MODIFICATION OF THE VECTOR-FIELD METHOD RELATED TO QUADARTICALLY PERTURBED WAVE EQUATIONS IN TWO SPACE DIMENSIONS

HIDEO KUBO

Abstract. The purpose of this paper is to shed light on the fact that the global solvability for the quadratically perturbed wave equation with small initial data in two space dimension can be shown by using only a restricted set of vector fields associated with the space-time translation and spatial rotations. As a by-product, we establish almost best possible decay estimates related to the above vector fields, as well as the tangential derivatives to the forward light cones.

Dedicated to Professor Nakao Hayashi on the occasion of his 60th birthday

1. Introduction

In this paper we consider precise decay property of solutions to the initial value problem for quasilinear wave equations:

$$\square_{g(\partial u)} u \equiv -g^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{1.1}$$

where $t = x^0$, $x = (x^1, \dots, x^n)$, and ∂_α $(\alpha = 0, 1, \dots, n)$ denotes the partial derivative with respect to x^α. Throughout this paper we will use the geometric conventions of raising and lowering indices with respect to the Minkowski metric $m = (m_{\alpha\beta}) = \mathrm{diag}(-1, 1, \dots, 1)$. In addition, here and in the following, we use the summation convention of repeated upper and lower indices. We denoted by $g^{\alpha\beta}(\partial u)$ the perturbation of the Minkowski metric m of the type

$$g^{\alpha\beta}(\partial u) = m^{\alpha\beta} + e^{\alpha\beta\gamma}\partial_\gamma u. \tag{1.2}$$

Then, denoting $F(\partial u) := e^{\alpha\beta\gamma}\partial_\gamma u \cdot \partial_\alpha \partial_\beta u$, we can rewrite (1.1) as

$$-(\partial_t^2 - \Delta)u = F(\partial u), \quad (t,x) \in (0,\infty) \times \mathbf{R}^n. \tag{1.3}$$

We prescribe the initial condition by

$$u(0,x) = \varepsilon\phi(x), \quad (\partial_t u)(0,x) = \varepsilon\psi(x), \quad x \in \mathbf{R}^n, \tag{1.4}$$

where ϕ and ψ are supposed to be compactly supported smooth functions (or the rapidly decreasing functions), and ε is supposed to be a small positive parameter.

1991 *Mathematics Subject Classification.* Primary: 35L72; Secondary: 35B45.

Key words and phrases. Nonlinear wave equation, global solution, vector-field method.

The author was partially supported by Grant-in-Aid for Science Research (No.24224003 and No.26220702), JSPS.

In the three space dimensional case, Klainerman [7] introduced the null condition on the nonlineraity:

$$e^{\alpha\beta\gamma}\hat{\omega}_\alpha\hat{\omega}_\beta\hat{\omega}_\gamma \equiv 0 \quad \text{for all } (\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) \in \{-1\} \times S^2, \tag{1.5}$$

in order to solve the problem (1.3)-(1.4). Actually, Christodoulou [2] and Klainerman [7] independently proved the global existence result, although the quadratic nonlinearity is on the critical level for the global solvability. We remark that the null condition (1.5) enables us to rewrite any bilinear form satisfying (1.5) in terms of the null forms:

$$Q(u,v) = \partial_t u \cdot \partial_t v - \nabla_x u \cdot \nabla_x v, \tag{1.6}$$

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \cdot \partial_\beta v - \partial_\beta u \cdot \partial_\alpha v \quad (\alpha, \beta = 0, 1, ..., n), \tag{1.7}$$

where ∇_x denotes the spatial gradient, and that the vector-field method due to [7] works well to deal with the null forms.

For the two space dimensional case, it would be more difficult to show the global existence result for the quadratically perturbed wave equations, because the decaying rate for the solution to the linear wave equation is $O\left(t^{-\frac{n-1}{2}}\right)$. Nevertheless, it was shown in Alinhac [1] that (1.1)-(1.4) admits a global smooth solution, provided that the null condition

$$e^{\alpha\beta\gamma}\hat{\omega}_\alpha\hat{\omega}_\beta\hat{\omega}_\gamma \equiv 0 \quad \text{for all } (\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_2) \in \{-1\} \times S^1 \tag{1.8}$$

holds. New ingredient to treat the sub-critical nonlinearity is an improved energy estimate, which was derived in the framework of the vector-field method.

In the vector-field method, the following vector fields are essentially used:

$$t\partial_j + x_j\partial_t \quad (j = 1, 2, ..., n), \quad t\partial_t + x \cdot \nabla_x, \tag{1.9}$$

which are closely related to the invariance of the homogeneous wave equation. However, if one wishes to handle the system of nonlinear wave equations with multiple propagation speeds, or the exterior problem for the nonlinear wave equations, then the above vector fields are unfavorable. Hence one needs to modify the vector-field method, that is, the full vector fields should be restricted to

$$\partial_t, \quad \nabla_x, \quad x_j\partial_k - x_k\partial_j \quad (1 \le j < k \le n).$$

In [5] one can find a review around this issue, and an alternative proof for the global existence result in three space dimensional case was given, by modifying the vector-field method. More concretely, it was explored that the lack of the vector fields (1.9) can be compensated by establishing a better decay estimate for tangential derivatives to the forward light cones:

$$T_j u(t,x) = \partial_j u(t,x) + \frac{x_j}{|x|}\partial_t u(t,x) \quad (j = 1, ..., n). \tag{1.10}$$

In fact, one can associate the null forms with the tangential derivatives as follows:

$$|Q(u,v)|+\sum_{\alpha,\beta=0}^{n}|Q_{\alpha\beta}(u,v)|\le C\left(|Tu||\partial v|+|\partial u||Tv|\right). \tag{1.11}$$

The aim of this paper is to give an alternative proof for the work of [1] in the same spirit as in [5]. To realize this, we derive an improved energy estimate (3.4) without using the vector fields (1.9). Moreover, we establish pointwise decay estimates (1.12), by employing an "exchange argument between regularity and decay". Actually, because the nonlinearity is on the sub-critical level, we have to start with very rough estimates, and then to improve them until we arrive at the desired decay estimate (1.12) (see step 2 through the final step in Section 5 for the details),

The precise statement of our global existence theorem is as follows.

