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ON ANALYTICITY OF THE $L^p$-STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

MARTIN BOLKART, YOSHIKAZU GIGA, TATSU-HIKO MIURA, TAKUYA SUZUKI, AND YOHEI TSUTSUI

Abstract. Consider the Stokes equations in a sector-like $C^3$ domain $\Omega \subset \mathbb{R}^3$. It is shown that the Stokes operator generates an analytic semigroup in $L^p_\sigma(\Omega)$ for $p \in [2, \infty)$. This includes domains where the $L^p$-Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in $VMO$ and $L^2$ by constructing a suitable non-Helmholtz projection to solenoidal spaces.

1. Introduction

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \text{div } v = 0 \quad \text{in } \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on } \partial\Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where $\Omega$ is a domain in $\mathbb{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a $C_0$-analytic semigroup in $L^p_\sigma$ ($1 < p < \infty$) for various domains like smooth bounded domains ([21], [35]). Here $L^p_\sigma = L^p_\sigma(\Omega)$ denotes the $L^p$-closure of $C_{\infty, \sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in $\Omega$. More recently, it has been proved in [20] that $S(t)$ always forms a $C_0$-analytic semigroup in $L^p_\sigma(\Omega)$ for any uniformly $C^2$-domain $\Omega$ provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \{ \nabla q \in L^p(\Omega) \mid q \in L^1_{loc}(\Omega) \}$. In [20] the $L^3$ maximal regularity in time with values in $L^p_\sigma(\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The $L^p$-Helmholtz decomposition holds for various domains like bounded or exterior domains with

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smooth boundary for $1 < p < \infty$ ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the $L^p$-Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely. Let $C(\vartheta)$ denote the cone of the form

$$C(\vartheta) = \{ x = (x', x_n) \in \mathbb{R}^n \mid -x_n \geq |x| \cos(\vartheta/2) \},$$

where $\vartheta \in (0, 2\pi)$ is the opening angle. When $n = 2$, we simply say that $C(\vartheta)$ is a sector. We say that a planar domain $\Omega$ is a sector-like domain if $\Omega \setminus \partial \Omega$ is a sector. We say that a planar domain $\Omega$ is a sector-like domain $(0)$ for some $R > 0$ (up to rotation and translation), where $B_R(0)$ is an open disk of radius $R$ centered at the origin.

It is known that the $L^p$-Helmholtz decomposition fails for a sector-like domain $\Omega$ when $p > q_0'$ or $p < q_0$ with $q_0 = 2/(1 + \pi/\vartheta)$, $1/q_0 + 1/q_0' = 1$ even if the boundary $\partial \Omega$ is smooth [28, Example 2, Fig. 5] while for $p \in (q_0, q_0')$ the $L^p$-Helmholtz decomposition holds. This means that if the opening angle $\vartheta$ is larger than $\pi$, there always exists $p > 2$ such that the $L^p$-Helmholtz decomposition fails.

It has been a longstanding open question whether or not the existence of the $L^p$-Helmholtz decomposition is necessary for $L^p$ analyticity of $S(t)$. In this paper, we give a negative answer for this question by proving that there is a domain $\Omega$ for which $S(t)$ is analytic in $L^p_0$ while the $L^p$-Helmholtz decomposition fails. This is a subtle problem since the existence of the $L^p$-Helmholtz projection is known to be necessary for $L^p$ solvability of the resolvent equation ([33]). However, in this statement the external force term is allowed to be in the more general space $L^q$ instead of $L^p_0$. Our problem is different from that in [33].

We say that $\Omega$ has a $\mathcal{C}^k$ graph boundary if $\Omega$ is of the form

$$\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x_n > h(x') \}$$

(up to translation and rotation) with some real-valued $\mathcal{C}^k$ function $h$ with variable $x' \in \mathbb{R}^{n-1}$.

**Theorem 1.1.** Let $\Omega$ be a sector-like domain in $\mathbb{R}^2$ having a $\mathcal{C}^1$ graph boundary. Then $S(t)$ forms a $\mathcal{C}_0$-analytic semigroup in $L^p_0(\Omega)$ for all $p \in [2, \infty)$.

Here is our strategy to prove Theorem 1.1. It is by now well-known that $S(t)$ forms an analytic semigroup in $L^p_0$, i.e., $L^p_0 = L^p_0 \cap L^2_0$ ($p \geq 2$), $L^p = L^p_0 + L^2_0$ ($1 < p < 2$) ([14], [15], [16]). Thus $S(t)v_0$ is well-defined for $v_0 \in \mathcal{C}_c(\Omega)$. To show Theorem 1.1, a key step is to prove the two estimates

(1.1) \[ \|S(t)v_0\|_p \leq C\|v_0\|_p \]

(1.2) \[ t \left\| \frac{d}{dt} S(t)v_0 \right\|_p \leq C\|v_0\|_p \]

for all $v_0 \in \mathcal{C}^{\infty}_{c,\sigma}(\Omega)$, $t \in (0,1)$, where $\|v_0\|_p$ denotes the $L^p$-norm of $v_0$. The constant $C$ should be taken independent of $t$ and $v_0$. We shall establish (1.1) and (1.2) by interpolation since both estimates are known for $p = 2$.

We are tempted to interpolate the $L^\infty$ type result obtained in [5] with the $L^2$-result. In fact, in [5] the estimates (1.1) and (1.2) with $p = \infty$ are established for all $v_0 \in \mathcal{C}_0(\Omega)$, the $L^\infty$-closure of $\mathcal{C}^{\infty}_{c,\sigma}(\Omega)$ for a $\mathcal{C}^2$ sector-like domain $\Omega$ in $\mathbb{R}^2$. However, it is not clear that the complex interpolation space $[L^2_0, \mathcal{C}_0]_\rho$ agrees with $L^p_0$ with $2/p = 1 - \rho$ although it is well-known as the Riesz-Thorin theorem that $[L^2, L^\infty]_\rho = L^p$. To interpolate, we would need a projection to solenoidal spaces.
which is almost impossible since such a projection involves the singular integral operator which is not bounded in $L^\infty$.

To circumvent this difficulty, we consider the Stokes semigroup $S(t)$ in $BMO$-type spaces as studied in [10], [11], [12]. For $p \in [1, \infty)$, $\mu \in (0, \infty]$ we define the $BMO$ seminorm

$$[f : BMO_\mu^p(\Omega)] := \sup \left\{ \left( \frac{1}{r} \int_{B_r(x)} |f(y)|^p \, dy \right)^{1/p} \left|_{x \in \partial \Omega, \ r > 0, \ B_r(x) \subset U_\nu(\partial \Omega)} \right. \right\},$$

where $f_B = \frac{1}{B} \int_B f \, dx$, the average of $f$ over $B$ and $B_r(x)$ denotes the closed ball of radius $r$ centered at $x$. It is well-known that one gets an equivalent seminorm when the ball $B_r(x)$ is replaced by a cube. We also need to control the boundary behavior. For $\nu \in (0, \infty]$ we define

$$[f : b_\nu^p(\Omega)] := \sup \left\{ \left( \frac{1}{r} \int_{B_r(x_0) \cap \Omega} |f(y)|^p \, dy \right)^{1/p} \left|_{x_0 \in \partial \Omega, \ r > 0, \ B_r(x_0) \subset U_\nu(\partial \Omega)} \right. \right\},$$

where $U_\nu(E)$ is a $\nu$-open neighborhood of $E$, i.e.,

$$U_\nu(E) = \{ x \in \mathbb{R}^n \ | \ dist(x, E) < \nu \}.$$

We shall often assume that $\nu < R^*$, where $R^*$ is the reach from the boundary. The $BMO$ norm we use is

$$\| f : BMO_\mu^p(\Omega) \| = [f : BMO_\mu^p(\Omega)] + [f : b_\nu^p(\Omega)].$$

If $p = 1$, we often drop $p$. The $BMO$ space we consider is

$$BMO_\mu^p(\Omega) = \left\{ f \in L^\infty(\Omega) \ \| f : BMO_\mu^p(\Omega) \| < \infty \right\}.$$

This space is independent of $p$ for sufficiently small $\nu$, i.e., $\nu < R^*$ ([11], [12]) and $BMO_\infty^{\infty, \infty}$ agrees with Miyachi $BMO$ space ([29]) for various domains including a half space and bounded $C^2$ domains ([12]). Although the $BMO_\mu^{\infty, \nu}(\Omega)$ norm is equivalent to the $BMO_\mu^{\infty, \infty}(\Omega)$ norm when $\Omega$ is bounded, there are many unbounded domains for which the $BMO_\mu^{\infty, \nu}(\Omega)$ norm is actually weaker than the $BMO_\mu^{\infty, \infty}(\Omega)$ norm when $\nu$ is finite. We define the solenoidal space $VMO_\mu^{\nu, \nu}(\Omega)$ as the $BMO_\mu^{\nu, \nu}$-closure of $\mathcal{C}_c^{\infty, \nu}(\Omega)$. In [10], [11] among other results the analyticity of $S(t)$ in $VMO_\mu^{\nu, \nu}(\Omega)$ has been established for a uniformly $C^3$ domain which is admissible in the sense of [2] provided that $\nu$ is sufficiently small.

**Theorem 1.2** ([10], [11]). *Let $\Omega$ be an admissible uniformly $C^3$ domain in $\mathbb{R}^n$. Then $S(t)$ forms a $C_0$-analytic semigroup in $VMO_\mu^{\nu, \nu}(\Omega)$ for any $\mu \in (0, \infty]$ and $\nu \in (0, \nu_0)$ with some $\nu_0$ depending only on $\mu$ and regularity of $\partial \Omega$.*

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace $L^p$ by $L^\infty$ or $BMO_\infty^{\nu, \nu}$, but even an estimate stronger than (1.2) with $p = \infty$, i.e.,

$$t \frac{dS(t)}{dt} \leq C \| v_0 : BMO_\mu^{\nu, \nu}(\Omega) \|, \ \mu, \nu \in (0, \infty],$$

which shows a regularizing effect.

It has been proved in [5] that a $C^2$ sector-like domain in $\mathbb{R}^2$ is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a $C^3$ sector-like
domain in $\mathbb{R}^2$ is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly $C^4$. On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a domain with $n \geq 3$ is not admissible ([8]).

In order to get the $L^p$ estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in $\Omega$.

**Theorem 1.3.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$, i.e., a domain having Lipschitz graph boundary. Let $T$ be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant $C$ such that

$$\|Tu\|_2 \leq C\|u\|_2$$

for $u \in C_c(\Omega)$. Then $\|Tu\|_p \leq C_*\|u\|_p$ for $u \in C_c(\Omega)$ with $C_*$ depending only on $C$, $h$ and $p \in (2, \infty)$.

