Evolutes of framed immersions in the Euclidean space

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September 9, 2016

Abstract

We consider a smooth curve with singular points in the Euclidean space. As a smooth curve with singular points, we have introduced a framed curve or a framed immersion. A framed immersion is a smooth curve with a moving frame and the pair is an immersion. We consider an evolute and a focal surface of a framed immersion in the Euclidean space. The evolutes and focal surfaces of framed immersions are generalizations of each object in the case of a regular space curve. We investigate relationships between singularities of the evolutes and of the focal surfaces. Moreover, we give properties of the evolutes and repeated evolutes.

1 Introduction

For regular space curves, the evolutes and focal surfaces are classical objects in differential geometry (cf. [3, 4, 7, 10]). The evolute is given by the locus of the center of the osculating spheres and the focal surface is given by the envelope of normal planes. The evolute is also given by the set of singular values of the focal surface. Moreover, these are given by bifurcation sets of distance squared functions. However, the evolute and focal surface may have singularities, even if the space curve is regular. Furthermore, the evolute may not be able to have the Frenet frame, we can not consider not only the curvature and torsion of the evolute but also repeated evolute.

In this paper, we consider an evolute and a focal surface of a space curve with singular points. In order to define the evolute and focal surface of a space curve with singular points, we apply a framed curve (a framed immersion) (cf. [8]). A framed curve is a smooth curve with a moving frame. If a framed curve is an immersion, we call a framed immersion. In §2, we give the definition of framed curves and basic notations. In §3, we define an evolute and a focal surface of the framed immersion under a condition. The evolutes and focal surfaces of framed immersions are generalizations of each objects in the case of regular space curves. We give properties of the evolutes and focal surfaces. For example, the evolute of a framed immersion is also a framed immersion, see Proposition 3.5. In addition, we give relationships between singularities of the evolutes and of the focal surfaces by using the criterion of singularities of smooth mappings (cf. [9]), see Theorem 3.10. In §4, we give properties of the evolutes of

2010 Mathematics Subject classification: 58K05, 53A04, 53A05, 57R45.
Key Words and Phrases. evolute, focal surface, framed immersion, singular point.

framed immersions. In particular, we introduce support functions and parallel curves. The discriminant and secondary discriminant sets of the support function are given by the focal surface and the evolute of the framed immersion, respectively. Moreover, the evolutes of the framed immersion and of the parallel curves coincide. We also give a relationship between the framed curves in the sphere and the evolutes. Furthermore, we consider contact between space curves and evolutes. Since the evolute of the framed immersion is also a framed immersion, we give the k-th evolute of a framed immersion in §5. We give examples of evolutes and focal surfaces of framed immersions in §6.

All maps and manifolds considered here are differential of class C^∞.

Acknowledgment. The second author was supported by JSPS KAKENHI Grant Number 26400078.

2 Basic definitions and notations

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3), \boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$. We define the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ by

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the canonical basis on $\mathbb{R}^3$. We define a set

$$\begin{aligned}\Delta &= \{\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{R}^3\times\mathbb{R}^3 \mid \nu_1\cdot\nu_1 = \nu_2\cdot\nu_2 = 1,\ \nu_1\cdot\nu_2 = 0\} \\ &= \{\boldsymbol{\nu} = (\nu_1, \nu_2) \in S^2\times S^2 \mid \nu_1\cdot\nu_2 = 0\},\end{aligned}$$

where $S^2 = \{\boldsymbol{x}\in\mathbb{R}^3 \mid |\boldsymbol{x}| = 1\}$. Then Δ is a 3-dimensional smooth manifold.

