On the Stokes resolvent estimates for cylindrical domains

Dedicated to Professor Jan Prüss on the occasion of his 65th birthday

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Abstract. This paper studies the analyticity of the Stokes semigroup in an infinite cylinder or more generally a cylindrical domain with several exits to infinity in the space $C_{0,\sigma}$, the L^∞-closure of all smooth compactly supported solenoidal vector fields. These domains are not strictly admissible in the sense of the first two authors (2014). However, it is shown that these domains are still admissible which yields the analyticity in $C_{0,\sigma}$. A new proof based on a blow-up argument is given to derive an L^∞-type resolvent estimate which enables us to conclude that the analyticity angle of the Stokes semigroup in $C_{0,\sigma}$ is $\pi/2$.

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1. Introduction

It is important to consider the Navier-Stokes equations in various types of domains since there is a huge variety of domains that a fluid might occupy. The analysis of the Stokes equations, a linearized version of the Navier-Stokes equations is fundamental. Especially, analyticity of the Stokes semigroup $S(t)$, the solution operator of the Stokes equation

$$v_t - \Delta v + \nabla q = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.1a}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.1b}$$

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.1c}$$

$$v|_{t=0} = v_0 \quad \text{on} \quad \Omega, \tag{1.1d}$$

i.e. $S(t)v_0 = v(\cdot,t)$, is important to measure a regularizing effect of the Stokes flow, where Ω is a uniformly C^2 domain in $\mathbf{R}^n$ $(n \geq 2)$. Here we impose the Dirichlet boundary condition (1.1c) to fix the idea.

It is well-known that $S(t)$ forms an analytic semigroup in $L^p_\sigma(\Omega)$ $(1 < p < \infty)$ for various kind of domains Ω including smoothly bounded domains [G81], [Sol77], where $L^p_\sigma = L^p_\sigma(\Omega)$ is the L^p-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. By now analyticity results are known for various type of unbounded domain not necessarily an exterior domain. For example, L^p-analyticity is proved for a layer domain [AS03], an aperture domain [FS96]. Moreover, these results are extended for the case of variable viscosity coefficients [AT09], [A10]. In fact, the analyticity of $S(t)$ in $L^p_\sigma(\Omega)$ holds for any uniformly C^2 domain provided that $L^p(\Omega)$ admits a topological direct sum decomposition, called the Helmholtz decomposition [GHHS]. It is also known that $S(t)$ forms analytic semigroup for any uniformly C^2 domain if one considers $\tilde{L}^p$ spaces, i.e., $\tilde{L}^p_\sigma = L^p_\sigma \cap L^2_\sigma$ (for $p > 2$) and $\tilde{L}^p_\sigma = L^p_\sigma + L^2_\sigma$ $(1 < p \leq 2)$ as developed by Farwig, Kozono and Sohr [FKS1], [FKS2], [FKS3].

However, if one considers the case $p = \infty$, the results are still limited since the Helmholtz projection is not bounded. If $\Omega = \mathbf{R}^n_+$, it is proved in [DHP], [Sol03] that $S(t)$ forms an analytic semigroup in $C_{0,\sigma} = C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$ based on an explicit representation formula. For a general domain it is proved [AG1] that the semigroup is analytic in $C_{0,\sigma}$ provided that the domain is "admissible" in the sense of [AG1]. Since it turns out that a bounded domain [AG1] and an exterior domain [AG2] are admissible (even strictly admissible), we conclude that $S(t)$ forms an analytic semigroup in such a domain; for improvement of these results see [AGH] where only C^2 regularity is used.

Recently, it turns out that $S(t)$ does not form an analytic semigroup on $C_{0,\sigma}(\Omega)$ in a layer domain $\{(x', x_n) \in \mathbf{R}^n \mid a < x_n < b\}$, $a, b \in \mathbf{R}$, $a < b$ provided that $n \geq 3$ as proved by von Below [B]. His result in particular implies that a layer domain is not admissible. Since a layer domain allows an L^r-Helmholtz decomposition [Mi], the Helmholtz decomposition does not imply admissibility. On the other hand, there is a planar non Helmholtz domain which is admissible so admissibility and Helmholtz domain is a different notion.

The main goal of this paper is to show that a cylindrical domain including a two-dimensional layer domain is admissible in the sense of [AG1]. By a cylindrical domain we mean there are finitely many outlets which are a half part of infinite cylinder (see Section 3 for a rigorous definition). An infinite cylinder $\mathbf{R} \times D$ with a bounded domain $D \subset \mathbf{R}^{n-1}$ is a typical example of a cylindrical domain. For this purpose, we consider the Neumann problem of the Laplace equation of the form

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{1.2a}$$

$$\frac{\partial u}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} g \quad \text{on} \quad \partial\Omega, \tag{1.2b}$$

where n_Ω is the exterior unit normal of $\partial\Omega$ and $\operatorname{div}_{\partial\Omega}$ is the surface divergence [G]. Admissibility (see Section 2.3) easily follows from the next weighted L^∞-estimate for the gradient of a solution of (1.2a), (1.2b).

Theorem 1.1. *Let Ω be a C^2 cylindrical domain in $\mathbf{R}^n$ $(n \geq 2)$. Then there exists a constant C such that*

$$\|d_\Omega \nabla u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)} \tag{1.3}$$

holds for all weak solution u of (1.2a), (1.2b) *with $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ for some $r \geq n$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ with $\mathrm{div}_{\partial\Omega}\, g \in C(\partial\Omega)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega$. Here $d_\Omega(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$.*

We shall prove this result in Theorem 2.7 for an infinite layer and in Theorem 3.7 for a general cylindrical domain.

This type of estimate is first proved in [AG1] for a C^3-bounded domain and a half space. For a C^3 exterior domain, a similar estimate is proved in [AG2]. In both cases we need not assume that $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$. We only need to assume that u is a (very) weak solution. Moreover, we need not assume the regularity of g. If (1.3) holds for all very weak solutions u having finite left-hand side of (1.3), we say that Ω is *strictly admissible* [AG2]. Note that an infinite cylinder is not strictly admissible because a linear function $u(x_1, x') = x_1$ solves (1.2a), (1.2b) with $g = 0$. The estimate (1.3) is independently proved by Kenig, Lin and Shen [KLS] for a bounded $C^{1,\gamma}$ domain for their study on homogenization of the Neumann problem.

Let us explain our idea of the proof of Theorem 1.1 when Ω is an infinite layer. We derive an estimate for $d_\Omega \nabla u$ in a domain $\Omega(S) = (-S, S) \times D$ uniformly in $S \geq 1$ of the form

$$d_{\Omega(S)}(x)\, |\nabla u(x)| \leq C \left(\|g\|_{L^\infty((-S,S)\times\partial D)} + \sum_{x_1=\pm S} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^{n-1}(D)} (x_1) \right), \tag{1.4}$$

where $\|f\|^p_{L^{n-1}(D)}(x_1) = \int_D |f(x_1, x')|^p \, dx'$ for $f = f(x_1, x')$. We shall establish the estimate (1.4) by a contradiction argument and derive a contradiction with uniqueness result of the Neumann problem under no flux condition. As we proved in Theorem 2.3, a solution u of (1.2a), (1.2b) with $g = 0$ satisfying $d_\Omega \nabla u \in L^\infty(\Omega)$ must be constant provided that there is no flux $\int_{x_1=R} \frac{\partial u}{\partial x_1} dx' = 0$. The no flux condition is essential since otherwise $u = x_1$ is a nontrivial solution which breaks the uniqueness. We shall prove such uniqueness essentially by strong maximum principles. Theorem 1.1 easily follows since the last term in (1.4) tends to zero as $S \to \infty$ by $\nabla u \in L^2(\Omega)$.

Once admissibility has been established, we have analyticity of the semigroup $S(t)$ by applying the main result of [AG1] if Ω is C^3. If one applies the resolvent estimate [AGH], we only need C^2 regularity. Although it is possible to extend the argument in [AGH] to admissible domains, we shall establish the resolvent estimate for general admissible domain without appealing Masuda-Stewart arguments used in [AGH]. We apply a blow-up argument to derive necessary resolvent estimates. We consider the resolvent equation

$$(\lambda - \Delta)v + \nabla p = f \quad \text{in} \quad \Omega, \tag{1.5a}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega, \tag{1.5b}$$

$$v = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5c}$$

with $\|d_\Omega \nabla p\|_\infty < \infty$, where $\lambda \in \Sigma_{\pi-\varepsilon}$ for a fixed $\varepsilon > 0$. Here $\Sigma_\varphi = \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \varphi\}$. We set $N(v, \lambda) = |\lambda|\,|v(x)| + |\lambda|^{1/2}\,|\nabla v(x)|$.

Theorem 1.2. *Let Ω be an admissible, uniformly C^2 domain in $\mathbf{R}^n$. For $\varepsilon \in (0, \pi/2)$ there exists a constant C and M (independent of f and λ) such that*

$$\|N(v, \lambda)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$$

for all $f \in C^\infty_{c,\sigma}(\Omega)$ and $\tilde{L}^r$-solution v $(r > n)$ of (1.5a)–(1.5c) *with $\lambda \in \Sigma_{\pi-\varepsilon}$ and $|\lambda| \geq M$.*

As we mentioned before, we appeal to a blow-up argument to prove Theorem 1.2 which was developed by the last author [Suzuki] for higher order elliptic problems under C^1-regularity of Ω. Since we have to control the pressure from vorticity, the present method seems to need C^2-regularity of Ω. The key equation to control the pressure from (1.5a), (1.5b) is

$$\Delta p = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial p}{\partial n_\Omega} = -\operatorname{div}_{\partial\Omega}(\omega \times n_\Omega) \quad \text{on} \quad \partial\Omega,$$

where $\omega = \operatorname{curl} v$ is the vorticity. Here we assume $n = 3$ for simplicity. The harmonicity of p is clear by taking the divergence of (1.5a) while the boundary condition is obtained by taking inner product of (1.5a) with n_Ω and use $n_\Omega \cdot \Delta v = -\operatorname{div}_{\partial\Omega}(\omega \times n)$ when $\operatorname{div} v = 0$. This relation may be well known but it is often implicit e.g. [JL, (1.3b)]. This pressure is sometimes called the Stokes pressure [LLP] because it reflects the effect of viscosity.

Theorem 1.2 is considered as an extension of the main resolvent estimate [AGH, Theorem 1.1] where the strict admissibility of Ω is assumed. As in [AGH] one is able to assert that the Stokes operator A defined as in [AGH] generates a C_0-analytic semigroup $S(t)$ on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$. We shall state this result which follows from Theorem 1.2 as in [AGH].

Corollary 1.3. *Let Ω be an admissible, uniformly C^2 domain in $\mathbf{R}^n$. Then the Stokes semigroup $S(t)$ forms an analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$. In particular, for a C^2 cylindrical domain this analyticity holds.*

The analyticity in $C_{0,\sigma}(\Omega)$ is one of key tools to estimate the lifespan from below of a solution to the Navier-Stokes equations starting from L^∞ type data as shown in Abe [A2] for bounded or exterior domains.

Note that the extension to L^∞_σ space is nontrivial in our setting. It is not clear whether or not the approximate limit is uniquely defined since its pressure is not well-controlled which is different from the case of a strictly admissible domain.

This type of result as well as [AG1], [AG2], [AGH] is concerned with regularizing effect locally-in-time. The boundedness of the Stokes semigroup

$S(t)$ for $t > 0$ is a different question. For a bounded domain, the semigroup is even exponentially decaying by the Poincaré estimate [AG1]. For exterior domains the global boundedness of $S(t)$ was proved in [Mar], which was extended to a global time derivative estimate in [HM]. Moreover, both results are even further extended to global boundedness in sectors of angle less than $\pi/2$ in [BH]. For the case of the half space, these results are already established in [Sol03] and [DHP].

This paper is organized as follows. In Section 2 we establish (1.4) and prove Theorem 1.1 for an infinite cylinder to clarify the idea. In Section 3 we extend this idea to a cylindrical domain. In Section 4 we prove Theorem 1.2 by a blow-up argument as well as Corollary 1.3. In Appendix A we recall an elliptic regularity theory for a very weak solution. In Appendix B we give a way to construct a cut-off function near the boundary which satisfies the Neumann boundary condition.

2. Infinite cylinders

2.1. Uniqueness under no flux condition

We begin with a uniqueness result for the Neumann problem

$$-\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad \partial\Omega \tag{2.1}$$

in a C^2 infinite cylinder Ω in $\mathbf{R}^n$ which means in this paper that $\Omega := \mathbf{R} \times D$ with a C^2 bounded domain D in $\mathbf{R}^{n-1}$ $(n \geq 2)$.

Lemma 2.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of (2.1) in a C^2 infinite cylinder Ω. Assume that u is bounded in Ω, i.e., $u \in L^\infty(\Omega)$. Then u is a constant function.*

Proof. We first observe that a flux

$$F(s) := \int_D \frac{\partial u}{\partial x_1}(s, x')\, dx' \tag{2.2}$$

is independent of $s \in \mathbf{R}$ since $0 = \int_{s_1}^{s_2} \int_D \Delta u \ dx = F(s_2) - F(s_1)$ for a solution u of (2.1). If we define

$$E(s) := \int_D u(s, x')\, dx', \tag{2.3}$$

this implies $dE(s)/ds$ is a constant function in s. Since u is assumed to be bounded, $E(s)$ must be a constant function, i.e. $E(s) \equiv c$. We may assume that $E(s) = 0$ for all $s \in \mathbf{R}$ by subtracting $c/|D|$, where $|D|$ is the Lebesgue measure of D.

