Remarks on numerical experiments of Allen–Cahn equations with constraint via Yosida approximation†

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Abstract. We consider a one-dimensional Allen–Cahn equation with constraint from the view-point of numerical analysis. Our constraint is the subdifferential of the indicator function on the closed interval, which is the multivalued function. Therefore, it is very difficult to make numerical experiments of our equation. In this paper we approximate our constraint by Yosida approximation. Then, we study the approximating system of our original model numerically. In particular, we give the criteria for the standard forward Euler method to give stable numerical experiments of our approximating equation. Moreover, we give some numerical experiments of approximating equation.

Key Words. Allen–Cahn equation, constraint, subdifferential, Yosida approximation, singular limit, numerical experiments.

1 Introduction

In this paper, for each $\varepsilon \in (0,1]$ we consider the following Allen–Cahn equation with constraint from the view-point of numerical analysis:

$$u^\varepsilon_t - u^\varepsilon_{xx} + \frac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni \frac{u^\varepsilon}{\varepsilon^2} \quad \text{in } Q := (0,T) \times (0,1), \tag{1.1}$$

$$u^\varepsilon_x(t,0) = u^\varepsilon_x(t,1) = 0, \qquad t \in (0,T), \tag{1.2}$$

$$u^\varepsilon(0,x) = u^\varepsilon_0(x), \qquad x \in (0,1), \tag{1.3}$$

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where $0 < T < +\infty$ and u_0^ε is a given initial data. Also, $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1,1]$ defined by

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z \in [-1,1], \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

More precisely, $\partial I_{[-1,1]}(\cdot)$ is a set-valued mapping defined by

$$\partial I_{[-1,1]}(z) := \begin{cases} \emptyset, & \text{if } z < -1 \text{ or } z > 1, \\ [0,\infty), & \text{if } z = 1, \\ \{0\}, & \text{if } -1 < z < 1 \\ (-\infty, 0], & \text{if } z = -1. \end{cases} \tag{1.5}$$

The Allen–Cahn equation was proposed to describe the macroscopic motion of phase boundaries. In the physical context, the function $u^\varepsilon = u^\varepsilon(t,x)$ in $(\mathrm{P})^\varepsilon$:=$\{$(1.1), (1.2), (1.3)$\}$ is the nonconserved order parameter that characterizes the physical structure. For instance, let $v = v(t,x)$ be the local ratio of the volume of pure liquid relative to that of pure solid at time t and position $x \in (0,1)$, defined by

$$v(t,x) := \lim_{r \downarrow 0} \frac{\text{the volume of pure liquid in } B_r(x) \text{ at time } t}{|B_r(x)|},$$

where $B_r(x)$ is the ball in $\mathbb{R}$ with center x and radius r and $|B_r(x)|$ denotes its volume. Put $u^\varepsilon(t,x) := 2v(t,x) - 1$ for any $(t,x) \in Q$. Then, we easily see that $u^\varepsilon(t,x)$ is the nonconserved order parameter that characterizes the physical structure:

$$\begin{cases} u^\varepsilon(t,x) = 1 & \text{on the pure liquid region,} \\ u^\varepsilon(t,x) = -1 & \text{on the pure solid region,} \\ -1 < u^\varepsilon(t,x) < 1 & \text{on the mixture region.} \end{cases}$$

There are vast literatures of Allen–Cahn equation with or without constraint $\partial I_{[-1,1]}(\cdot)$. For such works, we refer to [1, 3, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 22, 23], for instance. In particular, Chen and Elliott [8] considered the singular limit of $(\mathrm{P})^\varepsilon$ as $\varepsilon \to 0$ in the general bounded domain Ω ($\subset \mathbb{R}^N$ with $N \geq 1$).

Note that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to make numerical experiments of $(\mathrm{P})^\varepsilon$. Recently, Farshbaf-Shaker et. al [11] gave the results of the limit of a solution u^ε and an element of $\partial I_{[-1,1]}(u^\varepsilon)$, called the Lagrange multiplier, to $(\mathrm{P})^\varepsilon$ as $\varepsilon \to 0$. Moreover, Farshbaf-Shaker et. al [12] gave numerical experiments to $(\mathrm{P})^\varepsilon$ via the Lagrange multiplier in one dimension of space for sufficient small $\varepsilon \in (0,1]$. Also, they considered some approximating method. In fact, for $\delta > 0$, they use the following Yosida approximation $(\partial I_{[-1,1]})_\delta(\cdot)$ of $\partial I_{[-1,1]}(\cdot)$ defined by:

$$(\partial I_{[-1,1]})_\delta(z) = \frac{[z-1]^+ - [-1-z]^+}{\delta}, \quad \forall z \in \mathbb{R}, \tag{1.6}$$

where $[z]^+$ is the positive part of z. For each $\delta > 0$, they considered the following approximation problem of $(\mathrm{P})^\varepsilon$:

$$(\mathrm{P})^\varepsilon_\delta \begin{cases} (u^\varepsilon_\delta)_t - (u^\varepsilon_\delta)_{xx} + \dfrac{(\partial I_{[-1,1]})_\delta(u^\varepsilon_\delta)}{\varepsilon^2} = \dfrac{u^\varepsilon_\delta}{\varepsilon^2} \quad \text{in } Q := (0,T)\times(0,1), \\ (u^\varepsilon_\delta)_x(t,0) = (u^\varepsilon_\delta)_x(t,1) = 0, \quad t \in (0,T), \\ u^\varepsilon_\delta(0,x) = u^\varepsilon_0(x), \quad x \in (0,1). \end{cases}$$

Then, they gave the following numerical result to $(\mathrm{P})^\varepsilon_\delta$ by the standard explicit finite difference scheme to $(\mathrm{P})^\varepsilon_\delta$ (see [12, Remark 5.3]):

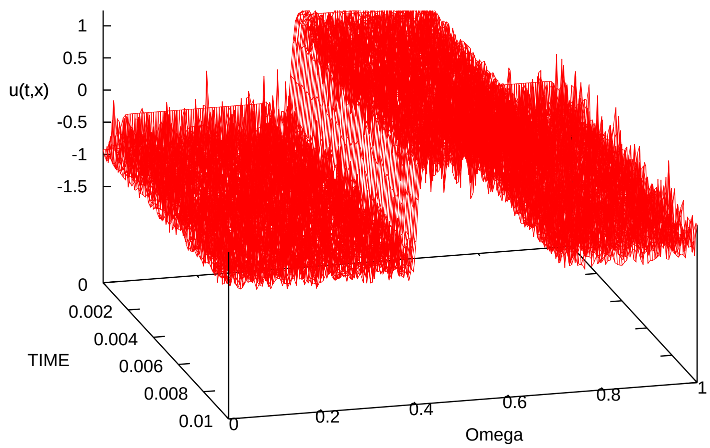

Figure 1: Behaviour of a solution to $(\mathrm{P})^\varepsilon_\delta$ with $\varepsilon = 0.007$ and $\delta = 0.01$.

From Figure 1, we easily see that we have to choose the suitable constants ε, δ and mesh size of time $\triangle t$ and space $\triangle x$ in order to get stable numerical results of $(\mathrm{P})^\varepsilon_\delta$. So, in this paper, for each $\varepsilon > 0$ and $\delta > 0$, we give the criteria for the standard explicit finite difference scheme to give stable numerical experiments of $(\mathrm{P})^\varepsilon_\delta$. To this end, we first consider the following ODE problem, denoted by $(\mathrm{E})^\varepsilon_\delta$:

$$(\mathrm{E})^\varepsilon_\delta \begin{cases} (u^\varepsilon_\delta)_t + \dfrac{(\partial I_{[-1,1]})_\delta(u^\varepsilon_\delta)}{\varepsilon^2} = \dfrac{u^\varepsilon_\delta}{\varepsilon^2} \quad \text{in } \mathbb{R}, \text{ for } t \in (0,T), \\ u^\varepsilon_\delta(0) = u^\varepsilon_0 \quad \text{in } \mathbb{R}. \end{cases}$$

Then, we give the criteria to get stable numerical experiments of $(\mathrm{E})^\varepsilon_\delta$. Also, we give some numerical experiments of $(\mathrm{E})^\varepsilon_\delta$. Moreover, we show the criteria to get stable numerical experiments of PDE problem $(\mathrm{P})^\varepsilon_\delta$. Therefore, the main novelties found in this paper are the following:

(a) We give the criteria to give stable numerical experiments of the ODE problem $(\mathrm{E})^\varepsilon_\delta$. Also, we give numerical experiments to $(\mathrm{E})^\varepsilon_\delta$ for sufficient small $\varepsilon \in (0,1]$.

(b) We give the criteria to give stable numerical experiments of the PDE problem $(\mathrm{P})_\delta^\varepsilon$. Also, we give numerical experiments to $(\mathrm{P})_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0,1]$.

The plan of this paper is as follows. In Section 2, we recall the solvability and convergence result of $(\mathrm{E})_\delta^\varepsilon$. In Section 3, we consider $(\mathrm{E})_\delta^\varepsilon$ numerically. Then, we prove the main result (Theorem 3.1) corresponding to the item (a) listed above. Also, we give numerical experiments to $(\mathrm{E})_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0,1]$ and $\delta \in (0,1]$. In Section 4, we recall the solvability and convergence result of $(\mathrm{P})_\delta^\varepsilon$. In the final Section 5, we consider $(\mathrm{P})_\delta^\varepsilon$ from the view-point of numerical analysis. Then, we prove the main result (Theorem 5.1) corresponding to the item (b) listed above. Also, we give numerical experiments to $(\mathrm{P})_\delta^\varepsilon$ for sufficient small $\varepsilon \in (0,1]$ and $\delta \in (0,1]$.

Notations and basic assumptions

Throughout this paper, we put $H := L^2(0,1)$ with usual real Hilbert space structure, and denote by $(\cdot,\cdot)_H$ the inner product in H. Also, we put $V := H^1(0,1)$ with the usual norm

$$|z|_V := \left\{ |z|_H^2 + |z_x|_H^2 \right\}^{\frac{1}{2}}, \quad z \in V.$$

In Sections 2 and 4, we use some techniques of proper (that is, not identically equal to infinity), l.s.c. (lower semi-continuous), convex functions and their subdifferentials, which are useful in the systematic study of variational inequalities. So, let us outline some notations and definitions. Let W be the real Hilbert space with the inner product $(\cdot,\cdot)_W$. For a proper, l.s.c. and convex function $\psi : W \to \mathbb{R} \cup \{+\infty\}$, the effective domain $D(\psi)$ is defined by

$$D(\psi) := \{z \in W;\ \psi(z) < \infty\}.$$

The subdifferential of ψ is a possibly multi-valued operator in W and is defined by $z^* \in \partial\psi(z)$ if and only if

$$z \in D(\psi) \quad \text{and} \quad (z^*, y - z)_W \le \psi(y) - \psi(z) \quad \text{for all } y \in W.$$

For various properties and related notions of the proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to a monograph by Brézis [4].

Finally, throughout this paper, $C_i = C_i(\cdot), i = 1, 2, 3, \cdots$, denotes positive (or nonnegative) constants depending only on its arguments.

2 Solvability and convergence results of $(\mathrm{E})_\delta^\varepsilon$

We begin by giving the rigorous definition of solutions to our problem $(\mathrm{E})_\delta^\varepsilon$ ($\varepsilon \in (0,1]$ and $\delta \in (0,1]$).

Definition 2.1. *Let $\varepsilon \in (0,1]$, $\delta \in (0,1]$ and $u_0^\varepsilon \in \mathbb{R}$. Then, a function $u_\delta^\varepsilon : [0,T] \to \mathbb{R}$ is called a solution to $(\mathrm{E})_\delta^\varepsilon$ on $[0,T]$, if the following conditions are satisfied:*

(i) $u_\delta^\varepsilon \in W^{1,2}(0,T)$.