Theorem 1.1. *Let $n=2$. Suppose that $F(\partial u)=e^{\alpha\beta\gamma}\partial_\gamma u\cdot\partial_\alpha\partial_\beta u$ satisfies* (1.8). *Then for any $\phi,\psi\in\mathcal{C}_0^\infty(\mathbf{R}^2)$, there exists a positive constant ε_0 such that the initial value problem* (1.1)-(1.4) *admits a unique global solution $u\in\mathcal{C}^\infty\left([0,\infty)\times\mathbf{R}^2\right)$ for any $\varepsilon\in(0,\varepsilon_0]$. Moreover, for a positive integer $s\ge 14$ and $\rho\in(0,1/2)$ there exists a positive constant M such that*

$$\begin{aligned}(1+t+r)^{\frac12}(1+|r-t|)^{\rho}&\sum_{|I|\le s+1}|Z^Iu(t,x)|\\ +(1+r)^{\frac12}(1+|r-t|)^{1+\rho}&\sum_{|I|\le s}|Z^I\partial u(t,x)|\\ +(1+r)^{\frac12}(1+r+t)(1+|r-t|)^{\rho}&\sum_{|I|\le s-1}|TZ^Iu(t,x)|\le M\varepsilon\end{aligned}\tag{1.12}$$

holds for $\varepsilon\in(0,\varepsilon_0]$.

As is mentioned above, the existence part of the theorem has been shown in [1]. But, the pointwise decay estimate such as (1.12) was not derived. We underline that the decaying orders are almost best possible, because even for the solution to the homogeneous wave equation, one can only show (1.12) with $\rho=1/2$. Moreover, since our proof is free from the vector fields (1.9), we can extend this result to systems of nonlinear wave equations with multiple speeds, without any essential difficulty, if the pointwise decay estimates given by Propositions 4.1 and 4.2 in [3] are adopted. It is also straightforward to put a semilinear term $\tilde e^{\alpha\beta}\partial_\alpha u\cdot\partial_\beta u$ on the right hand side of (1.1), if it satisfies the null condition

$$\tilde e^{\alpha\beta}\hat\omega_\alpha\hat\omega_\beta\equiv 0\quad\text{for}\quad(\hat\omega_0,\hat\omega_1,\hat\omega_2)\in\{-1\}\times S^1.$$

Concerning the exterior problem, some extra work is necessary and a corresponding result to Theorem 1.1 will be published elsewhere.

This paper is organized as follows: In Section 2, we introduce notation and basic lemmas. An improved energy estimate shall be derived in Section 3. In Section 4, we introduce known pointwise estimates and establish a better decay estimate for the tangential derivatives in Proposition 4.3. Section 5 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Let us start with some standard notation.

- We put $\langle y \rangle := \sqrt{1+|y|^2}$ for $y \in \mathbf{R}$.
- Let $A = A(z)$ and $B = B(z)$ be two positive functions of some variable z, such as $z = (t,x)$ or $z = x$, on suitable domains. We write $A \lesssim B$ if there exists a positive constant C such that $A(z) \le CB(z)$ for all z in the intersection of the domains of A and B.
- We denote by the Lebegue norm on $\mathbf{R}^2$ by $\|\cdot\|_{L^p}$ $(1 \le p \le \infty)$ as usual.
- For a time-space depending function u satisfying $u(t,\cdot) \in X$ for $0 \le t < T$ with a Banach space X, we put $\|u\|_{L^\infty_T X} := \sup_{0 \le t < T} \|u(t,\cdot)\|_X$. For the brevity of the description, we sometimes use the expression $\|h(s,y)\|_{L^\infty_t L^\infty}$ with dummy variables (s,y) for a function h on $[0,t) \times \mathbf{R}$, which means $\sup_{0 \le s < t} \|h(s,\cdot)\|_{L^\infty}$.

Next we introduce the vector fields:

$$Z_0 := \partial_t, \quad Z_1 := \partial_1, \quad Z_2 := \partial_2, \quad Z_3 \equiv \Omega := x_1\partial_2 - x_2\partial_1.$$

As is well known, we have

$$[Z_\alpha, Z_\beta] = \sum_{\gamma=0}^{3} c^{\alpha\beta\gamma} Z_\gamma \quad (\alpha, \beta = 0,1,2,3) \tag{2.1}$$

$$[\partial_\alpha, Z_\beta] = \sum_{\gamma=0}^{2} d^{\alpha\beta\gamma} \partial_\gamma \quad (\alpha = 0,1,2; \beta = 0,1,2,3) \tag{2.2}$$

with suitable constants $c^{\alpha\beta\gamma}$, $d^{\alpha\beta\gamma}$, and

$$[Z_\alpha, \partial_t^2 - \Delta] = 0 \quad (\alpha = 0,1,2,3). \tag{2.3}$$

Here we denoted $[A,B] := AB - BA$. We put $\partial = (\partial_0, \partial_1, \partial_2)$, $\nabla_x = (\partial_1, \partial_2)$, $Z = (Z_0, Z_1, Z_2, Z_3)$, and $\widetilde{Z} = (Z_1, Z_2, Z_3)$. The standard multi-index notation will be used for these sets of vector fields, such as $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $Z^\alpha = Z_0^{\alpha_0} \cdots Z_3^{\alpha_3}$ with $\alpha = (\alpha_0, \dots, \alpha_3)$.

Next we introduce the following Sobolev-type inequality due to Klainerman [6].

Lemma 2.1. *For $v \in \mathcal{C}_0^\infty(\mathbf{R}^2)$, we have*

$$\sup_{x \in \mathbf{R}^2} \langle r \rangle^{1/2} |v(x)| \lesssim \sum_{|\alpha| \le 2} \|\widetilde{Z}^\alpha v\|_{L^2}. \tag{2.4}$$

Next we derive useful estimates for the nonlinearity $F(\partial u) = e^{\alpha\beta\gamma}\partial_\gamma u \cdot \partial_\alpha \partial_\beta u$, provided that it satisfies the null condition (1.8).

