ON ANALYTICITY OF THE L^p-STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

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Abstract. Consider the Stokes equations in a sector-like C^3 domain $\Omega \subset \mathbf{R}^2$. It is shown that the Stokes operator generates an analytic semigroup in $L^p_\sigma(\Omega)$ for $p \in [2, \infty)$. This includes domains where the L^p-Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in VMO and L^2 by constructing a suitable non-Helmholtz projection to solenoidal spaces.

1. Introduction

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where Ω is a domain in $\mathbf{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a C_0-analytic semigroup in L^p_σ $(1 < p < \infty)$ for various domains like smooth bounded domains ([21], [35]). Here $L^p_\sigma = L^p_\sigma(\Omega)$ denotes the L^p-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. More recently, it has been proved in [20] that $S(t)$ always forms a C_0-analytic semigroup in $L^p_\sigma(\Omega)$ for any uniformly C^2-domain Ω provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \left\{\nabla q \in L^p(\Omega) \mid q \in L^1_{loc}(\Omega)\right\}$. In [20] the L^q maximal regularity in time with values in $L^p_\sigma(\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The L^p-Helmholtz decomposition holds for various domains like bounded or exterior domains with

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smooth boundary for $1 < p < \infty$ ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the L^p-Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely. Let $C(\vartheta)$ denote the cone of the form

$$C(\vartheta) = \{x = (x', x_n) \in \mathbf{R}^n \mid -x_n \geq |x| \cos(\vartheta/2)\},$$

where $\vartheta \in (0, 2\pi)$ is the opening angle. When $n = 2$, we simply say that $C(\vartheta)$ is a sector. We say that a planar domain Ω is a *sector-like domain* with opening angle ϑ if $\Omega \backslash B_R(0) = C(\vartheta) \backslash B_R(0)$ for some $R > 0$ (up to rotation and translation), where $B_R(0)$ is an open disk of radius R centered at the origin.

It is known that the L^p-Helmholtz decomposition fails for a sector-like domain Ω when $p > q'_\vartheta$ or $p < q_\vartheta$ with $q_\vartheta = 2/(1 + \pi/\vartheta)$, $1/q_\vartheta + 1/q'_\vartheta = 1$ even if the boundary $\partial\Omega$ is smooth [28, Example 2, Fig. 5] while for $p \in (q_\vartheta, q'_\vartheta)$ the L^p-Helmholtz decomposition holds. This means that if the opening angle ϑ is larger than π, there always exists $p > 2$ such that the L^p-Helmholtz decomposition fails.

It has been a longstanding open question whether or not the existence of the L^p-Helmholtz decomposition is necessary for L^p analyticity of $S(t)$. In this paper, we give a negative answer for this question by proving that there is a domain Ω for which $S(t)$ is analytic in L^p_σ while the L^p-Helmholtz decomposition fails. This is a subtle problem since the existence of the L^p-Helmholtz projection is known to be necessary for L^p solvability of the resolvent equation ([33]). However, in this statement the external force term is allowed to be in the more general space L^p instead of L^p_σ. Our problem is different from that in [33].

We say that Ω has a C^k graph boundary if Ω is of the form

$$\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$$

(up to translation and rotation) with some real-valued C^k function h with variable $x' \in \mathbf{R}^{n-1}$.

Theorem 1.1. *Let Ω be a sector-like domain in $\mathbf{R}^2$ having a C^3 graph boundary. Then $S(t)$ forms a C_0-analytic semigroup in $L^p_\sigma(\Omega)$ for all $p \in [2, \infty)$.*

Here is our strategy to prove Theorem 1.1. It is by now well-known that $S(t)$ forms an analytic semigroup in $\tilde{L}^p_\sigma$, i.e., $\tilde{L}^p_\sigma = L^p_\sigma \cap L^2_\sigma$ $(p \geq 2)$, $\tilde{L}^p = L^p_\sigma + L^2_\sigma$ $(1 < p < 2)$ ([14], [15], [16]). Thus $S(t)v_0$ is well-defined for $v_0 \in C^\infty_{c,\sigma}(\Omega)$. To show Theorem 1.1, a key step is to prove the two estimates

$$\|S(t)v_0\|_p \leq C\|v_0\|_p \tag{1.1}$$

$$t \left\| \frac{\mathrm{d}}{\mathrm{d}t} S(t)v_0 \right\|_p \leq C\|v_0\|_p \tag{1.2}$$

for all $v_0 \in C^\infty_{c,\sigma}(\Omega)$, $t \in (0, 1)$, where $\|v_0\|_p$ denotes the L^p-norm of v_0. The constant C should be taken independent of t and v_0. We shall establish (1.1) and (1.2) by interpolation since both estimates are known for $p = 2$.

We are tempted to interpolate the L^∞ type result obtained in [5] with the L^2-result. In fact, in [5] the estimates (1.1) and (1.2) with $p = \infty$ are established for all $v_0 \in C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$ for a C^2 sector-like domain Ω in $\mathbf{R}^2$. However, it is not clear that the complex interpolation space $\left[L^2_\sigma, C_{0,\sigma}\right]_\rho$ agrees with L^p_σ with $2/p = 1 - \rho$ although it is well-known as the Riesz-Thorin theorem that $\left[L^2, L^\infty\right]_\rho = L^p$. To interpolate, we would need a projection to solenoidal spaces which is almost impossible since such a projection involves the singular integral operator which is not bounded in L^∞.

To circumvent this difficulty, we consider the Stokes semigroup $S(t)$ in BMO-type spaces as studied in [10], [11], [12]. For $p \in [1,\infty)$, $\mu \in (0,\infty]$ we define the BMO seminorm

$$\left[f : BMO_p^\mu(\Omega)\right] := \sup\left\{ \left(\fint_{B_r(x)} \left|f(y) - f_{B_r(x)}\right|^p \mathrm{d}y \right)^{1/p} \,\middle|\, B_r(x) \subset \Omega,\ r < \mu \right\},$$

where $f_B = \fint_B f \mathrm{d}x$, the average of f over B and $B_r(x)$ denotes the closed ball of radius r centered at x. It is well-known that one gets an equivalent seminorm when the ball B_r is replaced by a cube. We also need to control the boundary behavior. For $\nu \in (0,\infty]$ we define

$$\left[f : b_p^\nu(\Omega)\right] := \sup\left\{ \left(\frac{1}{r^n} \int_{B_r(x_0)\cap\Omega} |f(y)|^p \mathrm{d}y \right)^{1/p} \,\middle|\, x_0 \in \partial\Omega,\ r > 0,\ B_r(x_0) \subset U_\nu(\partial\Omega) \right\},$$

where $U_\nu(E)$ is a ν-open neighborhood of E, i.e.,

$$U_\nu(E) = \{x \in \mathbf{R}^n \mid \mathrm{dist}(x,E) < \nu\}.$$

We shall often assume that $\nu < R^*$, where R^* is the reach from the boundary. The BMO norm we use is

$$\left\| f : BMO_{b,p}^{\mu,\nu}(\Omega) \right\| = \left[f : BMO_p^\mu(\Omega)\right] + \left[f : b_p^\nu(\Omega)\right].$$

If $p = 1$, we often drop p. The BMO space we consider is

$$BMO_{b,p}^{\mu,\nu}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \,\middle|\, \left\| f : BMO_{b,p}^{\mu,\nu}(\Omega) \right\| < \infty \right\}.$$

This space is independent of p for sufficiently small ν, i.e., $\nu < R^*$ ([11], [12]) and $BMO_b^{\infty,\infty}$ agrees with Miyachi BMO space ([29]) for various domains including a half space and bounded C^2 domains ([12]). Although the $BMO_b^{\infty,\nu}(\Omega)$ norm is equivalent to the $BMO_b^{\infty,\infty}(\Omega)$ norm when Ω is bounded, there are many unbounded domains for which the $BMO_b^{\infty,\nu}(\Omega)$ norm is actually weaker than the $BMO_b^{\infty,\infty}(\Omega)$ norm when ν is finite. We define the solenoidal space $VMO_{b,0,\sigma}^{\mu,\nu}$ as the $BMO_b^{\mu,\nu}$-closure of $C_{c,\sigma}^\infty(\Omega)$. In [10], [11] among other results the analyticity of $S(t)$ in $VMO_{b,0,\sigma}^{\infty,\nu}$ has been established for a uniformly C^3 domain which is admissible in the sense of [2] provided that ν is sufficiently small.

Theorem 1.2 ([10], [11])**.** *Let Ω be an admissible uniformly C^3 domain in $\mathbf{R}^n$. Then $S(t)$ forms a C_0-analytic semigroup in $VMO_{b,0,\sigma}^{\mu,\nu}$ for any $\mu \in (0,\infty]$ and $\nu \in (0,\nu_0)$ with some ν_0 depending only on μ and regularity of $\partial\Omega$.*

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace L^p by L^∞ or $BMO_b^{\infty,\nu}$, but even an estimate stronger than (1.2) with $p = \infty$, i.e.,

$$t \left\| \frac{\mathrm{d}S(t)}{\mathrm{d}t} v_0 \right\|_\infty \le C \left\| v_0 : BMO_b^{\mu,\nu}(\Omega) \right\|, \quad \mu,\nu \in (0,\infty] \tag{1.3}$$

which shows a regularizing effect.

It has been proved in [5] that a C^2 sector-like domain in $\mathbf{R}^2$ is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a C^2 sector-like domain in $\mathbf{R}^2$ is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly C^3. On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a layer domain with $n \geq 3$ is not admissible ([8]).

In order to get the L^p estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in Ω.

Theorem 1.3. *Let Ω be a Lipschitz half-space in $\mathbf{R}^n$, i.e., a domain having Lipschitz graph boundary. Let T be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant C such that*

$$\|Tu\|_2 \leq C\|u\|_2$$

$$[Tu : BMO^\infty(\Omega)] \leq C\|u\|_\infty$$

for $u \in C_c(\Omega)$. Then $\|Tu\|_p \leq C_*\|u\|_p$ for $u \in C_c(\Omega)$ with C_* depending only on C, h and $p \in (2,\infty)$.

There are a couple of such interpolation results between BMO and L^2, which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between L^p and BMO is discussed when Ω is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega) \cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with $BMO_A(\mathcal{X})$, where A is some operator. They worked on metric measure spaces of homogeneous type $(\mathcal{X}, d, \mu)$. In particular, in the case $\mathcal{X} = \Omega, d(x,y) = |x-y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO^\infty(\Omega)$.

Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the L^∞ case we do not know whether or not the complex interpolation space $\left[L^2_\sigma, VMO^{\infty,\nu}_{b,0,\sigma}\right]_\rho$ with $2/p = 1-\rho$ agrees with L^p_σ, although we know that $\left[L^2, BMO\right]_\rho = L^p$ for $\Omega = \mathbf{R}^n$ as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

Theorem 1.4. *Let Ω be a Lipschitz half-space in $\mathbf{R}^n$. Assume that $\nu \in (0,\infty]$. There is a linear operator Q from $C_c(\Omega)$ to $VMO^{\infty,\nu}_{b,0,\sigma}(\Omega) \cap L^2_\sigma(\Omega)$ such that*

$$\|Qu : BMO^{\infty,\nu}_b(\Omega)\| \leq C\|u\|_\infty$$

$$\|Qu\|_2 \leq C\|u\|_2$$

for all $u \in C_c(\Omega)$. Moreover, $Qu = u$ for $u \in C_c(\Omega) \cap L^2_\sigma(\Omega)$.

Since there may be no L^p-Helmholtz decomposition our Q should be different from the Helmholtz projection. We shall construct such an operator Q using the solution operator of the equation $\operatorname{div} u = f$ given by Solonnikov [36]. Although deriving the L^2 estimate is easy, to derive the BMO estimate is more involved since we have to estimate the b^ν type seminorm.

To derive (1.1), we actually interpolate

$$\|S(t)Qu\|_2 \leq C\|u\|_2$$

and

$$\|S(t)Qu : BMO^{\infty,\nu}_b\| \leq C\|u\|_\infty$$

for $u \in C_c(\Omega)$. Similarly, we derive (1.2) by interpolating the estimate for $t\frac{\mathrm{d}S}{\mathrm{d}t}Q$.

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 3, we construct the projection operator Q. In Section 4, we give a complete proof of Theorem 1.1.

2. $L^2 - BMO$ interpolation on a Lipschitz half-space

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

$$\Omega := \{(x', x_n) \in \mathbf{R}^n | x_n > h(x')\}$$

with a Lipschitz function h on $\mathbf{R}^{n-1}$.

By Q we mean a closed cube with sides parallel to the coordinate axes. Let $\ell(Q)$ be the side length of Q, and for $\tau > 0$, τQ a cube with the same center as Q and side length $\tau \ell(Q)$.

2.1. **Reduction to the half-space and extension.** Here, we prepare lemmas that are basic estimates for the proof. Since h is Lipschitz continuous, $F(x) := (x', x_n - h(x'))$ is a bi-Lipschitz map from Ω to $\mathbf{R}^n_+$. For a function u defined on $\mathbf{R}^n_+$ the pull-back function $F^*(u)$ of u on Ω is defined by $u \circ F$. We start with estimates for $(F^{-1})^*$ which is the pull-back function $(F^{-1})^*(v)$ of v on $\mathbf{R}^n_+$ defined by $v \circ F^{-1}$.

Lemma 2.1. *Let Ω be a Lipschitz half-space.*

(i):

$$\left[(F^{-1})^* v : BMO^\infty(\mathbf{R}^n_+)\right] \le c\,[v : BMO^\infty(\Omega)]\,.$$

(ii):

$$\left\|(F^{-1})^* v\right\|_{L^2(\mathbf{R}^n_+)} \le c\|v\|_{L^2(\Omega)}.$$

Here c is a constant depending only on Lipschitz bound of h and n.

Proof. (i): Because $\mathbf{R}^n_+$ is an open subset of $\mathbf{R}^n$, we know that for any $\tau > 2$,

$$\left[(F^{-1})^* v : BMO^\infty(\mathbf{R}^n_+)\right] \le c_\tau \sup_{\tau Q \subset \mathbf{R}^n_+} \inf_{d \in \mathbf{R}} \int_Q \left|(F^{-1})^* v - d\right| \mathrm{d}y,$$

where the supremum is taken over cubes Q, for which τQ is contained in $\mathbf{R}^n_+$, see [37]. Since F is a bi-Lipschitz map, it holds

$$c_1 \operatorname{dist}(y, \partial \mathbf{R}^n_+) \le \operatorname{dist}(F^{-1}(y), \partial\Omega) \le c_2 \operatorname{dist}(y, \partial \mathbf{R}^n_+)$$

with some constants $c_1, c_2 > 0$ for all $y \in \mathbf{R}^n_+$. Since $(\tau - 1)\ell(Q)/2 \le \operatorname{dist}(Q, \partial \mathbf{R}^n_+)$ for such cubes Q, we have the lower bound

$$c\tau\ell(Q) \le \operatorname{dist}(F^{-1}(Q), \partial\Omega)$$

with some $c > 0$, which depends on n and h. Therefore, taking large τ, we can find cubes $\{R_k\}_{k=1}^{c_*} \subset \Omega$, which have no intersection of interiors, so that $\cup_{k=1}^{c_*} R_k$ is connected and

$$\begin{cases} \circ\ \ell(R_k) = \ell(Q), \\ \circ\ F^{-1}(Q) \subset \cup_{k=1}^{c_*} R_k, \text{ where } c_* \in \mathbf{N} \text{ depends only on } h, \text{ and} \\ \circ \text{ if } R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega. \end{cases}$$

From these, one obtains that for cubes Q with $\tau Q \subset \mathbf{R}^n_+$,

$$\inf_{d \in \mathbf{R}} \frac{1}{|Q|} \int_Q \left|(F^{-1})^* v - d\right| \mathrm{d}y \le c \sum_{k=1}^{c_*} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_1}| \mathrm{d}y.$$

It is enough to show that

$$\frac{1}{|R_k|}\int_{R_k}|v-v_{R_j}|\mathrm{d}y \le c[v:BMO^\infty(\Omega)] \tag{2.1}$$

for the case $R_j\cap R_k\neq\emptyset$. To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let $\tilde{R}_k$ and $\tilde{R}_j$ be subcubes of R_k and R_j respectively so that $\ell(\tilde{R}_k)=\ell(R_k)/2$, $\ell(\tilde{R}_j)=\ell(R_j)/2$ and they touch each other. Moreover, denote by $\tilde{R}_{j,k}$ a cube satisfying $\ell(\tilde{R}_{j,k})=\ell(\tilde{R}_j)+\ell(\tilde{R}_k)$ and $\tilde{R}_j\cup\tilde{R}_k\subset\tilde{R}_{j,k}\subset R_{j,k}$. Hence, we have

$$\begin{aligned}\frac{1}{|R_k|}\int_{R_k}|v-v_{R_j}|\mathrm{d}y &\le \frac{1}{|R_k|}\int_{R_k}|v-v_{R_k}|\mathrm{d}y+|v_{R_k}-v_{R_j}|\\ &\le c[v:BMO^\infty(\Omega)]+c|v_{\tilde{R}_j}-v_{\tilde{R}_k}|\\ &\le c[v:BMO^\infty(\Omega)]+c\frac{1}{|\tilde{R}_{j,k}|}\int_{\tilde{R}_{j,k}}|v-v_{\tilde{R}_{j,k}}|\mathrm{d}y\\ &\le c[v:BMO^\infty(\Omega)].\end{aligned}$$

(ii): This is verified as follows

$$\|(F^{-1})^*v\|^2_{L^2(\mathbf{R}^n_+)}=\int_\Omega|v|^2J_F\mathrm{d}x\le c\int_\Omega|v|^2\mathrm{d}x,$$

where J_F is the modulus of the Jacobian of F which is bounded, because h is Lipschitz continuous. □

Next, we consider the even extension of functions on the half space. For a function f on $\mathbf{R}^n_+$, we extend f outside $\mathbf{R}^n_+$ by

$$E[f](x',-x_n):=f(x',x_n) \text{ for } x_n>0.$$

From elementary geometrical observation, we can see that the extension operator E is a BMO-extension operator for $\mathbf{R}^n_+$.

Lemma 2.2.

$$[E[f]:BMO^\infty(\mathbf{R}^n)]\le c\left[f:BMO^\infty(\mathbf{R}^n_+)\right].$$

Proof. It is sufficient to consider cubes $Q\subset\mathbf{R}^n$ with $Q\cap\mathbf{R}^n_+\neq\emptyset$ and $Q\cap\mathbf{R}^n_-\neq\emptyset$. For such Q, let Q' be a cube so that its center lies on $\partial\mathbf{R}^n_+$, $\ell(Q')=2\ell(Q)$ and $Q\subset Q'$. Further, let Q^* be the smallest cube in $\mathbf{R}^n_+$ containing the upper half of Q'. With these notations, the desired inequality is proved from

$$\inf_{d\in\mathbf{R}}\frac{1}{|Q|}\int_Q|E[f]-d|\,\mathrm{d}y\le c\inf_{d\in\mathbf{R}}\frac{1}{|Q^*|}\int_{Q^*}|f-d|\mathrm{d}y.$$

□

2.2. **Sharp maximal operator.** For the proof of Theorem 1.3, we make use of the sharp maximal operator $M^\sharp$ due to Fefferman and Stein ([18]). We define for $x\in\mathbf{R}^n$ and $f\in L^1_{loc}(\mathbf{R}^n)$ the function $M^\sharp f$ by

$$M^\sharp f(x):=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|\mathrm{d}y.$$

It is immediate from the definition that $[f : BMO^\infty(\mathbf{R}^n)] = \|M^\sharp f\|_{L^\infty(\mathbf{R}^n)}$. It is well-known that if $f \in L^{p_0}(\mathbf{R}^n)$ for some $p_0 \in (1,\infty)$, then for $p \in [p_0,\infty)$

$$\|f\|_{L^p(\mathbf{R}^n)} \leq c\|M^\sharp f\|_{L^p(\mathbf{R}^n)}, \tag{2.2}$$

which is applied below. (Both sides of (2.2) may be infinite.) This follows from $\|f\|_{L^p(\mathbf{R}^n)} \leq \|Mf\|_{L^p(\mathbf{R}^n)}$ and $\|Mf\|_{L^p(\mathbf{R}^n)} \leq c\|M^\sharp f\|_{L^p(\mathbf{R}^n)}$, where M is the Hardy-Littlewood maximal operator [18].

2.3. **Marcinkiewicz interpolation.** Here, we give a variant of the Marcinkiewicz interpolation theorem.

