ON ANALYTICITY OF THE $L^p$-STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

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Abstract. Consider the Stokes equations in a sector-like $C^3$ domain $\Omega \subset \mathbb{R}^3$. It is shown that the Stokes operator generates an analytic semigroup in $L^p_\sigma(\Omega)$ for $p \in [2, \infty)$. This includes domains where the $L^p$-Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in $VMO$ and $L^2$ by constructing a suitable non-Helmholtz projection to solenoidal spaces.

1. Introduction

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \text{div} \, v = 0 \quad \text{in} \quad \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where $\Omega$ is a domain in $\mathbb{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a $C_0$-analytic semigroup in $L^p_\sigma(\Omega)$ for various domains like smooth bounded domains ([21], [35]). Here $L^p_\sigma(\Omega)$ denotes the $L^p$-closure of $C_\infty^\infty(\Omega)$, the space of all solenoidal vector fields with compact support in $\Omega$. More recently, it has been proved in [20] that $S(t)$ always forms a $C_0$-analytic semigroup in $L^p_\sigma(\Omega)$ for any uniformly $C^2$-domain $\Omega$ provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \{ \nabla q \in L^p(\Omega) \mid q \in L^1_{loc}(\Omega) \}$. In [20] the $L^q$ maximal regularity in time with values in $L^p_\sigma(\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The $L^p$-Helmholtz decomposition holds for various domains like bounded or exterior domains with
smooth boundary for $1 < p < \infty$ ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the $L^p$-Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely. Let $C(\vartheta)$ denote the cone of the form

$$C(\vartheta) = \{ x = (x', x_n) \in \mathbb{R}^n \mid -x_n \geq |x| \cos(\vartheta/2) \} ,$$

where $\vartheta \in (0, 2\pi)$ is the opening angle. When $n = 2$, we simply say that $C(\vartheta)$ is a sector. We say that a planar domain $\Omega$ is a sector-like domain with opening angle $\vartheta$ if $\Omega \setminus B_R(0) = C(\vartheta) \setminus B_R(0)$ for some $R > 0$ (up to rotation and translation), where $B_R(0)$ is an open disk of radius $R$ centered at the origin.

It is known that the $L^p$-Helmholtz decomposition fails for a sector-like domain $\Omega$ when $p > q'_0$ or $p < q_\vartheta$ with $q_\vartheta = 2/(1 + \pi/\vartheta)$, $1/q_\vartheta + 1/q'_0 = 1$ even if the boundary $\partial \Omega$ is smooth [28, Example 2, Fig. 5] while for $p \in (q_\vartheta, q'_0)$ the $L^p$-Helmholtz decomposition holds. This means that if the opening angle $\vartheta$ is larger than $\pi$, there always exists $p > 2$ such that the $L^p$-Helmholtz decomposition fails.

It has been a longstanding open question whether or not the existence of the $L^p$-Helmholtz decomposition is necessary for $L^p$ analyticity of $S(t)$. In this paper, we give a negative answer for this question by proving that there is a domain $\Omega$ for which $S(t)$ is analytic in $L^p_0$ while the $L^p$-Helmholtz decomposition fails. This is a subtle problem since the existence of the $L^p$-Helmholtz projection is known to be necessary for $L^p$ solvability of the resolvent equation ([33]). However, in this statement the external force term is allowed to be in the more general space $L^p$ instead of $L^p_0$. Our problem is different from that in [33].

We say that $\Omega$ has a $C^k$ graph boundary if $\Omega$ is of the form

$$\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x_n > h(x') \}$$

(up to translation and rotation) with some real-valued $C^k$ function $h$ with variable $x' \in \mathbb{R}^{n-1}$.

**Theorem 1.1.** Let $\Omega$ be a sector-like domain in $\mathbb{R}^2$ having a $C^4$ graph boundary. Then $S(t)$ forms a $C_0$-analytic semigroup in $L^p_0(\Omega)$ for all $p \in [2, \infty)$.

Here is our strategy to prove Theorem 1.1. It is by now well-known that $S(t)$ forms an analytic semigroup in $L^p_0$, i.e., $L^p_0 = L^p_0 \cap L^2_0$ ($p \geq 2$), $L^p = L^p_0 + L^2_0$ ($1 < p < 2$) ([14], [15], [16]). Thus $S(t)v_0$ is well-defined for $v_0 \in C^\infty_{c, \sigma}(\Omega)$. To show Theorem 1.1, a key step is to prove the two estimates

$$\tag{1.1} \|S(t)v_0\|_p \leq C\|v_0\|_p$$

$$\tag{1.2} \frac{t}{\|D S(t)v_0\|_p} \leq C\|v_0\|_p$$

for all $v_0 \in C^\infty_{c, \sigma}(\Omega)$, $t \in (0, 1)$, where $\|v_0\|_p$ denotes the $L^p$-norm of $v_0$. The constant $C$ should be taken independent of $t$ and $v_0$. We shall establish (1.1) and (1.2) by interpolation since both estimates are known for $p = 2$.

We are tempted to interpolate the $L^\infty$ type result obtained in [5] with the $L^2$-result. In fact, in [5] the estimates (1.1) and (1.2) with $p = 2$ are established for all $v_0 \in C^\infty_{c, \sigma}(\Omega)$, the $L^\infty$-closure of $C^\infty_{c, \sigma}(\Omega)$ for a $C^2$ sector-like domain $\Omega$ in $\mathbb{R}^2$. However, it is not clear that the complex interpolation space $[L^2_0, C^\infty_{c, \sigma}]_\rho$ agrees with $L^p_0$ with $2/p = 1 - \rho$ although it is well-known as the Riesz-Thorin theorem that $[L^2, L^\infty]_\rho = L^p$. To interpolate, we would need a projection to solenoidal spaces...
which is almost impossible since such a projection involves the singular integral operator which is not bounded in $L^\infty$.

To circumvent this difficulty, we consider the Stokes semigroup $S(t)$ in $BMO$-type spaces as studied in [10], [11], [12]. For $p \in [1, \infty)$, $\mu \in (0, \infty]$ we define the $BMO$ seminorm

$$[f : BMO^\mu_p(\Omega)] := \sup \left\{ \left( \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p \, dy \right)^{1/p} \left| \begin{array}{c} B_r(x) \subset \Omega, \; r < \mu \\ x_0 \in \partial \Omega, \; r > 0, \; B_r(x_0) \subset U_\nu(\partial \Omega) \end{array} \right\},$$

where $f_B = \int_B f \, dx$, the average of $f$ over $B$ and $B_r(x)$ denotes the closed ball of radius $r$ centered at $x$. It is well-known that one gets an equivalent seminorm when the ball $B_r$ is replaced by a cube. We also need to control the boundary behavior. For $\nu \in (0, \infty]$ we define

$$[f : B_{\nu}^p(\Omega)] := \sup \left\{ \left( \frac{1}{r^p} \int_{B_r(x_0) \cap \Omega} |f(y)|^p \, dy \right)^{1/p} \left| \begin{array}{c} x_0 \in \partial \Omega, \; r > 0, \; B_r(x_0) \subset U_\nu(\partial \Omega) \end{array} \right\},$$

where $U_\nu(E)$ is a $\nu$-open neighborhood of $E$, i.e.,

$$U_\nu(E) = \{ x \in \mathbb{R}^n \mid \text{dist}(x, E) < \nu \}.$$

We shall often assume that $\nu < R^*$, where $R^*$ is the reach from the boundary. The $BMO$ norm we use is

$$\left\| f : BMO^\mu_{b,p}(\Omega) \right\| = [f : BMO^\mu_p(\Omega)] + [f : B_{\nu}^p(\Omega)].$$

If $p = 1$, we often drop $p$. The $BMO$ space we consider is

$$BMO^\mu_{b,p}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \left| \left\| f : BMO^\mu_{b,p}(\Omega) \right\| < \infty \right\}.$$ 

This space is independent of $p$ for sufficiently small $\nu$, i.e., $\nu < R^*$ ([11], [12]) and $BMO^{\infty,\infty}_b$ agrees with Miyachi $BMO$ space ([29]) for various domains including a half space and bounded $C^2$ domains ([12]). Although the $BMO^{\infty,\nu}_b(\Omega)$ norm is equivalent to the $BMO^{\infty,\infty}_b(\Omega)$ norm when $\Omega$ is bounded, there are many unbounded domains for which the $BMO^{\infty,\nu}_b(\Omega)$ norm is actually weaker than the $BMO^{\infty,\infty}_b(\Omega)$ norm when $\nu$ is finite. We define the solenoidal space $VMO^{\mu,\nu}_{b,0,\sigma}$ as the $BMO^{\mu,\nu}_{b,0}$-closure of $C^{\infty,\nu}_{\sigma}(\Omega)$. In [10], [11] among other results the analyticity of $S(t)$ in $VMO^{\mu,\nu}_{b,0,\sigma}$ has been established for a uniformly $C^3$ domain which is admissible in the sense of [2] provided that $\nu$ is sufficiently small.

**Theorem 1.2** ([10], [11]). Let $\Omega$ be an admissible uniformly $C^3$ domain in $\mathbb{R}^n$. Then $S(t)$ forms a $C_0$-analytic semigroup in $VMO^{\mu,\nu}_{b,0,\sigma}$ for any $\mu \in (0, \infty]$ and $\nu \in (0, \nu_0)$ with some $\nu_0$ depending only on $\mu$ and regularity of $\partial \Omega$.

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace $L^p$ by $L^\infty$ or $BMO^{\infty,\nu}_b$, but even an estimate stronger than (1.2) with $p = \infty$, i.e.,

$$t \left\| \frac{dS(t)}{dt} \right\|_{0, \nu_0} \leq C \| v_0 : BMO^{\mu,\nu}_b(\Omega) \|, \; \mu, \nu \in (0, \infty]$$

which shows a regularizing effect.

It has been proved in [5] that a $C^2$ sector-like domain in $\mathbb{R}^2$ is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a $C^3$ sector-like
domain in $\mathbb{R}^2$ is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly $C^4$. On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a domain with $n \geq 3$ is not admissible ([8]).

In order to get the $L^p$ estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in $\Omega$.

**Theorem 1.3.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$, i.e., a domain having Lipschitz graph boundary. Let $T$ be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant $C$ such that

$$||Tu||_2 \leq C||u||_2$$

$$[Tu : BMO^\infty(\Omega)] \leq C||u||_{\infty}$$

for $u \in C_c(\Omega)$. Then $||Tu||_p \leq C_\ast||u||_p$ for $u \in C_c(\Omega)$ with $C_\ast$ depending only on $C$, $h$ and $p \in (2, \infty)$.

There are a couple of such interpolation results between $BMO$ and $L^2$, which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between $L^p$ and $BMO$ is discussed when $\Omega$ is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega) \cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with $BMO_A(X)$, where $A$ is some operator. They worked on metric measure spaces of homogeneous type $(X, d, \mu)$. In particular, in the case $X = \Omega, d(x, y) = |x - y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO^\infty(\Omega)$.

Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the $L^\infty$ case we do not know whether or not the complex interpolation space $L^2(\Omega) \cap BMO^\infty(\Omega)$, with $2/p = 1 - \rho$ agrees with $L^p(\Omega)$, although we know that $[L^2, BMO]_{\rho} = L^p$ for $\Omega = \mathbb{R}^n$ as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

**Theorem 1.4.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$. Assume that $\nu \in (0, \infty]$. There is a linear operator $Q$ from $C_c(\Omega)$ to $VMO^\infty(\Omega) \cap L^2(\Omega)$ such that

$$||Qu : BMO^\infty(\Omega)|| \leq C||u||_{\infty}$$

$$||Qu||_2 \leq C||u||_2$$

for all $u \in C_c(\Omega)$. Moreover, $Qu = u$ for $u \in C_c(\Omega) \cap L^2(\Omega)$.

Since there may be no $L^p$-Helmholtz decomposition our $Q$ should be different from the Helmholtz projection. We shall construct such an operator $Q$ using the solution operator of the equation $\text{div} u = f$ given by Solonnikov [36]. Although deriving the $L^2$ estimate is easy, to derive the $BMO$ estimate is more involved since we have to estimate the $b^\nu$ type seminorm.

To derive (1.1), we actually interpolate

$$||S(t)Qu||_2 \leq C||u||_2$$

and

$$||S(t)Qu : BMO^\infty(\Omega)|| \leq C||u||_{\infty}$$

for $u \in C_c(\Omega)$. Similarly, we derive (1.2) by interpolating the estimate for $\frac{dS}{dt}Q$. 
ANALYTICITY OF THE $L^p$-STOKES SEMIGROUP

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 3, we construct the projection operator $Q$. In Section 4, we give a complete proof of Theorem 1.1.

2. $L^2 - BMO$ INTERPOLATION ON A LIPSCHITZ HALF-SPACE

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

$$\Omega := \{(x', x_n) \in \mathbb{R}^n | x_n > h(x')\}$$

with a Lipschitz function $h$ on $\mathbb{R}^{n-1}$.

By $Q$ we mean a closed cube with sides parallel to the coordinate axes. Let $\ell(Q)$ be the side length of $Q$, and for $\tau > 0$, $\tau Q$ a cube with the same length as $Q$ and side length $\tau \ell(Q)$.

2.1. Reduction to the half-space and extension. Here, we prepare lemmas that are basic estimates for the proof. Since $h$ is Lipschitz continuous, $F(x) := (x', x_n - h(x'))$ is a bi-Lipschitz map from $\Omega$ to $\mathbb{R}^n_+$. For a function $u$ defined on $\mathbb{R}^n_+$, the pull-back function $F^*(u)$ of $u$ on $\Omega$ is defined by $u \circ F$. We start with estimates for $(F^{-1})^*$ which is the pull-back function $(F^{-1})^*(v)$ of $v$ on $\mathbb{R}^n_+$ defined by $v \circ F^{-1}$.

**Lemma 2.1.** Let $\Omega$ be a Lipschitz half-space.

(i):

$$\left[(F^{-1})^*v : BMO^\infty(\mathbb{R}^n_+)\right] \leq c [v : BMO^\infty(\Omega)].$$

(ii):

$$\| (F^{-1})^*v \|_{L^2(\mathbb{R}^n_+)} \leq c \| v \|_{L^2(\Omega)}.$$  

Here $c$ is a constant depending only on Lipschitz bound of $h$ and $u$.

**Proof.** (i): Because $\mathbb{R}^n_+$ is an open subset of $\mathbb{R}^n$, we know that for any $\tau > 2$,

$$\left[(F^{-1})^*v : BMO^\infty(\mathbb{R}^n_+)\right] \leq c_\tau \sup_{\tau Q \subset \mathbb{R}^n_+} \inf_{d \in \mathbb{R}} \int_Q |(F^{-1})^*v - d| \, dy,$$

where the supremum is taken over cubes $Q$, for which $\tau Q$ is contained in $\mathbb{R}^n_+$, see [37]. Since $F$ is a bi-Lipschitz map, it holds

$$c_1 \text{dist}(y, \partial \mathbb{R}^n_+) \leq \text{dist}(F^{-1}(y), \partial \Omega) \leq c_2 \text{dist}(y, \partial \mathbb{R}^n_+)$$

with some constants $c_1, c_2 > 0$ for all $y \in \mathbb{R}^n_+$. Since $\tau = 1, 2, \ldots, \ell(Q)/2 \leq \text{dist}(Q, \partial \mathbb{R}^n_+)$ for such cubes $Q$, we have the lower bound

$$c_\tau \ell(Q) \leq \text{dist}(F^{-1}(Q), \partial \Omega)$$

with some $c > 0$, which depends on $n$ and $h$. Therefore, taking large $\tau$, we can find cubes $\{R_k\}_{k=1}^{c_\tau} \subset \Omega$, which have no intersection of interiors, so that $\cup_{k=1}^{c_\tau} R_k$ is connected and

$$\ell(R_k) = \ell(Q),$$

$$F^{-1}(Q) \subset \cup_{k=1}^{c_\tau} R_k,$$

and if $R_j \cap R_k \neq \emptyset$, the smallest cube $R_{j,k}$ including $R_j$ and $R_k$ is in $\Omega$.

From these, one obtains that for cubes $Q$ with $\tau Q \subset \mathbb{R}^n_+$,

$$\inf_{d \in \mathbb{R}} \frac{1}{|Q|} \int_Q |(F^{-1})^*v - d| \, dy \leq c \sum_{k=1}^{c_\tau} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| \, dy.$$
It is enough to show that
\begin{equation}
\frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| dy \leq c[v : BMO^\infty(\Omega)]
\end{equation}
for the case $R_j \cap R_k \neq \emptyset$. To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let $\tilde{R}_k$ and $\tilde{R}_j$ be subcubes of $R_k$ and $R_j$ respectively so that $\ell(\tilde{R}_k) = \ell(R_k)/2$, $\ell(\tilde{R}_j) = \ell(R_j)/2$ and they touch each other. Moreover, denote by $\tilde{R}_{j,k}$ a cube satisfying $\ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k)$ and $\tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k}$. Hence, we have
\begin{align*}
\frac{1}{|\tilde{R}_k|} \int_{\tilde{R}_k} |v - v_{R_k}| dy &\leq \frac{1}{|\tilde{R}_k|} \int_{\tilde{R}_k} |v - v_{R_k}| dy + |v_{R_k} - v_{R_j}| \\
&\leq c[v : BMO^\infty(\Omega)] + c|v_{R_k} - v_{R_j}| \\
&\leq c[v : BMO^\infty(\Omega)] + \frac{1}{|\tilde{R}_{j,k}|} \int_{\tilde{R}_{j,k}} |v - v_{\tilde{R}_{j,k}}| dy \\
&\leq c[v : BMO^\infty(\Omega)].
\end{align*}

(ii): This is verified as follows
\begin{align*}
\| (F^{-1})^* v \|_{L^2(R^+_\Omega)}^2 &= \int_{\Omega} |v|^2 J_F dx \\
&\leq c \int_{\Omega} |v|^2 dx,
\end{align*}
where $J_F$ is the modulus of the Jacobian of $F$ which is bounded, because $h$ is Lipschitz continuous.

Next, we consider the even extension of functions on the half space. For a function $f$ on $R^+_\Omega$, we extend $f$ outside $R^+_\Omega$ by
\[E[f](x', -x_n) := f(x', x_n) \quad \text{for} \quad x_n > 0.\]

From elementary geometrical observation, we can see that the extension operator $E$ is a $BMO$-extension operator for $R^+_\Omega$.

**Lemma 2.2.**
\[|E[f] : BMO^\infty(\Omega^n)| \leq c \left[ f : BMO^\infty(R^+_\Omega) \right].\]

**Proof.** It is sufficient to consider cubes $Q \subset R^\oplus$ with $Q \cap R^\oplus_+ \neq \emptyset$ and $Q \cap R^\ominus_+ \neq \emptyset$. For such $Q$, let $Q'$ be a cube so that its center lies on $\partial R^\oplus_+$, $\ell(Q') = 2\ell(Q)$ and $Q \subset Q'$. Further, let $Q^*$ be the smallest cube in $R^\oplus_+$ containing the upper half of $Q'$. With these notations, the desired inequality is proved from
\[
\inf_{d \in R} \frac{1}{|Q|} \int_Q |E[f] - d| dy \leq c \inf_{d \in R} \frac{1}{|Q^*|} \int_{Q^*} |f - d| dy.
\]

\[
2.2. \textbf{Sharp maximal operator.} \quad \text{For the proof of Theorem 1.3, we make use of the sharp maximal operator $M^2$ due to Fefferman and Stein ([18]). We define for $x \in R^n$ and $f \in L^1_{loc}(R^n)$ the function $M^2 f$ by}
\[
M^2 f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.
\]
It is immediate from the definition that \(|f : BMO^\infty(\mathbb{R}^n)| = \|M^2f\|_{L^\infty(\mathbb{R}^n)}\). It is well-known that if \(f \in L^{p_0}(\mathbb{R}^n)\) for some \(p_0 \in (1, \infty)\), then for \(p \in [p_0, \infty)\)

\[
\|f\|_{L^p(\mathbb{R}^n)} \leq c\|M^2f\|_{L^p(\mathbb{R}^n)},
\]

which is applied below. (Both sides of (2.2) may be infinite.) This follows from \(\|f\|_{L^p(\mathbb{R}^n)} \leq \|Mf\|_{L^p(\mathbb{R}^n)}\) and \(\|Mf\|_{L^p(\mathbb{R}^n)} \leq c\|M^2f\|_{L^p(\mathbb{R}^n)}\), where \(M\) is the Hardy-Littlewood maximal operator [18].

2.3. Marcinkiewicz interpolation. Here, we give a variant of the Marcinkiewicz interpolation theorem.

**Proposition 2.3.** Let \(D\) be an open subset of \(\mathbb{R}^n\) and \(S\) a sublinear operator from \(C_c(D)\) to \(L^2(\mathbb{R}^n)\). If

\[
\|S[f]\|_{L^2(\mathbb{R}^n)} \leq c\|f\|_{L^2(D)}
\]

\[
\|S[f]\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{L^\infty(D)}
\]

for \(f \in C_c(D)\), then \(\|S[f]\|_{L^p(\mathbb{R}^\infty)} \leq C\|f\|_{L^p(D)}\) for \(f \in C_c(D)\) with \(C\) depending only on \(c\) and \(p \in (2, \infty)\).