(ii) *The following equation holds:*

$$(u_\delta^\varepsilon)_t + \frac{(\partial I_{[-1,1]})_\delta(u_\delta^\varepsilon)}{\varepsilon^2} = \frac{u_\delta^\varepsilon}{\varepsilon^2} \quad \text{in } \mathbb{R}, \text{ for } t \in (0,T).$$

(iii) $u_\delta^\varepsilon(0) = u_0^\varepsilon$ *in* $\mathbb{R}$.

Now, let us recall the solvability result of $(\mathrm{E})_\delta^\varepsilon$ on $[0,T]$.

Proposition 2.1. *Let $\varepsilon \in (0,1]$, $\delta \in (0,1]$ and $u_0^\varepsilon \in \mathbb{R}$ with $|u_0^\varepsilon| \le 1$. Then, there exists a unique solution u_δ^ε to $(\mathrm{E})_\delta^\varepsilon$ on $[0,T]$ in the sense of Definition 2.1.*

Proof. We easily prove the uniqueness of solutions to $(\mathrm{E})_\delta^\varepsilon$ on $[0,T]$ by the quite standard arguments: monotonicity and Gronwall's inequality.

Also, we can show the existence of solutions to $(\mathrm{E})_\delta^\varepsilon$ on $[0,T]$ applying the abstract theory of evolution equations governed by subdifferentials. In fact, we define a function $(I_{[-1,1]})_\delta(\cdot)$ on $\mathbb{R}$ by

$$(I_{[-1,1]})_\delta(z) = \frac{|[z-1]^+|^2 + |[-1-z]^+|^2}{2\delta}, \quad \forall z \in \mathbb{R}. \tag{2.1}$$

Clearly, $(I_{[-1,1]})_\delta(\cdot)$ is proper, l.s.c. and convex on $\mathbb{R}$ with $\partial(I_{[-1,1]})_\delta(\cdot) = (\partial I_{[-1,1]})_\delta(\cdot)$ in $\mathbb{R}$, where $(\partial I_{[-1,1]})_\delta(\cdot)$ is a Yosida approximation of $\partial I_{[-1,1]}(\cdot)$ defined by (1.6).

We easily see that the problem $(\mathrm{E})_\delta^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$(\mathrm{EP})_\delta^\varepsilon \begin{cases} \dfrac{d}{dt}u_\delta^\varepsilon(t) + \dfrac{1}{\varepsilon^2}\partial(I_{[-1,1]})_\delta(u_\delta^\varepsilon(t)) - \dfrac{1}{\varepsilon^2}u_\delta^\varepsilon(t) = 0 \ \text{ in } \mathbb{R}, \ \text{ for } t \in (0,T), \\ u_\delta^\varepsilon(0) = u_0^\varepsilon \ \text{ in } \mathbb{R}. \end{cases} \tag{2.2}$$

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [5, 14, 21]), we can show the existence of a solution u_δ^ε to $(\mathrm{EP})_\delta^\varepsilon$ on $[0,T]$ for each $\varepsilon \in (0,1]$ and $\delta \in (0,1]$ in the sense of Definition 2.1. Thus, the proof of Proposition 2.1 has been completed. □

Next, we recall the convergence result of $(\mathrm{E})_\delta^\varepsilon$ as $\delta \to 0$. To this end, we recall a notion of convergence for convex functions, developed by Mosco [17].

Definition 2.2 (cf. [17])**.** *Let ψ, ψ_n ($n \in \mathbb{N}$) be proper, l.s.c. and convex functions on a Hilbert space W. Then, we say that ψ_n converges to ψ on W in the sense of Mosco [17] as $n \to \infty$, if the following two conditions are satisfied:*

(i) *for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in W as $k \to \infty$, then*

$$\liminf_{k\to\infty} \psi_{n_k}(z_k) \ge \psi(z);$$

(ii) *for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in W such that*

$$z_n \to z \text{ in } W \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \psi_n(z_n) = \psi(z).$$

It is well known that the following lemma holds. Therefore, we omit the detailed proof.

Lemma 2.1 (cf. [2, Section 5], [4, Chapter 2], [15, Section 2])**.**

$$(I_{[-1,1]})_\delta(\cdot) \longrightarrow I_{[-1,1]}(\cdot) \quad \text{on } \mathbb{R} \text{ in the sense of Mosco [17]} \tag{2.3}$$

as $\delta \to 0$.

By Lemma 2.1 and the general convergence theory of evolution equations, we easily get the following result.

Proposition 2.2 (cf. [2, Section 5], [15, Section 2])**.** *Let* $\varepsilon \in (0,1]$*,* $\delta \in (0,1]$ *and* $u_0^\varepsilon \in \mathbb{R}$ *with* $|u_0^\varepsilon| \le 1$*. Also, let* u_δ^ε *be the unique solution to* $(\mathrm{E})_\delta^\varepsilon$ *on* $[0,T]$*. Then, there exists a unique function* $u^\varepsilon \in W^{1,2}(0,T)$ *such that*

$$u_\delta^\varepsilon \longrightarrow u^\varepsilon \quad \text{strongly in } C([0,T]) \quad \text{as} \quad \delta \to 0$$

and u^ε *is the unique solution of the following problem* $(\mathrm{E})^\varepsilon$ *on* $[0,T]$*:*

$$(\mathrm{E})^\varepsilon \begin{cases} u_t^\varepsilon + \dfrac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni \dfrac{u^\varepsilon}{\varepsilon^2} \quad \text{in } \mathbb{R}, \text{ for } t \in (0,T), \\ u^\varepsilon(0) = u_0^\varepsilon \quad \text{in } \mathbb{R}. \end{cases}$$

3 Stable criteria and numerical experiments for $(\mathrm{E})_\delta^\varepsilon$

In this Section we consider $(\mathrm{E})_\delta^\varepsilon$ from the view-point of numerical analysis.

Remark 3.1. *Note from Proposition 2.2 that* $(\mathrm{E})_\delta^\varepsilon$ *is the approximating problem of* $(\mathrm{E})^\varepsilon$*. Also note from (1.5) that the constraint* $\partial I_{[-1,1]}(\cdot)$ *is the multivalued function. Therefore, it is very difficult to study* $(\mathrm{E})^\varepsilon$ *mumerically.*

In order to make numerical experiments of $(\mathrm{E})_\delta^\varepsilon$ via the standard forward Eular method, we consider the following explicit finite difference scheme to $(\mathrm{E})_\delta^\varepsilon$, denoted by of $(\mathrm{DE})_\delta^\varepsilon$:

$$(\mathrm{DE})_\delta^\varepsilon \begin{cases} \dfrac{u^{n+1}-u^n}{\triangle t} + \dfrac{(\partial I_{[-1,1]})_\delta(u^n)}{\varepsilon^2} = \dfrac{u^n}{\varepsilon^2} \quad \text{in } \mathbb{R}, \text{ for } n = 0,1,2,\cdots,N_t, \\ u^0 = u_0^\varepsilon \quad \text{in } \mathbb{R}, \end{cases}$$

where $\triangle t$ is the mesh size of time and N_t is the integer part of number $T/\triangle t$. We easily see that u^n is the approximating solution of $(\mathrm{E})_\delta^\varepsilon$ at the time $t = n\triangle t$

Clearly, the explicit finite difference scheme $(\mathrm{DE})_\delta^\varepsilon$ converges to $(\mathrm{E})_\delta^\varepsilon$ as $\triangle t \to 0$ since $(\mathrm{DE})_\delta^\varepsilon$ is the standard time discretization scheme for $(\mathrm{E})_\delta^\varepsilon$.

Here, we give the unstable numerical experiment of $(\mathrm{DE})_\delta^\varepsilon$ in the case when $T = 0.002$, $\varepsilon = 0.003$, $\delta = 0.01$, the initial data $u_0^\varepsilon = 0.1$ and the mesh size of time $\triangle t = 0.000001$:

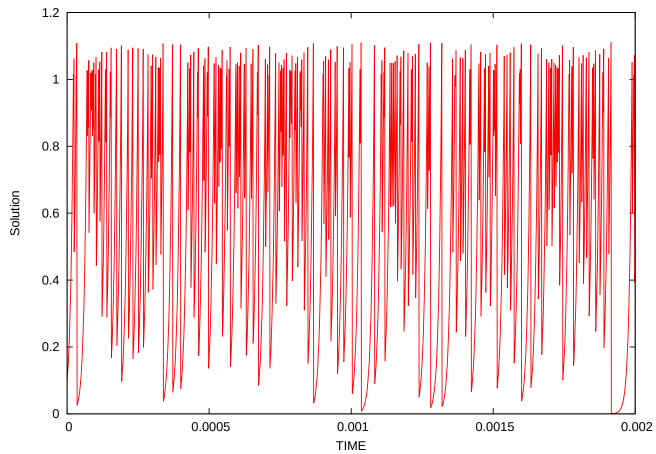

Figure 2: Behaviour of a solution u^n to $(\mathrm{DE})^{\varepsilon}_{\delta}$ with $\varepsilon = 0.003$ and $\delta = 0.01$.

From Figure 2, we easily see that we have to choose the suitable constants ε, δ and mesh size of time $\triangle t$ in order to get stable numerical results of $(\mathrm{DE})^{\varepsilon}_{\delta}$.

Now, let us mention our first main result in this paper, which is concerned with the criteria to give stable numerical experiments of $(\mathrm{DE})^{\varepsilon}_{\delta}$.

Theorem 3.1. *Let $\varepsilon \in (0,1]$, $\delta \in (0,1)$ and $\triangle t \in (0,1]$. Assume $u_0^{\varepsilon} \in (0,1]$ (resp. $u_0^{\varepsilon} \in [-1,0)$) and $T = \infty$. Let $\{u^n; n \geq 0\}$ be the solution to $(\mathrm{DE})^{\varepsilon}_{\delta}$. Then, we have:*

(i) If $\triangle t \in \big(0, \delta\varepsilon^2/(1-\delta)\big)$, u^n converges to $1/(1-\delta)$ (resp. $-1/(1-\delta)$) monotonically as $n \to \infty$.

(ii) If $\triangle t \in \big(\delta\varepsilon^2/(1-\delta), 2\delta\varepsilon^2/(1-\delta)\big)$, u^n oscillates and converges to $1/(1-\delta)$ (resp. $-1/(1-\delta)$) as $n \to \infty$.

Proof. We give the proof of Theorem 3.1 in the case of the initial value $u_0^{\varepsilon} \in (0,1]$.

For simplicity, we set:

$$f_\delta(z) := \big(\partial I_{[-1,1]}\big)_\delta (z) - z \qquad \text{for } z \in \mathbb{R}. \tag{3.1}$$

Then, we easily observe that

$$f_\delta(z) = \begin{cases} \dfrac{1+z}{\delta} - z, & \text{if } z \leq -1, \\ -z, & \text{if } z \in [-1,1], \\ \dfrac{z-1}{\delta} - z, & \text{if } z \geq 1 \end{cases} \tag{3.2}$$

and $z = 0, 1/(1-\delta), -1/(1-\delta)$ are zero points of $f_\delta(\cdot)$. Also, we observe that the difference equation of $(\mathrm{DE})^\varepsilon_\delta$ is reformulated in the following form:

$$u^{n+1} = u^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u^n) \ \text{ in } \mathbb{R}, \text{ for } n = 0, 1, 2, \cdots . \tag{3.3}$$

Note from (3.2) and (3.3) that if $u^n \in (0, 1]$ we have:

$$u^{n+1} = u^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u^n) = u^n - \frac{\triangle t}{\varepsilon^2} \cdot (-u^n) \geq u^n, \tag{3.4}$$

which implies that u^n is increasing with respect to n until $u^{n+1} \geq 1$.