We shall prove that $u \equiv 0$ by the strong maximum principle [PW, Section 3]. Assume that $u \not\equiv 0$. Then we may assume that $\sup_\Omega u > 0$ by considering $-u$ if necessary. This supremum is NOT attained in $\overline{\Omega}$. Indeed, if it were attained in the interior, then the strong maximum principle would imply that $u \equiv \sup u > 0$ which contradicts $E(s) \equiv 0$ by (2.3). If the maximum were taken on the boundary, again we obtain $u \equiv \sup u$ since otherwise the Hopf (boundary) lemma implies $\partial u/\partial n_\Omega > 0$ on $\partial\Omega$. This again contradicts $E(s) \equiv 0$.

Since the supremum $\sup u$ is not attained in $\overline{\Omega}$, we may assume that there is a sequence $x_m = (s_m, x'_m)$ such that $u(x_m) \to \sup u$ and $|s_m| \to \infty$ as $m \to \infty$. We may assume (by taking a subsequence) that $s_m \to \infty$ since the case $s_m \to -\infty$ can be treated similarly. Since $\overline{D}$ is compact, we may assume that $x'_m \to x^*$ for some $x^* \in \overline{D}$ by taking a subsequence if necessary. We shift u by defining

$$u_m(x) := u\left(x_1 + s_m, x'\right) \quad \text{for} \quad x = (x_1, x')\,. \tag{2.4}$$

Since u is bounded and satisfies (2.1), we observe that

$$\sup_{s\in\mathbf{R}} \left\| u_m : W^{2,q}\left((s, s+1) \times D\right)\right\| \le C_q \tag{2.5}$$

with $C_q > 0$ independent of m by elliptic regularity; see e.g. Theorem A.1 in the Appendix. By the Sobolev embedding and Rellich's compactness there is a subsequence of $\{u_m\}$ still denoted by $\{u_m\}$ such that u_m in (2.4) converges to some function v locally uniformly with its first derivatives so the boundary condition of (2.1) is inherited. Since v is weakly harmonic in Ω, Weyl's lemma implies that v is smooth and harmonic in Ω. Thus v is a solution of (2.1) with $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, $E(s) \equiv 0$ implies

$$\int_D v\left(s, x'\right) dx' = 0 \tag{2.6}$$

for all $s \in \mathbf{R}$. By definition of x_m we observe that v takes its maximum $\sup u$ at $(0, x^*)$. As before the Hopf lemma implies that $v \equiv \sup u$ is a positive constant which contradicts (2.6). We thus conclude $u \equiv 0$. $\square$

We shall prove a uniqueness result without assuming the boundedness but assuming $\|d_\Omega \nabla u\|_\infty := \|d_\Omega \nabla u\|_{L^\infty(\Omega)} < \infty$. Of course, $u(x) = x_1$ is a solution of (2.1) satisfying $\|d_\Omega \nabla u\|_\infty < \infty$ so to exclude such a solution we need some extra condition. For example, if we assume the no flux condition $F(s) = 0$ for F defined by (2.2), we are able to exclude such a solution. We prepare a lemma which asserts that no flux condition with $\|d_\Omega \nabla u\|_\infty < \infty$ implies boundedness of u.

Lemma 2.2. *Let Ω be a C^2 infinite cylinder in $\mathbf{R}^n$. For $S \in \mathbf{R}$ let $u \in C^2(\Omega_{>S}) \cap C^1(\overline{\Omega}_{>S})$ satisfy*

$$-\Delta u = 0 \quad \text{in} \quad \Omega_{>S}, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad (\partial\Omega)_{>S}, \tag{2.7}$$

where $U_{>S} = U \cap \{x_1 > S\}$ for a set $U \subset \mathbf{R}^n$. Assume that $d_\Omega \nabla u$ is bounded in $\Omega_{>S}$ and $F(s_1) = 0$ for some $s_1 > S$. Then u is bounded in $\Omega_{>S+\delta}$ for any $\delta > 0$.

Proof. As in the proof of Lemma 2.1 we observe that $F(s)$ is independent of s. Since $F(s_1) = 0$, $F(s) \equiv 0$ for $s > s_1$. Since $d_\Omega \nabla u$ is bounded and $F(s) = 0$ for all $s > s_1$, the integration of ∇u of one of x' variable implies

$$\sup_{s>S} \|u : L^q((s, s+1) \times D)\| < \infty$$

for any $q > 1$ (cf. [AG1], [AGH, (2.1)]). In a similar way to deriving (2.5), since u solves (2.7) by elliptic regularity (Appendix A), this implies

$$\sup_{s>S} \left\|u : W^{2,q}((s+\delta, s+1-\delta) \times D)\right\| < \infty$$

for any $\delta \in (0, 1/2)$. By the Sobolev inequality (for $q > n/2$) this implies that u is bounded in $(S + \delta, \infty) \times D$. □

Theorem 2.3. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy* (2.1) *in a C^2 infinite cylinder Ω. Assume that $F(s_1) = 0$ for some $s_1 \in \mathbf{R}$. If $\|d_\Omega \nabla u\|_\infty < \infty$, then u is a constant function.*

Proof. By Lemma 2.2 we observe that u is bounded. (The boundedness of u in $\{x_1 < 0\}$ follows from Lemma 2.2 by reflection with respect to $x_1 = 0$.) Thus Theorem 2.3 follows from Lemma 2.1 □

Corollary 2.4. *Let $u \in C^2(\Omega_{>0}) \cap C^1(\overline{\Omega}_{>0})$ satisfy* (2.1) *in $\Omega_{>0}$ when Ω is a C^2 infinite cylinder. Assume that $\partial u/\partial x_1 = 0$ at $x_1 = 0$ for $x' \in D$. If $\|d_\Omega \nabla u\|_\infty < \infty$, then u is a constant function.*

Proof. Since $\partial u/\partial x_1 = 0$ at $x_1 = 0$, we extend u for $x_1 < 0$ as an even function, i.e., $u(x_1, x') = u(-x_1, x')$. Then this extended function fulfills all assumptions of Theorem 2.3 since $F(0) = 0$. We apply Theorem 2.3 to conclude that u is a constant function. □

2.2. Weighted estimate for the Neumann problem

Our goal in this subsection is to establish a weighted L^∞ estimate of the form $\|d_\Omega \nabla u\|_\infty \leq C\|g\|_\infty$ for a weak solution u of

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} g \quad \text{on} \quad \partial\Omega, \tag{2.8}$$

with $\nabla u \in L^2(\Omega)$ when g is tangential, i.e. $g \cdot n_\Omega = 0$.

For this purpose we recall a Neumann problem associated to the Laplace equation. For a bounded Lipschitz domain U in $\mathbf{R}^n$ there exists a weak solution $v \in H^1(U)$ (unique up to an additive constant) of

$$\Delta v = 0 \quad \text{in} \quad U, \quad \frac{\partial v}{\partial n_U} = h \quad \text{on} \quad \partial U, \tag{2.9}$$

for any $h \in H^{-1/2}(\partial U)$ provided that $\int_{\partial U} h \, d\mathcal{H}^{n-1} = 0$, where H^s denotes the L^2-Sobolev space of order s. Such a solution is called an $\dot{H}^1$ weak solution v of (2.9), which means that $v \in L^1_{loc}(\Omega)$ with $\nabla v \in L^2(\Omega)$ fulfills

$$\int_U \nabla v \cdot \nabla \psi \, dx = \int_{\partial U} h \, \psi \, d\mathcal{H}^{n-1}$$

for all $\psi \in H^1(U)$. This definition applies to the case when U is unbounded by replacing H^1 by the homogeneous Sobolev space $\dot{H}^1$. The existence of a unique $\dot{H}^1$ weak solution for a bounded Lipschitz domain is guaranteed by the Lax-Milgram theorem [E] or the Riesz representation theorem for the Hilbert space

$$\mathring{H}^1(U) = \left\{ u \in H^1(U) \,\middle|\, \int_U u\, dx = 0 \right\}.$$

See e.g. [BF, Theorem III, 4.3]. For an unbounded domain the solution may not exist.

In our limiting procedure we have to handle these u such that ∇u may not be integrable near $\partial\Omega$. For this purpose it is convenient to recall the notion of a very weak solution of (2.9) for $U = \Omega(S)$ when $h = h_0$ in $\{\pm S\} \times D$ with $h_0 \in L^2(\{\pm S\} \times D)$ and $h = \operatorname{div}_{\partial\Omega} g \in C\left(\overline{\partial\Omega(S)}\right)$ on $(\partial\Omega)(S) = (-S, S) \times \partial D$ with $g = (g_1, g') \in C\left(\overline{(\partial\Omega)(S)}\right)$, where g is tangential. We say that $u \in L^1(\Omega(S))$ is a *very weak* solution of (2.9) in $\Omega(S)$ with such data h if

$$\begin{aligned}\int_{\Omega(S)} u\, \Delta\varphi\, dx = &\int_{(\partial\Omega)(S)} \nabla_{\partial\Omega}\varphi \cdot g\, d\mathcal{H}^{n-1} \\ &- \sum_{x_1=\pm S} \int_D (\varphi h_0)(x_1, x')dx' - \sum_{x_1=\pm S} (\operatorname{sgn} x_1) \int_{\partial D} (g_1\varphi)(x_1, x') d\mathcal{H}^{n-2}_{x'}\end{aligned}$$

for all $\varphi \in C^2\left(\overline{\Omega(S)}\right)$ with $\partial\varphi/\partial n_{\Omega(S)} = 0$ on $\partial(\Omega(S))$, where $\nabla_{\partial\Omega}$ denotes the tangential gradient [G]. Since g is tangential, $\nabla_{\partial\Omega}$ can be replaced by ∇. Such a notion of a very weak solution for (2.8) is introduced by [AG1], [AG2]. These types of notions of a very weak solution are elaborated by [MR] to handle the case when the Neumann data equals the Dirac measure which corresponds to the Green function of the Neumann problem.

Lemma 2.5. *Let Ω be a C^2 infinite cylinder in $\mathbf{R}^n$ $(n \geq 2)$. Then there exists a constant C (depending only on D and n but independent of dilation and translation) such that*

$$d_{\Omega(S)}(x)\, |\nabla u(x)| \leq C \left(\|g\|_{L^\infty((\partial\Omega)(S))} + \sum_{x_1=\pm S} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^{n-1}(D)} (x_1) \right) \quad \textit{for all } x \in \Omega(S) \qquad (2.10)$$

holds for all $S \geq 1$, and all $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega(S))$ of (2.9) in $\Omega(S)$ with $h = \operatorname{div}_{\partial\Omega} g \in C(\overline{\partial\Omega(S)})$ in $(\partial\Omega)(S)$, $h = \pm\partial u/\partial x_1$ at $x_1 = \pm S$, where $g \in C\left(\overline{(\partial\Omega)(S)}\right)$ satisfies $g \cdot n_\Omega = 0$ on $(\partial\Omega)(S)$ provided that $\|d_\Omega \nabla u\|_{L^\infty(\Omega(S))}$ is finite and that $\partial u/\partial x_1(\pm S, \cdot) \in L^{n-1}(D)$.

Proof. We first prove that u is a very weak solution of (2.9) with the same data. Since $\partial\varphi/\partial n_{\Omega(S)} = 0$, integration by parts yields

$$\int_{\Omega(S)} u\,\Delta\varphi\,dx = -\int_{\Omega(S)} \nabla u\cdot\nabla\varphi\,dx.$$

Since u is an $\dot{H}^1$ weak solution of (2.9), we see that

$$\int_{\Omega(S)} \nabla u\cdot\nabla\varphi\,dx = \int_{|x_1|=S} h_0\varphi\,dx' + \int_{(\partial\Omega)(S)} (\mathrm{div}_{\partial\Omega}\ g)\varphi\,d\mathcal{H}^{n-1}.$$

Since g is tangential, integration by parts yields

$$\begin{aligned}\int_{(\partial\Omega)(S)} (\mathrm{div}_{\partial\Omega}\ g)\varphi\,d\mathcal{H}^{n-1} = &-\int_{(\partial\Omega)(S)} \nabla_{\partial\Omega}\varphi\cdot g\,d\mathcal{H}^{n-1}\\ &+\sum_{x_1=\pm S}(\mathrm{sgn}\,x_1)\int_{\partial D}(g^1\varphi)(x_1,x')d\mathcal{H}^{n-2}_{x'}.\end{aligned}$$

Combining these three identities, we conclude that u is a very weak solution of (2.9).

As in [AG1], [AG2] we argue by contradiction. There exists a sequence $\{u_m, g_m, S_m\}_{m=1}^\infty$ such that

$$\begin{aligned}1 &= \|d_{\Omega_m}\nabla u_m\|_{L^\infty(\Omega_m)}\\ &> m\left(\|g_m\|_{L^\infty(\partial\Omega_m)} + \sum_{x_1=\pm S}\|h_{0m}\|_{L^{n-1}(D)}(x_1)\right)\end{aligned} \tag{2.11}$$

with $\Omega_m = \Omega(S_m)$ such that $u_m \in L^1_{loc}(\overline{\Omega}_m)$ is a weak solution of (2.9) with $\Omega = \Omega_m$. Here $g_m \in L^\infty(\partial\Omega_m)$ is assumed to satisfy $g_m\cdot n_\Omega = 0$ on $\partial\Omega_m$ and $h_{0m} = \pm\partial u_m/\partial x_1$ on $\{x_1 = \pm S_m\}$. We take $x_m \in \overline{\Omega}_m$ such that

$$|d_{\Omega_m}(x_m)\nabla u_m(x_m)| > 1/2. \tag{2.12}$$

We may assume that $u_m(x_m) = 0$ by adding a constant.

There are two cases depending on the behavior of $\{x_m\}_{m=1}^\infty$.

Case 1. There exists a subsequence still denoted by $\{x_m\}$ which converges to $\hat{x} \in \overline{\Omega}_\infty$ as $m\to\infty$.

Case 2. The sequence $\{x_m\}$ tends to infinity, i.e. $|x_m|\to\infty$.

