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# ON ANALYTICITY OF THE $L^p$ -STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

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ABSTRACT. Consider the Stokes equations in a sector-like  $C^3$  domain  $\Omega \subset \mathbf{R}^2$ . It is shown that the Stokes operator generates an analytic semigroup in  $L^p_\sigma(\Omega)$  for  $p \in [2, \infty)$ . This includes domains where the  $L^p$ -Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in  $VMO$  and  $L^2$  by constructing a suitable non-Helmholtz projection to solenoidal spaces.

## 1. INTRODUCTION

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator  $S(t) : v_0 \mapsto v(\cdot, t)$  of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \operatorname{div} v = 0 \quad \text{in } \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on } \partial\Omega \times (0, \infty)$$

and the initial condition  $v|_{t=0} = v_0$ , where  $\Omega$  is a domain in  $\mathbf{R}^n$  with  $n \geq 2$ . It is by now well-known that  $S(t)$  forms a  $C_0$ -analytic semigroup in  $L^p_\sigma$  ( $1 < p < \infty$ ) for various domains like smooth bounded domains ([21], [35]). Here  $L^p_\sigma = L^p_\sigma(\Omega)$  denotes the  $L^p$ -closure of  $C^\infty_{c,\sigma}(\Omega)$ , the space of all solenoidal vector fields with compact support in  $\Omega$ . More recently, it has been proved in [20] that  $S(t)$  always forms a  $C_0$ -analytic semigroup in  $L^p_\sigma(\Omega)$  for any uniformly  $C^2$ -domain  $\Omega$  provided that  $L^p(\Omega)$  admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

where  $G^p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L^1_{loc}(\Omega)\}$ . In [20] the  $L^q$  maximal regularity in time with values in  $L^p_\sigma(\Omega)$  was also established.

The Helmholtz decomposition holds for any domain if  $p = 2$ . The  $L^p$ -Helmholtz decomposition holds for various domains like bounded or exterior domains with

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smooth boundary for  $1 < p < \infty$  ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the  $L^p$ -Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely. Let  $C(\vartheta)$  denote the cone of the form

$$C(\vartheta) = \{x = (x', x_n) \in \mathbf{R}^n \mid -x_n \geq |x| \cos(\vartheta/2)\},$$

where  $\vartheta \in (0, 2\pi)$  is the opening angle. When  $n = 2$ , we simply say that  $C(\vartheta)$  is a sector. We say that a planar domain  $\Omega$  is a *sector-like domain* with opening angle  $\vartheta$  if  $\Omega \setminus B_R(0) = C(\vartheta) \setminus B_R(0)$  for some  $R > 0$  (up to rotation and translation), where  $B_R(0)$  is an open disk of radius  $R$  centered at the origin.

It is known that the  $L^p$ -Helmholtz decomposition fails for a sector-like domain  $\Omega$  when  $p > q'_\vartheta$  or  $p < q_\vartheta$  with  $q_\vartheta = 2/(1 + \pi/\vartheta)$ ,  $1/q_\vartheta + 1/q'_\vartheta = 1$  even if the boundary  $\partial\Omega$  is smooth [28, Example 2, Fig. 5] while for  $p \in (q_\vartheta, q'_\vartheta)$  the  $L^p$ -Helmholtz decomposition holds. This means that if the opening angle  $\vartheta$  is larger than  $\pi$ , there always exists  $p > 2$  such that the  $L^p$ -Helmholtz decomposition fails.

It has been a longstanding open question whether or not the existence of the  $L^p$ -Helmholtz decomposition is necessary for  $L^p$  analyticity of  $S(t)$ . In this paper, we give a negative answer for this question by proving that there is a domain  $\Omega$  for which  $S(t)$  is analytic in  $L^p_\sigma$  while the  $L^p$ -Helmholtz decomposition fails. This is a subtle problem since the existence of the  $L^p$ -Helmholtz projection is known to be necessary for  $L^p$  solvability of the resolvent equation ([33]). However, in this statement the external force term is allowed to be in the more general space  $L^p$  instead of  $L^p_\sigma$ . Our problem is different from that in [33].

We say that  $\Omega$  has a  $C^k$  graph boundary if  $\Omega$  is of the form

$$\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$$

(up to translation and rotation) with some real-valued  $C^k$  function  $h$  with variable  $x' \in \mathbf{R}^{n-1}$ .

**Theorem 1.1.** *Let  $\Omega$  be a sector-like domain in  $\mathbf{R}^2$  having a  $C^3$  graph boundary. Then  $S(t)$  forms a  $C_0$ -analytic semigroup in  $L^p_\sigma(\Omega)$  for all  $p \in [2, \infty)$ .*

Here is our strategy to prove Theorem 1.1. It is by now well-known that  $S(t)$  forms an analytic semigroup in  $\tilde{L}^p_\sigma$ , i.e.,  $\tilde{L}^p_\sigma = L^p_\sigma \cap L^2_\sigma$  ( $p \geq 2$ ),  $\tilde{L}^p = L^p_\sigma + L^2_\sigma$  ( $1 < p < 2$ ) ([14], [15], [16]). Thus  $S(t)v_0$  is well-defined for  $v_0 \in C^\infty_{c,\sigma}(\Omega)$ . To show Theorem 1.1, a key step is to prove the two estimates

$$(1.1) \quad \|S(t)v_0\|_p \leq C\|v_0\|_p$$

$$(1.2) \quad t \left\| \frac{d}{dt} S(t)v_0 \right\|_p \leq C\|v_0\|_p$$

for all  $v_0 \in C^\infty_{c,\sigma}(\Omega)$ ,  $t \in (0, 1)$ , where  $\|v_0\|_p$  denotes the  $L^p$ -norm of  $v_0$ . The constant  $C$  should be taken independent of  $t$  and  $v_0$ . We shall establish (1.1) and (1.2) by interpolation since both estimates are known for  $p = 2$ .

We are tempted to interpolate the  $L^\infty$  type result obtained in [5] with the  $L^2$ -result. In fact, in [5] the estimates (1.1) and (1.2) with  $p = \infty$  are established for all  $v_0 \in C_{0,\sigma}(\Omega)$ , the  $L^\infty$ -closure of  $C^\infty_{c,\sigma}(\Omega)$  for a  $C^2$  sector-like domain  $\Omega$  in  $\mathbf{R}^2$ . However, it is not clear that the complex interpolation space  $[L^2_\sigma, C_{0,\sigma}]_\rho$  agrees with  $L^p_\sigma$  with  $2/p = 1 - \rho$  although it is well-known as the Riesz-Thorin theorem that  $[L^2, L^\infty]_\rho = L^p$ . To interpolate, we would need a projection to solenoidal spaces

which is almost impossible since such a projection involves the singular integral operator which is not bounded in  $L^\infty$ .

To circumvent this difficulty, we consider the Stokes semigroup  $S(t)$  in  $BMO$ -type spaces as studied in [10], [11], [12]. For  $p \in [1, \infty)$ ,  $\mu \in (0, \infty]$  we define the  $BMO$  seminorm

$$[f : BMO_p^\mu(\Omega)] := \sup \left\{ \left( \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p dy \right)^{1/p} \mid B_r(x) \subset \Omega, r < \mu \right\},$$

where  $f_B = \int_B f dx$ , the average of  $f$  over  $B$  and  $B_r(x)$  denotes the closed ball of radius  $r$  centered at  $x$ . It is well-known that one gets an equivalent seminorm when the ball  $B_r$  is replaced by a cube. We also need to control the boundary behavior. For  $\nu \in (0, \infty]$  we define

$$[f : b_p^\nu(\Omega)] := \sup \left\{ \left( \frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |f(y)|^p dy \right)^{1/p} \mid x_0 \in \partial\Omega, r > 0, B_r(x_0) \subset U_\nu(\partial\Omega) \right\},$$

where  $U_\nu(E)$  is a  $\nu$ -open neighborhood of  $E$ , i.e.,

$$U_\nu(E) = \{x \in \mathbf{R}^n \mid \text{dist}(x, E) < \nu\}.$$

We shall often assume that  $\nu < R^*$ , where  $R^*$  is the reach from the boundary. The  $BMO$  norm we use is

$$\|f : BMO_{b,p}^{\mu,\nu}(\Omega)\| = [f : BMO_p^\mu(\Omega)] + [f : b_p^\nu(\Omega)].$$

If  $p = 1$ , we often drop  $p$ . The  $BMO$  space we consider is

$$BMO_{b,p}^{\mu,\nu}(\Omega) = \left\{ f \in L_{loc}^1(\Omega) \mid \|f : BMO_{b,p}^{\mu,\nu}(\Omega)\| < \infty \right\}.$$

This space is independent of  $p$  for sufficiently small  $\nu$ , i.e.,  $\nu < R^*$  ([11], [12]) and  $BMO_b^{\infty,\infty}$  agrees with Miyachi  $BMO$  space ([29]) for various domains including a half space and bounded  $C^2$  domains ([12]). Although the  $BMO_b^{\infty,\nu}(\Omega)$  norm is equivalent to the  $BMO_b^{\infty,\infty}(\Omega)$  norm when  $\Omega$  is bounded, there are many unbounded domains for which the  $BMO_b^{\infty,\nu}(\Omega)$  norm is actually weaker than the  $BMO_b^{\infty,\infty}(\Omega)$  norm when  $\nu$  is finite. We define the solenoidal space  $VMO_{b,0,\sigma}^{\mu,\nu}$  as the  $BMO_{b,p}^{\mu,\nu}$ -closure of  $C_{c,\sigma}^\infty(\Omega)$ . In [10], [11] among other results the analyticity of  $S(t)$  in  $VMO_{b,0,\sigma}^{\infty,\nu}$  has been established for a uniformly  $C^3$  domain which is admissible in the sense of [2] provided that  $\nu$  is sufficiently small.

**Theorem 1.2** ([10], [11]). *Let  $\Omega$  be an admissible uniformly  $C^3$  domain in  $\mathbf{R}^n$ . Then  $S(t)$  forms a  $C_0$ -analytic semigroup in  $VMO_{b,0,\sigma}^{\mu,\nu}$  for any  $\mu \in (0, \infty]$  and  $\nu \in (0, \nu_0)$  with some  $\nu_0$  depending only on  $\mu$  and regularity of  $\partial\Omega$ .*

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace  $L^p$  by  $L^\infty$  or  $BMO_b^{\infty,\nu}$ , but even an estimate stronger than (1.2) with  $p = \infty$ , i.e.,

$$(1.3) \quad t \left\| \frac{dS(t)}{dt} v_0 \right\|_\infty \leq C \|v_0 : BMO_b^{\mu,\nu}(\Omega)\|, \quad \mu, \nu \in (0, \infty]$$

which shows a regularizing effect.

It has been proved in [5] that a  $C^2$  sector-like domain in  $\mathbf{R}^2$  is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a  $C^2$  sector-like

domain in  $\mathbf{R}^2$  is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly  $C^3$ . On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a layer domain with  $n \geq 3$  is not admissible ([8]).

In order to get the  $L^p$  estimates we need an interpolation result. Let  $C_c(\Omega)$  denote the space of all continuous functions with compact support in  $\Omega$ .

