Envelopes of Legendre curves in the unit tangent bundle over the Euclidean plane

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Abstract

For singular plane curves, the classical definitions of envelopes are vague. In order to define envelopes for singular plane curves, we introduce a one-parameter family of Legendre curves in the unit tangent bundle over the Euclidean plane and the curvature. Then we give a definition of an envelope for the one-parameter family of Legendre curves. We investigate properties of the envelopes. For instance, the envelope is also a Legendre curve. Moreover, we consider bi-Legendre curves and give a relationship between envelopes.

1 Introduction

Envelopes are classical object in the differential geometry. There are many applications of envelopes to differential geometry, differential equations and physics, for instance [4, 5, 7, 9, 10, 15, 16, 18, 20]. An envelope of a family of curves in the plane is a curve that is "tangent" to each member of the family at some point. If the curves are regular, then the tangent is well-defined. However, the definitions of envelopes are vague for singular plane curves (smooth curves with singular points). In this paper, we would like to clarify the definition of the envelope for a family of singular curves. As singular curves, we consider Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see Appendix A (cf. [8]). The basic results on the singularity theory see [2, 4, 14, 17]. In §2, we quickly review on the definitions of envelopes which are given by implicit functions [3, 4, 12] and parametric curves [11, 19]. In §3, we consider one-parameter families of Legendre curves. We give a moving frame and the curvature of the one-parameter family of Legendre curves. Then we show that the existence and uniqueness theorem for one-parameter families of Legendre curves. In §4, we define an envelope of a one-parameter family of Legendre curves. Then the envelope is also a Legendre curve and hence we give a curvature of the envelope as a Legendre curve. Moreover, we give relationships between the envelopes given by implicit functions and one-parameter family of Legendre curves. In §5, we define a bi-Legendre curve as a special class of one-parameter family of Legendre curves and give a relationship between envelopes.

All maps and manifolds considered here are differential of class C^∞.

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2 Previous results

Let $\mathbb{R}^2$ be the Euclidean plane equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$, where $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$.

We review two definitions of envelopes for one-parameter family of plane curves. These are given by implicit functions and parametrized curves. Here we denote these envelopes by E_I and E_P respectively. Other related definitions of envelopes see [3, 19, 21].

Let $F : V \times \Lambda \to \mathbb{R}, (x, y, \lambda) \mapsto F(x, y, \lambda)$ be a smooth function, where V is a domain in $\mathbb{R}^2$, and Λ is an interval or $\mathbb{R}$. A family of curves in the plane is given by $\Gamma_\lambda = \{(x, y) \in V \mid F(x, y, \lambda) = 0\}$ for each $\lambda \in \Lambda$. Then one of the classical definition of the envelope is as follows, see for instance [3, 4]:

Definition 2.1 The *envelope* of the family F is the set E_I of points given by

$$E_I = \left\{(x, y) \in V \mid \text{for some } \lambda \in \Lambda, F(x, y, \lambda) = \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0\right\}.$$

If $F(x, y, \lambda) = (\partial F/\partial\lambda)(x, y, \lambda) = 0$, we say that $(x, y) \in E_I$ *with respect to* λ.

On the other hand, let $\gamma : I \times \Lambda \to \mathbb{R}^2$ be a one-parameter family of smooth parametrized curves, and let $e_p : U \to I \times \Lambda, e_p(u) = (t(u), \lambda(u))$ be a regular curve, where I, Λ and U are intervals or $\mathbb{R}$. We denote $\Gamma_\lambda(t) = \gamma(t, \lambda)$ and $E_P(u) = \gamma \circ e_p(u)$.

Definition 2.2 ([11, Page 138]) We call E_P an *envelope* (and e_p a *pre-envelope*) for the family γ, when the following conditions are satisfied.

(i) The function λ is non-constant on any non-trivial subinterval of U. (The Variability Condition.)

(ii) For all u, the curve E_P is tangent at u to the curve $\Gamma_{\lambda(u)}$ at the parameter $t(u)$, meaning that the tangent vectors $E_P'(u) = (dE_P/du)(u)$ and $\dot{\Gamma}_{\lambda(u)}(t(u)) = (d\Gamma_{\lambda(u)}/dt)(t(u))$ are linearly dependent. (The Tangency Condition.)

We say that the *singular set of* $\gamma : I \times \Lambda \to \mathbb{R}^2, \gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$ is the subset of the domain $I \times \Lambda$ defined by

$$\det\left(\gamma_t(t, \lambda), \gamma_\lambda(t, \lambda)\right) = \det\begin{pmatrix} x_t(t, \lambda) & y_t(t, \lambda) \\ x_\lambda(t, \lambda) & y_\lambda(t, \lambda) \end{pmatrix} = 0. \tag{1}$$

Here we denote $\gamma_t(t, \lambda) = (\partial\gamma/\partial t)(t, \lambda) = (x_t(t, \lambda), y_t(t, \lambda))$ and $\gamma_\lambda(t, \lambda) = (\partial\gamma/\partial\lambda)(t, \lambda) = (x_\lambda(t, \lambda), y_\lambda(t, \lambda))$. Then the envelope theorem is as follows:

Theorem 2.3 ([11, Page 140]) *Let $\gamma : I \times \Lambda \to \mathbb{R}^2$ be a family of parametrized curves, and let $e_p : U \to I \times \Lambda$ be a regular curve satisfying the variability condition. Then e_p is a pre-envelope of γ (and E_P is an envelope) if and only if the trace of e_p lies in the singular set of γ.*

We consider one-parameter families of 3/2-cusps as examples. Other examples see [3, 11].

Example 2.4 Let $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, F(x, y, \lambda) = (x - \lambda)^3 - y^2$. Since $(\partial F/\partial\lambda)(x, y, \lambda) = -3(x - \lambda)^2$, the envelope is given by $E_I = \{(\lambda, 0)|\lambda \in \mathbb{R}\}$.

Let $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2, \gamma(t, \lambda) = (t^2 + \lambda, t^3)$. Since (1), we have $-3t^2 = 0$. By Theorem 2.3, the pre-envelope and the envelope are given by $e_p : \mathbb{R} \to \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$ and $E_P : \mathbb{R} \to \mathbb{R}^2, E(u) = (u, 0)$.

Both cases, the envelopes are given by the x-axis, see Figure 1.

Example 2.5 Let $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, F(x, y, \lambda) = x^3 - (y - \lambda)^2$. Since $(\partial F/\partial\lambda)(x, y, \lambda) = -2(y - \lambda)$, the envelope is given by $E_I = \{(0, \lambda)|\lambda \in \mathbb{R}\}$.

Let $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2, \gamma(t, \lambda) = (t^2, t^3 + \lambda)$. Since (1), we have $2t = 0$. By Theorem 2.3, the pre-envelope and the envelope are given by $e_p : \mathbb{R} \to \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$ and $E_P : \mathbb{R} \to \mathbb{R}^2, E_P(u) = (0, u)$.

