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ON ANALYTICITY OF THE $L^p$-STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

MARTIN BOLKART, YOSHIKAZU GIGA, TATSU-HIKO MIURA, TAKUYA SUZUKI, AND YOHEI TSUTSUI

Abstract. Consider the Stokes equations in a sector-like $C^3$ domain $\Omega \subset \mathbb{R}^3$. It is shown that the Stokes operator generates an analytic semigroup in $L^p(\Omega)$ for $p \in [2, \infty)$. This includes domains where the $L^p$-Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in $\mathcal{VMO}$ and $L^2$ by constructing a suitable non-Helmholtz projection to solenoidal spaces.

1. Introduction

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \text{div } v = 0 \quad \text{in } \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on } \partial \Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where $\Omega$ is a domain in $\mathbb{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a $C_0$-analytic semigroup in $L^p_\sigma (1 < p < \infty)$ for various domains like smooth bounded domains ([21], [35]). Here $L^p_\sigma = L^p_\sigma (\Omega)$ denotes the $L^p$-closure of $C_c^\infty(\Omega)$, the space of all solenoidal vector fields with compact support in $\Omega$. More recently, it has been proved in [20] that $S(t)$ always forms a $C_0$-analytic semigroup in $L^p_\sigma (\Omega)$ for any uniformly $C^2$-domain $\Omega$ provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma (\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \{ \nabla q \in L^p(\Omega) \mid q \in L^1_{\text{loc}}(\Omega) \}$. In [20] the $L^p$ maximal regularity in time with values in $L^p_\sigma (\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The $L^p$-Helmholtz decomposition holds for various domains like bounded or exterior domains with
smooth boundary for $1 < p < \infty$ ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the $L^p$-Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely. Let $C(\vartheta)$ denote the cone of the form

$$C(\vartheta) = \{ x = (x', x_n) \in \mathbb{R}^n \mid -x_n \geq |x| \cos(\vartheta/2) \},$$

where $\vartheta \in (0, 2\pi)$ is the opening angle. When $n = 2$, we simply say that $C(\vartheta)$ is a sector. We say that a planar domain $\Omega$ is a sector-like domain if $\partial \Omega$ is a sector. We say that a planar domain $\Omega$ is a sector-like domain in $\mathbb{R}^2$ having a $C^k$ graph boundary.
where $U$.

We shall often assume that $\nu < R$, thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a $C^1$ seminorm we use is $BMO$-admissible in the sense of [2] provided that $\nu \in S$.

This space is independent of $p$ for sufficiently small $\nu$, i.e., $\nu < R^*$ ([11], [12]) and $BMO_b^{\infty,\infty}$ agrees with Miyachi $BMO$ space ([29]) for various domains including a half space and bounded $C^2$ domains ([12]). Although the $BMO_b^{\infty,\nu}(\Omega)$ norm is equivalent to the $BMO_b^{\infty,\infty}(\Omega)$ norm when $\Omega$ is bounded, there are many unbounded domains for which the $BMO_b^{\infty,\nu}(\Omega)$ norm is actually weaker than the $BMO_b^{\infty,\infty}(\Omega)$ norm when $\nu$ is finite. We define the solenoidal space $VMO_{b,0,\sigma}^{\mu,\nu}$ as the $BMO_b^{\mu,\nu}$-closure of $C_{\sigma,\sigma}(\Omega)$. In [10], [11] among other results the analyticity of $S(t)$ in $VMO_{b,0,\sigma}^{\mu,\nu}$ has been established for a uniformly $C^3$ domain which is admissible in the sense of [2] provided that $\nu$ is sufficiently small.

**Theorem 1.2** ([10], [11]). Let $\Omega$ be an admissible uniformly $C^3$ domain in $\mathbb{R}^n$. Then $S(t)$ forms a $C_0$-analytic semigroup in $VMO_{b,0,\sigma}^{\mu,\nu}$ for any $\mu \in (0,\infty]$ and $\nu \in (0,\nu_0)$ with some $\nu_0$ depending only on $\mu$ and regularity of $\partial \Omega$.

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace $L^p$ by $L^\infty$ or $BMO_b^{\infty,\nu}$, but even an estimate stronger than (1.2) with $p = \infty$, i.e.,

$$
t \int_0^\infty \frac{1}{\|dS(t)\|} = C \|v_0 : BMO_b^{\mu,\nu}(\Omega)\|, \quad \mu, \nu \in (0,\infty]$$

which shows a regularizing effect.

It has been proved in [5] that a $C^2$ sector-like domain in $\mathbb{R}^2$ is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a $C^3$ sector-like
domain in $\mathbb{R}^2$ is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly $C^4$. On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a layer domain with $n \geq 3$ is not admissible ([8]).

In order to get the $L^p$ estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in $\Omega$.

**Theorem 1.3.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$, i.e., a domain having Lipschitz graph boundary. Let $T$ be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant $C$ such that

$$
\|Tu\|_2 \leq C\|u\|_2
$$

for $u \in C_c(\Omega)$. Then $\|Tu\|_p \leq C\|u\|_p$ for $u \in C_c(\Omega)$ with $C$ depending only on $C, h$ and $p \in (2, \infty)$.

There are a couple of such interpolation results between $BMO$ and $L^2$, which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between $L^p$ and $BMO$ is discussed when $\Omega$ is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega) \cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with $BMO_A(\mathcal{X})$, where $A$ is some operator. They worked on metric measure spaces of homogeneous type $(\mathcal{X}, d, \mu)$. In particular, in the case $\mathcal{X} = \Omega, d(x, y) = |x - y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO(\Omega)$. Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the $L^\infty$ case we do not know whether or not the complex interpolation space $L^2_{\sigma, \tau}VMO^{\infty, \nu}_{b, 0, \sigma}$ with $2/p = 1 - \rho$ agrees with $L^p_{\sigma}$, although we know that $[L^2, BMO]_{\rho} = L^p$ for $\Omega = \mathbb{R}^n$ as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

**Theorem 1.4.** Let $\Omega$ be a Lipschitz half-space in $\mathbb{R}^n$. Assume that $\nu \in (0, \infty]$. There is a linear operator $Q$ from $C_c(\Omega)$ to $VMO^{\infty, \nu}_{b, 0, \sigma}(\Omega) \cap L^2_{\sigma}(\Omega)$ such that

$$
\|Qu : BMO^{\infty, \nu}_{b, \sigma}(\Omega)\| \leq C\|u\|_\infty
$$

$$
\|Qu\|_2 \leq C\|u\|_2
$$

for all $u \in C_c(\Omega)$. Moreover, $Qu = u$ for $u \in C_c(\Omega) \cap L^2_{\sigma}(\Omega)$.

Since there may be no $L^p$-Helmholtz decomposition our $Q$ should be different from the Helmholtz projection. We shall construct such an operator $Q$ using the solution operator of the equation $\text{div} u = f$ given by Solonnikov [36]. Although deriving the $L^2$ estimate is easy, to derive the $BMO$ estimate is more involved since we have to estimate the $b^\nu$ type seminorm.

To derive (1.1), we actually interpolate

$$
\|S(t)Qu\|_2 \leq C\|u\|_2
$$

and

$$
\|S(t)Qu : BMO^{\infty, \nu}_{b, \sigma}\| \leq C\|u\|_\infty
$$

for $u \in C_c(\Omega)$. Similarly, we derive (1.2) by interpolating the estimate for $t^{\frac{\nu}{2p}}Q$. 

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 3, we construct the projection operator $Q$. In Section 4, we give a complete proof of Theorem 1.1.

2. $L^2 - BMO$ INTERPOLATION ON A LIPSCHITZ HALF-SPACE

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

$$
\Omega := \{(x', x_n) \in \mathbb{R}^n | x_n > h(x')\}
$$

with a Lipschitz function $h$ on $\mathbb{R}^{n-1}$.

By $Q$ we mean a closed cube with sides parallel to the coordinate axes. Let $\ell(Q)$ be the side length of $Q$, and for $\tau > 0$, $\tau Q$ a cube with the same length as $Q$ and side length $\tau\ell(Q)$.

2.1. Reduction to the half-space and extension. Here, we prepare lemmas that are basic estimates for the proof. Since $h$ is Lipschitz continuous, $F(x) := (x', x_n - h(x'))$ is a bi-Lipschitz map from $\Omega$ to $\mathbb{R}^n_+$. For a function $u$ defined on $\mathbb{R}^n_+$, the pull-back function $F^*(u)$ of $u$ on $\Omega$ is defined by $u \circ F$. We start with estimates for $(F^{-1})^*$, which is the pull-back function $(F^{-1})^*(v)$ of $v$ on $\mathbb{R}^n_+$ defined by $v \circ F^{-1}$.

**Lemma 2.1.** Let $\Omega$ be a Lipschitz half-space.

(i): 
$$
[(F^{-1})^* v : BMO^\infty(\mathbb{R}^n_+)] \leq c [v : BMO^\infty(\Omega)].
$$

(ii): 
$$
\|(F^{-1})^* v\|_{L^2(\mathbb{R}^n_+)} \leq c\|v\|_{L^2(\Omega)}.
$$

Here $c$ is a constant depending only on Lipschitz bound of $h$ and $u$.