We quickly review on the evolutes and focal surfaces of regular space curves. Let I be an interval or $\mathbb{R}$. Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve with linear independent condition, that is, $\dot\gamma(t)$ and $\ddot\gamma(t)$ are linear independent for all $t\in I$, where $\dot\gamma(t) = (d\gamma/dt)(t)$ and $\ddot\gamma(t) = (d^2\gamma/dt^2)(t)$. We have an orthonormal frame

$$\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\} = \left\{\frac{\dot\gamma(t)}{|\dot\gamma(t)|}, \frac{(\dot\gamma(t)\times\ddot\gamma(t))\times\dot\gamma(t)}{|(\dot\gamma(t)\times\ddot\gamma(t))\times\dot\gamma(t)|}, \frac{\dot\gamma(t)\times\ddot\gamma(t)}{|\dot\gamma(t)\times\ddot\gamma(t)|}\right\}$$

along $\gamma(t)$, which is called the *Frenet frame*. Then we have the following Frenet-Serret formula:

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot\gamma(t)|\kappa(t) & 0 \\ -|\dot\gamma(t)|\kappa(t) & 0 & |\dot\gamma(t)|\tau(t) \\ 0 & -|\dot\gamma(t)|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix},$$

where

$$\kappa(t) = \frac{|\dot\gamma(t)\times\ddot\gamma(t)|}{|\dot\gamma(t)|^3},\ \tau(t) = \frac{\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t))}{|\dot\gamma(t)\times\ddot\gamma(t)|^2}.$$

We call $\kappa(t)$ the *curvature* and $\tau(t)$ the *torsion* of γ. Note that the curvature $\kappa(t)$ and torsion $\tau(t)$ are independent on the choice of a parametrization.

In this paper, we consider evolutes of space curves. The *evolute* $Ev(\gamma) : I \to \mathbb{R}^3$ of a regular space curve γ is given by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t) - \frac{\dot{\kappa}(t)}{|\dot{\gamma}(t)|\kappa^2(t)\tau(t)}\boldsymbol{b}(t),$$

away from the points where $\tau(t) = 0$ (cf. [3, 4]). The evolute $Ev(\gamma)$ of a regular space curve is given by the locus of the center of the osculating spheres. A regular space curve and its osculating sphere have contact of order at least three at a point.

However, we can not construct the Frenet frame at a singular point or a point which does not satisfies the linear independent condition. Moreover, the curvature and torsion may vanish at a singular point. It follows that if γ is not a regular curve with linear independent condition, then we can not define the evolute as the above. In this paper, we define an evolute of a space curve with singular points in the Euclidean space, see Definition 3.1. It is a generalization of the evolute of a regular space curve, see Example 3.2.

Moreover, the *focal surface* $FS(\gamma) : I \times \mathbb{R} \to \mathbb{R}^3$ of a regular space curve γ is given by

$$FS(\gamma)(t, \lambda) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t) + \lambda\boldsymbol{b}(t).$$

If $\tau(t) \neq 0$ for all $t \in I$, then the image of the set of singular points of the focal surface coincide with the image of the evolute. In this sense, the evolute and focal surface have an important relationship.

A framed curve in the Euclidean space is a curve with a moving frame (cf. [8]). It is a generalization of not only regular curves with linear independent condition, but also Legendre curves in the unit tangent bundle (cf. [5]).

Definition 2.1 We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *framed curve* if $\dot{\gamma}(t) \cdot \nu_1(t) = 0$ and $\dot{\gamma}(t) \cdot \nu_2(t) = 0$ for all $t \in I$. Moreover, if (γ, ν_1, ν_2) is an immersion, we call (γ, ν_1, ν_2) a *framed immersion*. We also say that $\gamma : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.

We have the Frenet-Serret type formula of a framed curve (γ, ν_1, ν_2) as follows. We put on $\boldsymbol{\mu}(t) = \nu_1(t) \times \nu_2(t)$. It follows that $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the framed base curve $\gamma(t)$. By the standard arguments, we have the Frenet-Serret type formula:

Proposition 2.2 ([8]) *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve. Then we have*

$$\begin{pmatrix} \dot{\nu}_1(t) \\ \dot{\nu}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\nu}_1(t) \cdot \nu_2(t)$, $m(t) = \dot{\nu}_1(t) \cdot \boldsymbol{\mu}(t)$ and $n(t) = \dot{\nu}_2(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\alpha(t)$ such that $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t)$.

We call the quadruplet (ℓ, m, n, α) *the curvature of the framed curve.* Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$. Moreover, we define a smooth function $f : I \to \mathbb{R}$ by

$$f(t) = \ell(t)(m^2(t) + n^2(t)) + m(t)\dot{n}(t) - \dot{m}(t)n(t).$$

If $f(t) = 0$ and $m^2(t) + n^2(t) \neq 0$ for all $t \in I$, $\gamma(t)$ is contained in a plane (cf. [8]).