Both cases, the envelopes are given by the y-axis, see Figure 2. However, in the sense of limit tangent of the 3/2-cusp, y-axis is not tangent to the 3/2-cusps. Moreover, as a solution of differential equations, the x-axis in Figure 1 is a singular solution of the ODE $-y+((2/3)y')^3 = 0$ and y-axis in Figure 2 is not a singular solution of the ODE $-x+((2/3)y')^2 = 0$ (cf. [15, 16, 20]). We would like to distinguish as envelopes, see Examples 4.2 and 4.3 below.

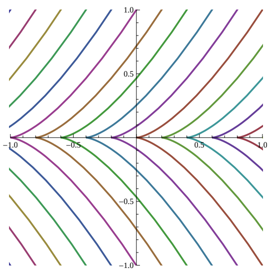


Figure 1.

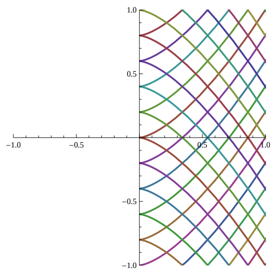


Figure 2.

3 One parameter families of Legendre curves

In this section, we consider one-parameter families of Legendre curves in the unit tangent bundle $T_1S^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$. The fundamental results for Legendre curves in the unit tangent bundle over $\mathbb{R}^2$ see the Appendix or [8].

Definition 3.1 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a smooth mapping. We say that (γ, ν) is a *one-parameter family of Legendre curves* if $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$.

Then $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve for each fixed parameter $\lambda \in \Lambda$, that is, $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda))$ is an integrable curve with respect to the canonical contact 1-form on $\mathbb{R}^2 \times S^1$. Therefore, $\gamma : I \times \Lambda \to \mathbb{R}^2$ is a one-parameter family of frontals.

We denote $J(\boldsymbol{a}) = (-a_2, a_1)$ the anticlockwise rotation by $\pi/2$ of a vector $\boldsymbol{a} = (a_1, a_2)$. We define $\boldsymbol{\mu}(t, \lambda) = J(\nu(t, \lambda))$. Since $\{\nu(t, \lambda), \boldsymbol{\mu}(t, \lambda)\}$ is a moving frame along $\gamma(t, \lambda)$ on $\mathbb{R}^2$, we have the Frenet type formula.

$$\begin{aligned}
\begin{pmatrix} \nu_t(t,\lambda) \\ \boldsymbol{\mu}_t(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t,\lambda) \\ -\ell(t,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t,\lambda) \\ \boldsymbol{\mu}(t,\lambda) \end{pmatrix}, \\
\begin{pmatrix} \nu_\lambda(t,\lambda) \\ \boldsymbol{\mu}_\lambda(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t,\lambda) \\ -m(t,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t,\lambda) \\ \boldsymbol{\mu}(t,\lambda) \end{pmatrix}
\end{aligned}$$

and

$$\gamma_t(t,\lambda)=\beta(t,\lambda)\boldsymbol{\mu}(t,\lambda),$$

where $\ell(t,\lambda)=\nu_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda), m(t,\lambda)=\nu_\lambda(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda)$ and $\beta(t,\lambda)=\gamma_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda)$. By the integrability condition $\nu_{t\lambda}(t,\lambda)=\nu_{\lambda t}(t,\lambda)$, ℓ and m satisfies the condition

$$\ell_\lambda(t,\lambda)=m_t(t,\lambda) \tag{2}$$

for all $(t,\lambda)\in I\times\Lambda$. We call the pair (ℓ,m,β) with the integrability condition (2) a *curvature of the one-parameter family of Legendre curves* (γ,ν).

Remark 3.2 *Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ,m,β). Then $(\gamma,-\nu)$ is also a one-parameter family of Legendre curves with the curvature $(\ell,m,-\beta)$.*

Example 3.3 (Example 2.4) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda)=(t^2+\lambda,t^3), \nu(t,\lambda)=(4+9t^2)^{-1/2}(-3t,2)$. Since $\gamma_t(t,\lambda)=(2t,3t^2), \nu_t(t,\lambda)=6(4+9t^2)^{-3/2}(-2,-3t), \nu_\lambda(t,\lambda)=0$ and $\boldsymbol{\mu}(t,\lambda)=(4+9t^2)^{-1/2}(-2,-3t)$, (γ,ν) is a one-parameter family of Legendre curves with the curvature $(\ell(t,\lambda),m(t,\lambda),\beta(t,\lambda))=(6(4+9t^2)^{-1},0,-t(4+9t^2)^{1/2})$.

Example 3.4 (Example 2.5) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda)=(t^2,t^3+\lambda), \nu(t,\lambda)=(4+9t^2)^{-1/2}(-3t,2)$. Since $\gamma_t(t,\lambda)=(2t,3t^2), \nu_t(t,\lambda)=6(4+9t^2)^{-3/2}(-2,-3t), \nu_\lambda(t,\lambda)=0$ and $\boldsymbol{\mu}(t,\lambda)=(4+9t^2)^{-1/2}(-2,-3t)$, (γ,ν) is a one-parameter family of Legendre curves with the curvature $(\ell(t,\lambda),m(t,\lambda),\beta(t,\lambda))=(6(4+9t^2)^{-1},0,-t(4+9t^2)^{1/2})$.

Definition 3.5 Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}):I\times\Lambda\to\mathbb{R}^2\times S^1$ be one-parameter families of Legendre curves. We say that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are *congruent as one-parameter family of Legendre curves* if there exist a constant rotation $A\in SO(2)$ and a smooth translation mapping $\boldsymbol{a}:\Lambda\to\mathbb{R}^2$ such that $\widetilde{\gamma}(t,\lambda)=A(\gamma(t,\lambda))+\boldsymbol{a}(\lambda)$ and $\widetilde{\nu}(t,\lambda)=A(\nu(t,\lambda))$ for all $(t,\lambda)\in I\times\Lambda$.

We give the existence and uniqueness theorems for one-parameter families of Legendre curves.