Proposition 2.3. *Let D be an open subset of $\mathbf{R}^n$ and S a sublinear operator from $C_c(D)$ to $L^2(\mathbf{R}^n)$. If*

$$\|S[f]\|_{L^2(\mathbf{R}^n)} \leq c\|f\|_{L^2(D)}$$

$$\|S[f]\|_{L^\infty(\mathbf{R}^n)} \leq c\|f\|_{L^\infty(D)}$$

for $f \in C_c(D)$, then $\|S[f]\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(D)}$ for $f \in C_c(D)$ with C depending only on c and $p \in (2,\infty)$.

Proof. For $\lambda > 0$ and $\alpha > 0$, we decompose f into two parts; $f = f_2 + f_\infty$ where

$$f_2(x) = \begin{cases} 0 \quad \text{if} \quad |f(x)| \leq \alpha\lambda \\ f(x) - \alpha\lambda \operatorname{sign}(f(x)) \quad \text{if} \quad |f(x)| > \alpha\lambda, \end{cases}$$

where sign $\xi = \xi/|\xi|$ for $\xi \neq 0$ and sign $\xi = 0$ for $\xi = 0$. Observe that $f_2,\ f_\infty \in BC(D)$, and then $f_2,\ f_\infty \in C_c(D)$. Therefore, the two inequalities of our assumption hold for f_2 and f_∞, respectively. We set $\alpha = \left(2\|S\|_{L^\infty(D)\to L^\infty(\mathbf{R}^n)}\right)^{-1}$ and observe that $|\{x \in \mathbf{R}^n \mid S[f_\infty](x) > \lambda/2\}| = 0$. We now conclude that

$$\begin{aligned}
\int_{\mathbf{R}^n} |S[f]|^p \, dx &\leq p\int_0^\infty \lambda^{p-1} \left|\{x \in \mathbf{R}^n \mid |S[f](x)| > \lambda\}\right| \mathrm{d}\lambda \\
&\leq p\int_0^\infty \lambda^{p-1} \left|\{x \in \mathbf{R}^n \mid |S[f_2](x)| > \lambda/2\}\right| \mathrm{d}\lambda \\
&\leq p\int_0^\infty \lambda^{p-1} \left(\frac{2}{\lambda}\|S\|_{L^2(D)\to L^2(\mathbf{R}^n)}\|f_2\|_{L^2(D)}\right)^2 \mathrm{d}\lambda \\
&\leq c\int_0^\infty \lambda^{p-3} \int_{\{|f|>\alpha\lambda\}} |f(x)|^2 \, \mathrm{d}x \, \mathrm{d}\lambda \\
&= 2c\int_0^\infty \lambda^{p-3} \left(\int_{\alpha\lambda}^\infty t \left|\{x \in \mathbf{R}^n \mid |f(x)| > t\}\right| \mathrm{d}t\right) \mathrm{d}\lambda \\
&= 2c\int_0^\infty t \left|\{x \in \mathbf{R}^n \mid |f(x)| > t\}\right| \left(\int_0^{t/\alpha} \lambda^{p-3} \mathrm{d}\lambda\right) \mathrm{d}t \\
&\leq c\|f\|_{L^p(D)}^p.
\end{aligned}$$

$\square$

2.4. **Proof of Theorem 1.3.** For simplicity, we write $g := Tf$. By changing variables, one obtains

$$\int_{\Omega} |g|^p \mathrm{d}x \le c \int_{\mathbf{R}^n_+} |(F^{-1})^* g|^p \mathrm{d}y \le c \int_{\mathbf{R}^n} |E[(F^{-1})^* g]|^p \mathrm{d}y \le c \int_{\mathbf{R}^n} |\Phi[f]|^p \mathrm{d}y,$$

where $\Phi[f] := M^\sharp \left(E[(F^{-1})^* g]\right)$. Here, because $E[(F^{-1})^* g] \in L^2(\mathbf{R}^n)$, we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see $L^2(\Omega) - L^2(\mathbf{R}^n)$ and $L^\infty(\Omega) - L^\infty(\mathbf{R}^n)$ estimates for Φ. The former estimate can be seen by L^2-boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The later one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

3. Non-Helmholtz projection

Our goal in this section is to prove Theorem 1.4.

3.1. **A solution operator to the divergence problem.** As in Section 2, let $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\, x_n > h(x')\}$ be a Lipschitz half-space in $\mathbf{R}^n$ with a Lipschitz continuous function h on $\mathbf{R}^{n-1}$. Then, there is a closed cone of the form

$$C_1 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ -x_n \ge |x| \cos(2\theta)\}$$

with an angle $\theta \in (0, \pi/4)$ (depending on the Lipschitz constant of h) such that

$$x + C_1 = \{y \in \mathbf{R}^n \mid y - x \in C_1\} \subset \Omega^c \; (:= \mathbf{R}^n \setminus \Omega) \quad \text{for all} \quad x \in \Omega^c.$$

In the notion of the introduction $C_1 = C(4\theta)$ so that the opening angle equals 4θ. With this angle we define a closed cone $C_0 = C(2\theta)$, i.e.,

$$C_0 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ -x_n \ge |x| \cos\theta\}.$$

The closed cone C_0 also satisfies

$$x + C_0 \subset \Omega^c \quad \text{for all} \quad x \in \Omega^c. \tag{3.1}$$

Let $L \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$\operatorname{supp} L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \qquad \int_{S^{n-1}} L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = 1. \tag{3.2}$$

Here $-C_0 = \{-y \mid y \in C_0\}$ and S^{n-1} is the unit sphere in $\mathbf{R}^n$. Then we define a vector field $K = (K_1, \dots, K_n)$ as

$$K(x) := \frac{x}{|x|^n} L\left(\frac{x}{|x|}\right), \quad x \in \mathbf{R}^n \setminus \{0\}. \tag{3.3}$$

Definition 3.1. For $f \in C_c^\infty(\Omega)$, we define a vector field $u = Sf$ as

$$u(x) = Sf(x) := (K * \bar{f})(x) = \int_{\mathbf{R}^n} K(x-y) \bar{f}(y)\, \mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Here $\bar{f}$ denotes the zero extension of f to $\mathbf{R}^n$ given by

$$\bar{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}$$

This operator was introduced by Solonnikov [36]. For a fixed $x \in \mathbf{R}^n$, since

$$\frac{x-y}{|x-y|} \in \operatorname{supp} L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)$$

implies $y \in x + C_0$, we can write

$$u(x) = \int_{x+C_0} K(x-y)\bar{f}(y)\,\mathrm{d}y.$$

This formula and the property (3.1) of Ω imply that $u(x) = 0$ for all $x \in \Omega^c$. In particular, u vanishes on $\partial\Omega$. However, the support of u may become unbounded although f is compactly supported in Ω.

By the change of variables $x - y = r\sigma$ with $r > 0$ and $\sigma \in S^{n-1}$ we have

$$u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma)\bar{f}(x-r\sigma)r^{n-1}\mathrm{d}\mathcal{H}^{n-1}(\sigma)\,\mathrm{d}r.$$

Hence if $f \in C_c^\infty(\Omega)$ is supported in $B_R(0)$ and $x \in B_a(0)$ $(R, a > 0)$, then

$$u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma)\bar{f}(x-r\sigma)r^{n-1}\mathrm{d}\mathcal{H}^{n-1}(\sigma)\,\mathrm{d}r,$$

which implies that $u = Sf$ is smooth in Ω. Moreover, $u = Sf$ vanishes near $\partial\Omega$ and thus it is smooth in the whole space $\mathbf{R}^n$, since f is compactly supported in Ω.

Lemma 3.2. *Let $p \in (1,\infty)$. There exists a constant $c > 0$ such that*

$$\|\nabla u\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)}$$

for all $f \in C_c^\infty(\Omega)$ and $u = Sf$.

Proof. Let u_i be the i-th component of u:

$$u_i(x) = (K_i * \bar{f})(x) = \int_{\mathbf{R}^n} K_i(z)\bar{f}(x-z)\,\mathrm{d}z.$$

Differentiating both sides with respect to the j-th variable, we have

$$\partial_j u_i(x) = \int_{\mathbf{R}^n} K_i(z)(\partial_j \bar{f})(x-z)\,\mathrm{d}z = \lim_{\varepsilon\to 0}\int_{\mathbf{R}^n\setminus B_\varepsilon(0)} K_i(z)(\partial_j\bar{f})(x-z)\,\mathrm{d}z$$

and, by changing variables $y = x - z$ and integrating by parts,

$$\partial_j u_i(x) =$$
$$\lim_{\varepsilon\to 0}\left(\int_{\partial B_\varepsilon(x)} K_i(x-y)\frac{x_j-y_j}{|x-y|}\bar{f}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\mathbf{R}^n\setminus B_\varepsilon(x)} (\partial_j K_i)(x-y)\bar{f}(y)\,\mathrm{d}y\right).$$

On the one hand, we change variables $x - y = \varepsilon\sigma$ with $\sigma \in S^{n-1}$ to get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{|x-y|=\varepsilon} K_i(x-y)\frac{x_j-y_j}{|x-y|}\bar{f}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&= \lim_{\varepsilon\to 0}\int_{|x-y|=\varepsilon} \frac{x_i-y_i}{|x-y|}\frac{x_j-y_j}{|x-y|}L\left(\frac{x-y}{|x-y|}\right)\bar{f}(y)\frac{1}{|x-y|^{n-1}}\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&= \lim_{\varepsilon\to 0}\int_{S^{n-1}} \sigma_i\sigma_j L(\sigma)\bar{f}(x-\varepsilon\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma)\\
&= \bar{f}(x)\int_{S^{n-1}} \sigma_i\sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma),
\end{aligned}$$

where the last equality follows from the fact that L is integrable on S^{n-1} and $\bar{f}$ is continuous at x. On the other hand, we differentiate K_i to obtain

$$\begin{aligned} K_{ij}(z) &:= \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n}, \\ k_{ij}(z) &:= (\delta_{ij} - n z_i z_j) L(z) + z_i (\partial_j L)(z) - z_i z_j \sum_{\ell=1}^{n} z_\ell (\partial_\ell L)(z) \end{aligned} \tag{3.4}$$

for $z \in \mathbf{R}^n \setminus \{0\}$. Then K_{ij} is homogeneous of degree $-n$ and there is a constant $c > 0$ such that

$$|K_{ij}(z)| \le \frac{c}{|z|^n} \quad \text{for all} \quad z \in \mathbf{R}^n \setminus \{0\}$$

by the smoothness of L on S^{n-1}. Moreover, for every R_1 and R_2 with $0 < R_1 < R_2$,

$$\begin{aligned} &\int_{R_1<|z|<R_2} K_{ij}(z)\,\mathrm{d}z = \int_{R_1<|z|<R_2} \partial_j K_i(z)\,\mathrm{d}z \\ &= \int_{|z|=R_2} K_i(z) \frac{z_j}{|z|}\, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} K_i(z) \frac{z_j}{|z|}\,\mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \int_{|z|=R_2} \frac{z_i}{|z|}\frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}}\,\mathrm{d}\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} \frac{z_i}{|z|}\frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}}\,\mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = 0. \end{aligned}$$