There are a couple of such interpolation results between $BMO$ and $L^2$, which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between $L^p$ and $BMO$ is discussed when $\Omega$ is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega) \cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with $BMO_A(\mathcal{X})$, where $A$ is some operator. They worked on metric measure spaces of homogeneous type $(\mathcal{X}, d, \mu)$. In particular, in the case $\mathcal{X} = \Omega$, $d(x, y) = |x - y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO^\infty(\Omega)$.

Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the $L^\infty$ case we do not know whether or not the complex interpolation space $[L^2, \text{VMO}^{\infty, \nu}_0, \rho]_{\rho}$ with $2/p = 1 - \rho$ agrees with $L^p_\rho$, although we know that $[L^2, BMO]_\rho = L^p$ for $\Omega = \mathbb{R}^n$ as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

**Theorem 1.4.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$. Assume that $\nu \in (0, \infty]$. There is a linear operator $Q$ from $C_c(\Omega)$ to $\text{VMO}^{\infty, \nu}_0(\Omega) \cap L^2_\rho(\Omega)$ such that

$$\|Qu : \text{VMO}^{\infty, \nu}_0(\Omega)\| \leq C\|u\|_\infty$$

$$\|Qu\|_2 \leq C\|u\|_2$$

for all $u \in C_c(\Omega)$. Moreover, $Qu = u$ for $u \in C_c(\Omega) \cap L^2_\rho(\Omega)$.

Since there may be no $L^p$-Helmholtz decomposition our $Q$ should be different from the Helmholtz projection. We shall construct such an operator $Q$ using the solution operator of the equation $\nabla u = f$ given by Solonnikov [36]. Although deriving the $L^2$ estimate is easy, to derive the $BMO$ estimate is more involved since we have to estimate the $b^\nu$ type seminorm.

To derive (1.1), we actually interpolate

$$\|S(t)Qu\|_2 \leq C\|u\|_2$$

and

$$\|S(t)Qu : \text{VMO}^{\infty, \nu}_0\| \leq C\|u\|_\infty$$

for $u \in C_c(\Omega)$. Similarly, we derive (1.2) by interpolating the estimate for $t^{\frac{\nu}{2}} Q$. 

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stamacechia type. In Section 3, we construct the projection operator $Q$. In Section 4, we give a complete proof of Theorem 1.1.

2. $L^2 - \text{BMO}$ interpolation on a Lipschitz half-space

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

$$\Omega := \{(x', x_n) \in \mathbb{R}^n | x_n > h(x')\}$$

with a Lipschitz function $h$ on $\mathbb{R}^{n-1}$.

By $Q$ we mean a closed cube with sides parallel to the coordinate axes. Let $\ell(Q)$ be the side length of $Q$, and for $\tau > 0$, $\tau Q$ a cube with the same radius as $Q$ and side length $\tau \ell(Q)$.

2.1. Reduction to the half-space and extension. Here, we prepare lemmas that are basic estimates for the proof. Since $h$ is Lipschitz continuous, $F(x) := (x', x_n - h(x'))$ is a bi-Lipschitz map from $\Omega$ to $\mathbb{R}_+^n$. For a function $u$ defined on $\mathbb{R}_+^n$, the pull-back function $F^*(u)$ of $u$ on $\Omega$ is defined by $u \circ F$. We start with estimates for $(F^{-1})^*$ which is the pull-back function $(F^{-1})^*(v)$ of $v$ on $\mathbb{R}_+^n$ defined by $v \circ F^{-1}$.

Lemma 2.1. Let $\Omega$ be a Lipschitz half-space.

(i):

$$\left[ (F^{-1})^* v : BMO^\infty(\mathbb{R}_+^n) \right] \leq c [v : BMO^\infty(\Omega)] .$$

(ii):

$$\| (F^{-1})^* v \|_{L^2(\mathbb{R}_+^n)} \leq c \| v \|_{L^2(\Omega)} .$$

Here $c$ is a constant depending only on Lipschitz bound of $h$ and $u$.

Proof. (i): Because $\mathbb{R}_+^n$ is an open subset of $\mathbb{R}^n$, we know that for any $\tau > 2$,

$$\left[ (F^{-1})^* v : BMO^\infty(\mathbb{R}_+^n) \right] \leq c_\tau \sup_{\tau Q \subset \mathbb{R}_+^n} \inf_{d \in \mathbb{R}} \int_{\tau Q} |(F^{-1})^* v - d| \, dy ,$$

where the supremum is taken over cubes $Q$, for which $\tau Q$ is contained in $\mathbb{R}_+^n$, see [37]. Since $F$ is a bi-Lipschitz map, it holds

$$c_1 \text{dist}(y, \partial \mathbb{R}_+^n) \leq \text{dist}(F^{-1}(y), \partial \Omega) \leq c_2 \text{dist}(y, \partial \mathbb{R}_+^n)$$

with some constants $c_1, c_2 > 0$ for all $y \in \mathbb{R}_+^n$. Since $(\tau - 1) \ell(Q)/2 \leq \text{dist}(Q, \partial \mathbb{R}_+^n)$ for such cubes $Q$, we have the lower bound

$$c \tau \ell(Q) \leq \text{dist}(F^{-1}(Q), \partial \Omega)$$

with some $c > 0$, which depends on $n$ and $h$. Therefore, taking large $\tau$, we can find cubes $\{R_k\}_{k=1}^c \subset \Omega$, which have no intersection of interiors, so that $\bigcup_{k=1}^c R_k$ is connected and

$$\begin{cases} 
\ell(R_k) = \ell(Q), \\
F^{-1}(Q) \subset \bigcup_{k=1}^c R_k, \text{ where } c_s \in \mathbb{N} \text{ depends only on } h, \text{ and} \\
\text{if } R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega .
\end{cases}$$

From these, one obtains that for cubes $Q$ with $\tau Q \subset \mathbb{R}_+^n$,

$$\inf_{d \in \mathbb{R}} \frac{1}{|Q|} \int_Q |(F^{-1})^* v - d| \, dy \leq c \sum_{k=1}^c \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| \, dy .$$
It is enough to show that
\[
\frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| \, dy \leq c[v : BMO^\infty(\Omega)]
\]
for the case $R_j \cap R_k \neq \emptyset$. To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let $\tilde{R}_k$ and $\tilde{R}_j$ be subcubes of $R_k$ and $R_j$ respectively so that $\ell(\tilde{R}_k) = \ell(R_k)/2$, $\ell(\tilde{R}_j) = \ell(R_j)/2$ and they touch each other. Moreover, denote by $\tilde{R}_{j,k}$ a cube satisfying $\ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k)$ and $\tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k}$. Hence, we have
\[
\frac{1}{|\tilde{R}_k|} \int_{\tilde{R}_k} |v - v_{R_j}| \, dy \leq \frac{1}{|\tilde{R}_k|} \int_{\tilde{R}_k} |v - v_{R_k}| \, dy + |v_{R_k} - v_{R_j}|
\leq c[v : BMO^\infty(\Omega)] + c|v_{\tilde{R}_j} - v_{R_k}|
\leq c[v : BMO^\infty(\Omega)] + \frac{1}{|\tilde{R}_{j,k}|} \int_{\tilde{R}_{j,k}} |v - v_{\tilde{R}_{j,k}}| \, dy
\leq c[v : BMO^\infty(\Omega)].
\]

(ii): This is verified as follows
\[
\| (F^{-1})^* v \|_{L^2(R^+_n)}^2 = \int_{\Omega} |v|^2 J_F \, dx \leq c \int_{\Omega} |v|^2 \, dx,
\]
where $J_F$ is the modulus of the Jacobian of $F$ which is bounded, because $h$ is Lipschitz continuous.

Next, we consider the even extension of functions on the half space. For a function $f$ on $R^+_n$, we extend $f$ outside $R^+_n$ by
\[
E[f](x', -x_n) := f(x', x_n) \text{ for } x_n > 0.
\]
From elementary geometrical observation, we can see that the extension operator $E$ is a $BMO$-extension operator for $R^+_n$.

**Lemma 2.2.**
\[
[E[f] : BMO^\infty(R^n)] \leq c [f : BMO^\infty(R^+_n)].
\]

**Proof.** It is sufficient to consider cubes $Q \subset R^n$ with $Q \cap R^+_n \neq \emptyset$ and $Q \cap R^n \neq \emptyset$. For such $Q$, let $Q'$ be a cube so that its center lies on $\partial R^+_n$, $\ell(Q') = 2\ell(Q)$ and $Q \subset Q'$. Further, let $Q^*$ be the smallest cube in $R^+_n$ containing the upper half of $Q'$. With these notations, the desired inequality is proved from
\[
\inf_{d \in R} \frac{1}{|Q|} \int_Q |E[f] - d| \, dy \leq c \inf_{d \in R} \frac{1}{|Q^*|} \int_{Q^*} |f - d| \, dy.
\]
\[\square\]

### 2.2. Sharp maximal operator

For the proof of Theorem 1.3, we make use of the sharp maximal operator $M^2$ due to Fefferman and Stein ([18]). We define for $x \in R^n$ and $f \in L^1_{loc}(R^n)$ the function $M^2 f$ by
\[
M^2 f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy.
\]
It is immediate from the definition that \([f : BMO^\infty(\mathbb{R}^n)] = \|M^2 f\|_{L^\infty(\mathbb{R}^n)}\). It is well-known that if \(f \in L^{p_0}(\mathbb{R}^n)\) for some \(p_0 \in (1, \infty)\), then for \(p \in [p_0, \infty)\)

\[
(2.2) \quad \|f\|_{L^p(\mathbb{R}^n)} \leq c\|M^2 f\|_{L^p(\mathbb{R}^n)},
\]

which is applied below. (Both sides of (2.2) may be infinite.) This follows from \(\|f\|_{L^p(\mathbb{R}^n)} \leq \|Mf\|_{L^p(\mathbb{R}^n)}\) and \(\|M^2 f\|_{L^p(\mathbb{R}^n)} \leq c\|M^2 f\|_{L^p(\mathbb{R}^n)}\), where \(M\) is the Hardy-Littlewood maximal operator [18].