Lemma 2.2. *Let N be a positive integer and let $|I| \le N$. If $F(\partial u)$ satisfies* (1.8)*, then we have*

$$\begin{aligned} & \left|Z^I F(\partial u) - e^{\alpha\beta\gamma}\partial_\gamma u \cdot \partial_\alpha \partial_\beta Z^I u\right| \\ \lesssim & \sum_{|J| \le \left[\frac{N-1}{2}\right]+1,\, |K| \le N} (|TZ^J u||\partial Z^K u| + |\partial Z^J u||TZ^K u|). \end{aligned} \tag{2.5}$$

In particular,

$$\begin{aligned} & \left|Z^I F(\partial u)\right| \\ \lesssim & \sum_{|J| \le \left[\frac{N-1}{2}\right]+1,\, |K| \le N+1} (|TZ^J u||\partial Z^K u| + |\partial Z^J u||TZ^K u|). \end{aligned} \tag{2.6}$$

Proof. As is well-known (see e.g. [7]), (1.8) implies that the bilinear form $e^{\alpha\beta\gamma}\partial_\gamma u \cdot \partial_\alpha \partial_\beta v$ can be expressed as a linear combination of the null forms:

$$e^{\alpha\beta\gamma}\partial_\gamma u \cdot \partial_\alpha \partial_\beta v = a^{\alpha\beta\gamma} Q_{\beta\gamma}(u, \partial_\alpha v) + b^\alpha Q(u, \partial_\alpha v), \tag{2.7}$$

where $a^{\alpha\beta\gamma}$ and b^α are suitable constants. Note that for any smooth functions $u = u(t,x)$ and $w = w(t,x)$, one can show

$$ZQ(u,w) = Q(Zu, w) + Q(u, Zw), \tag{2.8}$$

$$\partial Q_{\beta\gamma}(u,w) = Q_{\beta\gamma}(\partial u, w) + Q_{\beta\gamma}(u, \partial w), \tag{2.9}$$

$$\begin{aligned} \Omega Q_{\beta\gamma}(u,w) = & \; Q_{\beta\gamma}(\Omega u, w) + Q_{\beta\gamma}(u, \Omega w) \\ & - m_{\beta 1} Q_{\gamma 2}(u,w) + m_{\beta 2} Q_{\gamma 1}(u,w) \\ & + m_{\gamma 1} Q_{\beta 2}(u,w) - m_{\gamma 2} Q_{\beta 1}(u,w). \end{aligned} \tag{2.10}$$

Therefore, for $|I| \le N$, we get from (2.7) with $u = v$

$$\begin{aligned} & \left|Z^I F(\partial u) - \left(a^{\alpha\beta\gamma} Q_{\beta\gamma}\left(u, Z^I \partial_\alpha u\right) + b^\alpha Q\left(u, Z^I \partial_\alpha u\right)\right)\right| \\ \lesssim & \sum_{|J| \le \left[\frac{N-1}{2}\right]+1,\, |K| \le N} \left(\sum_{\beta,\gamma=0}^{2} \left|Q_{\beta\gamma}\left(Z^J u, Z^K u\right)\right| + \left|Q\left(Z^J u, Z^K u\right)\right| \right). \end{aligned} \tag{2.11}$$

Since $Q_{\beta\gamma}\left(u, Z^I \partial_\alpha u\right) - Q_{\beta\gamma}\left(u, \partial_\alpha Z^I u\right)$ and $Q\left(u, Z^I \partial_\alpha u\right) - Q\left(u, \partial_\alpha Z^I u\right)$ are evaluated by the right hand side of (2.11), we find (2.5) with the help of (2.7) with $v = Z^I u$.

It is easy to see that (2.6) follows from (2.5). This completes the proof. $\square$

3. Energy estimates

In this section we derive an improved energy estimate (3.4). For $\rho > 0$ we set $a(r) = \int_{-\infty}^r \langle r' \rangle^{-1-\rho}\, dr'$ and $A(r,t) = \exp[a(r-t)]$ for t, $r \ge 0$. Note that there exists a constant C such that $1 \le A(r,t) \le C$ for all t, $r \ge 0$.

Proposition 3.1. *For any smooth functions $u = u(t,x)$, $v = v(t,x)$, we have*

$$\Box_{g(\partial u)} v \cdot \partial_t v \cdot A(r,t) = \partial_t X_A + \mathrm{div} Y_A + Z_A + R_A \quad \text{in } (0,\infty) \times \mathbf{R}^2, \tag{3.1}$$

where we have set

$$\begin{aligned}
X_A &= -\frac{1}{2}\left(g^{00}(\partial u) \cdot (\partial_t v)^2 - g^{jk}(\partial u) \cdot \partial_j v \cdot \partial_k v\right) A(r,t), \\
Y_A^j &= -\left(g^{0i}(\partial u) \cdot (\partial_t v)^2 + g^{jk}(\partial u) \cdot \partial_t v \cdot \partial_k v\right) A(r,t), \quad j = 1,2, \\
Z_A &= -\frac{1}{2} g^{\alpha\beta}(\partial u) \cdot \partial_\alpha v \cdot \partial_\beta v \cdot \partial_t A(r,t) + g^{\alpha\beta}(\partial u) \cdot \partial_\alpha v \cdot \partial_t v \cdot \partial_\beta A(r,t), \\
R_A &= -\left(\frac{1}{2}\partial_t g^{\alpha\beta}(\partial u) \cdot \partial_\alpha v \cdot \partial_\beta v - \partial_\alpha g^{\alpha\beta} \cdot \partial_t v \cdot \partial_\beta v\right) A(r,t).
\end{aligned}$$

The identity (3.1) follows from a direct calculation and yields the following energy estimate (3.4). Notice that in the course of its proof, the favorable decay estimates (3.2), (3.3) play an essential role for avoiding the vector fields (1.9).

Corollary 3.2. *Let $\rho > 0$. Assume that*

$$\sum_{|I| \le 1} |\partial \partial^I u(t,x)| \lesssim \eta \langle r \rangle^{-1/2} \langle r - t \rangle^{-1-\rho}, \tag{3.2}$$

$$\sum_{|I| \le 1} |T \partial^I u(t,x)| \lesssim \eta (1+t)^{-1} \tag{3.3}$$

hold for $(t,x) \in [0,T] \times \mathbf{R}^2$. If the null condition (1.8) is satisfied, then there exists $\eta > 0$ such that for $0 < \eta \le \eta_0$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^2} |\partial v(t,x)|^2 \, dx + \int_0^t \int_{\mathbf{R}^2} \langle r - \tau \rangle^{-1-\rho} |T v(\tau,x)|^2 \, dx d\tau \\
\lesssim\ &\int_{\mathbf{R}^2} |\partial v(0,x)|^2 \, dx + \int_0^t \int_{\mathbf{R}^2} \eta (1+\tau)^{-1} |\partial v(\tau,x)|^2 \, dx d\tau \\
&+ \int_0^t \int_{\mathbf{R}^2} \eta^{-1} (1+\tau) \left|\Box_{g(\partial u)} v(t,x)\right|^2 \, dx d\tau
\end{aligned} \tag{3.4}$$

for $t \in [0,T]$.

