{1-\delta} - u_k^{n+1} \geq 0 \quad \text{for all } k = 1, 2, \dots, N_x - 1. \tag{5.9}$$

Similarly, we observe from (5.1), (5.3) and (5.4) that

$$\begin{aligned}
u_k^{n+1} + \frac{1}{1-\delta} &= r \left(u_{k-1}^n + \frac{1}{1-\delta} \right) + r \left(u_{k+1}^n + \frac{1}{1-\delta} \right) \\
&\quad + (1-2r) \left(u_k^n + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n) \\
&\geq (1-2r) \left(u_k^n + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(u_k^n) \tag{5.10}
\end{aligned}$$

for all $k = 1, 2, \dots, N_x - 1$.

By the similar arguments as above, we observe that the function $[-1/(1-\delta), 1/(1-\delta)] \ni z \rightarrow (1-2r)(z + 1/(1-\delta)) - \Delta t/\varepsilon^2 f_\delta(z)$ is non-negative and continuous. In fact, it follows from (3.2) that the function $[-1, 1/(1-\delta)] \ni z \rightarrow (1-2r)(z + 1/(1-\delta)) - \Delta t/\varepsilon^2 f_\delta(z)$ attains a minimum value at $z = -1$. Therefore, we observe from (3.2) and (5.1) that

$$\begin{aligned}
&(1-2r) \left(z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) \\
&\geq (1-2r) \left(-1 + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(-1) \\
&= (1-2r) \cdot \frac{\delta}{1-\delta} - \frac{\Delta t}{\varepsilon^2} \\
&\geq \frac{c_0 \delta}{1-\delta} - \frac{\Delta t}{\varepsilon^2} \\
&\geq 0 \quad \text{for all } z \in \left[-1, \frac{1}{1-\delta} \right]. \tag{5.11}
\end{aligned}$$

Also, for any $z \in [-1/(1-\delta), -1]$, we observe from (3.2) that

$$\begin{aligned}
&(1-2r) \left(z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) \\
&= (1-2r) \left(z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} \cdot \left(\frac{1+z}{\delta} - z \right) \\
&= \left[(1-2r) - \frac{1-\delta}{\delta \varepsilon^2} \Delta t \right] z + (1-2r) \frac{1}{1-\delta} - \frac{\Delta t}{\delta \varepsilon^2}. \tag{5.12}
\end{aligned}$$

Here we note from (5.1) that

$$(1-2r) - \frac{1-\delta}{\delta \varepsilon^2} \Delta t \geq c_0 - \frac{1-\delta}{\delta \varepsilon^2} \Delta t \geq 0.$$

Therefore, we infer from (5.12) that the function $[-1/(1-\delta), -1] \ni z \rightarrow (1-2r)(z + 1/(1-\delta)) - \Delta t/\varepsilon^2 f_\delta(z)$ is non-decreasing and attains a minimum value at $z = -1/(1-\delta)$. Hence, we have:

$$\begin{aligned}
(1-2r) \left(z + \frac{1}{1-\delta} \right) - \frac{\Delta t}{\varepsilon^2} f_\delta(z) &\geq -\frac{\Delta t}{\varepsilon^2} f_\delta \left(-\frac{1}{1-\delta} \right) = 0 \\
&\text{for all } z \in \left[-\frac{1}{1-\delta}, -1 \right]. \tag{5.13}
\end{aligned}$$

where we put

$$\mathbf{u}^{(n)} := \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N_x-1}^n \end{pmatrix}, \quad A = \begin{pmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & r & \\ & & \ddots & \ddots & \\ & & & r & 1-2r & r \\ & & & & r & 1-2r \end{pmatrix}.$$

Noting from (3.2) that

$$f_\delta(u_k^n) = \begin{cases} \frac{1-\delta}{\delta}u_k^n + \frac{1}{\delta} & \text{if } u_k^n \leq -1, \\ -u_k^n & \text{if } u_k^n \in [-1, 1], \\ \frac{1-\delta}{\delta}u_k^n - \frac{1}{\delta} & \text{if } u_k^n \geq 1. \end{cases} \quad (5.17)$$