2.3. Marcinkiewicz interpolation. Here, we give a variant of the Marcinkiewicz interpolation theorem.

**Proposition 2.3.** Let \(D\) be an open subset of \(\mathbb{R}^n\) and \(S\) a sublinear operator from \(C_c(D)\) to \(L^2(\mathbb{R}^n)\). If

\[
\|S[f]\|_{L^2(\mathbb{R}^n)} \leq c\|f\|_{L^2(D)}
\]

\[
\|S[f]\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{L^\infty(D)}
\]

for \(f \in C_c(D)\), then \(\|S[f]\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(D)}\) for \(f \in C_c(D)\) with \(C\) depending only on \(c\) and \(p \in (2, \infty)\).

**Proof.** For \(\lambda > 0\) and \(\alpha > 0\), we decompose \(f\) into two parts; \(f = f_2 + f_\infty\) where

\[
f_2(x) = \begin{cases} 
0 & \text{if } |f(x)| \leq \alpha \lambda \\
f(x) - \alpha \lambda \text{sgn}(f(x)) & \text{if } |f(x)| > \alpha \lambda ,
\end{cases}
\]

where \(\text{sign } \xi = \xi/|\xi|\) for \(\xi \neq 0\) and \(\text{sign } \xi = 0\) for \(\xi = 0\). Observe that \(f_2, f_\infty \in BC(D)\), and then \(f_2, f_\infty \in C_c(D)\). Therefore, the two inequalities of our assumption hold for \(f_2\) and \(f_\infty\), respectively. We set \(\alpha = \left(2\|S\|_{L^\infty(D) \to L^\infty(\mathbb{R}^n)}\right)^{-1}\) and observe that \(\{x \in \mathbb{R}^n : |S[f_\infty](x)| > \lambda/2\} = 0\). We now conclude that

\[
\int_{\mathbb{R}^n} |S[f]|^p \, dx \leq p \int_0^\infty \lambda^{p-1} \left|\{x \in \mathbb{R}^n : |S[f](x)| > \lambda\}\right| \, d\lambda \\
\leq p \int_0^\infty \lambda^{p-1} \left|\{x \in \mathbb{R}^n : |S[f_2](x)| > \lambda/2\}\right| \, d\lambda \\
\leq p \int_0^\infty \lambda^{p-1} \left(\frac{1}{\lambda} \|S\|_{L^2(D) \to L^2(\mathbb{R}^n)} \|f_2\|_{L^2(D)}\right)^2 \, d\lambda \\
\leq c \int_0^\infty \lambda^{p-3} \int_{\{|f| > \alpha \lambda\}} |f(x)|^2 \, dx \, d\lambda \\
= 2c \int_0^\infty \lambda^{p-3} \left(\int_{\alpha \lambda}^\infty t \left|\{x \in \mathbb{R}^n : |f(x)| > t\}\right| \, dt\right) \, d\lambda \\
= 2c \int_0^\infty t \left|\{x \in \mathbb{R}^n : |f(x)| > t\}\right| \left(\int_0^{t/\alpha} \lambda^{p-3} \, d\lambda\right) \, dt \\
\leq c\|f\|_{L^p(D)}^p.
\]

□
2.4. Proof of Theorem 1.3. For simplicity, we write \( g := Tf \). By changing variables, one obtains
\[
\int_{\Omega} |g|^p \, dx \leq c \int_{\mathbb{R}^n_+} |(F^{-1})^* g|^p \, dy \leq c \int_{\mathbb{R}^n} |E[(F^{-1})^* g]|^p \, dy \leq c \int_{\mathbb{R}^n} |\Phi[f]|^p \, dy,
\]
where \( \Phi[f] := M^p(E[(F^{-1})^* g]) \). Here, because \( E[(F^{-1})^* g] \in L^2(\mathbb{R}^n) \), we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see \( L^2(\Omega) \) estimates for \( \Phi \). The former estimate can be seen by \( L^2 \)-boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The later one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

3. Non-Helmholtz projection

Our goal in this section is to prove Theorem 1.4.

3.1. A solution operator to the divergence problem. As in Section 2, let \( \Omega = \{ (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n > h(x') \} \) be a Lipschitz half-space in \( \mathbb{R}^n \) with a Lipschitz continuous function \( h \) on \( \mathbb{R}^{n-1} \). Then, there is a closed cone of the form
\[
C_1 = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2\theta) \}
\]
with an angle \( \theta \in (0, \pi/4) \) (depending on the Lipschitz constant of \( h \)) such that
\[
x + C_1 = \{ y \in \mathbb{R}^n \mid y - x \in C_1 \} \subset \Omega^c := \mathbb{R}^n \setminus \Omega \quad \text{for all} \quad x \in \Omega^c.
\]
In the notion of the introduction \( C_1 = C(4\theta) \) so that the opening angle equals \( 4\theta \).

With this angle we define a closed cone \( C_0 = C(2\theta) \), i.e.,
\[
C_0 = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos \theta \}.
\]
The closed cone \( C_0 \) also satisfies
\[
(3.1) \quad x + C_0 \subset \Omega^c \quad \text{for all} \quad x \in \Omega^c.
\]

Let \( L \in C_c^\infty(\mathbb{R}^n) \) be a function such that
\[
(3.2) \quad \text{supp} \, L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1.
\]
Here \( -C_0 = \{ -y \mid y \in C_0 \} \) and \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). Then we define a vector field \( K = (K_1, \ldots, K_n) \) as
\[
(3.3) \quad K(x) := \frac{x}{|x|^n} L \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

**Definition 3.1.** For \( f \in C_c^\infty(\Omega) \), we define a vector field \( u = Tf \) as
\[
u(x) = \int_{\mathbb{R}^n} K(x - y)f(y) \, dy, \quad x \in \mathbb{R}^n.
\]
Here \( f \) denotes the zero extension of \( f \) to \( \mathbb{R}^n \) given by
\[
f(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}
\]
On the one hand, we change variables $x - y \in \text{supp } L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)$ implies $y \in x + C_0$, we can write
\[ u(x) = \int_{x+C_0} K(x-y)\hat{f}(y) \, dy. \]
This formula and the property (3.1) of $\Omega$ imply that $u(x) = 0$ for all $x \in \Omega^c$. In particular, $u$ vanishes on $\partial \Omega$. However, the support of $u$ may become unbounded although $f$ is compactly supported in $\Omega$.

By the change of variables $x - y = r \sigma$ with $r > 0$ and $\sigma \in S^{n-1}$ we have
\[ u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma)\hat{f}(r-r \sigma)r^{n-1} d\mathcal{H}^{n-1}(\sigma) \, dr. \]
Hence if $f \in C^\infty_c(\Omega)$ is supported in $B_R(0)$ and $x \in B_a(0)$ ($R, a > 0$), then
\[ u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma)\hat{f}(r-r \sigma)r^{n-1} d\mathcal{H}^{n-1}(\sigma) \, dr, \]
which implies that $u = Sf$ is smooth in $\Omega$. Moreover, $u = Sf$ vanishes near $\partial \Omega$ and thus it is smooth in the whole space $\mathbb{R}^n$, since $f$ is compactly supported in $\Omega$.

**Lemma 3.2.** Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that
\[ \|\nabla u\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)} \]
for all $f \in C^\infty_c(\Omega)$ and $u = Sf$.

**Proof.** Let $u_i$ be the $i$-th component of $u$:
\[ u_i(x) = (K_i * \hat{f})(x) = \int_{\mathbb{R}^n} K_i(z)\hat{f}(x-z) \, dz. \]
Differentiating both sides with respect to the $j$-th variable, we have
\[ \partial_j u_i(x) = \int_{\mathbb{R}^n} K_i(z) (\partial_j \hat{f})(x-z) \, dz = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} K_i(z) (\partial_j \hat{f})(x-z) \, dz \]
and, by changing variables $y = x - z$ and integrating by parts,
\[ \partial_j u_i(x) = \lim_{\varepsilon \to 0} \left( \int_{\partial B_\varepsilon(x)} K_i(x-y) \frac{x_j - y_j}{|x-y|} \hat{f}(y) \, d\mathcal{H}^{n-1}(y) + \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (\partial_j K_i)(x-y) \hat{f}(y) \, dy \right). \]
On the one hand, we change variables $x - y = \varepsilon \sigma$ with $\sigma \in S^{n-1}$ to get
\[ \lim_{\varepsilon \to 0} \int_{|x-y| = \varepsilon} K_i(x-y) \frac{x_j - y_j}{|x-y|} \hat{f}(y) \, d\mathcal{H}^{n-1}(y) \]
\[ = \lim_{\varepsilon \to 0} \int_{|x-y| = \varepsilon} \frac{x_i - y_i}{|x-y|} \frac{x_j - y_j}{|x-y|} L \left( \frac{x - y}{|x-y|} \right) \hat{f}(y) \frac{1}{|x-y|^{n-1}} \, d\mathcal{H}^{n-1}(y) \]
\[ = \lim_{\varepsilon \to 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \hat{f}(x - \varepsilon \sigma) \, d\mathcal{H}^{n-1}(\sigma) \]
\[ = \hat{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma), \]
where the last equality follows from the fact that $L$ is integrable on $S^{n-1}$ and $\bar{f}$ is continuous at $x$. On the other hand, we differentiate $K_i$ to obtain

$$K_{ij}(z) := \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n},$$

(3.4)

$$k_{ij}(z) := (\delta_{ij} - nz_i z_j)L(z) + z_i(\partial_j L)(z) - z_i z_j \sum_{\ell=1}^n z_{\ell}(\partial_{\ell} L)(z)$$

for $z \in \mathbb{R}^n \setminus \{0\}$. Then $K_{ij}$ is homogeneous of degree $-n$ and there is a constant $c > 0$ such that

$$|K_{ij}(z)| \leq \frac{c}{|z|^n} \quad \text{for all} \quad z \in \mathbb{R}^n \setminus \{0\}$$

by the smoothness of $L$ on $S^{n-1}$. Moreover, for every $R_1$ and $R_2$ with $0 < R_1 < R_2$,

$$\int_{R_1 < |z| < R_2} K_{ij}(z) \, dz = \int_{R_1 < |z| < R_2} \partial_j K_i(z) \, dz$$

$$= \int_{|z| = R_2} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z) - \int_{|z| = R_1} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z)$$

$$= \int_{|z| = R_2} \frac{z_i z_j}{|z|} L \left( \frac{z}{|z|} \right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z) - \int_{|z| = R_1} \frac{z_i z_j}{|z|} L \left( \frac{z}{|z|} \right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z)$$

$$= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0.$$

Thus we can apply the Calderón-Zygmund theory (see eg. [23, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel $K_{ij}$ and obtain the formula

$$\partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{f}(y) \, dy,$$

(3.6)

