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<td>Author(s)</td>
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ON ANALYTICITY OF THE $L^p$-STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

MARTIN BOLKART, YOSHIKAZU GIGA, TATSU-HIKO MIURA, TAKUYA SUZUKI, AND YOHEI TSUTSUI

Abstract. Consider the Stokes equations in a sector-like $C^3$ domain $\Omega \subset \mathbb{R}^3$. It is shown that the Stokes operator generates an analytic semigroup in $L^p(\Omega)$ for $p \in [2, \infty)$. This includes domains where the $L^p$-Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in $VMO$ and $L^2$ by constructing a suitable non-Helmholtz projection to solenoidal spaces.

1. Introduction

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \text{div} v = 0 \quad \text{in} \quad \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where $\Omega$ is a domain in $\mathbb{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a $C_0$-analytic semigroup in $L^p_\sigma (1 < p < \infty)$ for various domains like smooth bounded domains ([21], [35]). Here $L^p_\sigma = L^p_\sigma (\Omega)$ denotes the $L^p$-closure of $C_\infty c(\Omega)$, the space of all solenoidal vector fields with compact support in $\Omega$. More recently, it has been proved in [20] that $S(t)$ always forms a $C_0$-analytic semigroup in $L^p_\sigma (\Omega)$ for any uniformly $C^2$-domain $\Omega$ provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma (\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \{ \nabla q \in L^p(\Omega) \mid q \in L^1_{\text{loc}}(\Omega) \}$. In [20] the $L^p$ maximal regularity in time with values in $L^p_\sigma (\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The $L^p$-Helmholtz decomposition holds for various domains like bounded or exterior domains with

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smooth boundary for \(1 < p < \infty\) ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the \(L^p\)-Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely.

Let \(C(\vartheta)\) denote the cone of the form

\[
C(\vartheta) = \{ x = (x', x_n) \in \mathbb{R}^n \mid -x_n \geq |x| \cos(\vartheta/2) \},
\]

where \(\vartheta \in (0, 2\pi)\) is the opening angle. When \(n = 2\), we simply say that \(C(\vartheta)\) is a sector. We say that a planar domain \(\Omega\) is a sector-like domain if \(\Omega\) is smooth \([28, \text{Example } 2, \text{Fig. } 5]\) while for \(\Omega\) being a graph boundary if \(\Omega\) is of the form

\[
\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x_n > h(x')\}
\]

(up to translation and rotation) with some real-valued \(C^k\) function \(h\) with variable \(x' \in \mathbb{R}^{n-1}\).

**Theorem 1.1.** Let \(\Omega\) be a sector-like domain in \(\mathbb{R}^2\) having a \(C^4\) graph boundary. Then \(S(t)\) forms a \(C_0\)-analytic semigroup in \(L^p_0(\Omega)\) for all \(p \in [2, \infty)\).

Here is our strategy to prove Theorem 1.1. It is by now well-known that \(S(t)\) forms an analytic semigroup in \(L^p_0\), i.e., \(L^p_0 = L^p_\rho \cap L^2_\rho\) \((p \geq 2)\), \(L^p = L^p_\rho + L^2_\rho\) \((1 < p < 2)\) ([14], [15], [16]). Thus \(S(t)v_0\) is well-defined for \(v_0 \in C_{\infty,\sigma}(\Omega)\). To show Theorem 1.1, a key step is to prove the two estimates

\[
\|S(t)v_0\|_p \leq C\|v_0\|_p \quad (1.1)
\]

\[
t\left\| \frac{d}{dt} S(t)v_0 \right\|_p \leq C\|v_0\|_p \quad (1.2)
\]

for all \(v_0 \in C_{\infty,\sigma}(\Omega)\), \(t \in (0, 1)\), where \(\|v_0\|_p\) denotes the \(L^p\)-norm of \(v_0\). The constant \(C\) should be taken independent of \(t\) and \(v_0\). We shall establish (1.1) and (1.2) by interpolation since both estimates are known for \(p = 2\).

We are tempted to interpolate the \(L^\infty\) type result obtained in [5] with the \(L^2\)-result. In fact, in [5] the estimates (1.1) and (1.2) with \(p = \infty\) are established for all \(v_0 \in C_{0,\sigma}(\Omega)\), the \(L^\infty\)-closure of \(C_{\infty,\sigma}(\Omega)\) for a \(C^2\) sector-like domain \(\Omega\) in \(\mathbb{R}^2\). However, it is not clear that the complex interpolation space \([L^2_\rho, C_{0,\sigma}]_\rho\) agrees with \(L^p_0\) with \(2/p = 1 - \rho\) although it is well-known as the Riesz-Thorin theorem that \([L^2, L^\infty]_\rho = L^p\). To interpolate, we would need a projection to solenoidal spaces.
where $U_{\nu} < R$. We shall often assume that Theorem 1.2 applies to the setting of Theorem 1.1. Note that a $C^\infty_0$ which shows a regularizing effect.

Then $S_{\nu} = \nu \in S$. Theorem 1.2 ([10], [11]) thus

$$|f|_{BMO_{p}(\Omega)} := \sup \left\{ \left( \frac{1}{r^p} \int_{B_r(x)} |f(y)|^p \, dy \right)^{1/p} \mid x_0 \in \partial \Omega, \, r > 0, B_r(x_0) \subset U_{\nu}(\partial \Omega) \right\},$$

where $U_{\nu}(E)$ is a $\nu$-open neighborhood of $E$, i.e.,

$$U_{\nu}(E) = \{ x \in \mathbb{R}^n \mid \text{dist}(x, E) < \nu \}.$$

We shall often assume that $\nu < R^*$, where $R^*$ is the reach from the boundary. The $BMO$ norm we use is

$$\left\| f : BMO_{b, p}(\Omega) \right\| = \left[ f : BMO_{c, p}(\Omega) \right] + \left[ f : b_p(\Omega) \right].$$

If $p = 1$, we often drop $p$. The $BMO$ space we consider is

$$BMO_{b, p}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \mid \left\| f : BMO_{b, p}(\Omega) \right\| < \infty \right\}.$$

This space is independent of $p$ for sufficiently small $\nu$, i.e., $\nu < R^*$ ([11], [12]) and $BMO_{b, \infty}^{\infty, \infty}$ agrees with Miyachi $BMO$ space ([29]) for various domains including a half space and bounded $C^2$ domains ([12]). Although the $BMO_{b, \infty}^{\infty, \infty}(\Omega)$ norm is equivalent to the $BMO_{b}^{\infty, \infty}(\Omega)$ norm when $\Omega$ is bounded, there are many unbounded domains for which the $BMO_{b}^{\infty, \infty}(\Omega)$ norm is actually weaker than the $BMO_{b, \infty}^{\infty, \infty}(\Omega)$ norm when $\nu$ is finite. We define the solenoidal space $VMO_{b, \infty}^{\mu, \nu}$ as the $BMO_{b, \infty}^{\mu, \nu}$-closure of $C_{\nu, \sigma}(\Omega)$. In [10], [11] among other results the analyticity of $S(t)$ in $VMO_{b, \infty}^{\mu, \nu}$ has been established for a uniformly $C^3$ domain which is admissible in the sense of [2] provided that $\nu$ is sufficiently small.

**Theorem 1.2** ([10], [11]). Let $\Omega$ be an admissible uniformly $C^3$ domain in $\mathbb{R}^n$. Then $S(t)$ forms a $C_0$-analytic semigroup in $VMO_{b, \infty}^{\mu, \nu}$ for any $\mu \in (0, \infty]$ and $\nu \in (0, \nu_0)$ with some $\nu_0$ depending only on $\mu$ and regularity of $\partial \Omega$.

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace $L^p$ by $L^\infty$ or $BMO_{b, \infty}^{\infty, \nu}$, but even an estimate stronger than (1.2) with $p = \infty$, i.e.,

$$t \left\| \frac{dS(t)}{dt} \right\|_{0, \infty} \leq C \|f_0 : BMO_{b, \infty}^{\mu, \nu}(\Omega)\|, \quad \mu, \nu \in (0, \infty]$$

which shows a regularizing effect.

It has been proved in [5] that a $C^2$ sector-like domain in $\mathbb{R}^2$ is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a $C^3$ sector-like...
domain in $\mathbb{R}^2$ is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly $C^3$. On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a layer domain with $n \geq 3$ is not admissible ([8]).

In order to get the $L^p$ estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in $\Omega$.

**Theorem 1.3.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$, i.e., a domain having Lipschitz graph boundary. Let $T$ be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant $C$ such that

$$
\|Tu\|_2 \leq C\|u\|_2
$$

$$
[Tu : BMO^\infty(\Omega)] \leq C\|u\|_\infty
$$

for $u \in C_c(\Omega)$. Then $\|Tu\|_p \leq C_s\|u\|_p$ for $u \in C_c(\Omega)$ with $C_s$ depending only on $C$, $h$ and $p \in (2, \infty)$.

There are a couple of such interpolation results between $BMO$ and $L^2$, which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between $L^p$ and $BMO$ is discussed when $\Omega$ is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega) \cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with $BMO_A(\mathcal{X})$, where $A$ is some operator. They worked on metric measure spaces of homogeneous type $(\mathcal{X}, d, \mu)$. In particular, in the case $\mathcal{X} = \Omega$, $d(x, y) = |x - y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO^\infty(\Omega)$.

Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the $L^\infty$ case we do not know whether or not the complex interpolation space $[L^2_\sigma, VMO^\infty_{b,0,\sigma}]_\rho$ with $2/p = 1 - \rho$ agrees with $L^p_\sigma$, although we know that $[L^2, BMO]_\rho = L^p$ for $\Omega = \mathbb{R}^n$ as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

**Theorem 1.4.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$. Assume that $\nu \in (0, \infty]$. There is a linear operator $Q$ from $C_c(\Omega)$ to $VMO^\infty_{b,0,\sigma}(\Omega) \cap L^p_\sigma(\Omega)$ such that

$$
\|Qu : BMO^\infty_{b,\nu}(\Omega)\| \leq C\|u\|_\infty
$$

$$
\|Qu\|_2 \leq C\|u\|_2
$$

for all $u \in C_c(\Omega)$. Moreover, $Qu = u$ for $u \in C_c(\Omega) \cap L^2_\sigma(\Omega)$.