By putting

$$b_k^n := \begin{cases} -1 & \text{if } u_k^n \in [-1, 1], \\ \frac{1-\delta}{\delta} & \text{if } u_k^n \notin [-1, 1], \end{cases} \quad \tilde{b}_k^n := \begin{cases} \frac{1}{\delta} & \text{if } u_k^n \leq -1, \\ 0 & \text{if } u_k^n \in [-1, 1], \\ -\frac{1}{\delta} & \text{if } u_k^n \geq 1, \end{cases} \quad (5.18)$$

we observe from (5.17) and (5.18) that:

$$\begin{aligned} & \begin{pmatrix} ru_1^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u_1^n) \\ -\frac{\Delta t}{\varepsilon^2} f_\delta(u_2^n) \\ \vdots \\ ru_{N_x-1}^n - \frac{\Delta t}{\varepsilon^2} f_\delta(u_{N_x-1}^n) \end{pmatrix} \\ &= \begin{pmatrix} r - \frac{\Delta t}{\varepsilon^2} b_1^n & & & \\ & -\frac{\Delta t}{\varepsilon^2} b_2^n & & \\ & & \ddots & \\ & & & r - \frac{\Delta t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N_x-1}^n \end{pmatrix} + \begin{pmatrix} -\frac{\Delta t}{\varepsilon^2} \tilde{b}_1^n \\ -\frac{\Delta t}{\varepsilon^2} \tilde{b}_2^n \\ \vdots \\ -\frac{\Delta t}{\varepsilon^2} \tilde{b}_{N_x-1}^n \end{pmatrix}. \end{aligned}$$

By using the matrix as above, we observe that (5.15) can be rewritten as in the following form:

$$\mathbf{u}^{(n+1)} = A\mathbf{u}^{(n)} + B\mathbf{u}^{(n)} + \tilde{\mathbf{b}}^{(n)}, \quad (n \geq 0), \quad (5.19)$$

where we put

$$B := \begin{pmatrix} r - \frac{\Delta t}{\varepsilon^2} b_1^n & & & \\ & -\frac{\Delta t}{\varepsilon^2} b_2^n & & \\ & & \ddots & \\ & & & r - \frac{\Delta t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix}, \quad \tilde{\mathbf{b}}^{(n)} := \begin{pmatrix} -\frac{\Delta t}{\varepsilon^2} \tilde{b}_1^n \\ -\frac{\Delta t}{\varepsilon^2} \tilde{b}_2^n \\ \vdots \\ -\frac{\Delta t}{\varepsilon^2} \tilde{b}_{N_x-1}^n \end{pmatrix}.$$

By the general theory, we observe that the eigenvalue λ_j of matrix A is given by:

$$\lambda_j := 1 - 4r \sin^2 \left(\frac{j\pi}{2N_x} \right) \quad \text{for } j = 1, 2, \dots, N_x - 1, \quad (5.20)$$

which implies that λ_1 (rest. λ_{N_x-1}) is the maximum (rest. minimum) eigenvalue of A .

Now let $\{\tilde{\lambda}_j; j = 1, 2, \dots, N_x - 1\}$ be the set of all eigenvalues of matrix $\tilde{A} := A + B$ such that

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_{N_x-1}.$$

Also, let $\{\lambda_j^B; j = 1, 2, \dots, N_x - 1\}$ be the set of all eigenvalues of B such that

$$\lambda_1^B \geq \lambda_2^B \geq \dots \geq \lambda_{N_x-1}^B.$$

Then, by the abstract perturbation theory of matrix, we observe that:

$$\lambda_{N_x-1} + \lambda_j^B \leq \tilde{\lambda}_j \leq \lambda_1 + \lambda_j^B, \quad \forall j = 1, 2, \dots, N_x - 1. \quad (5.21)$$

Since B is a symmetric matrix, it follows from (5.1), (5.18) and (5.20) that

$$\begin{aligned} \lambda_{N_x-1} + \lambda_j^B &\geq 1 - 4r \sin^2 \left(\frac{(N_x - 1)\pi}{2N_x} \right) - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} \\ &\geq 1 - 4r - \frac{\Delta t(1 - \delta)}{\delta \varepsilon^2} \\ &\geq 1 - 4 \cdot \frac{1 - c_0}{2} - c_0 \\ &= -1 + c_0 \\ &> -1 \quad \text{for all } j = 1, 2, \dots, N_x - 1. \end{aligned}$$

Thus, we conclude from (5.21) and the above inequality that

$$\tilde{\lambda}_j > -1 \quad \text{for all } j = 1, 2, \dots, N_x - 1. \quad (5.22)$$