**Theorem 1.3.** *Let  $\Omega$  be a Lipschitz half-space in  $\mathbf{R}^n$ , i.e., a domain having Lipschitz graph boundary. Let  $T$  be a linear operator from  $C_c(\Omega)$  to  $L^2(\Omega)$ . Assume that there is a constant  $C$  such that*

$$\|Tu\|_2 \leq C\|u\|_2$$

$$\|Tu : BMO^\infty(\Omega)\| \leq C\|u\|_\infty$$

for  $u \in C_c(\Omega)$ . Then  $\|Tu\|_p \leq C_*\|u\|_p$  for  $u \in C_c(\Omega)$  with  $C_*$  depending only on  $C$ ,  $h$  and  $p \in (2, \infty)$ .

There are a couple of such interpolation results between  $BMO$  and  $L^2$ , which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between  $L^p$  and  $BMO$  is discussed when  $\Omega$  is a cube. However, in these results the original inequalities are assumed to hold for  $L^2(\Omega) \cap BMO(\Omega)$  and not for  $C_c(\Omega)$ . Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with  $BMO_A(\mathcal{X})$ , where  $A$  is some operator. They worked on metric measure spaces of homogeneous type  $(\mathcal{X}, d, \mu)$ . In particular, in the case  $\mathcal{X} = \Omega$ ,  $d(x, y) = |x - y|$  and  $\mu(E) = |E|$ , we can see that  $BMO_A(\Omega) \subset BMO^\infty(\Omega)$ .

Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the  $L^\infty$  case we do not know whether or not the complex interpolation space  $[L_\sigma^2, VMO_{b,0,\sigma}^{\infty,\nu}]_\rho$  with  $2/p = 1 - \rho$  agrees with  $L_\sigma^p$ , although we know that  $[L^2, BMO]_\rho = L^p$  for  $\Omega = \mathbf{R}^n$  as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

**Theorem 1.4.** *Let  $\Omega$  be a Lipschitz half-space in  $\mathbf{R}^n$ . Assume that  $\nu \in (0, \infty]$ . There is a linear operator  $Q$  from  $C_c(\Omega)$  to  $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega) \cap L_\sigma^2(\Omega)$  such that*

$$\|Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty$$

$$\|Qu\|_2 \leq C\|u\|_2$$

for all  $u \in C_c(\Omega)$ . Moreover,  $Qu = u$  for  $u \in C_c(\Omega) \cap L_\sigma^2(\Omega)$ .

Since there may be no  $L^p$ -Helmholtz decomposition our  $Q$  should be different from the Helmholtz projection. We shall construct such an operator  $Q$  using the solution operator of the equation  $\operatorname{div} u = f$  given by Solonnikov [36]. Although deriving the  $L^2$  estimate is easy, to derive the  $BMO$  estimate is more involved since we have to estimate the  $b^\nu$  type seminorm.

To derive (1.1), we actually interpolate

$$\|S(t)Qu\|_2 \leq C\|u\|_2$$

and

$$\|S(t)Qu : BMO_b^{\infty,\nu}\| \leq C\|u\|_\infty$$

for  $u \in C_c(\Omega)$ . Similarly, we derive (1.2) by interpolating the estimate for  $t \frac{dS}{dt} Q$ .

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 3, we construct the projection operator  $Q$ . In Section 4, we give a complete proof of Theorem 1.1.

## 2. $L^2 - BMO$ INTERPOLATION ON A LIPSCHITZ HALF-SPACE

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

$$\Omega := \{(x', x_n) \in \mathbf{R}^n | x_n > h(x')\}$$

with a Lipschitz function  $h$  on  $\mathbf{R}^{n-1}$ .

By  $Q$  we mean a closed cube with sides parallel to the coordinate axes. Let  $\ell(Q)$  be the side length of  $Q$ , and for  $\tau > 0$ ,  $\tau Q$  a cube with the same center as  $Q$  and side length  $\tau\ell(Q)$ .

**2.1. Reduction to the half-space and extension.** Here, we prepare lemmas that are basic estimates for the proof. Since  $h$  is Lipschitz continuous,  $F(x) := (x', x_n - h(x'))$  is a bi-Lipschitz map from  $\Omega$  to  $\mathbf{R}_+^n$ . For a function  $u$  defined on  $\mathbf{R}_+^n$  the pull-back function  $F^*(u)$  of  $u$  on  $\Omega$  is defined by  $u \circ F$ . We start with estimates for  $(F^{-1})^*$  which is the pull-back function  $(F^{-1})^*(v)$  of  $v$  on  $\mathbf{R}_+^n$  defined by  $v \circ F^{-1}$ .

**Lemma 2.1.** *Let  $\Omega$  be a Lipschitz half-space.*

(i):

$$[(F^{-1})^*v : BMO^\infty(\mathbf{R}_+^n)] \leq c[v : BMO^\infty(\Omega)].$$

(ii):

$$\|(F^{-1})^*v\|_{L^2(\mathbf{R}_+^n)} \leq c\|v\|_{L^2(\Omega)}.$$

Here  $c$  is a constant depending only on Lipschitz bound of  $h$  and  $n$ .

*Proof.* (i): Because  $\mathbf{R}_+^n$  is an open subset of  $\mathbf{R}^n$ , we know that for any  $\tau > 2$ ,

$$[(F^{-1})^*v : BMO^\infty(\mathbf{R}_+^n)] \leq c_\tau \sup_{\tau Q \subset \mathbf{R}_+^n} \inf_{d \in \mathbf{R}} \int_Q |(F^{-1})^*v - d| dy,$$

where the supremum is taken over cubes  $Q$ , for which  $\tau Q$  is contained in  $\mathbf{R}_+^n$ , see [37]. Since  $F$  is a bi-Lipschitz map, it holds

$$c_1 \text{dist}(y, \partial\mathbf{R}_+^n) \leq \text{dist}(F^{-1}(y), \partial\Omega) \leq c_2 \text{dist}(y, \partial\mathbf{R}_+^n)$$

with some constants  $c_1, c_2 > 0$  for all  $y \in \mathbf{R}_+^n$ . Since  $(\tau - 1)\ell(Q)/2 \leq \text{dist}(Q, \partial\mathbf{R}_+^n)$  for such cubes  $Q$ , we have the lower bound

$$c\tau\ell(Q) \leq \text{dist}(F^{-1}(Q), \partial\Omega)$$

with some  $c > 0$ , which depends on  $n$  and  $h$ . Therefore, taking large  $\tau$ , we can find cubes  $\{R_k\}_{k=1}^{c_*} \subset \Omega$ , which have no intersection of interiors, so that  $\cup_{k=1}^{c_*} R_k$  is connected and

$$\begin{cases} \circ \ell(R_k) = \ell(Q), \\ \circ F^{-1}(Q) \subset \cup_{k=1}^{c_*} R_k, \text{ where } c_* \in \mathbf{N} \text{ depends only on } h, \text{ and} \\ \circ \text{if } R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega. \end{cases}$$

From these, one obtains that for cubes  $Q$  with  $\tau Q \subset \mathbf{R}_+^n$ ,

$$\inf_{d \in \mathbf{R}} \frac{1}{|Q|} \int_Q |(F^{-1})^*v - d| dy \leq c \sum_{k=1}^{c_*} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_1}| dy.$$

It is enough to show that

$$(2.1) \quad \frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| dy \leq c[v : BMO^\infty(\Omega)]$$

for the case  $R_j \cap R_k \neq \emptyset$ . To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let  $\tilde{R}_k$  and  $\tilde{R}_j$  be subcubes of  $R_k$  and  $R_j$  respectively so that  $\ell(\tilde{R}_k) = \ell(R_k)/2$ ,  $\ell(\tilde{R}_j) = \ell(R_j)/2$  and they touch each other. Moreover, denote by  $\tilde{R}_{j,k}$  a cube satisfying  $\ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k)$  and  $\tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k}$ . Hence, we have

$$\begin{aligned} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| dy &\leq \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| dy + |v_{R_k} - v_{R_j}| \\ &\leq c[v : BMO^\infty(\Omega)] + c|v_{\tilde{R}_j} - v_{\tilde{R}_k}| \\ &\leq c[v : BMO^\infty(\Omega)] + c \frac{1}{|\tilde{R}_{j,k}|} \int_{\tilde{R}_{j,k}} |v - v_{\tilde{R}_{j,k}}| dy \\ &\leq c[v : BMO^\infty(\Omega)]. \end{aligned}$$

(ii): This is verified as follows

$$\|(F^{-1})^* v\|_{L^2(\mathbf{R}_+^n)}^2 = \int_{\Omega} |v|^2 J_F dx \leq c \int_{\Omega} |v|^2 dx,$$

where  $J_F$  is the modulus of the Jacobian of  $F$  which is bounded, because  $h$  is Lipschitz continuous.  $\square$

Next, we consider the even extension of functions on the half space. For a function  $f$  on  $\mathbf{R}_+^n$ , we extend  $f$  outside  $\mathbf{R}_+^n$  by

$$E[f](x', -x_n) := f(x', x_n) \text{ for } x_n > 0.$$

From elementary geometrical observation, we can see that the extension operator  $E$  is a  $BMO$ -extension operator for  $\mathbf{R}_+^n$ .

**Lemma 2.2.**

$$[E[f] : BMO^\infty(\mathbf{R}^n)] \leq c [f : BMO^\infty(\mathbf{R}_+^n)].$$

*Proof.* It is sufficient to consider cubes  $Q \subset \mathbf{R}^n$  with  $Q \cap \mathbf{R}_+^n \neq \emptyset$  and  $Q \cap \mathbf{R}_-^n \neq \emptyset$ . For such  $Q$ , let  $Q'$  be a cube so that its center lies on  $\partial \mathbf{R}_+^n$ ,  $\ell(Q') = 2\ell(Q)$  and  $Q \subset Q'$ . Further, let  $Q^*$  be the smallest cube in  $\mathbf{R}_+^n$  containing the upper half of  $Q'$ . With these notations, the desired inequality is proved from

$$\inf_{d \in \mathbf{R}} \frac{1}{|Q|} \int_Q |E[f] - d| dy \leq c \inf_{d \in \mathbf{R}} \frac{1}{|Q^*|} \int_{Q^*} |f - d| dy.$$

$\square$

**2.2. Sharp maximal operator.** For the proof of Theorem 1.3, we make use of the sharp maximal operator  $M^\sharp$  due to Fefferman and Stein ([18]). We define for  $x \in \mathbf{R}^n$  and  $f \in L_{loc}^1(\mathbf{R}^n)$  the function  $M^\sharp f$  by

$$M^\sharp f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is immediate from the definition that  $[f : BMO^\infty(\mathbf{R}^n)] = \|M^\sharp f\|_{L^\infty(\mathbf{R}^n)}$ . It is well-known that if  $f \in L^{p_0}(\mathbf{R}^n)$  for some  $p_0 \in (1, \infty)$ , then for  $p \in [p_0, \infty)$

$$(2.2) \quad \|f\|_{L^p(\mathbf{R}^n)} \leq c \|M^\sharp f\|_{L^p(\mathbf{R}^n)},$$

which is applied below. (Both sides of (2.2) may be infinite.) This follows from  $\|f\|_{L^p(\mathbf{R}^n)} \leq \|Mf\|_{L^p(\mathbf{R}^n)}$  and  $\|Mf\|_{L^p(\mathbf{R}^n)} \leq c \|M^\sharp f\|_{L^p(\mathbf{R}^n)}$ , where  $M$  is the Hardy-Littlewood maximal operator [18].