Now, we prove (i). To this end, we assume that $\triangle t \in \big(0, \delta\varepsilon^2/(1-\delta)\big)$. At first, by the mathematical induction, we show:

$$u^i \in \left(0, \frac{1}{1-\delta}\right) \quad \text{for all } i \geq 0. \tag{3.5}$$

Clearly (3.5) holds for $i = 0$ because of $u^0 = u_0^\varepsilon \in (0, 1]$.

Now, we assume that (3.5) holds for all $i = 0, 1, \cdots, n$. Suppose $u^n \in (0, 1]$. Then, we infer from (3.4) that

$$u^{n+1} = \left(1 + \frac{\triangle t}{\varepsilon^2}\right) u^n \leq 1 + \frac{\triangle t}{\varepsilon^2} < 1 + \frac{\delta}{1-\delta} = \frac{1}{1-\delta}.$$

Therefore, by (3.4) and the inequality as above, we obsersve that

$$u^{n+1} \in \left(0, \frac{1}{1-\delta}\right), \quad \text{if } u^n \in (0, 1]. \tag{3.6}$$

Next, if $u^n \in [1, 1/(1-\delta))$, we observe from (3.2) and (3.3) that

$$\begin{aligned} u^n \leq u^{n+1} &= u^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u^n) \\ &= u^n - \frac{\triangle t}{\varepsilon^2} \cdot \left(\frac{u^n - 1}{\delta} - u^n\right) \\ &= u^n + \frac{\triangle t}{\varepsilon^2} \cdot \frac{1 - (1-\delta)u^n}{\delta} \\ &< u^n + \frac{1 - (1-\delta)u^n}{1-\delta} = \frac{1}{1-\delta}, \end{aligned}$$

which implies that

$$u^{n+1} \in \left[1, \frac{1}{1-\delta}\right), \quad \text{if } u^n \in \left[1, \frac{1}{1-\delta}\right). \tag{3.7}$$

From (3.6) and (3.7) we infer that (3.5) holds for $i = n + 1$. Therefore, we conclude from the mathematical induction that (3.5) holds.

Also, by (3.2) and (3.5) we observe that $f_\delta(u^n) \le 0$ for all $n \ge 0$. Therefore, we observe from (3.3) that

$$u^{n+1} = u^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u^n) \ge u^n \quad \text{for all } n \ge 0. \tag{3.8}$$

Therefore, we infer from (3.5) and (3.8) that $\{u^n; n \ge 0\}$ is a bounded and increasing sequence with respect to n. Thus, there exist a subsequence $\{n_k\}$ of $\{n\}$ and a point $u^\infty \in \mathbb{R}$ such that $n_k \to \infty$ as $k \to \infty$ and

$$u^{n_k} \longrightarrow u^\infty \text{ in } \mathbb{R} \text{ as } k \to \infty. \tag{3.9}$$

By taking the limit in (3.3), we easily observe from the continuity of $f_\delta(\cdot)$ that $u^\infty = 1/(1-\delta)$, which is the zero point of $f_\delta(\cdot)$. Hence, taking into account of the uniqueness of the limit point, the proof of (i) has been completed.

Next, we show (ii). To this end, we assume that $\triangle t \in \big(\delta\varepsilon^2/(1-\delta), 2\delta\varepsilon^2/(1-\delta)\big)$. Then, we can find the minimal number $n_0 \in \mathbb{N}$ so that

$$u^{n_0} \in \left(1, \frac{1+\delta}{1-\delta}\right) \text{ and } u^i \in (0,1] \text{ for all } i = 0, 1, \cdots, n_0 - 1. \tag{3.10}$$

In fact, if $u^i \in (0,1]$ for all $i = 0, 1, \cdots, k$, we observe from (3.4) that

$$\begin{aligned} u^{k+1} &= u^k - \frac{\triangle t}{\varepsilon^2} f_\delta(u^k) = \left(1 + \frac{\triangle t}{\varepsilon^2}\right) u^k \\ &= \left(1 + \frac{\triangle t}{\varepsilon^2}\right)^2 u^{k-1} \\ &= \cdots \\ &= \left(1 + \frac{\triangle t}{\varepsilon^2}\right)^{k+1} u^0. \end{aligned} \tag{3.11}$$

Taking into account of (3.11), $u_0 \in (0,1]$ and

$$1 + \frac{\triangle t}{\varepsilon^2} > 1 + \frac{\delta}{1-\delta} > 1,$$

we can find the minimal number $n_0 \in \mathbb{N}$ so that

$$u^{n_0} > 1 \text{ and } u^i \in (0,1] \text{ for all } i = 0, 1, \cdots, n_0 - 1.$$

Also, by (3.4) we observe that

$$u^{n_0} = u^{n_0-1} - \frac{\triangle t}{\varepsilon^2} f_\delta(u^{n_0-1}) = \left(1 + \frac{\triangle t}{\varepsilon^2}\right) u^{n_0-1} < \left(1 + \frac{2\delta}{1-\delta}\right) \cdot 1 = \frac{1+\delta}{1-\delta},$$

thus, (3.10) holds.

To show (ii), we put

$$\triangle t := \frac{\delta\varepsilon^2}{1-\delta}\tau \quad \text{for some } \tau \in (1,2).$$

Then, we observe from (3.2) and (3.3) that

$$\begin{aligned} u^{n_0+1} &= u^{n_0} - \frac{\triangle t}{\varepsilon^2} f_\delta(u^{n_0}) \\ &= u^{n_0} + \frac{\triangle t}{\varepsilon^2} \cdot \frac{1-(1-\delta)u^{n_0}}{\delta} \\ &= (1-\tau)u^{n_0} + \frac{\tau}{1-\delta}. \end{aligned} \tag{3.12}$$

From (3.12) it follows that

$$\frac{u^{n_0+1} + (\tau-1)u^{n_0}}{\tau} = \frac{1}{1-\delta}. \tag{3.13}$$

Therefore, we observe from (3.13) and $\tau \in (1,2)$ that the zero point $1/(1-\delta)$ of $f_\delta(\cdot)$ is in the interval between u^{n_0} and u^{n_0+1}.

Also, by (3.12) we observe that

$$u^{n_0+1} = (1-\tau)u^{n_0} + \frac{\tau}{1-\delta} > (1-\tau)\frac{1+\delta}{1-\delta} + \frac{\tau}{1-\delta} = \frac{1+\delta-\tau\delta}{1-\delta} \geq 1$$

and

$$u^{n_0+1} = (1-\tau)u^{n_0} + \frac{\tau}{1-\delta} < (1-\tau)\cdot 1 + \frac{\tau}{1-\delta} = \frac{1-\delta+\tau\delta}{1-\delta} \leq \frac{1+\delta}{1-\delta},$$

which implies that

$$u^{n_0+1} \in \left(1, \frac{1+\delta}{1-\delta}\right).$$

Therefore, by (3.10), (3.13) and by repeating the procedure as above, we observe that

$$u^n \in \left(1, \frac{1+\delta}{1-\delta}\right) \qquad \text{for all } n \geq n_0 \tag{3.14}$$

and u^n oscillates around the zero point $1/(1-\delta)$ for all $n \geq n_0$.

Also, we observe from (3.12) and (3.14) that

$$\left|u^{n+1} - \frac{1}{1-\delta}\right| = |1-\tau| \left|u^n - \frac{1}{1-\delta}\right| \qquad \text{for all } n \geq n_0. \tag{3.15}$$

Therefore, by $\tau \in (1,2)$, (3.14) and (3.15), there exist a subsequence $\{n_k\}$ of $\{n\}$ such that u^{n_k} oscillates and converges to $1/(1-\delta)$ as $k \to \infty$. Hence, taking into account of the uniqueness of the limit point, the proof of (ii) has been completed. □

Remark 3.2. *Assume $\triangle t \in \left[2\delta\varepsilon^2/(1-\delta), \infty\right)$ and put $\triangle t := \delta\varepsilon^2\tau/(1-\delta)$ for some $\tau \geq 2$. Then, we observe that*

$$1 + \frac{\triangle t}{\varepsilon^2} > 1 + \frac{2\delta}{1-\delta} > 1 \qquad \textit{and} \qquad |1-\tau| \geq 1.$$

Therefore, we infer from the proof of Theorem 3.1 (cf. (3.11), (3.13), (3.15)) that the solution u^n to $(\mathrm{DE})^\varepsilon_\delta$ oscillates as $n \to \infty$, in general.

Remark 3.3. *By (3.3) we easily see that*

$$u^n \equiv 0 \quad \text{for all } n \geq 1, \ \text{if } u_0 = 0,$$

$$u^n \equiv \frac{1}{1-\delta} \quad \text{for all } n \geq 1, \ \text{if } u_0 = \frac{1}{1-\delta}$$

and

$$u^n \equiv \frac{-1}{1-\delta} \quad \text{for all } n \geq 1, \ \text{if } u_0 = \frac{-1}{1-\delta}.$$

By (ii) of Theorem 3.1, we observe that u_n oscillates and converges to the zero point of $f_\delta(\cdot)$ in the case when $\triangle t \in \big(\delta\varepsilon^2/(1-\delta), 2\delta\varepsilon^2/(1-\delta)\big)$. However, in the case when $\triangle t = 2\delta\varepsilon^2/(1-\delta)$, we have the following special case that the solution to $(\mathrm{DE})^\varepsilon_\delta$ does not oscillate and coincides with the zero point of $f_\delta(\cdot)$ after some finite number of iteration.

Corollary 3.1. *Let $\varepsilon \in (0,1]$, $\delta \in (0,1)$, $\triangle t = 2\delta\varepsilon^2/(1-\delta)$ and $n \in \mathbb{N}$. Assume $u_0^\varepsilon := (1-\delta)^{n-1}/(1+\delta)^n$. Then, the solution to $(\mathrm{DE})^\varepsilon_\delta$ is given by*

$$u^i = \begin{cases} \dfrac{(1-\delta)^{n-1-i}}{(1+\delta)^{n-i}}, & \text{if } i = 0, 1, \cdots, n-1, \\[2mm] \dfrac{1}{1-\delta}, & \text{if } i \geq n. \end{cases} \tag{3.16}$$

Proof. Note that $u_0^\varepsilon := (1-\delta)^{n-1}/(1+\delta)^n \in (0,1)$. Therefore, by (3.2) and (3.3) we observe that

$$u^1 = u^0 - \frac{\triangle t}{\varepsilon^2} f_\delta(u^0) = u_0^\varepsilon - \frac{2\delta}{1-\delta}(-u_0^\varepsilon) = \frac{1+\delta}{1-\delta} u_0^\varepsilon = \frac{(1-\delta)^{n-2}}{(1+\delta)^{n-1}}.$$

Similarly, we have:

$$u^2 = u^1 - \frac{\triangle t}{\varepsilon^2} f_\delta(u^1) = \frac{1+\delta}{1-\delta} u^1 = \frac{(1-\delta)^{n-3}}{(1+\delta)^{n-2}}.$$

Repeating this procedure, we easily observe that the solution to $(\mathrm{DE})^\varepsilon_\delta$ is given by (3.16). $\square$

Taking into account of Theorem 3.1, we give numerical experiments of $(\mathrm{DE})^\varepsilon_\delta$ as follows. To this end, we use the following numerical data:

Numerical data of $(\mathrm{DE})^\varepsilon_\delta$.

- $T = 0.002$;
- $\varepsilon = 0.01$;
- $\delta = 0.01$;
- The initial data $u_0^\varepsilon = 0.1$.