Theorem 3.6 (The Existence Theorem for one-parameter families of Legendre curves.) *Let $(\ell,m,\beta):I\times\Lambda\to\mathbb{R}^3$ be a smooth mapping with the integrability condition. There exists a one-parameter family of Legendre curves $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ whose associated curvature is (ℓ,m,β).*

Proof. Let $(t_0,\lambda_0)\in I\times\Lambda$ be fixed. We define a smooth mapping $\theta:I\times\Lambda\to\mathbb{R}$ by

$$\theta(t,\lambda)=\int_{t_0}^t \ell(t,\lambda)dt+\int_{\lambda_0}^\lambda m(t_0,\lambda)d\lambda.$$

Then θ satisfy the conditions $\theta_t(t,\lambda)=\ell(t,\lambda)$ and $\theta_\lambda(t,\lambda)=m(t,\lambda)$. We define a smooth mapping $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ by

$$\begin{aligned}\gamma(t,\lambda)&=\left(-\int\beta(t,\lambda)\sin\theta(t,\lambda)dt,\int\beta(t,\lambda)\cos\theta(t,\lambda)dt\right),\\ \nu(t,\lambda)&=(\cos\theta(t,\lambda),\sin\theta(t,\lambda)).\end{aligned}$$

By a direct calculation, (γ,ν) is a one-parameter family of Legendre curves with the curvature (ℓ,m,β). $\square$

Theorem 3.7 (The Uniqueness Theorem for one-parameter families of Legendre curves.) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be one-parameter families of Legendre curves with the curvatures (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ respectively. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as one-parameter family of Legendre curves if and only if (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ coincides.*

Proof. Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as one-parameter families of Legendre curves. By a direct calculation, we have

$$\begin{aligned}
\widetilde{\gamma}_t(t,\lambda) &= \frac{\partial}{\partial t}(A(\gamma(t,\lambda))+\boldsymbol{a}(\lambda)) = A(\gamma_t(t,\lambda)) = \beta(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \beta(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_t(t,\lambda) &= \frac{\partial}{\partial t}(A(\nu(t,\lambda))) = A(\nu_t(t,\lambda)) = \ell(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \ell(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_\lambda(t,\lambda) &= \frac{\partial}{\partial \lambda}(A(\nu(t,\lambda))) = A(\nu_\lambda(t,\lambda)) = m(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = m(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda).
\end{aligned}$$

Therefore the curvatures (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ coincides.

Conversely, suppose that (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ coincides. Let $(t_0,\lambda_0)\in I\times\Lambda$ be fixed. By using a congruence as one-parameter family of Legendre curves, we may assume $\gamma(t_0,\lambda_0) = \widetilde{\gamma}(t_0,\lambda_0)$ and $\nu(t_0,\lambda_0)=\widetilde{\nu}(t_0,\lambda_0)$. Moreover, we have $\theta(t,\lambda)=\widetilde{\theta}(t,\lambda)$ for all $(t,\lambda)\in I\times\Lambda$ in the proof of Theorem 3.6. It follows from the construction that we have $\nu(t,\lambda)=\widetilde{\nu}(t,\lambda)$, and $\gamma(t,\lambda)=\widetilde{\gamma}(t,\lambda)$ up to a smooth translation mapping $\boldsymbol{a}(\lambda)$ for all $(t,\lambda)\in I\times\Lambda$. □

4 Envelopes of Legendre curves

Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ,m,β), and let $e_L : U\to I\times\Lambda, e_L(u)=(t(u),\lambda(u))$ be a smooth curve. We denote $\Gamma_\lambda(t) = \gamma(t,\lambda)$ and $E_L=\gamma\circ e_L(u)$. Note that we don't assume e_L is a regular curve, see section 2.

Definition 4.1 We call E_L an *envelope* (and e_L a *pre-envelope*) for the family of Legendre curves (γ,ν), when the following conditions are satisfied.

(i) The function λ is non-constant on any non-trivial subinterval of U.(The Variability Condition.)

(ii) For all u the curve E_L is tangent at u to the curve $\Gamma_{\lambda(u)}$ at the parameter $t(u)$, meaning that $E_L'(u)$ and $\boldsymbol{\mu}(t(u),\lambda(u))$ are linearly dependent. (The Tangency Condition.)

Note that the tangency condition is equivalent to the condition $E_L'(u)\cdot\nu(e_L(u))=0$ for all $u\in U$.

Example 4.2 (Example 3.3) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda)=(t^2+\lambda,t^3), \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t,2)$. Let $e_L:\mathbb{R}\to\mathbb{R}\times\mathbb{R}, e_L(u)=(t(u),\lambda(u))=(0,u)$. Then $E_L(u) = \gamma\circ e_L(u)=(u,0)$. Since $\lambda'(u)=1$ and $E_L'(u)\cdot\nu(0,u)=0$, E_L is an envelope of (γ,ν).

Example 4.3 (Example 3.4) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda)=(t^2,t^3+\lambda), \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t,2)$. Let $e_L:\mathbb{R}\to\mathbb{R}\times\mathbb{R}, e_L(u)=(t(u),\lambda(u))=(0,u)$. Then $E_L(u) = \gamma\circ e_L(u)=(0,u)$ and $\lambda'(u)=1$. Since $E_L'(u)\cdot\nu(0,u)=1\neq 0$, E_L is not an envelope of (γ,ν).

Proposition 4.4 *Let $(\gamma,\nu): I\times\Lambda \to \mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ,m,β). Suppose that $e_L: U\to I\times\Lambda$ is a pre-envelope and $E_L=\gamma\circ e_L: U\to\mathbb{R}^2$ is an envelope of (γ,ν). Then E_L is a frontal. More preciously, $(E_L,\nu\circ e_L): U\to\mathbb{R}^2\times S^1$ is a Legendre curve with the curvature*

$$
\begin{aligned}
\ell_E(u) &= t'(u)\ell(e_L(u))+\lambda'(u)m(e_L(u)),\\
\beta_E(u) &= t'(u)\beta(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))\cdot\boldsymbol{\mu}(e_L(u)).
\end{aligned}
$$

Proof. We denote $e_L(u)=(t(u),\lambda(u))$. Since E_L is an envelope, $E_L'(u)\cdot\nu(e_L(u))=0$ for all $u\in U$. It follows that $(E_L,\nu\circ e_L): U\to\mathbb{R}^2\times S^1$ is a Legendre curve. Then $\ell_E(u)=(d/du)(\nu(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=(t'(u)\nu_t(e_L(u))+\lambda'(u)\nu_\lambda(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=t'(u)\ell(e_L(u))+\lambda'(u)m(e_L(u))$ and $\beta_E(u)=(d/du)(\gamma(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=(t'(u)\gamma_t(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=t'(u)\beta(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))\cdot\boldsymbol{\mu}(e_L(u))$. $\square$

We give the envelope theorem for one-parameter family of Legendre curves.

Theorem 4.5 *Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves, and let $e_L: U\to I\times\Lambda$ be a smooth curve satisfying the variability condition. Then e_L is a pre-envelope of (γ,ν) (and E_L is an envelope) if and only if $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$.*

Proof. Suppose that e_L is a pre-envelope of (γ,ν). By the tangency condition, there exists a function $c(u)\in\mathbb{R}$ such that $E_L'(u)=c(u)\boldsymbol{\mu}(e_L(u))$. By differentiate $E_L(u)=\gamma\circ e_L(u)$, we have $E_L'(u)=t'(u)\gamma_t(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))$. It follows from $\gamma_t(t,\lambda)=\beta(t,\lambda)\boldsymbol{\mu}(t,\lambda)$ that $(t'(u)\beta(e_L(u))-c(u))\boldsymbol{\mu}(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))=0$. Then we have $\lambda'(u)\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$. By the variability condition, we have $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$.