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) \right| \leq |\bar{f}(x)| \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \leq c \|\bar{f}\|_{L^p(\mathbb{R}^n)} = c \|f\|_{L^p(\Omega)}$$

with a positive constant $c$ independent of $f$. Hence the lemma follows. \qed

**Lemma 3.3.** For every $f \in C_0^\infty(\Omega)$ the vector field $u = Sf$ satisfies

$$\text{div } u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$
Proof. We have already observed that $u$ vanishes on the boundary. Let us compute
\[ \text{div } u = \sum_{i=1}^{n} \partial_i u_i \in \Omega. \]
By the formula (3.6) in the proof of Lemma 3.2,
\[
\text{div } u(x) = \bar{\nabla} f(x) \int_{S^{n-1}} \sum_{i=1}^{n} \sigma_i^2 L(\sigma_i) d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} \sum_{i=1}^{n} K_i(x-y) \bar{\nabla} f(y) dy.
\]
In this formula, we have
\[
\int_{S^{n-1}} \sum_{i=1}^{n} \sigma_i^2 L(\sigma_i) d\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1
\]
by (3.2) and, for all $z \in \mathbb{R}^n \setminus \{0\}$,
\[
\sum_{i=1}^{n} K_i(z) = \frac{1}{|z|^n} L \left( \frac{z}{|z|} \right) \sum_{i=1}^{n} \left( 1 - n \frac{z_i^2}{|z|^2} \right)
+ \frac{1}{|z|^n} \sum_{i=1}^{n} \frac{z_i}{|z|} (\partial_i L) \left( \frac{z}{|z|} \right) = \sum_{i=1}^{n} \frac{z_i^2}{|z|^n+2} \sum_{k=1}^{n} \frac{z_k}{|z|} (\partial_k L) \left( \frac{z}{|z|} \right) = 0.
\]
Hence \( \text{div } u(x) = \bar{\nabla} f(x) = f(x) \) for all \( x \in \Omega \). \( \square \)

Lemma 3.3 means that the operator \( S \) is a solution operator to the divergence problem with Dirichlet boundary condition. Note that \( S \) is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

**Definition 3.4.** For a vector field \( u \in C^\infty_c(\Omega) \), we define a vector field \( Tu \) as
\[
Tu(x) := \int_{\mathbb{R}^n} K(x-y) \overline{\text{div } u(y)} \, dy, \quad x \in \mathbb{R}^n.
\]
Here \( K \) is given by (3.3) and \( \overline{\text{div } u} \) denotes the zero extension of \( \text{div } u \) to \( \mathbb{R}^n \).

The above definition means that \( T \) is given by \( T = S \circ \text{div} \). Since \( u \in C^\infty_c(\Omega) \), its divergence is in \( C^\infty_c(\Omega) \) and thus \( Tu \) is smooth in the whole space \( \mathbb{R}^n \) and vanishes outside of \( \Omega \), as discussed right after Definition 3.1. Also, by Lemma 3.3 we have
\[
\text{div } Tu = \text{div } u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial \Omega.
\]
Clearly \( Tu = 0 \) in \( \mathbb{R}^n \) for \( u \in C^\infty_c(\Omega) \). Note that, as in the case of the operator \( S \), the support of \( Tu \) may be unbounded.

**Theorem 3.5.** Let \( \Omega \) be a Lipschitz half-space. Let \( p \in (1, \infty) \). There exists a constant \( c > 0 \) such that
\[
\|Tu\|_{L^p(\Omega)} \leq c \|u\|_{L^p(\Omega)}
\]
for all \( u \in C^\infty_c(\Omega) \).

**Proof.** Let us compute the \( i \)-th component \( (Tu)_i \) of \( Tu \) with \( i = 1, \ldots, n \) for compactly supported vector field \( u \) in \( \Omega \). As in the proof of Lemma 3.2, we integrate
by parts to get
\[
(Tu)_i(x) = \lim_{\epsilon \to 0} \int_{B_\epsilon(x)} K_i(x - y) \frac{x - y}{|x - y|} \cdot u(y) \, d\mathcal{H}^{n-1}(y)
+ \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (\nabla K_i)(x - y) \cdot u(y) \, dy
= \int_{S^{n-1}} \sigma_i L(\sigma) \{\sigma \cdot u(x)\} \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} (\nabla K_i)(x - y) \cdot u(y) \, dy,
\]
or equivalently,
\[
(Tu)_i(x) = \sum_{j=1}^n \{a_{ij} \bar{u}_j(x) + S_{ij} \bar{u}_j(x)\}, \quad x \in \mathbb{R}^n.
\]
Here \(u_j\) is the \(j\)-th component of \(u\) and
\[
a_{ij} = \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma), \quad S_{ij} \bar{u}_j(x) = \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{u}_j(y) \, dy,
\]
where \(K_{ij} = \partial_j K_i\) is given by (3.4). Since \(a_{ij}\) is a constant satisfying
\[
|a_{ij}| \leq \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1
\]
and \(S_{ij} \bar{u} = K_{ij} \ast \bar{u}\) is a singular integral (see the proof of Lemma 3.2), the Calderón-Zygmund theory yields the boundedness of the operator \(T\) on \(L^p(\Omega)\).

By Theorem 3.5, the operator \(T\) extends uniquely to a bounded linear operator on \(L^p(\Omega)\) with each \(p \in (1, \infty)\), which we again refer to as \(T\).

Our next goal is to estimate the \(BMO_b^{\infty, \nu}(\Omega)\)-norm of \(Tu\) for \(u \in C_c^\infty(\Omega)\) and \(\nu \in (0, \infty)\). To this end, we estimate each term of the right-hand side in (3.7) for \(u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)\). By (3.8) we have
\[
[a_{ij} \bar{u}_j : BMO^{\infty}(\Omega)] \leq [u_j : BMO^{\infty}(\Omega)], \quad [a_{ij} \bar{u}_j : b^\nu(\Omega)] \leq [u_j : b^\nu(\Omega)]
\]
and thus
\[
\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq \|u_j : BMO_b^{\infty, \nu}(\Omega)\|.
\]
Moreover, since
\[
[u_j : BMO^{\infty}(\Omega)] \leq 2\|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \leq \omega_n \|u_j\|_{L^\infty(\Omega)},
\]
where \(\omega_n = 2\pi^{n/2}/n\Gamma(n/2)\) is the volume of the unit ball \(B_1(0)\) in \(\mathbb{R}^n\) with the Gamma function \(\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx\), we have
\[
\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq (2 + \omega_n) \|u_j\|_{L^\infty(\Omega)}.
\]

Let us estimate \(S_{ij} \bar{u}_j = K_{ij} \ast \bar{u}_j, i, j = 1, \ldots, n\) in \(BMO_b^{\infty, \nu}(\Omega)\). Recall that the integral kernel \(K_{ij}\) is of the form
\[
K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
where \(k_{ij} \in C_c^\infty(\mathbb{R}^n)\) is given by (3.4) and satisfies
\[
\supp k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0,
\]
see (3.2) and (3.5). We first estimate the \(BMO^{\infty}\)-seminorm of \(S_{ij} \bar{u}_j\).
Lemma 3.6. Let \( K \) be a function defined on \( \mathbb{R}^n \setminus \{0\} \) such that
\[
|K(x-y) - K(x)| \leq A|y|^\delta |x|^{-n-\delta} \quad \text{whenever} \quad |x| \geq 2|y| > 0
\]
for some \( A, \delta > 0 \). Suppose that a convolution operator \( S \) with \( K \) is bounded on \( L^2(\mathbb{R}^n) \) with a norm \( B \). Then, there exists a dimensional constant \( c_n \) such that
\[
[Sf : BMO^{\infty}(\mathbb{R}^n)] \leq c_n(A + B)\|f\|_{L^\infty(\mathbb{R}^n)}
\]
for all \( f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \).

Proof. See [24, Theorem 3.4.9 and Corollary 3.4.10]. \( \square \)

Lemma 3.7. There exists a constant \( c > 0 \) such that
\[
[S_{ij} \tilde{u}_j : BMO^{\infty}(\Omega)] \leq c\|u_j\|_{L^\infty(\Omega)}
\]
for all \( u = (u_1, \ldots, u_n) \in C^\infty(\Omega) \) and \( i, j = 1, \ldots, n \).

Proof. We shall apply Lemma 3.6 to \( S = S_{ij} \). For this purpose it is sufficient to show that the function \( K = K_{ij} \) satisfies (3.10), since we already know that the convolution operator \( S_{ij} \) is bounded on \( L^2(\mathbb{R}^n) \), see the proof of Lemma 3.2. To this end, we differentiate \( K_{ij} \) to get
\[
\nabla K_{ij}(x) = -nk_{ij}(x/|x|) x/|x| + \frac{1}{|x|^{n+1}} (I_n - \frac{1}{|x|^2} x \otimes x) \nabla k_{ij} \left( \frac{x}{|x|} \right)
\]
for \( x \in \mathbb{R}^n \setminus \{0\} \), where \( I_n \) is the identity matrix of size \( n \) and \( x \otimes x := (x_i x_j)_{i,j} \) is the tensor product of \( x \). Since \( k_{ij} \) is smooth on \( S^{n-1} \), we have
\[
|\nabla K_{ij}(x)| \leq c/|x|^{n+1}, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]
Hence, for all \( x, y \in \mathbb{R}^n \setminus \{0\} \) with \( |x| \geq 2|y| > 0 \),
\[
|K(x-y) - K(x)| = \left| \int_0^1 \frac{d}{dt}(K(x-ty)) \, dt \right| = \left| \int_0^1 (-y) \cdot \nabla K(x-ty) \, dt \right|
\]
\[
\leq |y| \int_0^1 \frac{c}{|x-ty|^{n+1}} \, dt \leq |y| \int_0^1 \frac{c}{(|x| - |y|)^{n+1}} \, dt
\]
\[
\leq \frac{c|y|}{(|x| - |y|)^{n+1}} = \frac{2^{n+1}c|y|}{|x|^{n+1}}
\]
Thus \( K_{ij} \) satisfies (3.10) with \( \delta = 1 \) and we can apply Lemma 3.6 to obtain
\[
[S_{ij} \tilde{u}_j : BMO^{\infty}(\mathbb{R}^n)] \leq c\|\tilde{u}_j\|_{L^\infty(\mathbb{R}^n)} = c\|u_j\|_{L^\infty(\Omega)}
\]
with some constant \( c > 0 \).