Then, we easily observe that:

$$\frac{1}{1-\delta} = \frac{1}{1-0.01} = 1.010101010\cdots$$

and

$$\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\cdots .$$

3.1 The case when $\triangle t = 0.000001$

Now we consider the case when $\triangle t = 0.000001$. In this case, we have:

$$\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\cdots > \triangle t = 0.000001,$$

which implies that (i) of Theorem 3.1 holds. Thus, we have the following stable numerical result of $(\mathrm{DE})_\delta^\varepsilon$. Namely, the solution to $(\mathrm{DE})_\delta^\varepsilon$ converges to the stationary solution $1/(1-\delta) = 1/(1-0.01) = 1.010101010\cdots$.

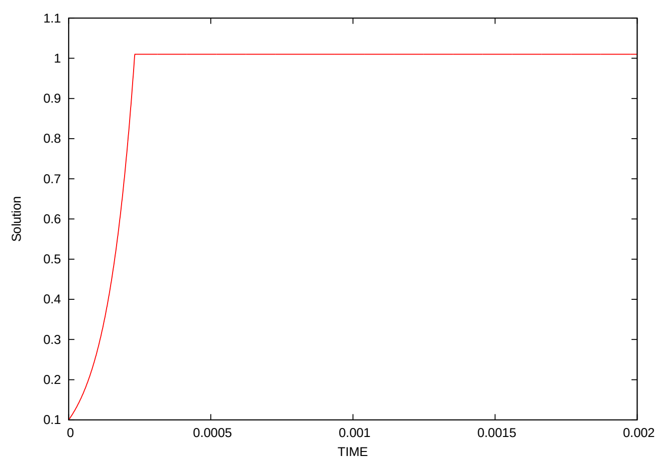

Figure 3: $\dfrac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\cdots > \triangle t = 0.000001.$

Table 1: Numerical data: $\triangle t = 000001$.

number of iterations i	the value of u^i
0	0.100000
1	0.101000
2	0.102010
3	0.103030
4	0.104060
5	0.105101
$\vdots$	$\vdots$
210	0.808144
211	0.816225
212	0.824387
213	0.832631
214	0.840957
215	0.849367
216	0.857861
217	0.866439
218	0.875104
219	0.883855
220	0.892693
221	0.901620
222	0.910636
223	0.919743
224	0.928940
225	0.938230
226	0.947612

number of iterations i	the value of u^i
227	0.957088
228	0.966659
229	0.976325
230	0.986089
231	0.995950
232	1.005909
233	1.010059
234	1.010101
235	1.010101
236	1.010101
237	1.010101
238	1.010101
239	1.010101
240	1.010101
241	1.010101
242	1.010101
243	1.010101
244	1.010101
245	1.010101
246	1.010101
247	1.010101
248	1.010101
249	1.010101
250	1.010101

3.2 The case when $\triangle t = 0.000002$

Now we consider the case when $\triangle t = 0.000002$. In this case, we have:

$$\frac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\cdots < \triangle t = 0.000002 < \frac{2\delta\varepsilon^2}{1-\delta},$$

which implies that (ii) of Theorem 3.1 holds. Thus, we have the following numerical result of $(\mathrm{DE})^\varepsilon_\delta$ that the solution to $(\mathrm{DE})^\varepsilon_\delta$ oscillates and converges to the stationary solution $1/(1-\delta) = 1/(1-0.01) = 1.010101010\cdots$.

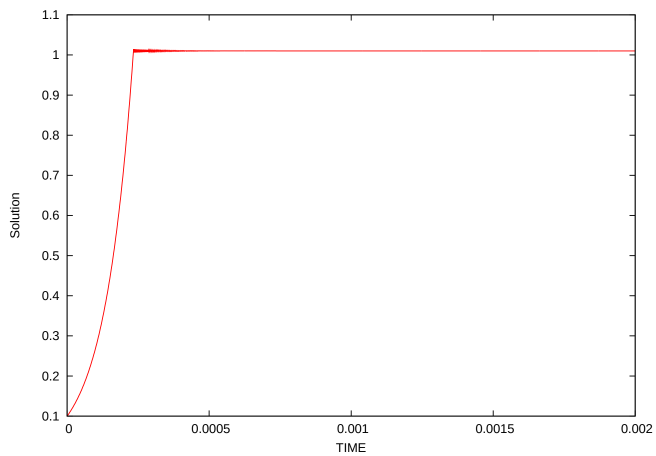

Figure 4: $\dfrac{\delta\varepsilon^2}{1-\delta} = 0.0000010101010\cdots < \triangle t = 0.000002 < \dfrac{2\delta\varepsilon^2}{1-\delta}$.

Table 2: Numerical data: $\triangle t = 0.000002$.

number of iterations i	the value of u^i
0	0.100000
1	0.102000
2	0.104040
3	0.106121
4	0.108243
5	0.110408
$\vdots$	$\vdots$
112	0.918796
113	0.937172
114	0.955916
115	0.975034
116	0.994535
117	1.014425
118	1.005863
119	1.014254
120	1.006031
121	1.014090
122	1.006192
123	1.013932
124	1.006347
125	1.013780
126	1.006496
127	1.013634
128	1.006638
129	1.013494
130	1.006775
131	1.013360
132	1.006907
133	1.013231
134	1.007034
135	1.013107
136	1.007155
137	1.012988
$\vdots$	$\vdots$

number of iterations i	the value of u^i
$\vdots$	$\vdots$
550	1.010100
551	1.010102
552	1.010100
553	1.010102
554	1.010100
555	1.010102
556	1.010100
557	1.010102
558	1.010100
559	1.010102
560	1.010100
561	1.010102
562	1.010100
563	1.010102
564	1.010100
565	1.010102
566	1.010101
567	1.010101
568	1.010101
569	1.010101
570	1.010101
571	1.010101
572	1.010101
573	1.010101
574	1.010101
575	1.010101
576	1.010101
577	1.010101
578	1.010101
579	1.010101
580	1.010101
581	1.010101
582	1.010101

3.3 The case when $\triangle t = 2\dfrac{\delta\varepsilon^2}{1-\delta}$

Now we consider the case when $\triangle t = 2\delta\varepsilon^2/(1-\delta) = 0.0000020202020\cdots$. In this case, we observe Remark 3.2. In fact, we have the following numerical result of $(\mathrm{DE})^{\varepsilon}_{\delta}$ that the solution to $(\mathrm{DE})^{\varepsilon}_{\delta}$ oscillates.

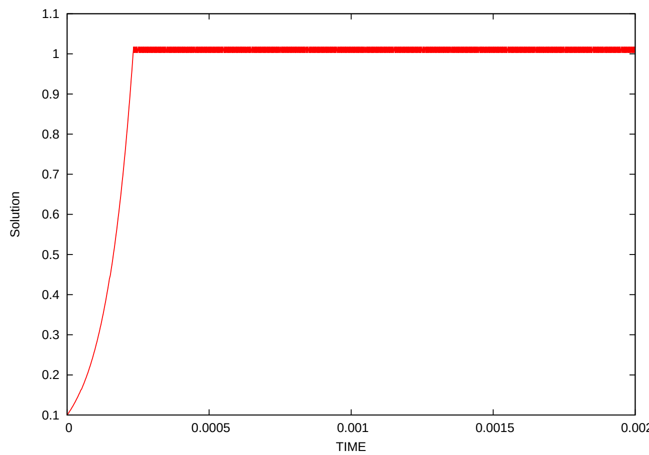

Figure 5: $\triangle t = 2\dfrac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\cdots$.

Table 3: Numerical data: $\triangle t = 2\dfrac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\cdots$.

number of iterations i	the value of u^i
0	0.100000
1	0.102020
2	0.104081
3	0.106184
4	0.108329
5	0.110517
$\vdots$	$\vdots$
100	0.738955
101	0.753883
102	0.769113
103	0.784651
104	0.800502
105	0.816674
106	0.833173
107	0.850004
108	0.867176
109	0.884695
110	0.902568
111	0.920801
112	0.939403
113	0.958381
114	0.977742
115	0.997495
116	1.017646
117	1.002556
118	1.017646
119	1.002556
120	1.017646

number of iterations i	the value of u^i
121	1.002556
122	1.017646
123	1.002556
124	1.017646
125	1.002556
126	1.017646
127	1.002556
128	1.017646
129	1.002556
130	1.017646
131	1.002556
132	1.017646
133	1.002556
134	1.017646
135	1.002556
136	1.017646
137	1.002556
138	1.017646
139	1.002556
140	1.017646
141	1.002556
142	1.017646
143	1.002556
144	1.017646
145	1.002556
146	1.017646
147	1.002556
148	1.017646

3.4 The case when $\triangle t = 0.000005$

Now we consider the case when $\triangle t = 0.000005$. In this case, we have:

$$2\frac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\cdots < \triangle t = 0.000005.$$

Therefore, we observe Remark 3.2. In fact, we have the following numerical result of $(\mathrm{DE})^{\varepsilon}_{\delta}$ that the solution to $(\mathrm{DE})^{\varepsilon}_{\delta}$ oscillates.

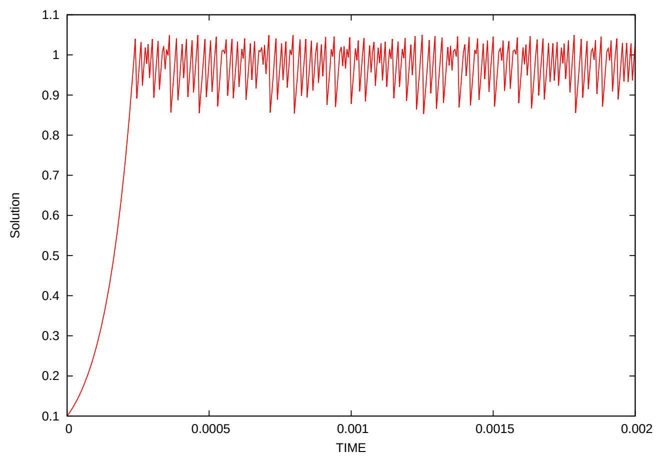

Figure 6: $2\frac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\cdots < \triangle t = 0.000005.$

Table 4: Numerical data: $\triangle t = 0.000005$.

number of iterations i	the value of u^i	number of iterations i	the value of u^i
0	0.100000	52	1.032021
1	0.105000	53	0.923517
2	0.110250	54	0.969693
3	0.115763	55	1.018178
4	0.121551	56	0.978198
5	0.127628	57	1.027108
$\vdots$	$\vdots$	58	0.942925
40	0.703999	59	0.990072
41	0.739199	60	1.039575
42	0.776159	61	0.893678
43	0.814967	62	0.938362
44	0.855715	63	0.985280
45	0.898501	64	1.034544
46	0.943426	65	0.913551
47	0.990597	66	0.959228
48	1.040127	67	1.007189
49	0.891498	68	1.021602
50	0.936073	69	0.964674
51	0.982877	70	1.012907

3.5 The case when $\triangle t = 15\dfrac{\delta\varepsilon^2}{1-\delta}$

Now we consider the case when $\triangle t = 15\delta\varepsilon^2/(1-\delta)$. In this case, we observe Remark 3.2. In fact, we have the following numerical result of $(\mathrm{DE})^{\varepsilon}_{\delta}$ that the solution to $(\mathrm{DE})^{\varepsilon}_{\delta}$ oscillates between three zero points of $f_\delta(\cdot)$.