**2.3. Marcinkiewicz interpolation.** Here, we give a variant of the Marcinkiewicz interpolation theorem.

**Proposition 2.3.** *Let  $D$  be an open subset of  $\mathbf{R}^n$  and  $S$  a sublinear operator from  $C_c(D)$  to  $L^2(\mathbf{R}^n)$ . If*

$$\|S[f]\|_{L^2(\mathbf{R}^n)} \leq c \|f\|_{L^2(D)}$$

$$\|S[f]\|_{L^\infty(\mathbf{R}^n)} \leq c \|f\|_{L^\infty(D)}$$

for  $f \in C_c(D)$ , then  $\|S[f]\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(D)}$  for  $f \in C_c(D)$  with  $C$  depending only on  $c$  and  $p \in (2, \infty)$ .

*Proof.* For  $\lambda > 0$  and  $\alpha > 0$ , we decompose  $f$  into two parts;  $f = f_2 + f_\infty$  where

$$f_2(x) = \begin{cases} 0 & \text{if } |f(x)| \leq \alpha\lambda \\ f(x) - \alpha\lambda \text{sign}(f(x)) & \text{if } |f(x)| > \alpha\lambda, \end{cases}$$

where  $\text{sign } \xi = \xi/|\xi|$  for  $\xi \neq 0$  and  $\text{sign } \xi = 0$  for  $\xi = 0$ . Observe that  $f_2, f_\infty \in BC(D)$ , and then  $f_2, f_\infty \in C_c(D)$ . Therefore, the two inequalities of our assumption hold for  $f_2$  and  $f_\infty$ , respectively. We set  $\alpha = (2\|S\|_{L^\infty(D) \rightarrow L^\infty(\mathbf{R}^n)})^{-1}$  and observe that  $|\{x \in \mathbf{R}^n \mid |S[f_\infty](x)| > \lambda/2\}| = 0$ . We now conclude that

$$\begin{aligned} \int_{\mathbf{R}^n} |S[f]|^p dx &\leq p \int_0^\infty \lambda^{p-1} |\{x \in \mathbf{R}^n \mid |S[f](x)| > \lambda\}| d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} |\{x \in \mathbf{R}^n \mid |S[f_2](x)| > \lambda/2\}| d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} \left( \frac{2}{\lambda} \|S\|_{L^2(D) \rightarrow L^2(\mathbf{R}^n)} \|f_2\|_{L^2(D)} \right)^2 d\lambda \\ &\leq c \int_0^\infty \lambda^{p-3} \int_{\{|f| > \alpha\lambda\}} |f(x)|^2 dx d\lambda \\ &= 2c \int_0^\infty \lambda^{p-3} \left( \int_{\alpha\lambda}^\infty t |\{x \in \mathbf{R}^n \mid |f(x)| > t\}| dt \right) d\lambda \\ &= 2c \int_0^\infty t |\{x \in \mathbf{R}^n \mid |f(x)| > t\}| \left( \int_0^{t/\alpha} \lambda^{p-3} d\lambda \right) dt \\ &\leq c \|f\|_{L^p(D)}^p. \end{aligned}$$

□



**2.4. Proof of Theorem 1.3.** For simplicity, we write  $g := Tf$ . By changing variables, one obtains

$$\int_{\Omega} |g|^p dx \leq c \int_{\mathbf{R}_+^n} |(F^{-1})^* g|^p dy \leq c \int_{\mathbf{R}^n} |E[(F^{-1})^* g]|^p dy \leq c \int_{\mathbf{R}^n} |\Phi[f]|^p dy,$$

where  $\Phi[f] := M^\sharp(E[(F^{-1})^* g])$ . Here, because  $E[(F^{-1})^* g] \in L^2(\mathbf{R}^n)$ , we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see  $L^2(\Omega) - L^2(\mathbf{R}^n)$  and  $L^\infty(\Omega) - L^\infty(\mathbf{R}^n)$  estimates for  $\Phi$ . The former estimate can be seen by  $L^2$ -boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The later one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

### 3. NON-HELMHOLTZ PROJECTION

Our goal in this section is to prove Theorem 1.4.

**3.1. A solution operator to the divergence problem.** As in Section 2, let  $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, x_n > h(x')\}$  be a Lipschitz half-space in  $\mathbf{R}^n$  with a Lipschitz continuous function  $h$  on  $\mathbf{R}^{n-1}$ . Then, there is a closed cone of the form

$$C_1 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \geq |x| \cos(2\theta)\}$$

with an angle  $\theta \in (0, \pi/4)$  (depending on the Lipschitz constant of  $h$ ) such that

$$x + C_1 = \{y \in \mathbf{R}^n \mid y - x \in C_1\} \subset \Omega^c \quad (:= \mathbf{R}^n \setminus \Omega) \quad \text{for all } x \in \Omega^c.$$

In the notion of the introduction  $C_1 = C(4\theta)$  so that the opening angle equals  $4\theta$ . With this angle we define a closed cone  $C_0 = C(2\theta)$ , i.e.,

$$C_0 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \geq |x| \cos \theta\}.$$

The closed cone  $C_0$  also satisfies

$$(3.1) \quad x + C_0 \subset \Omega^c \quad \text{for all } x \in \Omega^c.$$

Let  $L \in C_c^\infty(\mathbf{R}^n)$  be a function such that

$$(3.2) \quad \text{supp } L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1.$$

Here  $-C_0 = \{-y \mid y \in C_0\}$  and  $S^{n-1}$  is the unit sphere in  $\mathbf{R}^n$ . Then we define a vector field  $K = (K_1, \dots, K_n)$  as

$$(3.3) \quad K(x) := \frac{x}{|x|^n} L\left(\frac{x}{|x|}\right), \quad x \in \mathbf{R}^n \setminus \{0\}.$$

**Definition 3.1.** For  $f \in C_c^\infty(\Omega)$ , we define a vector field  $u = Sf$  as

$$u(x) = Sf(x) := (K * \bar{f})(x) = \int_{\mathbf{R}^n} K(x-y) \bar{f}(y) dy, \quad x \in \mathbf{R}^n.$$

Here  $\bar{f}$  denotes the zero extension of  $f$  to  $\mathbf{R}^n$  given by

$$\bar{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}$$

This operator was introduced by Solonnikov [36]. For a fixed  $x \in \mathbf{R}^n$ , since

$$\frac{x-y}{|x-y|} \in \text{supp } L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)$$

implies  $y \in x + C_0$ , we can write

$$u(x) = \int_{x+C_0} K(x-y)\bar{f}(y) \, dy.$$

This formula and the property (3.1) of  $\Omega$  imply that  $u(x) = 0$  for all  $x \in \Omega^c$ . In particular,  $u$  vanishes on  $\partial\Omega$ . However, the support of  $u$  may become unbounded although  $f$  is compactly supported in  $\Omega$ .

By the change of variables  $x-y = r\sigma$  with  $r > 0$  and  $\sigma \in S^{n-1}$  we have

$$u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma)\bar{f}(x-r\sigma)r^{n-1}d\mathcal{H}^{n-1}(\sigma) \, dr.$$

Hence if  $f \in C_c^\infty(\Omega)$  is supported in  $B_R(0)$  and  $x \in B_a(0)$  ( $R, a > 0$ ), then

$$u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma)\bar{f}(x-r\sigma)r^{n-1}d\mathcal{H}^{n-1}(\sigma) \, dr,$$

which implies that  $u = Sf$  is smooth in  $\Omega$ . Moreover,  $u = Sf$  vanishes near  $\partial\Omega$  and thus it is smooth in the whole space  $\mathbf{R}^n$ , since  $f$  is compactly supported in  $\Omega$ .

**Lemma 3.2.** *Let  $p \in (1, \infty)$ . There exists a constant  $c > 0$  such that*

$$\|\nabla u\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)}$$

for all  $f \in C_c^\infty(\Omega)$  and  $u = Sf$ .

*Proof.* Let  $u_i$  be the  $i$ -th component of  $u$ :

$$u_i(x) = (K_i * \bar{f})(x) = \int_{\mathbf{R}^n} K_i(z)\bar{f}(x-z) \, dz.$$

Differentiating both sides with respect to the  $j$ -th variable, we have

$$\partial_j u_i(x) = \int_{\mathbf{R}^n} K_i(z)(\partial_j \bar{f})(x-z) \, dz = \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n \setminus B_\varepsilon(0)} K_i(z)(\partial_j \bar{f})(x-z) \, dz$$

and, by changing variables  $y = x-z$  and integrating by parts,

$$\begin{aligned} \partial_j u_i(x) &= \\ \lim_{\varepsilon \rightarrow 0} \left( \int_{\partial B_\varepsilon(x)} K_i(x-y) \frac{x_j - y_j}{|x-y|} \bar{f}(y) \, d\mathcal{H}^{n-1}(y) + \int_{\mathbf{R}^n \setminus B_\varepsilon(x)} (\partial_j K_i)(x-y) \bar{f}(y) \, dy \right). \end{aligned}$$

On the one hand, we change variables  $x-y = \varepsilon\sigma$  with  $\sigma \in S^{n-1}$  to get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{|x-y|=\varepsilon} K_i(x-y) \frac{x_j - y_j}{|x-y|} \bar{f}(y) \, d\mathcal{H}^{n-1}(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|x-y|=\varepsilon} \frac{x_i - y_i}{|x-y|} \frac{x_j - y_j}{|x-y|} L \left( \frac{x-y}{|x-y|} \right) \bar{f}(y) \frac{1}{|x-y|^{n-1}} \, d\mathcal{H}^{n-1}(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \bar{f}(x - \varepsilon\sigma) \, d\mathcal{H}^{n-1}(\sigma) \\ &= \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma), \end{aligned}$$

where the last equality follows from the fact that  $L$  is integrable on  $S^{n-1}$  and  $\bar{f}$  is continuous at  $x$ . On the other hand, we differentiate  $K_i$  to obtain

$$(3.4) \quad \begin{aligned} K_{ij}(z) &:= \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n}, \\ k_{ij}(z) &:= (\delta_{ij} - nz_i z_j)L(z) + z_i(\partial_j L)(z) - z_i z_j \sum_{\ell=1}^n z_\ell(\partial_\ell L)(z) \end{aligned}$$

for  $z \in \mathbf{R}^n \setminus \{0\}$ . Then  $K_{ij}$  is homogeneous of degree  $-n$  and there is a constant  $c > 0$  such that

$$|K_{ij}(z)| \leq \frac{c}{|z|^n} \quad \text{for all } z \in \mathbf{R}^n \setminus \{0\}$$

by the smoothness of  $L$  on  $S^{n-1}$ . Moreover, for every  $R_1$  and  $R_2$  with  $0 < R_1 < R_2$ ,

$$\begin{aligned} &\int_{R_1 < |z| < R_2} K_{ij}(z) \, dz = \int_{R_1 < |z| < R_2} \partial_j K_i(z) \, dz \\ &= \int_{|z|=R_2} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z) \\ &= \int_{|z|=R_2} \frac{z_i}{|z|} \frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} \frac{z_i}{|z|} \frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z) \\ &= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0. \end{aligned}$$