By definition of the \( BMO^{\infty} \)-seminorm, we have
\[
[S_{ij} \tilde{u}_j : BMO^{\infty}(\Omega)] \leq [S_{ij} \tilde{u}_j : BMO^{\infty}(\mathbb{R}^n)].
\]
Hence the inequality (3.11) follows from (3.12). \( \square \)

Next, let us estimate the \( b' \)-part of \( S_{ij} \tilde{u}_j \). Recall the two closed cones
\( C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2^j \theta) \}, \quad j = 0, 1 \)
with opening angle \( \theta \in (0, \pi/4) \). For \( r > 0 \) and \( x_0 \in \mathbb{R}^n \), we define
\[
A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap (x_0 + C_1)^c \subset \mathbb{R}^n.
\]
Here \( x_0 + C_1 = \{ y \in \mathbb{R}^n \mid y - x_0 \in C_1 \} \) and \( x + C_0 \) is defined similarly.
Lemma 3.8. For all $r > 0$ and $x_0 \in \mathbb{R}^n$ we have $A_r(x_0) \subset B_{r/\sin \theta}(x_0)$. 

Proof. By translation, we may assume that $x_0 = 0$. Let $a := (0, \ldots, 0, r/\sin \theta) \in \mathbb{R}^n$. Suppose that 

(1) $B_r(0) \subset a + C_0,$

(2) $x + C_0 \subset a + C_0$ for all $x \in a + C_0,$

(3) $(a + C_0) \cap C_1^c \subset B_{r/\sin \theta}(0).$

Then, the statements (1) and (2) imply 

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.$$

Hence the statement (3) yields $A_r(0) \subset B_{r/\sin \theta}(0)$. Now let us prove the statements (1)-(3). Note that, since $\theta \in (0, \pi/4)$, the cones $C_0$ and $C_1$ are represented as 

$$C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n \leq 0, |x'| \leq (\tan(2^j \theta)), \ j = 0, 1.$$


(1) Let $x = (x', x_n) \in B_r(0)$. Then, $x - a = (x', x_n - r/\sin \theta)$ satisfies 

$$|x - a|_n = x_n - \frac{r}{\sin \theta} \leq r - \frac{r}{\sin \theta} < 0$$

and 

$$\left( \frac{r}{\sin \theta} - x_n \right) \tan^2 \theta - |x'|^2 \geq \left( \frac{r - x_n \sin \theta}{\cos^2 \theta} \right) - (r - x_n^2) = \left( \frac{r \sin \theta - x_n}{\cos^2 \theta} \right) \geq 0,$$

or equivalently, 

$$|x'|^2 \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta = -(x - a)_n \tan \theta.$$ 

Hence $x - a \in C_0$, that is, $x \in a + C_0$ and the statement (1) holds.

(2) Let $x \in a + C_0$. If $y \in x + C_0$, then $(y - a)_n = (y - x)_n + (x - a)_n \leq 0$ and 

$$|y'| \leq |x'| + |y' - x'| \leq -(x - a)_n \tan \theta - (y - x)_n \tan \theta = -(y - a)_n \tan \theta,$$

which means that $y \in a + C_0$. Hence the statement (2) holds.

(3) Let $x \in (a + C_0) \cap C_1^c$. Then we have 

$$|x|_n \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.$$ 

Hence 

$$|x|^2 \leq \left( \frac{r}{\sin \theta} - x_n \right)^2 \tan^2 \theta + x_n^2 =: f(x_n).$$ 

To estimate the right-hand side in the above inequality for $x \in (a + C_0) \cap C_1^c$, we derive the range of $x_n$ for $x \in (a + C_0) \cap C_1^c$. If $x_n \geq 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if the condition (3.14) is satisfied. Thus $x_n$ must satisfy 

$$0 \leq x_n \leq \frac{r}{\sin \theta}.$$ 

On the other hand, if $x_n < 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if 

$$(-x_n) \tan(2 \theta) < |x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.$$

Hence, in particular, if $x \in (a + C_0) \cap C_1^c$ and $x_n < 0$, then $x_n$ must satisfy 

$$(-x_n) \tan(2 \theta) < \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta,$$
which yields the inequality
\[-\frac{r}{\cos \theta} < (\tan(2\theta) - \tan \theta) x_n.\]

Since
\[
\tan(2\theta) - \tan \theta = \frac{1}{2} \tan(2\theta)(1 - \tan^2 \theta)
\]
\[= \frac{1}{2} \tan(2\theta)(1 + \tan^2 \theta) = \frac{\tan(2\theta)}{2 \cos^2 \theta} > 0 \quad (0 < \theta < \frac{\pi}{4}),
\]
the above inequality is equivalent to
\[-\frac{2r \cos \theta}{\tan(2\theta)} < x_n(<0).
\]
In summary, the range of $x_n$ for $x \in (a + C_0) \cap C_1^c$ is
\[
\alpha := -\frac{2r \cos \theta}{\tan(2\theta)} < x_n \leq \frac{r}{\sin \theta} =: \beta
\]
and thus we obtain
\[
|x|^2 \leq f(x_n) \leq \sup_{s \in (\alpha, \beta)} f(s) = \max\{f(\alpha), f(\beta)\},
\]
where the last equality follows from the fact that $f(x_n)$ is a concave parabola. On the one hand, we have $f(\beta) = \beta^2 = \frac{r^2}{\sin^2 \theta}$. On the other hand, since
\[
\alpha = -\frac{2r \cos \theta \cos(2\theta)}{\sin(2\theta)} = -\frac{r \cos(2\theta)}{\sin \theta} = r(1 - 2 \cos^2 \theta),
\]
we have
\[
f(\alpha) = \left( \frac{r}{\sin \theta} - \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta} \right)^2 \tan^2 \theta + \frac{r^2 \cos^2(2\theta)}{\sin^2 \theta} = \frac{r^2}{\sin^2 \theta} \left( 4 \tan^2 \theta \cos^4 \theta + \cos^2(2\theta) \right) = \frac{r^2}{\sin^2 \theta}.
\]
Hence $|x|^2 \leq \frac{r^2}{\sin^2 \theta}$ and thus $x \in B_{r/\sin \theta}(0)$ for every $x \in (a + C_0) \cap C_1^c$. Therefore, the statement (3) holds and the lemma follows.

Now we can estimate the $b^\nu$-part of $S_{ij}\bar{u}_j$.

**Lemma 3.9.** Let $\nu \in [0, \infty]$. There exists a constant $c > 0$ such that
\[
[S_{ij}\bar{u}_j : b^\nu(\Omega)] \leq \frac{c}{\sin^{\nu/2} \theta} \|u_j\|_{L^\infty(\Omega)}
\]
for all $u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \ldots, n$.

**Proof.** First we note that for all $f \in L^1_{loc}(\Omega)$ the inequality
\[
[f : b^\nu(\Omega)] \leq \frac{\omega_\nu^{1/2}}{\sin^{\nu/2} \theta} \|f\|_{L^\infty(\Omega)}
\]
holds by Hölder’s inequality. Hence, to prove (3.15), it is sufficient to show the inequality
\[
[S_{ij}\bar{u}_j : b_2^\nu(\Omega)] \leq \frac{c}{\sin^{\nu/2} \theta} \left[ u_j : b_2^\nu/\sin \theta(\Omega) \right] \leq \frac{c\omega_\nu^{1/2}}{\sin^{\nu/2} \theta} \|u_j\|_{L^\infty(\Omega)}.
\]
The second inequality of (3.16) follows from the definition of \( [ \cdot : b_2^{\nu/\sin \theta}(\Omega) ] \). Let us show the first inequality. The singular integral \( S_{ij} \bar{u}_j \) is of the form

\[
S_{ij} \bar{u}_j(x) = (K_{ij} \ast \bar{u}_j)(x) = \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{u}_j(y) \, dy, \quad x \in \mathbb{R}^n.
\]

Since \( \text{supp} \, K_{ij} \subset -C_0 \) (see (3.4) and (3.2)) and \( \text{supp} \, u \subset \Omega \), we can write

\[
S_{ij} \bar{u}_j(x) = \int_{(x + C_0) \cap \Omega} K_{ij}(x - y) \bar{u}_j(y) \, dy, \quad x \in \mathbb{R}^n.
\]

Hence, if we set

\[
W_r(x_0) := \bigcup_{x \in B_r(x_0) \cap \Omega} (x + C_0) \cap \Omega
\]

for each \( x_0 \in \partial \Omega \) and \( r > 0 \) with \( B_r(x_0) \subset U_r(\partial \Omega) \), then we have

\[
S_{ij} \bar{u}_j(x) = \int_{(x + C_0) \cap \Omega} K_{ij}(x - y)(\bar{u}_j | W_r(x_0))(y) \, dy = [K_{ij} \ast (\bar{u}_j | W_r(x_0))](x)
\]

for all \( x \in B_r(x_0) \cap \Omega \), where

\[
(\bar{u}_j | W_r(x_0))(x) := \begin{cases} \bar{u}_j(x), & x \in W_r(x_0), \\ 0, & x \notin W_r(x_0). \end{cases}
\]

Since \( K_{ij} \) is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that

\[
\int_{B_r(x_0) \cap \Omega} |S_{ij} \bar{u}_j(x)|^2 \, dx \leq \int_{B_r(x_0) \cap \Omega} [K_{ij} \ast (\bar{u}_j | W_r(x_0))](x)^2 \, dx
\]

\[
\leq c \int_{\mathbb{R}^n} |(\bar{u}_j | W_r(x_0))(x)|^2 \, dx = c \int_{W_r(x_0)} |\bar{u}_j(x)|^2 \, dx
\]

with some constant \( c > 0 \). Now we recall the property of the infinite cone \( C_1 \):

\[
x + C_1 \subset \Omega^c \iff \Omega \subset (x + C_1)^c \quad \text{for all} \quad x \in \Omega^c.
\]

By this property we have

\[
W_r(x_0) \subset \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r(x_0) \cap \Omega,
\]

where \( A_r(x_0) \) is given by (3.13), and thus Lemma 3.8 yields

\[
W_r(x_0) \subset A_r(x_0) \cap \Omega \subset B_{r/\sin \theta}(x_0) \cap \Omega.
\]

Hence we have

\[
\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |S_{ij} \bar{u}_j(x)|^2 \, dx \leq \frac{c}{r^n} \int_{W_r(x_0)} |\bar{u}_j(x)|^2 \, dx
\]

\[
\leq \frac{c}{r^n} \int_{B_{r/\sin \theta}(x_0) \cap \Omega} |\bar{u}_j(x)|^2 \, dx = \frac{c}{\sin^n \theta} \left( \frac{\sin \theta}{r} \right)^n \int_{B_{r/\sin \theta}(x_0) \cap \Omega} |u_j(x)|^2 \, dx
\]

\[
\leq \frac{c}{\sin^n \theta} \left[ u_j : b_2^{\nu/\sin \theta}(\Omega) \right]^2
\]

for every \( x_0 \in \partial \Omega \) and \( r > 0 \) with \( B_r(x_0) \subset U_r(\partial \Omega) \), which yields

\[
|S_{ij} \bar{u}_j : b_2^{\nu/\sin \theta}(\Omega)|^2 \leq \frac{c}{\sin^n \theta} \left[ u_j : b_2^{\nu/\sin \theta}(\Omega) \right]^2.
\]

The proof is complete. \( \Box \)
Now we obtain an estimate for the $BMO_b^c(\Omega)$-norm of $Tu$.