We discuss Case 1 which is divided into two cases, (a) $\hat{x}\in\Omega_\infty$ and (b) $\hat{x}\in\partial\Omega_\infty$. We first discuss case (a). We may assume that $\lim_{m\to\infty} S_m = S \in [1,\infty]$ by taking a subsequence. The estimate $\|d_{\Omega_m}\nabla u_m\|_{L^\infty(\Omega_m)} \le 1$ guarantees that $\{u_m\}$ is bounded in $L^q_{loc}(\overline{\Omega}_\infty)$ for any $q>1$ by (2.1). Indeed, since $u_m(x_m) = 0$ and $x_m \to \hat{x} \in \overline{\Omega}_\infty$, integrating $|\nabla u_m(x)| \le 1/d_{\Omega_m}(x)$ from x_m yields a bound in $L^q_{loc}(\overline{\Omega})$ (cf. [AG1], [AGH, (2.1)]). In fact, by [AGH, Proposition 2.1] we are able to prove that $\|u_m\|_{L^q(\Omega(K))} \le C_K$ for any $K(<S)$ since $u_m(x_m) = 0$. We shall discuss case (a) since $\{u_m\}$ is bounded in $L^q_{loc}(\overline{\Omega}_\infty)$. By a diagonal argument we see that $\{u_m\}$ converges to some $u \in L^q_{loc}(\overline{\Omega}_\infty)$ weakly for some $q>1$ by taking a subsequence. It is easy to see that u is a very weak solution of (2.9) in $\Omega_\infty = (-S,S)\times D$ with $g=0$ and

$h_0 = 0$ since $\|g_m\|_\infty \to 0$ and $\|h_{0m}\|_{L^1} \to 0$ by (2.11). (One should be a little bit careful since φ cannot be taken uniformly with respect to Ω_m when S_m depends on m since we request $\partial\varphi/\partial n_{\Omega_m} = 0$. However, it is easy to construct a sequence $\{\varphi_m\}$ converging to φ uniformly up to second derivatives so one concludes that u is a very weak solution.)

Since $\{u_m\}$ is bounded in $L^q_{loc}(\overline{\Omega}_\infty)$ and each u_m is harmonic, the Cauchy estimates for harmonic functions [E, 2.2.c] (quantitative version of Weyl's lemma) implies that all derivatives are locally bounded in Ω_∞. Thus the convergence $u_m \to u$ is locally uniform with its derivatives in Ω_∞ so that $u(\hat{x}) = 0$. This in particular implies $u \in C^\infty(\Omega_\infty)$. We have to prove that the limit u is C^1 up to the boundary. Since u is a very weak solution of (2.9) in Ω_∞ with zero boundary Neumann data, the elliptic regularity (Appendix A, Theorem A.1) implies that $u \in C^1(\overline{\Omega}_\infty)$. Note that there is a corner point of Ω_∞ if S is finite. In this case we can interpret this point as a regular point by a reflection argument since the Neumann data at $x_1 = \pm S$ is zero. (See Remark 2.6.)

If S is finite, the uniqueness of the homogeneous Neumann problem is still valid by a reflection argument since the problem is reduced to whole Ω with x_1-periodicity (Remark 2.6). Since $u(\hat{x}) = 0$, the uniqueness implies $u \equiv 0$. However, by (2.12) we have $|d_\Omega(\hat{x})\nabla u(\hat{x})| \geq 1/2$ which yields a contradiction.

If $S = \infty$ so that $\Omega_\infty = \mathbf{R} \times D$, we have to check the flux condition. We take a cut-off function $\chi \in C^2([0,\infty))$ such that $\chi(s) = 0$ on $[0, 1/2]$ and $\chi(s) = 1$ on $[1,\infty)$ such that $0 \leq \chi \leq 1$ and $\chi' \geq 0$. We set $\chi_k(x_1) = \chi(kx_1)$ $(k = 1, 2, \dots)$ and take it as a test function in the definition of a weak solution to get

$$\begin{aligned}\int_{\Omega_{>0}} u_m \Delta\chi_k dx = \int_{\partial\Omega_{>0}} \nabla_{\partial\Omega}\chi_k \cdot g_m d\mathcal{H}^{n-1} - \int_{|x_1|=S_m} h_{0m}\, \chi_k\, dx' \\ - \int_{\partial D} g^1_m(S_m, x') d\mathcal{H}^{n-1}_{x'} + \int_{\partial D} g^1_m(-S_m, x') d\mathcal{H}^{n-1}_{x'}.\end{aligned}$$

Since $\|g_m\|_\infty \to 0$ and $\|h_{0m}\|_{L^{n-1}} \to 0$ as $m \to \infty$, the right-hand side tends to zero. We thus obtain $\int_{\Omega_{>0}} u\Delta\chi_k dx = 0$. Integrating by parts yields

$$\int_0^{1/k} \left(\int_D \frac{\partial u}{\partial x_1}(x_1, x') dx' \right) \frac{\partial \chi_k}{\partial x_1} dx_1 = 0.$$

Letting $k \to \infty$ yields $F(0) = \int_D \frac{\partial u}{\partial x_1}(0, x') dx' = 0$. We are now able to apply Theorem 2.3 to get $u \equiv 0$ which is a contradiction.

The case (b) can be treated as in [AG1] by rescaling u_m as $v_m(x) = u_m(x_m + d_m x)$ with $d_m = d_{\Omega_m}(x_m)$. Thanks to the L^{n-1} norm of h in (2.10) the estimate (2.11) is preserved for v_m. Since v_m is bounded in $L^q_{loc}(\overline{\Omega'}_m)$ with $\Omega'_m = (\Omega_m - x_m)/d_m$, there is a weak limit v in $L^q_{loc}(\overline{\Omega})$ (for some $q > 1$) by taking a subsequence if necessary. We have to prove that v is a very weak solution with homogeneous data in a half space or a quadrant type space $\mathbf{R}^n_{+<S}$. We give a proof in the case of half space. In other words, we have

to prove $\int_{\mathbf{R}^n_+} v\Delta\varphi\ dx = 0$ for all $\varphi \in C^2_c(\overline{\mathbf{R}^n_+})$ satisfying $\frac{\partial\varphi}{\partial x_n} = 0$ on the boundary in the case that the limit domain is a half space. In [AG1, Proof of Theorem 2.5, Case 2] we use C^3-regularity of Ω to construct a sequence of test functions $\{\varphi_m\}$ (approximating $\varphi \in C^2(\overline{\mathbf{R}^n_+})$ with $\partial\varphi/\partial x_n = 0$) satisfying $\partial\varphi_m/\partial n_{\Omega_m} = 0$ on $\partial\Omega_m$, where Ω_m is the rescaled domain. The reason is that we appealed to the normal coordinate in [AG1, p. 12]. Here we use a different construction of $\{\varphi_m\}$ in Appendix B (Lemma B.3), which requires only C^2-regularity of Ω. In fact, by rotation and translation we may assume that the rescaled domain Ω_m converges to a half space $\mathbf{R}^n_{+,-1}$ of the form $\mathbf{R}^n_{+,-1} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -1\}$. Assume that $\varphi \in C^2_c(\overline{\mathbf{R}^n_{+,-1}})$ with $\operatorname{spt}\varphi \subset B_R(0)$ and $\partial\varphi/\partial x_n = 0$ on $x_n = -1$. Applying Lemma B.3 yields a sequence of functions $\varphi_m \in C^2_c(\overline{\Omega_m})$ such that

$$\frac{\partial\varphi_m}{\partial n_{\Omega_m}} = 0 \quad \text{on} \quad \partial\Omega_m, \quad \operatorname{spt}\varphi_m \subset B_{4R/3}(0)$$

and that φ_m converges to φ uniformly in $\Omega_m \cap \mathbf{R}^n_{+,-1}$, with its up to second derivatives. The desired condition $\int_{\mathbf{R}^n_{+,-1}} v\Delta\varphi\ dx = 0$ follows from the fact that v_m is a very weak solution of (2.8) in Ω_m with corresponding data g_m converging to zero. We apply the uniqueness result in a half space [AG1, Lemma 2.9] to get a contradiction. If S is finite, then there might be a chance that the rescaled limit space is not a half space but a quadrant type space like $\mathbf{R}^n_{+<S}$. In this case we extend a solution by an even reflexion outside $x_1 = S$ and the reduce the problem into the half space.

We next study Case 2, i.e. $|x_m| \to \infty$. If one writes $x_m = (s_m, x'_m)$, we may assume that $s_m \to \infty$ and $x'_m \to x^* \in \overline{D}$ since the case $s_m \to -\infty$ can be treated similarly. We shift u_m as in (2.4), i.e. $w_m(x) := u_m(x_1 + s_m, x')$. If $d_m = d_{\Omega_m}(x_m) \to 0$ by taking a subsequence, we rescale w_m to consider $v_m(x) := w_m(d_m x_1, d_m x' + x'_m)$. We are able to reduce this case to Case 1 (b). It remains to discuss the case that $\inf d_m > 0$. We may assume that $\lim_{m\to\infty}(S_m - s_m) = S_* \in [0, \infty]$ exists by taking a subsequence. As in Case 1 (a), by (2.10) we observe that there is a subsequence still denoted by $\{w_m\}$ converging to a weak solution w of (2.8) in $\Omega_\infty = (-\infty, S_*) \times D$ with $g = 0$. Moreover, the convergence is locally uniform with its derivatives in Ω_∞ so that $w(0, x^*) = 0$ since $\{w_m\}$ is a bounded sequence in $L^q_{loc}(\overline{\Omega})$ and w_m is harmonic. As before, the elliptic regularity implies $w \in C^2(\Omega_\infty) \cap C^1(\overline{\Omega}_\infty)$. In the case $S_* = \infty$ since the flux condition $F(0) = 0$ is fulfilled as in the proof of Case 1 (a), we apply Theorem 2.3 to get $w \equiv 0$. This would contradict $|d_\Omega((0, x^*))\nabla w(0, x^*)| \geq 1/2$. The case $S_* < \infty$ is easier and get a contradiction. We thus proved (2.10). □

Remark 2.6. Let $u \in L^1_{loc}(\Omega(S))$ be a very weak solution of (2.9) in $\Omega(S)$ with $g \in C\left(\overline{(\partial\Omega)(S)}\right)$ satisfying $g \cdot n_\Omega = 0$ with $\operatorname{div}_{\partial\Omega}\ g \in C\left(\overline{(\partial\Omega)(S)}\right)$ and $h_0 = 0$ on $\{x_1 = \pm S\}$. Then one can extend this solution u to $\tilde{u}$ by a reflection argument so that the extended $\tilde{u}$ is periodic in x_1 direction with period $4S$ and $\tilde{u}$ is a solution of (2.8) in Ω with periodic data $\tilde{g}$. Indeed, we extend

u evenly in x_1 with respect to $x_1 = S$ so that the extended function still denoted u is to be a function on $\Omega(-S, 3S)$. We then extended it periodically with respect to x_1 to get the desired $\tilde{u}$. Since the Neumann data at $x_1 = \pm S$ is zero, $\tilde{u}$ is harmonic in Ω. Moreover, $\tilde{u}$ solves (2.8) in Ω with the extended data $\tilde{g}$. One is always able to reduce the problem to the whole domain Ω having a periodicity in x_1.

Theorem 2.7. *Let Ω be a C^2 infinite cylinder in $\mathbf{R}^n$ $(n \geq 2)$. Then there exists a constant C (depending only on D and n) such that*

$$\|d_\Omega \nabla u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)} \tag{2.13}$$

holds for all $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ (with some $r \geq n$) of (2.9) *in Ω with $h = \operatorname{div}_{\partial\Omega}\ g \in C(\partial\Omega)$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfying $g \cdot n_\Omega = 0$ on $\partial\Omega$.*

Proof. We take $S > 1$. Since $\nabla u \in L^r(\Omega)$ with some $r \geq n$, as in [AG1] a mean value theorem implies that $\sup_{x\in\Omega(S)} d_\Omega(x)\,|\nabla u(x)| < \infty$.

Furthermore, we take S such that $\partial u/\partial x_1$ at $x_1 = \pm S$ is in $L^2(D)$. Since u is an $\dot{H}^1$ weak solution of (2.9) in $\Omega(S)$ with $h = \operatorname{div}_{\partial\Omega}\ g$ in $\partial\Omega$ and $h = \pm\partial u/\partial x_1$ at $x_1 = \pm S$ and since $\|d_\Omega \nabla u\|_{L^\infty(\Omega(S))}$ is finite, we are able to apply Lemma 2.5 to conclude that

$$\|d_{\Omega(S)} \nabla u\|_{L^\infty(\Omega(S))} \leq C_D \left(\|g\|_{L^\infty(\partial\Omega)(S)} + \sum_{x_1=\pm S} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^{n-1}(D)} (x_1) \right).$$

Since $\nabla u \in L^n(\Omega)$, we see that there is a subsequence $S_i \to \infty$ such that

$$\int_{|x_1|=S_i} \left| \frac{\partial u}{\partial x_1} \right|^{n-1} dx' \leq \left(\int_{|x_1|=S_i} \left| \frac{\partial u}{\partial x_1} \right|^n dx' \right)^{(n-1)/n} \left(\mathcal{H}^{n-1}(D)\right)^{1/n} \to 0.$$

The desired estimate now follows from the above estimate of $\|d_{\Omega(S)} \nabla u\|_{L^\infty(\Omega(S))}$ by sending $S = S_i \to \infty$. □

Remark 2.8. (i) The assumption $\operatorname{div}_{\partial\Omega}\ g \in C^0(\partial\Omega)$ is actually unnecessary if one observe that $\operatorname{div}_{\partial\Omega}\ g \in H^{-1/2}\left((\partial\Omega)(2S)\right)$ for any $S > 1$ which is enough to conclude that u is a very weak solution of (2.9) in $\Omega(S)$ with such a data.