Next, we now assume $\max_{1 \leq k \leq N_x-1} |u_k^n| \leq 1$. Then, we observe from (5.1) and (5.18) that

$$\begin{aligned} 1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n &= 1 - 2 \frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{\varepsilon^2} \\ &\geq 1 - 2 \cdot \frac{1 - c_0}{2} + \frac{\Delta t}{\varepsilon^2} \\ &> 0 \quad \text{for all } k = 1, 2, \dots, N_x - 1. \end{aligned}$$

Therefore, all components of $A + B$ are positive. Hence, the sum of all components in k -th row of $A + B$ is the following:

$$\left| 1 - r - \frac{\Delta t}{\varepsilon^2} b_k^n \right| + |r| = 1 - r - \frac{\Delta t}{\varepsilon^2} b_k^n + r = 1 + \frac{\Delta t}{\varepsilon^2} > 1 \quad (5.23)$$

for $k = 1$ or $N_x - 1$

and

$$|r| + \left| 1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n \right| + |r| = r + 1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n + r = 1 + \frac{\Delta t}{\varepsilon^2} > 1 \quad (5.24)$$

for all $k = 2, 3, \dots, N_x - 2$.

Therefore, we observe from (5.23)–(5.24) that $\max_{1 \leq k \leq N_x - 1} |u_k^n|$ is increasing with respect to n in the case when $\max_{1 \leq k \leq N_x - 1} |u_k^n| \leq 1$.

However, if $u_k^n \notin [-1, 1]$ for some $k = 1, 2, \dots, N_x - 1$, it follows from (5.1) and (5.18) that

$$\begin{aligned} 1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n &= 1 - 2 \frac{\Delta t}{(\Delta x)^2} - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} \\ &\geq 1 - 2 \cdot \frac{1 - c_0}{2} - c_0 \\ &= 0. \end{aligned}$$

Therefore, the sum of all components in k -th row of $A + B$ is the following:

$$\left| 1 - r - \frac{\Delta t}{\varepsilon^2} b_k^n \right| + |r| = 1 - r - \frac{\Delta t}{\varepsilon^2} b_k^n + r = 1 - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} < 1 \quad (5.25)$$

if $u_k^n \notin [-1, 1]$ for some $k = 1$ or $N_x - 1$

and

$$\left| r \right| + \left| 1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n \right| + |r| = r + 1 - 2r - \frac{\Delta t}{\varepsilon^2} b_k^n + r = 1 - \frac{\Delta t}{\varepsilon^2} \cdot \frac{1 - \delta}{\delta} < 1 \quad (5.26)$$

if $u_k^n \notin [-1, 1]$ for some $k = 2, 3, \dots, N_x - 2$.

Although $\max_{1 \leq k \leq N_x - 1} |u_k^n|$ is increasing with respect to n in the case when $\max_{1 \leq k \leq N_x - 1} |u_k^n| \leq 1$ (cf. (5.23)–(5.24)), we conclude from (5.22) and (5.25)–(5.26) that (ii) of Theorem 5.1 holds. Thus, the proof of Theorem 5.1 has been completed. \square

Remark 5.2. By (5.1) we get the suitable mesh size of space Δx . In fact, for each $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$ we take the constant $\tilde{c}_0 \in (0, 1)$, $\Delta t \in (0, 1]$, $\Delta x \in (0, 1]$ such that

$$\Delta t \leq \frac{\tilde{c}_0 \delta \varepsilon^2}{1 - \delta} \quad \text{and} \quad \frac{\Delta t}{(\Delta x)^2} = \frac{1 - \tilde{c}_0}{2}.$$

Then, we have:

$$\Delta t = \frac{1 - \tilde{c}_0}{2} \cdot (\Delta x)^2 \leq \frac{\tilde{c}_0 \delta \varepsilon^2}{1 - \delta}.$$

Thus, we have the following condition of Δx :

$$0 < \Delta x < \varepsilon \sqrt{\frac{2\tilde{c}_0\delta}{(1 - \tilde{c}_0)(1 - \delta)}}.$$

Remark 5.3. We can take $c_0 = 0$ in (5.1) for the explicit finite difference scheme to the following usual heat equation with Neumann boundary condition:

$$\begin{cases} u_t^\varepsilon - u_{xx}^\varepsilon = 0 & \text{in } Q := (0, T) \times (0, 1), \\ u_x^\varepsilon(t, 0) = u_x^\varepsilon(t, 1) = 0, & t \in (0, T), \\ u^\varepsilon(0, x) = u_0^\varepsilon(x), & x \in (0, 1). \end{cases}$$

Taking into account of Theorem 5.1, we give numerical experiments of $(DP)_\delta^\varepsilon$ as follows. To this end, we use the following numerical data:

Numerical data of $(DP)_\delta^\varepsilon$.