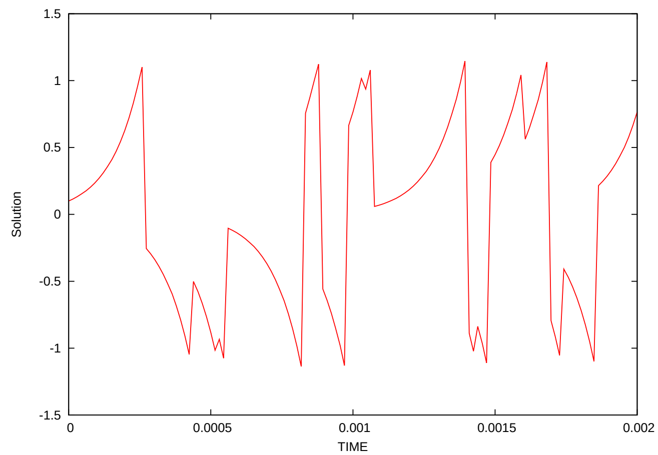

Figure 7: $\triangle t = 15\dfrac{\delta\varepsilon^2}{1-\delta}$.

Table 5: Numerical data: $\triangle t = 15\dfrac{\delta\varepsilon^2}{1-\delta}$.

number of iterations i	the value of u^i
0	0.100000
1	0.115152
2	0.132599
3	0.152689
4	0.175824
5	0.202464
$\vdots$	$\vdots$
6	0.233141
7	0.268465
8	0.309141
9	0.355981
10	0.409918
11	0.472026
12	0.543545
13	0.625901
14	0.720734
15	0.829936
16	0.955684
17	1.100485
18	-0.255276
19	-0.293954
20	-0.338493
21	-0.389780
22	-0.448837
23	-0.516843
24	-0.595152
25	-0.685327
26	-0.789164
27	-0.908735
28	-1.046422
29	-0.501608
30	-0.577609
31	-0.665126
32	-0.765902
33	-0.881948
34	-1.015576

number of iterations i	the value of u^i
35	-0.933445
36	-1.074876
37	-0.103248
38	-0.118892
39	-0.136906
40	-0.157649
41	-0.181536
42	-0.209041
43	-0.240714
44	-0.277186
45	-0.319183
46	-0.367545
47	-0.423233
48	-0.487359
49	-0.561202
50	-0.646232
51	-0.744146
52	-0.856896
53	-0.986728
54	-1.136232
55	0.755739
56	0.870245
57	1.002100
58	1.122108
59	-0.558000
60	-0.642546
61	-0.739901
62	-0.852007
63	-0.981099
64	-1.129751
65	0.664998
66	0.765755
67	0.881779
68	1.015381
69	0.936177
70	1.078022

3.6 Numerical result of Corollary 3.1

In this subsection, we consider Corollary 3.1 numerically. To this end, we use the follwing initial data:

$$u_0 := \frac{(1-\delta)^5}{(1+\delta)^6} = \frac{(1-0.01)^5}{(1+0.01)^6} = 0.8958756\cdots.$$

Then, we have the following numerical experiment of $(\mathrm{DE})^{\varepsilon}_{\delta}$ that Corollary 3.1 holds. Namely, we observe that (3.16) holds with $n = 6$:

Table 6: Numerical data: $\triangle t = 2\dfrac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\cdots$.

number of iterations i	the value of u^i
0	0.895876
1	0.913974
2	0.932438
3	0.951275
4	0.970493
5	0.990099
6	1.010101
7	1.010101
8	1.010101
9	1.010101
10	1.010101

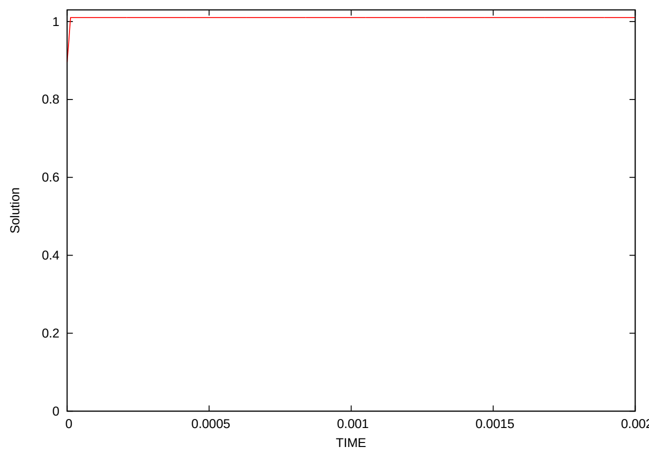

Figure 8: $\triangle t = 2\dfrac{\delta\varepsilon^2}{1-\delta} = 0.0000020202020\cdots$.

3.7 Conclusion of ODE problem $(\mathrm{DE})^{\varepsilon}_{\delta}$

By Theorem 3.1 and numerical experiments as above, we conclude that

(i) the mesh size of time $\triangle t$ must be smaller than $\delta\varepsilon^2/(1-\delta)$ in order to get the stable numerical experiments of $(\mathrm{DE})^{\varepsilon}_{\delta}$.

(ii) we have the stable numerical experiments of $(\mathrm{DE})^{\varepsilon}_{\delta}$ with the initial data $u^{\varepsilon}_0 := (1-\delta)^{n-1}/(1+\delta)^n$, even if the mesh size of time $\triangle t$ is equal to $2\delta\varepsilon^2/(1-\delta)$.

4 Solvability and convergence results for $(\mathrm{P})^{\varepsilon}_{\delta}$

We begin by giving the rigorous definition of solutions to our PDE problem $(\mathrm{P})^{\varepsilon}_{\delta}$ ($\varepsilon \in (0,1]$ and $\delta \in (0,1]$).

Definition 4.1. *Let $\varepsilon \in (0,1]$, $\delta \in (0,1]$ and $u^{\varepsilon}_0 \in H$. Then, a function $u^{\varepsilon}_{\delta} : [0,T] \to H$ is called a solution to $(\mathrm{P})^{\varepsilon}_{\delta}$ on $[0,T]$, if the following conditions are satisfied:*

(i) $u^{\varepsilon}_{\delta} \in W^{1,2}(0,T;H) \cap L^{\infty}(0,T;V)$.

(ii) *The following variational identity holds:*

$$((u^{\varepsilon}_{\delta})_t(t), z)_H + ((u^{\varepsilon}_{\delta})_x(t), z_x)_H + \left(\frac{(\partial I_{[-1,1]})_{\delta}(u^{\varepsilon}_{\delta}(t))}{\varepsilon^2}, z \right)_H = \left(\frac{u^{\varepsilon}_{\delta}(t)}{\varepsilon^2}, z \right)_H$$
$$\text{for all } z \in V \text{ and a.e. } t \in (0,T).$$

(iii) $u^\varepsilon_\delta(0) = u^\varepsilon_0$ *in* H.

Now, let us recall the solvability result of $(\mathrm{P})^\varepsilon_\delta$ on $[0, T]$.

Proposition 4.1. *Let* $\varepsilon \in (0, 1]$ *and* $\delta \in (0, 1]$. *Assume the following condition:*

(A) $u^\varepsilon_0 \in K := \{z \in V\ ;\ -1 \le z(x) \le 1\ a.e.\ x \in (0, 1)\}$.

Then, for each $u^\varepsilon_0 \in K$, *there exists a unique solution* u^ε_δ *to* $(\mathrm{P})^\varepsilon_\delta$ *on* $[0, T]$ *in the sense of Definition 4.1.*

Proof. By the same argument as in Proposition 2.1, we can show the existence-uniqueness of a solution u^ε_δ to $(\mathrm{P})^\varepsilon_\delta$ on $[0, T]$ for each $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$. In fact, we easily prove the uniqueness of solutions to $(\mathrm{P})^\varepsilon_\delta$ on $[0, T]$ by the quite standard arguments: monotonicity and Gronwall's inequality.

Also, we can show the existence of solutions to $(\mathrm{P})^\varepsilon_\delta$ on $[0, T]$ applying the abstract theory of evolution equations governed by subdifferentials. In fact, we define a functional $\varphi^\varepsilon_\delta$ on H by

$$\varphi^\varepsilon_\delta(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |z_x|^2 dx + \dfrac{1}{\varepsilon^2}\int_0^1 (I_{[-1,1]})_\delta(z(x))dx, & \text{if } z \in V, \\ \infty, & \text{otherwise}, \end{cases} \tag{4.1}$$

where $(I_{[-1,1]})_\delta(\cdot)$ is the function defined in (2.1). Clearly, $\varphi^\varepsilon_\delta$ is proper, l.s.c. and convex on H with the effective domain $D(\varphi) = V$.

We easily see that the problem $(\mathrm{P})^\varepsilon_\delta$ can be rewritten as in an abstract framework of the form:

$$(\mathrm{PP})^\varepsilon_\delta \begin{cases} \dfrac{d}{dt}u^\varepsilon_\delta(t) + \partial\varphi^\varepsilon_\delta(u^\varepsilon_\delta(t)) - \dfrac{1}{\varepsilon^2}u^\varepsilon_\delta(t) = 0 \ \text{ in } H, \ \text{for } t > 0, \\ u^\varepsilon(0) = u^\varepsilon_0 \ \text{ in } H. \end{cases}$$

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [5, 14, 21]), we can show the existence of a solution u^ε_δ to $(\mathrm{PP})^\varepsilon_\delta$, hence, $(\mathrm{P})^\varepsilon_\delta$, on $[0, T]$ for each $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$ in the sense of Definition 4.1. Thus, the proof of Proposition 4.1 has been completed. □

Next, we recall the convergence result of $(\mathrm{P})^\varepsilon_\delta$ as $\delta \to 0$. Taking account of Lemma 2.1 (cf. (2.3)), we easily observe that the following lemma holds.

Lemma 4.1 (cf. [2, Section 5], [4, Chapter 2], [15, Section 2])**.** *Let* $\varepsilon \in (0, 1]$, *and define a functional* φ^ε *on* H *by*

$$\varphi^\varepsilon(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |z_x|^2 dx + \dfrac{1}{\varepsilon^2}\int_0^1 I_{[-1,1]}(z(x))dx, & \textit{if } z \in V, \\ \infty, & \textit{otherwise}. \end{cases}$$

Then,

$$\varphi^\varepsilon_\delta(\cdot) \longrightarrow \varphi^\varepsilon(\cdot) \quad \textit{on } H \textit{ in the sense of Mosco [17]}$$

as $\delta \to 0$.

By Lemma 4.1 and the general convergence theory of evolution equations, we easily get the following result.

Proposition 4.2 (cf. [2, Section 5], [15, Section 2]). *Let $\varepsilon \in (0,1]$, $\delta \in (0,1]$ and $u_0^\varepsilon \in K$. Also, let u_δ^ε be the unique solution to $(\mathrm{P})_\delta^\varepsilon$ on $[0,T]$. Then, u_δ^ε converges to the unique function u^ε to $(\mathrm{P})^\varepsilon$ on $[0,T]$ in the sense that*

$$u_\delta^\varepsilon \longrightarrow u^\varepsilon \quad \textit{strongly in } C([0,T];H) \quad \textit{as} \quad \delta \to 0. \tag{4.2}$$

Proof. We easily observe that the problem $(\mathrm{P})^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$(\mathrm{PP})^\varepsilon \begin{cases} \dfrac{d}{dt}u^\varepsilon(t) + \partial\varphi^\varepsilon(u^\varepsilon(t)) - \dfrac{1}{\varepsilon^2}u^\varepsilon(t) \ni 0 \ \text{ in } H, \ \text{for } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \ \text{ in } H. \end{cases}$$

Therefore, by Lemma 4.1 and the abstract convergence theory of evolution equations (cf. [2, 15]), we observe that the solution u_δ^ε to $(\mathrm{PP})_\delta^\varepsilon$ converges to the unique solution u^ε to $(\mathrm{PP})^\varepsilon$ on $[0,T]$ as $\delta \to 0$ in the sense of (4.2). Note that u^ε (resp. u_δ^ε) is the unique solution to $(\mathrm{P})^\varepsilon$ (resp. $(\mathrm{P})_\delta^\varepsilon$) on $[0,T]$ (cf. Proposition 4.1). Thus, we conclude that Proposition 4.2 holds. □

5 Stable criteria and numerical experiments for $(\mathrm{P})_\delta^\varepsilon$

In this Section we consider $(\mathrm{P})_\delta^\varepsilon$ from the view-point of numerical analysis.