Since there may be no $L^p$-Helmholtz decomposition our $Q$ should be different from the Helmholtz projection. We shall construct such an operator $Q$ using the solution operator of the equation $\div u = f$ given by Solonnikov [36]. Although deriving the $L^2$ estimate is easy, to derive the $BMO$ estimate is more involved since we have to estimate the $b^\nu$ type seminorm.

To derive (1.1), we actually interpolate

$$
\|S(t)Qu\|_2 \leq C\|u\|_2
$$

and

$$
\|S(t)Qu : BMO^\infty_{b,\nu}\| \leq C\|u\|_\infty
$$

for $u \in C_c(\Omega)$. Similarly, we derive (1.2) by interpolating the estimate for $t^{\frac{\nu}2}Q$. 

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 3, we construct the projection operator \( Q \). In Section 4, we give a complete proof of Theorem 1.1.

2. \( L^2 - BMO \) interpolation on a Lipschitz half-space

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

\[
\Omega := \{ (x', x_n) \in \mathbb{R}^n | x_n > h(x') \}
\]

with a Lipschitz function \( h \) on \( \mathbb{R}^{n-1} \).

By \( Q \) we mean a closed cube with sides parallel to the coordinate axes. Let \( \ell(Q) \) be the side length of \( Q \), and for \( \tau > 0 \), \( \tau Q \) a cube with the same area as \( Q \) and side length \( \tau \ell(Q) \).

2.1. Reduction to the half-space and extension. Here, we prepare lemmas that are basic estimates for the proof. Since \( h \) is Lipschitz continuous, \( F(x) := (x', x_n - h(x')) \) is a bi-Lipschitz map from \( \Omega \) to \( \mathbb{R}^n_+ \). For a function \( u \) defined on \( \mathbb{R}^n_+ \), the pull-back function \( F^*(u) \) of \( u \) on \( \Omega \) is defined by \( u \circ F \). We start with estimates for \( (F^{-1})^* \) which is the pull-back function \( (F^{-1})^*(v) \) of \( v \) on \( \mathbb{R}^n_+ \) defined by \( v \circ F^{-1} \).

Lemma 2.1. Let \( \Omega \) be a Lipschitz half-space.

(i): \[ [(F^{-1})^* v : BMO^\infty(\mathbb{R}^n_+)] \leq c [v : BMO^\infty(\Omega)]. \]

(ii): \[ \|(F^{-1})^* v\|_{L^2(\mathbb{R}^n_+)} \leq c \|v\|_{L^2(\Omega)}. \]

Here \( c \) is a constant depending only on Lipschitz bound of \( h \) and \( u \).

Proof. (i): Because \( \mathbb{R}^n_+ \) is an open subset of \( \mathbb{R}^n \), we know that for any \( \tau > 2 \),

\[
[(F^{-1})^* v : BMO^\infty(\mathbb{R}^n_+)] \leq c_\tau \sup_{Q \subset \mathbb{R}^n_+} \inf_{d \in \mathbb{R}} \int_Q |(F^{-1})^* v - d| dy,
\]

where the supremum is taken over cubes \( Q \), for which \( \tau Q \) is contained in \( \mathbb{R}^n_+ \), see [37]. Since \( F \) is a bi-Lipschitz map, it holds

\[
c_1 \text{dist}(y, \partial \mathbb{R}^n_+) \leq \text{dist}(F^{-1}(y), \partial \Omega) \leq c_2 \text{dist}(y, \partial \mathbb{R}^n_+),
\]

with some constants \( c_1, c_2 > 0 \) for all \( y \in \mathbb{R}^n_+ \). Since \( (\tau - 1)\ell(Q)/2 \leq \text{dist}(Q, \partial \mathbb{R}^n_+) \) for such cubes \( Q \), we have the lower bound

\[
c\tau \ell(Q) \leq \text{dist}(F^{-1}(Q), \partial \Omega)
\]

with some \( c > 0 \), which depends on \( n \) and \( h \). Therefore, taking large \( \tau \), we can find cubes \( \{R_k\}_{k=1}^c \subset \Omega \), which have no intersection of interiors, so that \( \cup_{k=1}^c R_k \) is connected and

\[
\begin{cases}
\ell(R_k) = \ell(Q), \\
F^{-1}(Q) \subset \cup_{k=1}^c R_k, \text{ where } c_\ast \in \mathbb{N} \text{ depends only on } h, \text{ and} \\
of R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega.
\end{cases}
\]

From these, one obtains that for cubes \( Q \) with \( \tau Q \subset \mathbb{R}^n_+ \),

\[
\inf_{d \in \mathbb{R}} \frac{1}{|Q|} \int_Q |(F^{-1})^* v - d| dy \leq c \sum_{k=1}^c \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| dy.
\]
It is enough to show that
\[
\frac{1}{|R_k|} \int_{R_k} |v - v_{R_{j}}| dy \leq c[v : BMO^\infty(\Omega)]
\]
for the case \( R_{j} \cap R_{k} \neq \emptyset \). To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let \( \tilde{R}_k \) and \( \tilde{R}_j \) be subcubes of \( R_k \) and \( R_j \) respectively so that \( \ell(\tilde{R}_k) = \ell(R_k)/2 \), \( \ell(\tilde{R}_j) = \ell(R_j)/2 \) and they touch each other. Moreover, denote by \( \tilde{R}_{j,k} \) a cube satisfying \( \ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k) \) and \( \tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k} \). Hence, we have
\[
\frac{1}{|R_k|} \int_{R_k} |v - v_{R_{j}}| dy \leq \frac{1}{|R_k|} \int_{R_k} |v - v_{R_{j,k}}| dy + |v_{R_{k}} - v_{R_{j}}| \\
\leq c[v : BMO^\infty(\Omega)] + c[v_{R_{j,k}} - v_{R_{k}}] \\
\leq c[v : BMO^\infty(\Omega)] + c \frac{1}{|R_{j,k}|} \int_{R_{j,k}} |v - v_{R_{j,k}}| dy \\
\leq c[v : BMO^\infty(\Omega)].
\]

(ii): This is verified as follows
\[
\| (F^{-1})^* v \|_{L^2(\mathbb{R}^n_+)}^2 = \int_\Omega |v|^2 J_F dx \leq c \int_\Omega |v|^2 dx,
\]
where \( J_F \) is the modulus of the Jacobian of \( F \) which is bounded, because \( h \) is Lipschitz continuous. \( \square \)

Next, we consider the even extension of functions on the half space. For a function \( f \) on \( \mathbb{R}^n_+ \), we extend \( f \) outside \( \mathbb{R}^n_+ \) by
\[
E[f](x', -x_n) := f(x', x_n) \text{ for } x_n > 0.
\]
From elementary geometrical observation, we can see that the extension operator \( E \) is a \( BMO \)-extension operator for \( \mathbb{R}^n_+ \).

Lemma 2.2.
\[
[E[f] : BMO^\infty(\mathbb{R}^n)] \leq c [f : BMO^\infty(\mathbb{R}^n_+)].
\]

Proof. It is sufficient to consider cubes \( Q \subset \mathbb{R}^n \) with \( Q \cap \mathbb{R}^n_+ \neq \emptyset \) and \( Q \cap \mathbb{R}^n_- \neq \emptyset \). For such \( Q \), let \( Q' \) be a cube so that its center lies on \( \partial \mathbb{R}^n_+ \), \( \ell(Q') = 2 \ell(Q) \) and \( Q \subset Q' \). Further, let \( Q^* \) be the smallest cube in \( \mathbb{R}^n_+ \) containing the upper half of \( Q' \). With these notations, the desired inequality is proved from
\[
\inf_{d \in \mathbb{R}} \frac{1}{|Q|} \int_Q |E[f] - d| dy \leq c \inf_{d \in \mathbb{R}} \frac{1}{|Q^*|} \int_{Q^*} |f - d| dy.
\]
\( \square \)

2.2. Sharp maximal operator. For the proof of Theorem 1.3, we make use of the sharp maximal operator \( M^2 \) due to Fefferman and Stein ([18]). We define for \( x \in \mathbb{R}^n \) and \( f \in L^1_{loc}(\mathbb{R}^n) \) the function \( M^2 f \) by
\[
M^2 f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.
\]
It is immediate from the definition that \([f : BMO^\infty(\mathbb{R}^n)] = \|Mf\|_{L^\infty(\mathbb{R}^n)}\). It is well-known that if \(f \in L^{p_0}(\mathbb{R}^n)\) for some \(p_0 \in (1, \infty)\), then for \(p \in [p_0, \infty)\)

\[
(2.2) \quad \|f\|_{L^p(\mathbb{R}^n)} \leq c \|M^2f\|_{L^p(\mathbb{R}^n)},
\]

which is applied below. (Both sides of (2.2) may be infinite.) This follows from \(\|f\|_{L^p(\mathbb{R}^n)} \leq \|Mf\|_{L^p(\mathbb{R}^n)}\) and \(\|Mf\|_{L^p(\mathbb{R}^n)} \leq c \|M^2f\|_{L^p(\mathbb{R}^n)}\), where \(M\) is the Hardy-Littlewood maximal operator [18].

2.3. Marcinkiewicz interpolation. Here, we give a variant of the Marcinkiewicz interpolation theorem.

**Proposition 2.3.** Let \(D\) be an open subset of \(\mathbb{R}^n\) and \(S\) a sublinear operator from \(C_c(D)\) to \(L^p(\mathbb{R}^n)\). If

\[
\|S[f]\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L^2(D)}
\]

\[
\|S[f]\|_{L^\infty(\mathbb{R}^n)} \leq c \|f\|_{L^\infty(D)}
\]

for \(f \in C_c(D)\), then \(\|S[f]\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(D)}\) for \(f \in C_c(D)\) with \(C\) depending only on \(c\) and \(p \in (2, \infty)\).