(ii) In the proof of Theorem 2.7 we actually use a weaker assumption for u. We just invoke that u is an $\dot{H}^1$ weak solution of (2.9) in $\Omega(S)$ with $h = \operatorname{div}_{\partial\Omega} g$ on $\partial\Omega$ and $h = \pm\partial u/\partial x_1$ at $x_1 = \pm S$ for all S instead of assuming that u is an $\dot{H}^1$ weak solution of (2.9) in whole Ω with $h = \operatorname{div}_{\partial\Omega} g$ on Ω. This weaker assumption requiring only that u is a solution in $\Omega(S)$ with suitable boundary condition for all S is not enough to derive (2.13) if Ω is an aperture domain with $n \geq 3$ since there is a nontrivial u satisfying $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ (with some $r \geq n$) which is an $\dot{H}^1$ weak solution of (2.9) in $\Omega(S)$ with $g = 0$ for all S as shown in [FS96]. In fact, this u is not an $\dot{H}^1$ solution in Ω with $g = 0$ since such u must satisfy $\nabla u \in L^2_\sigma(\Omega)$ yielding $\nabla u \equiv 0$, by the L^2-Helmholtz decomposition.

In two-dimensional sector like domains one is able to prove that $\int_{\partial B_r \cap \Omega} \partial u/\partial r \; d\mathcal{H}^1 \to 0$ as $r \to \infty$ by taking a subsequence if $\nabla u \in L^2(\Omega)$ (Remark 2.13 (ii)), while in an aperture domain with $n \geq 3$ the corresponding flux may not decay under $\nabla u \in L^2(\Omega)$ (actually under $\nabla u \in L^p(\Omega)$ with $p > n/(n-1)$). In the two-dimensional setting it is likely that Theorem 2.7 extends to an aperture domain while for the higher dimensional aperture domain the present proof at least does not work although there might be a chance the statement still holds.

(iii) (Strict admissibility) We have proved that a C^3 bounded domain, a C^3 exterior domain are strictly admissible in [AG2]. The C^3 regularity was invoked to construct sequence $\{\varphi_m\}$ converging to φ in the rescaling procedure (Case 1 (b) of the proof of Lemma 2.5) as well as the uniqueness of very weak solutions of (2.8) in a given domain. This last uniqueness is easy to generalize to C^2 domain in the argument of the present paper. In fact, we get the assertion of Lemma 2.1 for C^2 bounded domain and exterior domain for u with $\|d\nabla u\|_\infty < \infty$. The approximation of φ by $\{\varphi_m\}$ requires only C^2 regularity as in Case 1 (b) of the proof of Lemma 2.5. Thus we conclude that C^2 bounded domain and C^2 exterior domain are strictly admissible. In [A1] it is proved that a C^3 perturbed half space is also strictly admissible. This also requires only C^2 regularity by the same reason.

2.3. Admissibility

Theorem 2.7 is enough to guarantee that Ω is admissible in the sense of [AG1]. We say that Ω is *admissible* if there exists $r \geq n$ and a constant C such that

$$\|d_\Omega \mathbf{Q}[\nabla \cdot f](x)\|_\infty \leq C \|f\|_{L^\infty(\partial\Omega)} \tag{2.14}$$

holds for all matrix-valued functions $f = (f_{ij})_{1 \leq i,\, j \leq n} \in C^1(\overline{\Omega})$ satisfying $\nabla \cdot f = \left(\sum_{j=0}^n \partial_j f_{ij} \right) \in (L^r \cap L^2)(\Omega)$ with $\operatorname{tr} f = 0$ and $\partial_\ell f_{ij} = \partial_j f_{i\ell}$ for $1 \leq i, j, \ell \leq n$ where $\mathbf{Q}$ is the L^2-Helmholtz projection from $L^2(\Omega)$ to $G^2(\Omega)$ the orthogonal space of $L^2_\sigma(\Omega)$.

Theorem 2.9. *A C^2 infinite cylinder is admissible.*

Proof. Since $\nabla w = \mathbf{Q}[\nabla \cdot f]$ solves the Neumann problem

$$-\Delta w = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial w}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} W$$

with $W = (f - f^T) \cdot n$ in $\dot{H}^1$ sense with $W \in C^1(\partial\Omega)$ we are able to apply Theorem 2.7 to deduce the estimate (2.14). □

Remark 2.10. (i) To deduce analyticity of the Stokes semigroup in $C_{0,\sigma}$ it suffices to restrict the class of f as $f_{ij} = \partial_j v^i$ with $\operatorname{div} v = 0$ and $f \in (L^r \cap L^2)(\Omega)$.

(ii) By Remark 2.8 we can prove (2.14) for $f \in C(\overline{\Omega})$ which may not be C^1. For all domains which we have considered so far, the restriction $f \in C^1(\Omega)$ is actually can be weaken for $f \in C(\Omega)$. This replacement enables us to conclude the analyticity of the Stokes semigroup in $C_{0,\sigma}$ even if Ω is C^2 under this stronger admissibility. For various domains we so far proved strict admissibility in the sense of [AG2] which then in turn yields admissibility of [AG1] with the replacement of $f \in C^1(\overline{\Omega})$ by $f \in C(\overline{\Omega})$ which is stronger than admissibility.

It turns out that the proof of [AGSS, Theorem 3.2] should be modified with slight modification of the statement. We shall give its rigorous statement.

Theorem 2.11. *Let Ω be a C^2 sector-like domain in $\mathbf{R}^2$. Then there exists a constant C such that the estimate* (2.13) *holds for all $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega)$ of* (2.9) *in Ω with $h = \mathrm{div}_{\partial\Omega}\ g \in C(\partial\Omega)$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega$.*

This is enough to prove the main theorem [AGSS, Theorem 1.3] claiming that Ω is admissible.

To show Theorem 2.11 the lemma [AGSS, Lemma 3.1] is too weak. It should be stated as Lemma 2.5 in the present paper. To state its explicit form let us fix the notations for a sector-like domain Ω with an opening angle θ. We may assume that $\Omega \backslash B_R = S_\theta \backslash B_R$ with some $R \in (0, 1)$ by dilation, rotation and translation, where $S_\theta = \left\{ x = (x_1, x_2) \in \mathbf{R}^2 \mid |\arg x| < \theta/2 \right\}$ and B_R is the closed ball centered at the origin. We set $\Omega_R = \mathrm{int}\, B_{2R} \cap \Omega$. Let us state the statement which is stronger than the lemma [AGSS, Lemma 3.1] and enough to prove Theorem 2.11. This lemma is considered as a variant of Lemma 2.5.

Lemma 2.12. *Let Ω be a C^2 sector-like domain in $\mathbf{R}^2$. Then there exists a constant C (depending only on Ω) such that*

$$\min\left(d_\Omega(x), \log(2R/|x|)\right) |\nabla u(x)| \leq C\|g\|_{L^\infty(\partial\Omega\cap B_{2R})} + \int_{\partial\Omega_R\cap\Omega} \left|\frac{\partial u}{\partial r}\right| d\mathcal{H}^1 \quad \text{for all} \quad x \in \Omega_R \tag{2.15}$$

holds for all $R \geq 1$ and all $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega_R)$ of (2.9) *in Ω_R with $h = \mathrm{div}_{\partial\Omega}\, g \in C(\partial\Omega)$ in $\partial\Omega \cap B_{2R}$, $h = \partial u/\partial r$ on $\partial\Omega_R \cap \Omega$, where $g \in C\left(\overline{\partial\Omega \cap B_{2R}}\right)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega \cap \Omega_R$ provided that $\|d\nabla u\|_{L^\infty(\Omega_R)}$ is finite and that $\partial u/\partial r|_{r=R} \in L^1(\partial\Omega_R \cap \Omega)$.*

Remark 2.13. (i) The proof is parallel to that of Lemma 2.5 with modification needed in the proof of [AGSS, Lemma 3.1], where $\partial u/\partial r = 0$ at $|x| = 2R$ is assumed. (In this case, $\min\left(d_\Omega(x), \log(2R/|x|)\right)$ can be replaced to $d_\Omega(x)$.) The quantity $\log(2R/|x|)$ is the distance from the boundary $S = \log 2R$ to a point $S_1 = \log|x|$ if we introduce the coordinates (s, φ) defined by $x_1 = e^s \cos\varphi$, $x_2 = e^s \sin\varphi$.

(ii) If $\nabla u \in L^2(\Omega)$, then $\int_{\partial B_R\cap\Omega} \left|\frac{\partial u}{\partial r}\right| d\mathcal{H}^1$ tends to zero as $R \to \infty$ if one takes a suitable subsequence. This is easy to prove if one notes that

(s,φ) is a coordinate system which is conformal to the original one in two-dimensional setting. In fact, for $S=\log R$

$$\int_{\Omega>R}|\nabla u|^2dx_1dx_2=\int_S^\infty\int_I|\nabla_{s,\varphi}\overline{u}|^2dsd\varphi,\quad I=(-\theta/2,\theta/2)$$

implies that

$$\int_I\left|\frac{\partial\overline{u}}{\partial s}\right|^2(S,\varphi)d\varphi\to 0$$

as $S\to\infty$ by taking a subsequence $\{S_j\}$, where $\overline{u}(s,\varphi)=u(x_1,x_2)$. This implies $\int_I\left|\frac{\partial\overline{u}}{\partial s}\right|(S,\varphi)d\varphi\to 0$, which implies $\int_{\partial B_R\cap\Omega}\left|\frac{\partial u}{\partial r}\right|d\mathcal{H}^1\to 0$ with $R=R_j=(\exp S_j)/2\to\infty$.

3. Cylindrical Domains

We say a domain $\Omega\subset\mathbf{R}^n$ is a C^2 *cylindrical domain with several exits to infinity* or, for short, C^2 *cylindrical domain*, if it has a C^2-boundary and there is an $m\in\mathbf{N}$ such that $\Omega=\bigcup_{i=0}^m\Omega^i$, where Ω^0 is a bounded domain and Ω^i, $i=1,\dots,m$ are disjoint semi-infinite cylinders, that is, up to rotation and translation,

$$\Omega^i=\{x^i=(x_1^i,\dots,x_n^i)\in\mathbf{R}^n:x_1^i>0,[x^i]'=(x_2^i,\dots,x_n^i)\in D^i\},\tag{3.1}$$

where $D^i\subseteq\mathbf{R}^{n-1}$, $i=1,\dots,m$ are bounded domains of class C^2 and $\Omega^i\cap\Omega^j=\emptyset$ for $i\neq j$, $i,j\leq m$. We may assume that the lateral boundary $\partial D^i\times(0,\infty)$ is also a part of the boundary $\partial\Omega$.

Our goal in this section is again to show that cylindrical domains are admissible domains. Before we come to that, we first state some properties of this class of domains.

Remark 3.1. 1. For each C^2 cylindrical domain $\Omega\subseteq\mathbf{R}^n$, there is an $R_\Omega>0$ and an $x_\Omega\in\Omega$ such that

$$\Omega\setminus B_{R_\Omega}(x_\Omega)=\bigcup_{i=1}^m\Omega^i\setminus B_{R_\Omega}(x_\Omega),\tag{3.2}$$

where Ω^i for $i=1,\dots,m$ are semi-infinite cylinders given in (3.1). Here $B_R(x)$ denotes the closed ball of radius R centered at x. Without loss of generality, we may assume $x_\Omega=0$ by translation.

2. The usual Sobolev embedding theorems hold, since Ω has minimally smooth boundary, and hence extension theorems for Sobolev spaces hold for Ω, see [A, Ch. 5,Thm. 2.4.5] and [T, Thm. 3.2.1].
3. For $1<q<\infty$, Poincaré's inequality holds, namely there is a $C>0$ such for all $u\in W_{q,0}^1(\Omega)$ it holds

$$\|u\|_{L_q(\Omega)}\leq C\|\nabla u\|_{L_q(\Omega)}.\tag{3.3}$$

This can be shown by using a suitable decomposition of unity and applying Poincaré's inequality in each cylinder and in the bounded part.

3.1. Uniqueness under no flux condition

Lemma 3.2. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a bounded solution of* (2.1) *in a C^2 cylindrical domain $\Omega \subseteq \mathbf{R}^n$. Then u is a constant function.*

Proof. The proof is similar to that of Lemma 2.1. For each semi-infinite cylinder Ω^i we define the flux by

$$F_i(s) := \int_{D^i} \frac{\partial u}{\partial x_1}(s, x')\,dx', \quad s > 0$$

where we write $x_1 = x_1^i$ and $x' = [x^i]'$ to increase readability. We also define

$$E_i(s) = \int_{D^i} u(s, x')\,dx'.$$

As in Lemma 2.1 $F_i(s)$ is independent of $s > 0$. Since u is bounded, this implies that $E_i(s)$ must be a constant c_i. We may assume that one of c_i, say $c_1 = 0$ by subtracting c_1/D from u.

Our goal is again to prove $u \equiv 0$ in Ω by the strong maximum principle [PW, Section 3]. Assume that $u \not\equiv 0$. Then we may assume $\sup u > 0$ by considering $-u$ if necessary.

We first show again that $\sup u$ is not attained in $\overline{\Omega}$. If it is attained in Ω, then u must be a constant $\sup u$. However, this must be zero since $c_1 = 0$. Also, u cannot attain its supremum on $\partial\Omega$ since, by the Hopf (boundary) lemma, this would imply $\partial u/\partial n_\Omega > 0$.