Proof. Since $g^{\alpha\beta}(\partial u) = m^{\alpha\beta} + e^{\alpha\beta\gamma} \partial_\gamma u$ with $m = \mathrm{diag}(-1,1,1)$, we get

$$\begin{aligned}
Z_A &= -\left(\frac{1}{2} m^{\alpha\beta} \cdot \partial_\alpha v \cdot \partial_\beta v \cdot \partial_t A(r,t) - m^{\alpha\beta} \cdot \partial_\alpha v \cdot \partial_t v \cdot \partial_\beta A(r,t)\right) \\
&\quad - \left(\frac{1}{2} e^{\alpha\beta\gamma} \partial_\gamma u \cdot \partial_\alpha v \cdot \partial_\beta v \cdot \partial_t A(r,t) - e^{\alpha\beta\gamma} \partial_\gamma u \cdot \partial_\alpha v \cdot \partial_t v \cdot \partial_\beta A\right) \\
&=: I_1 + I_2.
\end{aligned}$$

Since $\partial_t A(r,t) = A(r,t)\cdot a'(r-t)\cdot(-1)$ and $\partial_j A(r,t) = A(r,t)\cdot a'(r-t)\cdot\omega_j$ ($j = 1,2$), one can show

$$I_1 = \frac{1}{2}\sum_{j=1}^{2}(T_j v)^2\cdot A(r,t)\cdot a'(r-t). \tag{3.5}$$

On the other hand, thanks to the null condition (1.8), one can rewrite I_2 as

$$\begin{aligned} I_2 \;=\; & \frac{1}{2}e^{\alpha\beta\gamma}\left(T_\gamma u\cdot\partial_\alpha v\cdot\partial_\beta v - \hat{\omega}_\gamma\partial_t u\cdot(T_\alpha v\cdot T_\beta v - \hat{\omega}_\alpha\partial_t v T_\beta v\right. \\ & \left.-\,\hat{\omega}_\beta T_\alpha v\partial_t v)\right) A(r,t)\cdot a'(r-t)\cdot(-1) \\ & -e^{\alpha\beta\gamma}(T_\gamma u\cdot\partial_\alpha v\cdot\partial_t v - \hat{\omega}_\gamma\partial_t u\cdot T_\alpha v\cdot\partial_t v)A(r,t)a'(r-t)\hat{\omega}_\beta. \end{aligned}$$

Note that $\langle r\rangle^{-1/2}\langle r-t\rangle^{-1/2} \lesssim (1+t)^{-1/2}$. Indeed, if $0\le t\le 2r$ and $r\ge 1$, then $\langle r\rangle$ is equivalent to $(1+t)$. While, if $t\ge 2r$ or $0\le r\le 1$, then $\langle r-t\rangle$ is equivalent to $(1+t)$. Thus, using the assumptions (3.2) and (3.3), we have

$$\begin{aligned} |I_2| \;\lesssim\; & \eta(1+t)^{-1}|\partial v|^2 + \eta|Tv|^2 a'(r-t) \\ & +\eta(1+t)^{-1/2}|\partial v||Tv|a'(r-t) \\ \lesssim\; & \eta(1+t)^{-1}|\partial v|^2 + \eta|Tv|^2\langle r-t\rangle^{-1-\rho}. \end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned} R_A\cdot(A(r,t))^{-1} \;=\; & \frac{1}{2}e^{\alpha\beta\gamma}T_\gamma\partial_t u\cdot\partial_\alpha v\cdot\partial_\beta v \\ & -\frac{1}{2}e^{\alpha\beta\gamma}\hat{\omega}_\gamma\partial_t^2 u\cdot(T_\alpha v\cdot T_\beta v - \hat{\omega}_\alpha\partial_t v\cdot T_\beta v - \hat{\omega}_\beta T_\alpha v\cdot\partial_t v) \\ & -e^{\alpha\beta\gamma}\partial_\alpha\partial_\gamma u\cdot\partial_t v\cdot T_\beta v \\ & +e^{\alpha\beta\gamma}\hat{\omega}_\beta(T_\alpha\partial_\gamma u - \hat{\omega}_\alpha\partial_t T_\gamma u)(\partial_t v)^2, \end{aligned}$$

so that

$$\begin{aligned} |R_A| \;\lesssim\; & \eta(1+t)^{-1}|\partial v|^2 + \eta\langle r-t\rangle^{-1-\rho}|Tv|^2 \\ & +\eta(1+t)^{-1/2}\langle r-t\rangle^{-1/2-\rho}|\partial v||Tv| \\ \lesssim\; & \eta(1+t)^{-1}|\partial v|^2 + \eta\langle r-t\rangle^{-1-\rho}|Tv|^2. \end{aligned} \tag{3.7}$$

Therefore, integrating (3.1) over $[0,t]\times\mathbf{R}^2$ and using (3.6) and (3.7), we get

$$\begin{aligned} & \int_{\mathbf{R}_2} X_A(t,x)dx - \int_{\mathbf{R}^2} X_A(0,x)dx + \int_0^t\int_{\mathbf{R}^2} I_1(\tau,x)dxd\tau \\ \lesssim\; & \int_0^t\int_{\mathbf{R}^2}\left(\eta(1+\tau)^{-1}|\partial v(\tau,x)|^2 + \eta\langle r-t\rangle^{-1-\rho}|Tv(\tau,x)|^2\right)dxd\tau \\ & +\int_0^t\int_{\mathbf{R}^2}|\Box_{g(\partial u)}v(\tau,x)||\partial_t v(\tau,x)|dxd\tau. \end{aligned}$$

Since (3.2) yields $-g^{00}(\partial u)\ge 1/2$ and $g^{jk}(\partial u)\ge 1/2$ ($j,k=1,2$) for sufficiently small η, we obtain (3.4), in view of (3.5). This completes the proof. $\square$

4. Pointwise estimates

In this section we recall known decay estimates for solutions of the linear wave equations:

$$
\begin{aligned}
&(\partial_t^2-\Delta)v=h, && (t,x)\in(0,T)\times\mathbf{R}^2,\\
&v(0,x)=v_0(x), \quad (\partial_t v)(0,x)=v_1(x), && x\in\mathbf{R}^2.
\end{aligned}
\tag{4.1}
$$

We denote by $K_0[v_0,v_1](t,x)$ and $L_0[h](t,x)$ the solutions of (4.1) with $h=0$ and $v_0=v_1\equiv 0$, respectively. Then one can show the following estimates from Proposition 2.1 in [9] and Lemma 2.4 in [3].