**Proof.** (i): Because $\mathbb{R}^n_+$ is an open subset of $\mathbb{R}^n$, we know that for any $\tau > 2$,

$$
[(F^{-1})^* v : BMO^\infty(\mathbb{R}^n_+)] \leq c_\tau \sup_{Q \subset \mathbb{R}^n_+} \inf_{d \in \mathbb{R}} \int_Q |(F^{-1})^* v - d| dy,
$$

where the supremum is taken over cubes $Q$, for which $\tau Q$ is contained in $\mathbb{R}^n_+$, see [37]. Since $F$ is a bi-Lipschitz map, it holds

$$
c_1 \text{dist}(y, \partial \mathbb{R}^n_+) \leq \text{dist}(F^{-1}(y), \partial \Omega) \leq c_2 \text{dist}(y, \partial \mathbb{R}^n_+)
$$

with some constants $c_1, c_2 > 0$ for all $y \in \mathbb{R}^n_+$. Since $(\tau - 1)\ell(Q)/2 \leq \text{dist}(Q, \partial \mathbb{R}^n_+)$ for such cubes $Q$, we have the lower bound

$$
c_\tau \ell(Q) \leq \text{dist}(F^{-1}(Q), \partial \Omega)
$$

with some $c > 0$, which depends on $n$ and $h$. Therefore, taking large $\tau$, we can find cubes $\{R_k\}_{k=1}^\infty \subset \Omega$, which have no intersection of interiors, so that $\cup_{k=1}^\infty R_k$ is connected and

$$
\begin{align*}
\circ \ell(R_k) &= \ell(Q), \\
\circ F^{-1}(Q) &\subset \cup_{k=1}^\infty R_k, \text{ where } c_\ast \in \mathbb{N} \text{ depends only on } h, \text{ and} \\
\circ \text{if } R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega.
\end{align*}
$$

From these, one obtains that for cubes $Q$ with $\tau Q \subset \mathbb{R}^n_+$,

$$
\inf_{d \in \mathbb{R}} \frac{1}{|Q|} \int_Q |(F^{-1})^* v - d| dy \leq c \sum_{k=1}^{c_\ast} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| dy.
$$
It is enough to show that
\[
\frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| dy \leq c |v : BMO^\infty(\Omega)|
\]
for the case \(R_j \cap R_k \neq \emptyset\). To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let \(\tilde{R}_k\) and \(\tilde{R}_j\) be subcubes of \(R_k\) and \(R_j\) respectively so that \(\ell(\tilde{R}_k) = \ell(R_k)/2\), \(\ell(\tilde{R}_j) = \ell(R_j)/2\) and they touch each other. Moreover, denote by \(\tilde{R}_{j,k}\) a cube satisfying \(\ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k)\) and \(\tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k}\). Hence, we have
\[
\frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| dy \leq \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| dy + |v_{R_k} - v_{R_j}|
\]
\[
\leq c |v : BMO^\infty(\Omega)| + c |v_{\tilde{R}_j} - v_{\tilde{R}_k}|
\]
\[
\leq c |v : BMO^\infty(\Omega)| + c \frac{1}{|\tilde{R}_{j,k}|} \int_{\tilde{R}_{j,k}} |v - v_{\tilde{R}_{j,k}}| dy
\]
\[
\leq c |v : BMO^\infty(\Omega)|.
\]

(ii): This is verified as follows
\[
\| (F^{-1})^* v \|^2_{L^2(\mathbb{R}^n_+)} = \int_{\Omega} |v|^2 J_F dx \leq c \int_{\Omega} |v|^2 dx,
\]
where \(J_F\) is the modulus of the Jacobian of \(F\) which is bounded, because \(h\) is Lipschitz continuous. \(\square\)

Next, we consider the even extension of functions on the half space. For a function \(f\) on \(\mathbb{R}^n_+\), we extend \(f\) outside \(\mathbb{R}^n_+\) by
\[
E[f](x', -x_n) := f(x', x_n) \text{ for } x_n > 0.
\]
From elementary geometrical observation, we can see that the extension operator \(E\) is a \(BMO\)-extension operator for \(\mathbb{R}^n_+\).

**Lemma 2.2.**
\[
|E[f] : BMO^\infty(\mathbb{R}^n)| \leq c \left[ f : BMO^\infty(\mathbb{R}^n_+) \right].
\]

**Proof.** It is sufficient to consider cubes \(Q \subset \mathbb{R}^n\) with \(Q \cap \mathbb{R}^n_+ \neq \emptyset\) and \(Q \cap \mathbb{R}^n_\cup \neq \emptyset\). For such \(Q\), let \(Q'\) be a cube so that its center lies on \(\partial \mathbb{R}^n_+\), \(\ell(Q') = 2\ell(Q)\) and \(Q \subset Q'\). Further, let \(Q^*\) be the smallest cube in \(\mathbb{R}^n_+\) containing the upper half of \(Q'\). With these notations, the desired inequality is proved from
\[
\inf_{d \in \mathbb{R}} \frac{1}{|Q|} \int_Q |E[f] - d| dy \leq c \inf_{d \in \mathbb{R}} \frac{1}{|Q^*|} \int_{Q^*} |f - d| dy.
\]
\(\square\)

**2.2. Sharp maximal operator.** For the proof of Theorem 1.3, we make use of the sharp maximal operator \(M^2\) due to Fefferman and Stein ([18]). We define for \(x \in \mathbb{R}^n\) and \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) the function \(M^2 f\) by
\[
M^2 f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.
\]
It is immediate from the definition that \([f : BMO^\infty(\mathbb{R}^n)] = \|M^2 f\|_{L^\infty(\mathbb{R}^n)}^2\). It is well-known that if \(f \in L^{p_0}(\mathbb{R}^n)\) for some \(p_0 \in (1, \infty)\), then for \(p \in [p_0, \infty)\)

\[
(2.2) \quad \|f\|_{L^p(\mathbb{R}^n)} \leq c\|M^2 f\|_{L^p(\mathbb{R}^n)},
\]

which is applied below. (Both sides of (2.2) may be infinite.) This follows from \(\|f\|_{L^p(\mathbb{R}^n)} \leq \|M f\|_{L^p(\mathbb{R}^n)}\) and \(\|M f\|_{L^p(\mathbb{R}^n)} \leq c\|M^2 f\|_{L^p(\mathbb{R}^n)}\), where \(M\) is the Hardy-Littlewood maximal operator \([18]\).

### 2.3. Marcinkiewicz interpolation

Here, we give a variant of the Marcinkiewicz interpolation theorem.

**Proposition 2.3.** Let \(D\) be an open subset of \(\mathbb{R}^n\) and \(S\) a sublinear operator from \(C_c(D)\) to \(L^2(\mathbb{R}^n)\). If

\[
\|S[f]\|_{L^2(\mathbb{R}^n)} \leq c\|f\|_{L^2(D)}
\]

\[
\|S[f]\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{L^\infty(D)}
\]

for \(f \in C_c(D)\), then \(\|S[f]\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(D)}\) for \(f \in C_c(D)\) with \(C\) depending only on \(c\) and \(p \in (2, \infty)\).

**Proof.** For \(\lambda > 0\) and \(\alpha > 0\), we decompose \(f\) into two parts; \(f = f_2 + f_\infty\) where

\[
f_2(x) = \begin{cases} 0 & \text{if } |f(x)| \leq \alpha \lambda \\ f(x) - \alpha \lambda \text{sign}(f(x)) & \text{if } |f(x)| > \alpha \lambda, \end{cases}
\]

where \(\text{sign} \xi = \xi/|\xi|\) for \(\xi \neq 0\) and \(\text{sign} \xi = 0\) for \(\xi = 0\). Observe that \(f_2, f_\infty \in B(\mathbb{R}^n)\), and then \(f_2, f_\infty \in C_c(D)\). Therefore, the two inequalities of our assumption hold for \(f_2\) and \(f_\infty\), respectively. We set \(\alpha = (2\|S\|_{L^\infty(D) \to L^\infty(\mathbb{R}^n)})^{-1}\) and observe that \(\{|x \in \mathbb{R}^n | S[f_\infty](x) > \lambda/2\} = 0\). We now conclude that

\[
\int_{\mathbb{R}^n} |S[f]|^p \, dx \leq p \int_0^\infty \lambda^{p-1} \mathcal{N}([x \in \mathbb{R}^n | |S[f](x)| > \lambda]) \, d\lambda
\]

\[
\leq p \int_0^\infty \lambda^{p-1} \mathcal{N}([x \in \mathbb{R}^n | |S[f_2](x)| > \lambda/2]) \, d\lambda
\]

\[
\leq p \int_0^\infty \lambda^{p-1} \left\{ \frac{2}{\lambda} \mathcal{N}([S[f_2]]_{L^2(D) \to L^2(\mathbb{R}^n)} | f_2 |_{L^2(D)}) \right\} \, d\lambda
\]

\[
\leq c \int_0^\infty \lambda^{p-3} \int_{\{|f| > \alpha \lambda}\} |f(x)|^2 \, dx \, d\lambda
\]

\[
= 2c \int_0^\infty \lambda^{p-3} \left( \int_{\alpha \lambda}^\infty t \mathcal{N}([x \in \mathbb{R}^n | |f(x)| > t]) \, dt \right) \, d\lambda
\]

\[
= 2c \int_0^\infty t \mathcal{N}([x \in \mathbb{R}^n | |f(x)| > t]) \left( \int_0^{t/\alpha} \lambda^{p-3} \, d\lambda \right) \, dt
\]

\[
\leq c\|f\|_{L^p(D)}^p.
\]

\(\square\)
2.4. **Proof of Theorem 1.3.** For simplicity, we write $g := Tf$. By changing variables, one obtains

$$
\int_{\Omega} |g|^p \, dx \leq c \int_{\mathbb{R}^n_+} |(F^{-1})^* g|^p \, dy \leq c \int_{\mathbb{R}^n} |E[(F^{-1})^* g]|^p \, dy \leq c \int_{\mathbb{R}^n} |\Phi[f]|^p \, dy,
$$

where $\Phi[f] := M^2 (E[(F^{-1})^* g])$. Here, because $E[(F^{-1})^* g] \in L^2(\mathbb{R}^n)$, we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see $L^2(\Omega) - L^2(\mathbb{R}^n)$ and $L^\infty(\Omega) - L^\infty(\mathbb{R}^n)$ estimates for $\Phi$. The former estimate can be seen by $L^2$-boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The later one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

### 3. Non-Helmholtz projection

Our goal in this section is to prove Theorem 1.4.