In the fourth equality we changed variables  $z = R_2\sigma$  and  $z = R_1\sigma$  with  $\sigma \in S^{n-1}$ , respectively. This equality is equivalent to

$$(3.5) \quad \int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0.$$

Thus we can apply the Calderón-Zygmund theory (see eg. [23, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel  $K_{ij}$  and obtain the formula

$$(3.6) \quad \partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} K_{ij}(x-y) \bar{f}(y) \, dy,$$

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) \right| \leq |\bar{f}(x)| \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \leq c \|\bar{f}\|_{L^p(\mathbf{R}^n)} = c \|f\|_{L^p(\Omega)}$$

with a positive constant  $c$  independent of  $f$ . Hence the lemma follows.  $\square$

**Lemma 3.3.** *For every  $f \in C_c^\infty(\Omega)$  the vector field  $u = Sf$  satisfies*

$$\operatorname{div} u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

*Proof.* We have already observed that  $u$  vanishes on the boundary. Let us compute  $\operatorname{div} u = \sum_{i=1}^n \partial_i u_i$  in  $\Omega$ . By the formula (3.6) in the proof of Lemma 3.2,

$$\operatorname{div} u(x) = \bar{f}(x) \int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma) d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} \sum_{i=1}^n K_{ii}(x-y) \bar{f}(y) dy.$$

In this formula, we have

$$\int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma) d\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1$$

by (3.2) and, for all  $z \in \mathbf{R}^n \setminus \{0\}$ ,

$$\begin{aligned} \sum_{i=1}^n K_{ii}(z) &= \frac{1}{|z|^n} L\left(\frac{z}{|z|}\right) \sum_{i=1}^n \left(1 - n \frac{z_i^2}{|z|^2}\right) \\ &\quad + \frac{1}{|z|^n} \sum_{i=1}^n \frac{z_i}{|z|} (\partial_i L)\left(\frac{z}{|z|}\right) - \sum_{i=1}^n \frac{z_i^2}{|z|^{n+2}} \sum_{k=1}^n \frac{z_k}{|z|} (\partial_k L)\left(\frac{z}{|z|}\right) = 0. \end{aligned}$$

Hence  $\operatorname{div} u(x) = \bar{f}(x) = f(x)$  for all  $x \in \Omega$ .  $\square$

Lemma 3.3 means that the operator  $S$  is a solution operator to the divergence problem with Dirichlet boundary condition. Note that  $S$  is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

**Definition 3.4.** For a vector field  $u \in C_c^\infty(\Omega)$ , we define a vector field  $Tu$  as

$$Tu(x) := \int_{\mathbf{R}^n} K(x-y) \overline{\operatorname{div} u}(y) dy, \quad x \in \mathbf{R}^n.$$

Here  $K$  is given by (3.3) and  $\overline{\operatorname{div} u}$  denotes the zero extension of  $\operatorname{div} u$  to  $\mathbf{R}^n$ .

The above definition means that  $T$  is given by  $T = S \circ \operatorname{div}$ . Since  $u \in C_c^\infty(\Omega)$ , its divergence is in  $C_c^\infty(\Omega)$  and thus  $Tu$  is smooth in the whole space  $\mathbf{R}^n$  and vanishes outside of  $\Omega$ , as discussed right after Definition 3.1. Also, by Lemma 3.3 we have

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial\Omega.$$

Clearly  $Tu = 0$  in  $\mathbf{R}^n$  for  $u \in C_{c,\sigma}^\infty(\Omega)$ . Note that, as in the case of the operator  $S$ , the support of  $Tu$  may be unbounded.

**Theorem 3.5.** *Let  $\Omega$  be a Lipschitz half-space. Let  $p \in (1, \infty)$ . There exists a constant  $c > 0$  such that*

$$\|Tu\|_{L^p(\Omega)} \leq c \|u\|_{L^p(\Omega)}$$

for all  $u \in C_c^\infty(\Omega)$ .

*Proof.* Let us compute the  $i$ -th component  $(Tu)_i$  of  $Tu$  with  $i = 1, \dots, n$  for compactly supported vector field  $u$  in  $\Omega$ . As in the proof of Lemma 3.2, we integrate

by parts to get

$$\begin{aligned} (Tu)_i(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(x)} K_i(x-y) \frac{x-y}{|x-y|} \cdot \bar{u}(y) \, d\mathcal{H}^{n-1}(y) \\ &\quad + \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n \setminus B_\varepsilon(x)} (\nabla K_i)(x-y) \cdot \bar{u}(y) \, dy \\ &= \int_{S^{n-1}} \sigma_i L(\sigma) \{\sigma \cdot \bar{u}(x)\} \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} (\nabla K_i)(x-y) \cdot \bar{u}(y) \, dy, \end{aligned}$$

or equivalently,

$$(3.7) \quad (Tu)_i(x) = \sum_{j=1}^n \{a_{ij} \bar{u}_j(x) + S_{ij} \bar{u}_j(x)\}, \quad x \in \mathbf{R}^n.$$

Here  $u_j$  is the  $j$ -th component of  $u$  and

$$a_{ij} = \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma), \quad S_{ij} \bar{u}_j(x) = \int_{\mathbf{R}^n} K_{ij}(x-y) \bar{u}_j(y) \, dy,$$

where  $K_{ij} = \partial_j K_i$  is given by (3.4). Since  $a_{ij}$  is a constant satisfying

$$(3.8) \quad |a_{ij}| \leq \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1$$

and  $S_{ij} \bar{u} = K_{ij} * \bar{u}$  is a singular integral (see the proof of Lemma 3.2), the Calderón-Zygmund theory yields the boundedness of the operator  $T$  on  $L^p(\Omega)$ .  $\square$

By Theorem 3.5, the operator  $T$  extends uniquely to a bounded linear operator on  $L^p(\Omega)$  with each  $p \in (1, \infty)$ , which we again refer to as  $T$ .

Our next goal is to estimate the  $BMO_b^{\infty, \nu}(\Omega)$ -norm of  $Tu$  for  $u \in C_c^\infty(\Omega)$  and  $\nu \in (0, \infty]$ . To this end, we estimate each term of the right-hand side in (3.7) for  $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$ . By (3.8) we have

$$[a_{ij} \bar{u}_j : BMO^\infty(\Omega)] \leq [u_j : BMO^\infty(\Omega)], \quad [a_{ij} \bar{u}_j : b^\nu(\Omega)] \leq [u_j : b^\nu(\Omega)]$$

and thus

$$\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq \|u_j : BMO_b^{\infty, \nu}(\Omega)\|.$$

Moreover, since

$$[u_j : BMO^\infty(\Omega)] \leq 2\|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \leq \omega_n \|u_j\|_{L^\infty(\Omega)},$$

where  $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$  is the volume of the unit ball  $B_1(0)$  in  $\mathbf{R}^n$  with the Gamma function  $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} \, dx$ , we have

$$(3.9) \quad \|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq (2 + \omega_n) \|u_j\|_{L^\infty(\Omega)}.$$

Let us estimate  $S_{ij} \bar{u}_j = K_{ij} * \bar{u}_j$ ,  $i, j = 1, \dots, n$  in  $BMO_b^{\infty, \nu}(\Omega)$ . Recall that the integral kernel  $K_{ij}$  is of the form

$$K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbf{R}^n \setminus \{0\},$$

where  $k_{ij} \in C_c^\infty(\mathbf{R}^n)$  is given by (3.4) and satisfies

$$\text{supp } k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0,$$

see (3.2) and (3.5). We first estimate the  $BMO^\infty$ -seminorm of  $S_{ij} \bar{u}_j$ .

**Lemma 3.6.** *Let  $K$  be a function defined on  $\mathbf{R}^n \setminus \{0\}$  such that*

$$(3.10) \quad |K(x-y) - K(x)| \leq A|y|^\delta |x|^{-n-\delta} \quad \text{whenever } |x| \geq 2|y| > 0$$

for some  $A, \delta > 0$ . Suppose that a convolution operator  $S$  with  $K$  is bounded on  $L^2(\mathbf{R}^n)$  with a norm  $B$ . Then, there exists a dimensional constant  $c_n$  such that

$$[Sf : BMO^\infty(\mathbf{R}^n)] \leq c_n(A+B) \|f\|_{L^\infty(\mathbf{R}^n)}$$

for all  $f \in L^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ .

*Proof.* See [24, Theorem 3.4.9 and Corollary 3.4.10].  $\square$

**Lemma 3.7.** *There exists a constant  $c > 0$  such that*

$$(3.11) \quad [S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \leq c \|u_j\|_{L^\infty(\Omega)}$$

for all  $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$  and  $i, j = 1, \dots, n$ .

*Proof.* We shall apply Lemma 3.6 to  $S = S_{ij}$ . For this purpose it is sufficient to show that the function  $K = K_{ij}$  satisfies (3.10), since we already know that the convolution operator  $S_{ij}$  is bounded on  $L^2(\mathbf{R}^n)$ , see the proof of Lemma 3.2. To this end, we differentiate  $K_{ij}$  to get

$$\nabla K_{ij}(x) = -\frac{nk_{ij}(x/|x|)}{|x|^{n+1}} \frac{x}{|x|} + \frac{1}{|x|^{n+1}} \left( I_n - \frac{1}{|x|^2} x \otimes x \right) \nabla k_{ij} \left( \frac{x}{|x|} \right)$$

for  $x \in \mathbf{R}^n \setminus \{0\}$ , where  $I_n$  is the identity matrix of size  $n$  and  $x \otimes x := (x_i x_j)_{i,j}$  is the tensor product of  $x$ . Since  $k_{ij}$  is smooth on  $S^{n-1}$ , we have

$$|\nabla K_{ij}(x)| \leq \frac{c}{|x|^{n+1}}, \quad x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for all  $x, y \in \mathbf{R}^n \setminus \{0\}$  with  $|x| \geq 2|y| > 0$ ,

$$\begin{aligned} |K(x-y) - K(x)| &= \left| \int_0^1 \frac{d}{dt} (K(x-ty)) dt \right| = \left| \int_0^1 (-y) \cdot \nabla K(x-ty) dt \right| \\ &\leq |y| \int_0^1 \frac{c}{|x-ty|^{n+1}} dt \leq |y| \int_0^1 \frac{c}{(|x|-|y|)^{n+1}} dt \\ &\leq \frac{c|y|}{(|x|-|x|/2)^{n+1}} = \frac{2^{n+1}c|y|}{|x|^{n+1}}. \end{aligned}$$

Thus  $K_{ij}$  satisfies (3.10) with  $\delta = 1$  and we can apply Lemma 3.6 to obtain

$$(3.12) \quad [S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)] \leq c \|\bar{u}_j\|_{L^\infty(\mathbf{R}^n)} = c \|u_j\|_{L^\infty(\Omega)}$$

with some constant  $c > 0$ .