Remark 5.1. *Note from Proposition 4.2 that $(\mathrm{P})_\delta^\varepsilon$ is the approximating problem of $(\mathrm{P})^\varepsilon$. Also note from (1.5) that the constraint $\partial I_{[-1,1]}(\cdot)$ is the multivalued function. Therefore, it is very difficult to study $(\mathrm{P})^\varepsilon$ numerically.*

In order to make numerical experiments of $(\mathrm{P})_\delta^\varepsilon$, we consider the following explicit finite difference scheme to $(\mathrm{P})_\delta^\varepsilon$, denoted by of $(\mathrm{DP})_\delta^\varepsilon$:

$$(\mathrm{DP})_\delta^\varepsilon \begin{cases} \dfrac{u_k^{n+1} - u_k^n}{\triangle t} - \dfrac{u_{k-1}^n - 2u_k^n + u_{k+1}^n}{(\triangle x)^2} + \dfrac{(\partial I_{[-1,1]})_\delta(u_k^n)}{\varepsilon^2} = \dfrac{u_k^n}{\varepsilon^2} \\ \qquad \text{for } n = 0,1,2,\cdots,N_t \text{ and } k = 1,2,\cdots,N_x - 1, \\ \qquad u_0^n = u_1^n, \quad u_{N_x}^n = u_{N_x-1}^n \ \text{ for } n = 1,2,\cdots,N_t, \\ \qquad\qquad u_k^0 = u_0^\varepsilon(x_k) \ \text{ for } k = 0,1,2,\cdots,N_x, \end{cases}$$

where $\triangle t$ is the mesh size of time, $\triangle x$ is the mesh size of space, N_t is the integer part of number $T/\triangle t$, N_x is the integer part of number $1/\triangle x$ and $x_k := k\triangle x$. We easily see that u_k^n is the approximating solution of $(\mathrm{P})_\delta^\varepsilon$ at the time $t_n := n\triangle t$ and the position $x_k := k\triangle x$.

Clearly, the explicit finite difference scheme $(\mathrm{DP})_\delta^\varepsilon$ converges to $(\mathrm{P})_\delta^\varepsilon$ as $\triangle t \to 0$ and $\triangle x \to 0$.

From Figure 1, we easily see that we have to choose the suitable constants ε, δ, the mesh size of time $\triangle t$ and the mesh size of space $\triangle x$ in order to get stable numerical results of $(\mathrm{DP})_\delta^\varepsilon$. Now, let us mention our second main result in this paper, which is concerned with the stability of $(\mathrm{DE})_\delta^\varepsilon$.

Theorem 5.1. *Let $\varepsilon \in (0,1]$, $\delta \in (0,1)$, $\triangle t \in (0,1]$, $\triangle x \in (0,1]$, $T > 0$ and $u_0^\varepsilon \in K$, where K is the set of initial value defined in Proposition 4.1 (cf.* (A)*). Let N_x be the integer part of number $1/\triangle x$, and let $\{u_k^n; n \geq 0, k = 0, 1, \cdots, N_x\}$ be the solution to* $(\mathrm{DP})_\delta^\varepsilon$*. Also, let $c_0 \in (0,1)$ and assume that*

$$0 < \triangle t \leq \frac{c_0 \delta \varepsilon^2}{1-\delta} \quad \text{and} \quad 0 \leq \frac{\triangle t}{(\triangle x)^2} \leq \frac{1-c_0}{2}. \tag{5.1}$$

Then, we have:

(i) the solution to $(\mathrm{DP})_\delta^\varepsilon$ *is bounded in the following sense:*

$$\max_{0 \leq k \leq N_x} |u_k^n| \leq \frac{1}{1-\delta} \quad \text{for all } n \geq 0. \tag{5.2}$$

(ii) $\{u_k^n; k = 0, 1, \cdots, N_x\}$ does not oscillate with respect to $n \geq 0$.

Proof. We first show (i), i.e., (5.2) by the mathematical induction.

Clearly (5.2) holds for $n = 0$ because of $u_0^\varepsilon \in K$.

Now, we assume that

$$\max_{0 \leq k \leq N_x} \left|u_k^i\right| \leq \frac{1}{1-\delta} \quad \text{for all } i = 0, 1, \cdots, n. \tag{5.3}$$

We easily observe that the explicit finite difference problem $(\mathrm{DP})_\delta^\varepsilon$ can be reformulated as in the following form:

$$\begin{gathered} u_k^{n+1} = r u_{k-1}^n + r u_{k+1}^n + (1-2r) u_k^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u_k^n) \\ \text{for all } n = 0, 1, 2, \cdots, N_t \text{ and } k = 1, 2, \cdots, N_x - 1, \end{gathered} \tag{5.4}$$

where we put $r := \triangle t/(\triangle x)^2$ and $f_\delta(\cdot)$ is the function defined by (3.2).

We easily observe from (5.1), (5.3) and (5.4) that

$$\begin{aligned} \frac{1}{1-\delta} - u_k^{n+1} &= r\left(\frac{1}{1-\delta} - u_{k-1}^n\right) + r\left(\frac{1}{1-\delta} - u_{k+1}^n\right) \\ &\quad + (1-2r)\left(\frac{1}{1-\delta} - u_k^n\right) + \frac{\triangle t}{\varepsilon^2} f_\delta(u_k^n) \\ &\geq (1-2r)\left(\frac{1}{1-\delta} - u_k^n\right) + \frac{\triangle t}{\varepsilon^2} f_\delta(u_k^n) \\ &\qquad \text{for all } k = 1, 2, \cdots, N_x - 1. \end{aligned} \tag{5.5}$$

From (3.2), (5.1) and (5.3) we infer that the function $[-1/(1-\delta), 1/(1-\delta)] \ni z \to (1-2r)\left(1/(1-\delta) - z\right) + \triangle t/\varepsilon^2 f_\delta(z)$ is non-negative and continuous. In fact, it follows from (3.2) that the function $[-1/(1-\delta), 1] \ni z \to (1-2r)\left(1/(1-\delta) - z\right) + \triangle t/\varepsilon^2 f_\delta(z)$

attains a minimum value at $z = 1$. Therefore, we observe from (3.2) and (5.1) that

$$\begin{aligned}
&(1-2r)\left(\frac{1}{1-\delta}-z\right)+\frac{\triangle t}{\varepsilon^2}f_\delta(z)\\
\geq&(1-2r)\left(\frac{1}{1-\delta}-1\right)+\frac{\triangle t}{\varepsilon^2}f_\delta(1)\\
=&(1-2r)\cdot\frac{\delta}{1-\delta}-\frac{\triangle t}{\varepsilon^2}\\
\geq&\frac{c_0\delta}{1-\delta}-\frac{\triangle t}{\varepsilon^2}\\
\geq&0 \qquad \text{for all } z\in\left[-\frac{1}{1-\delta},1\right].
\end{aligned} \tag{5.6}$$

Also, for any $z \in [1, 1/(1-\delta)]$, we observe from (3.2) that

$$\begin{aligned}
&(1-2r)\left(\frac{1}{1-\delta}-z\right)+\frac{\triangle t}{\varepsilon^2}f_\delta(z)\\
=&(1-2r)\left(\frac{1}{1-\delta}-z\right)+\frac{\triangle t}{\varepsilon^2}\cdot\left(\frac{z-1}{\delta}-z\right)\\
=&\left[\frac{1-\delta}{\delta\varepsilon^2}\triangle t-(1-2r)\right]z+(1-2r)\frac{1}{1-\delta}-\frac{\triangle t}{\delta\varepsilon^2}.
\end{aligned} \tag{5.7}$$

Here we note from (5.1) that

$$\frac{1-\delta}{\delta\varepsilon^2}\triangle t-(1-2r)\leq\frac{1-\delta}{\delta\varepsilon^2}\triangle t-c_0\leq 0.$$

Therefore, we infer from (5.7) that the function $[1, 1/(1-\delta)] \ni z \to (1-2r)\,(1/(1-\delta)-z)+\triangle t/\varepsilon^2 f_\delta(z)$ is non-increasing and attains a minimum value at $z = 1/(1-\delta)$. Hence, we have:

$$\begin{gathered}
(1-2r)\left(\frac{1}{1-\delta}-z\right)+\frac{\triangle t}{\varepsilon^2}f_\delta(z)\geq\frac{\triangle t}{\varepsilon^2}f_\delta\left(\frac{1}{1-\delta}\right)=0\\
\text{for all } z\in\left[1,\frac{1}{1-\delta}\right].
\end{gathered} \tag{5.8}$$

Thus, we observe from (5.6) and (5.8) that

$$(1-2r)\left(\frac{1}{1-\delta}-z\right)+\frac{\triangle t}{\varepsilon^2}f_\delta(z)\geq 0,\quad \forall z\in\left[-\frac{1}{1-\delta},\frac{1}{1-\delta}\right],$$

which implies from (5.3) and (5.5) that

$$\frac{1}{1-\delta}-u_k^{n+1}\geq 0\qquad\text{for all } k=1,2,\cdots,N_x-1. \tag{5.9}$$

Similarly, we observe from (5.1), (5.3) and (5.4) that

$$
\begin{aligned}
u_k^{n+1}+\frac{1}{1-\delta}=&r\left(u_{k-1}^n+\frac{1}{1-\delta}\right)+r\left(u_{k+1}^n+\frac{1}{1-\delta}\right)\\
&+(1-2r)\left(u_k^n+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(u_k^n)\\
\geq&(1-2r)\left(u_k^n+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(u_k^n) \qquad (5.10)\\
&\text{for all } k=1,2,\cdots,N_x-1.
\end{aligned}
$$

By the similar arguments as above, we observe that the function $[-1/(1-\delta),1/(1-\delta)]\ni z\to(1-2r)\,(z+1/(1-\delta))-\triangle t/\varepsilon^2 f_\delta(z)$ is non-negative and continuous. In fact, it follows from (3.2) that the function $[-1,1/(1-\delta)]\ni z\to(1-2r)\,(z+1/(1-\delta))-\triangle t/\varepsilon^2 f_\delta(z)$ attains a minimum value at $z=-1$. Therefore, we observe from (3.2) and (5.1) that

$$
\begin{aligned}
&(1-2r)\left(z+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(z)\\
\geq&(1-2r)\left(-1+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(-1)\\
=&(1-2r)\cdot\frac{\delta}{1-\delta}-\frac{\triangle t}{\varepsilon^2}\\
\geq&\frac{c_0\delta}{1-\delta}-\frac{\triangle t}{\varepsilon^2}\\
\geq&0\quad\text{for all } z\in\left[-1,\frac{1}{1-\delta}\right]. \qquad (5.11)
\end{aligned}
$$

Also, for any $z\in[-1/(1-\delta),-1]$, we observe from (3.2) that

$$
\begin{aligned}
&(1-2r)\left(z+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(z)\\
=&(1-2r)\left(z+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}\cdot\left(\frac{1+z}{\delta}-z\right)\\
=&\left[(1-2r)-\frac{1-\delta}{\delta\varepsilon^2}\triangle t\right]z+(1-2r)\frac{1}{1-\delta}-\frac{\triangle t}{\delta\varepsilon^2}. \qquad (5.12)
\end{aligned}
$$

Here we note from (5.1) that

$$
(1-2r)-\frac{1-\delta}{\delta\varepsilon^2}\triangle t\geq c_0-\frac{1-\delta}{\delta\varepsilon^2}\triangle t\geq 0.
$$