Definition 2.3 Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as framed curves* if there exists a rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{a}$, $\widetilde{\nu}_1(t) = A(\nu_1(t))$ and $\widetilde{\nu}_2(t) = A(\nu_2(t))$ for all $t \in I$.

We have the existence and uniqueness for framed curves similarly to the case of regular space curves.

Theorem 2.4 (The Existence Theorem, [8]) *Let $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ whose curvature is (ℓ, m, n, α).*

Theorem 2.5 (The Uniqueness Theorem, [8]) *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvature (ℓ, m, n, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$, respectively. Then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as framed curves if and only if $(\ell, m, n, \alpha) = (\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$.*

Let I and $\overline{I}$ be intervals. A smooth function $s : \overline{I} \to I$ is a change of parameter when s is surjective and has a positive derivative at every point. It follows that s is a diffeomorphism map by calculus. For framed curves $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : \overline{I} \to \mathbb{R}^3 \times \Delta$ with the curvatures (ℓ, m, n, α) and $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$, respectively. Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are parametrically equivalent via the change of parameter $s : \overline{I} \to I$, that is, $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)(t) = (\gamma, \nu_1, \nu_2)(s(t))$ for all $t \in \overline{I}$. By differentiation, we have

$$\overline{\ell}(t) = \ell(s(t))\dot{s}(t),\ \overline{m}(t) = m(s(t))\dot{s}(t),\ \overline{n}(t) = n(s(t))\dot{s}(t),\ \overline{\alpha}(t) = \alpha(s(t))\dot{s}(t). \tag{1}$$

Therefore, the curvature of a framed curve is depend on a parametrization.

We give a relationship between regular space curves and framed curves.

Example 2.6 One of the typical example of framed curves is regular curves with linear independent condition. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve with linear independent condition. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve. It follows that $\boldsymbol{\mu}(t) = \nu_1(t) \times \nu_2(t) = \boldsymbol{t}(t)$. In this case, (γ, ν_1, ν_2) is a framed immersion, since γ is regular. Moreover, the relationships between the curvature of the framed curve $(\ell(t), m(t), n(t), \alpha(t))$, the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the regular curve are given by

$$|\alpha(t)|\kappa(t) = \sqrt{m^2(t) + n^2(t)},\ \alpha(t)(m^2(t) + n^2(t))\tau(t) = f(t).$$

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). For the normal plane of $\gamma(t)$, spanned by $\nu_1(t)$ and $\nu_2(t)$, there are other frames by rotations and reflections. We define $(\overline{\nu}_1(t), \overline{\nu}_2(t)) \in \Delta$ by

$$\begin{pmatrix} \overline{\nu}_1(t) \\ \overline{\nu}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed curve and $\overline{\boldsymbol{\mu}}(t) = \overline{\nu}_1(t) \times \overline{\nu}_2(t) = \nu_1(t) \times \nu_2(t) = \boldsymbol{\mu}(t)$. The curvature $(\overline{\ell}(t), \overline{m}(t), \overline{n}(t), \overline{\alpha}(t))$ of $(\gamma, \overline{\nu}_1, \overline{\nu}_2)$ is given by

$$\Big(\ell(t) - \dot{\theta}(t), m(t)\cos\theta(t) - n(t)\sin\theta(t), m(t)\sin\theta(t) + n(t)\cos\theta(t), \alpha(t)\Big).$$

We call the moving frame $\{\overline{\nu}_1(t), \overline{\nu}_2(t), \boldsymbol{\mu}(t)\}$ a *rotated frame* along $\gamma(t)$ by $\theta(t)$. If we take a smooth function $\theta : I \to \mathbb{R}$ which satisfies $\dot{\theta}(t) = \ell(t)$, then we call the rotated frame $\{\overline{\nu}_1(t), \overline{\nu}_2(t), \boldsymbol{\mu}(t)\}$ an *adapted frame* along $\gamma(t)$ by $\theta(t)$ (cf. [2, 8]).