**Theorem 3.10.** Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that
\[ \|Tu : BMO_b^c(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)} \]
for all $u \in C_c^\infty(\Omega)$.

**Proof.** Since the $i$-th component of $Tu$, $i = 1, \ldots, n$, is of the form (3.7), we have by (3.9), (3.11) and (3.15) that
\[
\|Tu : BMO_b^c(\Omega)\| \\
\leq c \sum_{i,j=1}^n \|a_{ij}u_j : BMO_b^c(\Omega)\| + |S_{ij}u_j : BMO^c(\Omega)| + |S_{ij} : b^c(\Omega)|
\]
\[
\leq c \sum_{j=1}^n \|u_j\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}
\]
with a positive constant $c$. \qed

3.2. Non-Helmholtz projection. As in the previous subsection, let $\Omega$ denote a Lipschitz half-space in $\mathbb{R}^n$.

**Definition 3.11.** For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field $Q'u$ on $\mathbb{R}^n$ as $Q'u := u - Tu$. Here the operator $T$ is given in Definition 3.4.

For a vector field $u \in C_c^\infty(\Omega)$, the vector field $Tu$ is smooth in $\mathbb{R}^n$ and
\[ \text{div} \, Tu = \text{div} \, u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial \Omega. \]
Moreover, $Tu = 0$ for all $u \in C_c^\infty(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbb{R}^n$ and
\[ \text{div} \, Q'u = 0 \quad \text{in} \quad \Omega, \quad Q'u = 0 \quad \text{on} \quad \partial \Omega \]
for all $u \in C_c^\infty(\Omega)$, and $Q'u = u$ for all $u \in C_c^\infty(\Omega)$. Note that $Q'$ is not a projection from $C_c^\infty(\Omega)$ onto $C_c^\infty(\Omega)$, since the support of $Tu$ may be unbounded and thus $Q'u$ is not in $C_c^\infty(\Omega)$ in general. However, $Q'$ maps $C_c^\infty(\Omega)$ into $L_b^p(\Omega)$.

**Lemma 3.12.** For all $u \in C_c^\infty(\Omega)$ and $p \in (1, \infty)$, we have $Q'u \in L_b^p(\Omega)$.

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1, \infty)$, let $G_p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L_{loc}^1(\Omega)\}$.

**Proposition 3.13.** Let $p \in (1, \infty)$. For every $\nabla q \in G_p(\Omega)$, there exists a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbb{R}^n)$ such that
\[ \lim_{k \to \infty} \|\nabla q - \nabla q_k\|_{L^p(\Omega)} = 0. \]

**Proof.** Since the restriction of $C_c^\infty(\mathbb{R}^n)$ on $\Omega$ is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G_p(\Omega)$ there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\Omega)$ such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space $\mathbb{R}^n_+$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x',x_n) \in \mathbb{R}^n \mid x_n > b(x')\}$. As in Section 2, let $F(x) := (x',x_n-h(x'))$ be a bi-Lipschitz map from $\Omega$ to $\mathbb{R}^n_+$. Let $\nabla q \in G_p(\Omega)$ and $\tilde{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y',y_n+h(y'))$ is the inverse mapping of $F$. Then, since $\nabla q(y) = \nabla F^{-1}(y)\nabla q(F^{-1}(y))$ for $y \in \mathbb{R}^n_+$ and each component
of $\nabla F^{-1}$ is bounded (because $h$ is Lipschitz continuous), we have $\nabla \tilde{q} \in G_p(\mathbb{R}_+^n)$. Hence, by our assumption that the claim is valid for $\mathbb{R}_+^n$, there is a sequence $\{\tilde{q}_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\mathbb{R}_+^n)$ such that $\lim_{k \to \infty} \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbb{R}_+^n)} = 0$.

Let $q_k := \tilde{q}_k \circ F$ for each $k \in \mathbb{N}$. Then, since

$$\nabla q(x) = \nabla F(x) \nabla \tilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x) \nabla \tilde{q}_k(F(x)),$$

and each component of $\nabla F$ is bounded, we have $q_k \in W^{1,p}(\Omega)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \leq c \|\nabla q - \nabla \tilde{q}_k\|_{L^p(\mathbb{R}_+^n)} \to 0$$
as $k \to \infty$. Thus the claim is valid for general Lipschitz half-spaces $\Omega$.

(2) Now we prove the claim for $\Omega = \mathbb{R}_+^n$. We follow the idea of the proof of the claim in the case $\Omega = \mathbb{R}^n$, see [34, Lemma 2.5.4]. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function such that

$$0 \leq \varphi \leq 1 \quad \text{in } \mathbb{R}^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } \mathbb{R}^n \setminus B_2(0)$$

and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Then, $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in \mathbb{R}^n$ and supp $\varphi_k \subset B_{2k}(0)$, supp $\varphi_k \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbb{N}$.

Let $\nabla q \in G_p(\mathbb{R}_+^n)$. Then $q \in W^{1,p}(\mathbb{R}_+^n)$, that is, $q \in W^{1,p}(U)$ for every bounded subset $U$ of $\mathbb{R}_+^n$; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting $G_k := \mathbb{R}_+^n \cap (B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbb{N}$, we have $q \in W^{1,p}(G_k)$ and thus there is a constant $a_k$ such that $\int_{G_k} (q - a_k) \, dx = 0$ for each $k \in \mathbb{N}$. From this equality and the change of variables $x = ky$ for $x \in G_k$ and $y \in G_1$ we have

$$\int_{G_k} (q(ky) - a_k) \, dy = k^{-n} \int_{G_k} (q(x) - a_k) \, dx = 0.$$}

Hence we can apply Poincaré’s inequality to $q(ky) - a_k$ on $G_1$ and get

$$\left( \int_{G_1} |q(ky) - a_k|^p \, dy \right)^{1/p} \leq c \left( \int_{G_1} |\nabla (q(ky))|^p \, dy \right)^{1/p}$$

with a constant $c > 0$ independent of $k$. In this inequality, we observe that

$$\int_{G_1} |q(ky) - a_k|^p \, dy = k^{-n} \int_{G_k} |q(x) - a_k|^p \, dx,$$

$$\int_{G_1} |\nabla (q(ky))|^p \, dy = k^n \int_{G_k} |\nabla q(x)|^p \, dx$$

by the change of variables $x = ky$ and thus

$$\|q - a_k\|_{L^p(G_k)} \leq c k \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbb{N}.$$}

For each $k \in \mathbb{N}$, let $q_k := \varphi_k(q - a_k)$ on $\mathbb{R}_+^n$. Then since supp $q_k \subset \mathbb{R}_+^n \cap B_{2k}(0)$ holds by the relation supp $\varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(\mathbb{R}_+^n)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\mathbb{R}_+^n)} \leq \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbb{R}_+^n)} + \|\nabla \varphi_k(q - a_k)\|_{L^p(\mathbb{R}_+^n)}.$$}

Since $0 \leq \varphi_k(x) \leq 1$ and $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in \mathbb{R}_+^n$ and $\nabla q \in L^p(\mathbb{R}_+^n)$, the dominated convergence theorem yields

$$\lim_{k \to \infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbb{R}_+^n)} = 0.$$

On the other hand, since $\nabla \varphi_k = k^{-1} (\nabla \varphi)_k$ and supp $\varphi_k \subset C_k$ for each $k \in \mathbb{N}$, it follows from (3.19) and the dominated convergence theorem that

$$\|\nabla \varphi_k(q - a_k)\|_{L^p(\mathbb{R}_+^n)} \leq c k^{-1} \|q - a_k\|_{L^p(G_k)} \leq c \|\nabla q\|_{L^p(\mathbb{R}_+^n)} \to 0$$

and so

$$\lim_{k \to \infty} \|\nabla \varphi_k(q - a_k)\|_{L^p(\mathbb{R}_+^n)} = 0.$$
as \( k \to \infty \). Applying (3.21) and (3.22) to (3.20) we have
\[
\lim_{k \to \infty} \| \nabla q - \nabla q_k \|_{L^p(\mathbb{R}^n)} = 0,
\]
where \( q_k \in W^{1,p}(\mathbb{R}^n_+) \) for all \( k \in \mathbb{N} \). Hence the claim is valid when \( \Omega = \mathbb{R}^n_+ \) and the proposition follows. \( \Box \)

**Proof of Lemma 3.12.** Let \( u \in C_c^\infty(\Omega) \) and \( p \in (1, \infty) \). Then, since \( Tu \in L_p(\Omega) \) by Theorem 3.5, we have \( Q'u = u - Tu \in L^p(\Omega) \). To show \( Q'u \in L^p_c(\Omega) \), we employ a characterization of elements of \( L^p_c(\Omega) \) ([19, Lemma III.2.1]): a vector field \( v \in L^p(\Omega) \) is in \( L^p_c(\Omega) \) if and only if
\[
\int_{\Omega} v \cdot \nabla q \, dx = 0 \quad \text{for all} \quad \nabla q \in G_p'(\Omega) \quad \left( p' := \frac{p}{p-1} \right).
\]
Let \( \nabla q \) be any element of \( G_p'(\Omega) \). From Proposition 3.13, there is a sequence \( \{ q_k \}_{k=1}^\infty \) of functions in \( C_c^\infty(\mathbb{R}^n) \) such that the equality (3.18) with \( p \) replaced by \( p' \) holds. Since \( Q'u \) is defined and smooth in \( \mathbb{R}^n \) for \( u \in C_c^\infty(\Omega) \) and \( q_k \in C_c^\infty(\mathbb{R}^n) \), integration by parts yields
\[
\int_{\Omega} Q'u \cdot \nabla q_k \, dx = - \int_{\Omega} q_k \, \text{div} \, Q'u \, dx + \int_{\partial \Omega} q_k \, Q'u \cdot \nu \, d\mathcal{H}^{n-1}
\]
for all \( k \in \mathbb{N} \), where \( \nu \) denotes the unit outer normal vector field of \( \partial \Omega \). We apply (3.17) to the right-hand side of this equality to get \( \int_{\Omega} Q'u \cdot \nabla q_k \, dx = 0 \) for all \( k \in \mathbb{N} \). Since \( Q'u \in L^p(\Omega) \) and (3.18) with \( p \) replaced by \( p' \) holds, the above equality implies that
\[
\int_{\Omega} Q'u \cdot \nabla q \, dx = \lim_{k \to \infty} \int_{\Omega} Q'u \cdot \nabla q_k \, dx = 0.
\]
Hence by the characterization of elements of \( L^p_c(\Omega) \) we conclude that \( Q'u \in L^p_c(\Omega) \) for all \( u \in C_c^\infty(\Omega) \). The proof is complete. \( \Box \)