By definition of the  $BMO^\infty$ -seminorm, we have

$$[S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \leq [S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)].$$

Hence the inequality (3.11) follows from (3.12).  $\square$

Next, let us estimate the  $b^\nu$ -part of  $S_{ij}\bar{u}_j$ . Recall the two closed cones

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \geq |x| \cos(2^j \theta)\}, \quad j = 0, 1$$

with opening angle  $\theta \in (0, \pi/4)$ . For  $r > 0$  and  $x_0 \in \mathbf{R}^n$ , we define

$$(3.13) \quad A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap (x_0 + C_1)^c \subset \mathbf{R}^n.$$

Here  $x_0 + C_1 = \{y \in \mathbf{R}^n \mid y - x_0 \in C_1\}$  and  $x + C_0$  is defined similarly.

**Lemma 3.8.** *For all  $r > 0$  and  $x_0 \in \mathbf{R}^n$  we have  $A_r(x_0) \subset B_{r/\sin\theta}(x_0)$ .*

*Proof.* By translation, we may assume that  $x_0 = 0$ . Let  $a := (0, \dots, 0, r/\sin\theta) \in \mathbf{R}^n$ . Suppose that

- (1)  $B_r(0) \subset a + C_0$ ,
- (2)  $x + C_0 \subset a + C_0$  for all  $x \in a + C_0$ ,
- (3)  $(a + C_0) \cap C_1^c \subset B_{r/\sin\theta}(0)$ .

Then, the statements (1) and (2) imply

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.$$

Hence the statement (3) yields  $A_r(0) \subset B_{r/\sin\theta}(0)$ . Now let us prove the statements (1)-(3). Note that, since  $\theta \in (0, \pi/4)$ , the cones  $C_0$  and  $C_1$  are represented as

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, x_n \leq 0, |x'| \leq (-x_n) \tan(2^j\theta)\}, \quad j = 0, 1.$$

(1) Let  $x = (x', x_n) \in B_r(0)$ . Then,  $x - a = (x', x_n - r/\sin\theta)$  satisfies

$$(x - a)_n = x_n - \frac{r}{\sin\theta} \leq r - \frac{r}{\sin\theta} < 0$$

and

$$\left(\frac{r}{\sin\theta} - x_n\right)^2 \tan^2\theta - |x'|^2 \geq \frac{(r - x_n \sin\theta)^2}{\cos^2\theta} - (r^2 - x_n^2) = \frac{(r \sin\theta - x_n)^2}{\cos^2\theta} \geq 0,$$

or equivalently,

$$|x'| \leq \left(\frac{r}{\sin\theta} - x_n\right) \tan\theta = -(x - a)_n \tan\theta.$$

Hence  $x - a \in C_0$ , that is,  $x \in a + C_0$  and the statement (1) holds.

(2) Let  $x \in a + C_0$ . If  $y \in x + C_0$ , then  $(y - a)_n = (y - x)_n + (x - a)_n \leq 0$  and

$$|y'| \leq |x'| + |y' - x'| \leq -(x - a)_n \tan\theta - (y - x)_n \tan\theta = -(y - a)_n \tan\theta,$$

which means that  $y \in a + C_0$ . Hence the statement (2) holds.

(3) Let  $x \in (a + C_0) \cap C_1^c$ . Then we have

$$(3.14) \quad (x - a)_n = x_n - r/\sin\theta \leq 0, \quad |x'| \leq \left(\frac{r}{\sin\theta} - x_n\right) \tan\theta.$$

Hence

$$|x|^2 \leq \left(\frac{r}{\sin\theta} - x_n\right)^2 \tan^2\theta + x_n^2 =: f(x_n).$$

To estimate the right-hand side in the above inequality for  $x \in (a + C_0) \cap C_1^c$ , we derive the range of  $x_n$  for  $x \in (a + C_0) \cap C_1^c$ . If  $x_n \geq 0$ , then  $x \in (a + C_0) \cap C_1^c$  holds if and only if the condition (3.14) is satisfied. Thus  $x_n$  must satisfy

$$0 \leq x_n \leq \frac{r}{\sin\theta}.$$

On the other hand, if  $x_n < 0$ , then  $x \in (a + C_0) \cap C_1^c$  holds if and only if

$$(-x_n) \tan(2\theta) < |x'| \leq \left(\frac{r}{\sin\theta} - x_n\right) \tan\theta.$$

Hence, in particular, if  $x \in (a + C_0) \cap C_1^c$  and  $x_n < 0$ , then  $x_n$  must satisfy

$$(-x_n) \tan(2\theta) < \left(\frac{r}{\sin\theta} - x_n\right) \tan\theta,$$

which yields the inequality

$$-\frac{r}{\cos \theta} < (\tan(2\theta) - \tan \theta) x_n.$$

Since

$$\begin{aligned} \tan(2\theta) - \tan \theta &= \tan(2\theta) - \frac{1}{2} \tan(2\theta)(1 + \tan^2 \theta) \\ &= \frac{1}{2} \tan(2\theta)(1 - \tan^2 \theta) = \frac{\tan(2\theta)}{2 \cos^2 \theta} > 0 \quad \left(0 < \theta < \frac{\pi}{4}\right), \end{aligned}$$

the above inequality is equivalent to

$$-\frac{2r \cos \theta}{\tan(2\theta)} < x_n (< 0).$$

In summary, the range of  $x_n$  for  $x \in (a + C_0) \cap C_1^c$  is

$$\alpha := -\frac{2r \cos \theta}{\tan(2\theta)} < x_n \leq \frac{r}{\sin \theta} =: \beta$$

and thus we obtain

$$|x|^2 \leq f(x_n) \leq \sup_{s \in (\alpha, \beta)} f(s) = \max\{f(\alpha), f(\beta)\},$$

where the last equality follows from the fact that  $f(x_n)$  is a concave parabola. On the one hand, we have  $f(\beta) = \beta^2 = r^2/\sin^2 \theta$ . On the other hand, since

$$\alpha = -\frac{2r \cos \theta \cos(2\theta)}{\sin(2\theta)} = -\frac{r \cos(2\theta)}{\sin \theta} = \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta},$$

we have

$$\begin{aligned} f(\alpha) &= \left( \frac{r}{\sin \theta} - \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta} \right)^2 \tan^2 \theta + \frac{r^2 \cos^2(2\theta)}{\sin^2 \theta} \\ &= \frac{r^2}{\sin^2 \theta} \{4 \tan^2 \theta \cos^4 \theta + \cos^2(2\theta)\} = \frac{r^2}{\sin^2 \theta}. \end{aligned}$$

Hence  $|x|^2 \leq r^2/\sin^2 \theta$  and thus  $x \in B_{r/\sin \theta}(0)$  for every  $x \in (a + C_0) \cap C_1^c$ . Therefore, the statement (3) holds and the lemma follows.  $\square$

Now we can estimate the  $b^\nu$ -part of  $S_{ij}\bar{u}_j$ .

**Lemma 3.9.** *Let  $\nu \in (0, \infty]$ . There exists a constant  $c > 0$  such that*

$$(3.15) \quad [S_{ij}\bar{u}_j : b^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \|u_j\|_{L^\infty(\Omega)}$$

for all  $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$  and  $i, j = 1, \dots, n$ .

*Proof.* First we note that for all  $f \in L_{loc}^1(\Omega)$  the inequality

$$[f : b^\nu(\Omega)] \leq \omega_n^{1/2} [f : b_2^\nu(\Omega)]$$

holds by Hölder's inequality. Hence, to prove (3.15), it is sufficient to show the inequality

$$(3.16) \quad [S_{ij}\bar{u}_j : b_2^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} [u_j : b_2^{\nu/\sin \theta}(\Omega)] \leq \frac{c\omega_n^{1/2}}{\sin^{n/2} \theta} \|u_j\|_{L^\infty}.$$



The second inequality of (3.16) follows from the definition of  $[\cdot : b_2^{\nu/\sin\theta}(\Omega)]$ . Let us show the first inequality. The singular integral  $S_{ij}\bar{u}_j$  is of the form

$$S_{ij}\bar{u}_j(x) = (K_{ij} * \bar{u}_j)(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\bar{u}_j(y) dy, \quad x \in \mathbf{R}^n.$$

Since  $\text{supp } K_{ij} \subset -C_0$  (see (3.4) and (3.2)) and  $\text{supp } u \subset \Omega$ , we can write

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)\bar{u}_j(y) dy, \quad x \in \mathbf{R}^n.$$

Hence, if we set

$$W_r(x_0) := \bigcup_{x \in B_r(x_0) \cap \Omega} (x + C_0) \cap \Omega$$

for each  $x_0 \in \partial\Omega$  and  $r > 0$  with  $B_r(x_0) \subset U_\nu(\partial\Omega)$ , then we have

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)(\bar{u}_j|_{W_r(x_0)})(y) dy = [K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)$$

for all  $x \in B_r(x_0) \cap \Omega$ , where

$$(\bar{u}_j|_{W_r(x_0)})(x) := \begin{cases} \bar{u}_j(x), & x \in W_r(x_0), \\ 0, & x \notin W_r(x_0). \end{cases}$$

Since  $K_{ij}$  is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that

$$\begin{aligned} \int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2 dx &= \int_{B_r(x_0)\cap\Omega} |[K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)|^2 dx \\ &\leq c \int_{\mathbf{R}^n} |(\bar{u}_j|_{W_r(x_0)})(x)|^2 dx = c \int_{W_r(x_0)} |\bar{u}_j(x)|^2 dx \end{aligned}$$

with some constant  $c > 0$ . Now we recall the property of the infinite cone  $C_1$ :

$$x + C_1 \subset \Omega^c \Leftrightarrow \Omega \subset (x + C_1)^c \quad \text{for all } x \in \Omega^c.$$

By this property we have

$$W_r(x_0) \subset \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r(x_0) \cap \Omega,$$

where  $A_r(x_0)$  is given by (3.13), and thus Lemma 3.8 yields

$$W_r(x_0) \subset A_r(x_0) \cap \Omega \subset B_{r/\sin\theta}(x_0) \cap \Omega.$$

Hence we have

$$\begin{aligned} \frac{1}{r^n} \int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2 dx &\leq \frac{c}{r^n} \int_{W_r(x_0)} |\bar{u}_j(x)|^2 dx \\ &\leq \frac{c}{r^n} \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |\bar{u}_j(x)|^2 dx = \frac{c}{\sin^n\theta} \left(\frac{\sin\theta}{r}\right)^n \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |u_j(x)|^2 dx \\ &\leq \frac{c}{\sin^n\theta} \left[ u_j : b_2^{\nu/\sin\theta}(\Omega) \right]^2 \end{aligned}$$

for every  $x_0 \in \partial\Omega$  and  $r > 0$  with  $B_r(x_0) \subset U_\nu(\partial\Omega)$ , which yields

$$[S_{ij}\bar{u}_j : b_2^{\nu}(\Omega)]^2 \leq \frac{c}{\sin^n\theta} \left[ u_j : b_2^{\nu/\sin\theta}(\Omega) \right]^2.$$

The proof is complete.  $\square$

Now we obtain an estimate for the  $BMO_b^{\infty, \nu}(\Omega)$ -norm of  $Tu$ .