On the other hand, we define $(\widetilde{\nu}_1(t), \widetilde{\nu}_2(t)) \in \Delta$ by

$$\begin{pmatrix} \widetilde{\nu}_1(t) \\ \widetilde{\nu}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed curve and $\widetilde{\boldsymbol{\mu}}(t) = \widetilde{\nu}_1(t) \times \widetilde{\nu}_2(t) = \nu_2(t) \times \nu_1(t) = -\boldsymbol{\mu}(t)$. The curvature $(\widetilde{\ell}(t), \widetilde{m}(t), \widetilde{n}(t), \widetilde{\alpha}(t))$ of $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is given by

$$\Big(-\ell(t) + \dot{\theta}(t), -m(t)\cos\theta(t) + n(t)\sin\theta(t), m(t)\sin\theta(t) + n(t)\cos\theta(t), -\alpha(t)\Big).$$

We call the moving frame $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), -\boldsymbol{\mu}(t)\}$ a *reflected frame* along $\gamma(t)$ by $\theta(t)$. By a direct calculation, we have the following result.

Proposition 2.7 *Under the above notations, we have the following:*

(1) $m^2(t) + n^2(t) = \overline{m}^2(t) + \overline{n}^2(t) = \widetilde{m}^2(t) + \widetilde{n}^2(t)$,

(2) $f(t) = \overline{f}(t) = -\widetilde{f}(t)$.

We now consider a special moving frame along a framed base curve under a condition. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Suppose that $m^2(t) + n^2(t) \neq 0$ for all $t \in I$. Then we define $(\boldsymbol{n}_1(t), \boldsymbol{n}_2(t)) \in \Delta$ by

$$\boldsymbol{n}_1(t) = \frac{1}{\sqrt{m^2(t) + n^2(t)}}(m(t)\nu_1(t) + n(t)\nu_2(t)),$$
$$\boldsymbol{n}_2(t) = \frac{1}{\sqrt{m^2(t) + n^2(t)}}(-n(t)\nu_1(t) + m(t)\nu_2(t)).$$

By a direct calculation, $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed immersion and $\boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t) = \boldsymbol{\mu}(t)$. We call the moving frame $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ a *Frenet type frame* along $\gamma(t)$. Then the Frenet-Serret type formula is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}_1(t) \\ \dot{\boldsymbol{n}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & L(t) & M(t) \\ -L(t) & 0 & 0 \\ -M(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t),$$

where

$$L(t) = \frac{f(t)}{m^2(t) + n^2(t)}, \ M(t) = \sqrt{m^2(t) + n^2(t)}.$$

Therefore, the curvature of the framed immersion $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is given by $(L, M, 0, \alpha)$.

Since the original frame $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ and the Frenet type frame $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ have the common unit vector $\boldsymbol{\mu}(t)$ and the orientation, the Frenet type frame is one of a rotated frame along $\gamma(t)$.

Let $\gamma : I \to \mathbb{R}^3$ be a regular curve with linear independent condition. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve. By a direct calculation, we have $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\} = \{-\boldsymbol{n}(t), -\boldsymbol{b}(t), \boldsymbol{t}(t)\}$. This is the reason why we call $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ the Frenet type frame along $\gamma(t)$.

3 Evolutes and focal surfaces of framed immersions

In this section, we consider evolutes and focal surfaces of framed immersions. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Suppose that $f(t) \neq 0$ for all $t \in I$. It follows that $m^2(t) + n^2(t) \neq 0$ for all $t \in I$ and hence (γ, ν_1, ν_2) is a framed immersion.

Definition 3.1 The *evolute* $\mathcal{E}v(\gamma) : I \to \mathbb{R}^3$ of the framed immersion (γ, ν_1, ν_2) with $f(t) \neq 0$ is given by

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t)}{f(t)}\nu_1(t) + \frac{\begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t)}{f(t)}\nu_2(t).$$

Example 3.2 Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve with the Frenet frame $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ and the torsion $\tau(t) \neq 0$. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then (γ, ν_1, ν_2) is a framed immersion with $f(t) \neq 0$, see Example 2.6. By a direct calculation, we have $\mathcal{E}v(\gamma)(t) = Ev(\gamma)(t)$. Hence, the definition of the evolute of a framed immersion is a generalization of the definition of the evolute of a regular space curve. Moreover, $(E_v(\gamma), \boldsymbol{t}, \boldsymbol{n})$ is a framed immersion.