#### 3.1. A solution operator to the divergence problem

As in Section 2, let $\Omega = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n > h(x')\}$ be a Lipschitz half-space in $\mathbb{R}^n$ with a Lipschitz continuous function $h$ on $\mathbb{R}^{n-1}$. Then, there is a closed cone of the form

$$
C_1 = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2\theta) \}
$$

with an angle $\theta \in (0, \pi/4)$ (depending on the Lipschitz constant of $h$) such that

$$
x + C_1 = \{ y \in \mathbb{R}^n \mid y - x \in C_1 \} \subset \Omega^c := \mathbb{R}^n \setminus \Omega \quad \text{for all} \quad x \in \Omega^c.
$$

In the notion of the introduction $C_1 = C(4\theta)$ so that the opening angle equals $4\theta$. With this angle we define a closed cone $C_0 = C(2\theta)$, i.e.,

$$
C_0 = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos \theta \}.
$$

The closed cone $C_0$ also satisfies

$$
(3.1) \quad x + C_0 \subset \Omega^c \quad \text{for all} \quad x \in \Omega^c.
$$

Let $L \in C_0^\infty(\mathbb{R}^n)$ be a function such that

$$
(3.2) \quad \text{supp } L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1.
$$

Here $-C_0 = \{-y \mid y \in C_0\}$ and $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Then we define a vector field $K = (K_1, \ldots, K_n)$ as

$$
(3.3) \quad K(x) := \frac{x}{|x|^n} L \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.
$$

**Definition 3.1.** For $f \in C_0^\infty(\Omega)$, we define a vector field $u = Sf$ as

$$
u(x) = Sf(x) := (K * f)(x) = \int_{\mathbb{R}^n} K(x - y)f(y) \, dy, \quad x \in \mathbb{R}^n.
$$

Here $f$ denotes the zero extension of $f$ to $\mathbb{R}^n$ given by

$$
\tilde{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}
$$
This operator was introduced by Solonnikov [36]. For a fixed $x \in \mathbb{R}^n$, since

$$\frac{x - y}{|x - y|} \in \text{supp} L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)$$

implies $y \in x + C_0$, we can write

$$u(x) = \int_{x + C_0} K(x - y) f(y) \, dy.$$ 

This formula and the property (3.1) of $\Omega$ imply that $u(x) = 0$ for all $x \in \Omega^c$. In particular, $u$ vanishes on $\partial \Omega$. However, the support of $u$ may become unbounded although $f$ is compactly supported in $\Omega$.

By the change of variables $x - y = r \sigma$ with $r > 0$ and $\sigma \in S^{n-1}$ we have

$$u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma)f(x - r \sigma) r^{n-1} \, d\mathcal{H}^{n-1}(\sigma) \, dr.$$ 

Hence if $f \in C_\alpha^\infty(\Omega)$ is supported in $B_R(0)$ and $x \in B_a(0)$ ($R, a > 0$), then

$$u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma)f(x - r \sigma) r^{-1} \, d\mathcal{H}^{n-1}(\sigma) \, dr,$$

which implies that $u = Sf$ is smooth in $\Omega$. Moreover, $u = Sf$ vanishes near $\partial \Omega$ and thus it is smooth in the whole space $\mathbb{R}^n$, since $f$ is compactly supported in $\Omega$.

**Lemma 3.2.** Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that

$$\|\nabla u\|_{L^p(\Omega)} \leq c \|f\|_{L^p(\Omega)}$$

for all $f \in C_\alpha^\infty(\Omega)$ and $u = Sf$.

**Proof.** Let $u_i$ be the $i$-th component of $u$:

$$u_i(x) = (K_i \ast f)(x) = \int_{\mathbb{R}^n} K_i(z) f(x - z) \, dz.$$ 

Differentiating both sides with respect to the $j$-th variable, we have

$$\partial_j u_i(x) = \int_{\mathbb{R}^n} K_i(z)(\partial_j f)(x - z) \, dz = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} K_i(z)(\partial_j f)(x - z) \, dz$$

and, by changing variables $y = x - z$ and integrating by parts,

$$\partial_j u_i(x) = \lim_{\varepsilon \to 0} \left( \int_{\partial B_\varepsilon(x)} K_i(x - y) \frac{x_j - y_j}{|x - y|} f(y) \, d\mathcal{H}^{n-1}(y) + \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (\partial_j K_i)(x - y) f(y) \, dy \right).$$

On the one hand, we change variables $x - y = \varepsilon \sigma$ with $\sigma \in S^{n-1}$ to get

$$\lim_{\varepsilon \to 0} \int_{|x - y| = \varepsilon} K_i(x - y) \frac{x_j - y_j}{|x - y|} f(y) \, d\mathcal{H}^{n-1}(y)$$

$$= \lim_{\varepsilon \to 0} \int_{|x - y| = \varepsilon} \frac{x_i - y_i}{|x - y|} \frac{x_j - y_j}{|x - y|} L \left( \frac{x - y}{|x - y|} \right) f(y) \frac{1}{|x - y|^{n-1}} \, d\mathcal{H}^{n-1}(y)$$

$$= \lim_{\varepsilon \to 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)f(x - \varepsilon \sigma) \, d\mathcal{H}^{n-1}(\sigma)$$

$$= \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma),$$
where the last equality follows from the fact that $L$ is integrable on $S^{n-1}$ and $\bar{f}$ is continuous at $x$. On the other hand, we differentiate $K_i$ to obtain

\begin{equation}
K_{ij}(z) := \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n},
\end{equation}

\begin{equation}
k_{ij}(z) := (\delta_{ij} - nz_iz_j)L(z) + z_i(\partial_j L)(z) - z_iz_j \sum_{\ell=1}^n z_\ell(\partial_\ell L)(z)
\end{equation}

for $z \in \mathbb{R}^n \setminus \{0\}$. Then $K_{ij}$ is homogeneous of degree $-n$ and there is a constant $c > 0$ such that

$$|K_{ij}(z)| \leq \frac{c}{|z|^n} \text{ for all } z \in \mathbb{R}^n \setminus \{0\}$$

by the smoothness of $L$ on $S^{n-1}$. Moreover, for every $R_1$ and $R_2$ with $0 < R_1 < R_2$,

$$\int_{R_1 < |z| < R_2} K_{ij}(z) \, dz = \int_{R_1 < |z| < R_2} \partial_j K_i(z) \, dz$$

$$= \int_{|z|=R_2} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z)$$

$$= \int_{|z|=R_2} \frac{z_i}{|z|} \frac{z_j}{|z|} L \left( \frac{z}{|z|} \right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} \frac{z_i}{|z|} \frac{z_j}{|z|} L \left( \frac{z}{|z|} \right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z)$$

$$= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0.$$

In the fourth equality we changed variables $z = R_2 \sigma$ and $z = R_1 \sigma$ with $\sigma \in S^{n-1}$, respectively. This equality is equivalent to

\begin{equation}
\int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0.
\end{equation}

Thus we can apply the Calderón-Zygmund theory (see eg. [23, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel $K_{ij}$ and obtain the formula

\begin{equation}
\partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{f}(y) \, dy,
\end{equation}

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \frac{f(x)}{\int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma)} \right| \leq |\bar{f}(x)| \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \leq c\|\bar{f}\|_{L^p(\mathbb{R}^n)} = c\|f\|_{L^p(\Omega)}$$

with a positive constant $c$ independent of $f$. Hence the lemma follows. \hfill \qedsymbol

Lemma 3.3. For every $f \in C_0^\infty(\Omega)$ the vector field $u = Sf$ satisfies

$$\text{div } u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$
Proof. We have already observed that $u$ vanishes on the boundary. Let us compute \( \text{div } u = \sum_{i=1}^{n} \partial_i u_i \) in $\Omega$. By the formula (3.6) in the proof of Lemma 3.2,

\[
\text{div } u(x) = \bar{f}(x) \int_{S^{n-1}} \sum_{i=1}^{n} \sigma_i^2 L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} \sum_{i=1}^{n} K_i(x-y) \bar{f}(y) \, dy.
\]

In this formula, we have

\[
\int_{S^{n-1}} \sum_{i=1}^{n} \sigma_i^2 L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1
\]

by (3.2) and, for all $z \in \mathbb{R}^n \setminus \{0\}$,

\[
\sum_{i=1}^{n} K_i(z) = \frac{1}{|z|^n} L \left( \frac{z}{|z|} \right) \sum_{i=1}^{n} \left( 1 - \frac{n z_i^2}{|z|^2} \right) + \frac{1}{|z|^n} \sum_{i=1}^{n} \frac{z_i}{|z|} (\partial_i L) \left( \frac{z}{|z|} \right) - \sum_{i=1}^{n} \frac{z_i}{|z|^{n+2}} \sum_{k=1}^{n} \frac{z_k}{|z|} (\partial_k L) \left( \frac{z}{|z|} \right) = 0.
\]

Hence \( \text{div } u(x) = \bar{f}(x) = f(x) \) for all $x \in \Omega$. \(\square\)

Lemma 3.3 means that the operator $S$ is a solution operator to the divergence problem with Dirichlet boundary condition. Note that $S$ is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

**Definition 3.4.** For a vector field $u \in C^\infty_c(\Omega)$, we define a vector field $Tu$ as

\[
Tu(x) := \int_{\mathbb{R}^n} K(x-y) \overline{\text{div } u(y)} \, dy, \quad x \in \mathbb{R}^n.
\]

Here $K$ is given by (3.3) and $\overline{\text{div } u}$ denotes the zero extension of $\text{div } u$ to $\mathbb{R}^n$.

The above definition means that $T$ is given by $T = S \circ \text{div}$. Since $u \in C^\infty_c(\Omega)$, its divergence is in $C^\infty_c(\Omega)$ and thus $Tu$ is smooth in the whole space $\mathbb{R}^n$ and vanishes outside of $\Omega$, as discussed right after Definition 3.1. Also, by Lemma 3.3 we have

\[
\text{div } Tu = \text{div } u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial \Omega.
\]

Clearly $Tu = 0$ in $\mathbb{R}^n$ for $u \in C^\infty_c(\Omega)$. Note that, as in the case of the operator $S$, the support of $Tu$ may be unbounded.

**Theorem 3.5.** Let $\Omega$ be a Lipschitz half-space. Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that

\[
\|Tu\|_{L^p(\Omega)} \leq c \|u\|_{L^p(\Omega)}
\]

for all $u \in C^\infty_c(\Omega)$.

**Proof.** Let us compute the $i$-th component $(Tu)_i$ of $Tu$ with $i = 1, \ldots, n$ for compactly supported vector field $u$ in $\Omega$. As in the proof of Lemma 3.2, we integrate
by parts to get
\[
(Tu)_i(x) = \lim_{\epsilon \to 0} \int_{B_\epsilon(x)} K_i(x - y) \frac{x - y}{|x - y|} \cdot \bar{u}(y) \, d\mathcal{H}^{n-1}(y)
\]
\[+ \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (\nabla K_i)(x - y) \cdot \bar{u}(y) \, dy
\]
\[= \int_{S^{n-1}} \sigma_i L(\sigma) \{ \sigma \cdot \bar{u}(x) \} \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbb{R}^n} (\nabla K_i)(x - y) \cdot \bar{u}(y) \, dy,
\]
or equivalently,
\[
(Tu)_i(x) = \sum_{j=1}^n \{a_{ij} \bar{u}_j(x) + S_{ij} \bar{u}_j(x)\}, \quad x \in \mathbb{R}^n.
\]
Here \(u_j\) is the \(j\)-th component of \(u\) and
\[
a_{ij} = \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma), \quad S_{ij} \bar{u}_j(x) = \int_{\mathbb{R}^n} K_{ij}(x - y) \bar{u}_j(y) \, dy,
\]
where \(K_{ij} = \partial_j K_i\) is given by (3.4). Since \(a_{ij}\) is a constant satisfying
\[
|a_{ij}| \leq \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 1
\]
and \(S_{ij} \bar{u} = K_{ij} * \bar{u}\) is a singular integral (see the proof of Lemma 3.2), the Calderón-Zygmund theory yields the boundedness of the operator \(T\) on \(L^p(\Omega)\).