Therefore, we may assume that there is a sequence $\{x_k\}$ with $x_k = (s_k, x_k')$ for $k \in \mathbf{N}$ such that $u(x_k) \to \sup u$ and $|s_k| \to \infty$ for $k \to \infty$. By the pigeonhole principle there is a subsequence of $\{x_k\}$ also denoted by $\{x_k\}$ such that $\{x_k\} \subset \Omega^i$ for some $i = 1, \ldots, m$. The assertion then follows by the proof of Lemma 2.1. □

Lemma 3.3. *Let Ω be a C^2 cylindrical domain in $\mathbf{R}^n$. For $S \geq 0$, let $u \in C^2(\Omega_{>S}) \cap C^1(\overline{\Omega}_{>S})$ satisfy*

$$-\Delta u = 0 \quad \text{in} \quad \Omega_{>S}, \qquad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad (\partial\Omega)_{>S}, \tag{3.4}$$

where $\Omega_{>S} := \bigcup_{i=1}^m \Omega^i \cap \{x_1^i > S\}$ and $\partial\Omega_{>S} := \bigcup_{i=1}^m \partial\Omega^i \cap \{x_1^i > S\}$. Assume that S is taken so that $\Omega_{>S}$ consists of mutually disjoint i semi (infinite) cylinder. Assume that $d_\Omega \nabla u$ is bounded in $\Omega_{>S}$ and $F_i(s_1) = 0$ for some $s_1 > S$. Then u is bounded in $\Omega_{>S+\delta}$ for any $\delta > 0$.

This is a trivial extension of Lemma 2.2.

Theorem 3.4. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy* (2.1) *in a C^2 cylindrical domain Ω in $\mathbf{R}^n$. Assume that $F_i(s_1) = 0$ for some $s_1 > 0$ and all $i = 1, \ldots, m$. If $\|d_\Omega \nabla u\|_\infty < \infty$, then u is a constant function.*

Proof. By Lemma 3.3 we observe that u is bounded in $\Omega_{>s_1}$. In $\Omega(s_1) := \Omega \setminus \Omega_{>s_1}$, it is clear that by the regularity assumption on $u \in C^1(\overline{\Omega(s_1)})$ and hence it attains its maximum there. Therefore u is bounded in Ω and Lemma

3.3 with Elliptic Regularity in Appendix A yields boundedness in $W^2_q(\Omega(s_1))$. Thus Theorem 3.4 follows from Lemma 3.2. $\square$

Cylindrical domains have one disadvantage compared to infinite cylinders: They are not mirror-reflexive at $\{x_1 = S\}$ for any $S \in \mathbf{R}$. To overcome this disadvantage, we need to also give a cut-off uniqueness result.

Corollary 3.5. *Let $\Omega \subseteq \mathbf{R}^n$ $(n \geq 2)$ be a C^2 cylindrical domain. Let $S > 0$ be given. Let $u \in C^2(\Omega(S)) \cap C^1(\overline{\Omega(S)})$ be a classical solution of*

$$-\Delta u = 0 \quad in \quad \Omega(S), \qquad \frac{\partial u}{\partial n_{\Omega(S)}} = 0 \quad on \quad \partial(\Omega(S)), \tag{3.5}$$

in $\Omega(S)$, where $\Omega(S) := \Omega_0 \cup \bigcup_{i=1}^m \Omega^i \cap \{x_1^i \leq S\}$ for $S \geq 0$. Then u is a constant function.

Proof. Since $\Omega(S)$ is bounded, $u \in C^1(\overline{\Omega(S)})$ attains its supremum, $\sup u$, on $\overline{\Omega(S)}$. Let $x \in \overline{\Omega(S)}$ be such that $u(x) = \sup u$. If $x \in \Omega(S)$ then the strong maximum principle, implies the assertion. If $x \in \partial\Omega \cap \partial\Omega(S)$, which means the C^2-part of the boundary, Hopf's Lemma contradicts the Neumann boundary condition. It remains to check the case $x \in \partial\Omega(S) \setminus \partial\Omega$. This implies there is an $i \in \{1, \dots, m\}$ such that $x \in \{x_1 = S\} \times \partial D^i$. We may evenly reflect once at $\{x_1 = S\} \times \partial D^i$ to obtain an extension $\tilde{u}$ of u which is harmonic in $\Omega(2S)$ and fulfills zero Neumann boundary conditions at $(\partial\Omega)(2S)$ and conclude again with Hopf's Lemma, that $x \notin \partial(\Omega(S)) \setminus \partial\Omega$. $\square$

3.2. Weighted estimate for the Neumann problem and Admissibility

For cylindrical domains we would now like to prove the equivalent of Theorem 2.7 in order to show that this class of domains is again an admissible. We continue to use the notation for $S \geq 0$,

$$\Omega_{>S} = \bigcup_{i=1}^m \Omega^i \cap \{x_1^i > S\}, \quad \Omega(S) = \Omega \setminus \Omega_{>S} = \Omega_0 \cup \bigcup_{i=1}^m \left(\Omega^i \cap \{x_1^i \leq S\}\right),$$

$$(\partial\Omega)(S) = \partial\Omega \setminus \partial\Omega_{>S} = \partial\Omega \cap \left(\partial\Omega_0 \cup \bigcup_{i=1}^m \left(\partial\Omega^i \cap \{x_1^i \leq S\}\right)\right).$$

Our goal in this subsection is to establish a weighted L^∞ estimate of the form $\|d_\Omega \nabla u\|_\infty \leq C\|g\|_\infty$ for a weak solution u of (2.8) where $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ is a cylindrical domain. For this class of domains, we also consider an $\dot{H}^1$ weak solutions of (2.9) bounded Lipschitz subdomains $U \subset \Omega$.

Also this time, in our limiting procedure we have to handle u such that ∇u may not be integrable near $\partial\Omega$. For this purpose it is convenient to recall notion of a very weak solution of (2.9) for $U = \Omega(S)$ when $h = h_0^i$ in $\{S\} \times D^i$ with $h_0^i \in L^2\left(\{\pm S\} \times D^i\right)$ for $i = 1, \dots, m$ and $h = \operatorname{div}_{\partial\Omega}\, g \in C\left(\overline{\partial\Omega(S)}\right)$ on $(\partial\Omega)(S)$ with $g = (g_1, g') \in C\left(\overline{(\partial\Omega)(S)}\right)$, where g is tangential. We say that $u \in L^1(\Omega(S))$ is a *very weak* solution of (2.9) in $\Omega(S))$ with such data h if

$$\int_{\Omega(S)} u\,\Delta\varphi\,dx = \int_{(\partial\Omega)(S)} \nabla_{\partial\Omega}\varphi\cdot g\,d\mathcal{H}^{n-1} - \sum_{i=1}^{m}\int_{D}^{i}(\varphi h_0^i)(S,x')dx' - \sum_{i=1}^{m}\int_{\partial D^i}(g_1\varphi)(S,x')d\mathcal{H}_{x'}^{n-2}$$

for all $\varphi \in C^2\left(\overline{\Omega(S)}\right)$ with $\partial\varphi/\partial n_{\Omega(S)} = 0$ on $\partial(\Omega(S))$. Since g is tangential, $\nabla_{\partial\Omega}$ can be replaced by ∇.

Lemma 3.6. *Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be a C^2 cylindrical domain. Then there exists a constant $C = C(D^i, n) > 0$ such that for all $x \in \Omega_{\leq S}$*

$$d_{\Omega(S)}(x)\,|\nabla u(x)| \leq C\left(\|g\|_{L^\infty((\partial\Omega)_{\leq S})} + \sum_{i=1}^{m}\left\|\frac{\partial u}{\partial x_1}\right\|_{L^{n-1}(D^i)}(S)\right) \tag{3.6}$$

holds for almost all $S \geq 1$ and all $\dot{H}^1$ weak solutions u of (2.9) *with boundary data $h \in H^{-1/2}(\partial(\Omega(S)))$, with*

$$h = \begin{cases} \operatorname{div}_{\partial\Omega}\, g & \text{in } (\partial\Omega)(S) \\ h_0^i := \partial u/\partial x_1 & \text{on } \{S\}\times D^i, (i \in \{1,\dots,m\}) \end{cases} \tag{3.7}$$

where $g \in C\left(\overline{(\partial\Omega(S)}\right)$ satisfies $g\cdot n_\Omega = 0$ on $(\partial\Omega)(S)$ and $\operatorname{div}_{\partial\Omega}\, g \in C(\overline{\partial\Omega(S)})$, provided that $\nabla u \in L^2(\Omega(S))$ in $\Omega(S)$ and $\|d_\Omega\nabla u\|_{L^\infty(\Omega(S))} < \infty$ and that $\partial u/\partial x_1(S,\cdot) \in L^{n-1}(D^i)$.

By a similar argumentation as in the proof of Lemma 2.5, u is also a very weak solution of (2.9).

Proof. We argue again by contradiction. Assume there is a sequence $\{u_k, g_k, S_k\}_{k=1}^\infty$ such that $1 = \|d_\Omega\nabla u_k\|_{L^\infty(\Omega_k)} > k\|g_k\|_{L^\infty(\partial\Omega_k)}$, $\Omega_k := \Omega(S_k)$. Let $\{x_k\}_{k=1}^\infty$ be a sequence such that $x_k \in \overline{\Omega_k}$, $|d_\Omega\nabla u_k(x_k)| > 1/2$ and again, without loss of generality $u_k(x_k) = 0$, $\lim_{k\to\infty} S_k = S_\infty \in [1,\infty]$. We begin to treat Case 1: $x_k \to \hat{x} \in \overline{\Omega}_\infty$, more specifically Case 1 (a) $\hat{x} \in \Omega_\infty$.

Similarly to Lemma 2.5, we obtain a weakly convergent subsequence converging to some u of (2.8) in $\Omega_\infty = \Omega(S_\infty)$ with $g = 0$ and $h_0 = 0$ as before. The convergence is again locally uniform with its derivatives in Ω: Again since $u_k(x_k) = 0$ and $x_m \to \hat{x} \in \Omega$ integrating over $|\nabla u_m(x)| \leq 1/d_\Omega(x)$ from x_k yields a bound in $L^q_{loc}(\Omega_k)$. By a diagonal argument, we see that $\{u_k\}$ converges to some $u \in L^q_{loc}(\overline{\Omega})$. This u is a very weak solution of (2.9) in $\Omega_\infty = \Omega(S_\infty)$ with zero Neumann data. Let $\phi \in C^2(\overline{\Omega_\infty})$ with $\partial\phi/\partial n_\Omega = 0$ on $\partial\Omega_\infty$. Let $(\chi_i)_{0\leq i\leq m_\Omega} \in C^\infty(\overline{\Omega})$ be a partition of unity in Ω, such that $\sum_{i=0}^m \chi_i = 1$ in $\overline{\Omega}$ with $0 \leq \chi^i \leq 1$, spt $\chi_i \subseteq \Omega^i, i \in \{1,\dots,m\}$ and spt $\chi_0 \subseteq \Omega(1)$. We write $\phi = \sum_{i=0}^m \chi_i\phi =: \sum_{i=0}^{m_\Omega}\phi^i$. For $i > 0$, it follows as in the infinite cylinder case that $\int_{\Omega_\infty} u_\infty \Delta\phi^i dx = 0$. For $i = 0$ it holds

$$\int_{\Omega_\infty} u_\infty \Delta\phi^0 dx = \int_{\Omega(1)} (u_\infty - u_m)\Delta\phi^0 dx + \int_{\Omega(1)} u_m \Delta\phi^0 dx.$$

For $m \to \infty$, the first summand tends to zero, since $u_m \to u_\infty$ weakly and the second summand tends to zero, since $[h_0]_m$ and g_m tend to zero for $m \to \infty$.

By the Cauchy estimates for harmonic functions, [E, 2.2.c], $u \in C^\infty(\Omega_\infty) \cap C^1(\overline{\Omega_\infty})$ and $u(\hat{x}) = 0$ and $|d_{\Omega_\infty} \nabla u(\hat{x})| > 1/2$.

For finite S_∞, Corollary 3.5 yields a contradiction to $|d_\Omega \nabla u(\hat{x})| > 1/2$. For $S_\infty = \infty$; checking the flux condition can be done analogously to Lemma 2.5 by replacing D by D^i to obtain $F(0) = 0$, then Theorem 3.4 yields the contradiction.

In the case (b), we can use the same argumentation of Lemma 2.5. If $\hat{x} \in \partial\Omega_\infty$, note that by the argumentation given in the aforementioned proof, C^2-regularity of the boundary is enough to deduce convergence to the half space or quadrant type space.

Case 2 can be dealt with analogously to the one in the proof of Lemma 2.5, since there is an $1 \le i \le m$ such that $x_k \in \Omega^i$ for sufficiently large k. □

Theorem 3.7. *Let Ω be a C^2 cylindrical domain in $\mathbf{R}^n$ $(n \ge 2)$. Then there exists a constant C (depending only on D^i, Ω_0 and n) such that*

$$\|d_\Omega \nabla u\|_{L^\infty(\Omega)} \le C\|g\|_{L^\infty(\partial\Omega)} \tag{3.8}$$

holds for all $\dot{H}^1$ weak solution $u \in L^1_{loc}(\overline{\Omega})$ of (2.8) in Ω with $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ (with some $r \ge n$) and $h = \operatorname{div}_{\partial\Omega} g \in C(\partial\Omega)$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfying $g \cdot n_\Omega = 0$ on $\partial\Omega$.

Proof. The proof is again analogous to the one given for Theorem 2.7. Instead of Lemma 2.5 we apply its cylindrical counterpart, Lemma 3.6 in order to estimate (3.8). □

Subsequently, we obtain analogously to Theorem 2.7 the following result.

Theorem 3.8. *A C^2 cylindrical domain Ω in $\mathbf{R}^n$ $(n \ge 2)$ is admissible.*

Proof. The proof is analogous to the infinite cylinder case by applying Theorem 3.7 instead of Theorem 2.7. □

4. Stokes resolvent estimate

Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. In this section we establish an a priori L^∞ estimate for solutions of the resolvent Stokes equations with zero Dirichlet condition (1.5a)–(1.5c) by a blow-up argument. The version we establish here is stronger than Theorem 1.2. We set

$$L^\infty_d(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \mid \|f\|_{L^\infty_d(\Omega)} := \operatorname{ess.sup}_{x\in\Omega} |d_\Omega(x) f(x)| < \infty \right\}.$$

We also use a standard notation $W^{m,r}$ for describing L^r-Sobolev space of order m and $W^{m,r}_{loc}$ for its localized version.