In the fourth equality we changed variables $z = R_2\sigma$ and $z = R_1\sigma$ with $\sigma \in S^{n-1}$, respectively. This equality is equivalent to

$$\int_{S^{n-1}} k_{ij}(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = 0. \tag{3.5}$$

Thus we can apply the Calderón-Zygmund theory (see eg. [23, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel K_{ij} and obtain the formula

$$\partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} K_{ij}(x-y) \bar{f}(y)\,\mathrm{d}y, \tag{3.6}$$

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) \right| \le |\bar{f}(x)| \int_{S^{n-1}} L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \le c \|\bar{f}\|_{L^p(\mathbf{R}^n)} = c\|f\|_{L^p(\Omega)}$$

with a positive constant c independent of f. Hence the lemma follows. □

Lemma 3.3. *For every $f \in C_c^\infty(\Omega)$ the vector field $u = Sf$ satisfies*

$$\operatorname{div} u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Proof. We have already observed that u vanishes on the boundary. Let us compute $\operatorname{div} u = \sum_{i=1}^n \partial_i u_i$ in Ω. By the formula (3.6) in the proof of Lemma 3.2,

$$\operatorname{div} u(x) = \bar{f}(x) \int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} \sum_{i=1}^n K_{ii}(x-y) \bar{f}(y)\, \mathrm{d}y.$$

In this formula, we have

$$\int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = 1$$

by (3.2) and, for all $z \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned}\sum_{i=1}^n K_{ii}(z) = {}& \frac{1}{|z|^n} L\left(\frac{z}{|z|}\right) \sum_{i=1}^n \left(1 - n\frac{z_i^2}{|z|^2}\right) \\ &+ \frac{1}{|z|^n} \sum_{i=1}^n \frac{z_i}{|z|} (\partial_i L)\left(\frac{z}{|z|}\right) - \sum_{i=1}^n \frac{z_i^2}{|z|^{n+2}} \sum_{k=1}^n \frac{z_k}{|z|} (\partial_k L)\left(\frac{z}{|z|}\right) = 0.\end{aligned}$$

Hence $\operatorname{div} u(x) = \bar{f}(x) = f(x)$ for all $x \in \Omega$. □

Lemma 3.3 means that the operator S is a solution operator to the divergence problem with Dirichlet boundary condition. Note that S is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

Definition 3.4. For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field Tu as

$$Tu(x) := \int_{\mathbf{R}^n} K(x-y) \overline{\operatorname{div} u}(y)\, \mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Here K is given by (3.3) and $\overline{\operatorname{div} u}$ denotes the zero extension of $\operatorname{div} u$ to $\mathbf{R}^n$.

The above definition means that T is given by $T = S \circ \operatorname{div}$. Since $u \in C_c^\infty(\Omega)$, its divergence is in $C_c^\infty(\Omega)$ and thus Tu is smooth in the whole space $\mathbf{R}^n$ and vanishes outside of Ω, as discussed right after Definition 3.1. Also, by Lemma 3.3 we have

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.$$

Clearly $Tu = 0$ in $\mathbf{R}^n$ for $u \in C_{c,\sigma}^\infty(\Omega)$. Note that, as in the case of the operator S, the support of Tu may be unbounded.

Theorem 3.5. *Let Ω be a Lipschitz half-space. Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that*

$$\|Tu\|_{L^p(\Omega)} \le c \|u\|_{L^p(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Let us compute the i-th component $(Tu)_i$ of Tu with $i = 1, \ldots, n$ for compactly supported vector field u in Ω. As in the proof of Lemma 3.2, we integrate by parts to get

$$\begin{aligned}(Tu)_i(x) &= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} K_i(x-y)\frac{x-y}{|x-y|}\cdot \bar{u}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad + \lim_{\varepsilon\to 0}\int_{\mathbf{R}^n\setminus B_\varepsilon(x)} (\nabla K_i)(x-y)\cdot \bar{u}(y)\,\mathrm{d}y\\ &= \int_{S^{n-1}} \sigma_i L(\sigma)\{\sigma\cdot \bar{u}(x)\}\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} (\nabla K_i)(x-y)\cdot \bar{u}(y)\,\mathrm{d}y,\end{aligned}$$

or equivalently,

$$(Tu)_i(x) = \sum_{j=1}^{n}\{a_{ij}\bar{u}_j(x) + S_{ij}\bar{u}_j(x)\}, \quad x\in\mathbf{R}^n. \tag{3.7}$$

Here u_j is the j-th component of u and

$$a_{ij} = \int_{S^{n-1}} \sigma_i\sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma), \quad S_{ij}\bar{u}_j(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\bar{u}_j(y)\,\mathrm{d}y,$$

where $K_{ij} = \partial_j K_i$ is given by (3.4). Since a_{ij} is a constant satisfying

$$|a_{ij}| \le \int_{S^{n-1}} L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = 1 \tag{3.8}$$

and $S_{ij}\bar{u} = K_{ij} * \bar{u}$ is a singular integral (see the proof of Lemma 3.2), the Calderón-Zygmund theory yields the boundedness of the operator T on $L^p(\Omega)$. $\square$

By Theorem 3.5, the operator T extends uniquely to a bounded linear operator on $L^p(\Omega)$ with each $p\in(1,\infty)$, which we again refer to as T.

Our next goal is to estimate the $BMO_b^{\infty,\nu}(\Omega)$-norm of Tu for $u\in C_c^\infty(\Omega)$ and $\nu\in(0,\infty]$. To this end, we estimate each term of the right-hand side in (3.7) for $u=(u_1,\dots,u_n)\in C_c^\infty(\Omega)$. By (3.8) we have

$$[a_{ij}\bar{u}_j : BMO^\infty(\Omega)] \le [u_j : BMO^\infty(\Omega)], \quad [a_{ij}\bar{u}_j : b^\nu(\Omega)] \le [u_j : b^\nu(\Omega)]$$

and thus

$$\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| \le \|u_j : BMO_b^{\infty,\nu}(\Omega)\|.$$

Moreover, since

$$[u_j : BMO^\infty(\Omega)] \le 2\|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \le \omega_n\|u_j\|_{L^\infty(\Omega)},$$

where $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$ is the volume of the unit ball $B_1(0)$ in $\mathbf{R}^n$ with the Gamma function $\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$, we have

$$\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| \le (2+\omega_n)\|u_j\|_{L^\infty(\Omega)}. \tag{3.9}$$

Let us estimate $S_{ij}\bar{u}_j = K_{ij} * \bar{u}_j$, $i,j=1,\dots,n$ in $BMO_b^{\infty,\nu}(\Omega)$. Recall that the integral kernel K_{ij} is of the form

$$K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x\in\mathbf{R}^n\setminus\{0\},$$

where $k_{ij}\in C_c^\infty(\mathbf{R}^n)$ is given by (3.4) and satisfies

$$\operatorname{supp} k_{ij} \subset (B_2(0)\setminus B_{1/2}(0))\cap(-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma)\,\mathrm{d}\mathcal{H}^{n-1} = 0,$$

see (3.2) and (3.5). We first estimate the BMO^∞-seminorm of $S_{ij}\bar{u}_j$.

Lemma 3.6. *Let K be a function defined on $\mathbf{R}^n \setminus \{0\}$ such that*

$$|K(x-y)-K(x)| \le A|y|^\delta |x|^{-n-\delta} \quad \text{whenever} \quad |x| \ge 2|y| > 0 \tag{3.10}$$

for some $A, \delta > 0$. Suppose that a convolution operator S with K is bounded on $L^2(\mathbf{R}^n)$ with a norm B. Then, there exists a dimensional constant c_n such that

$$[Sf : BMO^\infty(\mathbf{R}^n)] \le c_n(A+B)\|f\|_{L^\infty(\mathbf{R}^n)}$$

for all $f \in L^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

Proof. See [24, Theorem 3.4.9 and Corollary 3.4.10]. □

Lemma 3.7. *There exists a constant $c > 0$ such that*

$$[S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \le c\|u_j\|_{L^\infty(\Omega)} \tag{3.11}$$

for all $u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \ldots, n$.

Proof. We shall apply Lemma 3.6 to $S = S_{ij}$. For this purpose it is sufficient to show that the function $K = K_{ij}$ satisfies (3.10), since we already know that the convolution operator S_{ij} is bounded on $L^2(\mathbf{R}^n)$, see the proof of Lemma 3.2. To this end, we differentiate K_{ij} to get

$$\nabla K_{ij}(x) = -\frac{nk_{ij}(x/|x|)}{|x|^{n+1}}\frac{x}{|x|} + \frac{1}{|x|^{n+1}}\left(I_n - \frac{1}{|x|^2}x \otimes x\right)\nabla k_{ij}\left(\frac{x}{|x|}\right)$$

for $x \in \mathbf{R}^n \setminus \{0\}$, where I_n is the identity matrix of size n and $x \otimes x := (x_i x_j)_{i,j}$ is the tensor product of x. Since k_{ij} is smooth on S^{n-1}, we have

$$|\nabla K_{ij}(x)| \le \frac{c}{|x|^{n+1}}, \quad x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for all $x, y \in \mathbf{R}^n \setminus \{0\}$ with $|x| \ge 2|y| > 0$,

$$\begin{aligned}
|K(x-y)-K(x)| &= \left|\int_0^1 \frac{d}{dt}(K(x-ty))\,\mathrm{d}t\right| = \left|\int_0^1 (-y)\cdot\nabla K(x-ty)\,\mathrm{d}t\right| \\
&\le |y|\int_0^1 \frac{c}{|x-ty|^{n+1}}\,\mathrm{d}t \le |y|\int_0^1 \frac{c}{(|x|-|y|)^{n+1}}\,\mathrm{d}t \\
&\le \frac{c|y|}{(|x|-|x|/2)^{n+1}} = \frac{2^{n+1}c|y|}{|x|^{n+1}}.
\end{aligned}$$

Thus K_{ij} satisfies (3.10) with $\delta = 1$ and we can apply Lemma 3.6 to obtain

$$[S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)] \le c\|\bar{u}_j\|_{L^\infty(\mathbf{R}^n)} = c\|u_j\|_{L^\infty(\Omega)} \tag{3.12}$$

with some constant $c > 0$.