Remark 3.3 Let $s : \overline{I} \to I$ be a change of parameter, where I and $\overline{I}$ are intervals. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : \overline{I} \to \mathbb{R}^3 \times \Delta$ be framed immersions with $f(t) \neq 0$. Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are parametrically equivalent via the change of parameter s, that is, $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)(t) = (\gamma, \nu_1, \nu_2)(s(t))$ for all $t \in \overline{I}$. By equation (1), we have $\overline{f}(t) = (\dot{s}(t))^3 f(s(t)) \neq 0$ and $\mathcal{E}v(\overline{\gamma})(t) = \mathcal{E}v(\gamma)(s(t))$. Hence, the evolute of a framed immersion is independent on parameterizations.

Remark 3.4 Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) \neq 0$. We consider a framed immersion $(\gamma, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$, where $\{\overline{\nu}_1(t), \overline{\nu}_2(t), \boldsymbol{\mu}(t)\}$ is a rotated frame by $\theta(t)$. Since Proposition 2.7, we have $\overline{f}(t) \neq 0$. By a direct calculation, the evolute of (γ, ν_1, ν_2) coincide with the evolute of $(\gamma, \overline{\nu}_1, \overline{\nu}_2)$. Moreover, we consider a framed immersion $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$, where $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), -\boldsymbol{\mu}(t)\}$ is a reflected frame by $\theta(t)$. Since Proposition 2.7, we have $\widetilde{f}(t) \neq 0$. By a direct calculation, the evolute of (γ, ν_1, ν_2) coincide with the evolute of framed immersion $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$. Hence, the evolute of a framed immersion is independent on rotations and reflections of a moving frame.

Proposition 3.5 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with the curvature (ℓ, m, n, α) and $f(t) \neq 0$. Then the evolute $\mathcal{E}v(\gamma)$ is also a framed base curve. More precisely, $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1) : I \to \mathbb{R}^3 \times \Delta$ is a framed immersion with the curvature $(-M, 0, L, \alpha_{\mathcal{E}v})$, where*

$$\boldsymbol{n}_1 = \frac{1}{\sqrt{m^2+n^2}}(m\nu_1 + n\nu_2), \ M = \sqrt{m^2+n^2}, \ L = \frac{f}{m^2+n^2},$$

$$\begin{aligned}\alpha_{\mathcal{E}v} =& \frac{1}{\sqrt{m^2+n^2}}\left(n\left(\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha & n \\ \dot{\alpha} & \dot{n} \end{vmatrix} + \alpha\ell m}{f}\right) + \ell\left(\frac{\begin{vmatrix} \alpha & m \\ \dot{\alpha} & \dot{m} \end{vmatrix} - \alpha\ell n}{f}\right)\right)\right. \\ &\left. + m\left(\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha & m \\ \dot{\alpha} & \dot{m} \end{vmatrix} - \alpha\ell n}{f}\right) - \ell\left(\frac{\begin{vmatrix} \alpha & n \\ \dot{\alpha} & \dot{n} \end{vmatrix} + \alpha\ell m}{f}\right)\right)\right).\end{aligned}$$

Proof. By using the Frenet-Serret type formula of a framed curve in Proposition 2.2, we have

$$\dot{\mathcal{E}v}(\gamma) = \left(-\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha & n \\ \dot{\alpha} & \dot{n} \end{vmatrix} + \alpha\ell m}{f} \right) - \left(\frac{\begin{vmatrix} \alpha & m \\ \dot{\alpha} & \dot{m} \end{vmatrix} - \alpha\ell n}{f} \right)\ell \right)\nu_1$$
$$+ \left(\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha & m \\ \dot{\alpha} & \dot{m} \end{vmatrix} - \alpha\ell n}{f} \right) - \left(\frac{\begin{vmatrix} \alpha & n \\ \dot{\alpha} & \dot{n} \end{vmatrix} + \alpha\ell m}{f} \right)\ell \right)\nu_2. \tag{2}$$