By Theorem 3.5, the operator \(T\) extends uniquely to a bounded linear operator on \(L^p(\Omega)\) with each \(p \in (1, \infty)\), which we again refer to as \(T\).

Our next goal is to estimate the \(BMO_b^{\infty, \nu}(\Omega)\)-norm of \(Tu\) for \(u \in C_c^\infty(\Omega)\) and \(\nu \in (0, \infty)\). To this end, we estimate each term of the right-hand side in (3.7) for \(u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)\). By (3.8) we have
\[
[a_{ij} \bar{u}_j : BMO^\infty(\Omega)] \leq [u_j : BMO^\infty(\Omega)], \quad [a_{ij} \bar{u}_j : b^\nu(\Omega)] \leq [u_j : b^\nu(\Omega)]
\]
and thus
\[
\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq \|u_j : BMO_b^{\infty, \nu}(\Omega)\|
\]
Moreover, since
\[
[u_j : BMO^\infty(\Omega)] \leq 2 \|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \leq \omega_n \|u_j\|_{L^\infty(\Omega)},
\]
where \(\omega_n = 2\pi^{n/2}/n! (n/2)\) is the volume of the unit ball \(B_1(0)\) in \(\mathbb{R}^n\) with the Gamma function \(\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} \, dx\), we have
\[
\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq (2 + \omega_n) \|u_j\|_{L^\infty(\Omega)}.
\]
Let us estimate \(S_{ij} \bar{u}_j = K_{ij} * \bar{u}_j, i, j = 1, \ldots, n\) in \(BMO_b^{\infty, \nu}(\Omega)\). Recall that the integral kernel \(K_{ij}\) is of the form
\[
K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
where \(k_{ij} \in C_c^\infty(\mathbb{R}^n)\) is given by (3.4) and satisfies
\[
\text{supp } k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1} = 0,
\]
see (3.2) and (3.5). We first estimate the \(BMO^\infty\)-seminorm of \(S_{ij} \bar{u}_j\).
Lemma 3.6. Let $K$ be a function defined on $\mathbb{R}^n \setminus \{0\}$ such that
\begin{equation}
|K(x - y) - K(x)| \leq A|y|^\delta |x|^{-n-\delta} \quad \text{whenever} \quad |x| \geq 2|y| > 0
\end{equation}
for some $A, \delta > 0$. Suppose that a convolution operator $S$ with $K$ is bounded on $L^2(\mathbb{R}^n)$ with a norm $B$. Then, there exists a dimensional constant $c_n$ such that
\begin{equation}
[Sf : BMO^{\infty}(\mathbb{R}^n)] \leq c_n(A + B)\|f\|_{L^\infty(\mathbb{R}^n)}
\end{equation}
for all $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. See [24, Theorem 3.4.9 and Corollary 3.4.10].

Lemma 3.7. There exists a constant $c > 0$ such that
\begin{equation}
[S_{ij} \bar{u}_j : BMO^{\infty}(\Omega)] \leq c\|u_j\|_{L^\infty(\Omega)}
\end{equation}
for all $u = (u_1, \ldots, u_n) \in C^\infty_\text{c}(\Omega)$ and $i, j = 1, \ldots, n$.

Proof. We shall apply Lemma 3.6 to $S = S_{ij}$. For this purpose it is sufficient to show that the function $K = K_{ij}$ satisfies (3.10), since we already know that the convolution operator $S_{ij}$ is bounded on $L^2(\mathbb{R}^n)$, see the proof of Lemma 3.2. To this end, we differentiate $K_{ij}$ to get
\begin{equation}
\nabla K_{ij}(x) = -\frac{n k_{ij}(x/|x|)}{|x|^{n+1}} \frac{x}{|x|} + \frac{1}{|x|^{n+1}} \left( I_n - \frac{1}{|x|^2} x \otimes x \right) \nabla k_{ij} \left( \frac{x}{|x|} \right)
\end{equation}
for $x \in \mathbb{R}^n \setminus \{0\}$, where $I_n$ is the identity matrix of size $n$ and $x \otimes x := (x, x)_i, j$ is the tensor product of $x$. Since $k_{ij}$ is smooth on $S^{n-1}$, we have
\begin{equation}
|\nabla K_{ij}(x)| \leq \frac{c}{|x|^{n+1}}, \quad x \in \mathbb{R}^n \setminus \{0\}.
\end{equation}
Hence, for all $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \geq 2|y| > 0$,
\begin{align*}
|K(x - y) - K(x)| &= \left| \int_0^1 \frac{d}{dt} (K(x - ty)) \, dt \right| = \left| \int_0^1 (-y) \cdot \nabla K(x - ty) \, dt \right| \\
&\leq |y| \int_0^1 \frac{c}{|x - ty|^{n+1}} \, dt \leq |y| \int_0^1 \frac{c}{(|x| - |y|)^{n+1}} \, dt \\
&\leq \frac{c |y|}{(|x| - |y|)^{n+1}} = \frac{2^{n+1} c |y|}{|x|^{n+1}}.
\end{align*}
Thus $K_{ij}$ satisfies (3.10) with $\delta = 1$ and we can apply Lemma 3.6 to obtain
\begin{equation}
[S_{ij} \bar{u}_j : BMO^{\infty}(\mathbb{R}^n)] \leq c\|u_j\|_{L^\infty(\mathbb{R}^n)} = c\|u_j\|_{L^\infty(\Omega)}
\end{equation}
with some constant $c > 0$.

By definition of the $BMO^{\infty}$-seminorm, we have
\begin{equation}
[S_{ij} \bar{u}_j : BMO^{\infty}(\Omega)] \leq [S_{ij} \bar{u}_j : BMO^{\infty}(\mathbb{R}^n)].
\end{equation}
Hence the inequality (3.11) follows from (3.12).

Next, let us estimate the $b'$-part of $S_{ij} \bar{u}_j$. Recall the two closed cones
\begin{equation}
C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, -x_n \geq |x| \cos(2^j \theta) \}, \quad j = 0, 1
\end{equation}
with opening angle $\theta \in (0, \pi/4)$. For $r > 0$ and $x_0 \in \mathbb{R}^n$, we define
\begin{equation}
A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)} (x + C_0) \cap (x_0 + C_1) \subset \mathbb{R}^n.
\end{equation}
Here $x_0 + C_1 = \{ y \in \mathbb{R}^n \mid y - x_0 \in C_1 \}$ and $x + C_0$ is defined similarly.
Lemma 3.8. For all $r > 0$ and $x_0 \in \mathbb{R}^n$ we have $A_r(x_0) \subset B_{r/\sin \theta}(x_0)$.

Proof. By translation, we may assume that $x_0 = 0$. Let $a := (0, \ldots, 0, r/\sin \theta) \in \mathbb{R}^n$. Suppose that

1. $B_r(0) \subset a + C_0$,
2. $x + C_0 \subset a + C_0$ for all $x \in a + C_0$,
3. $(a + C_0) \cap C_1^r \subset B_{r/\sin \theta}(0)$.

Then, the statements (1) and (2) imply

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^r} (x + C_0) \cap C_1^r \subset (a + C_0) \cap C_1^r.$$  

Hence the statement (3) yields $A_r(0) \subset B_{r/\sin \theta}(0)$. Now let us prove the statements (1)-(3). Note that, since $\theta \in (0, \pi/4)$, the cones $C_0$ and $C_1$ are represented as

$$C_j = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, x_n \leq 0, |x'| \leq (-x_n) \tan(2^j \theta) \}, \quad j = 0, 1.$$

(1) Let $x = (x', x_n) \in B_r(0)$. Then, $x - a = (x', x_n-r/\sin \theta)$ satisfies

$$(x-a)_n = x_n - \frac{r}{\sin \theta} \leq r - \frac{r}{\sin \theta} < 0$$

and

$$\left( \frac{r}{\sin \theta} - x_n \right)^2 \tan^2 \theta - |x'|^2 \geq \frac{(r-x_n \sin \theta)^2}{\cos^2 \theta} - (r^2 - x_n^2) = \frac{(r \sin \theta - x_n)^2}{\cos^2 \theta} \geq 0,$$

or equivalently,

$$|x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta = -(x-a)_n \tan \theta.$$

Hence $x-a \in C_0$, that is, $x \in a + C_0$ and the statement (1) holds.

(2) Let $x \in a + C_0$. If $y \in x + C_0$, then $(y-a)_n = (y-x)_n + (x-a)_n \leq 0$ and

$$|y'| \leq |x'| + |y'-x'| \leq -(x-a)_n \tan \theta - (y-x)_n \tan \theta = -(y-a)_n \tan \theta,$$

which means that $y \in a + C_0$. Hence the statement (2) holds.