By definition of the BMO^∞-seminorm, we have

$$[S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \le [S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)].$$

Hence the inequality (3.11) follows from (3.12). □

Next, let us estimate the b^ν-part of $S_{ij}\bar{u}_j$. Recall the two closed cones

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ -x_n \ge |x|\cos(2^j\theta)\}, \quad j = 0, 1$$

with opening angle $\theta \in (0, \pi/4)$. For $r > 0$ and $x_0 \in \mathbf{R}^n$, we define

$$A_r(x_0) := \bigcup_{x \in B_r(x_0)\cap(x_0+C_1)^c} (x + C_0) \cap (x_0 + C_1)^c \subset \mathbf{R}^n. \tag{3.13}$$

Here $x_0 + C_1 = \{y \in \mathbf{R}^n \mid y - x_0 \in C_1\}$ and $x + C_0$ is defined similarly.

Lemma 3.8. *For all $r > 0$ and $x_0 \in \mathbf{R}^n$ we have $A_r(x_0) \subset B_{r/\sin\theta}(x_0)$.*

Proof. By translation, we may assume that $x_0 = 0$. Let $a := (0, \dots, 0, r/\sin\theta) \in \mathbf{R}^n$. Suppose that

(1) $B_r(0) \subset a + C_0$,
(2) $x + C_0 \subset a + C_0$ for all $x \in a + C_0$,
(3) $(a + C_0) \cap C_1^c \subset B_{r/\sin\theta}(0)$.

Then, the statements (1) and (2) imply

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.$$

Hence the statement (3) yields $A_r(0) \subset B_{r/\sin\theta}(0)$. Now let us prove the statements (1)-(3). Note that, since $\theta \in (0, \pi/4)$, the cones C_0 and C_1 are represented as

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ x_n \leq 0,\ |x'| \leq (-x_n)\tan(2^j\theta)\}, \quad j = 0, 1.$$

(1) Let $x = (x', x_n) \in B_r(0)$. Then, $x - a = (x', x_n - r/\sin\theta)$ satisfies

$$(x - a)_n = x_n - \frac{r}{\sin\theta} \leq r - \frac{r}{\sin\theta} < 0$$

and

$$\left(\frac{r}{\sin\theta} - x_n\right)^2 \tan^2\theta - |x'|^2 \geq \frac{(r - x_n \sin\theta)^2}{\cos^2\theta} - (r^2 - x_n^2) = \frac{(r\sin\theta - x_n)^2}{\cos^2\theta} \geq 0,$$

or equivalently,

$$|x'| \leq \left(\frac{r}{\sin\theta} - x_n\right)\tan\theta = -(x - a)_n \tan\theta.$$

Hence $x - a \in C_0$, that is, $x \in a + C_0$ and the statement (1) holds.

(2) Let $x \in a + C_0$. If $y \in x + C_0$, then $(y - a)_n = (y - x)_n + (x - a)_n \leq 0$ and

$$|y'| \leq |x'| + |y' - x'| \leq -(x - a)_n \tan\theta - (y - x)_n \tan\theta = -(y - a)_n \tan\theta,$$

which means that $y \in a + C_0$. Hence the statement (2) holds.

(3) Let $x \in (a + C_0) \cap C_1^c$. Then we have

$$(x - a)_n = x_n - r/\sin\theta \leq 0, \quad |x'| \leq \left(\frac{r}{\sin\theta} - x_n\right)\tan\theta. \tag{3.14}$$

Hence

$$|x|^2 \leq \left(\frac{r}{\sin\theta} - x_n\right)^2 \tan^2\theta + x_n^2 =: f(x_n).$$

To estimate the right-hand side in the above inequality for $x \in (a + C_0) \cap C_1^c$, we derive the range of x_n for $x \in (a + C_0) \cap C_1^c$. If $x_n \geq 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if the condition (3.14) is satisfied. Thus x_n must satisfy

$$0 \leq x_n \leq \frac{r}{\sin\theta}.$$

On the other hand, if $x_n < 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if

$$(-x_n)\tan(2\theta) < |x'| \leq \left(\frac{r}{\sin\theta} - x_n\right)\tan\theta.$$

Hence, in particular, if $x \in (a + C_0) \cap C_1^c$ and $x_n < 0$, then x_n must satisfy

$$(-x_n)\tan(2\theta) < \left(\frac{r}{\sin\theta} - x_n\right)\tan\theta,$$

which yields the inequality

$$-\frac{r}{\cos\theta} < (\tan(2\theta) - \tan\theta)\, x_n.$$

Since

$$\begin{aligned}\tan(2\theta) - \tan\theta &= \tan(2\theta) - \frac{1}{2}\tan(2\theta)(1-\tan^2\theta)\\ &= \frac{1}{2}\tan(2\theta)(1+\tan^2\theta) = \frac{\tan(2\theta)}{2\cos^2\theta} > 0 \quad \left(0<\theta<\frac{\pi}{4}\right),\end{aligned}$$

the above inequality is equivalent to

$$-\frac{2r\cos\theta}{\tan(2\theta)} < x_n (< 0).$$

In summary, the range of x_n for $x \in (a + C_0) \cap C_1^c$ is

$$\alpha := -\frac{2r\cos\theta}{\tan(2\theta)} < x_n \le \frac{r}{\sin\theta} =: \beta$$

and thus we obtain

$$|x|^2 \le f(x_n) \le \sup_{s\in(\alpha,\beta]} f(s) = \max\{f(\alpha), f(\beta)\},$$

where the last equality follows from the fact that $f(x_n)$ is a concave parabola. On the one hand, we have $f(\beta) = \beta^2 = r^2/\sin^2\theta$. On the other hand, since

$$\alpha = -\frac{2r\cos\theta\cos(2\theta)}{\sin(2\theta)} = -\frac{r\cos(2\theta)}{\sin\theta} = \frac{r(1-2\cos^2\theta)}{\sin\theta},$$

we have

$$\begin{aligned}f(\alpha) &= \left(\frac{r}{\sin\theta} - \frac{r(1-2\cos^2\theta)}{\sin\theta}\right)^2\tan^2\theta + \frac{r^2\cos^2(2\theta)}{\sin^2\theta}\\ &= \frac{r^2}{\sin^2\theta}\{4\tan^2\theta\cos^4\theta + \cos^2(2\theta)\} = \frac{r^2}{\sin^2\theta}.\end{aligned}$$

Hence $|x|^2 \le r^2/\sin^2\theta$ and thus $x \in B_{r/\sin\theta}(0)$ for every $x \in (a + C_0) \cap C_1^c$. Therefore, the statement (3) holds and the lemma follows. □

Now we can estimate the b^ν-part of $S_{ij}\bar{u}_j$.

Lemma 3.9. *Let $\nu \in (0,\infty]$. There exists a constant $c > 0$ such that*

$$[S_{ij}\bar{u}_j : b^\nu(\Omega)] \le \frac{c}{\sin^{n/2}\theta}\|u_j\|_{L^\infty(\Omega)} \tag{3.15}$$

for all $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \dots, n$.

Proof. First we note that for all $f \in L^1_{loc}(\Omega)$ the inequality

$$[f : b^\nu(\Omega)] \le \omega_n^{1/2}\,[f : b_2^\nu(\Omega)]$$

holds by Hölder's inequality. Hence, to prove (3.15), it is sufficient to show the inequality

$$[S_{ij}\bar{u}_j : b_2^\nu(\Omega)] \le \frac{c}{\sin^{n/2}\theta}\left[u_j : b_2^{\nu/\sin\theta}(\Omega)\right] \le \frac{c\omega_n^{1/2}}{\sin^{n/2}\theta}\|u_j\|_{L^\infty}. \tag{3.16}$$

The second inequality of (3.16) follows from the definition of $[\,\cdot : b_2^{\nu/\sin\theta}(\Omega)]$. Let us show the first inequality. The singular integral $S_{ij}\bar{u}_j$ is of the form

$$S_{ij}\bar{u}_j(x) = (K_{ij} * \bar{u}_j)(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\bar{u}_j(y)\,\mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Since $\operatorname{supp} K_{ij} \subset -C_0$ (see (3.4) and (3.2)) and $\operatorname{supp} u \subset \Omega$, we can write

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)\bar{u}_j(y)\,\mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Hence, if we set

$$W_r(x_0) := \bigcup_{x \in B_r(x_0)\cap\Omega} (x+C_0)\cap\Omega$$

for each $x_0 \in \partial\Omega$ and $r > 0$ with $B_r(x_0) \subset U_\nu(\partial\Omega)$, then we have

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)(\bar{u}_j|_{W_r(x_0)})(y)\,dy = [K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)$$

for all $x \in B_r(x_0)\cap\Omega$, where

$$(\bar{u}_j|_{W_r(x_0)})(x) := \begin{cases} \bar{u}_j(x), & x \in W_r(x_0), \\ 0, & x \notin W_r(x_0). \end{cases}$$

Since K_{ij} is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that

$$\begin{aligned} \int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2\,\mathrm{d}x &= \int_{B_r(x_0)\cap\Omega} |[K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)|^2\,\mathrm{d}x \\ &\le c\int_{\mathbf{R}^n} |(\bar{u}_j|_{W_r(x_0)})(x)|^2\,\mathrm{d}x = c\int_{W_r(x_0)} |\bar{u}_j(x)|^2\,\mathrm{d}x \end{aligned}$$

with some constant $c > 0$. Now we recall the property of the infinite cone C_1:

$$x + C_1 \subset \Omega^c \Leftrightarrow \Omega \subset (x+C_1)^c \quad \text{for all} \quad x \in \Omega^c.$$

By this property we have

$$W_r(x_0) \subset \bigcup_{x \in B_r(x_0)\cap(x_0+C_1)^c} (x+C_0)\cap((x_0+C_1)^c\cap\Omega) = A_r(x_0)\cap\Omega,$$

where $A_r(x_0)$ is given by (3.13), and thus Lemma 3.8 yields

$$W_r(x_0) \subset A_r(x_0)\cap\Omega \subset B_{r/\sin\theta}(x_0)\cap\Omega.$$

Hence we have

$$\begin{aligned} &\frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2\,\mathrm{d}x \le \frac{c}{r^n}\int_{W_r(x_0)} |\bar{u}_j(x)|^2\,\mathrm{d}x \\ &\le \frac{c}{r^n}\int_{B_{r/\sin\theta}(x_0)\cap\Omega} |\bar{u}_j(x)|^2\,\mathrm{d}x = \frac{c}{\sin^n\theta}\left(\frac{\sin\theta}{r}\right)^n \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |u_j(x)|^2\,\mathrm{d}x \\ &\le \frac{c}{\sin^n\theta}\left[u_j : b_2^{\nu/\sin\theta}(\Omega)\right]^2 \end{aligned}$$

for every $x_0 \in \partial\Omega$ and $r > 0$ with $B_r(x_0) \subset U_\nu(\partial\Omega)$, which yields

$$[S_{ij}\bar{u}_j : b_2^{\nu}(\Omega)]^2 \le \frac{c}{\sin^n\theta}\left[u_j : b_2^{\nu/\sin\theta}(\Omega)\right]^2.$$

The proof is complete. $\square$

Now we obtain an estimate for the $BMO_b^{\infty,\nu}(\Omega)$-norm of Tu.