**Theorem 3.10.** *Let  $\nu \in (0, \infty]$ . There exists a constant  $c > 0$  such that*

$$\|Tu : BMO_b^{\infty, \nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all  $u \in C_c^\infty(\Omega)$ .

*Proof.* Since the  $i$ -th component of  $Tu$ ,  $i = 1, \dots, n$ , is of the form (3.7), we have by (3.9), (3.11) and (3.15) that

$$\begin{aligned} & \|Tu : BMO_b^{\infty, \nu}(\Omega)\| \\ & \leq c \sum_{i,j=1}^n (\|a_{ij}\bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| + [S_{ij}\bar{u}_j : BMO^\infty(\Omega)] + [S_{ij}\bar{u}_j : b^\nu(\Omega)]) \\ & \leq c \sum_{j=1}^n \|u_j\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)} \end{aligned}$$

with a positive constant  $c$ . □

**3.2. Non-Helmholtz projection.** As in the previous subsection, let  $\Omega$  denote a Lipschitz half-space in  $\mathbf{R}^n$ .

**Definition 3.11.** For a vector field  $u \in C_c^\infty(\Omega)$ , we define a vector field  $Q'u$  on  $\mathbf{R}^n$  as  $Q'u := u - Tu$ . Here the operator  $T$  is given in Definition 3.4.

For a vector field  $u \in C_c^\infty(\Omega)$ , the vector field  $Tu$  is smooth in  $\mathbf{R}^n$  and

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial\Omega.$$

Moreover,  $Tu = 0$  for all  $u \in C_{c,\sigma}^\infty(\Omega)$ , see the argument after Definition 3.4. Thus  $Q'u = u - Tu$  is also smooth in  $\mathbf{R}^n$  and

$$(3.17) \quad \operatorname{div} Q'u = 0 \quad \text{in } \Omega, \quad Q'u = 0 \quad \text{on } \partial\Omega$$

for all  $u \in C_c^\infty(\Omega)$ , and  $Q'u = u$  for all  $u \in C_{c,\sigma}^\infty(\Omega)$ . Note that  $Q'$  is not a projection from  $C_c^\infty(\Omega)$  onto  $C_{c,\sigma}^\infty(\Omega)$ , since the support of  $Tu$  may be unbounded and thus  $Q'u$  is not in  $C_{c,\sigma}^\infty(\Omega)$  in general. However,  $Q'$  maps  $C_c^\infty(\Omega)$  into  $L_\sigma^p(\Omega)$ .

**Lemma 3.12.** *For all  $u \in C_c^\infty(\Omega)$  and  $p \in (1, \infty)$ , we have  $Q'u \in L_\sigma^p(\Omega)$ .*

We shall first prove an auxiliary proposition for the above lemma. For  $p \in (1, \infty)$ , let  $G_p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L_{loc}^1(\Omega)\}$ .

**Proposition 3.13.** *Let  $p \in (1, \infty)$ . For every  $\nabla q \in G_p(\Omega)$ , there exists a sequence  $\{q_k\}_{k=1}^\infty$  of functions in  $C_c^\infty(\mathbf{R}^n)$  such that*

$$(3.18) \quad \lim_{k \rightarrow \infty} \|\nabla q - \nabla q_k\|_{L^p(\Omega)} = 0.$$

*Proof.* Since the restriction of  $C_c^\infty(\mathbf{R}^n)$  on  $\Omega$  is dense in  $W^{1,p}(\Omega)$ , it is sufficient to show that for every  $\nabla q \in G_p(\Omega)$  there is a sequence  $\{q_k\}_{k=1}^\infty$  of functions in  $W^{1,p}(\Omega)$  such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space  $\mathbf{R}_+^n$  and show the claim for general Lipschitz half-spaces  $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$ . As in Section 2, let  $F(x) := (x', x_n - h(x'))$  be a bi-Lipschitz map from  $\Omega$  to  $\mathbf{R}_+^n$ . Let  $\nabla q \in G_p(\Omega)$  and  $\tilde{q} := q \circ F^{-1}$ , where  $F^{-1}(y) := (y', y_n + h(y'))$  is the inverse mapping of  $F$ . Then, since  $\nabla \tilde{q}(y) = \nabla F^{-1}(y) \nabla q(F^{-1}(y))$  for  $y \in \mathbf{R}_+^n$  and each component

of  $\nabla F^{-1}$  is bounded (because  $h$  is Lipschitz continuous), we have  $\nabla \tilde{q} \in G_p(\mathbf{R}_+^n)$ . Hence, by our assumption that the claim is valid for  $\mathbf{R}_+^n$ , there is a sequence  $\{\tilde{q}_k\}_{k=1}^\infty$  of functions in  $W^{1,p}(\mathbf{R}_+^n)$  such that  $\lim_{k \rightarrow \infty} \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbf{R}_+^n)} = 0$ .

Let  $q_k := \tilde{q}_k \circ F$  for each  $k \in \mathbf{N}$ . Then, since

$$\nabla q(x) = \nabla F(x) \nabla \tilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x) \nabla \tilde{q}_k(F(x)), \quad x \in \Omega$$

and each component of  $\nabla F$  is bounded, we have  $q_k \in W^{1,p}(\Omega)$  and

$$\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \leq c \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbf{R}_+^n)} \rightarrow 0$$

as  $k \rightarrow \infty$ . Thus the claim is valid for general Lipschitz half-spaces  $\Omega$ .

(2) Now we prove the claim for  $\Omega = \mathbf{R}_+^n$ . We follow the idea of the proof of the claim in the case  $\Omega = \mathbf{R}^n$ , see [34, Lemma 2.5.4]. Let  $\varphi \in C_c^\infty(\mathbf{R}^n)$  be a function such that

$$0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R}^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } \mathbf{R}^n \setminus B_2(0)$$

and  $\varphi_k(x) := \varphi(k^{-1}x)$  for  $k \in \mathbf{N}$  and  $x \in \mathbf{R}^n$ . Then,  $\lim_{k \rightarrow \infty} \varphi_k(x) = 1$  for all  $x \in \mathbf{R}^n$  and  $\text{supp } \varphi_k \subset B_{2k}(0)$ ,  $\text{supp } \nabla \varphi_k \subset B_{2k}(0) \setminus B_k(0)$  for  $k \in \mathbf{N}$ .

Let  $\nabla q \in G_p(\mathbf{R}_+^n)$ . Then  $q \in W_{loc}^{1,p}(\overline{\mathbf{R}_+^n})$ , that is,  $q \in W^{1,p}(U)$  for every bounded subset  $U$  of  $\mathbf{R}_+^n$ ; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting  $G_k := \mathbf{R}_+^n \cap (B_{2k}(0) \setminus B_k(0))$  for  $k \in \mathbf{N}$ , we have  $q \in W^{1,p}(G_k)$  and thus there is a constant  $a_k$  such that  $\int_{G_k} (q - a_k) dx = 0$  for each  $k \in \mathbf{N}$ . From this equality and the change of variables  $x = ky$  for  $x \in G_k$  and  $y \in G_1$  we have

$$\int_{G_1} (q(ky) - a_k) dy = k^{-n} \int_{G_k} (q(x) - a_k) dx = 0.$$

Hence we can apply Poincaré's inequality to  $q(ky) - a_k$  on  $G_1$  and get

$$\left( \int_{G_1} |q(ky) - a_k|^p dy \right)^{1/p} \leq c \left( \int_{G_1} |\nabla(q(ky))|^p dy \right)^{1/p}$$

with a constant  $c > 0$  independent of  $k$ . In this inequality, we observe that

$$\begin{aligned} \int_{G_1} |q(ky) - a_k|^p dy &= k^{-n} \int_{G_k} |q(x) - a_k|^p dx, \\ \int_{G_1} |\nabla(q(ky))|^p dy &= k^p \int_{G_1} |(\nabla q)(ky)|^p dy = k^{p-n} \int_{G_k} |\nabla q(x)|^p dx \end{aligned}$$

by the change of variables  $x = ky$  and thus

$$(3.19) \quad \|q - a_k\|_{L^p(G_k)} \leq ck \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbf{N}.$$

For each  $k \in \mathbf{N}$ , let  $q_k := \varphi_k(q - a_k)$  on  $\mathbf{R}_+^n$ . Then since  $\text{supp } q_k \subset \mathbf{R}_+^n \cap B_{2k}(0)$  holds by the relation  $\text{supp } \varphi_k \subset B_{2k}(0)$ , it follows that  $q_k \in W^{1,p}(\mathbf{R}_+^n)$  and

$$(3.20) \quad \|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}_+^n)} \leq \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}_+^n)} + \|(\nabla \varphi_k)(q - a_k)\|_{L^p(\mathbf{R}_+^n)}.$$

Since  $0 \leq \varphi_k(x) \leq 1$  and  $\lim_{k \rightarrow \infty} \varphi_k(x) = 1$  for all  $x \in \mathbf{R}_+^n$  and  $\nabla q \in L^p(\mathbf{R}_+^n)$ , the dominated convergence theorem yields

$$(3.21) \quad \lim_{k \rightarrow \infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}_+^n)} = 0.$$

On the other hand, since  $\nabla \varphi_k = k^{-1}(\nabla \varphi)_k$  and  $\text{supp } \nabla \varphi_k|_{\mathbf{R}_+^n} \subset \overline{G_k}$  for each  $k \in \mathbf{N}$ , it follows from (3.19) and the dominated convergence theorem that

$$(3.22) \quad \|(\nabla \varphi_k)(q - a_k)\|_{L^p(\mathbf{R}_+^n)} \leq ck^{-1} \|q - a_k\|_{L^p(G_k)} \leq c \|\nabla q\|_{L^p(G_k)} \rightarrow 0$$

as  $k \rightarrow \infty$ . Applying (3.21) and (3.22) to (3.20) we have

$$\lim_{k \rightarrow \infty} \|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}_+^n)} = 0,$$

where  $q_k \in W^{1,p}(\mathbf{R}_+^n)$  for all  $k \in \mathbf{N}$ . Hence the claim is valid when  $\Omega = \mathbf{R}_+^n$  and the proposition follows.  $\square$

*Proof of Lemma 3.12.* Let  $u \in C_c^\infty(\Omega)$  and  $p \in (1, \infty)$ . Then, since  $Tu \in L_p(\Omega)$  by Theorem 3.5, we have  $Q'u = u - Tu \in L^p(\Omega)$ . To show  $Q'u \in L_\sigma^p(\Omega)$ , we employ a characterization of elements of  $L_\sigma^p(\Omega)$  ([19, Lemma III.2.1]): a vector field  $v \in L^p(\Omega)$  is in  $L_\sigma^p(\Omega)$  if and only if

$$\int_{\Omega} v \cdot \nabla q \, dx = 0 \quad \text{for all } \nabla q \in G_{p'}(\Omega) \quad \left( p' := \frac{p}{p-1} \right).$$