Therefore, we infer from (5.12) that the function $[-1/(1-\delta),-1]\ni z\to(1-2r)\,(z+1/(1-\delta))-\triangle t/\varepsilon^2 f_\delta(z)$ is non-decreasing and attains a minimum value at $z=-1/(1-\delta)$. Hence, we have:

$$
\begin{aligned}
&(1-2r)\left(z+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(z)\geq-\frac{\triangle t}{\varepsilon^2}f_\delta\left(-\frac{1}{1-\delta}\right)=0\\
&\text{for all } z\in\left[-\frac{1}{1-\delta},-1\right]. \qquad (5.13)
\end{aligned}
$$

Thus, we observe from (5.11) and (5.13) that

$$(1-2r)\left(z+\frac{1}{1-\delta}\right)-\frac{\triangle t}{\varepsilon^2}f_\delta(z)\geq 0,\qquad \forall z\in\left[-\frac{1}{1-\delta},\frac{1}{1-\delta}\right],$$

which implies from (5.3) and (5.10) that

$$u_k^{n+1}+\frac{1}{1-\delta}\geq 0\quad \text{for all } k=1,2,\cdots,N_x-1. \tag{5.14}$$

Taking into account of Neumann boundary condition, namely, by $u_0^{n+1}=u_1^{n+1}$ and $u_{N_x}^{n+1}=u_{N_x-1}^{n+1}$, we observe from (5.9) and (5.14) that

$$\max_{0\leq k\leq N_x}\left|u_k^{n+1}\right|\leq\frac{1}{1-\delta},$$

which implies that (5.3) holds for $i=n+1$. Therefore, we conclude from the mathematical induction that (5.2) holds. Hence, the proof of (i) of Theorem 5.1 has been completed.

Next, we show (ii) by the standard arguments. Namely, we reformulate $(\mathrm{DP})_\delta^\varepsilon$ as in the following form:

$$\begin{pmatrix} u_1^{n+1}\\ u_2^{n+1}\\ \vdots\\ u_{N_x-1}^{n+1}\end{pmatrix}=\begin{pmatrix} 1-2r & r & & & & \\ r & 1-2r & r & & & \\ & r & 1-2r & r & & \\ & & \ddots & \ddots & & \\ & & & r & 1-2r & r\\ & & & & r & 1-2r\end{pmatrix}\begin{pmatrix} u_1^{n}\\ u_2^{n}\\ \vdots\\ u_{N_x-1}^{n}\end{pmatrix}$$
$$+r\begin{pmatrix} u_0^n\\ 0\\ \vdots\\ 0\\ u_{N_x}^n\end{pmatrix}+\begin{pmatrix} -\frac{\triangle t}{\varepsilon^2}f_\delta(u_1^n)\\ -\frac{\triangle t}{\varepsilon^2}f_\delta(u_2^n)\\ \vdots\\ -\frac{\triangle t}{\varepsilon^2}f_\delta(u_{N_x-1}^n)\end{pmatrix}. \tag{5.15}$$

Here by taking into account of Neumann boundary condition and initial condition, namely, by $u_0^n=u_1^n$ and $u_{N_x}^n=u_{N_x-1}^n$ for all $n\geq 0$, we observe that (5.15) is reformulated as in the following form:

$$\boldsymbol{u}^{(n+1)}=A\boldsymbol{u}^{(n)}+\begin{pmatrix} ru_1^n-\frac{\triangle t}{\varepsilon^2}f_\delta(u_1^n)\\ -\frac{\triangle t}{\varepsilon^2}f_\delta(u_2^n)\\ \vdots\\ ru_{N_x-1}^n-\frac{\triangle t}{\varepsilon^2}f_\delta(u_{N_x-1}^n)\end{pmatrix}, \tag{5.16}$$

where we put

$$\boldsymbol{u}^{(n)} := \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N_x-1}^n \end{pmatrix}, \qquad A = \begin{pmatrix} 1-2r & r & & & & \\ r & 1-2r & r & & & \\ & r & 1-2r & r & & \\ & & \ddots & \ddots & & \\ & & & r & 1-2r & r \\ & & & & r & 1-2r \end{pmatrix}.$$

Noting from (3.2) that

$$f_\delta(u_k^n) = \begin{cases} \dfrac{1-\delta}{\delta} u_k^n + \dfrac{1}{\delta} & \text{if } u_k^n \le -1, \\ -u_k^n & \text{if } u_k^n \in [-1, 1], \\ \dfrac{1-\delta}{\delta} u_k^n - \dfrac{1}{\delta} & \text{if } u_k^n \ge 1. \end{cases} \tag{5.17}$$

By putting

$$b_k^n := \begin{cases} -1 & \text{if } u_k^n \in [-1,1], \\ \dfrac{1-\delta}{\delta} & \text{if } u_k^n \notin [-1,1], \end{cases} \qquad \tilde{b}_k^n := \begin{cases} \dfrac{1}{\delta} & \text{if } u_k^n \le -1, \\ 0 & \text{if } u_k^n \in [-1,1], \\ -\dfrac{1}{\delta} & \text{if } u_k^n \ge 1, \end{cases} \tag{5.18}$$

we observe from (5.17) and (5.18) that:

$$\begin{pmatrix} ru_1^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u_1^n) \\ -\frac{\triangle t}{\varepsilon^2} f_\delta(u_2^n) \\ \vdots \\ ru_{N_x-1}^n - \frac{\triangle t}{\varepsilon^2} f_\delta(u_{N_x-1}^n) \end{pmatrix}$$

$$= \begin{pmatrix} r - \frac{\triangle t}{\varepsilon^2} b_1^n & & & \\ & -\frac{\triangle t}{\varepsilon^2} b_2^n & & \\ & & \ddots & \\ & & & r - \frac{\triangle t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N_x-1}^n \end{pmatrix} + \begin{pmatrix} -\frac{\triangle t}{\varepsilon^2} \tilde{b}_1^n \\ -\frac{\triangle t}{\varepsilon^2} \tilde{b}_2^n \\ \vdots \\ -\frac{\triangle t}{\varepsilon^2} \tilde{b}_{N_x-1}^n \end{pmatrix}.$$

By using the matrix as above, we observe that (5.15) can be rewritten as in the following form:

$$\boldsymbol{u}^{(n+1)} = A\boldsymbol{u}^{(n)} + B\boldsymbol{u}^{(n)} + \tilde{\boldsymbol{b}}^{(n)}, \quad (n \ge 0), \tag{5.19}$$

where we put

$$B := \begin{pmatrix} r - \frac{\triangle t}{\varepsilon^2} b_1^n & & & \\ & -\frac{\triangle t}{\varepsilon^2} b_2^n & & \\ & & \ddots & \\ & & & r - \frac{\triangle t}{\varepsilon^2} b_{N_x-1}^n \end{pmatrix}, \qquad \tilde{\boldsymbol{b}}^{(n)} := \begin{pmatrix} -\frac{\triangle t}{\varepsilon^2} \tilde{b}_1^n \\ -\frac{\triangle t}{\varepsilon^2} \tilde{b}_2^n \\ \vdots \\ -\frac{\triangle t}{\varepsilon^2} \tilde{b}_{N_x-1}^n \end{pmatrix}.$$

By the general theory, we observe that the eigenvalue λ_j of matrix A is given by:

$$\lambda_j := 1 - 4r\sin^2\left(\frac{j\pi}{2N_x}\right) \qquad \text{for } j = 1, 2, \cdots, N_x - 1, \tag{5.20}$$

which implies that λ_1 (rest. λ_{N_x-1}) is the maximum (rest. minimum) eigenvalue of A.

Now let $\{\tilde{\lambda}_j; j = 1, 2, \cdots, N_x - 1\}$ be the set of all eigenvalues of matrix $\tilde{A} := A + B$ such that

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{N_x-1}.$$

Also, let $\{\lambda_j^B; j = 1, 2, \cdots, N_x - 1\}$ be the set of all eigenvalues of B such that

$$\lambda_1^B \geq \lambda_2^B \geq \cdots \geq \lambda_{N_x-1}^B.$$

Then, by the abstract perturbation theory of matrix, we observe that:

$$\lambda_{N_x-1} + \lambda_j^B \leq \tilde{\lambda}_j \leq \lambda_1 + \lambda_j^B, \quad \forall j = 1, 2, \cdots, N_x - 1. \tag{5.21}$$

Since B is a symmetric matrix, it follows from (5.1), (5.18) and (5.20) that

$$\begin{aligned}
\lambda_{N_x-1} + \lambda_j^B &\geq 1 - 4r\sin^2\left(\frac{(N_x-1)\pi}{2N_x}\right) - \frac{\triangle t}{\varepsilon^2}\cdot\frac{1-\delta}{\delta} \\
&\geq 1 - 4r - \frac{\triangle t(1-\delta)}{\delta\varepsilon^2} \\
&\geq 1 - 4\cdot\frac{1-c_0}{2} - c_0 \\
&= -1 + c_0 \\
&> -1 \quad \text{for all } j = 1, 2, \cdots, N_x - 1.
\end{aligned}$$

Thus, we conclude from (5.21) and the above inequality that

$$\tilde{\lambda}_j > -1 \quad \text{for all } j = 1, 2, \cdots, N_x - 1. \tag{5.22}$$

Next, we now assume $\max_{1\leq k\leq N_x-1}|u_k^n| \leq 1$. Then, we observe from (5.1) and (5.18) that

$$\begin{aligned}
1 - 2r - \frac{\triangle t}{\varepsilon^2}b_k^n &= 1 - 2\frac{\triangle t}{(\triangle x)^2} + \frac{\triangle t}{\varepsilon^2} \\
&\geq 1 - 2\cdot\frac{1-c_0}{2} + \frac{\triangle t}{\varepsilon^2} \\
&> 0 \qquad \text{for all } k = 1, 2, \cdots, N_x - 1.
\end{aligned}$$

Therefore, all components of $A + B$ are positive. Hence, the sum of all components in k-th row of $A + B$ is the following:

$$\begin{gathered}
\left|1 - r - \frac{\triangle t}{\varepsilon^2}b_k^n\right| + |r| = 1 - r - \frac{\triangle t}{\varepsilon^2}b_k^n + r = 1 + \frac{\triangle t}{\varepsilon^2} > 1 \\
\text{for } k = 1 \text{ or } N_x - 1
\end{gathered} \tag{5.23}$$

and

$$|r| + \left|1 - 2r - \frac{\triangle t}{\varepsilon^2} b_k^n\right| + |r| = r + 1 - 2r - \frac{\triangle t}{\varepsilon^2} b_k^n + r = 1 + \frac{\triangle t}{\varepsilon^2} > 1 \quad \text{for all } k = 2, 3, \cdots, N_x - 2. \tag{5.24}$$

Therefore, we observe from (5.23)–(5.24) that $\max_{1 \leq k \leq N_x - 1} |u_k^n|$ is increasing with respect to n in the case when $\max_{1 \leq k \leq N_x - 1} |u_k^n| \leq 1$.