- $T = 0.01$;
- $\delta = 0.01$;
- $\Delta x = 0.005$;
- $c_0 := 0.6$.

Also, we consider the following initial data $u_0^\varepsilon(x)$ defined by

$$u_0^\varepsilon(x) = \begin{cases} -0.5, & \text{if } x \in [0.00, 0.25], \\ -0.5 \sin(2\pi x), & \text{if } x \in [0.25, 0.75], \\ 0.5, & \text{if } x \in [0.75, 1.00]. \end{cases} \quad (5.27)$$

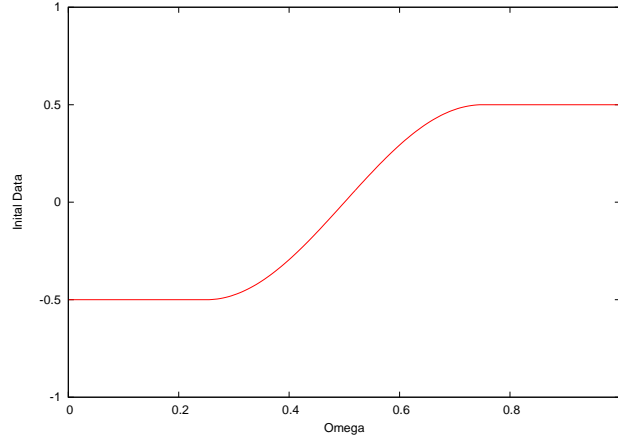


Figure 9: The graph of initial data $u_0^\varepsilon(x)$ defined by (5.27).

5.1 The case when $\varepsilon = 0.05$ and $\Delta t = 0.000005$

Now, we consider the case when $\varepsilon = 0.05$ and $\Delta t = 0.000005$. In this case, we easily observe that:

$$\frac{\Delta t}{(\Delta x)^2} = \frac{0.000005}{(0.005)^2} = 0.2 = \frac{1 - c_0}{2},$$

and

$$\frac{c_0 \delta \varepsilon^2}{1 - \delta} = \frac{0.6 \times 0.01 \times (0.05)^2}{1 - 0.01} = 0.0000151515151515 \dots$$

Therefore, we have

$$\frac{c_0 \delta \varepsilon^2}{1 - \delta} = 0.0000151515151515 \dots > \Delta t,$$

which implies that the criteria condition (5.1) holds. Thus, we have the following stable numerical experiment of $(DP)_\delta^\varepsilon$:

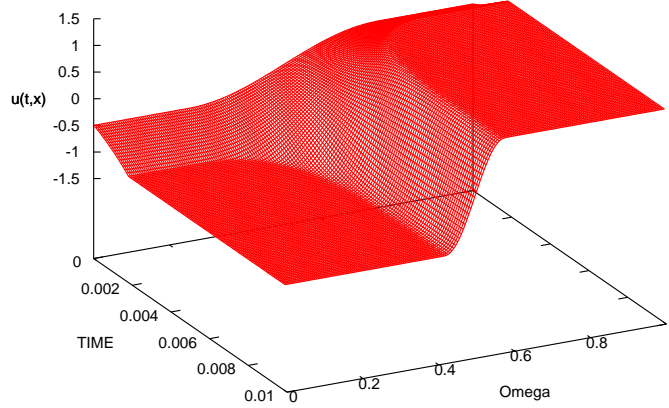


Figure 10: $\varepsilon = 0.05$, $\Delta t = 0.000005$, $\Delta x = 0.005$, $\delta = 0.01$.