**Proof.** For \(\lambda > 0\) and \(\alpha > 0\), we decompose \(f\) into two parts; \(f = f_2 + f_\infty\) where

\[
f_2(x) = \begin{cases} 
0 & \text{if } |f(x)| \leq \alpha \lambda \\
 f(x) - \alpha \lambda \text{sign}(f(x)) & \text{if } |f(x)| > \alpha \lambda,
\end{cases}
\]

where \(\text{sign } \xi = \xi/|\xi|\) for \(\xi \neq 0\) and \(\text{sign } \xi = 0\) for \(\xi = 0\). Observe that \(f_2, f_\infty \in BC(D)\), and then \(f_2, f_\infty \in C_c(D)\). Therefore, the two inequalities of our assumption hold for \(f_2\) and \(f_\infty\), respectively. We set \(\alpha = (2\|S\|_{L^\infty(\mathbb{R}^n)}^{L^\infty(\mathbb{R}^n)})^{-1}\) and observe that \(|\{x \in \mathbb{R}^n \mid S[f_\infty](x) > \lambda/2\}| = 0\). We now conclude that

\[
\int_{\mathbb{R}^n} |S[f]|^p \, dx \leq p \int_0^\infty \lambda^{p-1} \|\{x \in \mathbb{R}^n \mid |S[f](x)| > \lambda\}\| \, d\lambda \\
\leq p \int_0^\infty \lambda^{p-1} \|\{x \in \mathbb{R}^n \mid |S[f_2](x)| > \lambda/2\}\| \, d\lambda \\
\leq p \int_0^\infty \lambda^{p-1} \left(\frac{2}{\lambda^p} \|S\|_{L^2(D) \rightarrow L^2(\mathbb{R}^n)} \|f_2\|_{L^2(D)}\right)^2 \, d\lambda \\
\leq c \int_0^\infty \lambda^{p-3} \int_{\{|f| > \alpha \lambda\}} |f(x)|^2 \, dx \, d\lambda \\
= 2c \int_0^\infty \lambda^{p-3} \left(\int_0^\infty t \|\{x \in \mathbb{R}^n \mid |f(x)| > t\}\| \, dt\right) \, d\lambda \\
= 2c \int_0^\infty t \|\{x \in \mathbb{R}^n \mid |f(x)| > t\}\| \left(\int_0^{t/\alpha} \lambda^{p-3} \, d\lambda\right) \, dt \\
\leq c \|f\|^p_{L^p(D)}.
\]

\(\square\).
2.4. Proof of Theorem 1.3. For simplicity, we write \( g := Tf \). By changing variables, one obtains
\[
\int_{\Omega} |g|^p \, dx \leq c \int_{\mathbb{R}^n_+} |\Phi[F^{-1}]^*g|^p \, dy \leq c \int_{\mathbb{R}^n} |\Phi[F^{-1}]^*g|^p \, dy \leq c \int_{\mathbb{R}^n} |\Phi[f]|^p \, dy,
\]
where \( \Phi[f] := M^2(E[(F^{-1})^*g]) \). Here, because \( E[(F^{-1})^*g] \in L^2(\mathbb{R}^n) \), we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see \( L^2(\Omega) - L^2(\mathbb{R}^n) \) and \( L^\infty(\Omega) - L^\infty(\mathbb{R}^n) \) estimates for \( \Phi \). The former estimate can be seen by \( L^2 \)-boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The later one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

3. Non-Helmholtz projection

Our goal in this section is to prove Theorem 1.4.

3.1. A solution operator to the divergence problem. As in Section 2, let \( \Omega = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n > h(x')\} \) be a Lipschitz half-space in \( \mathbb{R}^n \) with a Lipschitz continuous function \( h \) on \( \mathbb{R}^{n-1} \). Then, there is a closed cone of the form
\[
C_1 = \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2\theta)\}
\]
with an angle \( \theta \in (0, \pi/4) \) (depending on the Lipschitz constant of \( h \)) such that
\[
x + C_1 = \{y \in \mathbb{R}^n \mid y - x \in C_1\} \subset \Omega^c := \mathbb{R}^n \setminus \Omega \quad \text{for all} \quad x \in \Omega^c.
\]
In the notion of the introduction \( C_1 = C(4\theta) \) so that the opening angle equals \( 4\theta \).

With this angle we define a closed cone \( C_0 = C(2\theta) \), i.e.,
\[
C_0 = \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos \theta\}.
\]
The closed cone \( C_0 \) also satisfies
\[
(3.1) \quad x + C_0 \subset \Omega^c \quad \text{for all} \quad x \in \Omega^c.
\]

Let \( L \in C^\infty_c(\mathbb{R}^n) \) be a function such that
\[
(3.2) \quad \text{supp} L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{\mathbb{S}^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1.
\]

Here \( -C_0 = \{-y \mid y \in C_0\} \) and \( \mathbb{S}^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). Then we define a vector field \( K = (K_1, \ldots, K_n) \) as
\[
(3.3) \quad K(x) := \frac{x}{|x|^n} L \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

**Definition 3.1.** For \( f \in C^\infty_c(\Omega) \), we define a vector field \( u = Sf \) as
\[
u(x) = Sf(x) := (K \ast \hat{f})(x) = \int_{\mathbb{R}^n} K(x-y) \hat{f}(y) \, dy, \quad x \in \mathbb{R}^n.
\]

Here \( \hat{f} \) denotes the zero extension of \( f \) to \( \mathbb{R}^n \) given by
\[
\hat{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}
\]
This operator was introduced by Solonnikov [36]. For a fixed $x \in \mathbb{R}^n$, since
\[
\frac{x - y}{|x - y|} \in \text{supp } L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)
\]
implies $y \in x + C_0$, we can write
\[
u(x) = \int_{x + C_0} K(x - y) \tilde{f}(y) \, dy.
\]
This formula and the property (3.1) of $\Omega$ imply that $u(x) = 0$ for all $x \in \Omega^c$. In particular, $u$ vanishes on $\partial \Omega$. However, the support of $u$ may become unbounded although $f$ is compactly supported in $\Omega$.

By the change of variables $x - y = r\sigma$ with $r > 0$ and $\sigma \in S^{n-1}$ we have
\[
u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma)\tilde{f}(x - r\sigma)r^{n-1}d\mathcal{H}^{n-1}(\sigma) \, dr.
\]
Hence if $f \in C_c^\infty(\Omega)$ is supported in $B_R(0)$ and $x \in B_a(0)$ ($R, a > 0$), then
\[
u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma)\tilde{f}(x - r\sigma)r^{n-1}d\mathcal{H}^{n-1}(\sigma) \, dr,
\]
which implies that $u = Sf$ is smooth in $\Omega$. Moreover, $u = Sf$ vanishes near $\partial \Omega$ and thus it is smooth in the whole space $\mathbb{R}^n$, since $f$ is compactly supported in $\Omega$.

**Lemma 3.2.** Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that
\[
\|\nabla u\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)}
\]
for all $f \in C_c^\infty(\Omega)$ and $u = Sf$.

**Proof.** Let $u_i$ be the $i$-th component of $u$:
\[
u_i(x) = (K_i \ast \tilde{f})(x) = \int_{\mathbb{R}^n} K_i(z)\tilde{f}(x - z) \, dz.
\]
Differentiating both sides with respect to the $j$-th variable, we have
\[
\partial_j u_i(x) = \int_{\mathbb{R}^n} K_i(z)(\partial_j \tilde{f})(x - z) \, dz = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(0)} K_i(z)(\partial_j \tilde{f})(x - z) \, dz
\]
and, by changing variables $y = x - z$ and integrating by parts,
\[
\partial_j u_i(x) = \lim_{\varepsilon \to 0} \left( \int_{\partial B_{\varepsilon}(x)} K_i(x - y) \frac{x_j - y_j}{|x - y|} \tilde{f}(y) \, d\mathcal{H}^{n-1}(y) + \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} (\partial_j K_i)(x - y) \tilde{f}(y) \, dy \right).
\]
On the one hand, we change variables $x - y = \varepsilon \sigma$ with $\sigma \in S^{n-1}$ to get
\[
\lim_{\varepsilon \to 0} \int_{|x - y| = \varepsilon} K_i(x - y) \frac{x_j - y_j}{|x - y|} \tilde{f}(y) \, d\mathcal{H}^{n-1}(y)
\]
\[
= \lim_{\varepsilon \to 0} \int_{|x - y| = \varepsilon} \frac{x_i - y_i}{|x - y|} \frac{x_j - y_j}{|x - y|} L \left( \frac{x - y}{|x - y|} \right) \tilde{f}(y) \frac{1}{|x - y|^{n-1}} \, d\mathcal{H}^{n-1}(y)
\]
\[
= \lim_{\varepsilon \to 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \tilde{f}(x - \varepsilon \sigma) \, d\mathcal{H}^{n-1}(\sigma)
\]
\[
= \tilde{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma),
\]
Lemma 3.3. For every \( f \in C_0^\infty(\Omega) \) the vector field \( u = Sf \) satisfies
\[
\text{div} u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.
\]
Proof. We have already observed that \( u \) vanishes on the boundary. Let us compute \( \text{div} \, u = \sum_{i=1}^{n} \partial_i u_i \) in \( \Omega \). By the formula (3.6) in the proof of Lemma 3.2,

\[
\text{div} \, u(x) = \tilde{f}(x) \int_{S^{n-1}} \sum_{i=1}^{n} \sigma_i^2 L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} \sum_{i=1}^{n} K_i(x-y) \tilde{f}(y) \, dy.
\]

In this formula, we have

\[
\int_{S^{n-1}} \sum_{i=1}^{n} \sigma_i^2 L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1
\]

by (3.2) and, for all \( z \in \mathbb{R}^n \setminus \{0\} \),

\[
\sum_{i=1}^{n} K_i(z) = \frac{1}{|z|^n} L\left(\frac{z}{|z|}\right) \sum_{i=1}^{n} \left(1 - n \frac{z_i^2}{|z|^2}\right) + \frac{1}{|z|^n} \sum_{i=1}^{n} \frac{z_i}{|z|} (\partial_i L) \left(\frac{z}{|z|}\right) - \sum_{i=1}^{n} \frac{z_i}{|z|^{n+2}} \sum_{k=1}^{n} \frac{z_k}{|z|^2} (\partial_k L) \left(\frac{z}{|z|}\right) = 0.
\]

Hence \( \text{div} \, u(x) = \tilde{f}(x) = f(x) \) for all \( x \in \Omega \).

Lemma 3.3 means that the operator \( S \) is a solution operator to the divergence problem with Dirichlet boundary condition. Note that \( S \) is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

**Definition 3.4.** For a vector field \( u \in C_\infty^c(\Omega) \), we define a vector field \( Tu \) as

\[
Tu(x) := \int_{\mathbb{R}^n} K(x-y) \overline{\text{div} \, u(y)} \, dy, \quad x \in \mathbb{R}^n.
\]

Here \( K \) is given by (3.3) and \( \overline{\text{div} \, u} \) denotes the zero extension of \( \text{div} \, u \) to \( \mathbb{R}^n \).