(3) Let $x \in (a + C_0) \cap C_1^r$. Then we have

$$(3.14) \quad (x-a)_n = x_n - r/\sin \theta \leq 0, \quad |x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.$$

Hence

$$|x|^2 \leq \left( \frac{r}{\sin \theta} - x_n \right)^2 \tan^2 \theta + x_n^2 =: f(x_n).$$

To estimate the right-hand side in the above inequality for $x \in (a + C_0) \cap C_1^r$, we derive the range of $x_n$ for $x \in (a + C_0) \cap C_1^r$. If $x_n \geq 0$, then $x \in (a + C_0) \cap C_1^r$ holds if and only if the condition (3.14) is satisfied. Thus $x_n$ must satisfy

$$0 \leq x_n \leq \frac{r}{\sin \theta}.$$  

On the other hand, if $x_n < 0$, then $x \in (a + C_0) \cap C_1^r$ holds if and only if

$$-(x_n) \tan(2\theta) < |x'| \leq \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta.$$

Hence, in particular, if $x \in (a + C_0) \cap C_1^r$ and $x_n < 0$, then $x_n$ must satisfy

$$-(x_n) \tan(2\theta) < \left( \frac{r}{\sin \theta} - x_n \right) \tan \theta,$$
which yields the inequality
\[-\frac{r}{\cos \theta} < (\tan(2\theta) - \tan \theta) x_n.\]

Since
\[
\tan(2\theta) - \tan \theta = \frac{\tan(2\theta) - 1}{\tan(2\theta)(1 - \tan^2 \theta)} = \frac{\tan(2\theta)(1 + \tan^2 \theta)}{2\cos^2 \theta} > 0 \quad (0 < \theta < \frac{\pi}{4}),
\]
the above inequality is equivalent to
\[-\frac{2r \cos \theta}{\tan(2\theta)} < x_n (< 0).\]

In summary, the range of \(x_n\) for \(x \in (a + C_0) \cap C_1^c\) is
\[\alpha := -\frac{2r \cos \theta}{\tan(2\theta)} < x_n \leq \frac{r}{\sin \theta} =: \beta\]
and thus we obtain
\[|x|^2 \leq f(x_n) \leq \sup_{s \in (\alpha, \beta)} f(s) = \max\{f(\alpha), f(\beta)\},\]
where the last equality follows from the fact that \(f(x_n)\) is a concave parabola. On the one hand, we have \(f(\beta) = \beta^2 = \frac{r^2}{\sin^2 \theta}\). On the other hand, since
\[\alpha = -\frac{2r \cos \theta \cos(2\theta)}{\sin(2\theta)} = -\frac{r \cos(2\theta)}{\sin \theta} = \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta},\]
we have
\[f(\alpha) = \left(\frac{r}{\sin \theta} - \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta}\right)^2 \tan^2 \theta + \frac{r^2 \cos^2(2\theta)}{\sin^2 \theta} = \frac{r^2}{\sin^2 \theta} \left(4 \tan^2 \theta \cos^4 \theta + \cos^2(2\theta)\right) = \frac{r^2}{\sin^2 \theta} \cdot \frac{r^2}{\sin^2 \theta} = \frac{r^2}{\sin^2 \theta} \left(4 \tan^2 \theta \cos^4 \theta + \cos^2(2\theta)\right).
\]

Hence \(|x|^2 \leq r^2/\sin^2 \theta\) and thus \(x \in B_{r/\sin \theta}(0)\) for every \(x \in (a + C_0) \cap C_1^c\). Therefore, the statement (3) holds and the lemma follows. \(\square\)

Now we can estimate the \(b^\nu\)-part of \(S_{ij} \bar{u}_j\).

**Lemma 3.9.** Let \(\nu \in [0, \infty]\). There exists a constant \(c > 0\) such that
\[(3.15) \quad [S_{ij} \bar{u}_j : b^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \|u_j\|_{L^\infty(\Omega)},\]
for all \(u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)\) and \(i, j = 1, \ldots, n\).

**Proof.** First we note that for all \(f \in L^1_{loc}(\Omega)\) the inequality
\[(f : b^\nu(\Omega)) \leq \omega_\nu^{1/2}\|f : b^\nu_0(\Omega)\|\]
holds by Hölder’s inequality. Hence, to prove (3.15), it is sufficient to show the inequality
\[(3.16) \quad [S_{ij} \bar{u}_j : b^\nu_0(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \left[ u_j : b^\nu_0/\sin \theta(\Omega) \right] \leq \frac{c\omega_\nu^{1/2}}{\sin^{n/2} \theta} \|u_j\|_{L^\infty}.\]
The second inequality of (3.16) follows from the definition of \([ \cdot : b_2^n/\sin \theta (\Omega) \]). Let us show the first inequality. The singular integral \(S_{ij} \tilde{u}_j\) is of the form

\[
S_{ij} \tilde{u}_j(x) = (K_{ij} \ast \tilde{u}_j)(x) = \int_{\mathbb{R}^n} K_{ij}(x-y) \tilde{u}_j(y) \, dy, \quad x \in \mathbb{R}^n.
\]

Since \(\text{supp} \ K_{ij} \subset (-C_0, 0)\) (see (3.4) and (3.2)) and \(\text{supp} \ u \subset \Omega\), we can write

\[
S_{ij} \tilde{u}_j(x) = \int_{(x+C_0) \cap \Omega} K_{ij}(x-y) \tilde{u}_j(y) \, dy, \quad x \in \mathbb{R}^n.
\]

Hence, if we set

\[
W_r(x_0) := \bigcup_{x \in B_r(x_0) \cap \Omega} (x + C_0) \cap \Omega
\]

for each \(x_0 \in \partial \Omega\) and \(r > 0\) with \(B_r(x_0) \subset U_r(\partial \Omega)\), then we have

\[
S_{ij} \tilde{u}_j(x) = \int_{(x+C_0) \cap \Omega} K_{ij}(x-y)(\tilde{u}_j | W_r(x_0))(y) \, dy = [K_{ij} \ast (\tilde{u}_j | W_r(x_0))](x)
\]

for all \(x \in B_r(x_0) \cap \Omega\), where

\[
(\tilde{u}_j | W_r(x_0))(x) := \begin{cases} \tilde{u}_j(x), & x \in W_r(x_0), \\ 0, & x \not\in W_r(x_0). \end{cases}
\]

Since \(K_{ij}\) is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that

\[
\int_{B_r(x_0) \cap \Omega} |S_{ij} \tilde{u}_j(x)|^2 \, dx = \int_{B_r(x_0) \cap \Omega} |[K_{ij} \ast (\tilde{u}_j | W_r(x_0))](x)|^2 \, dx
\]

\[
\leq c \int_{B_r(x_0) \cap \Omega} |(\tilde{u}_j | W_r(x_0))(x)|^2 \, dx = c \int_{W_r(x_0)} |\tilde{u}_j(x)|^2 \, dx
\]

with some constant \(c > 0\). Now we recall the property of the infinite cone \(C_1\):

\[
x + C_1 \subset \Omega^c \Leftrightarrow \Omega \subset (x + C_1)^c \quad \text{for all} \quad x \in \Omega^c.
\]

By this property we have

\[
W_r(x_0) \subset \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r(x_0) \cap \Omega,
\]

where \(A_r(x_0)\) is given by (3.13), and thus Lemma 3.8 yields

\[
W_r(x_0) \subset A_r(x_0) \cap \Omega \subset B_r/\sin \theta(x_0) \cap \Omega.
\]

Hence we have

\[
\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |S_{ij} \tilde{u}_j(x)|^2 \, dx \leq \frac{c}{r^n} \int_{W_r(x_0)} |\tilde{u}_j(x)|^2 \, dx
\]

\[
\leq \frac{c}{r^n} \int_{B_r/\sin \theta(x_0) \cap \Omega} |\tilde{u}_j(x)|^2 \, dx = \frac{c}{\sin^n \theta} \left( \frac{\sin \theta}{r} \right)^n \int_{B_r/\sin \theta(x_0) \cap \Omega} |u_j(x)|^2 \, dx
\]

\[
\leq \frac{c}{\sin^n \theta} \left[ u_j : b_2^n/\sin \theta (\Omega) \right]^2
\]

for every \(x_0 \in \partial \Omega\) and \(r > 0\) with \(B_r(x_0) \subset U_r(\partial \Omega)\), which yields

\[
|S_{ij} \tilde{u}_j : b_2^n(\Omega)|^2 \leq \frac{c}{\sin^n \theta} \left[ u_j : b_2^n/\sin \theta (\Omega) \right]^2.
\]

The proof is complete. \(\Box\)
Now we obtain an estimate for the $BMO_{b}^{\infty,\nu}(\Omega)$-norm of $Tu$.

**Theorem 3.10.** Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that

$$
\|Tu : BMO_{b}^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^{\infty}(\Omega)}
$$

for all $u \in C_{c}^{\infty}(\Omega)$.

**Proof.** Since the $i$-th component of $Tu$, $i = 1, \ldots, n$, is of the form (3.7), we have by (3.9), (3.11) and (3.15) that

$$
\|Tu : BMO_{b}^{\infty,\nu}(\Omega)\|
\leq c \sum_{i,j=1}^{n} \|a_{ij}\|_{BMO_{b}^{\infty,\nu}(\Omega)} + \|S_{ij}\|_{BMO^{\infty}(\Omega)} + \|b_{\nu}\|_{BMO^{\infty}(\Omega)}
\leq c \sum_{j=1}^{n} \|u_{j}\|_{L^{\infty}(\Omega)} \leq c\|u\|_{L^{\infty}(\Omega)}
$$

with a positive constant $c$. $\square$

### 3.2. Non-Helmholtz projection.

As in the previous subsection, let $\Omega$ denote a Lipschitz half-space in $\mathbb{R}^{n}$.

**Definition 3.11.** For a vector field $u \in C_{c}^{\infty}(\Omega)$, we define a vector field $Q'u$ on $\mathbb{R}^{n}$ as $Q'u := u - Tu$. Here the operator $T$ is given in Definition 3.4.

For a vector field $u \in C_{c}^{\infty}(\Omega)$, the vector field $Tu$ is smooth in $\mathbb{R}^{n}$ and

$$
\text{div}Tu = \text{div}u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.
$$

Moreover, $Tu = 0$ for all $u \in C_{c}^{\infty}(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbb{R}^{n}$ and

$$
\text{div}Q'u = 0 \quad \text{in} \quad \Omega, \quad Q'u = 0 \quad \text{on} \quad \partial\Omega
$$

for all $u \in C_{c}^{\infty}(\Omega)$, and $Q'u = u$ for all $u \in C_{c}^{\infty,\sigma}(\Omega)$. Note that $Q'$ is not a projection from $C_{c}^{\infty}(\Omega)$ onto $C_{c}^{\infty,\sigma}(\Omega)$, since the support of $Tu$ may be unbounded and thus $Q'u$ is not in $C_{c}^{\infty,\sigma}(\Omega)$ in general. However, $Q'$ maps $C_{c}^{\infty}(\Omega)$ into $L_{\nu}^{p}(\Omega)$.

**Lemma 3.12.** For all $u \in C_{c}^{\infty}(\Omega)$ and $p \in (1, \infty)$, we have $Q'u \in L_{\nu}^{p}(\Omega)$.

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1, \infty)$, let $G_{p}(\Omega) = \{\nabla q \in L^{p}(\Omega) \mid q \in L_{\nu}^{1}(\Omega)\}$.