Let  $\nabla q$  be any element of  $G_{p'}(\Omega)$ . From Proposition 3.13, there is a sequence  $\{q_k\}_{k=1}^\infty$  of functions in  $C_c^\infty(\mathbf{R}^n)$  such that the equality (3.18) with  $p$  replaced by  $p'$  holds. Since  $Q'u$  is defined and smooth in  $\mathbf{R}^n$  for  $u \in C_c^\infty(\Omega)$  and  $q_k \in C_c^\infty(\mathbf{R}^n)$ , integration by parts yields

$$\int_{\Omega} Q'u \cdot \nabla q_k \, dx = - \int_{\Omega} q_k \operatorname{div} Q'u \, dx + \int_{\partial\Omega} q_k Q'u \cdot \nu \, d\mathcal{H}^{n-1}$$

for all  $k \in \mathbf{N}$ , where  $\nu$  denotes the unit outer normal vector field of  $\partial\Omega$ . We apply (3.17) to the right-hand side of this equality to get  $\int_{\Omega} Q'u \cdot \nabla q_k \, dx = 0$  for all  $k \in \mathbf{N}$ . Since  $Q'u \in L^p(\Omega)$  and (3.18) with  $p$  replaced by  $p'$  holds, the above equality implies that

$$\int_{\Omega} Q'u \cdot \nabla q \, dx = \lim_{k \rightarrow \infty} \int_{\Omega} Q'u \cdot \nabla q_k \, dx = 0.$$

Hence by the characterization of elements of  $L_\sigma^p(\Omega)$  we conclude that  $Q'u \in L_\sigma^p(\Omega)$  for all  $u \in C_c^\infty(\Omega)$ . The proof is complete.  $\square$

**Remark 3.14.**

- (1) Let  $p \in (1, \infty)$ . By Theorem 3.5 and Lemma 3.12, we have  $Q'u \in L_\sigma^p(\Omega)$  and  $\|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}$  for all  $u \in C_c^\infty(\Omega)$ . Moreover,  $Q'u = u$  holds for all  $u \in C_{c,\sigma}^\infty(\Omega)$ . Hence, by the density argument,  $Q'$  extends uniquely to a bounded linear operator on  $L^p(\Omega)$  that is a projection onto  $L_\sigma^p(\Omega)$ .
- (2) The projection onto  $L_\sigma^p(\Omega)$  given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each  $u \in C_c^\infty(\Omega)$  there would exist  $\pi \in L_{loc}^1(\Omega)$  such that  $(I - Q')u = \nabla\pi$  holds. Since  $(I - Q')u = Tu = K * \operatorname{div} u$  for  $u \in C_c^\infty(\Omega)$ , the existence of such  $\pi$  would imply that  $\partial_j(K_i * \operatorname{div} u) = \partial_i(K_j * \operatorname{div} u)$  for all  $i, j = 1, \dots, n$ . For each  $f \in C_c^\infty(\Omega)$  with  $\int_{\Omega} f \, dx = 0$  there is  $u \in C_c^\infty(\Omega)$  satisfying  $f = \operatorname{div} u$ . This is possible since we are able to apply Bogovskii's lemma to a bounded Lipschitz domain  $D \subset \Omega$  containing the support of  $f$  (see [19, Theorem III.3.3]). Thus the above equality would imply that  $\partial_j K_i = \partial_i K_j + c$  with some constant  $c$  for all  $i, j = 1, \dots, n$  as a distribution. This contradicts the fact that  $\partial_j K_i \neq \partial_i K_j + c$  for  $i \neq j$  as observed in (3.4).
- (3) It is possible to prove the characterization

$$L_\sigma^p(\Omega) = \{u \in L^p(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \nu = 0 \text{ on } \partial\Omega\}$$

if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains

(see [19, Section III.2]). However, for a Lipschitz half-space, it is less popular. A proof can be found in [30, Lemma 2.1].

The linear operator  $Q'$  also maps  $C_c^\infty(\Omega)$  into  $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ .

**Lemma 3.15.** *Let  $\Omega$  be a Lipschitz half-space. For all  $u \in C_c^\infty(\Omega)$  and  $\nu \in (0, \infty]$ , we have  $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ .*

We shall prove two auxiliary propositions for the above lemma. For  $p \in (1, \infty)$ , let  $W_{0,\sigma}^{1,p}(\Omega)$  be the  $W^{1,p}$ -closure of  $C_{c,\sigma}^\infty(\Omega)$ .

**Proposition 3.16.** *Let  $\Omega$  be a Lipschitz half-space. For all  $p \in (1, \infty)$  we have  $L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega) \subset W_{0,\sigma}^{1,p}(\Omega)$ . Thus  $L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega) = W_{0,\sigma}^{1,p}(\Omega)$ .*

*Proof.* Let  $\rho \in C_c^\infty(\mathbf{R}^n)$  be a function such that

$$0 \leq \rho \leq 1 \quad \text{in } \mathbf{R}^n, \quad \text{supp } \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \, dx = 1$$

and  $\rho_\delta(x) := \delta^{-n} \rho(\delta^{-1}x)$  for  $\delta > 0$ ,  $x \in \mathbf{R}^n$ . Let  $u \in L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega)$ . Then there is a sequence  $\{u_k\}_{k=1}^\infty$  of functions in  $C_{c,\sigma}^\infty(\Omega)$  such that  $\lim_{k \rightarrow \infty} \|u - u_k\|_{L^p(\Omega)} = 0$ . For  $a > 0$ , we define a vector field  $u^a$  on  $\Omega$  as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ 0, & h(x') < x_n \leq h(x') + a \end{cases}$$

and  $u_k^a = (u_k)^a$  similarly. Then it is clear that  $u^a \in W_0^{1,p}(\Omega)$  and  $u_k^a \in C_{c,\sigma}^\infty(\Omega)$  for all  $a > 0$ . Moreover, we have

$$\|u^a - u_k^a\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all } a > 0, \quad \lim_{a \rightarrow 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0.$$

By the second equality and the fact that  $W_{0,\sigma}^{1,p}(\Omega)$  is closed in  $W^{1,p}(\Omega)$ , it is sufficient for showing  $u \in W_{0,\sigma}^{1,p}(\Omega)$  to prove  $u^a \in W_{0,\sigma}^{1,p}(\Omega)$  for all  $a > 0$ .

For each  $a > 0$ , there is a constant  $d = d(a) > 0$  such that  $\text{dist}(\text{supp } u_k^a, \partial\Omega) \geq d$  for all  $k \in \mathbf{N}$ . Then, for a given  $\varepsilon > 0$ , we can take  $\delta \in (0, d/2)$  so small that

$$\|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2},$$

since  $u^a \in W_0^{1,p}(\Omega)$ . Also, since  $\nabla \rho_\delta = \delta^{-1}(\nabla \rho)_\delta$ , we have

$$\begin{aligned} & \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \\ & \leq c(\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{L^p(\Omega)} + \|u^a * \nabla \rho_\delta - u_k^a * \nabla \rho_\delta\|_{L^p(\Omega)}) \\ & = c(\|(u^a - u_k^a) * \rho_\delta\|_{L^p(\Omega)} + \delta^{-1}\|(u^a - u_k^a) * (\nabla \rho)_\delta\|_{L^p(\Omega)}) \\ & \leq c(1 + \delta^{-1})\|u^a - u_k^a\|_{L^p(\Omega)} = c(1 + \delta^{-1})\|u - u_k\|_{L^p(\Omega)} \end{aligned}$$

with a constant  $c > 0$  independent of  $\varepsilon$  and  $\delta$ . Hence by taking  $k \in \mathbf{N}$  so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})},$$

we have  $\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2$  and thus

$$\|u^a - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \leq \|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon.$$

On the other hand, since  $\text{dist}(\text{supp } u_k^a, \partial\Omega) > d$  and  $\delta \in (0, d/2)$ , the function  $u_k^a * \rho_\delta$  is smooth and compactly supported in  $\Omega$ . Moreover, we have

$$\text{div}(u_k^a * \rho_\delta) = (\text{div } u_k^a) * \rho_\delta = 0 \quad \text{in } \Omega.$$

Thus  $u_k^a * \rho_\delta \in C_{c,\sigma}^\infty(\Omega)$  and  $u^a$  is approximated by elements of  $C_{c,\sigma}^\infty(\Omega)$  in  $W^{1,p}(\Omega)$ , which means that  $u^a \in W_{0,\sigma}^{1,p}(\Omega)$ . Hence  $u \in W_{0,\sigma}^{1,p}(\Omega)$  and the proof is now complete.  $\square$

**Proposition 3.17.** *Let  $\nu \in (0, \infty]$ . If  $p > n$ , then  $W_{0,\sigma}^{1,p}(\Omega) \subset VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ .*

*Proof.* Let  $u \in W_{0,\sigma}^{1,p}(\Omega)$  and  $u_k \in C_{c,\sigma}^\infty(\Omega)$  such that  $\lim_{k \rightarrow \infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0$ . Since  $p > n$  and  $u, u_k \in W_{0,\sigma}^{1,p}(\Omega)$ , Morrey's inequality (see e.g. [7, Theorem 4.12]) implies

$$\|u - u_k\|_{L^\infty(\Omega)} \leq c \|u - u_k\|_{W^{1,p}(\Omega)}$$

with a positive constant  $c$  independent of  $u$  and  $u_k$ . Thus we have

$$\|u - u_k : BMO_b^{\infty,\nu}(\Omega)\| \leq (2 + \omega_n) \|u - u_k\|_{L^\infty(\Omega)} \leq c \|u - u_k\|_{W^{1,p}(\Omega)} \rightarrow 0$$

as  $k \rightarrow \infty$ . Hence  $u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$  and the proof is now complete.  $\square$

*Proof of Lemma 3.15.* Since  $u \in C_c^\infty(\Omega)$  and thus  $\partial_i u \in C_c^\infty(\Omega)$  for all  $i = 1, \dots, n$ , it follows from Lemma 3.12 that  $Q'u \in L_\sigma^r(\Omega)$  and  $\partial_i Q'u = Q'(\partial_i u) \in L^r(\Omega)$  for all  $r \in (1, \infty)$  and  $i = 1, \dots, n$ . From this fact and the equality (3.17), we have  $Q'u \in L_\sigma^r(\Omega) \cap W_0^{1,r}(\Omega)$  for all  $r \in (1, \infty)$ . Hence, by taking  $r > n$ , we can apply Proposition 3.16 and Proposition 3.17 to obtain  $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ .  $\square$

**Remark 3.18.** Let  $\nu \in (0, \infty]$ . Theorem 3.10 and Lemma 3.15 imply that  $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$  and  $\|Q'u : BMO_b^{\infty,\nu}(\Omega)\| \leq c \|u\|_{L^\infty(\Omega)}$  for all  $u \in C_c^\infty(\Omega)$ . Also, we have  $Q'u = u$  for all  $u \in C_{c,\sigma}^\infty(\Omega)$ . Hence  $Q'$  extends uniquely to a bounded linear operator (again referred to as  $Q'$ ) from  $C_0(\Omega)$ , which is the  $L^\infty$ -closure of  $C_c^\infty(\Omega)$ , into  $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$  that satisfies  $Q'u = u$  for all  $u \in C_{0,\sigma}(\Omega)$ .