**Proof.** For \(\lambda > 0\) and \(\alpha > 0\), we decompose \(f\) into two parts; \(f = f_2 + f_\infty\) where

\[
f_2(x) = \begin{cases} 
0 & \text{if } |f(x)| \leq \alpha \lambda \\
 f(x) - \alpha \lambda \text{sign}(f(x)) & \text{if } |f(x)| > \alpha \lambda,
\end{cases}
\]

where \(\text{sign} \xi = \xi/|\xi|\) for \(\xi \neq 0\) and \(\text{sign} \xi = 0\) for \(\xi = 0\). Observe that \(f_2, f_\infty \in BC(D)\), and then \(f_2, f_\infty \in C_c(D)\). Therefore, the two inequalities of our assumption hold for \(f_2\) and \(f_\infty\), respectively. We set \(\alpha = (2\|S\|_{L^\infty(D) \to L^\infty(\mathbb{R}^n)})^{-1}\) and observe that \(|\{x \in \mathbb{R}^n \mid S[f_\infty](x) > \lambda/2\}| = 0\). We now conclude that

\[
\int_{\mathbb{R}^n} |S[f]|^p \, dx \leq p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^n \mid |S[f_2](x)| > \lambda\}| \, d\lambda
\]

\[
\leq p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^n \mid |S[f_2](x)| > \lambda/2\}| \, d\lambda
\]

\[
\leq p \int_0^\infty \lambda^{p-1} \left(\frac{2}{\lambda} \|S\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \|f_2\|_{L^2(D)}\right)^2 \, d\lambda
\]

\[
\leq c \int_0^\infty \lambda^{p-3} \left(\int_{\{f(x) > \alpha \lambda\}} |f(x)|^2 \, dx\right) \, d\lambda
\]

\[
= 2c \int_0^\infty \lambda^{p-3} \left(\int_{\alpha \lambda}^\infty t \{x \in \mathbb{R}^n \mid |f(x)| > t\} \, dt\right) \, d\lambda
\]

\[
= 2c \int_0^\infty t \{x \in \mathbb{R}^n \mid |f(x)| > t\} \left(\int_0^{t/\alpha} \lambda^{p-3} \, d\lambda\right) \, dt
\]

\[
\leq c\|f\|_{L^p(D)}^p.
\]

\[\square\]
2.4. Proof of Theorem 1.3. For simplicity, we write $g := Tf$. By changing variables, one obtains

$$
\int_{\Omega} |g|^p dx \leq c \int_{\mathbb{R}^n_+} |(F^{-1})^*g|^p dy \leq c \int_{\mathbb{R}^n} |E[(F^{-1})^*g]|^p dy \leq c \int_{\mathbb{R}^n} |\Phi[f]|^p dy,
$$

where $\Phi[f] := M^p (E[(F^{-1})^*g])$. Here, because $E[(F^{-1})^*g] \in L^2(\mathbb{R}^n)$, we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see $L^2(\Omega) - L^2(\mathbb{R}^n)$ and $L^\infty(\Omega) - L^\infty(\mathbb{R}^n)$ estimates for $\Phi$. The former estimate can be seen by $L^2$-boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The latter one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

3. Non-Helmholtz projection

Our goal in this section is to prove Theorem 1.4.

3.1. A solution operator to the divergence problem. As in Section 2, let $\Omega = \{(x', x_n) \in \mathbb{R}^n | x' \in \mathbb{R}^{n-1}, x_n > h(x')\}$ be a Lipschitz half-space in $\mathbb{R}^n$ with a Lipschitz continuous function $h$ on $\mathbb{R}^{n-1}$. Then, there is a closed cone of the form

$$
C_1 = \{x = (x', x_n) \in \mathbb{R}^n | x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(\theta)\}
$$

with an angle $\theta \in (0, \pi/4)$ (depending on the Lipschitz constant of $h$) such that

$$
x + C_1 = \{y \in \mathbb{R}^n | y - x \in C_1\} \subset \Omega^c := \mathbb{R}^n \setminus \Omega \quad \text{for all} \quad x \in \Omega^c.
$$

In the notion of the introduction $C_1 = C(4\theta)$ so that the opening angle equals $4\theta$. With this angle we define a closed cone $C_0 = C(2\theta)$, i.e.,

$$
C_0 = \{x = (x', x_n) \in \mathbb{R}^n | x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos \theta\}.
$$

The closed cone $C_0$ also satisfies

$$
x + C_0 \subset \Omega^c \quad \text{for all} \quad x \in \Omega^c.
$$

Let $L \in C_\infty_c(\mathbb{R}^n)$ be a function such that

$$
supp L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{\mathbb{S}^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1.
$$

Here $-C_0 = \{-y | y \in C_0\}$ and $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Then we define a vector field $K = (K_1, \ldots, K_n)$ as

$$
K(x) := \frac{x}{|x|^n} L \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.
$$

Definition 3.1. For $f \in C_\infty_c(\Omega)$, we define a vector field $u = Sf$ as

$$
u(x) = Sf(x) := (K * \tilde{f})(x) = \int_{\mathbb{R}^n} K(x - y)\tilde{f}(y) dy, \quad x \in \mathbb{R}^n.
$$

Here $\tilde{f}$ denotes the zero extension of $f$ to $\mathbb{R}^n$ given by

$$
\tilde{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}
$$
This operator was introduced by Solonnikov [36]. For a fixed \( x \in \mathbb{R}^n \), since
\[
\frac{x - y}{|x - y|} \in \text{supp} \, L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)
\]
implies \( y \in x + C_0 \), we can write
\[
u(x) = \int_{x+C_0} K(x - y)\tilde{f}(y) \, dy.
\]
This formula and the property (3.1) of \( \Omega \) imply that \( u(x) = 0 \) for all \( x \in \Omega^c \). In particular, \( u \) vanishes on \( \partial \Omega \). However, the support of \( u \) may become unbounded although \( f \) is compactly supported in \( \Omega \).

By the change of variables \( x - y = r\sigma \) with \( r > 0 \) and \( \sigma \in S^{n-1} \) we have
\[
u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma)\tilde{f}(x - r\sigma)r^{n-1} \, d\mathcal{H}^{n-1}(\sigma) \, dr.
\]
Hence if \( f \in C_\infty^c(\Omega) \) is supported in \( B_R(0) \) and \( x \in B_\sigma(0) \) \( (R, \sigma > 0) \), then
\[
u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma)\tilde{f}(x - r\sigma)r^{n-1} \, d\mathcal{H}^{n-1}(\sigma) \, dr,
\]
which implies that \( u = Sf \) is smooth in \( \Omega \). Moreover, \( u = Sf \) vanishes near \( \partial \Omega \) and thus it is smooth in the whole space \( \mathbb{R}^n \), since \( f \) is compactly supported in \( \Omega \).

**Lemma 3.2.** Let \( p \in (1, \infty) \). There exists a constant \( c > 0 \) such that
\[
\|\nabla u\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)}
\]
for all \( f \in C_\infty^c(\Omega) \) and \( u = Sf \).

**Proof.** Let \( u_i \) be the \( i \)-th component of \( u \):
\[
u_i(x) = (K_i \ast \tilde{f})(x) = \int_{\mathbb{R}^n} K_i(z)\tilde{f}(x - z) \, dz.
\]
Differentiating both sides with respect to the \( j \)-th variable, we have
\[
\partial_j \nu_i(x) = \int_{\mathbb{R}^n} K_i(z)(\partial_j \tilde{f})(x - z) \, dz = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} K_i(z)(\partial_j \tilde{f})(x - z) \, dz
\]
and, by changing variables \( y = x - z \) and integrating by parts,
\[
\partial_j \nu_i(x) = 
\lim_{\varepsilon \to 0} \left( \int_{\partial B_\varepsilon(x)} K_i(x - y) \frac{x_j - y_j}{|x - y|} \tilde{f}(y) \, d\mathcal{H}^{n-1}(y) + \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (\partial_j K_i)(x - y) \tilde{f}(y) \, dy \right).
\]
On the one hand, we change variables \( x - y = \varepsilon \sigma \) with \( \sigma \in S^{n-1} \) to get
\[
\lim_{\varepsilon \to 0} \int_{|x - y| = \varepsilon} K_i(x - y) \frac{x_j - y_j}{|x - y|} \tilde{f}(y) \, d\mathcal{H}^{n-1}(y)
\]
\[
= \lim_{\varepsilon \to 0} \int_{|x - y| = \varepsilon} \frac{x_i - y_i}{|x - y|} \frac{x_j - y_j}{|x - y|} L \left( \frac{x - y}{|x - y|} \right) \tilde{f}(y) \frac{1}{|x - y|^{n-1}} \, d\mathcal{H}^{n-1}(y)
\]
\[
= \lim_{\varepsilon \to 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \tilde{f}(x - \varepsilon \sigma) \, d\mathcal{H}^{n-1}(\sigma)
\]
\[
= \tilde{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma),
\]
where the last equality follows from the fact that $L$ is integrable on $S^{n-1}$ and $\bar{f}$ is continuous at $x$. On the other hand, we differentiate $K_i$ to obtain

$$K_{ij}(z) := \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n},$$

(3.4)

$$k_{ij}(z) := (\delta_{ij} - nz_i z_j) L(z) + z_i (\partial_j L)(z) - z_j (\partial_i L)(z)$$

for $z \in \mathbb{R}^n \setminus \{0\}$. Then $K_{ij}$ is homogeneous of degree $-n$ and there is a constant $c > 0$ such that

$$|K_{ij}(z)| \leq \frac{c}{|z|^n} \text{ for all } z \in \mathbb{R}^n \setminus \{0\}$$

by the smoothness of $L$ on $S^{n-1}$. Moreover, for every $R_1$ and $R_2$ with $0 < R_1 < R_2$,

$$\int_{|z| < R_2} K_{ij}(z) \, dz = \int_{R_1 < |z| < R_2} \partial_j K_i(z) \, dz$$

$$= \int_{|z| = R_2} K_i(z) \frac{z_j}{|z|} \, dH^{n-1}(z) - \int_{|z| = R_1} K_i(z) \frac{z_j}{|z|} \, dH^{n-1}(z)$$

$$= \int_{|z| = R_2} \frac{z_i z_j}{|z|^n} L \left( \frac{z}{|z|} \right) \frac{1}{|z|^{n-1}} \, dH^{n-1}(z) - \int_{|z| = R_1} \frac{z_i z_j}{|z|^n} L \left( \frac{z}{|z|} \right) \frac{1}{|z|^{n-1}} \, dH^{n-1}(z)$$

$$= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, dH^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, dH^{n-1}(\sigma) = 0.$$ 

In the fourth equality we changed variables $z = R_2 \sigma$ and $z = R_1 \sigma$ with $\sigma \in S^{n-1}$, respectively. This equality is equivalent to

$$\int_{S^{n-1}} k_{ij} \sigma \, dH^{n-1}(\sigma) = 0.$$ 

(3.5)

Thus we can apply the Calderón-Zygmund theory (see eg. [23, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel $K_{ij}$ and obtain the formula

$$\partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, dH^{n-1}(\sigma) + \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{f}(y) \, dy,$$

(3.6)