Theorem 3.10. *Let $\nu \in (0,\infty]$. There exists a constant $c > 0$ such that*

$$\|Tu : BMO_b^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Since the i-th component of Tu, $i = 1, \dots, n$, is of the form (3.7), we have by (3.9), (3.11) and (3.15) that

$$\begin{aligned}
&\|Tu : BMO_b^{\infty,\nu}(\Omega)\| \\
&\leq c \sum_{i,j=1}^{n} (\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| + [S_{ij}\bar{u}_j : BMO^\infty(\Omega)] + [S_{ij}\bar{u}_j : b^\nu(\Omega)]) \\
&\leq c \sum_{j=1}^{n} \|u_j\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}
\end{aligned}$$

with a positive constant c. □

3.2. **Non-Helmholtz projection.** As in the previous subsection, let Ω denote a Lipschitz half-space in $\mathbf{R}^n$.

Definition 3.11. For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field $Q'u$ on $\mathbf{R}^n$ as $Q'u := u - Tu$. Here the operator T is given in Definition 3.4.

For a vector field $u \in C_c^\infty(\Omega)$, the vector field Tu is smooth in $\mathbf{R}^n$ and

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover, $Tu = 0$ for all $u \in C_{c,\sigma}^\infty(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbf{R}^n$ and

$$\operatorname{div} Q'u = 0 \quad \text{in} \quad \Omega, \quad Q'u = 0 \quad \text{on} \quad \partial\Omega \tag{3.17}$$

for all $u \in C_c^\infty(\Omega)$, and $Q'u = u$ for all $u \in C_{c,\sigma}^\infty(\Omega)$. Note that Q' is not a projection from $C_c^\infty(\Omega)$ onto $C_{c,\sigma}^\infty(\Omega)$, since the support of Tu may be unbounded and thus $Q'u$ is not in $C_{c,\sigma}^\infty(\Omega)$ in general. However, Q' maps $C_c^\infty(\Omega)$ into $L_\sigma^p(\Omega)$.

Lemma 3.12. *For all $u \in C_c^\infty(\Omega)$ and $p \in (1,\infty)$, we have $Q'u \in L_\sigma^p(\Omega)$.*

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1,\infty)$, let $G_p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L_{loc}^1(\Omega)\}$.

Proposition 3.13. *Let $p \in (1,\infty)$. For every $\nabla q \in G_p(\Omega)$, there exists a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbf{R}^n)$ such that*

$$\lim_{k\to\infty} \|\nabla q - \nabla q_k\|_{L^p(\Omega)} = 0. \tag{3.18}$$

Proof. Since the restriction of $C_c^\infty(\mathbf{R}^n)$ on Ω is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G_p(\Omega)$ there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\Omega)$ such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space $\mathbf{R}_+^n$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$. As in Section 2, let $F(x) := (x', x_n - h(x'))$ be a bi-Lipschitz map from Ω to $\mathbf{R}_+^n$. Let $\nabla q \in G_p(\Omega)$ and $\widetilde{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y', y_n + h(y'))$ is the inverse mapping of F. Then, since $\nabla\widetilde{q}(y) = \nabla F^{-1}(y)\nabla q(F^{-1}(y))$ for $y \in \mathbf{R}_+^n$ and each component

of ∇F^{-1} is bounded (because h is Lipschitz continuous), we have $\nabla \widetilde{q} \in G_p(\mathbf{R}^n_+)$. Hence, by our assumption that the claim is valid for $\mathbf{R}^n_+$, there is a sequence $\{\widetilde{q}_k\}_{k=1}^{\infty}$ of functions in $W^{1,p}(\mathbf{R}^n_+)$ such that $\lim_{k\to\infty} \|\nabla \widetilde{q} - \nabla \widetilde{q}_k\|_{L^p(\mathbf{R}^n_+)} = 0$.

Let $q_k := \widetilde{q}_k \circ F$ for each $k \in \mathbf{N}$. Then, since

$$\nabla q(x) = \nabla F(x) \nabla \widetilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x) \nabla \widetilde{q}_k(F(x)), \quad x \in \Omega$$

and each component of ∇F is bounded, we have $q_k \in W^{1,p}(\Omega)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \le c \|\nabla \widetilde{q} - \nabla \widetilde{q}_k\|_{L^p(\mathbf{R}^n_+)} \to 0$$

as $k \to \infty$. Thus the claim is valid for general Lipschitz half-spaces Ω.

(2) Now we prove the claim for $\Omega = \mathbf{R}^n_+$. We follow the idea of the proof of the claim in the case $\Omega = \mathbf{R}^n$, see [34, Lemma 2.5.4]. Let $\varphi \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$0 \le \varphi \le 1 \quad \text{in } \mathbf{R}^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } \mathbf{R}^n \setminus B_2(0)$$

and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbf{N}$ and $x \in \mathbf{R}^n$. Then, $\lim_{k\to\infty} \varphi_k(x) = 1$ for all $x \in \mathbf{R}^n$ and $\operatorname{supp} \varphi_k \subset B_{2k}(0)$, $\operatorname{supp} \nabla \varphi_k \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbf{N}$.

Let $\nabla q \in G_p(\mathbf{R}^n_+)$. Then $q \in W^{1,p}_{loc}(\overline{\mathbf{R}^n_+})$, that is, $q \in W^{1,p}(U)$ for every bounded subset U of $\mathbf{R}^n_+$; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting $G_k := \mathbf{R}^n_+ \cap (B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbf{N}$, we have $q \in W^{1,p}(G_k)$ and thus there is a constant a_k such that $\int_{G_k} (q - a_k)\, dx = 0$ for each $k \in \mathbf{N}$. From this equality and the change of variables $x = ky$ for $x \in G_k$ and $y \in G_1$ we have

$$\int_{G_1} (q(ky) - a_k)\, \mathrm{d}y = k^{-n} \int_{G_k} (q(x) - a_k)\, \mathrm{d}x = 0.$$

Hence we can apply Poincaré's inequality to $q(ky) - a_k$ on G_1 and get

$$\left(\int_{G_1} |q(ky) - a_k|^p\, \mathrm{d}y \right)^{1/p} \le c \left(\int_{G_1} |\nabla(q(ky))|^p\, \mathrm{d}y \right)^{1/p}$$

with a constant $c > 0$ independent of k. In this inequality, we observe that

$$\int_{G_1} |q(ky) - a_k|^p\, \mathrm{d}y = k^{-n} \int_{G_k} |q(x) - a_k|^p\, \mathrm{d}x,$$

$$\int_{G_1} |\nabla(q(ky))|^p\, \mathrm{d}y = k^p \int_{G_1} |(\nabla q)(ky)|^p\, \mathrm{d}y = k^{p-n} \int_{G_k} |\nabla q(x)|^p\, \mathrm{d}x$$

by the change of variables $x = ky$ and thus

$$\|q - a_k\|_{L^p(G_k)} \le ck \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbf{N}. \tag{3.19}$$

For each $k \in \mathbf{N}$, let $q_k := \varphi_k(q - a_k)$ on $\mathbf{R}^n_+$. Then since $\operatorname{supp} q_k \subset \mathbf{R}^n_+ \cap B_{2k}(0)$ holds by the relation $\operatorname{supp} \varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(\mathbf{R}^n_+)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}^n_+)} \le \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}^n_+)} + \|(\nabla \varphi_k)(q - a_k)\|_{L^p(\mathbf{R}^n_+)}. \tag{3.20}$$

Since $0 \le \varphi_k(x) \le 1$ and $\lim_{k\to\infty} \varphi_k(x) = 1$ for all $x \in \mathbf{R}^n_+$ and $\nabla q \in L^p(\mathbf{R}^n_+)$, the dominated convergence theorem yields

$$\lim_{k\to\infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}^n_+)} = 0. \tag{3.21}$$

On the other hand, since $\nabla \varphi_k = k^{-1}(\nabla \varphi)_k$ and $\operatorname{supp} \nabla \varphi_k|_{\mathbf{R}^n_+} \subset \overline{G_k}$ for each $k \in \mathbf{N}$, it follows from (3.19) and the dominated convergence theorem that

$$\|(\nabla \varphi_k)(q - a_k)\|_{L^p(\mathbf{R}^n_+)} \le ck^{-1} \|q - a_k\|_{L^p(G_k)} \le c \|\nabla q\|_{L^p(G_k)} \to 0 \tag{3.22}$$

as $k \to \infty$. Applying (3.21) and (3.22) to (3.20) we have

$$\lim_{k\to\infty} \|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}^n_+)} = 0,$$

where $q_k \in W^{1,p}(\mathbf{R}^n_+)$ for all $k \in \mathbf{N}$. Hence the claim is valid when $\Omega = \mathbf{R}^n_+$ and the proposition follows. □

Proof of Lemma 3.12. Let $u \in C_c^\infty(\Omega)$ and $p \in (1,\infty)$. Then, since $Tu \in L_p(\Omega)$ by Theorem 3.5, we have $Q'u = u - Tu \in L^p(\Omega)$. To show $Q'u \in L^p_\sigma(\Omega)$, we employ a characterization of elements of $L^p_\sigma(\Omega)$ ([19, Lemma Ⅲ.2.1]): a vector field $v \in L^p(\Omega)$ is in $L^p_\sigma(\Omega)$ if and only if

$$\int_\Omega v \cdot \nabla q \, \mathrm{d}x = 0 \quad \text{for all} \quad \nabla q \in G_{p'}(\Omega) \ \left(p' := \frac{p}{p-1}\right).$$

Let ∇q be any element of $G_{p'}(\Omega)$. From Proposition 3.13, there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbf{R}^n)$ such that the equality (3.18) with p replaced by p' holds. Since $Q'u$ is defined and smooth in $\mathbf{R}^n$ for $u \in C_c^\infty(\Omega)$ and $q_k \in C_c^\infty(\mathbf{R}^n)$, integration by parts yields

$$\int_\Omega Q'u \cdot \nabla q_k \, \mathrm{d}x = -\int_\Omega q_k \operatorname{div} Q'u \, \mathrm{d}x + \int_{\partial\Omega} q_k \, Q'u \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}$$

for all $k \in \mathbf{N}$, where ν denotes the unit outer normal vector field of $\partial\Omega$. We apply (3.17) to the right-hand side of this equality to get $\int_\Omega Q'u \cdot \nabla q_k \, dx = 0$ for all $k \in \mathbf{N}$. Since $Q'u \in L^p(\Omega)$ and (3.18) with p replaced by p' holds, the above equality implies that

$$\int_\Omega Q'u \cdot \nabla q \, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega Q'u \cdot \nabla q_k \, \mathrm{d}x = 0.$$

Hence by the characterization of elements of $L^p_\sigma(\Omega)$ we conclude that $Q'u \in L^p_\sigma(\Omega)$ for all $u \in C_c^\infty(\Omega)$. The proof is complete. □

Remark 3.14.