Conversely, suppose that $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$. Since $E_L'(u)\cdot\nu(e_L(u))=(t'(u)\gamma_t(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u)))\cdot\nu(e_L(u))=0$, e_L is a pre-envelope of (γ,ν). $\square$

Example 4.6 Let i,j,m,n,j,k be natural numbers with $j=i+h, n=m+k$. Moreover, we take $h=1$ or $k=1$, or h,k are relatively prime numbers. Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1$,

$$\gamma(t,\lambda)=\left(\frac{t^m}{m}+\frac{\lambda^i}{i},\frac{t^n}{n}+\frac{\lambda^j}{j}\right),\ \nu(t,\lambda)=\frac{1}{\sqrt{t^{2k}+1}}(-t^k,1).$$

Since $\gamma_t(t,\lambda)=(t^{m-1},t^{n-1})$, we have $\gamma_t(t,\lambda)\cdot\nu(t,\lambda)=0$ for all $(t,\lambda)\in\mathbb{R}\times\mathbb{R}$. Moreover, since $\gamma_\lambda(t,\lambda)=(\lambda^{i-1},\lambda^{j-1})$, we have $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda)=(\lambda^{i-1}/\sqrt{t^{2k}+1})(-t^k+\lambda^h)$. If we take $e_L:\mathbb{R}\to\mathbb{R}\times\mathbb{R}, e_L(u)=(u^h,u^k)$, then the variability condition holds. Furthermore, since

$$\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=\frac{u^{k(i-1)}}{\sqrt{u^{2kh}+1}}(-u^{hk}+u^{hk})=0,$$

e_L is a pre-envelope of (γ,ν) by Theorem 4.5. Hence, the envelope $(E_L,\nu_L):\mathbb{R}\to\mathbb{R}^2\times S^1$ is given by

$$E_L(u)=\left(\frac{u^{mh}}{m}+\frac{u^{ik}}{i},\frac{u^{nh}}{n}+\frac{u^{jk}}{j}\right),\nu_L(u)=\frac{1}{\sqrt{u^{2kh}+1}}(-u^{kh},1).$$

Proposition 4.7 *Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves. Suppose that $e_L: U\to I\times\Lambda$ is a pre-envelope and $E_L=\gamma\circ e_L$ is an envelope of (γ,ν). Then $e_L: U\to I\times\Lambda$ is also a pre-envelope and $E_L=\gamma\circ e_L$ is also an envelope of $(\gamma,-\nu)$.*

Proof. By Remark 3.2, $(\gamma, -\nu)$ is also a one-parameter family of Legendre curves. It follows from Theorem 4.5 that we have the same pre-envelopes and the envelopes of (γ, ν) and $(\gamma, -\nu)$. $\Box$

Definition 4.8 We say that a map $\Phi : \widetilde{I} \times \widetilde{\Lambda} \to I \times \Lambda$ is a one-parameter family of parameter change if Φ is a diffeomorphism and given by the form $\Phi(s, k) = (\phi(s, k), \varphi(k))$.

Proposition 4.9 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves. Suppose that $e_L : U \to I \times \Lambda$ is a pre-envelope, $E_L = \gamma \circ e_L$ is an envelope and $\Phi : \widetilde{I} \times \widetilde{\Lambda} \to I \times \Lambda$ is a one-parameter family of parameter change. Then $(\widetilde{\gamma}, \widetilde{\nu}) = (\gamma \circ \Phi, \nu \circ \Phi) : \widetilde{I} \times \widetilde{\Lambda} \to \mathbb{R}^2 \times S^1$ is also a one-parameter family of Legendre curves. Moreover, $\Phi^{-1} \circ e_L : U \to \widetilde{I} \times \widetilde{\lambda}$ is a pre-envelope and E_L is also an envelope of $(\widetilde{\gamma}, \widetilde{\nu})$.*

Proof. Since $\widetilde{\gamma}_s(s, k) = \phi_s(s, k)\gamma_t(\Phi(s, k))$ and $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$, we have $\widetilde{\gamma}_s(s, k) \cdot \widetilde{\nu}(s, k) = 0$ for all $(s, k) \in \widetilde{I} \times \widetilde{\Lambda}$. Therefore, $(\widetilde{\gamma}, \widetilde{\nu})$ is a one-parameter family of Legendre curves. By the form of the diffeomorphism $\Phi(s, k) = (\phi(s, k), \varphi(k))$, $\Phi^{-1} : I \times \Lambda \to \widetilde{I} \times \widetilde{\Lambda}$ is given by the form $\Phi^{-1}(t, \lambda) = (\psi(t, \lambda), \varphi^{-1}(\lambda))$. It follows that $\Phi^{-1} \circ e_L(u) = (\phi(t(u), \lambda(u)), \varphi^{-1}(\lambda(u)))$. Since $(d/du)\varphi^{-1}(\lambda(u)) = \varphi_\lambda^{-1}(\lambda(u))\lambda'(u)$, the variability condition holds. Moreover, we have $\widetilde{\gamma}_k(s, k) \cdot \widetilde{\nu}(s, k) = (\gamma_t(\Phi(s, k))\phi_k(s, k) + \gamma_\lambda(\Phi(s, k))\varphi'(k)) \cdot \nu(\Phi(s, k)) = \varphi'(k)\gamma_\lambda(\Phi(s, k)) \cdot \nu(\Phi(s, k))$. It follows that $\widetilde{\gamma}_k(\Phi^{-1} \circ e_L(u)) \cdot \widetilde{\nu}(\Phi^{-1} \circ e_L(u)) = \varphi'(\varphi^{-1}(\lambda(u)))\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$. By Theorem 4.5, $\Phi^{-1} \circ e_L$ is a pre-envelope of $(\widetilde{\gamma}, \widetilde{\nu})$. Therefore, $\widetilde{\gamma} \circ \Phi^{-1} \circ e_L = \gamma \circ \Phi \circ \Phi^{-1} \circ e_L = \gamma \circ e_L = E_L$ is also an envelope of $(\widetilde{\gamma}, \widetilde{\nu})$. $\Box$

We give a relationship between envelopes which are given by implicit functions (Definition 2.1) and one-parameter families of Legendre curves.

Proposition 4.10 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $F(x, y, \lambda) = 0$ be an implicit function of the one-parameter family of frontals, that is, $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, where $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$. If $e_L : U \to I \times \Lambda$ is a pre-envelope and $E_L : U \to \mathbb{R}^2$ is an envelope of (γ, ν), then $E_L(U) \subset E_I$.*

Proof. By differentiate $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Equivalently, $\gamma_t(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = 0$ and $\gamma_\lambda(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0$. Since (γ, ν) is a one-parameter family of Legendre curves, there exists a function $c(t, \lambda)$ such that $(F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = c(t, \lambda)\nu(t, \lambda)$. Moreover, $e_L(u) = (t(u), \lambda(u))$ is a pre-envelope of (γ, ν), we have $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. It follows that $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$. Therefore, we have $E(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$ for all $u \in U$. $\Box$

In order to consider the converse result, we need the following lemma and proposition.