Theorem 4.1 (L^∞ a priori estimate). *Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. Let $c_* > 0$. For $\varepsilon \in (0, \pi/2)$ there exist positive constants C and M depending only on Ω, c_*, ε such that*

$$\|N(v,\lambda)\|_\infty \leq C\|f\|_\infty \tag{4.1}$$

for all $(v,p) \in \left(W^{1,\infty}(\Omega) \cap W^{2,r}_{loc}(\Omega)\right) \times L^\infty_d(\Omega)$ (with some $r > 1$) solving (1.5a)–(1.5c) *with $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|\lambda| \geq M\}$ and $f \in L^\infty_\sigma(\Omega)$ provided that*

$$\|\nabla p\|_{L^\infty_d(\Omega)} \leq c_* \|\nabla v\|_\infty. \tag{4.2}$$

Since admissibility of Ω implies (4.2), Theorem 4.1 yields Theorem 1.2. To prove (4.1) we argue by contradiction and apply a blow up argument as in [AG1], [AG2] but not for evolution equation (1.1a)–(1.1d) but the resolvent equations (1.5a)–(1.5c) as in [Suzuki] where elliptic problems are discussed by a blow-up argument. We construct a blow up sequence and prove strong convergence to a nontrivial solution with homogeneous problem which contradicts the uniqueness.

4.1. Construction of a blow up sequence

We argue by contradiction to prove (4.1). Suppose that (4.1) were not valid. Then there are $\varepsilon > 0$ and $\lambda_k \in \Sigma_{\pi-\varepsilon}$, $|\lambda_k| \geq k$, $f_k \in L^\infty_\sigma(\Omega)$, $(v_k, \nabla p_k)$ which solve (1.5a)–(1.5c) with

$$k\,\|f_k\|_{L^\infty(\Omega)} < \|N(v_k,\lambda_k)\|_\infty < \infty, \quad \|\nabla p_k\|_{L^\infty_d(\Omega)} \leq c_*\,\|\nabla v_k\|_\infty\,.$$

We normalize (v_k, q_k) as

$$u_k = |\lambda_k| v_k / \|N(v_k,\lambda_k)\|_\infty\,, \quad q_k = p_k / \|N(v_k,\lambda_k)\|_\infty \quad \text{and} \quad \lambda_k = |\lambda_k| e^{i\theta_k}$$

and observe that (u_k, q_k) solves

$$\begin{aligned} \left(e^{i\theta_k} - \frac{\Delta}{|\lambda_k|}\right) u_k + \nabla q_k &= \tilde{f}_k \quad \text{in} \quad \Omega \\ \operatorname{div} u_k &= 0 \quad \text{in} \quad \Omega \\ u_k &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where $\tilde{f}_k = f_k / \|N(v_k,\lambda_k)\|_\infty$. By definition we notice that

$$\left\|N\left(\frac{u_k}{|\lambda_k|}, \lambda_k\right)\right\|_\infty = 1, \quad \left\|\tilde{f}_k\right\|_\infty < 1/k, \quad |\lambda_k| \geq k, \quad |\theta_k| \leq \pi - \varepsilon$$

We next rescale (u_k, q_k) around the point x_k where $N\left(u_k/|\lambda_k|, \lambda_k\right)(x_k)$ is close to 1. We take a sequence $\{x_k\} \subset \Omega$ such that

$$|u_k(x_k)| + |\lambda_k|^{-1/2}\,|\nabla u_k(x_k)| > 1/2.$$

We rescale (u_k, q_k) and $\tilde{f}_k$ as

$$w_k(x) = u_k\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right) \quad \varpi_k = |\lambda_k|^{1/2} q_k\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right)$$

$$g_k = \tilde{f}_k \left(x_k + \frac{x}{|\lambda_k|^{1/2}} \right).$$

Then we observe that rescaled (w_k, ϖ_k) solves

$$\left(e^{i\theta_k} - \Delta\right) w_k + \nabla \varpi_k = g_k \quad \text{in} \quad \Omega_k, \tag{4.3a}$$

$$\operatorname{div} w_k = 0 \quad \text{in} \quad \Omega_k, \tag{4.3b}$$

$$w_k = 0 \quad \text{on} \quad \partial\Omega_k. \tag{4.3c}$$

One can translate the estimates for (u_k, q_k) by (w_k, ϖ_k).

$$\sup_{x \in \Omega_k} |w_k(x)| + |\nabla w_k(x)| = 1, \quad |w_k(0)| + |\nabla w_k(0)| > 1/2, \quad \|g_k\|_\infty \leq 1/k, \tag{4.4}$$

where $\Omega_k = |\lambda_k|^{1/2}(\Omega - x_k)$. The pressure estimates becomes

$$\|d_{\Omega_k} \nabla \varpi_k\| \leq c_* \|\nabla w_k\|_\infty. \tag{4.5}$$

4.2. Convergence

We shall divide the situation into two cases depending on whether or not $\hat{d} = \limsup_{k \to \infty} d_k$ with $d_k = d(0, \partial\Omega_k) = |\lambda_k|^{1/2} d(x_k, \partial\Omega)$ is infinite.

Lemma 4.2 (Case 1, $\hat{d} = \infty$). *Let $\{w_k\}$ be the blow up sequence in Section 4.1. Then w_k converges to some $w \in W^{1,\infty}(\mathbf{R}^n)$ locally uniformly as $k \to \infty$ with $\|\nabla w_k\|_\infty \leq 1$ by taking a subsequence. Moreover, w solves a resolvent Laplace equation in the sense that there exists θ_m with $|\theta_\infty| \leq \pi - \varepsilon$ such that w satisfies*

$$\int_{\mathbf{R}^n} \nabla w \cdot \nabla \varphi \, dx + \int_{\mathbf{R}^n} e^{i\theta_\infty} w \varphi \, dx = 0 \tag{4.6}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n)$.

Proof. We may assume that $d_k \to \infty$ and $\theta_k \to \theta_\infty$ by taking a subsequence if necessary. Since Ω_k converges to $\mathbf{R}^n$, for $\varphi \in C_c^\infty(\mathbf{R}^n)$ we may assume that φ vanishes in neighborhood of $\partial\Omega_k$ if k is sufficiently large. For such k since (w_k, ϖ_k) solves (4.3a), we see that

$$\int_{\mathbf{R}^n} \nabla w_k \cdot \nabla \varphi \, dx + \int_{\mathbf{R}^n} e^{i\theta_k} w_k \varphi \, dx + \int_{\mathbf{R}^n} \nabla \pi_k \cdot \varphi \, dx = \int_{\mathbf{R}^n} g_k \cdot \varphi \, dx. \tag{4.7}$$

By (4.4) we note that g_k converges to zero uniformly. Moreover, since $\|\nabla w_k\|_\infty \leq 1$ by (4.4), the estimate (4.5) implies that $\nabla \varpi_k$ tends to zero locally uniformly in $\mathbf{R}^n$. This implies that

$$\int_{\mathbf{R}^n} \nabla w_k \cdot \nabla \varphi \, dx + \int_{\mathbf{R}^n} e^{i\theta_k} w_k \cdot \varphi \, dx \to 0 \quad \text{as} \quad k \to \infty. \tag{4.8}$$

Since $\{w_k\}$ is bounded in $W^{1,\infty}(\Omega_k)$ by (4.4), there is a limit w such that $\nabla w_k \to \nabla w$ $*$ weakly in L^∞ and $w_k \to w$ locally uniformly by taking a subsequence; the latter convergence is by Ascoli-Arzelà theorem. The convergence (4.8) now yields (4.6). $\square$

The case $\hat{d} < \infty$ (Case 2) is more involved. The rescaled domain Ω_k converges to a half space of the form (up to rotation)

$$\mathbf{R}^n_{+,-c} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c\}$$

with some $c > 0$. Indeed, we take a nearest boundary point $y_k \in \partial\Omega_k$ from 0. This y_k converges to some $\hat{y}$ and $y_k/|y_k|$ converges to some unit vector e by taking a subsequence. We rotate the coordinate so that $e = (0, \dots, -1)$ and that $\hat{y}_n = -c$ with $c \geq 0$. By this choice of coordinates one can prove that $\partial\Omega_k$ converges to $\partial\mathbf{R}^n_{+,-c}$ in the sense of C^2 graphs in a big cube.

Lemma 4.3 (Case 2, $\hat{d} < \infty$). *Assume that Ω_k converges to $\mathbf{R}^n_{+,-c}$. Let $\{(w_k, \varpi_k)\}$ be the blow up sequence in Section 4.1. Then w_k converges to some $w \in W^{1,\infty}(\mathbf{R}^n_{+,-c})$ locally uniformly as $k \to \infty$ with $\|\nabla w_k\|_\infty \leq 1$ and $\nabla\varpi_k$ converges to some $\nabla\varpi \in L^\infty_d(\mathbf{R}^n_{+,-c})$ $*$ weakly in $L^\infty(\mathbf{R}^n_{+,b})$ (with $\varpi \in L^1_{loc}(\mathbf{R}^n_{+,-c})$) for any $b > -c$ by taking a subsequence. Moreover, (w, ϖ) solves the resolvent Stokes equation in $\mathbf{R}^n_{+,-c}$ in the sense that there exists θ_∞ with $|\theta_\infty| \leq \pi - \varepsilon$ such that (w, ϖ) satisfies*

$$\int_{\mathbf{R}^n_{+,-c}} \nabla w \cdot \nabla\varphi \, dx + \int_{\mathbf{R}^n_{+,-c}} e^{i\theta_\infty} w \cdot \varphi \, dx + \int_{\mathbf{R}^n_{+,-c}} \nabla\varpi \cdot \varphi \, dx = 0 \tag{4.9}$$

for all $\varphi \in C^\infty_c(\mathbf{R}^n_+)$. Furthermore, w satisfies $\operatorname{div} w = 0$ *in $\mathbf{R}^n_{+,-c}$ as well as $w = 0$ on $\partial\mathbf{R}^n_{+,-c}$.*

Proof. The proof parallels that of Lemma 4.2. We may assume that $d_k \to \hat{d}$, $\theta_k \to \theta_\infty$. Moreover, for $\varphi \in C^\infty_c(\mathbf{R}^n_{+,-c})$ we may assume that φ vanishes in a neighborhood of $\partial\Omega_k$. Since (4.5) (with (4.4)) now only yields a bound on $\|\nabla\varpi_k\|_{L^\infty(\operatorname{spt}\varpi)}$ uniformly for large k, the last term of the left-hand side of (4.7) does not vanish. Letting $k \to \infty$ in (4.7) yields (4.9) for a limit of (w_k, ϖ_k). The remaining assertion is easy to verify. $\square$

4.3. Uniform convergence near the origin

We shall prove that w_k converges to w uniformly together with its first derivative near the origin. For this purpose we recall $W^{2,r}$ estimates for the generalized Stokes system. Here L^r_{av} is the space of all L^r-function in Ω whose average over Ω equals zero.

Proposition 4.4 ($W^{2,r}$ estimates). *Let Ω be a C^2 bounded domain in $\mathbf{R}^n$ and let $r \in (1, \infty)$. Then there exists a constant $c > 0$ depending only on r, n and the C^2-regularity of $\partial\Omega$ such that*

$$\|u\|_{W^{2,r}(\Omega)} + \|p\|_{W^{2,r}(\Omega)} \leq C \left(\|f\|_{L^r(\Omega)} + \|g\|_{W^{1,r}(\Omega)}\right)$$

for all solutions $(u, p) \in \left(W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)\right) \times \left(W^{1,r}(\Omega) \cap L^r_{\mathrm{av}}(\Omega)\right)$ of the Stokes system

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = g \quad \text{in} \quad \Omega$$

with $u = 0$ on $\partial\Omega$.

This type of L^r estimates is by now very popular [BF, Section b.3], [Ga, Section 4.6] and went back to R. Farwig and H. Sohr [FS94], where the existence of solution is also established.

Lemma 4.5. *Let $\{w_k\}$ and its limit w be as in Lemma 4.2 or Lemma 4.3. Then ∇w_k converges to ∇w locally uniformly near the origin as $k \to \infty$. In particular, $|w(0)| + |\nabla w(0)| \geq 1/2$.*

Proof. Let $\zeta \in C_c^\infty(\mathbf{R}^n)$ be a cut-off function of the form such that $\zeta(x) = 1$ on $B_1(0)$ and $\zeta(x) = 0$ outside $B_2(0)$ where $B_r(x)$ denotes the closed ball of radius r centered at x. We localize w_k by defining $W_k = \zeta w_k$. Then W_k solves

$$\begin{aligned} -\Delta W_k + \nabla \Pi_{k,0} = \zeta g_k - e^{i\theta_k} W_k + E \quad &\text{in} \quad \Omega_k \cap B_2(0) \\ \operatorname{div} W_k = w_k \cdot \nabla \zeta \quad &\text{in} \quad \Omega_k \cap B_2(0) \\ W_k = 0 \quad &\text{on} \quad (\partial \Omega_k \cap B_2(0)) \cup (\partial B_2(0) \cap \Omega_k), \end{aligned}$$

where $\varpi_{k,0} = \varpi_k - ⨍_{\Omega_k \cap B_2(0)} \varpi_k dx$, $\Pi_{k,0} = \zeta \varpi_{k,0}$ and E is the lower order terms of w_k and $\varpi_{k,0}$. It can be calculated as

$$E = -2 \operatorname{div}(w \nabla \zeta) + w_k \Delta \zeta + \varpi_{k,0} \nabla \zeta$$

since $\Delta W_k = (\Delta w)\zeta + 2 \operatorname{div}(w_k \zeta) - w_k \Delta \zeta$ and $\nabla(\varpi_{0,k}\zeta) = (\nabla \pi_{0,k})\zeta + \pi_{0,k}\nabla\zeta$ by Leibniz rule.