(1) Let $p \in (1,\infty)$. By Theorem 3.5 and Lemma 3.12, we have $Q'u \in L^p_\sigma(\Omega)$ and $\|Q'u\|_{L^p(\Omega)} \le c\|u\|_{L^p(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Moreover, $Q'u = u$ holds for all $u \in C^\infty_{c,\sigma}(\Omega)$. Hence, by the density argument, Q' extends uniquely to a bounded linear operator on $L^p(\Omega)$ that is a projection onto $L^p_\sigma(\Omega)$.

(2) The projection onto $L^p_\sigma(\Omega)$ given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each $u \in C_c^\infty(\Omega)$ there would exist $\pi \in L^1_{loc}(\Omega)$ such that $(I - Q')u = \nabla\pi$ holds. Since $(I - Q')u = Tu = K * \operatorname{div} u$ for $u \in C_c^\infty(\Omega)$, the existence of such π would imply that $\partial_j(K_i * \operatorname{div} u) = \partial_i(K_j * \operatorname{div} u)$ for all $i, j = 1, \dots, n$. For each $f \in C_c^\infty(\Omega)$ with $\int_\Omega f \mathrm{d}x = 0$ there is $u \in C_c^\infty(\Omega)$ satisfying $f = \operatorname{div} u$. This is possible since we are able to apply Bogovskiĭ's lemma to a bounded Lipschitz domain $D \subset \Omega$ containing the support of f (see [19, Theorem Ⅲ.3.3]). Thus the above equality would imply that $\partial_j K_i = \partial_i K_j + c$ with some constant c for all $i, j = 1, \dots, n$ as a distribution. This contradicts the fact that $\partial_j K_i \ne \partial_i K_j + c$ for $i \ne j$ as observed in (3.4).

(3) It is possible to prove the characterization

$$L^p_\sigma(\Omega) = \{u \in L^p(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial\Omega\}$$

if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains (see [19, Section III.2]). However, for a Lipschitz half-space, it is less popular. A proof can be found in [30, Lemma 2.1].

The linear operator Q' also maps $C_c^\infty(\Omega)$ into $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.

Lemma 3.15. *Let Ω be a Lipschitz half-space. For all $u \in C_c^\infty(\Omega)$ and $\nu \in (0,\infty]$, we have $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.*

We shall prove two auxiliary propositions for the above lemma. For $p \in (1,\infty)$, let $W_{0,\sigma}^{1,p}(\Omega)$ be the $W^{1,p}$-closure of $C_{c,\sigma}^\infty(\Omega)$.

Proposition 3.16. *Let Ω be a Lipschitz half-space. For all $p \in (1,\infty)$ we have $L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega) \subset W_{0,\sigma}^{1,p}(\Omega)$. Thus $L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega) = W_{0,\sigma}^{1,p}(\Omega)$.*

Proof. Let $\rho \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$0 \le \rho \le 1 \quad \text{in } \mathbf{R}^n, \quad \operatorname{supp} \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \, \mathrm{d}x = 1$$

and $\rho_\delta(x) := \delta^{-n}\rho(\delta^{-1}x)$ for $\delta > 0$, $x \in \mathbf{R}^n$. Let $u \in L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega)$. Then there is a sequence $\{u_k\}_{k=1}^\infty$ of functions in $C_{c,\sigma}^\infty(\Omega)$ such that $\lim_{k\to\infty} \|u - u_k\|_{L^p(\Omega)} = 0$. For $a > 0$, we define a vector field u^a on Ω as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ 0, & h(x') < x_n \le h(x') + a \end{cases}$$

and $u_k^a = (u_k)^a$ similarly. Then it is clear that $u^a \in W_0^{1,p}(\Omega)$ and $u_k^a \in C_{c,\sigma}^\infty(\Omega)$ for all $a > 0$. Moreover, we have

$$\|u^a - u_k^a\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all} \quad a > 0, \quad \lim_{a\to 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0.$$

By the second equality and the fact that $W_{0,\sigma}^{1,p}(\Omega)$ is closed in $W^{1,p}(\Omega)$, it is sufficient for showing $u \in W_{0,\sigma}^{1,p}(\Omega)$ to prove $u^a \in W_{0,\sigma}^{1,p}(\Omega)$ for all $a > 0$.

For each $a > 0$, there is a constant $d = d(a) > 0$ such that $\operatorname{dist}(\operatorname{supp} u_k^a, \partial\Omega) \ge d$ for all $k \in \mathbf{N}$. Then, for a given $\varepsilon > 0$, we can take $\delta \in (0, d/2)$ so small that

$$\|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2},$$

since $u^a \in W_0^{1,p}(\Omega)$. Also, since $\nabla\rho_\delta = \delta^{-1}(\nabla\rho)_\delta$, we have

$$\begin{aligned} &\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \\ &\le c(\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{L^p(\Omega)} + \|u^a * \nabla\rho_\delta - u_k^a * \nabla\rho_\delta\|_{L^p(\Omega)}) \\ &= c(\|(u^a - u_k^a) * \rho_\delta\|_{L^p(\Omega)} + \delta^{-1}\|(u^a - u_k^a) * (\nabla\rho)_\delta\|_{L^p(\Omega)}) \\ &\le c(1 + \delta^{-1})\|u^a - u_k^a\|_{L^p(\Omega)} = c(1 + \delta^{-1})\|u - u_k\|_{L^p(\Omega)} \end{aligned}$$

with a constant $c > 0$ independent of ε and δ. Hence by taking $k \in \mathbf{N}$ so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})},$$

we have $\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2$ and thus

$$\|u^a - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \le \|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon.$$

On the other hand, since $\operatorname{dist}(\operatorname{supp} u_k^a, \partial\Omega) > d$ and $\delta \in (0, d/2)$, the function $u_k^a * \rho_\delta$ is smooth and compactly supported in Ω. Moreover, we have

$$\operatorname{div}(u_k^a * \rho_\delta) = (\operatorname{div} u_k^a) * \rho_\delta = 0 \quad \text{in} \quad \Omega.$$

Thus $u_k^a * \rho_\delta \in C_{c,\sigma}^\infty(\Omega)$ and u^a is approximated by elements of $C_{c,\sigma}^\infty(\Omega)$ in $W^{1,p}(\Omega)$, which means that $u^a \in W_{0,\sigma}^{1,p}(\Omega)$. Hence $u \in W_{0,\sigma}^{1,p}(\Omega)$ and the proof is now complete. □

Proposition 3.17. *Let $\nu \in (0,\infty]$. If $p > n$, then $W_{0,\sigma}^{1,p}(\Omega) \subset VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.*

Proof. Let $u \in W_{0,\sigma}^{1,p}(\Omega)$ and $u_k \in C_{c,\sigma}^\infty(\Omega)$ such that $\lim_{k\to\infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0$. Since $p > n$ and $u, u_k \in W_0^{1,p}(\Omega)$, Morrey's inequality (see e.g. [7, Theorem 4.12]) implies

$$\|u - u_k\|_{L^\infty(\Omega)} \le c\|u - u_k\|_{W^{1,p}(\Omega)}$$

with a positive constant c independent of u and u_k. Thus we have

$$\|u - u_k : BMO_b^{\infty,\nu}(\Omega)\| \le (2 + \omega_n)\|u - u_k\|_{L^\infty(\Omega)} \le c\|u - u_k\|_{W^{1,p}(\Omega)} \to 0$$

as $k \to \infty$. Hence $u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ and the proof is now complete. □

Proof of Lemma 3.15. Since $u \in C_c^\infty(\Omega)$ and thus $\partial_i u \in C_c^\infty(\Omega)$ for all $i = 1, \dots, n$, it follows from Lemma 3.12 that $Q'u \in L_\sigma^r(\Omega)$ and $\partial_i Q'u = Q'(\partial_i u) \in L^r(\Omega)$ for all $r \in (1,\infty)$ and $i = 1, \dots, n$. From this fact and the equality (3.17), we have $Q'u \in L_\sigma^r(\Omega) \cap W_0^{1,r}(\Omega)$ for all $r \in (1,\infty)$. Hence, by taking $r > n$, we can apply Proposition 3.16 and Proposition 3.17 to obtain $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$. □

Remark 3.18. Let $\nu \in (0,\infty]$. Theorem 3.10 and Lemma 3.15 imply that $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ and $\|Q'u : BMO_b^{\infty,\nu}(\Omega)\| \le c\|u\|_{L^\infty(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Also, we have $Q'u = u$ for all $u \in C_{c,\sigma}^\infty(\Omega)$. Hence Q' extends uniquely to a bounded linear operator (again referred to as Q') from $C_0(\Omega)$, which is the L^∞-closure of $C_c^\infty(\Omega)$, into $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ that satisfies $Q'u = u$ for all $u \in C_{0,\sigma}(\Omega)$.