**Proposition 3.13.** Let $p \in (1, \infty)$. For every $\nabla q \in G_{p}(\Omega)$, there exists a sequence $\{q_{k}\}_{k=1}^{\infty}$ of functions in $C_{c}^{\infty}(\mathbb{R}^{n})$ such that

$$
\lim_{k \to \infty} \|\nabla q - \nabla q_{k}\|_{L^{p}(\Omega)} = 0.
$$

**Proof.** Since the restriction of $C_{c}^{\infty}(\mathbb{R}^{n})$ on $\Omega$ is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G_{p}(\Omega)$ there is a sequence $\{q_{k}\}_{k=1}^{\infty}$ of functions in $W^{1,p}(\Omega)$ such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space $\mathbb{R}_{+}^{n}$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x', x_{n}) \in \mathbb{R}^{n} \mid x_{n} > b(x')\}$. As in Section 2, let $F(x) := (x', x_{n} - h(x'))$ be a bi-Lipschitz map from $\Omega$ to $\mathbb{R}_{+}^{n}$. Let $\nabla q \in G_{p}(\Omega)$ and $\tilde{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y', y_{n} + h(y'))$ is the inverse mapping of $F$. Then, since $\nabla \tilde{q}(y) = \nabla F^{-1}(y) \nabla q(F^{-1}(y))$ for $y \in \mathbb{R}_{+}^{n}$ and each component
of $\nabla F^{-1}$ is bounded (because $h$ is Lipschitz continuous), we have $\nabla \tilde{q} \in G_p(\mathbb{R}^n_+)$. Hence, by our assumption that the claim is valid for $\mathbb{R}^n_+$, there is a sequence $\{\tilde{q}_k\}_{k=1}^{\infty}$ of functions in $W^{1,p}(\mathbb{R}^n_+)$ such that $\lim_{k \to \infty} \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbb{R}^n_+)} = 0$.

Let $q_k := \tilde{q}_k \circ F$ for each $k \in \mathbb{N}$. Then, since

$$
\nabla q(x) = \nabla F(x) \nabla \tilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x) \nabla \tilde{q}_k(F(x)), \quad x \in \Omega
$$

and each component of $\nabla F$ is bounded, we have $q_k \in W^{1,p}(\Omega)$ and

$$
\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \leq c \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbb{R}^n_+)} \to 0
$$
as $k \to \infty$. Thus the claim is valid for general Lipschitz half-spaces $\Omega$.

(2) Now we prove the claim for $\Omega = \mathbb{R}^n_+$. We follow the idea of the proof of the claim in the case $\Omega = \mathbb{R}^n$, see [34, Lemma 2.5.4]. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function such that

$$
0 \leq \varphi \leq 1 \quad \text{in } \mathbb{R}^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } \mathbb{R}^n \setminus B_2(0)
$$

and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Then, $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in \mathbb{R}^n$ and $\sup k \varphi_k \subset B_{2k}(0)$, $\sup k \varphi \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbb{N}$.

Let $q \in G_p(\mathbb{R}^n_+)$. Then $q \in W^{1,p}(\mathbb{R}^n_+)$, that is, $q \in W^{1,p}(U)$ for every bounded subset $U$ of $\mathbb{R}^n_+$; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting $G_k := \mathbb{R}^n_+ \cap (B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbb{N}$, we have $q \in W^{1,p}(G_k)$ and thus there is a constant $a_k$ such that $\int_{G_k} (q - a_k) dx = 0$ for each $k \in \mathbb{N}$. From this equality and the change of variables $x = ky$ for $x \in G_k$ and $y \in G_1$ we have

$$
\int_{G_1} (q(ky) - a_k) dy = k^{-n} \int_{G_k} (q(x) - a_k) dx = 0.
$$

Hence we can apply Poincaré’s inequality to $q(ky) - a_k$ on $G_1$ and get

$$
\left( \int_{G_1} |q(ky) - a_k|^p dy \right)^{1/p} \leq c \left( \int_{G_1} |\nabla (q(ky))|^p dy \right)^{1/p}
$$

with a constant $c > 0$ independent of $k$. In this inequality, we observe that

$$
\int_{G_1} |q(ky) - a_k|^p dy = k^{-n} \int_{G_k} |q(x) - a_k|^p dx,
$$

$$
\int_{G_1} |\nabla (q(ky))|^p dy = k^p \int_{G_1} |(\nabla q)(ky)|^p dy = k^{p-n} \int_{G_k} |\nabla q(x)|^p dx
$$

by the change of variables $x = ky$ and thus

$$
(3.19) \quad \|q - a_k\|_{L^p(G_k)} \leq ck \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbb{N}.
$$

For each $k \in \mathbb{N}$, let $q_k := \varphi_k(q - a_k)$ on $\mathbb{R}^n_+$. Then since $\sup k \varphi_k \subset \mathbb{R}^n_+ \cap B_{2k}(0)$ holds by the relation $\sup \varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(\mathbb{R}^n_+)$ and

$$
(3.20) \quad \|\nabla q - \nabla q_k\|_{L^p(\mathbb{R}^n_+)} \leq \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbb{R}^n_+)} + \|\nabla \varphi_k(q - a_k)^p\|_{L^p(\mathbb{R}^n_+)}
$$

Since $0 \leq \varphi_k(x) \leq 1$ and $\lim_{k \to \infty} \varphi_k(x) = 1$ for all $x \in \mathbb{R}^n_+$ and $q \in L^p(\mathbb{R}^n_+)$, the dominated convergence theorem yields

$$
(3.21) \quad \lim_{k \to \infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbb{R}^n_+)} = 0.
$$

On the other hand, since $\nabla \varphi_k = k^{-1}(\nabla \varphi)_k$ and $\sup k \varphi_k \subset \mathbb{R}^n_+$ for each $k \in \mathbb{N}$, it follows from (3.19) and the dominated convergence theorem that

$$
(3.22) \quad \|\nabla \varphi_k(q - a_k)^p\|_{L^p(\mathbb{R}^n_+)} \leq c k^{-1} \|q - a_k\|_{L^p(G_k)} \leq c \|\nabla q\|_{L^p(G_k)} \to 0.
$$
Proof of Lemma 3.12. Let $u \in C_c^\infty(\Omega)$ and $p \in (1, \infty)$. Then, since $Tu \in L_p(\Omega)$ by Theorem 3.5, we have $Q'u = u - Tu \in L^p(\Omega)$. To show $Q'u \in L^p_\sigma(\Omega)$, we employ a characterization of elements of $L^p_\sigma(\Omega)$ ([19, Lemma III.2.1]): a vector field $v \in L^p(\Omega)$ is in $L^p_\sigma(\Omega)$ if and only if
\[
\int_\Omega v \cdot \nabla q \, dx = 0 \quad \text{for all} \quad \nabla q \in G_{p'}(\Omega) \quad \left( p' := \frac{p}{p-1} \right).
\]
Let $\nabla q$ be any element of $G_{p'}(\Omega)$. From Proposition 3.13, there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbb{R}^n)$ such that the equality (3.18) with $p$ replaced by $p'$ holds. Since $Q'u$ is defined and smooth in $\mathbb{R}^n$ for $u \in C_c^\infty(\Omega)$ and $q_k \in C_c^\infty(\mathbb{R}^n)$, integration by parts yields
\[
\int_\Omega Q'u \cdot \nabla q_k \, dx = - \int_\Omega q_k \text{div} Q'u \, dx + \int_{\partial\Omega} q_k (Q'u \cdot \nu) \, d\mathcal{H}^{n-1}
\]
for all $k \in \mathbb{N}$, where $\nu$ denotes the unit outer normal vector field of $\partial\Omega$. We apply (3.17) to the right-hand side of this equality to get $\int_\Omega Q'u \cdot \nabla q_k \, dx = 0$ for all $k \in \mathbb{N}$. Since $Q'u \in L^p(\Omega)$ and (3.18) with $p$ replaced by $p'$ holds, the above equality implies that
\[
\int_\Omega Q'u \cdot \nabla q \, dx = \lim_{k \to \infty} \int_\Omega Q'u \cdot \nabla q_k \, dx = 0.
\]
Hence by the characterization of elements of $L^p_\sigma(\Omega)$ we conclude that $Q'u \in L^p_\sigma(\Omega)$ for all $u \in C_c^\infty(\Omega)$. The proof is complete. \hfill \Box

Remark 3.14.

1. Let $p \in (1, \infty)$. By Theorem 3.5 and Lemma 3.12, we have $Q'u \in L^p_\sigma(\Omega)$ and $\|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Moreover, $Q'u = u$ holds for all $u \in C_c^\infty(\Omega)$. Hence, by the density argument, $Q'$ extends uniquely to a bounded linear operator on $L^p(\Omega)$ that is a projection onto $L^p_\sigma(\Omega)$.

2. The projection onto $L^p_\sigma(\Omega)$ given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each $u \in C_c^\infty(\Omega)$ there would exist $\pi \in L^1_{loc}(\Omega)$ such that $(I - Q')u = \nabla \pi$ holds. Since $(I - Q')u = Tu = K * \text{div} u$ for $u \in C_c^\infty(\Omega)$, the existence of such $\pi$ would imply that $\partial_j(K_i * \text{div} u) = \partial_i(K_j * \text{div} u)$ for all $i, j = 1, \ldots, n$. For each $f \in C_c^\infty(\Omega)$ with $\int_\Omega f \, dx = 0$ there is $u \in C_c^\infty(\Omega)$ satisfying $f = \text{div} u$. This is possible since we are able to apply Bogovskii’s lemma to a bounded Lipschitz domain $D \subset \Omega$ containing the support of $f$ (see [19, Theorem III.3.3]). Thus the above equality would imply that $\partial_j K_i = \partial_i K_j + c$ with some constant $c$ for all $i, j = 1, \ldots, n$ as a distribution. This contradicts the fact that $\partial_j K_i \neq \partial_i K_j + c$ for $i \neq j$ as observed in (3.4).

3. It is possible to prove the characterization
\[
L^p_\sigma(\Omega) = \{ u \in L^p(\Omega) \mid \text{div} u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial\Omega \}
\]
if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains.
is smooth and compactly supported in Ω. Moreover, we have

\[ \text{dist}(\text{supp}u, \Omega) \leq c \] for all \( k \in \mathbb{N} \) and \( u \in C_c^\infty(\Omega) \) with a constant \( c \).