Lemma 4.11 *Let $\boldsymbol{a}, \boldsymbol{b} : U \to \mathbb{R}^2$ be smooth maps. Suppose that the set of non-zero points of smooth function $k : U \to \mathbb{R}$ is dense in U. If $k(u)\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$ are linearly dependent, then $\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$ are linearly dependent for all $u \in U$.*

Proof. Since $\det(k(u)\boldsymbol{a}(u), \boldsymbol{b}(u)) = 0$, we have $k(u)\det(\boldsymbol{a}(u), \boldsymbol{b}(u)) = 0$. By the condition and continuous property, we have $\det(\boldsymbol{a}(u), \boldsymbol{b}(u)) = 0$ for all $u \in U$. $\square$

Proposition 4.12 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2$ be a one-parameter family of Legendre curves, and let $e_L : U \to I \times \Lambda$ be a smooth curve satisfying the variability condition. If the set of regular points of γ on $e_L(U)$ is dense in U and the trace of e_L lies in the singular set of γ, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Proof. Since $e_L(u)$ belong to the singular set of γ, we have $\det(\gamma_t(e_L(u)), \gamma_\lambda(e_L(u))) = 0$ for all $u \in U$. Therefore $\gamma_t(e_L(u)) = \beta(e_L(u))\boldsymbol{\mu}(e_L(u))$ and $\gamma_\lambda(e_L(u))$ are linearly dependent. By the assumption, the set of non-zero points of $\beta \circ e_L$ is dense in U. It follows from Lemma 4.11 that $\boldsymbol{\mu}(e_L(u))$ and $\gamma_\lambda(e_L(u))$ are linearly dependent. Therefore $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. By Theorem 4.5, e_L is a pre-envelope of (γ, ν). $\square$

Proposition 4.13 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $F(x, y, \lambda) = 0$ be an implicit function of the one-parameter family of frontals, that is, $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, where $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$. Let $e_L : U \to I \times \Lambda, e(u) = (t(u), \lambda(u))$ be a smooth curve satisfying the variability condition. If the set of regular points of γ on $e_L(U)$ is dense in U, $E_L(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$ and*

$$(F_x, F_y)(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \neq (0, 0)$$

for all $u \in U$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).

Proof. By differentiate $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Since $E_L(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$, we have $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$. It follows that

$$\begin{pmatrix} x_t(t(u), \lambda(u)) & y_t(t(u), \lambda(u)) \\ x_\lambda(t(u), \lambda(u)) & y_\lambda(t(u), \lambda(u)) \end{pmatrix} \begin{pmatrix} F_x(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \\ F_y(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then the trace of e_L lies in the singular set of γ. By Proposition 4.12, e_L is a pre-envelope of (γ, ν). $\square$

We give interesting examples of envelopes of one-parameter families of Legendre curves by using two Legendre curves. Also see [6, 9, 10].

Let $(\boldsymbol{p}, \nu_p) : I \to \mathbb{R}^2 \times S^1$ and $(\boldsymbol{q}, \nu_q) : \Lambda \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvature (ℓ_p, β_p) and (ℓ_q, β_q) respectively, see Appendix A. We denote

$$\boldsymbol{p}(t) = (p_1(t), p_2(t)), \nu_p(t) = (\nu_{p1}(t), \nu_{p2}(t)), \boldsymbol{\mu}_p(t) = (-\nu_{p2}(t), \nu_{p1}(t)),$$

$$\boldsymbol{q}(\lambda) = (q_1(\lambda), q_2(\lambda)), \nu_q(\lambda) = (\nu_{q1}(\lambda), \nu_{q2}(\lambda)), \boldsymbol{\mu}_q(\lambda) = (-\nu_{q2}(\lambda), \nu_{q1}(\lambda)),$$

respectively. Suppose that $\boldsymbol{p}(0) = (0,0)$ and $\nu_p(0) = (0,1)$.

We define $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ by

$$\gamma(t,\lambda) = \boldsymbol{q}(\lambda) + A(\theta(\lambda))\boldsymbol{p}(t), \nu(t,\lambda) = A(\theta(\lambda))\nu_p(t), \tag{3}$$

where $\theta : \Lambda \to \mathbb{R}$ and

$$A(\theta(\lambda)) = \begin{pmatrix} \cos\theta(\lambda) & -\sin\theta(\lambda) \\ \sin\theta(\lambda) & \cos\theta(\lambda) \end{pmatrix}.$$

By a direct calculation, (γ, ν) is a one-parameter family of Legendre curves.

First, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the both unit normal vectors coincide. Suppose that $\nu(0,\lambda) = \nu_q(\lambda)$. This means that the unit normal vector of γ at $(0,\lambda)$ coincide with the unit normal vector of $\boldsymbol{q}$ at λ. It follows that $\cos\theta(\lambda) = \nu_{q2}(\lambda)$ and $\sin\theta(\lambda) = -\nu_{q1}(\lambda)$. By a direct calculation, we have $\gamma_\lambda(t,\lambda) \cdot \nu(t,\lambda) = -\beta_q(\lambda)\nu_{p1}(t) - \ell_q(\lambda)\boldsymbol{p}(t) \cdot \boldsymbol{\mu}_p(t)$. By a corollary of Theorem 4.5, we have the following.

Corollary 4.14 *Under the above notations, let (γ, ν) be given by* (3) *with the conditions $\boldsymbol{p}(0) = (0,0), \nu_p(0) = (0,1)$ and $\nu(0,\lambda) = \nu_q(\lambda)$. Let $e_L : U \to I \times \Lambda$ be a smooth curve satisfying the variability condition. If $\beta_q(\lambda(u))\nu_{p1}(t(u)) + \ell_q(\lambda(u))\boldsymbol{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Note that $e_L(u) = (0,u)$ is a pre-envelope of (γ, ν). Thus, $E_L(u) = \boldsymbol{q}(\lambda)$ is always an envelope of (γ, ν).

Second, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the unit normal vector of $\boldsymbol{p}$ coincide with the tangent vector of $\boldsymbol{q}$. Suppose that $\nu(0,\lambda) = \boldsymbol{\mu}_q(\lambda)$. This means that the unit normal vector of γ at $(0,\lambda)$ coincide with the unit tangent vector of $\boldsymbol{q}$ at λ. It follows that $\cos\theta(\lambda) = \nu_{q1}(\lambda)$ and $\sin\theta(\lambda) = \nu_{q2}(\lambda)$. By a direct calculation, we have $\gamma_\lambda(t,\lambda) \cdot \nu(t,\lambda) = \beta_q(\lambda)\nu_{p2}(t) - \ell_q(\lambda)\boldsymbol{p}(t) \cdot \boldsymbol{\mu}_p(t)$. By a corollary of Theorem 4.5, we have the following.