5.2 The case when $\varepsilon = 0.007$ and $\Delta t = 0.000005$

Now, we consider the case when $\varepsilon = 0.007$ and $\Delta t = 0.000005$. In this case, we easily observe that:

$$\frac{\Delta t}{(\Delta x)^2} = \frac{0.000005}{(0.005)^2} = 0.2 = \frac{1 - c_0}{2},$$

and

$$\frac{c_0 \delta \varepsilon^2}{1 - \delta} = \frac{0.6 \times 0.01 \times (0.007)^2}{1 - 0.01} = 0.00000029696969 \dots$$

Therefore, we have

$$\frac{c_0 \delta \varepsilon^2}{1 - \delta} = 0.00000029696969 \dots < \Delta t,$$

which implies that the criteria condition (5.1) does not hold. Therefore, we have the following unstable numerical experiment of $(DP)_\delta^\varepsilon$:

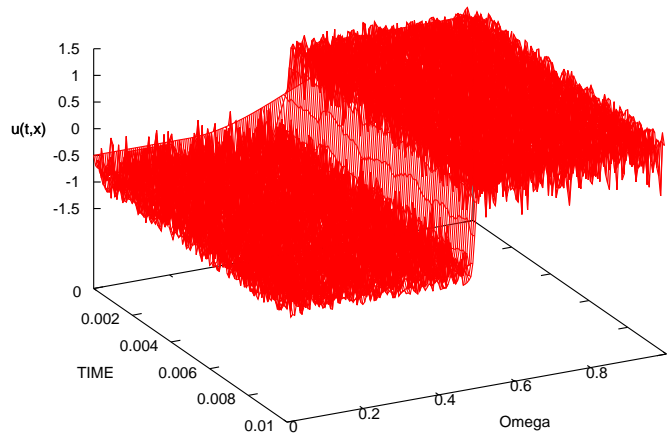


Figure 11: $\varepsilon = 0.007$, $\Delta t = 0.000005$, $\Delta x = 0.005$, $\delta = 0.01$.

5.3 The case when $\varepsilon = 0.007$ and $\Delta t = 0.0000002$

Now, we consider the case when $\varepsilon = 0.007$ and $\Delta t = 0.0000002$. In this case, we have

$$\frac{\Delta t}{(\Delta x)^2} = \frac{0.0000002}{(0.005)^2} = \frac{1}{125} \leq 0.2 = \frac{1 - c_0}{2}$$

and

$$\frac{c_0 \delta \varepsilon^2}{1 - \delta} = 0.00000029696969 \dots > \Delta t,$$

which implies that the criteria condition (5.1) holds. Therefore, we have the following stable numerical experiment of $(DP)_\delta^\varepsilon$:

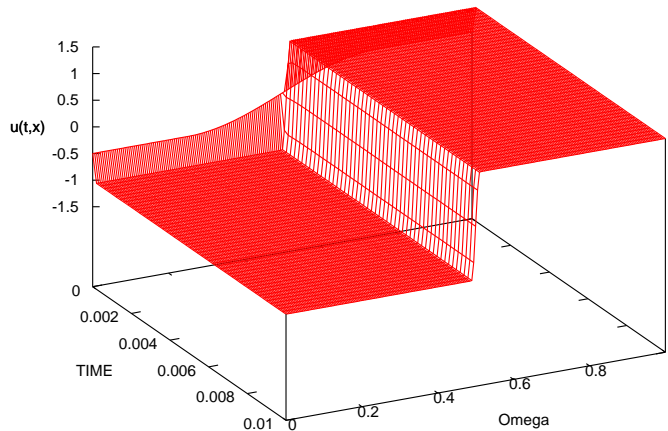


Figure 12: $\varepsilon = 0.007$, $\Delta t = 0.0000002$, $\Delta x = 0.005$, $\delta = 0.01$.

Remark 5.4. We observe from Theorem 5.1 that in order to get stable numerical results of $(DP)_\delta^\varepsilon$, we have to choose the suitable constants ε , δ and the mesh size of time Δt and space Δx . Therefore, if we make a numerical experiment of $(P)^\varepsilon$ for sufficient small ε , we had better consider the original problem $(P)^\varepsilon$ by using a primal-dual active set method in [3], a Lagrange multiplier method in [12] and so on.