The linear operator \( Q' \) also maps \( C_c^\infty(\Omega) \) into \( \text{VMO}_{b,0,\sigma}(\Omega) \).

**Lemma 3.15.** Let \( \Omega \) be a Lipschitz half-space. For all \( u \in C_c^\infty(\Omega) \) and \( \nu \in (0, \infty] \), we have \( Q'u \in \text{VMO}_{b,0,\sigma}(\Omega) \).

We shall prove two auxiliary propositions for the above lemma. For \( p \in (1, \infty) \), let \( W_0^{1,p}(\Omega) \) be the \( W^{1,p} \)-closure of \( C_c^\infty(\Omega) \).

**Proposition 3.16.** Let \( \Omega \) be a Lipschitz half-space. For all \( p \in (1, \infty) \) we have \( \text{Lip}_0^p(\Omega) \cap W_0^{1,p}(\Omega) \subset W^{1,p}_{0,\sigma}(\Omega) \). Thus \( \text{Lip}_0^p(\Omega) \cap W_0^{1,p}(\Omega) = W^{1,p}_{0,\sigma}(\Omega) \).

**Proof.** Let \( \rho \in C_c^\infty(\mathbb{R}^n) \) be a function such that

\[ 0 \leq \rho \leq 1 \quad \text{in} \quad \mathbb{R}^n, \quad \text{supp} \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \, dx = 1 \]

and \( \rho_\delta(x) := \rho(\delta^{-1}x) \) for \( \delta > 0 \), \( x \in \mathbb{R}^n \). Let \( u \in L_0^p(\Omega) \cap W_0^{1,p}(\Omega) \). Then there is a sequence \( \{u_k\}_{k=1}^\infty \) of functions in \( C_c^\infty(\Omega) \) such that \( \lim_{k \to \infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0 \).

For \( a > 0 \), we define a vector field \( u^a \) on \( \Omega \) as

\[ u^a(x) := \begin{cases} u(x', x_n - a), & x_n \leq h(x') + a, \\ 0, & h(x') < x_n \leq h(x') + a \end{cases} \]

and \( u_k^a = (u_k)^a \) similarly. Then it is clear that \( u^a \in W_0^{1,p}(\Omega) \) and \( u_k^a \in C_c^{\infty}(\Omega) \) for all \( a > 0 \). Moreover, we have

\[ \|u^a - u_k^a\|_{W^{1,p}(\Omega)} = \|u - u_k\|_{W^{1,p}(\Omega)} \quad \text{for all} \quad a > 0, \quad \lim_{a \to 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0. \]

By the second equality and the fact that \( W_0^{1,p}(\Omega) \) is closed in \( W^{1,p}(\Omega) \), it is sufficient for showing \( u \in W_0^{1,p}(\Omega) \) to prove \( u^a \in W_0^{1,p}(\Omega) \) for all \( a > 0 \).

For each \( a > 0 \), there is a constant \( d = d(a) > 0 \) such that \( \text{dist}(\text{supp} u_k^a, \partial \Omega) \geq d \) for all \( k \in \mathbb{N} \). Then, for a given \( \varepsilon > 0 \), we can take \( \delta \in (0, d/2) \) so small that

\[ \|u^a - u^a \ast \rho_\delta\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2} \]

since \( u^a \in W_0^{1,p}(\Omega) \). Also, since \( \nabla \rho_\delta = \delta^{-1} (\nabla \rho)_\delta \), we have

\[ \|u^a \ast \rho_\delta - u_k^a \ast \rho_\delta\|_{W^{1,p}(\Omega)} \leq c\|u^a \ast \rho_\delta - u_k^a \ast \nabla \rho_\delta\|_{L^p(\Omega)} + c\|u^a \ast \nabla \rho_\delta - u_k^a \ast \nabla \rho_\delta\|_{L^p(\Omega)} \]

\[ = c\|u^a - u_k^a\|_{W^{1,p}(\Omega)} + \delta^{-1}\|u^a - u_k^a\|_{L^p(\Omega)} \]

\[ \leq c(1 + \delta^{-1})\|u^a - u_k^a\|_{L^p(\Omega)} = c(1 + \delta^{-1})\|u - u_k\|_{L^p(\Omega)} \]

with a constant \( c > 0 \) independent of \( \varepsilon \) and \( \delta \). Hence by taking \( k \in \mathbb{N} \) so large that

\[ \|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})} \]

we have \( \|u^a \ast \rho_\delta - u_k^a \ast \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2 \) and thus

\[ \|u^a - u_k^a\|_{W^{1,p}(\Omega)} \leq \|u^a - u_k^a \ast \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a \ast \rho_\delta - u_k^a \ast \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon. \]

On the other hand, since \( \text{dist}(\text{supp} u_k^a, \partial \Omega) > d \) and \( \delta \in (0, d/2) \), the function \( u_k^a \ast \rho_\delta \) is smooth and compactly supported in \( \Omega \). Moreover, we have

\[ \text{div}(u_k^a \ast \rho_\delta) = (\text{div} u_k^a) \ast \rho_\delta = 0 \quad \text{in} \quad \Omega. \]
Thus $u_b^\delta * \rho_b \in C_c^\infty(\Omega)$ and $u^\delta$ is approximated by elements of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, which means that $u^\delta \in W^{1,p}_{0,\sigma}(\Omega)$. Hence $u \in W^{1,p}_{0,\sigma}(\Omega)$ and the proof is now complete. \qed

**Proposition 3.17.** Let $\nu \in (0, \infty]$. If $p > n$, then $W^{1,p}_{0,\sigma}(\Omega) \subset VMO_0^{\infty,\nu}(\Omega)$.

**Proof.** Let $u \in W^{1,p}_{0,\sigma}(\Omega)$ and $u_k \in C_c^\infty(\Omega)$ such that $\lim_{k \to \infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0$.

Since $p > n$ and $u, u_k \in W^{1,p}(\Omega)$, Morrey’s inequality (see e.g. [7, Theorem 4.12]) implies

$$\|u - u_k\|_{L^\infty(\Omega)} \leq c\|u - u_k\|_{W^{1,p}(\Omega)}$$

with a positive constant $c$ independent of $u$ and $u_k$. Thus we have

$$\|u - u_k\|_{L^\infty(\Omega)} \leq 2 + \omega_0,\quad \|u - u_k\|_{L^\infty(\Omega)} \leq c\|u - u_k\|_{W^{1,p}(\Omega)} \to 0$$

as $k \to \infty$. Hence $u \in VMO_0^{\infty,\nu}(\Omega)$ and the proof is now complete. \qed

**Proof of Lemma 3.15.** Since $u \in C_c^\infty(\Omega)$ and thus $\partial_i u \in C_c^\infty(\Omega)$ for all $i = 1, \ldots, n$, it follows from Lemma 3.12 that $Q' u \in L_p^*(\Omega)$ and $\partial_i Q' u = Q' (\partial_i u) \in L_p^*(\Omega)$ for all $r \in (1, \infty)$ and $i = 1, \ldots, n$. From this fact and the equality (3.17), we have $Q' u \in L_p^*(\Omega) \cap W^{1,p}_{0,\sigma}(\Omega)$ for all $r \in (1, \infty)$. Hence, by taking $r > n$, we can apply Proposition 3.16 and Proposition 3.17 to obtain $Q' u \in VMO_0^{\infty,\nu}(\Omega)$. \qed

**Remark 3.18.** Let $\nu \in (0, \infty]$. Theorem 3.10 and Lemma 3.15 imply that $Q' u \in VMO_0^{\infty,\nu}(\Omega)$ and $\|Q' u : BMO_0^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Also, we have $Q' u = u$ for all $u \in C_c^\infty(\Omega)$. Hence $Q'$ extends uniquely to a bounded linear operator (again referred to as $Q'$) from $C_0(\Omega)$, which is the $L^\infty$-closure of $C_c^\infty(\Omega)$, into $VMO_0^{\infty,\nu}(\Omega)$ that satisfies $Q' u = u$ for all $u \in C_0(\Omega)$.

Now let us extend $Q'$ to a linear operator that gives the projection mentioned in Theorem 1.4. For $p \in (1, \infty)$, we define a Banach space $X_p$ and its norm as

$$X_p := L^p(\Omega) \cap C_0(\Omega), \quad \|u\|_{X_p} := \max\{\|u\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}\}.$$  

Note that the Banach space $C_0(\Omega)$ consists of all continuous functions $f$ on $\Omega$ such that the set $\{x \in \Omega \mid |f(x)| \geq \varepsilon\}$ is compact in $\Omega$ for every $\varepsilon > 0$ (see e.g. [32, Theorem 3.17]).

**Lemma 3.19.** For each $p \in (1, \infty)$, the linear subspace $C_c^\infty(\Omega)$ is dense in $X_p$.

**Proof.** The proof is more or less standard (see e.g. [27, Corollary 19.24]). We give it for completeness. Let $u \in X_p$ and $\Omega_k := \{x \in \Omega \mid |x| \leq k, \text{dist}(x, \partial \Omega) \geq k/k\}$ for $k \in \mathbb{N}$. For any given $\varepsilon > 0$, the set $\{x \in \Omega \mid |u(x)| \geq \varepsilon/2\}$ is compact in $\Omega$ since $u \in C_0(\Omega)$. Moreover, since $u \in L^p(\Omega)$, we can take $k \in \mathbb{N}$ so large that

$$\|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}. \quad \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}.$$  

Let $\varphi \in C_c^\infty(\Omega)$ be a continuous cut-off function such that

$$0 \leq \varphi \leq 1 \quad \text{in} \quad \Omega, \quad \varphi = 1 \quad \text{in} \quad \Omega_k, \quad \varphi = 0 \quad \text{in} \quad \Omega \setminus \Omega_{2k}.$$  

Since $u - \varphi u = 0$ in $\Omega_k$ and $|u - \varphi u| \leq |u|$ in $\Omega \setminus \Omega_k$, it follows from (3.23) that

$$\|u - \varphi u\|_{L^1(\Omega)} \leq \|u\|_{L^1(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}. \quad \|u - \varphi u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}.$$  