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, dH^{n-1}(\sigma) \right| \leq |\bar{f}(x)| \int_{S^{n-1}} L(\sigma) \, dH^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \leq c \|\bar{f}\|_{L^p(\mathbb{R}^n)} = c \|\bar{f}\|_{L^p(\Omega)}$$

with a positive constant $c$ independent of $f$. Hence the lemma follows. \hfill \square

**Lemma 3.3.** For every $f \in C^\infty_c(\Omega)$ the vector field $u = Sf$ satisfies

$$\text{div } u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$
Proof. We have already observed that \( u \) vanishes on the boundary. Let us compute \( \text{div} \, u = \sum_{i=1}^{n} \partial_{i} u_{i} \) in \( \Omega \). By the formula (3.6) in the proof of Lemma 3.2,

\[
\text{div} \, u(x) = \tilde{f}(x) \int_{S^{n-1}} \sum_{i=1}^{n} \sigma_{i}^{2} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^{n}} \sum_{i=1}^{n} K_{ii}(x-y) \, \bar{f}(y) \, dy.
\]

In this formula, we have

\[
\int_{S^{n-1}} \sum_{i=1}^{n} \sigma_{i}^{2} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) \bigg|_{\sigma = \sigma_{0}} = \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1
\]

by (3.2) and, for all \( z \in \mathbb{R}^{n} \setminus \{0\}, \)

\[
\sum_{i=1}^{n} K_{ii}(z) = \frac{1}{|z|^{n}} L \left( \frac{z}{|z|} \right) \sum_{i=1}^{n} \left( 1 - n \frac{z_{i}^{2}}{|z|^{2}} \right)
\]

\[
+ \frac{1}{|z|^{n}} \sum_{i=1}^{n} \frac{z_{i}}{|z|} (\partial_{j} L) \left( \frac{z}{|z|} \right) - \sum_{i=1}^{n} \frac{z_{i}}{|z|^{n+2}} \sum_{k=1}^{n} \frac{z_{k}}{|z|} (\partial_{k} L) \left( \frac{z}{|z|} \right) = 0.
\]

Hence \( \text{div} \, u(x) = \tilde{f}(x) = f(x) \) for all \( x \in \Omega \). \( \Box \)

Lemma 3.3 means that the operator \( S \) is a solution operator to the divergence problem with Dirichlet boundary condition. Note that \( S \) is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

Definition 3.4. For a vector field \( u \in C_{c}^{\infty}(\Omega) \), we define a vector field \( Tu \) as

\[
Tu(x) := \int_{\mathbb{R}^{n}} K(x-y) \overline{\text{div} \, u(y)} \, dy, \quad x \in \mathbb{R}^{n}.
\]

Here \( K \) is given by (3.3) and \( \overline{\text{div} \, u} \) denotes the zero extension of \( \text{div} \, u \) to \( \mathbb{R}^{n} \).

The above definition means that \( T \) is given by \( T = S \circ \text{div} \). Since \( u \in C_{c}^{\infty}(\Omega) \), its divergence is in \( C_{c}^{\infty}(\Omega) \) and thus \( Tu \) is smooth in the whole space \( \mathbb{R}^{n} \) and vanishes outside of \( \Omega \), as discussed right after Definition 3.1. Also, by Lemma 3.3 we have

\[
\text{div} \, Tu = \text{div} \, u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial \Omega.
\]

Clearly \( Tu = 0 \) in \( \mathbb{R}^{n} \) for \( u \in C_{c}^{\infty}(\Omega) \). Note that, as in the case of the operator \( S \), the support of \( Tu \) may be unbounded.

Theorem 3.5. Let \( \Omega \) be a Lipschitz half-space. Let \( p \in (1, \infty) \). There exists a constant \( c > 0 \) such that

\[
\|Tu\|_{L^{p}(\Omega)} \leq c \|u\|_{L^{p}(\Omega)}
\]

for all \( u \in C_{c}^{\infty}(\Omega) \).

Proof. Let us compute the \( i \)-th component \((Tu)_{i}\) of \( Tu \) with \( i = 1, \ldots, n \) for compactly supported vector field \( u \) in \( \Omega \). As in the proof of Lemma 3.2, we integrate
Let us estimate \((3.9)\)

\[\Gamma(z)\]

Moreover, since \(12 M. BOLKART, Y. GIGA, T.-H. MIURA, T. SUZUKI, AND Y. TSUTSUI\)

\[\text{BMO}\]

see \((3.2)\) and \((3.5)\). We first estimate the \(\text{BMO}\)

\[\text{L}\]

on \(\text{Zygmund theory yields the boundedness of the operator } T \text{ on } L^p(\Omega). \]

By Theorem 3.5, the operator \(T\) extends uniquely to a bounded linear operator on \(L^p(\Omega)\) with each \(p \in (1, \infty)\), which we again refer to as \(T\).

Our next goal is to estimate the \(\text{BMO}^{\infty,\nu}(\Omega)\)-norm of \(Tu\) for \(u \in C_c^\infty(\Omega)\) and \(\nu \in (0, \infty]\). To this end, we estimate each term of the right-hand side in \((3.7)\) for \(u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)\). By \((3.8)\) we have

\[\|a_{ij}u_j : \text{BMO}^{\infty,\nu}(\Omega)\| \leq \|u_j : \text{BMO}^{\infty,\nu}(\Omega)\|, \quad [a_{ij}u_j : b^\nu(\Omega)] \leq \|u_j : b^\nu(\Omega)\|\]

and thus

\[\|a_{ij}u_j : \text{BMO}^{\infty,\nu}(\Omega)\| \leq \|u_j : \text{BMO}^{\infty,\nu}(\Omega)\| \]

Moreover, since

\[\|u_j : \text{BMO}^{\infty}(\Omega)\| \leq 2\|u_j : L^\infty(\Omega)\|, \quad [u_j : b^\nu(\Omega)] \leq \omega_n\|u_j\|_{L^\infty(\Omega)},\]

where \(\omega_n = 2\pi^{n/2}/n! \Gamma(n/2)\) is the volume of the unit ball \(B_1(0)\) in \(\mathbb{R}^n\) with the Gamma function \(\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx\), we have

\[\|a_{ij}u_j : \text{BMO}^{\infty,\nu}(\Omega)\| \leq (2 + \omega_n)\|u_j\|_{L^\infty(\Omega)}.\]

Let us estimate \(S_{ij}\bar{u}_j = K_{ij} * \bar{u}_j, i, j = 1, \ldots, n\) in \(\text{BMO}^{\infty,\nu}(\Omega)\). Recall that the integral kernel \(K_{ij}\) is of the form

\[K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},\]

where \(k_{ij} \in C_c^\infty(\mathbb{R}^n)\) is given by \((3.4)\) and satisfies

\[\text{supp } k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma) d\mathcal{H}^{n-1} = 0,\]

see \((3.2)\) and \((3.5)\). We first estimate the \(\text{BMO}^{\infty}\)-seminorm of \(S_{ij}\bar{u}_j\).
Lemma 3.6. Let $K$ be a function defined on $\mathbb{R}^n \setminus \{0\}$ such that
\begin{equation}
|K(x-y) - K(x)| \leq A|y|^\delta |x|^{-n-\delta} \text{ whenever } |x| \geq 2|y| > 0
\end{equation}
for some $A, \delta > 0$. Suppose that a convolution operator $S$ with $K$ is bounded on $L^2(\mathbb{R}^n)$ with a norm $B$. Then, there exists a dimensional constant $c_n$ such that
\[ |Sf|_{BMO^\infty(\mathbb{R}^n)} \leq c_n(A + B)\|f\|_{L^\infty(\mathbb{R}^n)} \]
for all $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. See [24, Theorem 3.4.9 and Corollary 3.4.10].

Lemma 3.7. There exists a constant $c > 0$ such that
\begin{equation}
[S_{ij}u_j : BMO^\infty(\Omega)] \leq c\|u_j\|_{L^\infty(\Omega)}
\end{equation}
for all $u = (u_1, \ldots, u_n) \in C^\infty(\Omega)$ and $i, j = 1, \ldots, n$.

Proof. We shall apply Lemma 3.6 to $S = S_{ij}$. For this purpose it is sufficient to show that the function $K = K_{ij}$ satisfies (3.10), since we already know that the convolution operator $S_{ij}$ is bounded on $L^2(\mathbb{R}^n)$, see the proof of Lemma 3.2. To this end, we differentiate $K_{ij}$ to get
\[ \nabla K_{ij}(x) = -nk_{ij}(x/|x|) x/|x| + 1/|x|^{n+1} \left( I_n - 1/|x|^2 x \otimes x \right) \nabla k_{ij}(x/|x|) \]
for $x \in \mathbb{R}^n \setminus \{0\}$, where $I_n$ is the identity matrix of size $n$ and $x \otimes x := (x, x)_{i,j}$ is the tensor product of $x$. Since $k_{ij}$ is smooth on $S^{n-1}$, we have
\[ |\nabla K_{ij}(x)| \leq \frac{c}{|x|^{n+1}}, \quad x \in \mathbb{R}^n \setminus \{0\}. \]

Hence, for all $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \geq 2|y| > 0$,\begin{align*}
|K(x-y) - K(x)| &= \left| \int_0^1 \frac{d}{dt}(K(x-ty)) \right| dt = \left| \int_0^1 (-y) \cdot \nabla K(x-ty) dt \right| \\
&\leq |y| \int_0^1 \frac{1}{|x-ty|} \frac{c}{|x|^{n+1}} dt \leq |y| \int_0^1 \frac{c}{(|x| - |y|)^{n+1}} dt \\
&\leq \frac{c|y|}{(|x| - |y|)^{n+1}} = \frac{2^{n+1}c|y|}{(|x| - |y|)^{n+1}}.
\end{align*}

Thus $K_{ij}$ satisfies (3.10) with $\delta = 1$ and we can apply Lemma 3.6 to obtain
\begin{equation}
[S_{ij}u_j : BMO^\infty(\mathbb{R}^n)] \leq c\|u_j\|_{L^\infty(\mathbb{R}^n)} = c\|u_j\|_{L^\infty(\Omega)}
\end{equation}
with some constant $c > 0$.

By definition of the $BMO^\infty$-seminorm, we have
\[ [S_{ij}u_j : BMO^\infty(\Omega)] \leq [S_{ij}u_j : BMO^\infty(\mathbb{R}^n)]. \]

Hence the inequality (3.11) follows from (3.12). 

Next, let us estimate the $B'$-part of $S_{ij}u_j$. Recall the two closed cones
\[ C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2^j \theta) \}, \quad j = 0, 1 \]
with opening angle $\theta \in (0, \pi/4)$. For $r > 0$ and $x_0 \in \mathbb{R}^n$, we define
\begin{equation}
A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)} (x + C_0) \cap (x_0 + C_1) \subset \mathbb{R}^n.
\end{equation}
Here $x_0 + C_1 = \{ y \in \mathbb{R}^n \mid y - x_0 \in C_1 \}$ and $x + C_0$ is defined similarly.
Lemma 3.8. For all \( r > 0 \) and \( x_0 \in \mathbb{R}^n \) we have \( A_r(x_0) \subset B_{r/\sin \theta}(x_0) \).