**Remark 3.14.**

1. Let \( p \in (1, \infty) \). By Theorem 3.5 and Lemma 3.12, we have \( Q'u \in L^p_c(\Omega) \) and \( \| Q'u \|_{L^p(\Omega)} \leq c \| u \|_{L^p(\Omega)} \) for all \( u \in C_c^\infty(\Omega) \). Moreover, \( Q'u = u \) holds for all \( u \in C_c^\infty(\Omega) \). Hence, by the density argument, \( Q' \) extends uniquely to a bounded linear operator on \( L^p(\Omega) \) that is a projection onto \( L^p_c(\Omega) \).
2. The projection onto \( L^p_c(\Omega) \) given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each \( u \in C_c^\infty(\Omega) \) there would exist \( \pi \in L^1_{\text{loc}}(\Omega) \) such that \( (I - Q')u = \nabla \pi \) holds. Since \( (I - Q')u = Tu = K * \text{div} \, u \) for \( u \in C_c^\infty(\Omega) \), the existence of such \( \pi \) would imply that \( \partial_i (K_i * \text{div} \, u) = \partial_i (K_j * \text{div} \, u) \) for all \( i, j = 1, \ldots, n \). For each \( f \in C_c^\infty(\Omega) \) with \( \int_{\Omega} f \, dx = 0 \) there is \( u \in C_c^\infty(\Omega) \) satisfying \( f = \text{div} \, u \). This is possible since we are able to apply Bogovskii’s lemma to a bounded Lipschitz domain \( D \subset \Omega \) containing the support of \( f \) (see [19, Theorem III.3.3]). Thus the above equality would imply that \( \partial_i K_i = \partial_i K_j + c \) with some constant \( c \) for all \( i, j = 1, \ldots, n \) as a distribution. This contradicts the fact that \( \partial_i K_i \neq \partial_i K_j + c \) for \( i \neq j \) as observed in (3.4).
3. It is possible to prove the characterization
\[
L^p_c(\Omega) = \{ u \in L^p(\Omega) \mid \text{div} \, u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial \Omega \}
\]
if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains.
with a constant $c > 20 M$. BOLKART, Y. GIGA, T.-H. MIURA, T. SUZUKI, AND Y. TSUTSUI

We shall prove two auxiliary propositions for the above lemma. For $p \in (1, \infty)$, let $W^{1, p}_0(\Omega)$ be the $W^{1, p}$-closure of $C_0^\infty(\Omega)$.

**Proposition 3.16.** Let $\Omega$ be a Lipschitz half-space. For all $p \in (1, \infty)$ we have $L^p_0(\Omega) \cap W^{1, p}_0(\Omega) \subset W^{1, p}_0(\Omega)$. Thus $L^p_0(\Omega) \cap W^{1, p}_0(\Omega) = W^{1, p}_0(\Omega)$.

**Proof.** Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a function such that

$$0 \leq \rho \leq 1 \text{ in } \mathbb{R}^n, \quad \text{supp } \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \, dx = 1$$

and $\rho\delta(x) := \delta^{-n} \rho(\delta^{-1}x)$ for $\delta > 0$, $x \in \mathbb{R}^n$. Let $u \in L^p_0(\Omega) \cap W^{1, p}_0(\Omega)$. Then there is a sequence $\{u_k\}_{k=1}^\infty$ of functions in $C^\infty(\Omega)$ such that $\lim_{k \to \infty} \|u - u_k\|_{L^p(\Omega)} = 0$.

For $a > 0$, we define a vector field $u^a$ on $\Omega$ as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ 0, & h(x') < x_n \leq h(x') + a \end{cases}$$

and $u_k^a = (u_k)^a$ similarly. Then it is clear that $u^a \in W^{1, p}_0(\Omega)$ and $u_k^a \in C^\infty(\Omega)$ for all $a > 0$. Moreover, we have

$$\|u^a - u_k^a\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all } a > 0, \quad \lim_{a \to 0} \|u - u^a\|_{W^{1, p}(\Omega)} = 0.$$

By the second equality and the fact that $W^{1, p}_0(\Omega)$ is closed in $W^{1, p}(\Omega)$, it is sufficient for showing $u \in W^{1, p}_0(\Omega)$ to prove $u^a \in W^{1, p}_0(\Omega)$ for all $a > 0$.

For each $a > 0$, there is a constant $d = d(a) > 0$ such that $\text{dist}(\text{supp } u_k^a, \partial \Omega) \geq d$ for all $k \in \mathbb{N}$. Then, for a given $\varepsilon > 0$, we can take $\delta \in (0, d/2)$ so small that

$$\|u^a - u_k^a\|_{W^{1, p}(\Omega)} < \varepsilon$$

since $u^a \in W^{1, p}_0(\Omega)$. Also, since $\nabla \rho_\delta = \delta^{-1}(\nabla \rho)_\delta$, we have

$$\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1, p}(\Omega)} \leq c(\|u^a - u_k^a\|_{L^p(\Omega)} + \|u^a * \nabla \rho_\delta - u_k^a * \nabla \rho_\delta\|_{L^p(\Omega)})$$

$$= c(\|u^a - u_k^a\|_{L^p(\Omega)} + \delta^{-1}(\|u^a - u_k^a\|_{L^p(\Omega)} - \|u^a - u_k^a\|_{L^p(\Omega)} \oplus \delta^{-1}(\nabla \rho_\delta)\|_{L^p(\Omega)}))$$

$$\leq c(1 + \delta^{-1})(\|u^a - u_k^a\|_{L^p(\Omega)} = c(1 + \delta^{-1})\|u - u_k\|_{L^p(\Omega)})$$

with a constant $c > 0$ independent of $\varepsilon$ and $\delta$. Hence by taking $k \in \mathbb{N}$ so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})},$$

we have $\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1, p}(\Omega)} < \varepsilon/2$ and thus

$$\|u^a - u_k^a\|_{W^{1, p}(\Omega)} \leq \|u^a - u_k^a\|_{W^{1, p}(\Omega)} + \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1, p}(\Omega)} < \varepsilon.$$
Thus $u^n \ast \rho_k \in C_c^\infty(\Omega)$ and $u^n$ is approximated by elements of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, which means that $u^n \in W^{1,p}_0(\Omega)$. Hence $u \in W^{1,p}_0(\Omega)$ and the proof is now complete.

\[ \square \]

**Proposition 3.17.** Let $\nu \in (0, \infty]$. If $p > n$, then $W^{1,p}_0(\Omega) \subset \text{VMO}^{\infty,\nu}_{b,0,\nu}(\Omega)$.

**Proof.** Let $u \in W^{1,p}_0(\Omega)$ and $u_k \in C_c^\infty(\Omega)$ such that $\lim_{k \to \infty} \| u - u_k \|_{W^{1,p}(\Omega)} = 0$. Since $p > n$ and $u, u_k \in W^{1,p}_0(\Omega)$, Morrey’s inequality (see e.g. [7, Theorem 4.12]) implies

\[ \| u - u_k \|_{L^\infty(\Omega)} \leq c \| u - u_k \|_{W^{1,p}(\Omega)} \]

with a positive constant $c$ independent of $u$ and $u_k$. Thus we have

\[ \| u - u_k : \text{BMO}^{\infty,\nu}(\Omega) \| \leq (2 + \omega_n) \| u - u_k \|_{L^\infty(\Omega)} \leq c \| u - u_k \|_{W^{1,p}(\Omega)} \to 0 \]

as $k \to \infty$. Hence $u \in \text{VMO}^{\infty,\nu}_{b,0,\nu}(\Omega)$ and the proof is now complete. \[ \square \]

**Proof of Lemma 3.15.** Since $u \in C_c^\infty(\Omega)$ and thus $\partial_i u \in C_c^\infty(\Omega)$ for all $i = 1, \ldots, n$, it follows from Lemma 3.12 that $Q' u \in L^\infty_c(\Omega)$ and $\partial_i Q' u = Q'(\partial_i u) \in L^\infty(\Omega)$ for all $\tau \in (1, \infty)$ and $i = 1, \ldots, n$. From this fact and the equality (3.17), we have $Q' u \in L^\infty_c(\Omega) \cap W^{1,p}_0(\Omega)$ for all $\tau \in (1, \infty)$. Hence, by taking $\tau > n$, we can apply Proposition 3.16 and Proposition 3.17 to obtain $Q' u \in \text{VMO}^{\infty,\nu}_{b,0,\nu}(\Omega)$. \[ \square \]

**Remark 3.18.** Let $\nu \in (0, \infty]$. Theorem 3.10 and Lemma 3.15 imply that $Q' u \in \text{VMO}^{\infty,\nu}_{b,0,\nu}(\Omega)$ and $\| Q' u : \text{BMO}^{\infty,\nu}(\Omega) \| \leq c \| u \|_{L^\infty(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Also, we have $Q' u = u$ for all $u \in C_c^\infty(\Omega)$. Hence $Q'$ extends uniquely to a bounded linear operator (again referred to as $Q'$) from $C_0(\Omega)$, which is the $L^\infty$-closure of $C_c^\infty(\Omega)$, into $\text{VMO}^{\infty,\nu}_{b,0,\nu}(\Omega)$ that satisfies $Q' u = u$ for all $u \in C_0(\Omega)$.

Now let us extend $Q'$ to a linear operator that gives the projection mentioned in Theorem 1.4. For $p \in (1, \infty)$, we define a Banach space $X_p$ and its norm as

\[ X_p := L^p(\Omega) \cap C_0(\Omega), \quad \| u \|_{X_p} := \max\{\| u \|_{L^p(\Omega)}, \| u \|_{L^\infty(\Omega)}\}. \]

Note that the Banach space $C_0(\Omega)$ consists of all continuous functions $f$ on $\Omega$ such that the set $\{ x \in \Omega \mid | f(x) | \geq \epsilon \}$ is compact in $\Omega$ for every $\epsilon > 0$ (see e.g. [32, Theorem 3.17]).