Now let us extend Q' to a linear operator that gives the projection mentioned in Theorem 1.4. For $p \in (1,\infty)$, we define a Banach space X_p and its norm as

$$X_p := L^p(\Omega) \cap C_0(\Omega), \quad \|u\|_{X_p} := \max\{\|u\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}\}.$$

Note that the Banach space $C_0(\Omega)$ consists of all continuous functions f on Ω such that the set $\{x \in \Omega \mid |f(x)| \ge \varepsilon\}$ is compact in Ω for every $\varepsilon > 0$ (see e.g. [32, Theorem 3.17]).

Lemma 3.19. *For each $p \in (1,\infty)$, the linear subspace $C_c^\infty(\Omega)$ is dense in X_p.*

Proof. The proof is more or less standard (see e.g. [27, Corollary 19.24]). We give it for completeness. Let $u \in X_p$ and $\Omega_k := \{x \in \Omega \mid |x| \le k, \operatorname{dist}(x, \partial\Omega) \ge 1/k\}$ for $k \in \mathbf{N}$. For any given $\varepsilon > 0$, the set $\{x \in \Omega \mid |u(x)| \ge \varepsilon/2\}$ is compact in Ω since $u \in C_0(\Omega)$. Moreover, since $u \in L^p(\Omega)$, we can take $k \in \mathbf{N}$ so large that

$$\|u\|_{L^p(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}, \quad \|u\|_{L^\infty(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}. \tag{3.23}$$

Let $\varphi \in C_c^\infty(\Omega)$ be a continuous cut-off function such that

$$0 \le \varphi \le 1 \quad \text{in } \Omega, \quad \varphi = 1 \quad \text{in } \Omega_k, \quad \varphi = 0 \quad \text{in } \Omega \setminus \Omega_{2k}.$$

Since $u - \varphi u = 0$ in Ω_k and $|u - \varphi u| \le |u|$ in $\Omega \setminus \Omega_k$, it follows from (3.23) that

$$\|u - \varphi u\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}, \quad \|u - \varphi u\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}. \tag{3.24}$$

Let ρ_δ be a mollifier as in the beginning of the proof of Proposition 3.16. Since

$$\varphi u \in L^p(\Omega), \quad \operatorname{dist}(\operatorname{supp}(\varphi u), \partial\Omega) \geq \frac{1}{2k},$$

we can take $\delta \in (0, 1/4k)$ so small that

$$u_\delta := \rho_\delta * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}. \tag{3.25}$$

On the other hand, since φu is uniformly continuous on Ω_{4k}, we can again choose $\delta \in (0, 1/4k)$ so small that $\|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \varepsilon/2$. Moreover, since $\operatorname{supp}(\varphi u) \subset \Omega_{2k}$ and $\delta \in (0, 1/4k)$, we have $\varphi u = u_\delta = 0$ outside of Ω_{4k} and thus

$$\|\varphi u - u_\delta\|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}. \tag{3.26}$$

Combining (3.24), (3.25) and (3.26), we obtain $u_\delta \in C_c^\infty(\Omega)$ and

$$\|u - u_\delta\|_{X_p} = \max\{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon.$$

Hence the lemma follows. □

Let $Y_p := L^p_\sigma(\Omega) \cap VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$ for $p \in (1, \infty)$, $\nu \in (0, \infty]$. Since $L^p_\sigma(\Omega)$ and $VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$ are closed in $L^p(\Omega)$ and $BMO^{\infty,\nu}_b(\Omega)$, respectively, Y_p becomes a Banach space under the norm $\|v\|_{Y_p} := \max\{\|v\|_{L^p(\Omega)}, \|v : BMO^{\infty,\nu}_b(\Omega)\|\}$.

Theorem 3.20. *Let $p \in (1, \infty)$ and $\nu \in (0, \infty]$. The linear operator Q' given in Definition 3.11 extends uniquely to a bounded linear operator Q_p from X_p into Y_p. Moreover, there exists a constant $c > 0$ such that*

$$\|Q_p u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q_p u : BMO^{\infty,\nu}_b(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)} \tag{3.27}$$

for all $u \in X_p$ and $Q_p u = u$ holds for all u in the X_p-closure of $C^\infty_{c,\sigma}(\Omega)$.

Proof. Let $u \in C_c^\infty(\Omega)$. Then we have $Q'u \in Y_p$ by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant $c > 0$ independent of u such that

$$\|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q'u : BMO^{\infty,\nu}_b(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}. \tag{3.28}$$

Hence we have $Q'u \in Y_p$ and $\|Q'u\|_{Y_p} \leq c\|u\|_{X_p}$ for all $u \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in X_p by Lemma 3.19, the operator Q' extends uniquely to a bounded linear operator Q_p from X_p into Y_p. Also, it follows from (3.28) that the inequality (3.27) holds for all $u \in X_p$. Since $Q'u = u$ holds for all $u \in C^\infty_{c,\sigma}(\Omega)$ as observed after Definition 3.11, by the density argument we have $Q_p u = u$ for all u in the X_p-closure of $C^\infty_{c,\sigma}(\Omega)$. The proof is complete. □

Finally, Theorem 1.4 follows from Theorem 3.20 with $p = 2$, that is, the linear operator Q in Theorem 1.4 is given by $Q = Q_2$.

4. Analyticity in L^p

In this section we shall give a complete proof of Theorem 1.1.

Proof of Theorem 1.1. Let $S(t)$ be the Stokes semigroup in $\tilde{L}^p_\sigma$ constructed by [14], [16]. To show that $S(t)$ forms an analytic semigroup in L^p_σ ($2 \leq p < \infty$) it suffices to prove that there exists a constant C that

$$\|S(t)v_0\|_p \leq C\|v_0\|_p \tag{4.1}$$

$$\left\| t \frac{\mathrm{d}}{\mathrm{d}t} S(t) v_0 \right\|_p \le C \|v_0\|_p \tag{4.2}$$

for all $v_0 \in C^\infty_{c,\sigma}(\Omega)$ and for all $t \in (0,1)$. Let Q be the operator in Theorem 1.4. Since Q is bounded in L^2 and maps L^2 to L^2_σ and $S(t)$ fulfills (4.1) and (4.2) for $p = 2$, we have

$$\|S(t)Qu\|_2 \le C\|u\|_2 \tag{4.3}$$

$$\left\| t \frac{\mathrm{d}}{\mathrm{d}t} S(t) Q u \right\|_2 \le C \|u\|_2 \tag{4.4}$$

for all $u \in C_c(\Omega)$ and $t \in (0,1)$. Since Ω is admissible as proved in [5], $S(t)$ forms an analytic semigroup in $VMO^{\infty,\nu}_{b,0,\sigma}$ by Theorem 1.2. We conclude that

$$\|S(t)Qu : BMO^{\infty,\nu}_b(\Omega)\| \le C\|u\|_\infty \tag{4.5}$$

$$\left\| t \frac{\mathrm{d}}{\mathrm{d}t} S(t) Q u : BMO^{\infty,\nu}_b(\Omega) \right\| \le C \|u\|_\infty \tag{4.6}$$

for all $u \in C_c(\Omega)$ and $t \in (0,1)$ since Q fulfills

$$\|Qu : BMO^{\infty,\nu}_b(\Omega)\| \le C\|u\|_\infty, \; Qu \in VMO^{\infty,\nu}_{b,0,\sigma}$$

for all $u \in C_c(\Omega)$ by Theorem 1.4. (Note that we have a stronger statement than (4.6) by replacing the BMO_b type norm by the L^∞ norm since we have the regularizing estimate (1.3).) We apply an interpolation result (Theorem 1.3) to (4.3) and (4.5) and to (4.4) and (4.6) to get, respectively

$$\|S(t)Qu\|_p \le C\|u\|_p \tag{4.7}$$

$$\left\| t \frac{\mathrm{d}}{\mathrm{d}t} S(t) Q u \right\|_p \le C \|u\|_p \tag{4.8}$$

for all $u \in C_c(\Omega)$ and for all $t \in (0,1)$. Since $Qu = u$ for $u \in C^\infty_{c,\sigma}(\Omega)$ this yields (4.1) and (4.2).

It remains to prove that $S(t)$ is a C_0-semigroup in L^p_σ. Since $C^\infty_{c,\sigma}(\Omega)$ is dense in L^p_σ, for $v_0 \in L^p_\sigma$ there is $v_{0m} \in C^\infty_{c,\sigma}$ such that $\|v_0 - v_{0m}\|_p \to 0$ as $m \to \infty$. By (4.1) we observe that

$$\begin{aligned}\|S(t)v_0 - v_0\|_p \le& \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p \\ \le& C\|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p.\end{aligned}$$

Sending $t \downarrow 0$, we get

$$\varlimsup_{t\downarrow 0} \|S(t)v_0 - v_0\|_p \le C\|v - v_{0m}\|_p,$$

since $S(t)v_{0m} \to v_{0m}$ in $\tilde{L}^p_\sigma$ as $t \downarrow 0$ by [14], [16]. Sending $m \to \infty$, we conclude that $S(t)v_0 \to v_0$ in L^p_σ as $t \downarrow 0$. □

Remark 4.1. In a similar way as we derived (4.5) and (4.6) we are able to derive from the L^∞-BMO estimates in [10] that

$$t \left\|\nabla^2 S(t)Qu : BMO^{\infty,\nu}_b(\Omega)\right\| \le C\|u\|_\infty$$

$$t^{1/2} \left\|\nabla S(t)Qu : BMO^{\infty,\nu}_b(\Omega)\right\| \le C\|u\|_\infty$$

for all $u \in C_c(\Omega)$ and $t \in (0,1)$.

Note that L^2 results

$$t\left\|\nabla^2 S(t)Qu\right\|_2 \leq C\|u\|_2$$

$$t^{1/2}\left\|\nabla S(t)Qu\right\|_2 \leq C\|u\|_2$$

easily follow from the analyticity of $S(t)$ in L^2_σ and L^2-boundedness of Q if one observes that $\|\nabla u\|_2^2 = (Au, u)_{L^2}$ and

$$\|\nabla^2 u\|_2 \leq C\left(\|Au\|_2 + \|\nabla u\|_2 + \|u\|_2\right)$$

(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where A is the Stokes operator in L^2_σ.

Interpolating the L^2 results and the above L^∞-BMO results, we are able to prove that there is $C_p > 0$ satisfying

$$t\left\|\nabla^2 S(t)v_0\right\|_p \leq C_p\|v_0\|_p$$

$$t^{1/2}\left\|\nabla S(t)v_0\right\|_p \leq C_p\|v_0\|_p$$

for all $v_0 \in L^p_\sigma(\Omega)$ and $t \in (0,1)$ with $p \in (2,\infty)$.

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