Lemma 4.1. *Let s be a nonnegative integer. For any $(v_0,v_1)\in\mathcal{C}_0^\infty(\mathbf{R}^2)\times\mathcal{C}_0^\infty(\mathbf{R}^2)$, it holds that for any $\mu>0$ we have*

$$
\begin{aligned}
&\langle t+|x|\rangle^{1/2}\langle t-|x|\rangle^{1/2}\sum_{|I|\le s}|Z^IK_0[v_0,v_1](t,x)|\lesssim\mathcal{A}_{2+\mu,s}[v_0,v_1],\\
&\langle t+|x|\rangle^{1/2}\langle t-|x|\rangle^{3/2}\sum_{|I|\le s}|Z^I\partial K_0[v_0,v_1](t,x)|\lesssim\mathcal{A}_{3+\mu,s+1}[v_0,v_1],
\end{aligned}
\tag{4.2}
$$

for $(t,x)\in[0,T)\times\mathbf{R}^2$, where we put

$$
\mathcal{A}_{\rho,s}[v_0,v_1]:=\sum_{|I|\le s+1}\|\langle\cdot\rangle^\rho\widetilde{Z}^Iv_0\|_{L^\infty}+\sum_{|I|\le s}\|\langle\cdot\rangle^\rho\widetilde{Z}^Iv_1\|_{L^\infty}.
$$

As for the solution to the inhomogeneous wave equation, we have the following estimates which can be deduced from [8] and Proposition 4.2 in [3], with the help of (2.3).

Lemma 4.2. *Let s be a nonnegative integer. Then for $0<\rho<1/2$ and $\mu>0$, we have*

$$
\begin{aligned}
&\langle t+|x|\rangle^{1/2}\langle t-|x|\rangle^{\rho}\sum_{|I|\le s}|Z^IL_0[h](t,x)|\\
&\quad\lesssim\sum_{|I|\le s}\|\langle y\rangle^{1/2}W_{1+\rho,1+\mu}(s,y)Z^Ih(s,y)\|_{L_t^\infty L^\infty},
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&\langle r\rangle^{1/2}\langle r-t\rangle^{1+\rho}\sum_{|I|\le s-1}|Z^I\partial L_0[h](t,x)|\\
&\quad\lesssim\sum_{|I|\le s}\|\langle y\rangle^{1/2}W_{1+\rho+\mu,1}(s,y)Z^Ih(s,y)\|_{L_t^\infty L^\infty}
\end{aligned}
\tag{4.4}
$$

for $(t,x)\in[0,T)\times\mathbf{R}^2$. Here for ν, $\kappa>0$ we put

$$
W_{\nu,\kappa}(t,x)=\langle t+|x|\rangle^\nu\left(\min\{\langle x\rangle,\langle t-|x|\rangle\}\right)^\kappa.
$$

Next we evaluate the tangential derivative T_ju $(j=1,2)$.

Proposition 4.3. *Let N be a positive integer and let $|I| \le N-1$. If u is the solution to* (1.1)*, then we have*

$$\begin{aligned}(4.5)\qquad |TZ^Iu(t,x)| \lesssim & \langle t+r\rangle^{-1/2}\,|(\partial_L(r^{1/2}Z^Iu))(0,(r+t))\omega)| \\ & + \langle t+r\rangle^{-2}\int_0^t \sum_{|I|\le N+1} |Z^Iu(\tau,(r+(t-\tau))\omega)|d\tau \\ & + \int_0^t \sum_{|I|\le N-1} |Z^IF(\partial u(\tau,(r+(t-\tau))\omega))|d\tau \\ & + \langle t+r\rangle^{-1}\sum_{|I|\le N}|Z^Iu(t,x)|\end{aligned}$$

for $0\le t\le 2r$ and $r\ge 1$. Here we put $\partial_L=\partial_t+\partial_r$ with $\partial_r=(x/|x|)\cdot\nabla_x$.

Proof. Let $|I|\le N-1$ and let $0\le t\le 2r$ and $r\ge 1$ in the following. Then r is equivalent to $\langle t+r\rangle$. Therefore, in view of the identities:

$$T_1=\omega_1\partial_L-\frac{\omega_2}{r}\Omega,\quad T_2=\omega_2\partial_L+\frac{\omega_1}{r}\Omega,$$

we see that (4.5) follows from the fact that $|(\partial_t+\partial_r)Z^Iu(t,x)|$ is estimated by the right hand side of (4.5). Observe that we have

$$(4.6)\qquad r^{1/2}(\partial_t^2-\Delta)v=\partial_{\underline{L}}\partial_L(r^{1/2}v)-r^{-3/2}(v/4+\Omega^2v)$$

for any smooth function $v=v(t,x)$, where we put $\partial_{\underline{L}}=\partial_t-\partial_r$. Taking $v=Z^Iu$ in (4.6) with u being the solution to (1.1), i.e., (1.3), we get from (2.3)

$$(4.7)\qquad \partial_{\underline{L}}\partial_L(r^{1/2}v)=r^{-3/2}(Z^Iu/4+\Omega^2Z^Iu)+r^{1/2}Z^IF(\partial u).$$

Therefore, we get

$$(4.8)\quad |\partial_{\underline{L}}\partial_L(r^{1/2}v)|\lesssim\langle t+r\rangle^{-3/2}\sum_{|I|\le N+1}|Z^Iu|+\langle t+r\rangle^{1/2}\sum_{|I|\le N-1}|Z^IF(\partial u)|.$$

Now, if we fix $(t_0,x_0)=(t_0,r_0\,\omega_0)$ so that $0\le t_0\le 2r_0$ and $r_0\ge 1$, and integrate (4.8) along a ray $\{(\tau,(r_0+(t_0-\tau))\omega_0)\mid\tau\in[0,t_0]\}$, then we find

$$\begin{aligned}(4.9)\qquad & |(\partial_L(r^{1/2}v))(t_0,x_0)| \\ \lesssim & |(\partial_L(r^{1/2}v))(0,(r_0+t_0)\omega_0)| \\ & + \langle t_0+r_0\rangle^{-3/2}\int_0^{t_0}\sum_{|I|\le N+1}|Z^Iu(\tau,(r_0+(t_0-\tau))\omega_0)|d\tau \\ & + \langle t_0+r_0\rangle^{1/2}\int_0^{t_0}\sum_{|I|\le N-1}|Z^IF(\partial u(\tau,(r_0+(t_0-\tau))\omega_0))|d\tau.\end{aligned}$$