Proof. By translation, we may assume that \( x_0 = 0 \). Let \( a := (0, \ldots, 0, r/\sin \theta) \in \mathbb{R}^n \). Suppose that

1. \( B_r(0) \subset a + C_0 \),
2. \( x + C_0 \subset a + C_0 \) for all \( x \in a + C_0 \),
3. \( (a + C_0) \cap C_1^c \subset B_{r/\sin \theta}(0) \).

Then, the statements (1) and (2) imply

\[
A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.
\]

Hence the statement (3) yields \( A_r(0) \subset B_{r/\sin \theta}(0) \). Now let us prove the statements (1)-(3). Note that, since \( \theta \in (0, \pi/4) \), the cones \( C_0 \) and \( C_1 \) are represented as

\[
C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n \leq 0, |x'| \leq (-x_n) \tan(2j \theta) \}, \quad j = 0, 1.
\]

(1) Let \( x = (x', x_n) \in B_r(0) \). Then, \( x - a = (x', x_n - r/\sin \theta) \) satisfies

\[
(x - a)_n = x_n - \frac{r}{\sin \theta} \leq r - \frac{r}{\sin \theta} < 0
\]

and

\[
\left( \frac{r}{\sin \theta} - x_n \right)^2 \tan^2 \theta - |x'|^2 \geq \left( \frac{r - x_n \sin \theta}{\cos^2 \theta} \right)^2 - (r - x_n^2) = \left( \frac{r \sin \theta - x_n}{\cos \theta} \right)^2 \geq 0,
\]

or equivalently,

\[
|x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta = -(x - a)_n \tan \theta.
\]

Hence \( x - a \in C_0 \), that is, \( x \in a + C_0 \) and the statement (1) holds.

(2) Let \( x \in a + C_0 \). If \( y \in x + C_0 \), then \( (y - a)_n = (y - x)_n + (x - a)_n \leq 0 \) and

\[
|y'| \leq |x'| + |y' - x'| \leq -(x - a)_n \tan \theta - (y - x)_n \tan \theta = -(y - a)_n \tan \theta,
\]

which means that \( y \in a + C_0 \). Hence the statement (2) holds.

(3) Let \( x \in (a + C_0) \cap C_1^c \). Then we have

\[
(x - a)_n = x_n - r/\sin \theta \leq 0, \quad |x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.
\]

Hence

\[
|x|^2 \leq \left( \frac{r}{\sin \theta} - x_n \right)^2 \tan^2 \theta + x_n^2 =: f(x_n).
\]

To estimate the right-hand side in the above inequality for \( x \in (a + C_0) \cap C_1^c \), we derive the range of \( x_n \) for \( x \in (a + C_0) \cap C_1^c \). If \( x_n \geq 0 \), then \( x \in (a + C_0) \cap C_1^c \) holds if and only if the condition (3.14) is satisfied. Thus \( x_n \) must satisfy

\[
0 \leq x_n \leq \frac{r}{\sin \theta}.
\]

On the other hand, if \( x_n < 0 \), then \( x \in (a + C_0) \cap C_1^c \) holds if and only if

\[
(-x_n) \tan(2\theta) < |x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.
\]

Hence, in particular, if \( x \in (a + C_0) \cap C_1^c \) and \( x_n < 0 \), then \( x_n \) must satisfy

\[
(-x_n) \tan(2\theta) < \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta,
\]
which yields the inequality

\[-\frac{r}{\cos \theta} < (\tan(2\theta) - \tan \theta) x_n.\]

Since

\[\tan(2\theta) - \tan \theta = \tan(2\theta) - \frac{1}{2} \tan(2\theta)(1 - \tan^2 \theta)\]
\[= \frac{1}{2} \tan(2\theta)(1 + \tan^2 \theta) = \frac{\tan(2\theta)}{2 \cos^2 \theta} > 0 \quad (0 < \theta < \frac{\pi}{4}),\]

the above inequality is equivalent to

\[-\frac{2r \cos \theta}{\tan(2\theta)} < x_n (< 0).\]

In summary, the range of \(x_n\) for \(x \in (a + C_0) \cap C^1\) is

\[\alpha := -\frac{2r \cos \theta}{\tan(2\theta)} < x_n \leq \frac{r}{\sin \theta} =: \beta\]

and thus we obtain

\[|x|^2 \leq f(x_n) \leq \sup_{s \in (\alpha, \beta)} f(s) = \max\{f(\alpha), f(\beta)\},\]

where the last equality follows from the fact that \(f(x_n)\) is a concave parabola. On the one hand, we have \(f(\beta) = \beta^2 = \frac{r^2}{\sin^2 \theta}\). On the other hand, since

\[\alpha = -\frac{2r \cos \theta \cos(2\theta)}{\sin(2\theta)} = -\frac{r \cos(2\theta)}{\sin \theta} = \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta},\]

we have

\[f(\alpha) = \left(\frac{r}{\sin \theta} - \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta}\right)^2 \tan^2 \theta + \frac{r^2 \cos^2(2\theta)}{\sin^2 \theta}\]
\[= \frac{r^2}{\sin^2 \theta} \left\{4 \tan^2 \theta \cos^4 \theta + \cos^2(2\theta)\right\} = \frac{r^2}{\sin^2 \theta}.\]

Hence \(|x|^2 \leq r^2/\sin^2 \theta\) and thus \(x \in B_{r/\sin \theta}(0)\) for every \(x \in (a + C_0) \cap C^1\). Therefore, the statement (3) holds and the lemma follows.

Now we can estimate the \(b^\nu\)-part of \(S_{ij}\bar{u}_j\).

**Lemma 3.9.** Let \(\nu \in (0, \infty]\). There exists a constant \(c > 0\) such that

\[(3.15) \quad [S_{ij}\bar{u}_j : b^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \|u_j\|_{L^\infty(\Omega)}\]

for all \(u = (u_1, \ldots, u_n) \in C^\infty_c(\Omega)\) and \(i, j = 1, \ldots, n\).

**Proof.** First we note that for all \(f \in L^1_{loc}(\Omega)\) the inequality

\[[f : b^\nu(\Omega)] \leq \omega_n^{1/2} [f : b^\nu_2(\Omega)]\]

holds by Hölder’s inequality. Hence, to prove (3.15), it is sufficient to show the inequality

\[(3.16) \quad [S_{ij}\bar{u}_j : b^\nu_2(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \|u_j\|_{L^\infty}.\]
The second inequality of (3.16) follows from the definition of $\cdot : h_2^{\nu} / \sin \theta (\Omega)$. Let us show the first inequality. The singular integral $S_{ij} \bar{u}_j$ is of the form

$$S_{ij} \bar{u}_j (x) = (K_{ij} * \bar{u}_j)(x) = \int_{\mathbb{R}^n} K_{ij}(x-y) \bar{u}_j(y) \, dy, \quad x \in \mathbb{R}^n.$$ 

Since $\text{supp} \ K_{ij} \subset -C_0$ (see (3.4) and (3.2)) and $\text{supp} \ u \subset \Omega$, we can write

$$S_{ij} \bar{u}_j (x) = \int_{(x+C_0) \cap \Omega} K_{ij}(x-y) \bar{u}_j(y) \, dy, \quad x \in \mathbb{R}^n.$$ 

Hence, we have

$$W_r (x_0) := \bigcup_{x \in B_r (x_0) \cap \Omega} (x+C_0) \cap \Omega$$

for each $x_0 \in \partial \Omega$ and $r > 0$ with $B_r (x_0) \subset U_r (\partial \Omega)$, then we have

$$S_{ij} \bar{u}_j (x) = \int_{(x+C_0) \cap \Omega} K_{ij}(x-y) \bar{u}_j|W_r (x_0))(y) \, dy = [K_{ij} * (\bar{u}_j|W_r (x_0)](x)$$

for all $x \in B_r (x_0) \cap \Omega$, where

$$\bar{u}_j|W_r (x_0))(x) := \begin{cases} \bar{u}_j(x), & x \in W_r (x_0), \\ 0, & x \notin W_r (x_0). \end{cases}$$

Since $K_{ij}$ is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that

$$\int_{B_r (x_0) \cap \Omega} |S_{ij} \bar{u}_j (x)|^2 \, dx = \int_{B_r (x_0) \cap \Omega} [K_{ij} * (\bar{u}_j|W_r (x_0)](x)|^2 \, dx$$

$$\leq c \int_{\mathbb{R}^n} |(\bar{u}_j|W_r (x_0))(x)|^2 \, dx = c \int_{W_r (x_0)} |\bar{u}_j(x)|^2 \, dx$$

with some constant $c > 0$. Now we recall the property of the infinite cone $C_1$:

$$x + C_1 \subset \Omega^c \Leftrightarrow \Omega \subset (x + C_1)^c \quad \text{for all} \quad x \in \Omega^c.$$

By this property we have

$$W_r (x_0) \subset \bigcup_{x \in B_r (x_0) \cap \Omega} (x+C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r (x_0) \cap \Omega,$$

where $A_r (x_0)$ is given by (3.13), and thus Lemma 3.8 yields

$$W_r (x_0) \subset A_r (x_0) \cap \Omega \subset B_{\nu / \sin \theta} (x_0) \cap \Omega.$$ 

Hence we have

$$\frac{1}{r^n} \int_{B_r (x_0) \cap \Omega} |S_{ij} \bar{u}_j (x)|^2 \, dx \leq \frac{c}{r^n} \int_{W_r (x_0)} |\bar{u}_j(x)|^2 \, dx$$

$$\leq \frac{c}{r^n} \int_{B_{\nu / \sin \theta} (x_0) \cap \Omega} |\bar{u}_j(x)|^2 \, dx = \frac{c}{\sin^n \theta} \left( \frac{\sin \theta}{r} \right)^n \int_{B_{\nu / \sin \theta} (x_0) \cap \Omega} |u_j(x)|^2 \, dx$$

$$\leq \frac{c}{\sin^n \theta} \left[ u_j : b_2^{\nu / \sin \theta} (\Omega) \right]^2$$

for every $x_0 \in \partial \Omega$ and $r > 0$ with $B_r (x_0) \subset U_r (\partial \Omega)$, which yields

$$|S_{ij} \bar{u}_j : b_2^{\nu / \sin \theta} (\Omega)|^2 \leq \frac{c}{\sin^n \theta} \left[ u_j : b_2^{\nu / \sin \theta} (\Omega) \right]^2.$$ 

The proof is complete. \qed
Now we obtain an estimate for the $\text{BMO}^{c,\nu}_{b}(\Omega)$-norm of $Tu$. 