The above definition means that \( T \) is given by \( T = S \circ \text{div} \). Since \( u \in C_\infty^c(\Omega) \), its divergence is in \( C_\infty^c(\Omega) \) and thus \( Tu \) is smooth in the whole space \( \mathbb{R}^n \) and vanishes outside of \( \Omega \), as discussed right after Definition 3.1. Also, by Lemma 3.3 we have

\[
\text{div} \, Tu = \text{div} \, u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.
\]

Clearly \( Tu = 0 \) in \( \mathbb{R}^n \) for \( u \in C_\infty^c(\Omega) \). Note that, as in the case of the operator \( S \), the support of \( Tu \) may be unbounded.

**Theorem 3.5.** Let \( \Omega \) be a Lipschitz half-space. Let \( p \in (1, \infty) \). There exists a constant \( c > 0 \) such that

\[
\| Tu \|_{L^p(\Omega)} \leq c \| u \|_{L^p(\Omega)}
\]

for all \( u \in C_\infty^c(\Omega) \).

**Proof.** Let us compute the \( i \)-th component \( (Tu)_i \) of \( Tu \) with \( i = 1, \ldots, n \) for compactly supported vector field \( u \) in \( \Omega \). As in the proof of Lemma 3.2, we integrate
by parts to get
\[
(Tu)_i(x) = \lim_{\varepsilon \to 0} \int_{B_{\varepsilon}(x)} K_i(x-y) \frac{x-y}{|x-y|} \cdot \bar{u}(y) \, d\mathcal{H}^{n-1}(y)
+ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} (\nabla K_i)(x-y) \cdot \bar{u}(y) \, dy
= \int_{S^{n-1}} \sigma_i L(\sigma) \{ \sigma \cdot \bar{u}(x) \} \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} (\nabla K_i)(x-y) \cdot \bar{u}(y) \, dy,
\]
or equivalently,
\[
(Tu)_i(x) = \sum_{j=1}^n \{ a_{ij} \bar{u}_j(x) + S_{ij} \bar{u}_j(x) \}, \quad x \in \mathbb{R}^n.
\]
Here \(u_j\) is the \(j\)-th component of \(u\) and
\[
a_{ij} = \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma), \quad S_{ij} \bar{u}_j(x) = \int_{\mathbb{R}^n} K_{ij}(x-y) \bar{u}_j(y) \, dy,
\]
where \(K_{ij} = \partial_j K_i\) is given by (3.4). Since \(a_{ij}\) is a constant satisfying
\[
|a_{ij}| \leq \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1
\]
and \(S_{ij} \bar{u} = K_{ij} \ast \bar{u}\) is a singular integral (see the proof of Lemma 3.2), the Calderón-Zygmund theory yields the boundedness of the operator \(T\) on \(L^p(\Omega)\).

By Theorem 3.5, the operator \(T\) extends uniquely to a bounded linear operator on \(L^p(\Omega)\) with each \(p \in (1, \infty)\), which we again refer to as \(T\).

Our next goal is to estimate the \(BMO^\infty(\Omega)\)-norm of \(Tu\) for \(u \in C_0^\infty(\Omega)\) and \(\nu \in (0, \infty]\). To this end, we estimate each term of the right-hand side in (3.7) for \(u = (u_1, \ldots, u_n) \in C_0^\infty(\Omega)\). By (3.8) we have
\[
[a_{ij} \bar{u}_j : BMO^\infty(\Omega)] \leq [u_j : BMO^\infty(\Omega)], \quad [a_{ij} \bar{u}_j : b^\nu(\Omega)] \leq [u_j : b^\nu(\Omega)]
\]
and thus
\[
\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq \|u_j : BMO_b^{\infty, \nu}(\Omega)\|.
\]
Moreover, since
\[
[u_j : BMO^\infty(\Omega)] \leq 2\|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \leq \omega_n \|u_j\|_{L^\infty(\Omega)},
\]
where \(\omega_n = 2\pi^{n/2}/n\Gamma(n/2)\) is the volume of the unit ball \(B_1(0)\) in \(\mathbb{R}^n\) with the Gamma function \(\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} \, dx\), we have
\[
\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq (2 + \omega_n) \|u_j\|_{L^\infty(\Omega)}.
\]
Let us estimate \(S_{ij} \bar{u}_j = K_{ij} \ast \bar{u}_j, i, j = 1, \ldots, n\) in \(BMO_b^{\infty, \nu}(\Omega)\). Recall that the integral kernel \(K_{ij}\) is of the form
\[
K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
where \(k_{ij} \in C_0^\infty(\mathbb{R}^n)\) is given by (3.4) and satisfies
\[
\text{supp} \, k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1} = 0,
\]
see (3.2) and (3.5). We first estimate the \(BMO^\infty\)-seminorm of \(S_{ij} \bar{u}_j\).
Lemma 3.6. Let $K$ be a function defined on $\mathbb{R}^n \setminus \{0\}$ such that
\begin{equation}
K(x) - K(y) - K(x) \leq A |y|^\delta |x|^{-\delta - \epsilon}
\end{equation}
for some $A, \delta > 0$. Suppose that a convolution operator $S$ with $K$ is bounded on $L^2(\mathbb{R}^n)$ with a norm $B$. Then, there exists a dimensional constant $c_0$ such that
\begin{equation}
[Sf : BMO^\infty(\mathbb{R}^n)] \leq c_0(A + B)\|f\|_{L^\infty(\mathbb{R}^n)}
\end{equation}
for all $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. See [24, Theorem 3.4.9 and Corollary 3.4.10].

Lemma 3.7. There exists a constant $c > 0$ such that
\begin{equation}
[S_{ij} \tilde u_j : BMO^\infty(\Omega)] \leq c \|u_j\|_{L^\infty(\Omega)}
\end{equation}
for all $u = (u_1, \ldots, u_n) \in C_\text{c}^\infty(\Omega)$ and $i, j = 1, \ldots, n$.

Proof. We shall apply Lemma 3.6 to $S = S_{ij}$. For this purpose it is sufficient to show that the function $K = K_{ij}$ satisfies (3.10), since we already know that the convolution operator $S_{ij}$ is bounded on $L^2(\mathbb{R}^n)$, see the proof of Lemma 3.2. To this end, we differentiate $K_{ij}$ to get
\begin{align*}
\nabla K_{ij}(x) &= -nk_{ij}(x) \frac{x}{|x|} + \frac{1}{|x|^{n+1}} \left( I_n - \frac{1}{|x|^2} x \otimes x \right) \nabla k_{ij} \left( \frac{x}{|x|} \right)
\end{align*}
for $x \in \mathbb{R}^n \setminus \{0\}$, where $I_n$ is the identity matrix of size $n$ and $x \otimes x := (x_i x_j)_{i,j}$ is the tensor product of $x$. Since $k_{ij}$ is smooth on $S^{n-1}$, we have
\begin{equation}
|\nabla K_{ij}(x)| \leq \frac{c}{|x|^{n+1}}, \quad x \in \mathbb{R}^n \setminus \{0\}.
\end{equation}
Hence, for all $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \geq 2|y| > 0$,
\begin{align*}
|K(x) - K(y)| &= \left| \int_0^1 \frac{d}{dt} (K(x - ty)) \, dt \right| = \left| \int_0^1 (y) \cdot \nabla K(x - ty) \, dt \right| \\
&\leq |y| \int_0^1 \frac{c}{|x - ty|^{n+1}} \, dt \leq |y| \int_0^1 \frac{c}{(|x| - |y|)^{n+1}} \, dt \\
&\leq \frac{c |y|}{(|x| - |y|)^{n+1}} \leq \frac{2^{n+1} |y|}{|x|^{n+1}}.
\end{align*}
Thus $K_{ij}$ satisfies (3.10) with $\delta = 1$ and we can apply Lemma 3.6 to obtain
\begin{equation}
[S_{ij} \tilde u_j : BMO^\infty(\mathbb{R}^n)] \leq c \|u_j\|_{L^\infty(\mathbb{R}^n)} = c \|u_j\|_{L^\infty(\Omega)}
\end{equation}
with some constant $c > 0$.

By definition of the $BMO^\infty$-seminorm, we have
\begin{equation}
[S_{ij} \tilde u_j : BMO^\infty(\Omega)] \leq [S_{ij} \tilde u_j : BMO^\infty(\mathbb{R}^n)].
\end{equation}
Hence the inequality (3.11) follows from (3.12).

Next, let us estimate the $b'$-part of $S_{ij} \tilde u_j$. Recall the two closed cones
\begin{equation}
C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2^j \theta) \}, \quad j = 0, 1
\end{equation}
with opening angle $\theta \in (0, \pi/4)$. For $r > 0$ and $x_0 \in \mathbb{R}^n$, we define
\begin{equation}
A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap (x_0 + C_1)^c \subset \mathbb{R}^n.
\end{equation}
Here $x_0 + C_1 = \{ y \in \mathbb{R}^n \mid y - x_0 \in C_1 \}$ and $x + C_0$ is defined similarly.
Lemma 3.8. For all $r > 0$ and $x_0 \in \mathbb{R}^n$ we have $A_r(x_0) \subset B_{r/\sin \theta}(x_0)$.

Proof. By translation, we may assume that $x_0 = 0$. Let $a := (0, \ldots, 0, r/\sin \theta) \in \mathbb{R}^n$. Suppose that

1. $B_r(0) \subset a + C_0$,
2. $x + C_0 \subset a + C_0$ for all $x \in a + C_0$,
3. $(a + C_0) \cap C_1^c \subset B_{r/\sin \theta}(0)$.