**Lemma 3.19.** For each $p \in (1, \infty)$, the linear subspace $C_c^\infty(\Omega)$ is dense in $X_p$.

**Proof.** The proof is more or less standard (see e.g. [27, Corollary 19.24]). We give it for completeness. Let $u \in X_p$ and $\Omega_k := \{ x \in \Omega \mid | x | \leq k, \text{dist}(x, \partial \Omega) \geq 1/k \}$ for $k \in \mathbb{N}$. For any given $\epsilon > 0$, the set $\{ x \in \Omega \mid | u(x) | \geq \epsilon/2 \}$ is compact in $\Omega$ since $u \in C_0(\Omega)$. Moreover, since $u \in L^p(\Omega)$, we can take $k \in \mathbb{N}$ so large that

\[ \| u \|_{L^p(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}, \quad \| u \|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}. \]

Let $\varphi \in C_c^\infty(\Omega)$ be a continuous cut-off function such that

\[ 0 \leq \varphi \leq 1 \quad \text{in} \quad \Omega, \quad \varphi = 1 \quad \text{in} \quad \Omega_k, \quad \varphi = 0 \quad \text{in} \quad \Omega \setminus \Omega_{2k}. \]

Since $u - \varphi u = 0$ in $\Omega_k$ and $| u - \varphi u | \leq | u |$ in $\Omega \setminus \Omega_k$, it follows from (3.23) that

\[ \| u - \varphi u \|_{L^1(\Omega)} \leq \| u \|_{L^p(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}, \quad \| u - \varphi u \|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}, \quad \| u - \varphi u \|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}. \]
Let $\rho_\delta$ be a mollifier as in the beginning of the proof of Proposition 3.16. Since
\[ \varphi u \in L^p(\Omega), \quad \text{dist}(\text{supp} (\varphi u), \partial \Omega) \geq \frac{1}{2k}, \]
we can take $\delta \in (0, 1/4k)$ so small that
\[ u_\delta := \rho_\delta * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}. \]
(3.25)

On the other hand, since $\varphi u$ is uniformly continuous on $\Omega_{4k}$, we can again choose $\delta \in (0, 1/4k)$ so small that $\| \varphi u - u_\delta \|_{L^\infty(\Omega_{4k})} < \varepsilon/2$. Moreover, since $\text{supp} (\varphi u) \subset \Omega_{2k}$ and $\delta \in (0, 1/4k)$, we have $\varphi u = u_\delta = 0$ outside of $\Omega_{4k}$ and thus
\[ \| \varphi u - u_\delta \|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}. \]
(3.26)

Combining (3.24), (3.25) and (3.26), we obtain $u_\delta \in C_c^\infty(\Omega)$ and
\[ \| u - u_\delta \|_{X_p} = \max\{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon. \]
Hence the lemma follows.

Let $Y_p := L^p_b(\Omega) \cap VMO_b^{\infty,p}(\Omega)$ for $p \in (1, \infty)$, $\nu \in (0, \infty]$. Since $L^p_b(\Omega)$ and $VMO_b^{\infty,p}(\Omega)$ are closed in $L^p(\Omega)$ and $BMO_b^{\infty,p}(\Omega)$, respectively, $Y_p$ becomes a Banach space under the norm $\|v\|_{Y_p} := \max\{\|v\|_{L^p(\Omega)}, \|v : BMO_b^{\infty,p}(\Omega)\|\}$.

**Theorem 3.20.** Let $p \in (1, \infty)$ and $\nu \in (0, \infty]$. The linear operator $Q'$ given in Definition 3.11 extends uniquely to a bounded linear operator $Q_p$ from $X_p$ into $Y_p$. Moreover, there exists a constant $c > 0$ such that
\[ \|Q_p u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q_p u : BMO_b^{\infty,p}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)} \]
for all $u \in X_p$ and $Q_p u = u$ holds for all $u$ in the $X_p$-closure of $C_c^{\infty}(\Omega)$.

**Proof.** Let $u \in C_c^{\infty}(\Omega)$. Then we have $Q'u \in Y_p$ by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant $c > 0$ independent of $u$ such that
\[ \|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q'u : BMO_b^{\infty,p}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}. \]
(3.28)

Hence we have $Q'u \in Y_p$ and $\|Q'u\|_{Y_p} \leq c\|u\|_{X_p}$ for all $u \in C_c^{\infty}(\Omega)$. Since $C_c^{\infty}(\Omega)$ is dense in $X_p$ by Lemma 3.19, the operator $Q'$ extends uniquely to a bounded linear operator $Q_p$ from $X_p$ into $Y_p$. Also, it follows from (3.28) that the inequality (3.27) holds for all $u \in X_p$. Since $Q'u = u$ holds for all $u$, the density argument we have $Q_p u = u$ for all $u$ in the $X_p$-closure of $C_c^{\infty}(\Omega)$. The proof is complete.

Finally, Theorem 1.4 follows from Theorem 3.20 with $p = 2$, that is, the linear operator $Q$ in Theorem 1.4 is given by $Q = Q_2$.

4. **Analyticity in $L^p$**

In this section we shall give a complete proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $S(t)$ be the Stokes semigroup in $\tilde{L}^p_b$ constructed by [14], [16]. To show that $S(t)$ forms an analytic semigroup in $L^p_b$ ($2 \leq p < \infty$) it suffices to prove that there exists a constant $C$ that
\[ \|S(t)u_0\|_p \leq C\|v_0\|_p \]
(4.1)
for all \( v_0 \in C^\infty_c(\Omega) \) and for all \( t \in (0, 1) \). Let \( Q \) be the operator in Theorem 1.4. Since \( Q \) is bounded in \( L^2 \) and maps \( L^2 \) to \( L^2_{\sigma} \) and \( S(t) \) fulfills (4.1) and (4.2) for \( p = 2 \), we have

\[
\|S(t)Qu\|_2 \leq C\|u\|_2
\]

for all \( u \in C_c(\Omega) \) and \( t \in (0, 1) \). Since \( \Omega \) is admissible as proved in [5], \( S(t) \) forms an analytic semigroup in \( VMO^{\infty,\nu}_{b,0,\sigma} \) by Theorem 1.2. We conclude that

\[
\|S(t)Qu : BMO^{\infty,\nu}_{b}(\Omega)\| \leq C\|u\|_\infty
\]

(4.5)

\[ t \frac{d}{dt} S(t)Qu \mid_{C_c(\Omega)} : BMO^{\infty,\nu}_{b,0}(\Omega) \mid \leq C\|u\|_\infty \]

(4.6)

for all \( u \in C_c(\Omega) \) and \( t \in (0, 1) \) since \( Q \) fulfills

\[
\|Qu : BMO^{\infty,\nu}_{b,0}(\Omega)\| \leq C\|u\|_\infty, \; Qu \in VMO^{\infty,\nu}_{b,0,\sigma}
\]

for all \( u \in C_c(\Omega) \) by Theorem 1.4. (Note that we have a stronger statement than (4.6) by replacing the \( BMO_b \) type norm by the \( L^\infty \) norm since we have the regularizing estimate (1.3)). We apply an interpolation result (Theorem 1.3) to (4.3) and (4.5) and to (4.4) and (4.6) to get, respectively

\[
\|S(t)Qu\|_p \leq C\|u\|_p
\]

(4.7)

\[
\left\| t \frac{d}{dt} S(t)Qu \right\|_p \leq C\|u\|_p
\]

(4.8)

for all \( u \in C_c(\Omega) \) and for all \( t \in (0, 1) \). Since \( Qu = u \) for \( u \in C^\infty_c(\Omega) \) this yields (4.1) and (4.2).

It remains to prove that \( S(t) \) is a \( C_0 \)-semigroup in \( L^2_{\sigma} \). Since \( C^\infty_c(\Omega) \) is dense in \( L^2_{\sigma} \), for \( v_0 \in L^2_{\sigma} \), there is \( v_{0m} \in C^\infty_c(\Omega) \) such that \( \|v_0 - v_{0m}\|_p \to 0 \) as \( m \to \infty \). By (4.1) we observe that

\[
\|S(t)v_0 - v_0\|_p \leq \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p \\
\leq C\|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p.
\]

(4.1)

Sending \( t \downarrow 0 \), we get

\[
\lim_{t \downarrow 0}\|S(t)v_0 - v_0\|_p \leq C\|v - v_{0m}\|_p,
\]

since \( S(t)v_{0m} \to v_{0m} \) in \( L^2_{\sigma} \) as \( t \downarrow 0 \) by [14], [16]. Sending \( m \to \infty \), we conclude that \( S(t)v_0 \to v_0 \) in \( L^2_{\sigma} \) as \( t \downarrow 0 \).

\[ \square \]

**Remark 4.1.** In a similar way as we derived (4.5) and (4.6) we are able to derive from the \( L^\infty-BMO \) estimates in [10] that

\[
t\|\nabla^2 S(t)Qu : BMO^{\infty,\nu}_{b}(\Omega)\| \leq C\|u\|_\infty
\]

\[
t^{1/2} \|\nabla S(t)Qu : BMO^{\infty,\nu}_{b}(\Omega)\| \leq C\|u\|_\infty
\]

for all \( u \in C_c(\Omega) \) and \( t \in (0, 1) \).
Note that $L^2$ results
\[
t \| \nabla^2 S(t)u \|_2 \leq C \| u \|_2
\]
and
\[
t^{1/2} \| \nabla S(t)u \|_2 \leq C \| u \|_2
\]
easily follow from the analyticity of $S(t)$ in $L^2$ and $L^2$-boundedness of $Q$ if one observes that $\| \nabla u \|_2 = (Au, u)_{L^2}$ and
\[
\| \nabla^2 u \|_2 \leq C (\| Au \|_2 + \| \nabla u \|_2 + \| u \|_2)
\]
(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where $A$ is the Stokes operator in $L^2$. Interpolating the $L^2$ results and the above $L^\infty$-$BMO$ results, we are able to prove that there is $C_p > 0$ satisfying
\[
t \| \nabla^2 S(t)v_0 \|_p \leq C_p \| v_0 \|_p
\]
and
\[
t^{1/2} \| \nabla S(t)v_0 \|_p \leq C_p \| v_0 \|_p
\]
for all $v_0 \in L^p(\Omega)$ and $t \in (0,1)$ with $p \in (2, \infty)$.

References

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