**Theorem 3.10.** Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that 

$$
\|Tu : \text{BMO}^{c,\nu}_{b}(\Omega)\| \leq c\|u\|_{L^{\infty}(\Omega)}
$$

for all $u \in C^{\infty}_{c}(\Omega)$.

**Proof.** Since the $i$-th component of $Tu$, $i = 1, \ldots, n$, is of the form (3.7), we have by (3.9), (3.11) and (3.15) that

$$
\|Tu : \text{BMO}^{c,\nu}_{b}(\Omega)\|
\leq c \sum_{i,j=1}^{n} \|a_{ij}u_{j} : \text{BMO}^{c,\nu}_{b}(\Omega)\| + \|S_{ij}u_{j} : \text{BMO}^{\infty}(\Omega)\| + \|S_{ij}u_{j} : b'_{\nu}(\Omega)\|
\leq c \sum_{j=1}^{n} \|u_{j}\|_{L^{\infty}(\Omega)} \leq c\|u\|_{L^{\infty}(\Omega)}
$$

with a positive constant $c$. \hfill \Box

### 3.2. Non-Helmholtz projection

As in the previous subsection, let $\Omega$ denote a Lipschitz half-space in $\mathbb{R}^n$.

**Definition 3.11.** For a vector field $u \in C^\infty_{c}(\Omega)$, we define a vector field $Q'u$ on $\mathbb{R}^n$ as $Q'u := u - Tu$. Here the operator $T$ is given in Definition 3.4. Moreover, $Tu = 0$ for all $u \in C^\infty_{c}(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbb{R}^n$ and

$$
\text{div } Tu = \text{div } u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial \Omega.
$$

Moreover, $Tu = 0$ for all $u \in C^\infty_{c}(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbb{R}^n$ and

$$
\text{div } Q'u = 0 \quad \text{in } \Omega, \quad Q'u = 0 \quad \text{on } \partial \Omega
$$

for all $u \in C^\infty_{c}(\Omega)$, and $Q'u = u$ for all $u \in C^\infty_{c,\sigma}(\Omega)$. Note that $Q'$ is not a projection from $C^\infty_{c}(\Omega)$ onto $C^\infty_{c,\sigma}(\Omega)$, since the support of $Tu$ may be unbounded and thus $Q'u$ is not in $C^\infty_{c,\sigma}(\Omega)$ in general. However, $Q'$ maps $C^\infty_{c}(\Omega)$ into $L^p_{c}(\Omega)$.

**Lemma 3.12.** For all $u \in C^\infty_{c}(\Omega)$ and $p \in (1, \infty)$, we have $Q'u \in L^p_{c}(\Omega)$.

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1, \infty)$, let $G_{p}(\Omega) = \{ \nabla q \in L^{p}(\Omega) \mid q \in L^{1}_{\text{loc}}(\Omega)\}$.

**Proposition 3.13.** Let $p \in (1, \infty)$. For every $\nabla q \in G_{p}(\Omega)$, there exists a sequence $\{q_{k}\}_{k=1}^{\infty}$ of functions in $C^\infty_{c}(\mathbb{R}^n)$ such that

$$
\lim_{k \to \infty} \|\nabla q - \nabla q_{k}\|_{L^{p}(\Omega)} = 0. \tag{3.18}
$$

**Proof.** Since the restriction of $C^\infty_{c}(\mathbb{R}^n)$ on $\Omega$ is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G_{p}(\Omega)$ there is a sequence $\{q_{k}\}_{k=1}^{\infty}$ of functions in $W^{1,p}(\Omega)$ such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space $\mathbb{R}^n_+$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x_n > h(x')\}$. As in Section 2, let $F(x) := (x', x_n - h(x'))$ be a bi-Lipschitz map from $\Omega$ to $\mathbb{R}^n_+$. Let $\nabla q \in G_{p}(\Omega)$ and $\tilde{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y', y_n + h(y'))$ is the inverse mapping of $F$. Then, since $\nabla \tilde{q}(y) = \nabla F^{-1}(y) \nabla q(F^{-1}(y))$ for $y \in \mathbb{R}^n_+$ and each component...
of $\nabla F^{-1}$ is bounded (because $h$ is Lipschitz continuous), we have $\nabla \tilde{q} \in G_{q}(R^n)$. Hence, by our assumption that the claim is valid for $R^n$, there is a sequence $\{\tilde{q}_k\}_{k=1}^{\infty}$ of functions in $W^{1,p}(R^n)$ such that $\lim_{k \to \infty} \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(R^n)} = 0$.

Let $q_k := \tilde{q}_k \ast F$ for each $k \in \mathbf{N}$. Then, since

$$\nabla q(x) = \nabla F(x) \nabla \tilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x) \nabla \tilde{q}_k(F(x)),$$

and each component of $\nabla F$ is bounded, we have $q_k \in W^{1,p}(\Omega)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \leq c \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(R^n)} \to 0$$
as $k \to \infty$. Thus the claim is valid for general Lipschitz half-spaces $\Omega$.

(2) Now we prove the claim for $\Omega = R^n$. We follow the idea of the proof of the claim in the case $\Omega = R^d$, see [34, Lemma 2.5.4]. Let $\varphi \in C^\infty_c(R^n)$ be a function such that

$$0 \leq \varphi \leq 1 \quad \text{in } R^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } R^n \setminus B_2(0)$$

and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbf{N}$ and $x \in R^n$. Then, $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in R^n$ and supp $\varphi_k \subset B_{2k}(0)$, supp $\nabla \varphi_k \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbf{N}$.

Let $q \in G_{q}(R^n)$. Then $q \in W^{1,p}(R^n)$, that is, $q \in W^{1,p}(U)$ for every bounded subset $U$ of $R^n$; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting $G_k := R^n \cap (B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbf{N}$, we have $q \in W^{1,p}(G_k)$ and thus there is a constant $a_k$ such that $\int_{G_k} (q - a_k) \, dx = 0$ for each $k \in \mathbf{N}$. From this equality and the change of variables $x = ky$ for $x \in G_k$ and $y \in G_1$ we have

$$\int_{G_k} (q(ky) - a_k) \, dy = \int_{G_1} (q(x) - a_k) \, dx = 0.$$Hence we can apply Poincaré’s inequality to $q(ky) - a_k$ on $G_1$ and get

$$\left( \int_{G_1} |q(ky) - a_k|^p \, dy \right)^{1/p} \leq c \left( \int_{G_1} |\nabla (q(ky))|^p \, dy \right)^{1/p}$$

with a constant $c > 0$ independent of $k$. In this inequality, we observe that

$$\int_{G_1} |q(ky) - a_k|^p \, dy = k^{-n} \int_{G_k} |q(x) - a_k|^p \, dx,$$

$$\int_{G_1} |\nabla (q(ky))|^p \, dy = k^p \int_{G_1} |(\nabla q)(ky)|^p \, dy = k^{p-n} \int_{G_k} |\nabla q(x)|^p \, dx$$

by the change of variables $x = ky$ and thus

$$\|q - a_k\|_{L^p(G_k)} \leq ck \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbf{N}.$$For each $k \in \mathbf{N}$, let $q_k := \varphi_k(q - a_k)$ on $R^n$. Then since supp $q_k \subset R^n \cap B_{2k}(0)$ holds by the relation supp $\varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(R^n)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(R^n)} \leq \|\nabla - \varphi_k \nabla q\|_{L^p(R^n)} + \| (\nabla \varphi_k)(q - a_k)\|_{L^p(R^n)}.$$Since $0 \leq \varphi_k(x) \leq 1$ and $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in R^n$ and $\nabla q \in L^p(R^n)$, the dominated convergence theorem yields

$$\lim_{k \to \infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(R^n)} = 0.$$On the other hand, since $\nabla \varphi_k = k^{-1}(\nabla \varphi)_k$ and supp $\varphi_k \subset \overline{G_k}$ for each $k \in \mathbf{N}$, it follows from (3.19) and the dominated convergence theorem that

$$\| (\nabla \varphi_k)(q - a_k)\|_{L^p(R^n)} \leq ck^{-1} \|q - a_k\|_{L^p(G_k)} \leq c \|\nabla q\|_{L^p(G_k)} \to 0$$

(3.22)
as \( k \to \infty \). Applying (3.21) and (3.22) to (3.20) we have
\[
\lim_{k \to \infty} \| \nabla q - \nabla q_k \|_{L^p(\mathbb{R}^n_+)} = 0,
\]
where \( q_k \in W^{1,p}(\mathbb{R}^n_+) \) for all \( k \in \mathbb{N} \). Hence the claim is valid when \( \Omega = \mathbb{R}^n_+ \) and the proposition follows. \( \square \)

**Proof of Lemma 3.12.** Let \( u \in C_c^\infty(\Omega) \) and \( p \in (1, \infty) \). Then, since \( Tu \in L_p(\Omega) \) by Theorem 3.5, we have \( Q'u = u - Tu \in L_p(\Omega) \). To show \( Q'u \in L^p(\Omega) \), we employ a characterization of elements of \( L^p(\Omega) \) ([19, Lemma III.2.1]): a vector field \( v \in L^p(\Omega) \) is in \( L^p(\Omega) \) if and only if
\[
\int_\Omega v \cdot \nabla q \, dx = 0 \quad \text{for all} \quad \nabla q \in G_p(\Omega) \quad \left( p' := \frac{p}{p-1} \right).
\]
Let \( \nabla q \) be any element of \( G_p(\Omega) \). From Proposition 3.13, there is a sequence \( \{q_k\}_{k=1}^\infty \) of functions in \( C_c^\infty(\mathbb{R}^n) \) such that the equality (3.18) with \( p \) replaced by \( p' \) holds. Since \( Q'u \) is defined and smooth in \( \mathbb{R}^n \) for \( u \in C_c^\infty(\Omega) \) and \( q_k \in C_c^\infty(\mathbb{R}^n) \), integration by parts yields
\[
\int_\Omega Q'u \cdot \nabla q_k \, dx = - \int_\Omega q_k \div Q'u \, dx + \int_{\partial\Omega} \pi Q'u \cdot \nu \, H^{n-1}
\]
for all \( k \in \mathbb{N} \), where \( \nu \) denotes the unit outer normal vector field of \( \partial \Omega \). We apply (3.17) to the right-hand side of this equality to get \( \int_\Omega Q'u \cdot \nabla q_k \, dx = 0 \) for all \( k \in \mathbb{N} \). Since \( Q'u \in L^p(\Omega) \) and (3.18) with \( p \) replaced by \( p' \) holds, the above equality implies that
\[
\int_\Omega Q'u \cdot \nabla q \, dx = \lim_{k \to \infty} \int_\Omega Q'u \cdot \nabla q_k \, dx = 0.
\]
Hence by the characterization of elements of \( L^p(\Omega) \) we conclude that \( Q'u \in L^p(\Omega) \) for all \( u \in C_c^\infty(\Omega) \). The proof is complete. \( \square \)

**Remark 3.14.**

(1) Let \( p \in (1, \infty) \). By Theorem 3.5 and Lemma 3.12, we have \( Q'u \in L^p(\Omega) \) and \( \|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)} \) for all \( u \in C_c^\infty(\Omega) \). Moreover, \( Q'u = u \) holds for all \( u \in C_c^{\infty,\sigma}(\Omega) \). Hence, by the density argument, \( Q' \) extends uniquely to a bounded linear operator on \( L^p(\Omega) \) that is a projection onto \( L^p(\Omega) \).