Since $\partial_L(r^{1/2}v(t,x))=r^{1/2}\partial_LZ^Iu(t,x)+(1/2)r^{-1/2}Z^Iu(t,x)$, we have

$$|\partial_LZ^Iu(t,x)|\le\langle t+r\rangle^{-1/2}\,|\partial_L(r^{1/2}v(t,x))|+\langle t+r\rangle^{-1}\,|Z^Iu(t,x)|.$$

Thus we obtain the needed estimate. This completes the proof. □

5. Proof of Theorem 1.1

Since the local existence for the initial value problem (1.1) can be proved by a standard argument, we have only to deduce a suitable *apriori* estimate. Let u be a smooth solution to (1.1) on $[0,T)\times\mathbf{R}^2$ and let $\rho\in(1/2,1)$. In the following, we always assume $(t,x)\in[0,T)\times\mathbf{R}^2$.

For a nonnegative integer $s\geq 14$, we set

$$e_s[u](t,x)=\sum_{|I|\leq s+1}\langle t+r\rangle^{1/2}\langle r-t\rangle^{\rho}|Z^Iu(t,x)|$$
$$+\sum_{|I|\leq s}\langle r\rangle^{1/2}\langle r-t\rangle^{1+\rho}|Z^I\partial u(t,x)|+\sum_{|I|\leq s-1}\langle r\rangle^{1/2}\langle r+t\rangle\langle r-t\rangle^{\rho}|TZ^Iu(t,x)|.$$

Assume that

$$\|e_s[u]\|_{L^\infty_TL^\infty}\leq M\varepsilon \tag{5.1}$$

holds for some large $M(>1)$ and small $\varepsilon(>0)$ such that $M\varepsilon$ is sufficiently small. Our aim is to show that we can replace M by $M/2$ in (5.1), provided that the null condition (1.8) is satisfied. Once such an estimate is derived, we find from the so-called bootstrap argument that $\|e_s[u]\|_{L^\infty_TL^\infty}$ stays bounded as far as the solution exists.

Step 1. We evaluate $\sum_{|I|\leq 2s-3}\|\partial Z^Iu(t)\|_{L^2}$. Since (5.1) implies

$$\sum_{|I|\leq s-1}|Z^I\partial u(t,x)|\lesssim M\varepsilon\langle r\rangle^{-1/2}\langle r-t\rangle^{-1-\rho}, \tag{5.2}$$

$$\sum_{|I|\leq s-1}|TZ^Iu(t,x)|\lesssim M\varepsilon(1+t)^{-1} \tag{5.3}$$

with $s\geq 15$, we can apply (3.4) as $\eta=M\varepsilon$ and $v=Z^Iu$ with $|I|\leq 2s-3$. Note that by (2.3) and (1.1) we have

$$\Box_{g(\partial u)}Z^Iu=Z^IF(\partial u)-e^{\alpha\beta\gamma}\partial_\gamma u\cdot\partial_\alpha\partial_\beta Z^Iu,$$

so that for $|I|\leq 2s-3$, (2.5), (5.2), and (5.3) yield

$$|\Box_{g(\partial u)}Z^Iu|\lesssim M\varepsilon(1+t)^{-1}\sum_{|I|\leq 2s-3}|\partial Z^Iu(t,x)|$$
$$+M\varepsilon(1+t)^{-1/2}\langle r-t\rangle^{-(1/2)-\rho}\sum_{|I|\leq 2s-3}|TZ^Iu(t,x)|.$$

Thus, choosing $M\varepsilon$ is suitably small, we get

$$\begin{aligned}&\sum_{|I|\le 2s-3}\|\partial Z^I u(t)\|_{L^2}^2+\int_0^t\int_{\mathbf{R}^2}\langle r-\tau\rangle^{-1-\rho}\sum_{|I|\le 2s-3}\left|TZ^Iu(\tau,x)\right|^2dxd\tau\\ \lesssim\ &\varepsilon^2+\int_0^t M\varepsilon(1+\tau)^{-1}\sum_{|I|\le 2s-3}\|\partial Z^Iu(\tau)\|_{L^2}^2d\tau.\end{aligned}$$

Now, the Gronwall inequality leads to

$$\sum_{|I|\le 2s-3}\|\partial Z^Iu(t)\|_{L^2}^2\lesssim\varepsilon^2\exp(C_*M\varepsilon\log(1+t))$$

where C_* is a positive constant, independent of M and ε. Putting $\delta=C_*M\varepsilon$ which is assumed to be sufficiently small, we obtain

$$\sum_{|I|\le 2s-3}\|\partial Z^Iu(t)\|_{L^2}\lesssim M\varepsilon(1+t)^{\delta}. \tag{5.4}$$

Step 2. We evaluate $\sum_{|I|\le 2s-6}|Z^Iu(t,x)|$ and $\sum_{|I|\le 2s-7}|Z^I\partial u(t,x)|$. For $|I|\le N-1$, by the Leibniz rule, we have

$$|Z^IF(\partial u(t,x))|\lesssim\sum_{|J|\le[N/2]}|Z^J\partial u(t,x)|\sum_{|K|\le N}|Z^K\partial u(t,x)|. \tag{5.5}$$