Then, the statements (1) and (2) imply

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.$$ 

Hence the statement (3) yields $A_r(0) \subset B_{r/\sin \theta}(0)$. Now let us prove the statements (1)-(3). Note that, since $\theta \in (0, \pi/4)$, the cones $C_0$ and $C_1$ are represented as

$$C_j = \{ x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n \leq 0, |x'| \leq (-x_n) \tan(2^j \theta) \}, \quad j = 0, 1.$$

(1) Let $x = (x', x_n) \in B_r(0)$. Then, $x - a = (x', x_n - r/\sin \theta)$ satisfies

$$(x - a)_n = x_n - \frac{r}{\sin \theta} \leq r - \frac{r}{\sin \theta} < 0$$

and

$$\left( r \frac{\tan \theta}{\sin \theta} - x_n \right)^2 + \frac{r - x_n \sin \theta}{\cos^2 \theta} - (r^2 - x_n^2) = \frac{(r \sin \theta - x_n)^2}{\cos^2 \theta} \geq 0,$$

or equivalently,

$$|x'| \leq \left( r \frac{\tan \theta}{\sin \theta} - x_n \right) \tan \theta = -(x - a)_n \tan \theta.$$

Hence $x - a \in C_0$, that is, $x \in a + C_0$ and the statement (1) holds.

(2) Let $x \in a + C_0$. If $y \in x + C_0$, then $(y - a)_n = (y - x)_n + (x - a)_n \leq 0$ and

$$|y'| \leq |x'| + |y' - x'| \leq -(x - a)_n \tan \theta - (y - x)_n \tan \theta = -(y - a)_n \tan \theta,$$

which means that $y \in a + C_0$. Hence the statement (2) holds.

(3) Let $x \in (a + C_0) \cap C_1^c$. Then we have

$$\tag{3.14} (x - a)_n = x_n - r/\sin \theta \leq 0, \quad |x'| \leq \left( r \frac{\tan \theta}{\sin \theta} - x_n \right) \tan \theta.$$

Hence

$$|x|^2 \leq \left( r \frac{\tan \theta}{\sin \theta} - x_n \right)^2 \tan^2 \theta + x_n^2 =: f(x_n).$$

To estimate the right-hand side in the above inequality for $x \in (a + C_0) \cap C_1^c$, we derive the range of $x_n$ for $x \in (a + C_0) \cap C_1^c$. If $x_n \geq 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if the condition (3.14) is satisfied. Thus $x_n$ must satisfy

$$0 \leq x_n \leq \frac{r}{\sin \theta}.$$

On the other hand, if $x_n < 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if

$$(-x_n) \tan(2 \theta) < |x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.$$

Hence, in particular, if $x \in (a + C_0) \cap C_1^c$ and $x_n < 0$, then $x_n$ must satisfy

$$(-x_n) \tan(2 \theta) < \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.$$
which yields the inequality
\[ -\frac{r}{\cos \theta} < (\tan(2\theta) - \tan \theta) x_n. \]

Since
\[ \tan(2\theta) - \tan \theta = \tan(2\theta) - \frac{1}{2} \tan(2\theta)(1 - \tan^2 \theta) \]
\[ = \frac{1}{2} \tan(2\theta)(1 + \tan^2 \theta) = \frac{\tan(2\theta)}{2\cos^2 \theta} > 0 \quad (0 < \theta < \frac{\pi}{4}), \]

the above inequality is equivalent to
\[ -\frac{2r \cos \theta}{\tan(2\theta)} < x_n (< 0). \]

In summary, the range of \( x_n \) for \( x \in (a + C_0) \cap C_1^c \) is
\[ \alpha := -\frac{2r \cos \theta}{\tan(2\theta)} < x_n \leq \frac{r}{\sin \theta} =: \beta \]
and thus we obtain
\[ |x|^2 \leq f(x_n) \leq \sup_{s \in (\alpha, \beta]} f(s) = \max\{f(\alpha), f(\beta)\}, \]
where the last equality follows from the fact that \( f(x_n) \) is a concave parabola. On
the one hand, we have \( f(\beta) = \beta^2 = \frac{r^2}{\sin^2 \theta} \). On the other hand, since
\[ \alpha = -\frac{2r \cos \theta \cos(2\theta)}{\sin(2\theta)} = -\frac{r \cos(2\theta)}{\sin \theta} = \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta}, \]
we have
\[ f(\alpha) = \left( \frac{r}{\sin \theta} - \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta} \right)^2 \tan^2 \theta + \frac{r^2 \cos^2(2\theta)}{\sin^2 \theta} \]
\[ = \frac{r^2}{\sin^2 \theta} \left( 4\tan^2 \theta \cos^4 \theta + \cos^2(2\theta) \right) = \frac{r^2}{\sin^2 \theta}. \]
Hence \( |x|^2 \leq \frac{r^2}{\sin^2 \theta} \) and thus \( x \in B_{r/\sin \theta}(0) \) for every \( x \in (a + C_0) \cap C_1^c \).
Therefore, the statement (3) holds and the lemma follows.

Now we can estimate the \( b^\nu \)-part of \( S_{ij} \bar{u}_j \).

**Lemma 3.9.** Let \( \nu \in (0, \infty] \). There exists a constant \( c > 0 \) such that
\[ (3.15) \quad [S_{ij} \bar{u}_j : b^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \|u_j\|_{L^\infty(\Omega)} \]
for all \( u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega) \) and \( i, j = 1, \ldots, n \).

**Proof.** First we note that for all \( f \in L^1_{loc}(\Omega) \) the inequality
\[ [f : b^\nu(\Omega)] \leq \omega_n^{1/2} [f : b^\nu_2(\Omega)] \]
holds by Hölder’s inequality. Hence, to prove (3.15), it is sufficient to show the inequality
\[ (3.16) \quad [S_{ij} \bar{u}_j : b^\nu_2(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \left[ u_j : b^\nu_{2/\sin \theta}(\Omega) \right] \leq \frac{c\omega_n^{1/2}}{\sin^{n/2} \theta} \|u_j\|_{L^\infty}. \]
The second inequality of (3.16) follows from the definition of \([ \cdot : b_2^{\nu/sin\theta}(\Omega) ]\). Let us show the first inequality. The singular integral \( S_{ij}(\bar{u}_j) \) is of the form
\[
S_{ij}(\bar{u}_j)(x) = (K_{ij} \ast \bar{u}_j)(x) = \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{u}_j(y) \ dy, \quad x \in \mathbb{R}^n.
\]
Since \( \text{supp} \ K_{ij} \subset -C_0 \) (see (3.4) and (3.2)) and \( \text{supp} \ u \subset \Omega \), we can write
\[
S_{ij}(\bar{u}_j)(x) = \int_{(x + C_0) \cap \Omega} K_{ij}(x - y) \bar{u}_j(y) \ dy, \quad x \in \mathbb{R}^n.
\]
Hence, if we set
\[
W_r(x_0) := \bigcup_{x \in B_r(x_0) \cap \Omega} (x + C_0) \cap \Omega
\]
for each \( x_0 \in \partial \Omega \) and \( r > 0 \) with \( B_r(x_0) \subset U_x(\partial \Omega) \), then we have
\[
S_{ij}(\bar{u}_j)(x) = \int_{(x + C_0) \cap \Omega} K_{ij}(x - y) (\bar{u}_j|W_r(x_0))(y) \ dy = [K_{ij} \ast (\bar{u}_j|W_r(x_0))](x)
\]
for all \( x \in B_r(x_0) \cap \Omega \), where
\[
(\bar{u}_j|W_r(x_0))(x) := \begin{cases} \bar{u}_j(x), & x \in W_r(x_0), \\ 0, & x \notin W_r(x_0). \end{cases}
\]
Since \( K_{ij} \) is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that
\[
\int_{B_r(x_0) \cap \Omega} |S_{ij}(\bar{u}_j)(x)|^2 \ dx = \int_{B_r(x_0) \cap \Omega} |[K_{ij} \ast (\bar{u}_j|W_r(x_0))](x)|^2 \ dx
\]
\[
\leq c \int_{\mathbb{R}^n} |(\bar{u}_j|W_r(x_0))(x)|^2 \ dx = c \int_{W_r(x_0)} |\bar{u}_j(x)|^2 \ dx
\]
with some constant \( c > 0 \). Now we recall the property of the infinite cone \( C_1: \)
\[
x + C_1 \subset \Omega^c \Rightarrow \Omega \subset (x + C_1)^c \quad \text{for all} \quad x \in \Omega^c.
\]
By this property we have
\[
W_r(x_0) \subset \bigcup_{x \in B_r(x_0) \cap \Omega^c} (x + C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r(x_0) \cap \Omega,
\]
where \( A_r(x_0) \) is given by (3.13), and thus Lemma 3.8 yields
\[
W_r(x_0) \subset A_r(x_0) \cap \Omega \subset B_{r/sin\theta}(x_0) \cap \Omega.
\]
Hence we have
\[
\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |S_{ij}(\bar{u}_j)(x)|^2 \ dx \leq c \int_{W_r(x_0)} |\bar{u}_j(x)|^2 \ dx
\]
\[
\leq c \int_{B_{r/sin\theta}(x_0) \cap \Omega} |\bar{u}_j(x)|^2 \ dx = c \frac{n}{sin\theta} \left( \frac{sin\theta}{r} \right)^n \int_{B_{r/sin\theta}(x_0) \cap \Omega} |u_j(x)|^2 \ dx
\]
\[
\leq c \frac{1}{sin^n \theta} \left[ u_j : b_2^{\nu/sin\theta}(\Omega) \right]^2
\]
for every \( x_0 \in \partial \Omega \) and \( r > 0 \) with \( B_r(x_0) \subset U_x(\partial \Omega) \), which yields
\[
|S_{ij}(\bar{u}_j) : b_2^{\nu/sin\theta}(\Omega)|^2 \leq c \frac{1}{sin^n \theta} \left[ u_j : b_2^{\nu/sin\theta}(\Omega) \right]^2.
\]
The proof is complete. \( \square \)
Now we obtain an estimate for the $BMO_b^{\infty,\nu}(\Omega)$-norm of $Tu$.

**Theorem 3.10.** Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that
\[
\|Tu : BMO_b^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}
\]
for all $u \in C_c^\infty(\Omega)$.

**Proof.** Since the $i$-th component of $Tu$, $i = 1, \ldots, n$, is of the form (3.7), we have by (3.9), (3.11) and (3.15) that
\[
\|Tu : BMO_b^{\infty,\nu}(\Omega)\|
\leq c\sum_{i,j=1}^n (\|a_{ij} u_j : BMO_b^{\infty,\nu}(\Omega)\| + |S_{ij} u_j : BMO^{\infty}(\Omega)| + |S_{ij} u_j : \nu(\Omega)|)
\leq c\sum_{j=1}^n \|u_j\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}
\]
with a positive constant $c$. \hfill \Box

3.2. **Non-Helmholtz projection.** As in the previous subsection, let $\Omega$ denote a Lipschitz half-space in $\mathbb{R}^n$.