(2) The projection onto \( L^p(\Omega) \) given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each \( u \in C_c^\infty(\Omega) \) there would exist \( \pi \in L^1_{loc}(\Omega) \) such that \( (I - Q')u = \nabla \pi \) holds. Since \( (I - Q')u = Tu = K * \div u \) for \( u \in C_c^\infty(\Omega) \), the existence of such \( \pi \) would imply that \( \partial_i(K_i * \div u) = \partial_i(K_i * \div u) \) for all \( i, j = 1, \ldots, n \). For each \( f \in C^\infty(\Omega) \) with \( \int_\Omega f \, dx = 0 \) there is \( u \in C_c^\infty(\Omega) \) satisfying \( f = \div u \). This is possible since we are able to apply Bogovskii’s lemma to a bounded Lipschitz domain \( D \subset \Omega \) containing the support of \( f \) (see [19, Theorem III.3.3]). Thus the above equality would imply that \( \partial_i K_i = \partial_i K_j + c \) with some constant \( c \) for all \( i, j = 1, \ldots, n \) as a distribution. This contradicts the fact that \( \partial_i K_i \neq \partial_i K_j + c \) for \( i \neq j \) as observed in (3.4).

(3) It is possible to prove the characterization
\[
L^p(\Omega) = \{ u \in L^p(\Omega) \mid \div u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial \Omega \}
\]
if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains.
(see [19, Section III.2]). However, for a Lipschitz half-space, it is less popular. A proof can be found in [30, Lemma 2.1].

The linear operator $Q'$ also maps $C^\infty_c(\Omega)$ into $VMO_{b,0,\sigma}(\Omega)$.

**Lemma 3.15.** Let $\Omega$ be a Lipschitz half-space. For all $u \in C^\infty_c(\Omega)$ and $\nu \in (0, \infty]$, we have $Q'u \in VMO_{b,0,\sigma}(\Omega)$.

We shall prove two auxiliary propositions for the above lemma. For $p \in (1, \infty)$, let $W_{0,0,\sigma}(\Omega)$ be the $W^{1,p}$-closure of $C^\infty_c(\Omega)$.

**Proposition 3.16.** Let $\Omega$ be a Lipschitz half-space. For all $p \in (1, \infty)$ we have $L^p_0(\Omega) \cap W^{1,p}_{0,0,\sigma}(\Omega) \subset W^{1,p}_{0,0,\sigma}(\Omega)$. Thus $L^p_0(\Omega) \cap W^{1,p}_{0,0,\sigma}(\Omega) = W^{1,p}_{0,0,\sigma}(\Omega)$.

**Proof.** Let $\rho \in C^\infty_c(\mathbb{R}^n)$ be a function such that

$$0 \leq \rho(x) \leq 1 \quad \text{in } \mathbb{R}^n, \quad \text{supp } \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \, dx = 1$$

and $\rho_\delta(x) := \delta^{-n} \rho(\delta^{-1}x)$ for $\delta > 0, x \in \mathbb{R}^n$. Let $u \in L^p_0(\Omega) \cap W^{1,p}_{0,0,\sigma}(\Omega)$. Then there is a sequence $\{u_k\}_{k=1}^\infty$ of functions in $C^\infty_c(\Omega)$ such that $\lim_{k \to \infty} \|u - u_k\|_{L^p(\Omega)} = 0$. For $a > 0$, we define a vector field $u^a$ on $\Omega$ as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ 0, & h(x') < x_n \leq h(x') + a \end{cases}$$

and $u^a_k = (u_k)^a$ similarly. Then it is clear that $u^a \in W^{1,p}_{0,0,\sigma}(\Omega)$ and $u^a_k \in C^\infty_c(\Omega)$ for all $a > 0$. Moreover, we have

$$\|u^a - u^a_k\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all } a > 0, \quad \lim_{a \to 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0.$$

By the second equality and the fact that $W^{1,p}_{0,0,\sigma}(\Omega)$ is closed in $W^{1,p}(\Omega)$, it is sufficient for showing $u \in W^{1,p}_{0,0,\sigma}(\Omega)$ to prove $u^a \in W^{1,p}_{0,0,\sigma}(\Omega)$ for all $a > 0$.

For each $a > 0$, there is a constant $d = d(a) > 0$ such that $\text{dist}(\text{supp } u^a_k, \partial \Omega) \geq d$ for all $k \in \mathbb{N}$. Then, for a given $\varepsilon > 0$, we can take $\delta \in (0, d/2)$ so small that

$$\|u^a - u^a_k \ast \rho_\delta\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2},$$

since $u^a \in W^{1,p}_{0,0,\sigma}(\Omega)$. Also, since $\nabla \rho_\delta = \delta^{-1}(\nabla \rho)_\delta$, we have

$$\|u^a \ast \rho_\delta - u^a_k \ast \rho_\delta\|_{W^{1,p}(\Omega)}$$

$$\leq c(\|u^a \ast \rho_\delta - u^a_k \ast \rho_\delta\|_{L^p(\Omega)} + \|u^a \ast \nabla \rho_\delta - u^a_k \ast \nabla \rho_\delta\|_{L^p(\Omega)})$$

$$= c(\|(u^a - u^a_k) \ast \rho_\delta\|_{L^p(\Omega)} + \delta^{-1} \|(u^a - u^a_k) \ast (\nabla \rho)_\delta\|_{L^p(\Omega)})$$

$$\leq c(1 + \delta^{-1}) \|u^a - u^a_k\|_{L^p(\Omega)} = c(1 + \delta^{-1}) \|u - u_k\|_{L^p(\Omega)}$$

with a constant $c > 0$ independent of $\varepsilon$ and $\delta$. Hence by taking $k \in \mathbb{N}$ so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})},$$

we have $\|u^a \ast \rho_\delta - u^a_k \ast \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2$ and thus

$$\|u^a - u^a_k \ast \rho_\delta\|_{W^{1,p}(\Omega)} \leq \|u^a - u^a_k \ast \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a \ast \rho_\delta - u^a_k \ast \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon.$$

On the other hand, since $\text{dist}(\text{supp } u^a_k, \partial \Omega) > d$ and $\delta \in (0, d/2)$, the function $u^a_k \ast \rho_\delta$ is smooth and compactly supported in $\Omega$. Moreover, we have

$$\text{div}(u^a_k \ast \rho_\delta) = (\text{div } u^a_k) \ast \rho_\delta = 0 \quad \text{in } \Omega.$$
Thus \( u^a \rho \in C^\infty_c(\Omega) \) and \( u^a \) is approximated by elements of \( C^\infty_c(\Omega) \) in \( W^{1,p}(\Omega) \), which means that \( u^a \in W^{1,p}_{0,\sigma}(\Omega) \). Hence \( u \in W^{1,p}_{0,\sigma}(\Omega) \) and the proof is now complete.

\[ \square \]

**Proposition 3.17.** Let \( \nu \in (0,\infty] \). If \( p > n \), then \( W^{1,p}_{0,\sigma}(\Omega) \subset VM\!O_{b,0,\sigma}^\infty(\Omega) \).

**Proof.** Let \( u \in W^{1,p}_{0,\sigma}(\Omega) \) and \( u_k \in C^\infty_c(\Omega) \) such that \( \lim_{k \to \infty} \| u - u_k \|_{W^{1,p}(\Omega)} = 0 \).

Since \( p > n \) and \( u, u_k \in W^{1,p}_{0,\sigma}(\Omega) \), Morrey’s inequality (see e.g. \([7, \text{Theorem 4.12}]) implies

\[ \| u - u_k \|_{L^\infty(\Omega)} \leq c \| u - u_k \|_{W^{1,p}(\Omega)} \]

with a positive constant \( c \) independent of \( u \) and \( u_k \). Thus we have

\[ \| u - u_k : VM\!O_{b,0,\sigma}^\infty(\Omega) \| \leq (2 + \omega_n) \| u - u_k \|_{L^\infty(\Omega)} \leq c \| u - u_k \|_{W^{1,p}(\Omega)} \to 0 \]

as \( k \to \infty \). Hence \( u \in VM\!O_{b,0,\sigma}^\infty(\Omega) \) and the proof is now complete. \( \square \)

**Proof of Lemma 3.15.** Since \( u \in C^\infty_c(\Omega) \) and thus \( \partial_i u \in C^\infty_c(\Omega) \) for all \( i = 1,\ldots,n \), it follows from Lemma 3.12 that \( Q' u \in L^p_c(\Omega) \) and \( \partial_i Q' u = Q' (\partial_i u) \in L^p(\Omega) \) for all \( r \in (1,\infty) \) and \( i = 1,\ldots,n \). From this fact and the equality (3.17), we have \( Q' u \in L^p_c(\Omega) \cap W^{1,p}_0(\Omega) \) for all \( r \in (1,\infty) \). Hence, by taking \( r > n \), we can apply Proposition 3.16 and Proposition 3.17 to obtain \( Q' u \in VM\!O_{b,0,\sigma}^\infty(\Omega) \). \( \square \)

**Remark 3.18.** Let \( \nu \in (0,\infty] \). Theorem 3.10 and Lemma 3.15 imply that \( Q' u \in VM\!O_{b,0,\sigma}^\infty(\Omega) \) and \( \| Q' u : VM\!O_{b,0,\sigma}^\infty(\Omega) \| \leq c \| u \|_{L^\infty(\Omega)} \) for all \( u \in C^\infty_c(\Omega) \). Also, we have \( Q' u = u \) for all \( u \in C^\infty_c(\Omega) \). Hence \( Q' \) extends uniquely to a bounded linear operator (again referred to as \( Q' \)) from \( C_0(\Omega) \), which is the \( L^\infty \)-closure of \( C^\infty_c(\Omega) \), into \( VM\!O_{b,0,\sigma}^\infty(\Omega) \) that satisfies \( Q' u = u \) for all \( u \in C_0(\Omega) \).

Now let us extend \( Q' \) to a linear operator that gives the projection mentioned in Theorem 1.4. For \( p \in (1,\infty) \), we define a Banach space \( X_p \) and its norm as

\[ X_p : = L^p(\Omega) \cap C_0(\Omega), \quad \| u \|_{X_p} : = \max \{ \| u \|_{L^p(\Omega)}, \| u \|_{L^\infty(\Omega)} \} \].

Note that the Banach space \( C_0(\Omega) \) consists of all continuous functions \( f \) on \( \Omega \) such that the set \( \{ x \in \Omega \mid |f(x)| \geq \varepsilon \} \) is compact in \( \Omega \) for every \( \varepsilon > 0 \) (see e.g. \([32, \text{Theorem 3.17}] \)).