However, if $u_k^n \notin [-1, 1]$ for some $k = 1, 2, \cdots, N_x - 1$, it follows from (5.1) and (5.18) that

$$\begin{aligned} 1 - 2r - \frac{\triangle t}{\varepsilon^2} b_k^n &= 1 - 2\frac{\triangle t}{(\triangle x)^2} - \frac{\triangle t}{\varepsilon^2} \cdot \frac{1-\delta}{\delta} \\ &\geq 1 - 2 \cdot \frac{1 - c_0}{2} - c_0 \\ &= 0. \end{aligned}$$

Therefore, the sum of all components in k-th row of $A + B$ is the following:

$$\left|1 - r - \frac{\triangle t}{\varepsilon^2} b_k^n\right| + |r| = 1 - r - \frac{\triangle t}{\varepsilon^2} b_k^n + r = 1 - \frac{\triangle t}{\varepsilon^2} \cdot \frac{1-\delta}{\delta} < 1 \quad \text{if } u_k^n \notin [-1, 1] \text{ for some } k = 1 \text{ or } N_x - 1 \tag{5.25}$$

and

$$|r| + \left|1 - 2r - \frac{\triangle t}{\varepsilon^2} b_k^n\right| + |r| = r + 1 - 2r - \frac{\triangle t}{\varepsilon^2} b_k^n + r = 1 - \frac{\triangle t}{\varepsilon^2} \cdot \frac{1-\delta}{\delta} < 1 \quad \text{if } u_k^n \notin [-1, 1] \text{ for some } k = 2, 3, \cdots, N_x - 2. \tag{5.26}$$

Althought $\max_{1 \leq k \leq N_x - 1} |u_k^n|$ is increasing with respect to n in the case when $\max_{1 \leq k \leq N_x - 1} |u_k^n| \leq 1$ (cf. (5.23)–(5.24)), we conculde from (5.22) and (5.25)–(5.26) that (ii) of Theorem 5.1 holds. Thus, the proof of Theorem 5.1 has been completed. □

Remark 5.2. *By (5.1) we get the suitable mesh size of space $\triangle x$. In fact, for each $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$ we take the constant $\tilde{c}_0 \in (0, 1)$, $\triangle t \in (0, 1]$, $\triangle x \in (0, 1]$ such that*

$$\triangle t \leq \frac{\tilde{c}_0 \delta \varepsilon^2}{1 - \delta} \quad \textit{and} \quad \frac{\triangle t}{(\triangle x)^2} = \frac{1 - \tilde{c}_0}{2}.$$

Then, we have:

$$\triangle t = \frac{1 - \tilde{c}_0}{2} \cdot (\triangle x)^2 \leq \frac{\tilde{c}_0 \delta \varepsilon^2}{1 - \delta}.$$

Thus, we have the following condition of $\triangle x$:

$$0 < \triangle x < \varepsilon \sqrt{\frac{2\tilde{c}_0 \delta}{(1 - \tilde{c}_0)(1 - \delta)}}.$$

Remark 5.3. *We can take $c_0 = 0$ in (5.1) for the explicit finite difference scheme to the following usual heat equation with Neumann boundary condition:*

$$\begin{cases} u^\varepsilon_t - u^\varepsilon_{xx} = 0 \quad \text{in } Q := (0,T) \times (0,1), \\ u^\varepsilon_x(t,0) = u^\varepsilon_x(t,1) = 0, \qquad t \in (0,T), \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), \qquad x \in (0,1). \end{cases}$$

Taking into account of Theorem 5.1, we give numerical experiments of $(\mathrm{DP})^\varepsilon_\delta$ as follows. To this end, we use the following numerical data:

<u>**N**umerical data of $(\mathrm{DP})^\varepsilon_\delta$.</u>

- $T = 0.01$;
- $\delta = 0.01$;
- $\triangle x = 0.005$;
- $c_0 := 0.6$.

Also, we consider the following initial data $u^\varepsilon_0(x)$ defined by

$$u^\varepsilon_0(x) = \begin{cases} -0.5, & \text{if } x \in [0.00, 0.25], \\ -0.5\sin(2\pi x), & \text{if } x \in [0.25, 0.75], \\ 0.5, & \text{if } x \in [0.75, 1.00]. \end{cases} \tag{5.27}$$

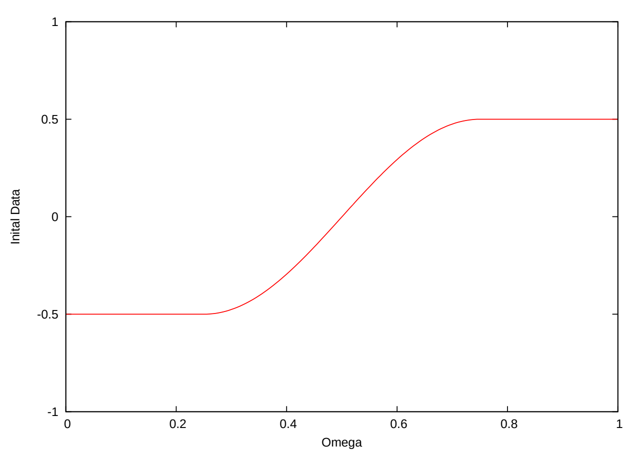

Figure 9: The graph of initial data $u^\varepsilon_0(x)$ defined by (5.27).

5.1 The case when $\varepsilon = 0.05$ and $\triangle t = 0.000005$

Now, we consider the case when $\varepsilon = 0.05$ and $\triangle t = 0.000005$. In this case, we easily observe that:

$$\frac{\triangle t}{(\triangle x)^2} = \frac{0.000005}{(0.005)^2} = 0.2 = \frac{1-c_0}{2},$$

and

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} = \frac{0.6 \times 0.01 \times (0.05)^2}{1-0.01} = 0.00001515151515\cdots.$$

Therefore, we have

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} = 0.00001515151515\cdots > \triangle t,$$

which implies that the criteria condition (5.1) holds. Thus, we have the following stable numerical experiment of $(\mathrm{DP})^{\varepsilon}_{\delta}$:

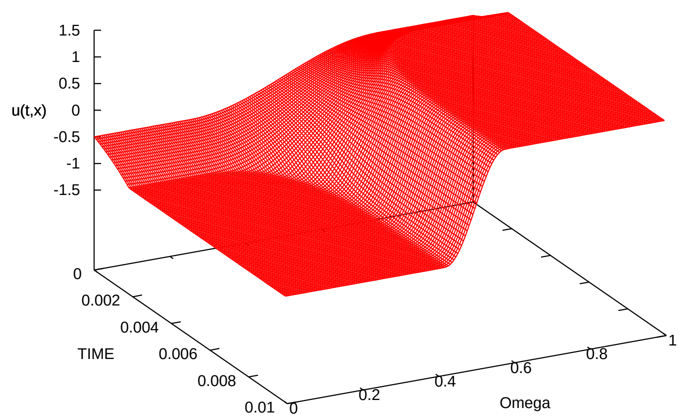

Figure 10: $\varepsilon = 0.05$, $\triangle t = 0.000005$, $\triangle x = 0.005$, $\delta = 0.01$.

5.2 The case when $\varepsilon = 0.007$ and $\triangle t = 0.000005$

Now, we consider the case when $\varepsilon = 0.007$ and $\triangle t = 0.000005$. In this case, we easily observe that:

$$\frac{\triangle t}{(\triangle x)^2} = \frac{0.000005}{(0.005)^2} = 0.2 = \frac{1-c_0}{2},$$

and

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} = \frac{0.6 \times 0.01 \times (0.007)^2}{1-0.01} = 0.00000029696969\cdots.$$

Therefore, we have

$$\frac{c_0 \delta \varepsilon^2}{1-\delta} = 0.00000029696969\cdots < \triangle t,$$

which implies that the criteria condition (5.1) does not hold. Therefore, we have the following unstable numerical experiment of $(\mathrm{DP})_{\delta}^{\varepsilon}$:

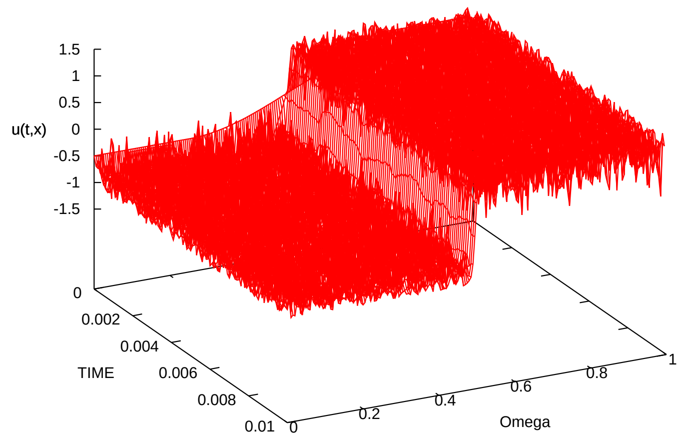

Figure 11: $\varepsilon = 0.007$, $\triangle t = 0.000005$, $\triangle x = 0.005$, $\delta = 0.01$.

5.3 The case when $\varepsilon = 0.007$ and $\triangle t = 0.0000002$

Now, we consider the case when $\varepsilon = 0.007$ and $\triangle t = 0.0000002$. In this case, we have

$$\frac{\triangle t}{(\triangle x)^2} = \frac{0.0000002}{(0.005)^2} = \frac{1}{125} \leq 0.2 = \frac{1 - c_0}{2}$$

and

$$\frac{c_0 \delta \varepsilon^2}{1 - \delta} = 0.00000029696969 \cdots > \triangle t,$$

which implies that the criteria condition (5.1) holds. Therefore, we have the following stable numerical experiment of $(\mathrm{DP})_{\delta}^{\varepsilon}$:

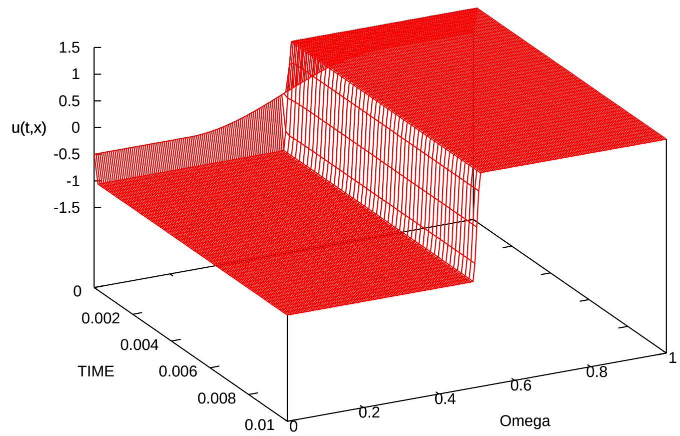

Figure 12: $\varepsilon = 0.007,\ \ \triangle t = 0.0000002,\ \triangle x = 0.005,\ \delta = 0.01.$

Remark 5.4. *We observe from Theorem 5.1 that in order to get stable numerical results of* $(\mathrm{DP})^{\varepsilon}_{\delta}$*, we have to choose the suitable constants* ε*,* δ *and the mesh size of time* $\triangle t$ *and space* $\triangle x$*. Therefore, if we make a numerical experiment of* $(\mathrm{P})^{\varepsilon}$ *for sufficient small* ε*, we had better consider the original problem* $(\mathrm{P})^{\varepsilon}$ *by using a primal-dual active set method in [3], a Lagrange multiplier method in [12] and so on.*

5.4 Conclusion of PDE problem $(\mathrm{DP})^{\varepsilon}_{\delta}$

By Theorem 5.1 and numerical experiments as above, we conclude that the mesh size of time $\triangle t$ and space $\triangle x$ must be satisfied

$$0 < \triangle t \le \frac{c_0 \delta \varepsilon^2}{1-\delta}, \quad 0 \le \frac{\triangle t}{(\triangle x)^2} \le \frac{1-c_0}{2} \quad \text{for some constant } c_0 \in (0,1),$$

in order to get the stable numerical experiments of $(\mathrm{DP})^{\varepsilon}_{\delta}$. Also, by Theorems 3.1 and 5.1, we conclude that the value $\delta\varepsilon^2/(1-\delta)$ is very important to make numerical experiments of $(\mathrm{DE})^{\varepsilon}_{\delta}$ and $(\mathrm{DP})^{\varepsilon}_{\delta}$.

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