Therefore, for $|I|\le 2s-6$, we see from (5.1), (2.4), and (5.4) that

$$\begin{aligned}&|Z^IF(\partial u(t,x))|\\ \lesssim&M\varepsilon\langle r\rangle^{-1/2}\langle t+r\rangle^{-1/2}\left(\langle r-t\rangle^{-1-\rho}+\langle r\rangle^{-1-\rho}\right)\sum_{|K|\le 2s-3}\|Z^K\partial u(t)\|_{L^2}\\ \lesssim&M^2\varepsilon^2\langle r\rangle^{-1/2}\langle t+r\rangle^{-1/2}\left(\langle r-t\rangle^{-1-\delta}+\langle r\rangle^{-1-\delta}\right)\langle t+r\rangle^{\delta},\end{aligned}$$

because we may assume $\delta<\rho$. Then we get from (4.3) and (4.4) with $\rho=\mu=\delta$,

$$\begin{aligned}&\langle t+|x|\rangle^{1/2}\langle t-|x|\rangle^{\delta}\sum_{|I|\le 2s-6}|Z^IL_0[F(\partial u)](t,x)|\\ &\lesssim\sum_{|I|\le 2s-6}\|\langle y\rangle^{1/2}W_{1+\delta,1+\delta}(s,y)Z^IF(\partial u(s,y))\|_{L_t^\infty L^\infty}\\ &\lesssim M^2\varepsilon^2\langle t+r\rangle^{(1/2)+2\delta},\end{aligned}$$

and

$$\begin{aligned}&\langle r\rangle^{1/2}\langle r-t\rangle^{1+\delta}\sum_{|I|\le 2s-7}|Z^I\partial L_0[F(\partial u)](t,x)|\\ &\lesssim\sum_{|I|\le 2s-6}\|\langle y\rangle^{1/2}W_{1+2\delta,1}(s,y)Z^IF(\partial u(s,y))\|_{L_t^\infty L^\infty}\\ &\lesssim M^2\varepsilon^2\langle t+r\rangle^{(1/2)+3\delta}.\end{aligned}$$

Combining these estimates with (4.2), we get

$$\sum_{|I|\le 2s-6} |Z^I u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{2\delta}, \tag{5.6}$$

and if $0 \le t \le 2r$ and $r \ge 1$, then

$$\sum_{|I|\le 2s-7} |Z^I \partial u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{3\delta} \langle r-t\rangle^{-1-\delta}, \tag{5.7}$$

while, if $t \ge 2r$ or $0 \le r \le 1$, then

$$\sum_{|I|\le 2s-7} |Z^I \partial u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-(1/2)+2\delta} \langle r\rangle^{-1/2}. \tag{5.8}$$

Step 3. We evaluate $\sum_{|I|\le 2s-8} |TZ^I u(t,x)|$. Let $|I| \le 2s-8$. From (5.5) with $N = 2s-7$, (5.1), and (5.7) we get

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{-(1/2)+3\delta} \langle r-t\rangle^{-2-\rho-\delta}$$

for $0 \le t \le 2r$ and $r \ge 1$. Since we assumed that the initial data ϕ, ψ are compactly supported (or rapidly decreasing), we see that $(\partial_L(r^{1/2}Z^I u))(0,(r+t))\omega)$ can be estimated by $C\varepsilon \langle t+r\rangle^{-3/2}$. Therefore, using (4.5) with $N = 2s-7$ and (5.6), we get

$$\begin{aligned}|TZ^I u(t,x)| \lesssim & \varepsilon \langle t+r\rangle^{-2} + M\varepsilon \langle t+r\rangle^{-2} \int_0^t \langle t+r\rangle^{2\delta} d\tau \\ & + \int_0^t M^2\varepsilon^2 \langle t+r\rangle^{-(1/2)+3\delta} \langle r+t-2\tau\rangle^{-2-\rho-\delta} d\tau \\ & + M\varepsilon \langle t+r\rangle^{-1+2\delta},\end{aligned}$$

which yields

$$\sum_{|I|\le 2s-8} |TZ^I u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-(1/2)+3\delta} \tag{5.9}$$

for $0 \le t \le 2r$ and $r \ge 1$.

Step 4. We shall improve the estimate in the previous step, by using the null structure. Let $|I| \le 2s-9$. From (2.6), (5.1), (5.7), (5.9), we get

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{-1+3\delta} \langle r-t\rangle^{-1-\delta}$$

for $0 \le t \le 2r$ and $r \ge 1$. Repeating the argument for getting (5.9), we obtain

$$\sum_{|I|\le 2s-9} |TZ^I u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-1+3\delta} \tag{5.10}$$

for $0 \le t \le 2r$ and $r \ge 1$.

Step 5. We shall improve the estimates (5.6) through (5.8), by using the null structure. Let $|I| \le 2s-10$. From (2.6), (5.1), (5.7), (5.10), we get

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{-(3/2)+3\delta} \langle r-t\rangle^{-1-\delta}$$

for $0 \le t \le 2r$ and $r \ge 1$. On the other hand, it follows from (5.5) with $N = 2s - 9$, (5.1), and (5.8) that

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle r\rangle^{-1-\rho} \langle t+r\rangle^{-(3/2)+2\delta}$$

for $t \ge 2r$ or $0 \le r \le 1$. Then we get

$$\sum_{|I|\le 2s-10} |Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle r\rangle^{-1/2} \langle t+r\rangle^{-1+3\delta} (\langle r-t\rangle^{-1-\delta} + \langle r\rangle^{-1-\delta}).$$

Therefore, applying (4.3) and (4.4) with $\rho = \mu = \delta$, we obtain

$$\langle t+|x|\rangle^{1/2} \langle t-|x|\rangle^{\delta} \sum_{|I|\le 2s-10} |Z^I L_0[F(\partial u)](t,x)| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{4\delta},$$

and

$$\langle r\rangle^{1/2} \langle r-t\rangle^{1+\delta} \sum_{|I|\le 2s-11} |Z^I \partial L_0[F(\partial u)](t,x)| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{5\delta}.$$

Combining these estimates with (4.2), we get

$$\sum_{|I|\le 2s-10} |Z^I u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-(1/2)+4\delta}, \tag{5.11}$$

and if $0 \le t \le 2r$ and $r \ge 1$, then

$$\sum_{|I|\le 2s-11} |Z^I \partial u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-(1/2)+5\delta} \langle r-t\rangle^{-1-\delta}, \tag{5.12}$$

while, if $t \ge 2r$ or $0 \le r \le 1$, then

$$\sum_{|I|\le 2s-11} |Z^I \partial u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-1+4\delta} \langle r\rangle^{-1/2}. \tag{5.13}$$

Step 6. We shall further improve the estimate (5.10). Let $|I| \le 2s-12$. From (2.6), (5.1), (5.12), and (5.10), we get

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{-(3/2)+5\delta} \langle r-t\rangle^{-1-\delta}$$

for $0 \le t \le 2r$ and $r \ge 1$. Using (4.5), (5.11), we obtain

$$\sum_{|I|\le 2s-12} |T Z^I u(t,x)| \lesssim M\varepsilon \langle t+r\rangle^{-(3/2)+5\delta} \tag{5.14}$$

for $0 \le t \le 2r$ and $r \ge 1$.