**Definition 3.11.** For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field $Q'u$ on $\mathbb{R}^n$ as $Q'u := u - Tu$. Here the operator $T$ is given in Definition 3.4.

For a vector field $u \in C_c^\infty(\Omega)$, the vector field $Tu$ is smooth in $\mathbb{R}^n$ and
\[
\text{div}Tu = \text{div}u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.
\]
Moreover, $Tu = 0$ for all $u \in C_c^\infty(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbb{R}^n$ and
\[
(3.17) \quad \text{div}Q'u = 0 \quad \text{in} \quad \Omega, \quad Q'u = 0 \quad \text{on} \quad \partial\Omega
\]
for all $u \in C_c^\infty(\Omega)$, and $Q'u = u$ for all $u \in C_c^\infty(\Omega)$. Note that $Q'$ is not a projection from $C_c^\infty(\Omega)$ onto $C_c^\infty(\Omega)$, since the support of $Tu$ may be unbounded and thus $Q'u$ is not in $C_c^\infty(\Omega)$ in general. However, $Q'$ maps $C_c^\infty(\Omega)$ into $L_2^p(\Omega)$.

**Lemma 3.12.** For all $u \in C_c^\infty(\Omega)$ and $p \in (1, \infty)$, we have $Q'u \in L^p(\Omega)$.

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1, \infty)$, let $G_p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L^1_{loc}(\Omega)\}$.

**Proposition 3.13.** Let $p \in (1, \infty)$. For every $\nabla q \in G_p(\Omega)$, there exists a sequence \(\{q_k\}_{k=1}^\infty\) of functions in $C_c^\infty(\mathbb{R}^n)$ such that
\[
(3.18) \quad \lim_{k \to \infty} \|\nabla q - \nabla q_k\|_{L^p(\Omega)} = 0.
\]

**Proof.** Since the restriction of $C_c^\infty(\mathbb{R}^n)$ on $\Omega$ is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G_p(\Omega)$ there is a sequence \(\{q_k\}_{k=1}^\infty\) of functions in $W^{1,p}(\Omega)$ such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space $\mathbb{R}^n_+$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x_n > b(x')\}$. As in Section 2, let $F(x) := (x', x_n - b(x'))$ be a bi-Lipschitz map from $\Omega$ to $\mathbb{R}^n_+$. Let $\nabla q \in G_p(\Omega)$ and $\bar{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y', y_n + h(y'))$ is the inverse mapping of $F$. Then, since $\nabla (\bar{q}(y)) = \nabla F^{-1}(y)\nabla q(F^{-1}(y))$ for $y \in \mathbb{R}^n_+$ and each component
of $\nabla F^{-1}$ is bounded (because $h$ is Lipschitz continuous), we have $\nabla \tilde{q} \in G_q(R^+_n)$. Hence, by our assumption that the claim is valid for $R^+_n$, there is a sequence $\{\tilde{q}_k\}_{k=1}^{\infty}$ of functions in $W^{1,p}(R^+_n)$ such that $\lim_{k \to \infty} \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(R^+_n)} = 0$.

Let $q_k := \tilde{q}_k \circ F$ for each $k \in \mathbb{N}$. Then, since
\[
\nabla q(x) = \nabla F(x)\nabla \tilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x)\nabla \tilde{q}_k(F(x)), \quad x \in \Omega
\]
and each component of $\nabla F$ is bounded, we have $q_k \in W^{1,p}(\Omega)$ and
\[
\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \leq c\|\nabla q - \nabla \tilde{q}_k\|_{L^p(R^+_n)} \to 0
\]
as $k \to \infty$. Thus the claim is valid for general Lipschitz half-spaces $\Omega$.

(2) Now we prove the claim for $\Omega = R^n$. We follow the idea of the proof of the claim in the case $\Omega = \Omega^*$, see [34, Lemma 2.5.4]. Let $\varphi \in C_c^\infty(R^n)$ be a function such that
\[
0 \leq \varphi \leq 1 \quad \text{in } R^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } R^n \setminus B_2(0)
\]
and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbb{N}$ and $x \in R^n$. Then, $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in R^n$ and supp $\varphi_k \subset B_{2k}(0)$, supp $\nabla \varphi_k \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbb{N}$.

Let $q \in G^1_p(R^n)$. Then $q \in W^{1,p}(R^n)$, that is, $q \in W^{1,p}(U)$ for every bounded subset $U$ of $R^n$; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting $G_k := R^n \cap (B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbb{N}$, we have $q \in W^{1,p}(G_k)$ and thus there is a constant $a_k$ such that $\int_{G_k} (q - a_k) \, dx = 0$ for each $k \in \mathbb{N}$. From this equality and the change of variables $x = ky$ for $x \in G_k$ and $y \in G_1$ we have
\[
\int_{G_1} (q(ky) - a_k) \, dy = k^{-n} \int_{G_k} (q(x) - a_k) \, dx = 0.
\]

Hence we can apply Poincaré’s inequality to $q(ky) - a_k$ on $G_1$ and get
\[
\left( \int_{G_1} |q(ky) - a_k|^p \, dy \right)^{1/p} \leq c \left( \int_{G_1} |\nabla (q(ky))|^p \, dy \right)^{1/p}
\]
with a constant $c > 0$ independent of $k$. In this inequality, we observe that
\[
\int_{G_1} |q(ky) - a_k|^p \, dy = k^{-n} \int_{G_k} |q(x) - a_k|^p \, dx,
\]
\[
\int_{G_1} |\nabla (q(ky))|^p \, dy = k^p \int_{G_1} |\nabla (q(y))|^p \, dy = k^{p-n} \int_{G_k} |\nabla q(x)|^p \, dx
\]
by the change of variables $x = ky$ and thus
\[
\|q - a_k\|_{L^p(G_k)} \leq c_k \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbb{N}.
\]

For each $k \in \mathbb{N}$, let $q_k := \varphi_k(q - a_k)$ on $R^n$. Then since supp $q_k \subset R^n \cap B_{2k}(0)$ holds by the relation supp $\varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(R^n)$ and
\[
\|\nabla q - \nabla q_k\|_{L^p(R^n)} \leq \|\nabla q - \varphi_k \nabla q\|_{L^p(R^n)} + \|\nabla \varphi_k(q - a_k)\|_{L^p(R^n)}.
\]

Since $0 \leq \varphi_k(x) \leq 1$ and $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in R^n$ and $\nabla q \in L^p(R^n)$, the dominated convergence theorem yields
\[
\lim_{k \to \infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(R^n)} = 0.
\]

On the other hand, since $\nabla \varphi_k = k^{-1}(\nabla \varphi)_k$ and supp $\varphi_k R^n \subset \overline{G_k}$ for each $k \in \mathbb{N}$, it follows from (3.19) and the dominated convergence theorem that
\[
\|(\nabla \varphi_k)(q - a_k)\|_{L^p(R^n)} \leq c k^{-1} \|q - a_k\|_{L^p(G_k)} \leq c \|\nabla q\|_{L^p(G_k)} \to 0
\]
as \( k \to \infty \). Applying (3.21) and (3.22) to (3.20) we have
\[
\lim_{k \to \infty} \| \nabla q - \nabla q_k \|_{L^p(\mathbb{R}^n)} = 0,
\]
where \( q_k \in W^{1,p}(\mathbb{R}^n) \) for all \( k \in \mathbb{N} \). Hence the claim is valid when \( \Omega = \mathbb{R}^n_+ \) and the proposition follows. \( \square \)

Proof of Lemma 3.12. Let \( u \in C_c^\infty(\Omega) \) and \( p \in (1, \infty) \). Then, since \( Tu \in L_p(\Omega) \) by Theorem 3.5, we have \( Q' u = u - Tu \in L_p(\Omega) \). To show \( Q' u \in L_p(\Omega) \), we employ a characterization of elements of \( L_p(\Omega) \) ([19, Lemma III.2.1]): a vector field \( v \in L_p(\Omega) \) is in \( L_p(\Omega) \) if and only if
\[
\int_\Omega v \cdot \nabla q \, dx = 0 \quad \text{for all} \quad \nabla q \in G_p'(\Omega) \quad \left( p' := \frac{p}{p-1} \right).
\]
Let \( \nabla q \) be any element of \( G_p'(\Omega) \). From Proposition 3.13, there is a sequence \( \{q_k\}_{k=1}^\infty \) of functions in \( C_c^\infty(\mathbb{R}^n) \) such that the equality (3.18) with \( p \) replaced by \( p' \) holds. Since \( Q' u \) is defined and smooth in \( \mathbb{R}^n \) for \( u \in C_c^\infty(\Omega) \) and \( q_k \in C_c^\infty(\mathbb{R}^n) \), integration by parts yields
\[
\int_\Omega Q' u \cdot \nabla q_k \, dx = - \int_\Omega q_k \, \text{div} \, Q' u \, dx + \int_{\partial \Omega} q_k \, Q' u \cdot \nu \, d\mathcal{H}^{n-1}
\]
for all \( k \in \mathbb{N} \), where \( \nu \) denotes the unit outer normal vector field of \( \partial \Omega \). We apply (3.17) to the right-hand side of this equality to get \( \int_\Omega Q' u \cdot \nabla q_k \, dx = 0 \) for all \( k \in \mathbb{N} \). Since \( Q' u \in L_p(\Omega) \) and (3.18) with \( p \) replaced by \( p' \) holds, the above equality implies that
\[
\int_\Omega Q' u \cdot \nabla q \, dx = \lim_{k \to \infty} \int_\Omega Q' u \cdot \nabla q_k \, dx = 0.
\]
Hence by the characterization of elements of \( L_p(\Omega) \) we conclude that \( Q' u \in L_p(\Omega) \) for all \( u \in C_c^\infty(\Omega) \). The proof is complete. \( \square \)

Remark 3.14.

(1) Let \( p \in (1, \infty) \). By Theorem 3.5 and Lemma 3.12, we have \( Q' u \in L_p(\Omega) \) and \( \|Q' u\|_{L_p(\Omega)} \leq c\|u\|_{L_p(\Omega)} \) for all \( u \in C_c^\infty(\Omega) \). Moreover, \( Q' u = u \) holds for all \( u \in C_c^\infty(\Omega) \). Hence, by the density argument, \( Q' \) extends uniquely to a bounded linear operator on \( L_p(\Omega) \) that is a projection onto \( L_p(\Omega) \).