Corollary 4.15 *Under the above notations, let (γ, ν) be given by* (3) *with the conditions $\boldsymbol{p}(0) = (0,0), \nu_p(0) = (0,1)$ and $\nu(0,\lambda) = \boldsymbol{\mu}_q(\lambda)$. Let $e_L : U \to I \times \Lambda$ be a smooth curve satisfying the variability condition. If $\beta_q(\lambda(u))\nu_{p2}(t(u)) - \ell_q(\lambda(u))\boldsymbol{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Example 4.16 Let $(\boldsymbol{p}, \nu_p) : [0, 2\pi) \to \mathbb{R}^2 \times S^1$ be an astroid $\boldsymbol{p}(t) = (\cos^3 t - 1, \sin^3 t), \nu_p(t) = (\sin t, \cos t)$ and $(\boldsymbol{q}, \nu_q) : [0, 2\pi) \to \mathbb{R}^2 \times S^1$ be the unit circle $\boldsymbol{q}(\lambda) = (\cos\lambda, \sin\lambda), \nu_q(\lambda) = (\cos\lambda, \sin\lambda)$, see Figure 3. Then we have $\beta_p(t) = 3\cos t \sin t, \ell_p(t) = -1, \beta_q(\lambda) = 1$ and $\ell_q(\lambda) = 1$. Moreover, the conditions $\boldsymbol{p}(0) = (0,0)$ and $\nu_p(0) = (0,1)$ are satisfied.

First, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the both unit normal vectors coincide. By (3) and the condition $\nu(0,\lambda) = \nu_q(\lambda)$, the one-parameter family of Legendre curves $(\gamma, \nu) : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^2 \times S^1$ is given by

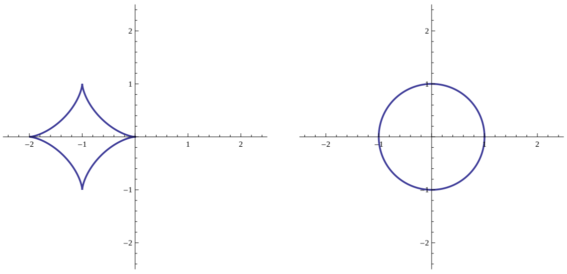
$$\begin{aligned} \gamma(t,\lambda) &= \begin{pmatrix} \cos\lambda \\ \sin\lambda \end{pmatrix} + \begin{pmatrix} \sin\lambda & \cos\lambda \\ -\cos\lambda & \sin\lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\ \nu(t,\lambda) &= \begin{pmatrix} \sin\lambda & \cos\lambda \\ -\cos\lambda & \sin\lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

By a direct calculation, we have $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda) = -4\cos(t-(\pi/4))\cos((t/2)-(\pi/4))\sin t/2$. It follows that $e_L : [0,2\pi) \to [0,2\pi)\times[0,2\pi)$, $e_L(u) = (0,u), (3\pi/4,u), (3\pi/2,u), (7\pi/4,u)$ are pre-envelopes of (γ,ν) respectively, by Corollary 4.14. Therefore, the envelopes $E_L : [0,2\pi) \to \mathbb{R}^2$ of (γ,ν) are given by $E_L(u) = (\cos u, \sin u), (\sqrt{2}+(1/2))(\cos(u+\pi/4), \sin(u+\pi/4)), (\cos(u+\pi/2), \sin(u+\pi/2)), (\sqrt{2}-(1/2))(\cos(u+\pi/4), \sin(u+\pi/4))$, respectively see Figure 4 left.

Second, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the unit normal vector of $\boldsymbol{p}$ coincide with the tangent vector of $\boldsymbol{q}$. By (3) and the condition $\nu(0,\lambda) = \boldsymbol{\mu}_q(\lambda) = (-\sin\lambda, \cos\lambda)$, the one-parameter family of Legendre curves $(\gamma,\nu) : [0,2\pi)\times[0,2\pi) \to \mathbb{R}^2\times S^1$ is given by

$$\begin{aligned}
\gamma(t,\lambda) &= \begin{pmatrix} \cos\lambda \\ \sin\lambda \end{pmatrix} + \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\
\nu(t,\lambda) &= \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.
\end{aligned}$$

By a direct calculation, we have $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda) = \cos 2t$. It follows that $e_L : [0,2\pi) \to [0,2\pi)\times[0,2\pi)$, $e_L(u) = (\pi/4,u), (3\pi/4,u), (5\pi/4,u), (7\pi/4,u)$ are pre-envelopes of (γ,ν) respectively, by Corollary 4.15. Therefore the envelopes $E_L : [0,2\pi) \to \mathbb{R}^2$ of (γ,ν) are given by $E_L(u) = (1/2)(\cos(u+\pi/4), \sin(u+\pi/4)), (1/2)(\cos(u+3\pi/4), \sin(u+3\pi/4)), (1/2)(\cos(u+5\pi/4), \sin(u+5\pi/4)), (1/2)(\cos(u+7\pi/4), \sin(u+7\pi/4))$, respectively see Figure 4 right.



an astroid the unit circle

Figure 3.

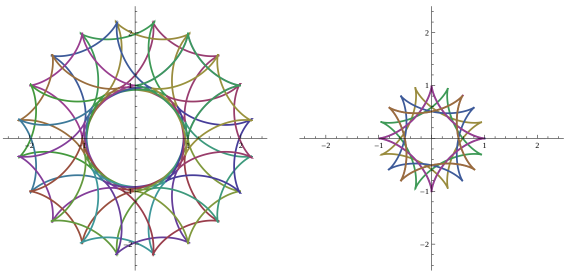


Figure 4.

5 Bi-Legendre curves and envelopes

We consider a special class of one-parameter families of Legendre curves. Let $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be a smooth mapping.

Definition 5.1 We say that $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ is a *bi-Legendre curve* if $\gamma_t(t,\lambda)\cdot\nu(t,\lambda) = 0$ and $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda) = 0$ for all $(t,\lambda) \in I\times\Lambda$.

Then (γ, ν) is a one-parameter family of Legendre curves with respect to both parameters t and λ. We define $\boldsymbol{\mu}(t,\lambda) = J(\nu(t,\lambda))$. Since $\{\nu(t,\lambda), \boldsymbol{\mu}(t,\lambda)\}$ is a moving frame along $\gamma(t,\lambda)$, we have the Frenet type formula.

$$
\begin{aligned}
\begin{pmatrix} \nu_t(t,\lambda) \\ \boldsymbol{\mu}_t(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t,\lambda) \\ -\ell(t,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t,\lambda) \\ \boldsymbol{\mu}(t,\lambda) \end{pmatrix}, \\
\begin{pmatrix} \nu_\lambda(t,\lambda) \\ \boldsymbol{\mu}_\lambda(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t,\lambda) \\ -m(t,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t,\lambda) \\ \boldsymbol{\mu}(t,\lambda) \end{pmatrix}, \\
\gamma_t(t,\lambda) &= \beta(t,\lambda)\boldsymbol{\mu}(t,\lambda), \\
\gamma_\lambda(t,\lambda) &= \alpha(t,\lambda)\boldsymbol{\mu}(t,\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
&\ell(t,\lambda) = \nu_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda),\ m(t,\lambda) = \nu_\lambda(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda), \\
&\beta(t,\lambda) = \gamma_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda),\ \alpha(t,\lambda) = \gamma_\lambda(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda).
\end{aligned}
$$

By the integrability conditions $\nu_{t\lambda}(t,\lambda) = \nu_{\lambda t}(t,\lambda), \gamma_{t\lambda}(t,\lambda) = \gamma_{\lambda t}(t,\lambda)$, ℓ, m, β, α satisfies the conditions

$$
\ell_\lambda(t,\lambda) = m_t(t,\lambda), \beta_\lambda(t,\lambda) = \alpha_t(t,\lambda), \ell(t,\lambda)\alpha(t,\lambda) = m(t,\lambda)\beta(t,\lambda) \tag{4}
$$

for all $(t,\lambda) \in I \times \Lambda$. We call the pair (ℓ, m, β, α) with the integrability conditions (4) a *curvature of the bi-Legendre curve* (γ, ν).