5.4 Conclusion of PDE problem $(DP)_\delta^\varepsilon$

By Theorem 5.1 and numerical experiments as above, we conclude that the mesh size of time Δt and space Δx must be satisfied

$$0 < \Delta t \leq \frac{c_0 \delta \varepsilon^2}{1 - \delta}, \quad 0 \leq \frac{\Delta t}{(\Delta x)^2} \leq \frac{1 - c_0}{2} \quad \text{for some constant } c_0 \in (0, 1),$$

in order to get the stable numerical experiments of $(DP)_\delta^\varepsilon$. Also, by Theorems 3.1 and 5.1, we conclude that the value $\delta \varepsilon^2 / (1 - \delta)$ is very important to make numerical experiments of $(DE)_\delta^\varepsilon$ and $(DP)_\delta^\varepsilon$.

References

- [1] S. Allen and J. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metall., **27**(1979), 1084-1095.

- [2] H. Attouch, “Variational Convergence for Functions and Operators,” Pitman Advanced Publishing Program, Boston-London-Melbourne, 1984.
- [3] L. Blank, H. Garcke L. Sarbu and V. Styles, Primal-dual active set methods for Allen-Cahn variational inequalities with nonlocal constraints, *Numer. Methods Partial Differential Equations*, **29**(2013), 999–1030.
- [4] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.
- [5] H. Brézis, M. G. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach space, *Comm. Pure Appl. Math.*, **23**(1970), 123–144.
- [6] L. Bronsard and R.V. Kohn, Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, *J. Differential Equations*, **90**(1991), 211–237.
- [7] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations, *J. Differential Equations*, **96**(1992), 116–141.
- [8] X. Chen and C. M. Elliott, Asymptotics for a parabolic double obstacle problem, *Proc. Roy. Soc. London Ser. A*, **444**(1994), 429–445.
- [9] P. de Mottoni and M. Schatzman, Evolution géométrique d’interfaces, *C. R. Acad. Sci. Paris Ser. I Math.*, **309**(1989), 453–458.
- [10] P. C. Fife, Dynamics of internal layers and diffusive interfaces, *CBMS-NSF Regional Conf. Ser. in Appl. Math.*, **53**, SIAM, Philadelphia, 1988.
- [11] M. H. Farshbaf-Shaker, T. Fukao and N. Yamazaki, Singular limit of Allen-Cahn equation with constraints and its Lagrange multiplier, (to appear).
- [12] M. H. Farshbaf-Shaker, T. Fukao and N. Yamazaki, Lagrange multiplier and singular limit of double obstacle problems for Allen-Cahn Equation with constraints, submitted.
- [13] A. Ito, Asymptotic stability of Allen-Cahn model for nonlinear Laplacian with constraints, *Adv. Math. Sci. Appl.*, **9**(1999), 137–161.
- [14] A. Ito, N. Yamazaki and N. Kenmochi, Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces, *Dynamical systems and differential equations, Vol. I (Springfield, MO, 1996)*, *Discrete Contin. Dynam. Systems* 1998, Added Volume I, 327–349.
- [15] N. Kenmochi, Monotonicity and compactness methods for nonlinear variational inequalities, *Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. 4*, ed. M. Chipot, Chapter 4, 203–298, North Holland, Amsterdam, 2007.
- [16] N. Kenmochi and K. Shirakawa, Stability for a parabolic variational inequality associated with total variation functional, *Funkcial. Ekvac.*, **44**(2001), 119–137.

- [17] U. Mosco, Convergence of convex sets and of solutions variational inequalities, *Advances Math.*, **3**(1969), 510–585.
- [18] T. Ohtsuka, Motion of interfaces by an Allen-Cahn type equation with multiple-well potentials. *Asymptot. Anal.*, **56**(2008), 87–123.
- [19] T. Ohtsuka, K. Shirakawa and N. Yamazaki, Optimal control problems of singular diffusion equation with constraint, *Adv. Math. Sci. Appl.*, **18**(2008), 1–28.
- [20] T. Ohtsuka, K. Shirakawa and N. Yamazaki, Optimal control problem for Allen-Cahn type equation associated with total variation energy, *Discrete Contin. Dyn. Syst. Ser. S*, **5**(2012), 159–181.
- [21] M. Ôtani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, *J. Differential Equations*, **46**(1982), 268–299.
- [22] K. Shirakawa and M. Kimura, Stability analysis for Allen-Cahn type equation associated with the total variation energy, *Nonlinear Anal.*, **60**(2005), 257–282.
- [23] Y. Tonegawa, Integrality of varifolds in the singular limit of reaction-diffusion equations, *Hiroshima Math. J.*, **33**(2003), 323–341.