(2) The projection onto \( L_p(\Omega) \) given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each \( u \in C_c^\infty(\Omega) \) there would exist \( \pi \in L_p^1(\Omega) \) such that \( (I - Q')u = \nabla \pi \) holds. Since \( (I - Q')u = Tu = K \ast \text{div} \, u \) for \( u \in C_c^\infty(\Omega) \), the existence of such \( \pi \) would imply that \( \partial_i (K \ast \text{div} \, u) = \partial_i (K_j \ast \text{div} \, u) \) for all \( i, j = 1, \ldots, n \). For each \( f \in C_c^\infty(\Omega) \) with \( \int_\Omega f \, dx = 0 \) there is \( u \in C_c^\infty(\Omega) \) satisfying \( f = \text{div} \, u \). This is possible since we are able to apply Bogovskii’s lemma to a bounded Lipschitz domain \( D \subset \Omega \) containing the support of \( f \) (see [19, Theorem III.3.3]). Thus the above equality would imply that \( \partial_i K_i = \partial_i K_j + c \) with some constant \( c \) for all \( i, j = 1, \ldots, n \) as a distribution. This contradicts the fact that \( \partial_i K_i \neq \partial_i K_j + c \) for \( i \neq j \) as observed in (3.4).

(3) It is possible to prove the characterization
\[
L_p(\Omega) = \{ u \in L_p(\Omega) \mid \text{div} \, u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial \Omega \}
\]
if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains.
(see [19, Section III.2]). However, for a Lipschitz half-space, it is less popular. A proof can be found in [30, Lemma 2.1].

The linear operator $Q'$ also maps $C^\infty_c(\Omega)$ into $VMO_{b,0,\sigma}(\Omega)$.

**Lemma 3.15.** Let $\Omega$ be a Lipschitz half-space. For all $u \in C^\infty_c(\Omega)$ and $\nu \in (0, \infty]$, we have $Q'u \in VMO_{b,0,\sigma}(\Omega)$.

We shall prove two auxiliary propositions for the above lemma. For $p \in (1, \infty)$, let $W^{1,p}_0(\Omega)$ be the $W^{1,p}$-closure of $C^\infty_c(\Omega)$.

**Proposition 3.16.** Let $\Omega$ be a Lipschitz half-space. For all $p \in (1, \infty)$ we have $L^p_0(\Omega) \cap W^{1,p}_0(\Omega) \subset W^{1,p}_0(\Omega)$. Thus $L^p_0(\Omega) \cap W^{1,p}_0(\Omega) = W^{1,p}_0(\Omega)$.

**Proof.** Let $\rho \in C^\infty_c(\mathbb{R}^n)$ be a function such that $0 \leq \rho \leq 1$ in $\mathbb{R}^n$, supp $\rho \subset B_1(0)$, $\int_{B_1(0)} \rho \, dx = 1$

and $\rho_\delta(x) := \delta^{-n} \rho(\delta^{-1}x)$ for $\delta > 0$, $x \in \mathbb{R}^n$. Let $u \in L^\infty_0(\Omega) \cap W^{1,p}_0(\Omega)$. Then there is a sequence $\{u_k\}_{k=1}^\infty$ of functions in $C^\infty_{c,\sigma}(\Omega)$ such that $\lim_{k \to \infty} \|u-u_k\|_{L^p(\Omega)} = 0$.

For $a > 0$, we define a vector field $u^a$ on $\Omega$ as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ u(x'), & h(x') \leq x_n \leq h(x') + a \\ 0, & \end{cases}$$

and $u^a_k = (u_k)^a$ similarly. Then it is clear that $u^a \in W^{1,p}_0(\Omega)$ and $u^a_k \in C^\infty_{c,\sigma}(\Omega)$ for all $a > 0$. Moreover, we have

$$\|u^a - u^a_k\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all } a > 0, \lim_{a \to 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0.$$

By the second equality and the fact that $W^{1,p}_0(\Omega)$ is closed in $W^{1,p}(\Omega)$, it is sufficient for showing $u \in W^{1,p}_0(\Omega)$ to prove $u^a \in W^{1,p}_0(\Omega)$ for all $a > 0$.

For each $a > 0$, there is a constant $d = d(a) > 0$ such that dist(supp $u^a_k$, $\partial \Omega$) $\geq d$ for all $k \in \mathbb{N}$. Then, for a given $\varepsilon > 0$, we can take $\delta \in (0, d/2)$ so small that

$$\|u^a - u^a_k\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2},$$

since $u^a \in W^{1,p}_0(\Omega)$. Also, since $\nabla \rho_\delta = \delta^{-1}(\nabla \rho)_\delta$, we have

$$\|u^a * \rho_\delta - u^a_k * \rho_\delta\|_{W^{1,p}(\Omega)} \leq c(\|u^a - u^a_k\|_{L^p(\Omega)} + \|u^a * \nabla \rho_\delta - u^a_k * \nabla \rho_\delta\|_{L^p(\Omega)}) = c(\|u^a - u^a_k\|_{L^p(\Omega)} + \delta^{-1}(\|u^a - u^a_k\|_{L^p(\Omega)})) \leq c(1 + \delta^{-1})\|u^a - u^a_k\|_{L^p(\Omega)} = c(1 + \delta^{-1})\|u - u_k\|_{L^p(\Omega)}$$

with a constant $c > 0$ independent of $\varepsilon$ and $\delta$. Hence by taking $k \in \mathbb{N}$ so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})},$$

we have $\|u^a * \rho_\delta - u^a_k * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2$ and thus

$$\|u^a - u^a_k\|_{W^{1,p}(\Omega)} \leq \|u^a - u^a_k * \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a * \rho_\delta - u^a_k * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon.$$

On the other hand, since dist(supp $u^a_k$, $\partial \Omega$) $\geq d$ and $\delta \in (0, d/2)$, the function $u^a_k * \rho_\delta$ is smooth and compactly supported in $\Omega$. Moreover, we have

$$\text{div}(u^a_k * \rho_\delta) = (\text{div} u^a_k) * \rho_\delta = 0 \quad \text{in } \Omega.$$
Thus $u_k^\ast \rho_\delta \in C_c^\infty(\Omega)$ and $u^\ast$ is approximated by elements of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, which means that $u^\ast \in W^{1,p}_{0,\sigma}(\Omega)$. Hence $u \in W^{1,p}_{0,\sigma}(\Omega)$ and the proof is now complete.

\(\square\)

**Proposition 3.17.** Let $\nu \in (0, \infty]$. If $p > n$, then $W^{1,p}_{0,\sigma}(\Omega) \subset \text{VMO}^{\infty,\nu}_{b,0,\sigma}(\Omega)$.

**Proof.** Let $u \in W^{1,p}_{0,\sigma}(\Omega)$ and $u_k \in C_c^\infty(\Omega)$ such that $\lim_{k \to \infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0$. Since $p > n$ and $u, u_k \in W^{1,p}_{0,\sigma}(\Omega)$, Morrey’s inequality (see e.g. [7, Theorem 4.12]) implies

$$\|u - u_k\|_{L^\infty(\Omega)} \leq c \|u - u_k\|_{W^{1,p}(\Omega)}$$

with a positive constant $c$ independent of $u$ and $u_k$. Thus we have

$$\|u - u_k\|_{L^\infty(\Omega)} \leq (2 + \omega_n)\|u - u_k\|_{L^\infty(\Omega)} \leq c \|u - u_k\|_{W^{1,p}(\Omega)} \to 0$$

as $k \to \infty$. Hence $u \in \text{VMO}^{\infty,\nu}_{b,0,\sigma}(\Omega)$ and the proof is now complete. \(\square\)

**Proof of Lemma 3.15.** Since $u \in C_c^\infty(\Omega)$ and thus $\partial_i u \in C_c^\infty(\Omega)$ for all $i = 1, \ldots, n$, it follows from Lemma 3.12 that $Q' u \in L^p_r(\Omega)$ and $\partial_i Q' u = Q'(\partial_i u) \in L^p_r(\Omega)$ for all $r \in (1, \infty)$ and $i = 1, \ldots, n$. From this fact and the equality (3.17), we have $Q' u \in L^p_r(\Omega) \cap W^{1,p}_{0,\sigma}(\Omega)$ for all $r \in (1, \infty)$. Hence, by taking $r > n$, we can apply Proposition 3.16 and Proposition 3.17 to obtain $Q' u \in \text{VMO}^{\infty,\nu}_{b,0,\sigma}(\Omega)$. \(\square\)

**Remark 3.18.** Let $\nu \in (0, \infty]$. Theorem 3.10 and Lemma 3.15 imply that $Q' u \in \text{VMO}^{\infty,\nu}_{b,0,\sigma}(\Omega)$ and $\|Q' u : \text{BMO}^{\infty,\nu}_{b}(\Omega)\| \leq c \|u\|_{L^\infty(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Also, we have $Q' u = u$ for all $u \in C_c^\infty(\Omega)$. Hence $Q'$ extends uniquely to a bounded linear operator (again referred to as $Q'$) from $C_0(\Omega)$, which is the $L^\infty$-closure of $C_c^\infty(\Omega)$, into $\text{VMO}^{\infty,\nu}_{b,0,\sigma}(\Omega)$ that satisfies $Q' u = u$ for all $u \in C_0(\Omega)$.

Now let us extend $Q'$ to a linear operator that gives the projection mentioned in Theorem 1.4. For $p \in (1, \infty)$, we define a Banach space $X_p$ and its norm as

$$X_p := L^p(\Omega) \cap C_0(\Omega), \quad \|u\|_{X_p} := \max\{\|u\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}\}.$$ 

Note that the Banach space $C_0(\Omega)$ consists of all continuous functions $f$ on $\Omega$ such that the set $\{x \in \Omega \mid |f(x)| \geq \epsilon\}$ is compact in $\Omega$ for every $\epsilon > 0$ (see e.g. [32, Theorem 3.17]).