Since $u - \varphi u = 0$ in $\Omega_k$ and $|u - \varphi u| \leq |u|$ in $\Omega \setminus \Omega_k$, it follows from (3.23) that

$$\|u - \varphi u\|_{L^1(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}. \quad \|u - \varphi u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}. $$
Let \( \rho_b \) be a mollifier as in the beginning of the proof of Proposition 3.16. Since
\[
\varphi u \in L^p(\Omega), \quad \text{dist}(\text{supp}(\varphi u), \partial \Omega) \geq \frac{1}{2k},
\]
we can take \( \delta \in (0, 1/4k) \) so that
\[
(3.25) \quad u_\delta := \rho_b * (\varphi u) \in C_0^\infty(\Omega), \quad \| \varphi u - u_\delta \|_{L^p(\Omega)} \leq \frac{\varepsilon}{2}.
\]
On the other hand, since \( \varphi u \) is uniformly continuous on \( \Omega_{4k} \), we can again choose \( \delta \in (0, 1/4k) \) so that \( \| \varphi u - u_\delta \|_{L^\infty(\Omega_{4k})} < \varepsilon/2 \). Moreover, since \( \supp(\varphi u) \subset \Omega_{2k} \) and \( \delta \in (0, 1/4k) \), we have \( \varphi u = u_\delta = 0 \) outside of \( \Omega_{4k} \) and thus
\[
(3.26) \quad \| \varphi u - u_\delta \|_{L^\infty(\Omega)} = \| \varphi u - u_\delta \|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}.
\]
Combining (3.24), (3.25) and (3.26), we obtain \( u_\delta \in C_0^\infty(\Omega) \) and
\[
\| u - u_\delta \|_{X_p} = \max\{ \| u - u_\delta \|_{L^p(\Omega)}, \| u - u_\delta \|_{L^\infty(\Omega)} \} < \varepsilon.
\]
Hence the lemma follows. \( \square \)

Let \( Y_p := L^p_b(\Omega) \cap \text{VMO}^{\infty,\nu}_{0,0}(\Omega) \) for \( p \in (1, \infty), \nu \in (0, \infty) \). Since \( L^p_b(\Omega) \) and \( \text{VMO}^{\infty,\nu}_{0,0}(\Omega) \) are closed in \( L^p(\Omega) \) and \( \text{BMO}^{\infty,\nu}_b(\Omega) \), respectively, \( Y_p \) becomes a Banach space under the norm \( \| v \|_{Y_p} := \max\{ \| v \|_{L^p(\Omega)}, \| v : \text{BMO}^{\infty,\nu}_b(\Omega) \| \} \).

**Theorem 3.20.** Let \( p \in (1, \infty) \) and \( \nu \in (0, \infty) \). The linear operator \( Q' \) given in Definition 3.11 extends uniquely to a bounded linear operator \( Q'_p \) from \( X_p \) into \( Y_p \).
Moreover, there exists a constant \( c > 0 \) such that
\[
(3.27) \quad \| Q'_p u \|_{L^p(\Omega)} \leq c \| u \|_{L^p(\Omega)}, \quad \| Q'_p : \text{BMO}^{\infty,\nu}_b(\Omega) \| \leq c \| u \|_{L^\infty(\Omega)}
\]
for all \( u \in X_p \) and \( Q'_p u = u \) holds for all \( u \) in the \( X_p \)-closure of \( C_0^\infty(\Omega) \).

**Proof.** Let \( u \in C_0^\infty(\Omega) \). Then we have \( Q'u \in Y_p \) by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant \( c > 0 \) independent of \( u \) such that
\[
(3.28) \quad \| Q'u \|_{L^p(\Omega)} \leq c \| u \|_{L^p(\Omega)}, \quad \| Q'_p : \text{BMO}^{\infty,\nu}_b(\Omega) \| \leq c \| u \|_{L^\infty(\Omega)}.
\]
Hence we have \( Q'u \in Y_p \) and \( \| Q'u \|_{Y_p} \leq c \| u \|_{X_p} \) for all \( u \in C_0^\infty(\Omega) \). Since \( C_0^\infty(\Omega) \) is dense in \( X_p \) by Lemma 3.19, the operator \( Q' \) extends uniquely to a bounded linear operator \( Q_p \) from \( X_p \) into \( Y_p \). Also, it follows from (3.28) that the inequality (3.27) holds for all \( u \in X_p \). Since \( Q'u = u \) holds for all \( u \in C_0^\infty(\Omega) \) as observed after Definition 3.11, by the density argument we have \( Q_p u = u \) for all \( u \) in the \( X_p \)-closure of \( C_0^\infty(\Omega) \). The proof is complete. \( \square \)

Finally, Theorem 1.4 follows from Theorem 3.20 with \( p = 2 \), that is, the linear operator \( Q \) in Theorem 1.4 is given by \( Q = Q_2 \).

4. **Analyticity in \( L^p \)**

In this section we shall give a complete proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( S(t) \) be the Stokes semigroup in \( \tilde{L}^p_b \) constructed by [14], [16]. To show that \( S(t) \) forms an analytic semigroup in \( L^p_b \) \((2 \leq p < \infty)\) it suffices to prove that there exists a constant \( C \) that
\[
(4.1) \quad \| S(t)u_0 \|_p \leq C\| v_0 \|_p
\]
for all $v_0 \in C_{c,\sigma}^{\infty}(\Omega)$ and for all $t \in (0, 1)$. Let $Q$ be the operator in Theorem 1.4. Since $Q$ is bounded in $L^2$ and maps $L^2$ to $L^{2}_\sigma$ and $S(t)$ fulfills (4.1) and (4.2) for $p = 2$, we have

\begin{equation}
\left\| \frac{d}{dt} S(t)v_0 \right\|_2 \leq C\|v_0\|_p
\end{equation}

(4.2)

for all $u \in C_{c}(\Omega)$ and $t \in (0, 1)$. Since $\Omega$ is admissible as proved in [5], $S(t)$ forms an analytic semigroup in $VMO_{b,0,\sigma}^{\infty}$ by Theorem 1.2. We conclude that

\begin{equation}
\|S(t)Qu : BMO_{b}^{\infty,\nu}(\Omega)\| \leq C\|u\|_{\infty}
\end{equation}

(4.5)

\begin{equation}
\left\| t \frac{d}{dt} S(t)Qu : BMO_{b}^{\infty,\nu}(\Omega) \right\| \leq C\|u\|_{\infty}
\end{equation}

(4.6)

for all $u \in C_{c}(\Omega)$ and $t \in (0, 1)$ since $Q$ fulfills

\begin{equation*}
\|Qu : BMO_{b}^{\infty,\nu}(\Omega)\| \leq C\|u\|_{\infty}, \; Qu \in VMO_{b,0,\sigma}^{\infty}
\end{equation*}

for all $u \in C_{c}(\Omega)$ by Theorem 1.4. (Note that we have a stronger statement than (4.6) by replacing the $BMO_b$ type norm by the $L^\infty$ norm since we have the regularizing estimate (1.3).) We apply an interpolation result (Theorem 1.3) to (4.3) and (4.4) and (4.5) to get, respectively

\begin{equation}
\|S(t)Qu\|_p \leq C\|u\|_p
\end{equation}

(4.7)

\begin{equation}
\left\| t \frac{d}{dt} S(t)Qu \right\|_p \leq C\|u\|_p
\end{equation}

(4.8)

for all $u \in C_{c}(\Omega)$ and for all $t \in (0, 1)$. Since $Qu = u$ for $u \in C_{c,\sigma}^{\infty}(\Omega)$ this yields (4.1) and (4.2).

It remains to prove that $S(t)$ is a $C_0$-semigroup in $L^p_\sigma$. Since $C_{c,\sigma}^{\infty}(\Omega)$ is dense in $L^p_\sigma$, for $v_0 \in L^p_\sigma$ there is $v_{0m} \in C_{c,\sigma}^{\infty}$ such that $\|v_0 - v_{0m}\|_p \to 0$ as $m \to \infty$. By (4.1) we observe that

\begin{equation*}
\|S(t)v_0 - v_0\|_p \leq \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p
\end{equation*}

\begin{equation*}
\leq C\|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p.
\end{equation*}

Sending $t \downarrow 0$, we get

\begin{equation*}
\lim_{t \downarrow 0} \|S(t)v_0 - v_0\|_p \leq C\|v - v_{0m}\|_p,
\end{equation*}

since $S(t)v_{0m} \to v_{0m}$ in $\tilde{L}^p_\sigma$ as $t \downarrow 0$ by [14], [16]. Sending $m \to \infty$, we conclude that $S(t)v_0 \to v_0$ in $L^p_\sigma$ as $t \downarrow 0$.

**Remark 4.1.** In a similar way as we derived (4.5) and (4.6) we are able to derive from the $L^\infty$-$BMO$ estimates in [10] that

\begin{equation*}
t\|\nabla^2 S(t)Qu : BMO_{b}^{\infty,\nu}(\Omega)\| \leq C\|u\|_{\infty}
\end{equation*}

\begin{equation*}
t^{1/2}\|\nabla S(t)Qu : BMO_{b}^{\infty,\nu}(\Omega)\| \leq C\|u\|_{\infty}
\end{equation*}

for all $u \in C_{c}(\Omega)$ and $t \in (0, 1)$.
Note that \( L^2 \) results

\[
    t \left\| \nabla^2 S(t) Q u \right\|_2 \leq C \| u \|_2 \\
    t^{1/2} \left\| \nabla S(t) Q u \right\|_2 \leq C \| u \|_2
\]

easily follow from the analyticity of \( S(t) \) in \( L^2 \sigma \) and \( L^2 \)-boundedness of \( Q \) if one observes that

\[
    \left\| \nabla u \right\|_2 = (Au, u)_{L^2} \\
    \left\| \nabla^2 u \right\|_2 \leq C \left( \| Au \|_2 + \| \nabla u \|_2 + \| u \|_2 \right)
\]

(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where \( A \) is the Stokes operator in \( L^2 \sigma \).

Interpolating the \( L^2 \) results and the above \( L^\infty -BMO \) results, we are able to prove that there is \( C_p > 0 \) satisfying

\[
    t \left\| \nabla^2 S(t) v_0 \right\|_p \leq C_p \| v_0 \|_p \\
    t^{1/2} \left\| \nabla S(t) v_0 \right\|_p \leq C_p \| v_0 \|_p
\]

for all \( v_0 \in L^p_\sigma(\Omega) \) and \( t \in (0,1) \) with \( p \in (2, \infty) \).

References

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