We mollify $\partial(\Omega_k \cap B_k(0))$ in an open neighborhood of $\partial(\Omega_k) \cap \partial(B_k(0))$ so that the boundary $\partial\Omega'_k$ of Ω'_k is C^2 its C^2-regularity is uniform in k. Moreover, we may take Ω'_k so that

$$\Omega_k \cap B_2(0) \subset \Omega'_k \subset \Omega_k \cap B_3(0).$$

We now apply the $W^{2,r}$ estimate (Proposition 4.4) to get

$$\begin{aligned} &\|W_k\|_{W^{2,r}(\Omega'_k)} + \|\nabla \Pi_k\|_{L^r(\Omega'_k)} \\ &\qquad \leq C \left(\| - e^{i\theta_k} W_k + \zeta g_k + E \|_{L^r(\Omega'_k)} + \|w_k \nabla \zeta\|_{W^{1,r}(\Omega'_k)} \right) \\ &\qquad\qquad \leq C \, |B_2(0)|^{\frac{1}{r}} \left(\|g_k\|_\infty + \|w_k\|_{W^{1,\infty}(\Omega_k)} \right) + C \|\varpi_{k,0}\|_{L^r(\Omega'_k)}. \end{aligned}$$

Here we modify the definition of $\varpi_{k,0}$ by $\varpi_k - ⨍_{\Omega'_k} \varpi_k dx$. By (4.4) we observe that

$$\|g_k\|_\infty + \|w_k\|_{W^{1,\infty}} \leq 1 + 1/k \leq 2.$$

By the Poincaré type inequality [AGH, (2.1)] we have

$$\|\varpi_{k,0}\|_{L^r(\Omega'_k)} \leq C \|\nabla \varpi_k\|_{L^\infty_d(\Omega'_k)}.$$

This together with (4.5) and bound on $\|\nabla W_k\|_\infty$ yields $\|\varpi_{k,0}\|_{L^r(\Omega'_k)} \leq M$ with M independent of k. We thus observe that $\|W_k\|_{W^{2,r}(\Omega'_k)} + \|\nabla \Pi_k\|_{L^r(\Omega'_k)}$ is bounded in k. We take $r > n$ to get uniform bound for $\|w_k\|_{C^{1+\mu}(\Omega_k \cap B_1(0))}$ with $\mu = 1 - n/r$ from the above $W^{2,r}$ (Proposition 4.4 bound by Morrey's inequality [E, 5.6.2]. By Ascoli-Arzelà theorem we conclude that $w_k \to w$ uniformly (with its first derivative) in $B_1(0) \cap \mathbf{R}^n_{+,-c}$ (Case 2) or in $B_2(0)$ (Case 1) since w_k itself converges to w locally uniformly.

Since $|w_k(0)| + |\nabla w_k(0)| \geq 1/2$ by (4.4), this implies that $|w(0)| + |\nabla w(0)| \geq 1/2$. □

4.4. Uniqueness of the limit problem

In this subsection we give a uniqueness result for the resolvent Laplace equation in $\mathbf{R}^n$ and the resolvent Stokes equations in $\mathbf{R}^n_+$ so that the limit w in Lemma 4.2 and Lemma 4.3 is identically zero.

Lemma 4.6 (Uniqueness in $\mathbf{R}^n$). *For $\mu \in \mathbf{C}\backslash(-\infty, 0]$ assume that $w \in L^\infty(\mathbf{R}^n)$ satisfies*

$$\int_{\mathbf{R}^n} w(\mu - \Delta)\eta \, dx = 0 \tag{4.10}$$

for all $\eta \in C_c^\infty(\mathbf{R}^n)$. Then $w = 0$.

Lemma 4.7 (Uniqueness in $\mathbf{R}^n_+$). *For $\mu \in \mathbf{C}\backslash(-\infty, 0]$ assume that $w \in W^{1,\infty}(\mathbf{R}^n_+)$ satisfies*

$$\int_{\mathbf{R}^n_+} w(\mu - \Delta)\eta \, dx + \int_{\mathbf{R}^n_+} \nabla q \cdot \eta \, dx = 0$$

with some $q \in L^1_{loc}(\overline{\mathbf{R}^n_+})$ such that $\nabla q \in L^\infty_d(\mathbf{R}^n_+)$ for all $\eta \in C_c^\infty(\mathbf{R}^n_+)$. If $w = 0$ on $\partial\mathbf{R}^n_+$, then $w = 0$, $\nabla q = 0$ in $\mathbf{R}^n_+$.

Proof of Theorem 4.1 *admitting Lemma* 4.6 *and* 4.7*.* Suppose that (4.1) were false. Then we have w as a blow up limit of a sequence of solutions as in Lemma 4.2 and 4.3. By Lemma 4.5 we know that $|w(0)| + |\nabla w(0)| \geq 1/4$. However, since w solves the resolvent Laplace equation (4.6) in $\mathbf{R}^n$ or the resolvent Stokes equation (4.9), we are able to apply the uniqueness results (Lemma 4.6 for the case $\hat{d} = \infty$ and Lemma 4.7 for the case $\hat{d} < \infty$) to get $w = 0$ which is a contradiction. The proof is now complete. □

The rest of this subsection is devoted to the proof of Lemma 4.6 and 4.7.

Proof of Lemma 4.6*.* Since $C_c^\infty(\mathbf{R}^n)$ is dense in $S(\mathbf{R}^n)$, the space of rapidly decaying functions in the sense of Schwartz, one may assume $\eta \in S(\mathbf{R}^n)$ in (4.10).

For a given $\psi \in C_c^\infty(\mathbf{R}^n)$ there is a solution $\eta \in S(\mathbf{R}^n)$ of $(\mu - \Delta)\eta = \psi$. This is easy to prove since the Fourier transform $\hat{\eta}$ of η is given by $\hat{\eta}(\zeta) = \left(\mu + |\zeta|^2\right)^{-1} \hat{\psi}$ and the Fourier transform is bijective on $S(\mathbf{R}^n)$. We thus observe that

$$\int_{\mathbf{R}^n} w \, \psi \, dx = 0$$

for all $\psi \in C_c^\infty(\mathbf{R}^n)$. By a fundamental theorem of calculus of variation this implies that $w = 0$ a.e. □

Lemma 4.7 can be proved as in [Sol03], where the uniqueness of the nonstationary problem has been proved. Note that Lemma 4.7 can be proved directly by a duality argument by the following existence result which can be proved by the Laplace transform of L^1 theory of the Stokes flow in a half space [GMS]. We will not give its proof.

Lemma 4.8 (Existence in L^1). *Let $f \in C^\infty_{c,\sigma}(\mathbf{R}^n_+)$ be a vector field of the form $f(x) = \sum_{j=1}^{n-1} \partial_j \psi_j(x)$ for $\psi = (\psi_j) \in C^\infty_{c,\sigma}(\mathbf{R}^n_+)$. For $\mu \in \mathbf{C}\backslash(-\infty, 0]$ there exists $\eta \in W^{2,1}(\mathbf{R}^n_+)$, $\pi \in L^1_{loc}(\mathbf{R}^n_+)$ with $\nabla\pi \in L^1(\mathbf{R}^n_+)$ satisfying*

$$(\mu - \Delta)\eta + \nabla\pi = f, \quad \operatorname{div}\eta = 0 \quad in \quad \mathbf{R}^n_+$$

with $\eta(x', 0) = 0$. Moreover, there exists a constant $C = C(\mu)$ such that

$$\|\eta\|_{W^{2,1}} \leq C\|\psi\|_{W^{2,1}}.$$

Appendix A. Elliptic local regularity

We shall prove a local regularity up to the boundary for an L^r (very) weak solution of the Neumann problem. Such regularity results are more or less known but it is not easy to find exact reference to our setting so we give a proof for the reader's convenience and completeness.

Theorem A.1. *Let Ω be a domain in $\mathbf{R}^n$ $(n \geq 2)$. Let x_0 be a point on $\partial\Omega$. Then there exists a constant C depending only on $n, r \in (1,\infty)$, $R > 0$ and C^2-regularity of $\partial\Omega$ in $B_{2R}(x_0)$ such that*

$$\|\nabla^2 u\|_{L^r(\Omega_R)} \leq C\|u\|_{L^r(\Omega_{2R})}$$

for all $u \in L^r(\Omega_{2R})$ which satisfies $\Delta u = 0$ in Ω_{2R} and $\partial u/\partial n_\Omega = 0$ on $\operatorname{int} B_{2R}(x_0) \cap \partial\Omega$ in a (very) weak sense provided that $\partial\Omega$ is C^2 in $B_{2R}(x_0)$. Here $\Omega_R = \operatorname{int} B_R(x_0) \cap \Omega$.

To prove this statement we need to recall unique solvability of $W^{2,q}$ solution and uniqueness of L^r (very) weak solution.

Lemma A.2. *Let Ω be a C^1 bounded domain in $\mathbf{R}^n$ and $q \in (1,\infty)$.*

(i). *For a given $f \in W^{-1,q}(\Omega)$ satisfying $\int_\Omega f\, dx = 0$ there exists a unique solution $u \in W^{1,q}(\Omega)$ with $\int_\Omega u\, dx = 0$ (a standard weak solution based on bilinear form) of*

$$-\Delta u = f \quad in \quad \Omega \quad and \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad on \quad \partial\Omega.$$

Moreover, there exists a constant C depending only on q, n and C^1-regularity of $\partial\Omega$ such that

$$\|\nabla u\|_{L^q(\Omega)} \leq C\|f\|_{W^{-1,q}(\Omega)}.$$

(ii). *Assume that Ω is C^2. If f is in $L^q(\Omega)$, then $u \in W^{2,q}(\Omega)$. Moreover,*

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)}.$$

The first statement under C^1-regularity assumption is taken from Simader and Sohr [SS]. Both statements with C^2-regularity is rather standard and classical; see e.g. [LM, Teor. 4.1].

Lemma A.3 (Uniqueness of L^r-solution). *Let Ω be a C^2 bounded domain in $\mathbf{R}^n$. For $r \in (1,\infty)$ let $u \in L^r(\Omega)$ be a (very) weak solution to the homogeneous Neumann problem: $\Delta u = 0$ in Ω, $\partial u/\partial n_\Omega = 0$ on $\partial\Omega$. Then u must be a constant.*

Proof of Lemma A.3. This can be proved by a duality argument. We may assume $r < n/(n-1)$ since Ω is bounded. For a given $f \in C_c^\infty(\Omega)$ with $\int_\Omega f \; dx = 0$, there is a unique $W^{2,r'}$ solution v for $-\Delta v = f$ in Ω and $\partial v/\partial n_\Omega = 0$ on $\partial\Omega$ with $\int_\Omega v \; dx = 0$ by Lemma A.2 (ii) where r' is the conjugate exponent of r, i.e., $1/r + 1/r' = 1$. By definition of a very weak solution we see that

$$\int_\Omega u\Delta v \; dx = 0.$$

In the definition we need $v \in C^2(\overline{\Omega})$ but it can be replaced by $v \in C^1(\overline{\Omega}) \cap W^{2,r'}(\Omega)$ by an approximation. Indeed, for $v \in C^1(\overline{\Omega}) \cap W^{2,r'}(\Omega)$ with $\partial v/\partial n_\Omega = 0$ on $\partial\Omega$ and $\int_\Omega v \; dx = 0$, we approximate the boundary value $v \in W^{2-1/r',r'}(\partial\Omega)$ by $w_m \in C^2(\partial\Omega)$ such that $\|v - w_m\|_{W^{2-1/r',r'}} \to 0$. Since the boundary is C^2, there is a bounded linear extension operator E from $W^{2-1/r',r'}(\partial\Omega) \times W^{1-1/r',r'}(\partial\Omega)$ to $W^{2,r'}(\Omega)$ such that $E(w,g)|_{\partial\Omega} = w$, $(\partial/\partial n_\Omega)\; E(w,g)|_{\partial\Omega} = g$ and moreover, $E(w,g) \in C^2(\overline{\Omega})$ if $w \in C^2(\partial\Omega)$, $g \in C^1(\partial\Omega)$; see [Ad]. (Such an extension operator is for example found in Appendix B (Lemma B.1 and Lemma B.2.) We set $v_m = E(w_m,0) - \fint E(w_m,0)dx$ and observe that $\|v_m - v\|_{W^{2,r'}} \to 0$ and that $v_m \in C^2(\overline{\Omega})$ with $\int_\Omega v_m dx = 0$. The desired identity $\int_\Omega u\Delta v \; dx = 0$ follows from $\int_\Omega u\Delta v_m dx = 0$ since $v_m \to v$ in $W^{2,r'}(\Omega)$. We thus conclude that $\int_\Omega uf \; dx = 0$ for all $f \in C_c^\infty(\Omega)$ with the average zero condition. This implies that u is a constant. $\square$

Proof of Theorem A.1. We use a cut-off function φ satisfying the homogeneous Neumann condition constructed in Lemma B.2. Since Ω_{2R} is not C^2, we consider a C^2 bounded domain which is slightly larger domain $\tilde{\Omega}_{2R}$. We consider $v = u\varphi$ and observe that v is an L^r very weak solution of

$$-\Delta v = f \quad \text{in} \quad \tilde{\Omega}_{2R} \quad \text{and} \quad \partial v/\partial n_\Omega = 0 \quad \text{on} \quad \partial\tilde{\Omega}_{2R}$$

with $f = -2\,\mathrm{div}(u\nabla\varphi) - u\Delta\varphi \in W^{-1,r}(\tilde{\Omega}_{2R})$. By the existence (Lemma A.2 (i)) and the uniqueness (Lemma A.3) there exists a constant C' depending on φ such that

$$\|\nabla u\|_{L^r(\tilde{\Omega}_{2R})} \le C' \|u\|_{L^r(\tilde{\Omega}_{2R})}.$$

In particular, $v \in W^{1,r}(\tilde{\Omega}_{2R})$. We may take the cut-off function so that $\varphi \equiv 1$ on $\Omega_{4R/3}$, then we observe that $u \in W^{1,r}(\Omega_{4R/3})$.