**Lemma 3.19.** For each $p \in (1, \infty)$, the linear subspace $C_c^\infty(\Omega)$ is dense in $X_p$.

**Proof.** The proof is more or less standard (see e.g. [27, Corollary 19.24]). We give it for completeness. Let $u \in X_p$ and $\Omega_k := \{x \in \Omega \mid |x| \leq k, \text{dist}(x, \partial \Omega) \geq 1/k\}$ for $k \in \mathbb{N}$. For any given $\epsilon > 0$, the set $\{x \in \Omega \mid |u(x)| \geq \epsilon\}$ is compact in $\Omega$ since $u \in C_0(\Omega)$. Moreover, since $u \in L^p(\Omega)$, we can take $k \in \mathbb{N}$ so large that

$$\|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}, \quad \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}.$$ 

Let $\varphi \in C_c^\infty(\Omega)$ be a continuous cut-off function such that

$$0 \leq \varphi \leq 1 \quad \text{in} \quad \Omega, \quad \varphi = 1 \quad \text{in} \quad \Omega_k, \quad \varphi = 0 \quad \text{in} \quad \Omega \setminus \Omega_{2k}.$$ 

Since $u - \varphi u = 0$ in $\Omega_k$ and $|u - \varphi u| \leq |u|$ in $\Omega \setminus \Omega_k$, it follows from (3.23) that

$$\|u - \varphi u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}, \quad \|u - \varphi u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\epsilon}{2}.$$
Let $\rho_\delta$ be a mollifier as in the beginning of the proof of Proposition 3.16. Since

$$\varphi u \in L^p(\Omega), \quad \text{dist} (\text{supp} (\varphi u), \partial \Omega) \geq \frac{1}{2k},$$

we can take $\delta \in (0, 1/4k)$ so that

$$u_\delta := \rho_\delta * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}.$$  \hfill (3.25)

On the other hand, since $\varphi u$ is uniformly continuous on $\Omega_{4k}$, we can again choose $\delta \in (0, 1/4k)$ so that $\|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \varepsilon/2$. Moreover, since $\text{supp}(\varphi u) \subset \Omega_{2k}$ and $\delta \in (0, 1/4k)$, we have $\varphi u = u_\delta = 0$ outside of $\Omega_{4k}$ and thus

$$\|\varphi u - u_\delta\|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}.$$  \hfill (3.26)

Combining (3.24), (3.25) and (3.26), we obtain $u_\delta \in C_c^\infty(\Omega)$ and

$$\|u - u_\delta\|_{X_p} = \max \{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon.$$

Hence the lemma follows. \hfill \Box

Let $Y_p := L^p_b(\Omega) \cap VMO_b(\Omega)$ for $p \in (1, \infty)$, $\nu \in (0, \infty]$. Since $L^p_b(\Omega)$ and $VMO_b(\Omega)$ are closed in $L^p(\Omega)$ and $BMO_b(\Omega)$, respectively, $Y_p$ becomes a Banach space under the norm $\|v\|_{Y_p} := \max \{\|v\|_{L^p(\Omega)}, \|v : BMO_b(\Omega)\|\}$.

**Theorem 3.20.** Let $p \in (1, \infty)$ and $\nu \in (0, \infty]$. The linear operator $Q'$ given in Definition 3.11 extends uniquely to a bounded linear operator $Q_p$ from $X_p$ into $Y_p$. Moreover, there exists a constant $c > 0$ such that

$$\|Q_p u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q_p u : BMO_b^{\infty, \nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all $u \in X_p$ and $Q_p u = u$ holds for all $u$ in the $X_p$-closure of $C_c^{\infty, \nu}(\Omega)$.

**Proof.** Let $u \in C_c^{\infty}(\Omega)$. Then we have $Q' u \in Y_p$ by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant $c > 0$ independent of $u$ such that

$$\|Q' u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q' u : BMO_b^{\infty, \nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}.$$  \hfill (3.28)

Hence we have $Q' u \in Y_p$ and $Q' u|_{Y_p} \leq c\|u\|_{X_p}$ for all $u \in C_c^{\infty}(\Omega)$. Since $C_c^{\infty}(\Omega)$ is dense in $X_p$ by Lemma 3.19, the operator $Q'$ extends uniquely to a bounded linear operator $Q_p$ from $X_p$ into $Y_p$. Also, it follows from (3.28) that the inequality (3.27) holds for all $u \in X_p$. Since $Q' u = u$ holds for all $u \in C_c^{\infty}(\Omega)$ as observed after Definition 3.11, by the density argument we have $Q_p u = u$ for all $u$ in the $X_p$-closure of $C_c^{\infty, \nu}(\Omega)$. The proof is complete. \hfill \Box

Finally, Theorem 1.4 follows from Theorem 3.20 with $p = 2$, that is, the linear operator $Q$ in Theorem 1.4 is given by $Q = Q_2$.

4. **Analyticity in $L^p$**

In this section we shall give a complete proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $S(t)$ be the Stokes semigroup in $\tilde{L}_c^p$ constructed by [14], [16]. To show that $S(t)$ forms an analytic semigroup in $L_c^p$ ($2 \leq p < \infty$) it suffices to prove that there exists a constant $C$ that

$$\|S(t)u_0\|_p \leq C\|v_0\|_p$$  \hfill (4.1)
for all \( v_0 \in C^\infty_{c,\sigma}(\Omega) \) and for all \( t \in (0, 1) \). Let \( Q \) be the operator in Theorem \( 1.4. \) Since \( Q \) is bounded in \( L^2 \) and maps \( L^2 \) to \( L^2_\sigma \) and \( S(t) \) fulfills \( (4.1) \) and \( (4.2) \) for \( p = 2 \), we have
\[
\| S(t)Qu \|_p \leq C\| u \|_p
\]
for all \( u \in C_c(\Omega) \) and \( t \in (0, 1) \). Since \( \Omega \) is admissible as proved in \([5]\), \( S(t) \) forms an analytic semigroup in \( \text{VMO}^{\infty,\nu}_b \) by Theorem \( 1.2. \) We conclude that
\[
\| S(t)Qu : \text{BMO}^{\infty,\nu}_b(\Omega) \| \leq C\| u \|_\infty
\]
for all \( u \in C_c(\Omega) \) and \( t \in (0, 1) \) since \( Q \) fulfills
\[
\| Qu : \text{BMO}^{\infty,\nu}_b(\Omega) \| \leq C\| u \|_\infty, \; Qu \in \text{VMO}^{\infty,\nu}_b
\]
for all \( u \in C_c(\Omega) \) by Theorem \( 1.4. \) (Note that we have a stronger statement than \( (4.6) \) by replacing the \( \text{BMO}_b \) type norm by the \( L^\infty \) norm since we have the regularizing estimate \( (1.3) \).) We apply an interpolation result (Theorem \( 1.3. \)) to \( (4.3) \) and \( (4.4) \) and \( (4.5) \) to \( (4.6) \) to get, respectively
\[
\| S(t)Qu \|_p \leq C\| u \|_p
\]
(4.7)
\[
\| t \frac{d}{dt} S(t)Qu \|_p \leq C\| u \|_p
\]
(4.8)
for all \( u \in C_c(\Omega) \) and for all \( t \in (0, 1) \). Since \( Qu = u \) for \( u \in C^\infty_{c,\sigma}(\Omega) \) this yields \( (4.1) \) and \( (4.2). \)

It remains to prove that \( S(t) \) is a \( C_0 \)-semigroup in \( L^p_\sigma \). Since \( C^\infty_{c,\sigma}(\Omega) \) is dense in \( L^p_\sigma \), for \( v_0 \in L^p_\sigma \) there is \( v_{0m} \in C^\infty_{c,\sigma} \) such that \( \| v_0 - v_{0m} \|_p \rightarrow 0 \) as \( m \rightarrow \infty \). By \( (4.1) \) we observe that
\[
\| S(t)v_0 - v_0 \|_p \leq \| S(t)(v_0 - v_{0m}) \|_p + \| S(t)v_{0m} - v_{0m} \|_p + \| v_{0m} - v_0 \|_p \leq C\| v_0 - v_{0m} \|_p + \| S(t)v_{0m} - v_{0m} \|_p.
\]
Sending \( t \downarrow 0 \), we get
\[
\lim_{t \downarrow 0} \| S(t)v_0 - v_0 \|_p \leq C\| v - v_{0m} \|_p,
\]
since \( S(t)v_{0m} \rightarrow v_{0m} \) in \( L^p_\sigma \) as \( t \downarrow 0 \) by \([14],[16]\). Sending \( m \rightarrow \infty \), we conclude that \( S(t)v_0 \rightarrow v_0 \) in \( L^p_\sigma \) as \( t \downarrow 0 \). \( \square \)

**Remark 4.1.** In a similar way as we derived \( (4.5) \) and \( (4.6) \) we are able to derive from the \( L^\infty-\text{BMO} \) estimates in \([10]\) that
\[
t \| \nabla^2 S(t)Qu : \text{BMO}^{\infty,\nu}_b(\Omega) \| \leq C\| u \|_\infty
\]
\[
t^{1/2} \| \nabla S(t)Qu : \text{BMO}^{\infty,\nu}_b(\Omega) \| \leq C\| u \|_\infty
\]
for all \( u \in C_c(\Omega) \) and \( t \in (0, 1) \).
Note that $L^2$ results

$$t \left\| \nabla^2 S(t)Qu \right\|_2 \leq C \|u\|_2$$

$$t^{1/2} \| \nabla S(t)Qu \|_2 \leq C \|u\|_2$$

easily follow from the analyticity of $S(t)$ in $L^2_\sigma$ and $L^2$-boundedness of $Q$ if one observes that $\|\nabla u\|_2^2 = (Au, u)_{L^2}$ and

$$\|\nabla^2 u\|_2 \leq C (\|Au\|_2 + \|\nabla u\|_2 + \|u\|_2)$$

(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where $A$ is the Stokes operator in $L^2_\sigma$.

Interpolating the $L^2$ results and the above $L^\infty$-$BMO$ results, we are able to prove that there is $C_p > 0$ satisfying

$$t \left\| \nabla^2 S(t)v_0 \right\|_p \leq C_p \|v_0\|_p$$

$$t^{1/2} \| \nabla S(t)v_0 \|_p \leq C_p \|v_0\|_p$$

for all $v_0 \in L^p_\sigma(\Omega)$ and $t \in (0, 1)$ with $p \in (2, \infty)$.

References


ANALYTICITY OF THE $L^p$-STOKES SEMIGROUP