Definition 5.2 Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be bi-Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as bi-Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a}$ on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t,\lambda) = A(\gamma(t,\lambda)) + \boldsymbol{a}$ and $\widetilde{\nu}(t,\lambda) = A(\nu(t,\lambda))$ for all $(t,\lambda) \in I \times \Lambda$.

Theorem 5.3 (The Existence Theorem for bi-Legendre curves.) *Let $(\ell, m, \beta, \alpha) : I \times \Lambda \to \mathbb{R}^4$ be a smooth mapping with the integrability conditions. There exists a bi-Legendre curve (γ, ν) : $I \times \Lambda \to \mathbb{R}^2 \times S^1$ whose associated curvature is (ℓ, m, β, α).*

Proof. Let $(t_0, \lambda_0) \in I \times \Lambda$ be fixed. We define a smooth mapping $\theta : I \times \Lambda \to \mathbb{R}$ by

$$
\theta(t,\lambda) = \int_{t_0}^{t} \ell(t,\lambda)dt + \int_{\lambda_0}^{\lambda} m(t_0,\lambda)d\lambda.
$$

Then θ satisfy the conditions $\theta_t(t,\lambda) = \ell(t,\lambda)$ and $\theta_\lambda(t,\lambda) = m(t,\lambda)$. We also define a smooth mapping $(x, y) : I \times \Lambda \to \mathbb{R}^2$ by

$$
\begin{aligned}
x(t,\lambda) &= -\int_{t_0}^{t} \beta(t,\lambda)\sin\theta(t,\lambda)dt - \int_{\lambda_0}^{\lambda} \alpha(t_0,\lambda)\sin\theta(t_0,\lambda)d\lambda \\
y(t,\lambda) &= \int_{t_0}^{t} \beta(t,\lambda)\cos\theta(t,\lambda)dt + \int_{\lambda_0}^{\lambda} \alpha(t_0,\lambda)\cos\theta(t_0,\lambda)d\lambda.
\end{aligned}
$$

By the integrability condition (4), we have

$$
x_t(t,\lambda) = -\beta(t,\lambda)\sin\theta(t,\lambda), x_\lambda(t,\lambda) = -\alpha(t,\lambda)\sin\theta(t,\lambda),
$$

$$y_t(t,\lambda) = \beta(t,\lambda)\cos\theta(t,\lambda), y_\lambda(t,\lambda) = \alpha(t,\lambda)\cos\theta(t,\lambda).$$

We define a smooth mapping $(\gamma,\nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ by

$$\gamma(t,\lambda) = (x(t,\lambda), y(t,\lambda)), \nu(t,\lambda) = (\cos\theta(t,\lambda), \sin\theta(t,\lambda)).$$

By a direct calculation, (γ,ν) is a bi-Legendre curve with the curvature (ℓ, m, β, α). $\Box$

Theorem 5.4 (The Uniqueness Theorem for bi-Legendre curves.) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be bi-Legendre curves with the curvatures (ℓ, m, β, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{\beta}, \widetilde{\alpha})$ respectively. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as bi-Legendre curves if and only if (ℓ, m, β, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{\beta}, \widetilde{\alpha})$ coincides.*

Proof. Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as bi-Legendre curves. By a direct calculation, we have

$$\begin{aligned}
\widetilde{\gamma}_t(t,\lambda) &= \frac{\partial}{\partial t}(A(\gamma(t,\lambda)) + \boldsymbol{a}) = A(\gamma_t(t,\lambda)) = \beta(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \beta(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\gamma}_\lambda(t,\lambda) &= \frac{\partial}{\partial \lambda}(A(\gamma(t,\lambda)) + \boldsymbol{a}) = A(\gamma_\lambda(t,\lambda)) = \alpha(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \alpha(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_t(t,\lambda) &= \frac{\partial}{\partial t}A(\nu(t,\lambda)) = A(\nu_t(t,\lambda)) = \ell(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \ell(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_\lambda(t,\lambda) &= \frac{\partial}{\partial \lambda}A(\nu(t,\lambda)) = A(\nu_\lambda(t,\lambda)) = m(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = m(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda).
\end{aligned}$$

Therefore the curvatures (ℓ, m, β, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{\beta}, \widetilde{\alpha})$ coincides.

Conversely, suppose that (ℓ, m, β, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{\beta}, \widetilde{\alpha})$ coincides. Let $(t_0, \lambda_0) \in I \times \Lambda$ be fixed. By using a congruence as bi-Legendre curves, $\gamma(t_0,\lambda_0) = \widetilde{\gamma}(t_0,\lambda_0)$ and $\nu(t_0,\lambda_0) = \widetilde{\nu}(t_0,\lambda_0)$. Moreover, we have $\theta(t,\lambda) = \widetilde{\theta}(t,\lambda)$ in the proof of Theorem 5.3. It follows from the construction that $\nu(t,\lambda) = \widetilde{\nu}(t,\lambda)$ and $\gamma(t,\lambda) = \widetilde{\gamma}(t,\lambda)$ for all $(t,\lambda) \in I \times \Lambda$. $\Box$

Let $(\gamma,\nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a bi-Legendre curve. Then (γ,ν) is a one-parameter family of Legendre curves with respect to the parameter λ. We denote a smooth map $e_L : U \to I \times \Lambda, e_L(u) = (t(u), \lambda(u))$. Since $\gamma_\lambda(t,\lambda) \cdot \nu(t,\lambda) = 0$ for all $(t,\lambda) \in I \times \Lambda$, we have $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. If the function λ is non-constant on any non-trivial subinterval of U, then $E_L = \gamma \circ e_L$ is an envelope of (γ,ν) with respect to the parameter λ by Theorem 4.5. Moreover, (γ,ν) is also a one-parameter family of Legendre curves with respect to the parameter t. Since $\gamma_t(t,\lambda)\cdot\nu(t,\lambda) = 0$ for all $(t,\lambda) \in I \times \Lambda$, we have $\gamma_t(e_L(u))\cdot\nu(e_L(u)) = 0$ for all $u \in U$. If the function t is non-constant on any non-trivial subinterval of U, then $E_L = \gamma \circ e_L$ is an envelope of (γ,ν) with respect to the parameter t by Theorem 4.5. Summary we have the following result.