We repeat this argument in $\tilde{\Omega}_{4R/3}$ with a cut-off function such that $\varphi \equiv 1$ on $\tilde{\Omega}_R$ with $\varphi \in C_c^2(B_{4R/3})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$ (Lemma B.2). We now apply Lemma A.2 (ii) to get the desired $W^{2,r}$-estimate in Ω_R. Such an argument is often called a bootstrap argument. $\square$

Appendix B. Construction of cut-off functions

We shall construct cut-off functions which keep the Neumann boundary condition. We begin with an extension lemma. This is by now a standard way to extend which is for example found in [BF, 2.4].

Lemma B.1. *Let $k \geq 1$ be a natural number and R, $R' > 0$. Let $f \in C_c^k(\mathbf{R}^{n-1})$, $g \in C_c^{k-1}(\mathbf{R}^{n-1})$ satisfy* $\operatorname{spt} f$, $\operatorname{spt} g \subset \operatorname{int} B_{2R}^{n-1}$ *and $g = 0$ on $B_{4R/3}^{n-1}$. Then there exists $\psi \in C_c^k(\mathbf{R}^n)$ such that*

$$\psi(x', 0) = f(x'), \quad \frac{\partial \psi}{\partial x_n}(x', 0) = g(x'), \quad x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$$

and $\operatorname{spt} \psi \subset \operatorname{int} B_{2R}^{n-1} \times (-2R', 2R')$ *and $\psi(x', x_n) = f(x')$ on $B_R^{n-1} \times [-R', R']$. If $0 \leq f \leq 1$, then ψ can be taken such that $0 \leq \psi \leq 1$.*

Proof. Let ρ_ε be a symmetric Friedrichs' mollifier in $\mathbf{R}^{n-1}$. In other words, we take a non-increasing function $\eta \in C_c^\infty[0, \infty)$ such that $\eta = 1$ on $[0, 1/2]$ and $\eta = 0$ on $[2, \infty)$ and $0 \leq \eta \leq 1$ and that $\int_0^2 \eta(s) ds = 1$ and define $\rho_\varepsilon(x') = \eta\left(|x'|/\varepsilon\right) \varepsilon^{1-n}$ for $\varepsilon > 0$. Let a and b be small positive parameters to be determined later. We set

$$\psi(x', x_n) := f(x') \eta(x_n/R') + \left(\rho_{a|x_n|} * g\right)(x') \eta(x_n/b) x_n, \ x_n \neq 0$$

and $\psi(x', 0) = f(x')$, when $*$ denotes the convolution in $\mathbf{R}^{n-1}$. The parameter $a > 0$ is taken small so that $\rho_{a|x_n|} * g = 0$ on $B_R^{n-1} \times [-R', R']$. The parameter $b > 0$ is taken small so that $0 \leq f \leq 1$ implies $0 \leq \psi \leq 1$. Since ψ is C^k outside $x_n = 0$, it suffices to prove that ψ is C^k at $x_n = 0$ and satisfies $\partial \psi / \partial x_n = g$ at $x_n = 0$.

To show C^1 property at $x_n = 0$ at $\partial \psi / \partial x_n = g$ it suffices to prove that

$$\lim_{x_n \to 0} |x_n| \left\| \partial_{x_n} \left(\rho_{a|x_n|} * g\right) \right\|_\infty = 0, \tag{B.1}$$

$$\lim_{x_n \to 0} |x_n| \left\| \partial_{x'} \left(\rho_{a|x_n|} * g\right) \right\|_\infty = 0, \tag{B.2}$$

where $\| \cdot \|_\infty$ denotes the L^∞ norm in $\mathbf{R}^{n-1}$. We first prove (B.1) when $g \in C_c^1(\mathbf{R}^{n-1})$. Since $\rho_s(x') = s^{1-n} \eta\left(|x'|/s\right)$, we observe that $\partial_s \rho_s = -s^{-1} \operatorname{div}'(x' \rho_s)$, where div' is the divergence in x' variable. This implies

$$|x_n| \partial_{x_n} (\rho_{a|x_n|} * g) = -\operatorname{div}'(x' \rho_s) * g = -x' \rho_s * \nabla' g, \ s = a|x_n|$$

where ∇' denotes the gradient in x' variable. Since $\|x' \rho_s\|_{L^1(\mathbf{R}^{n-1})} \leq C|x_n|$ with some C independent of x_n, we observe that

$$\left\| |x_n| \partial_{x_n} (\rho_{a|x_n|} * g) \right\|_\infty \leq C|x_n| \|\nabla' g\|_\infty \to 0 \quad \text{as} \quad x_n \to 0$$

so (B.1) is proved for $g \in C_c^1(\mathbf{R}^{n-1})$.

For general $g \in C_c(\mathbf{R}^{n-1})$, we approximate g by $g_m \in C_c^1(\mathbf{R}^{n-1})$ so that $\|g - g_m\|_\infty \to 0$ as $m \to 0$. Since $\left\| |x_n| \, \partial_{x_n} \left(\rho_{a|x_n|} * g\right) \right\|_\infty \leq C \|g\|_\infty$, with C independent of g and $x_n \neq 0$ and $\left\| |x_n| \, \partial_{x_n} \left(\rho_{a|x_n|} * g_m\right) \right\|_\infty \to 0$ as $x_n \to 0$ for $g_m \in C^1_c(\mathbf{R}^{n-1})$, we conclude that

$$\varlimsup_{x_n \to 0} \left\| |x_n| \, \partial_{x_n} \left(\rho_{a|x_n|} * g\right) \right\|_\infty \leq \varlimsup_{x_n \to 0} \left\| |x_n| \, \partial_{x_n} \left(\rho_{a|x_n|} * g_m\right) \right\|_\infty$$
$$+ \varlimsup_{x_n \to 0} \left\| |x_n| \, \partial_{x_n} \left(\rho_{a|x_n|} (g - g_m)\right) \right\|_\infty \leq 0 + C \left\| g - g_m \right\|_\infty .$$

This yields (B.1) by sending $m \to \infty$. The proof for (B.2) is similar. A similar argument yields higher regularity if $k \geq 2$; we omit the detail. □

We now construct a cut-off function near the boundary such that the normal derivative equals zero. We are tempting to use the normal coordinates but to get a C^2 cut-off function we need C^3 regularity of the boundary. We won't use the normal coordinates here. For a given $h \in C^k\left(B^{n-1}_{2R}\right)$ with $h(0) = 0$ let

$$\Omega_h = \left\{ (y', y_n) \mid y_n > h(y'), \ y' \in B^{n-1}_{2R} \right\}. \tag{B.3}$$

Lemma B.2. *Let $k \geq 1$ be a natural number and $R > 0$. Let $\Omega = \Omega_h$ be as in* (B.3)*. Then there exists $\varphi \in C^k_c\left(\operatorname{int} B_{2R} \cap \overline{\Omega}\right)$ such that $0 \leq \varphi \leq 1$ in $B_{2R} \cap \Omega$ and $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega \cap B_{2R}$ with $\varphi = 1$ on $B_R \cap \Omega$, where $B_R = B^n_R$.*

Proof. We flatten the boundary by

$$x_n = y_n - h(y'), \quad x' = y'. \tag{B.4}$$

This transformation $x = T(y)$ is C^k. We write φ with new independent variables x and still denoted by φ. The condition $\partial\varphi/\partial n_\Omega = 0$ is transformed into

$$\frac{\partial \varphi}{\partial x_n} - \frac{\nabla' h}{\sqrt{1 + |\nabla' h|^2}} \cdot \nabla' \varphi = 0 \quad \text{at} \quad x_n = 0. \tag{B.5}$$

Let f be $C^k_c\left(\mathbf{R}^{n-1}\right)$ such that $f = 1$ on $T\left(B_{4R/3} \cap \Omega\right)$ and $f = 0$ on $T\left(B_{2R} \cap \Omega\right)$. We set

$$g = \frac{\nabla' h}{\sqrt{1 + |\nabla' h|^2}} \cdot \nabla' f \in C^{k-1}_c\left(\mathbf{R}^{n-1}\right).$$

Applying C^k-extension lemma (Lemma B.1) yields desired φ by choosing R and R' appropriately. Note that the numbers R, $4R/3$, $2R$ do not have a particular meaning. We may take R, $C_1 R$, $C_2 R$ for $1 < C_1 < C_2$ in Lemma B.1 to apply to construct φ so that φ satisfies $\varphi = 1$ in $B_R \cap \Omega$; we shall cut-off outside B_{2R} if necessary to fulfill $\operatorname{spt} \varphi \subset \operatorname{int} B_{2R} \cap \Omega$. □

We conclude this section to give a way to approximate test functions φ in $\mathbf{R}^n_+$ satisfying $\partial\varphi/\partial x_n = 0$ on the boundary by a similar test function in a domain approximating $\mathbf{R}^n_+$.

Lemma B.3. *Assume the same hypothesis of Lemma* B.2*. Let $\varphi \in C^k_c(\overline{\mathbf{R}^n_+})$ satisfy $\partial\varphi/\partial x_n = 0$ at $x_n = 0$ and $\operatorname{spt} \varphi \subset B_R$. Then there exists $\varphi_h \in C^k_c(\overline{\Omega}_h)$ which fulfills $\partial\varphi_h/\partial n_{\Omega_h} = 0$ on $\partial\Omega_h$ with $\operatorname{spt} \varphi_h \subset B_{4R/3}$ such that*

$$\|\varphi_h - \varphi\|_{L^\infty(\Omega_{h+})} \leq C_1 \|\nabla h\|_\infty$$

$$\|\nabla\varphi_h - \nabla\varphi\|_{L^\infty(\Omega_{h+})} \le C_1\|\nabla h\|_\infty$$

with a constant C_1 depending only on n and a bound for $\|\nabla h\|_\infty$. The mapping $\varphi \mapsto \varphi_h$ can be taken a linear operator. If $k \ge 2$, then

$$\|\nabla^2\varphi_h - \nabla^2\varphi\|_{L^\infty(\Omega_{h+})} \le C_2\left(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty\right)$$

with C_2 depending only on a bound for $\|\nabla h\|_\infty$ and $\|\nabla^2 h\|_\infty$. Here $\Omega_{h+} = \Omega_h \cap \mathbf{R}^n_+$.

Proof. We again use the transformation T given in (B.4). Since T is a C^k-transformation, it suffices to construct $\Phi_h \in C^k_c(\overline{\mathbf{R}^n_+})$ satisfying (B.5) with $\varphi = \Phi_h$ and $\operatorname{spt}\Phi_h \subset B_{R_0}$ with R_0 slightly bigger than R so that $T^{-1}(B_{R_0}) \subset B_{4R/3}$ as well as following estimates:

$$\|\Phi_h - \varphi\|_{L^\infty(\mathbf{R}^n_+)} \le C_1\|\nabla h\|_\infty \tag{B.6}$$

$$\|\nabla\Phi_h - \nabla\varphi\|_{L^\infty(\mathbf{R}^n_+)} \le C_1\|\nabla h\|_\infty \tag{B.7}$$

$$\|\nabla^2\Phi_h - \nabla^2\varphi\|_{L^\infty(\mathbf{R}^n_+)} \le C_2\left(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty\right) \quad \text{if} \quad k \ge 2. \tag{B.8}$$

For this purpose we construct

$$\Phi_h(x', x_n) = \varphi(x', x_n) + \left(\rho_{a|x_n|} * g\right)\eta(x_n/b)x_n$$

with

$$g = \frac{\nabla h'}{\sqrt{1+|\nabla' h|^2}} \cdot \nabla'\varphi \in C^{k-1}(\mathbf{R}^{n-1}),$$

where $\rho,\ \eta$ is taken as in the proof of Lemma B.1. The positive parameter a and b are taken small so that $\operatorname{spt}\rho_{a|x_m|} * g \subset B_{R_0}$. This Φ_h which linearly depends on φ fulfills all desired properties as shown below.

Since $\partial\varphi/\partial x_n = 0$ and $\Phi_h = \varphi$ on the boundary $\{x_n = 0\}$, Φ_h fulfills

$$\frac{\partial\Phi_h}{\partial x_n} = g = \frac{\nabla h'}{\sqrt{1+|\nabla' h|^2}} \cdot \nabla'\Phi_h \quad \text{at} \quad x_n = 0,$$

which is nothing but (B.5). By the Hausdorff-Young inequality one observes that the operator defined by $U(x_n)g = \rho_{a|x_n|} * g = g * \rho_{a|x_n|}$ fullfills

$$\|U(x_n)g\|_\infty \le C\|g\|_\infty$$

$$\||x_n|\partial_{x_n}U(x_n)g\|_\infty \le C\|g\|_\infty, \quad \||x_n|\nabla' U_{x_n}(x_n)g\|_\infty \le C\|g\|_\infty \tag{B.9}$$

with C independent of $x_n \in \mathbf{R},\ x_n \ne 0$. This implies (B.6) and (B.7) since $\Phi_h - \varphi = (U(x_n)g)\,\eta(x_n/b)x_n$. Similarly, the estimate (B.8) follows from above estimate as well as

$$\||x_n|\nabla\nabla' U(x_n)g\|_\infty \le C\|\nabla' g\|, \quad \left\||x_n|\partial^2_{x_n}U(x_n)g\right\|_\infty \le C\|\nabla' g\|_\infty.$$

The first one follows from (B.9) while the second one follows from the observation that

$$\left(U(x_n)g\right)(x') = a^{1-n}\int_{\mathbf{R}^{n-1}} \eta\left(\frac{z'}{a}\right) g\left(x' - |x_n|z'\right) dz'.$$

As in the proof of (B.1) the function $U(x_n)g \in C^k_c(\overline{\mathbf{R}^n_+})$ which implies that $\Phi_h \in C^k_c(\overline{\mathbf{R}^n_+})$. □

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