Step 7. We shall further improve the estimates (5.11) through (5.13). Let $|I| \le 2s - 13$. From (2.6), (5.1), (5.12), and (5.14), we get

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle t+r\rangle^{-2+5\delta} \langle r-t\rangle^{-1-\delta}$$

for $0 \le t \le 2r$ and $r \ge 1$. On the other hand, it follows from (5.5) with $N = 2s - 12$, (5.1), and (5.13) that

$$|Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle r\rangle^{-1-\rho} \langle t+r\rangle^{-2+4\delta}$$

for $t \geq 2r$ or $0 \leq r \leq 1$. Then we get

$$\sum_{|I|\leq 2s-13} |Z^I F(\partial u(t,x))| \lesssim M^2\varepsilon^2 \langle r\rangle^{-1/2} \langle t+r\rangle^{-(3/2)+5\delta} (\langle r-t\rangle^{-1-\delta} + \langle r\rangle^{-1-\delta}).$$

Using (4.3) and (4.4) with $\rho = (1/2) - 6\delta$, $\mu = \delta$, we get

$$\langle t+|x|\rangle^{1/2} \langle t-|x|\rangle^{(1/2)-6\delta} \sum_{|I|\leq 2s-13} |Z^I L_0[F(\partial u)](t,x)| \lesssim M^2\varepsilon^2,$$

and

$$\langle r\rangle^{1/2} \langle r-t\rangle^{(3/2)-6\delta} \sum_{|I|\leq 2s-14} |Z^I \partial L_0[F(\partial u)](t,x)| \lesssim M^2\varepsilon^2.$$

Now we fix δ so small that $(1/2) - 6\delta > \rho$. Then, combining these estimates with (4.2), we get

$$\langle t+r\rangle^{1/2} \langle r-t\rangle^{\rho} \sum_{|I|\leq 2s-13} |Z^I u(t,x)| \lesssim \varepsilon + M^2\varepsilon^2 \tag{5.15}$$

and

$$\langle r\rangle^{1/2} \langle r-t\rangle^{1+\rho} \sum_{|I|\leq 2s-14} |Z^I \partial u(t,x)| \lesssim \varepsilon + M^2\varepsilon^2 \tag{5.16}$$

for all $(t,x) \in [0,T) \times \mathbf{R}^2$.

Final step. We shall finalize the improvement on the estimate (5.14). Let $|I| \leq 2s - 15$. From (2.6), (5.1), (5.16), and (5.14), we get

$$|Z^I F(\partial u(t,x))| \lesssim M\varepsilon^2 \langle t+r\rangle^{-2+5\delta} \langle r-t\rangle^{-1-\delta}$$

for $0 \leq t \leq 2r$ and $r \geq 1$. Using (4.5), (5.15), we obtain

$$\begin{aligned}
|TZ^I u(t,x)| \lesssim & \varepsilon \langle t+r\rangle^{-2} + M\varepsilon \langle t+r\rangle^{-(5/2)} \int_0^t \langle r+t-2\tau\rangle^{-\rho} d\tau \\
& + M^2\varepsilon^2 \langle t+r\rangle^{-2+5\delta} \int_0^t \langle r+t-2\tau\rangle^{-1-\delta} d\tau \\
& + M\varepsilon \langle t+r\rangle^{-(3/2)} \langle r-t\rangle^{-\rho} \\
\lesssim & \varepsilon \langle t+r\rangle^{-2} + M\varepsilon \langle t+r\rangle^{-(3/2)-\rho} + M^2\varepsilon^2 \langle t+r\rangle^{-2+5\delta} \\
& + M\varepsilon \langle t+r\rangle^{-(3/2)} \langle r-t\rangle^{-\rho}
\end{aligned}$$

for $0 \leq t \leq 2r$ and $r \geq 1$. Since $\rho < (1/2) - 5\delta$, we thus get

$$\langle t+r\rangle^{3/2} \langle r-t\rangle^{\rho} \sum_{|I|\leq 2s-15} |TZ^I u(t,x)| \lesssim \varepsilon + M^2\varepsilon^2$$

for $0 \leq t \leq 2r$ and $r \geq 1$. Combing this with (5.16), we get

$$\langle r\rangle^{1/2} \langle t+r\rangle \langle r-t\rangle^{\rho} \sum_{|I|\leq 2s-15} |TZ^I u(t,x)| \lesssim \varepsilon + M^2\varepsilon^2 \tag{5.17}$$

for all $(t,x) \in [0,T) \times \mathbf{R}^2$. Now, from (5.15), (5.16), and (5.17), we find

$$e_s[u](t,x) \le C(\varepsilon + M^2\varepsilon^2)$$

for all $(t,x) \in [0,T) \times \mathbf{R}^2$, because $s \ge 14$. Here C is a positive constant, independent M and ε. Finally, if we fix M large enough to satisfy $C \le M/4$ and choose ε to be sufficiently small so that $CM^2\varepsilon \le 1/4$, then the desired estimate follows. This completes the proof of Theorem 1.1.

References

[1] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math., **145** (2001), 597–618.

[2] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math., **39**, (1986), 267–282.

[3] A. Hoshiga and H. Kubo, *Global solvability for systems of nonlinear wave equations with multiple speeds in two space dimensions*, Diff. Integral Eqs., **17** (2004), 593–622.

[4] S. Katayama and H. Kubo, *An alternative proof of global existence for nonlinear wave equations in an exterior domain*, J. Math. Soc. Japan, **60** (2008), 1135–1170.

[5] S. Katayama and H. Kubo, *Decay estimates of a tangential derivative to the light cone for the wave equation and their application*, SIAM J. Math. Anal., **39** (2008), 1851–1862.

[6] S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math., **38** (1985), 321–332.

[7] S. Klainerman, *The null condition and global existence to nonlinear wave equations,* in "Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1" (Santa Fe, N.M., 1984), Lectures in Appl. Math., **23**, Amer. Math. Soc., Providence, RI, (1986), 293–326.

[8] H. Kubo, *Almost global existence for nonlinear wave equations in an exterior domain in two space dimensions*, J. Differential Equations, 257 (2014), 2765-2800.

[9] K. Kubota, *Existence of a global solutions to a semi-linear wave equation with initial data of noncompact support in low space dimensions*, Hokkaido Math. J., **22** (1993), 123–180.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: `kubo@math.sci.hokudai.ac.jp`