Proposition 5.5 *Let $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be a bi-Legendre curve. If $e_L : U \to I\times\Lambda, e_L(u) = (t(u), \lambda(u))$ satisfy the conditions that the functions t and λ are non-constant on any non-trivial subinterval of U, then $E_L = \gamma \circ e_L$ is an envelope of (γ,ν) with respect to the both parameter t and λ respectively.*

Let $(\gamma,\nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a bi-Legendre curve. Since $\gamma_t(t,\lambda) = \beta(t,\lambda)\boldsymbol{\mu}(t,\lambda)$ and $\gamma_\lambda(t,\lambda) = \alpha(t,\lambda)\boldsymbol{\mu}(t,\lambda)$, we have $\det(\gamma_t(t,\lambda), \gamma_\lambda(t,\lambda)) = 0$ for all $(t,\lambda) \in I \times \Lambda$. It follows that for any $(t,\lambda) \in I \times \Lambda$ are singular points of $\gamma : I \times \Lambda \to \mathbb{R}^2$. Hence, at a rank 1 point, the image of γ is a curve at least locally. We give a concrete example of bi-Legendre curves.

Example 5.6 Let k, n be natural numbers. We define $(\ell, m, \beta, \alpha) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$ by $\ell(t,\lambda) = \lambda^{k+1}t^k, m(t,\lambda) = \lambda^k t^{k+1}, \beta(t,\lambda) = \lambda^{n+1}t^n, \alpha(t,\lambda) = \lambda^n t^{n+1}$. Then the integrability conditions $\ell_\lambda(t,\lambda) = m_t(t,\lambda), \beta_\lambda(t,\lambda) = \alpha_t(t,\lambda), \alpha(t,\lambda)\ell(t,\lambda) = \beta(t,\lambda)m(t,\lambda)$ hold for all $(t,\lambda) \in \mathbb{R} \times \mathbb{R}$. It follows that $\theta(t,\lambda) = \lambda^{k+1}t^{k+1}/(k+1)$. By the construction in the proof of Theorem 5.3, we give a bi-Legendre curve $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$,

$$\begin{aligned}
\gamma(t,\lambda) &= (x(t,\lambda), y(t,\lambda)) = \left(-\int_0^t \lambda^{n+1}t^n \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt, \int_0^\lambda \lambda^{n+1}t^n \cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt \right), \\
\nu(t,\lambda) &= (\cos\theta(t,\lambda), \sin\theta(t,\lambda)) = \left(\cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right), \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) \right).
\end{aligned}$$

A Legendre curves in the unit tangent bundle

We quickly review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see detail [8]. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal if there exists $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve. Examples of Legendre curves see [13, 14]. We have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. Then we call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a frontal (or, Legendre curve),

$$\begin{pmatrix} \dot\nu(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot\nu(t) \cdot \boldsymbol{\mu}(t)$ and $\beta(t) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ, β) *the curvature of the Legendre curve.*

Definition A.1 Let (γ, ν) and $(\widetilde\gamma, \widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde\gamma, \widetilde\nu)$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a}$ on $\mathbb{R}^2$ such that $\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{a}$ and $\widetilde\nu(t) = A(\nu(t))$ for all $t \in I$.

Theorem A.2 (The Existence Theorem for Legendre curves.) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem A.3 (The Uniqueness Theorem for Legendre curves.) *Let (γ, ν) and $(\widetilde\gamma, \widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ, β) and $(\widetilde\ell, \widetilde\beta)$. Then (γ, ν) and $(\widetilde\gamma, \widetilde\nu)$ are congruent as Legendre curves if and only if (ℓ, β) and $(\widetilde\ell, \widetilde\beta)$ coincides.*

References

[1] V. I. Arnol'd, *Singularities of Caustics and Wave Fronts.* Mathematics and Its Applications **62** Kluwer Academic Publishers (1990).

[2] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I.* Birkhäuser (1986).

[3] J. W. Bruce and P. J. Giblin, What is an envelope? *Math. Gaz.* **65** (1981), 186–192.

[4] J. W. Bruce and P. J. Giblin, *Curves and Singularities. A geometrical introduction to singularity theory. Second edition.* Cambridge University Press, Cambridge, 1992.

[5] J. W. Bruce, P. J. Giblin and C. G. Gibson, Caustics through the looking glass. *Math. Intelligencer* **6** (1984), 47–58.

[6] G. Capitanio, On the envelope of 1-parameter families of curves tangent to a semicubic cusp. *C. R. Math. Acad. Sci. Paris.* **335** (2002), 249–254.

[7] S. Ei, K. Fujii, and T. Kunihiro, Renormalization-group method for reduction of evolution equations; invariant manifolds and envelopes. *Ann. Physics.* **280** (2000), 236–298.

[8] T. Fukunaga and M. Takahashi, Existence and uniqueness for Legendre curves. *J. Geom.* **104** (2013), 297–307.

[9] T. Fukunaga and M. Takahashi, Evolutes of fronts in the Euclidean plane. *J. Singul.* **10** (2014), 92–107.

[10] T. Fukunaga and M. Takahashi, Evolutes and involutes of frontals in the Euclidean plane. *Demonstr. Math.* **48** (2015), 147–166.

[11] C. G. Gibson, *Elementary Geometry of Differentiable Curves. An undergraduate introduction.* Cambridge University Press, Cambridge, 2001.

[12] A. Gray, E. Abbena, and S. Salamon, *Modern differential geometry of curves and surfaces with Mathematica. Third edition.* Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL, 2006

[13] G. Ishikawa, Zariski's moduli problem for plane branches and the classification of Legendre curve singularities. *Real and complex singularities, World Sci. Publ., Hackensack, NJ,* (2007), 56–84.

[14] G. Ishikawa, *Singularities of Curves and Surfaces in Various Geometric Problems.* CAS Lecture Notes 10, Exact Sciences. 2015.

[15] S. Izumiya, Singular solutions of first-order differential equations, *Bull. London Math. Soc.* **26** (1994), 69–74.

[16] S. Izumiya, On Clairaut-type equations, *Publ. Math. Debrecen* **45** (1995), 159–166.

[17] S. Izumiya, M. C. Romero-Fuster, M. A. S. Ruas, F. Tari, *Differential Geometry from a Singularity Theory Viewpoint.* World Scientific Pub. Co Inc. 2015.

[18] T. Kunihiro, A geometrical formulation of the renormalization group method for global analysis. *Progr. Theoret. Phys.* **94** (1995), 503–514.

[19] J. W. Rutter, *Geometry of Curves.* Chapman & Hall/CRC Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2000.

[20] M. Takahashi, On completely integrable first order ordinary differential equations, *Proceedings of the Australian-Japanese Workshop on Real and Complex singularities.* (2007), 388–418.

[21] R. Thom, Sur la thorie des enveloppes. *J. Math. Pures Appl.* (9) **41** (1962), 177–192.

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