Now let us extend  $Q'$  to a linear operator that gives the projection mentioned in Theorem 1.4. For  $p \in (1, \infty)$ , we define a Banach space  $X_p$  and its norm as

$$X_p := L^p(\Omega) \cap C_0(\Omega), \quad \|u\|_{X_p} := \max\{\|u\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}\}.$$

Note that the Banach space  $C_0(\Omega)$  consists of all continuous functions  $f$  on  $\Omega$  such that the set  $\{x \in \Omega \mid |f(x)| \geq \varepsilon\}$  is compact in  $\Omega$  for every  $\varepsilon > 0$  (see e.g. [32, Theorem 3.17]).

**Lemma 3.19.** *For each  $p \in (1, \infty)$ , the linear subspace  $C_c^\infty(\Omega)$  is dense in  $X_p$ .*

*Proof.* The proof is more or less standard (see e.g. [27, Corollary 19.24]). We give it for completeness. Let  $u \in X_p$  and  $\Omega_k := \{x \in \Omega \mid |x| \leq k, \text{dist}(x, \partial\Omega) \geq 1/k\}$  for  $k \in \mathbf{N}$ . For any given  $\varepsilon > 0$ , the set  $\{x \in \Omega \mid |u(x)| \geq \varepsilon/2\}$  is compact in  $\Omega$  since  $u \in C_0(\Omega)$ . Moreover, since  $u \in L^p(\Omega)$ , we can take  $k \in \mathbf{N}$  so large that

$$(3.23) \quad \|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}, \quad \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}.$$

Let  $\varphi \in C_c^\infty(\Omega)$  be a continuous cut-off function such that

$$0 \leq \varphi \leq 1 \quad \text{in } \Omega, \quad \varphi = 1 \quad \text{in } \Omega_k, \quad \varphi = 0 \quad \text{in } \Omega \setminus \Omega_{2k}.$$

Since  $u - \varphi u = 0$  in  $\Omega_k$  and  $|u - \varphi u| \leq |u|$  in  $\Omega \setminus \Omega_k$ , it follows from (3.23) that

$$(3.24) \quad \|u - \varphi u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}, \quad \|u - \varphi u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}.$$

Let  $\rho_\delta$  be a mollifier as in the beginning of the proof of Proposition 3.16. Since

$$\varphi u \in L^p(\Omega), \quad \text{dist}(\text{supp}(\varphi u), \partial\Omega) \geq \frac{1}{2k},$$

we can take  $\delta \in (0, 1/4k)$  so small that

$$(3.25) \quad u_\delta := \rho_\delta * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}.$$

On the other hand, since  $\varphi u$  is uniformly continuous on  $\Omega_{4k}$ , we can again choose  $\delta \in (0, 1/4k)$  so small that  $\|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \varepsilon/2$ . Moreover, since  $\text{supp}(\varphi u) \subset \Omega_{2k}$  and  $\delta \in (0, 1/4k)$ , we have  $\varphi u = u_\delta = 0$  outside of  $\Omega_{4k}$  and thus

$$(3.26) \quad \|\varphi u - u_\delta\|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}.$$

Combining (3.24), (3.25) and (3.26), we obtain  $u_\delta \in C_c^\infty(\Omega)$  and

$$\|u - u_\delta\|_{X_p} = \max\{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon.$$

Hence the lemma follows.  $\square$

Let  $Y_p := L_\sigma^p(\Omega) \cap VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$  for  $p \in (1, \infty)$ ,  $\nu \in (0, \infty]$ . Since  $L_\sigma^p(\Omega)$  and  $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$  are closed in  $L^p(\Omega)$  and  $BMO_b^{\infty,\nu}(\Omega)$ , respectively,  $Y_p$  becomes a Banach space under the norm  $\|v\|_{Y_p} := \max\{\|v\|_{L^p(\Omega)}, \|v\|_{BMO_b^{\infty,\nu}(\Omega)}\}$ .

**Theorem 3.20.** *Let  $p \in (1, \infty)$  and  $\nu \in (0, \infty]$ . The linear operator  $Q'$  given in Definition 3.11 extends uniquely to a bounded linear operator  $Q_p$  from  $X_p$  into  $Y_p$ . Moreover, there exists a constant  $c > 0$  such that*

$$(3.27) \quad \|Q_p u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q_p u : BMO_b^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all  $u \in X_p$  and  $Q_p u = u$  holds for all  $u$  in the  $X_p$ -closure of  $C_{c,\sigma}^\infty(\Omega)$ .

*Proof.* Let  $u \in C_c^\infty(\Omega)$ . Then we have  $Q' u \in Y_p$  by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant  $c > 0$  independent of  $u$  such that

$$(3.28) \quad \|Q' u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q' u : BMO_b^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}.$$

Hence we have  $Q' u \in Y_p$  and  $\|Q' u\|_{Y_p} \leq c\|u\|_{X_p}$  for all  $u \in C_c^\infty(\Omega)$ . Since  $C_c^\infty(\Omega)$  is dense in  $X_p$  by Lemma 3.19, the operator  $Q'$  extends uniquely to a bounded linear operator  $Q_p$  from  $X_p$  into  $Y_p$ . Also, it follows from (3.28) that the inequality (3.27) holds for all  $u \in X_p$ . Since  $Q' u = u$  holds for all  $u \in C_{c,\sigma}^\infty(\Omega)$  as observed after Definition 3.11, by the density argument we have  $Q_p u = u$  for all  $u$  in the  $X_p$ -closure of  $C_{c,\sigma}^\infty(\Omega)$ . The proof is complete.  $\square$

Finally, Theorem 1.4 follows from Theorem 3.20 with  $p = 2$ , that is, the linear operator  $Q$  in Theorem 1.4 is given by  $Q = Q_2$ .

#### 4. ANALYTICITY IN $L^p$

In this section we shall give a complete proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let  $S(t)$  be the Stokes semigroup in  $\tilde{L}_\sigma^p$  constructed by [14], [16]. To show that  $S(t)$  forms an analytic semigroup in  $L_\sigma^p$  ( $2 \leq p < \infty$ ) it suffices to prove that there exists a constant  $C$  that

$$(4.1) \quad \|S(t)v_0\|_p \leq C\|v_0\|_p$$

$$(4.2) \quad \left\| t \frac{d}{dt} S(t)v_0 \right\|_p \leq C \|v_0\|_p$$

for all  $v_0 \in C_{c,\sigma}^\infty(\Omega)$  and for all  $t \in (0, 1)$ . Let  $Q$  be the operator in Theorem 1.4. Since  $Q$  is bounded in  $L^2$  and maps  $L^2$  to  $L_\sigma^2$  and  $S(t)$  fulfills (4.1) and (4.2) for  $p = 2$ , we have

$$(4.3) \quad \|S(t)Qu\|_2 \leq C \|u\|_2$$

$$(4.4) \quad \left\| t \frac{d}{dt} S(t)Qu \right\|_2 \leq C \|u\|_2$$

for all  $u \in C_c(\Omega)$  and  $t \in (0, 1)$ . Since  $\Omega$  is admissible as proved in [5],  $S(t)$  forms an analytic semigroup in  $VMO_{b,0,\sigma}^{\infty,\nu}$  by Theorem 1.2. We conclude that

$$(4.5) \quad \|S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C \|u\|_\infty$$

$$(4.6) \quad \left\| t \frac{d}{dt} S(t)Qu : BMO_b^{\infty,\nu}(\Omega) \right\| \leq C \|u\|_\infty$$

for all  $u \in C_c(\Omega)$  and  $t \in (0, 1)$  since  $Q$  fulfills

$$\|Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C \|u\|_\infty, \quad Qu \in VMO_{b,0,\sigma}^{\infty,\nu}$$

for all  $u \in C_c(\Omega)$  by Theorem 1.4. (Note that we have a stronger statement than (4.6) by replacing the  $BMO_b$  type norm by the  $L^\infty$  norm since we have the regularizing estimate (1.3).) We apply an interpolation result (Theorem 1.3) to (4.3) and (4.5) and to (4.4) and (4.6) to get, respectively

$$(4.7) \quad \|S(t)Qu\|_p \leq C \|u\|_p$$

$$(4.8) \quad \left\| t \frac{d}{dt} S(t)Qu \right\|_p \leq C \|u\|_p$$

for all  $u \in C_c(\Omega)$  and for all  $t \in (0, 1)$ . Since  $Qu = u$  for  $u \in C_{c,\sigma}^\infty(\Omega)$  this yields (4.1) and (4.2).

It remains to prove that  $S(t)$  is a  $C_0$ -semigroup in  $L_\sigma^p$ . Since  $C_{c,\sigma}^\infty(\Omega)$  is dense in  $L_\sigma^p$ , for  $v_0 \in L_\sigma^p$  there is  $v_{0m} \in C_{c,\sigma}^\infty$  such that  $\|v_0 - v_{0m}\|_p \rightarrow 0$  as  $m \rightarrow \infty$ . By (4.1) we observe that

$$\begin{aligned} \|S(t)v_0 - v_0\|_p &\leq \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p \\ &\leq C \|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p. \end{aligned}$$

Sending  $t \downarrow 0$ , we get

$$\overline{\lim}_{t \downarrow 0} \|S(t)v_0 - v_0\|_p \leq C \|v_0 - v_{0m}\|_p,$$

since  $S(t)v_{0m} \rightarrow v_{0m}$  in  $\tilde{L}_\sigma^p$  as  $t \downarrow 0$  by [14], [16]. Sending  $m \rightarrow \infty$ , we conclude that  $S(t)v_0 \rightarrow v_0$  in  $L_\sigma^p$  as  $t \downarrow 0$ .  $\square$

**Remark 4.1.** In a similar way as we derived (4.5) and (4.6) we are able to derive from the  $L^\infty$ - $BMO$  estimates in [10] that

$$\begin{aligned} t \|\nabla^2 S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| &\leq C \|u\|_\infty \\ t^{1/2} \|\nabla S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| &\leq C \|u\|_\infty \end{aligned}$$

for all  $u \in C_c(\Omega)$  and  $t \in (0, 1)$ .



Note that  $L^2$  results

$$t \|\nabla^2 S(t)Qu\|_2 \leq C\|u\|_2$$

$$t^{1/2} \|\nabla S(t)Qu\|_2 \leq C\|u\|_2$$

easily follow from the analyticity of  $S(t)$  in  $L^2_\sigma$  and  $L^2$ -boundedness of  $Q$  if one observes that  $\|\nabla u\|_2^2 = (Au, u)_{L^2}$  and

$$\|\nabla^2 u\|_2 \leq C (\|Au\|_2 + \|\nabla u\|_2 + \|u\|_2)$$

(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where  $A$  is the Stokes operator in  $L^2_\sigma$ .

Interpolating the  $L^2$  results and the above  $L^\infty$ -BMO results, we are able to prove that there is  $C_p > 0$  satisfying

$$t \|\nabla^2 S(t)v_0\|_p \leq C_p \|v_0\|_p$$

$$t^{1/2} \|\nabla S(t)v_0\|_p \leq C_p \|v_0\|_p$$

for all  $v_0 \in L^p_\sigma(\Omega)$  and  $t \in (0, 1)$  with  $p \in (2, \infty)$ .

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