**Lemma 3.19.** For each \( p \in (1,\infty) \), the linear subspace \( C^\infty_c(\Omega) \) is dense in \( X_p \).

**Proof.** The proof is more or less standard (see e.g. \([27, \text{Corollary 19.24}] \)). We give it for completeness. Let \( u \in X_p \) and \( \Omega_k : = \{ x \in \Omega \mid |x| \leq k, \text{dist}(x, \partial \Omega) \geq 1/k \} \) for \( k \in \mathbb{N} \). For any given \( \varepsilon > 0 \), the set \( \{ x \in \Omega \mid |u(x)| \geq \varepsilon/2 \} \) is compact in \( \Omega \) since \( u \in C_0(\Omega) \). Moreover, since \( u \in L^p(\Omega) \), we can take \( k \in \mathbb{N} \) so large that

\[ \| u \|_{L^p(\Omega) \cap \Omega_k} < \varepsilon/2, \quad \| u \|_{L^\infty(\Omega) \cap \Omega_k} < \varepsilon/2 \]

Let \( \varphi \in C^\infty_c(\Omega) \) be a continuous cut-off function such that

\[ 0 \leq \varphi \leq 1 \quad \text{in} \quad \Omega, \quad \varphi = 1 \quad \text{in} \quad \Omega_k, \quad \varphi = 0 \quad \text{in} \quad \Omega \setminus \Omega_{2k}. \]

Since \( u - \varphi u = 0 \) in \( \Omega_k \) and \( |u - \varphi u| \leq |u| \) in \( \Omega \setminus \Omega_k \), it follows from (3.23) that

\[ \| u - \varphi u \|_{L^p(\Omega)} \leq \| u \|_{L^p(\Omega) \cap \Omega_k} < \varepsilon/2, \quad \| u - \varphi u \|_{L^\infty(\Omega)} \leq \| u \|_{L^\infty(\Omega) \cap \Omega_k} < \varepsilon/2. \]
Let \( \rho_4 \) be a mollifier as in the beginning of the proof of Proposition 3.16. Since 
\[
\varphi u \in L^p(\Omega), \quad \text{dist}(\text{supp}(\varphi u), \partial \Omega) \geq \frac{1}{2k},
\]
we can take \( \delta \in (0, 1/4k) \) so that 
\[
\tag{3.25} u_\delta := \rho_4 * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}.
\]
On the other hand, since \( \varphi u \) is uniformly continuous on \( \Omega_{4k} \), we can again choose \( \delta \in (0, 1/4k) \) so that \( \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \varepsilon/2 \). Moreover, since \( \text{supp}(\varphi u) \subset \Omega_{2k} \) and \( \delta \in (0, 1/4k) \), we have \( \varphi u = u_\delta = 0 \) outside of \( \Omega_{4k} \) and thus 
\[
\|\varphi u - u\|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}.
\]
Combining (3.24), (3.25) and (3.26), we obtain \( u_\delta \in C_c^\infty(\Omega) \) and 
\[
\|u - u_\delta\|_{X_p} = \max\{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon.
\]
Hence the lemma follows. \( \square \)

Let \( Y_p := L^p_b(\Omega) \cap VMO_{b,0,0}(\Omega) \) for \( p \in (1, \infty) \), \( \nu \in (0, \infty) \). Since \( L^p_b(\Omega) \) and \( VMO_{b,0,0}(\Omega) \) are closed in \( L^p(\Omega) \) and \( BMO_{b,0,0}(\Omega) \), respectively, \( Y_p \) becomes a Banach space under the norm \( \|v\|_{Y_p} := \max\{\|v\|_{L^p(\Omega)}, \|v : BMO_{b,0,0}(\Omega)\|\} \).

**Theorem 3.20.** Let \( p \in (1, \infty) \) and \( \nu \in (0, \infty) \). The linear operator \( Q' \) given in Definition 3.11 extends uniquely to a bounded linear operator \( Q_p \) from \( X_p \) into \( Y_p \). Moreover, there exists a constant \( c > 0 \) such that 
\[
\|Q_p u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q_p u : BMO_{b,0,0}^\infty(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}
\]
for all \( u \in X_p \) and \( Q_p u = u \) holds for all \( u \) in the \( X_p \)-closure of \( C_{c,\sigma}^\infty(\Omega) \).

**Proof.** Let \( u \in C_{c,\sigma}^\infty(\Omega) \). Then we have \( Q' u \in Y_p \) by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant \( c > 0 \) independent of \( u \) such that 
\[
\|Q' u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q' u : BMO_{b,0,0}^\infty(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}.
\]
Hence we have \( Q' u \in Y_p \) and \( \|Q' u\|_{Y_p} \leq c\|u\|_{X_p} \) for all \( u \in C_{c,\sigma}^\infty(\Omega) \). Since \( C_{c,\sigma}^\infty(\Omega) \) is dense in \( X_p \) by Lemma 3.19, the operator \( Q' \) extends uniquely to a bounded linear operator \( Q_p \) from \( X_p \) into \( Y_p \). Also, it follows from (3.28) that the inequality (3.27) holds for all \( u \in X_p \). Since \( Q' u = u \) holds for all \( u \in C_{c,\sigma}^\infty(\Omega) \) as observed after Definition 3.11, by the density argument we have \( Q_p u = u \) for all \( u \) in the \( X_p \)-closure of \( C_{c,\sigma}^\infty(\Omega) \). The proof is complete. \( \square \)

Finally, Theorem 1.4 follows from Theorem 3.20 with \( p = 2 \), that is, the linear operator \( Q \) in Theorem 1.4 is given by \( Q = Q_2 \).

4. Analyticity in \( L^p \)

In this section we shall give a complete proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( S(t) \) be the Stokes semigroup in \( L^p(\Omega) \) constructed by [14], [16]. To show that \( S(t) \) forms an analytic semigroup in \( L^p(\Omega) \) \( (2 \leq p < \infty) \) it suffices to prove that there exists a constant \( C \) that 
\[
\|S(t)u_0\|_p \leq C\|v_0\|_p
\]
for all \( u_0 \in C^\infty_0(\Omega) \) and for all \( t \in (0,1) \). Let \( Q \) be the operator in Theorem 1.4. Since \( Q \) is bounded in \( L^2 \) and maps \( L^2 \) to \( L^2_\sigma \) and \( S(t) \) fulfills (4.1) and (4.2) for \( p = 2 \), we have

\[
\|S(t)Qu\|_2 \leq C\|u\|_2
\]

(4.3)

\[
\left\| \frac{d}{dt} S(t)Qu \right\|_2 \leq C\|u\|_2
\]

(4.4)

for all \( u \in C_c(\Omega) \) and \( t \in (0,1) \). Since \( \Omega \) is admissible as proved in [5], \( S(t) \) forms an analytic semigroup in \( VMO_{b,0,\sigma} \) by Theorem 1.2. We conclude that

\[
\|S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty
\]

(4.5)

\[
\left\| \frac{d}{dt} S(t)Qu : BMO_b^{\infty,\nu}(\Omega) \right\| \leq C\|u\|_\infty
\]

(4.6)

for all \( u \in C_c(\Omega) \) and \( t \in (0,1) \) since \( Q \) fulfills

\[
\|Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty, \quad Qu \in VMO_{b,0,\sigma}^{\infty,\nu}
\]

for all \( u \in C_c(\Omega) \) by Theorem 1.4. (Note that we have a stronger statement than (4.6) by replacing the \( BMO_b \) type norm by the \( L^\infty \) norm since we have the regularizing estimate (1.3).) We apply an interpolation result (Theorem 1.3) to (4.3) and (4.5) and to (4.4) and (4.6) to get, respectively

\[
\|S(t)Qu\|_p \leq C\|u\|_p
\]

(4.7)

\[
\left\| \frac{d}{dt} S(t)Qu \right\|_p \leq C\|u\|_p
\]

(4.8)

for all \( u \in C_c(\Omega) \) and for all \( t \in (0,1) \). Since \( Qu = u \) for \( u \in C^\infty_c(\Omega) \) this yields (4.1) and (4.2).

It remains to prove that \( S(t) \) is a \( C_\sigma \)-semigroup in \( L^p_\sigma \). Since \( C^\infty_c(\Omega) \) is dense in \( L^p_\sigma \), for \( v_0 \in L^p_\sigma \) there is \( v_{0m} \in C^\infty_\sigma \) such that \( \|v_0 - v_{0m}\|_p \to 0 \) as \( m \to \infty \). By (4.1) we observe that

\[
\|S(t)v_0 - v_0\|_p \leq \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p \leq C\|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p.
\]

Sending \( t \downarrow 0 \), we get

\[
\lim_{t \downarrow 0} \|S(t)v_0 - v_0\|_p \leq C\|v - v_{0m}\|_p,
\]

since \( S(t)v_{0m} \to v_{0m} \) in \( \tilde{L}^p_\sigma \) as \( t \downarrow 0 \) by [14], [16]. Sending \( m \to \infty \), we conclude that \( S(t)v_0 \to v_0 \) in \( L^p_\sigma \) as \( t \downarrow 0 \).

**Remark 4.1.** In a similar way as we derived (4.5) and (4.6) we are able to derive from the \( L^\infty - BMO \) estimates in [10] that

\[
t \left\| \nabla^2 S(t)Qu : BMO_b^{\infty,\nu}(\Omega) \right\| \leq C\|u\|_\infty
\]

(4.9)

\[
t^{1/2} \left\| \nabla S(t)Qu : BMO_b^{\infty,\nu}(\Omega) \right\| \leq C\|u\|_\infty
\]

(4.10)

for all \( u \in C_c(\Omega) \) and \( t \in (0,1) \).
Note that $L^2$ results

$$t \left\| \nabla^2 S(t)Qu \right\|_2 \leq C\|u\|_2$$

$$t^{1/2} \left\| \nabla S(t)Qu \right\|_2 \leq C\|u\|_2$$

easily follow from the analyticity of $S(t)$ in $L^2_\sigma$ and $L^2$-boundedness of $Q$ if one observes that $\| \nabla u \|_2^2 = (Au, u)_{L^2}$ and

$$\| \nabla^2 u \|_2 \leq C (\|Au\|_2 + \| \nabla u \|_2 + \| u \|_2)$$

(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where $A$ is the Stokes operator in $L^2_\sigma$.

Interpolating the $L^2$ results and the above $L^\infty$-$BMO$ results, we are able to prove that there is $C_p > 0$ satisfying

$$t \left\| \nabla^2 S(t)v_0 \right\|_p \leq C_p\|v_0\|_p$$

$$t^{1/2} \left\| \nabla S(t)v_0 \right\|_p \leq C_p\|v_0\|_p$$

for all $v_0 \in L^p_\sigma(\Omega)$ and $t \in (0, 1)$ with $p \in (2, \infty)$.

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