It follows that $\dot{\mathcal{E}v}(\gamma)(t)\cdot\boldsymbol{\mu}(t) = 0$ and $\dot{\mathcal{E}v}(\gamma)(t)\cdot\boldsymbol{n}_1(t) = 0$. Moreover, since $\boldsymbol{\mu}(t)\cdot\boldsymbol{n}_1(t) = 0$ for all $t \in I$, $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ is a framed curve. Then we have a moving frame $\{\boldsymbol{\mu}(t), \boldsymbol{n}_1(t), \boldsymbol{n}_2(t)\}$ along $\mathcal{E}v(\gamma)$, where $\boldsymbol{n}_2(t) = \boldsymbol{\mu}(t)\times\boldsymbol{n}_1(t)$. Since $\dot{\mathcal{E}v}(\gamma)(t) = \alpha_{\mathcal{E}v}(t)\boldsymbol{n}_2(t)$, the curvature of the framed curve $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ is given by $(-M(t), 0, L(t), \alpha_{\mathcal{E}v}(t))$. By the assumption, we have $M(t) \neq 0$ and $L(t) \neq 0$. Therefore $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ is a framed immersion. □

By Proposition 3.5, t_0 is a singular point of the evolute $\mathcal{E}v(\gamma)$ if and only if $\alpha_{\mathcal{E}v}(t_0) = 0$. Moreover, since $f_{\mathcal{E}v}(t) = -L^2(t)M(t) \neq 0$, the framed immersion $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1) : I \to \mathbb{R}^3 \times \Delta$ has the evolute $\mathcal{E}v(\mathcal{E}v(\gamma))$, see §5.

On the other hand, for a framed immersion $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ with $m^2(t) + n^2(t) \neq 0$, we define a focal surface.

Definition 3.6 The *focal surface* $\mathcal{FS}(\gamma) : I \times J \to \mathbb{R}^3$ of the framed immersion (γ, ν_1, ν_2) is given by

$$\mathcal{FS}(\gamma)(t, a, b) = \gamma(t) - a\nu_1(t) - b\nu_2(t),$$

where $J = \{(a, b) \in \mathbb{R}^2 \mid \alpha(t) - am(t) - bn(t) = 0 \text{ for all } t \in I\}$.

At least locally, we may assume $m(t) \neq 0$ or $n(t) \neq 0$. If $m(t) \neq 0$, the focal surface $\mathcal{FS}(\gamma) : I \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\mathcal{FS}(\gamma)(t, b) = \gamma(t) - \left(\frac{\alpha(t)}{m(t)} - b\frac{n(t)}{m(t)} \right)\nu_1(t) - b\nu_2(t).$$

If $n(t) \neq 0$, the focal surface $\mathcal{FS}(\gamma) : I \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\mathcal{FS}(\gamma)(t, a) = \gamma(t) - a\nu_1(t) - \left(\frac{\alpha(t)}{n(t)} - a\frac{m(t)}{n(t)} \right)\nu_2(t).$$

Remark 3.7 Let $\gamma : I \to \mathbb{R}^3$ be a regular curve with the Frenet frame $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then (γ, ν_1, ν_2) is a framed immersion, see Example 2.6. Since $m(t) = -|\dot{\gamma}(t)|\kappa(t)$, $n(t) = 0$ and $\alpha(t) = |\dot{\gamma}(t)|$, the focal surface $\mathcal{FS}(\gamma) : I \times J \to \mathbb{R}^3$ is given by $\mathcal{FS}(\gamma)(t, a, b) = \gamma(t) - a\boldsymbol{n}(t) - b\boldsymbol{b}(t)$. Here $J = \{(a, b) \in \mathbb{R}^2 \mid a = -1/\kappa(t) \text{ for all } t \in I\}$. Thus, $\mathcal{FS}(\gamma)(t, a, b)$ coincides with $FS(\gamma)(t, \lambda)$. For this reason, the definition of the focal surface of the framed immersion is a generalization of the definition of the focal surface of the regular curve.

Remark 3.8 Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : \overline{I} \to \mathbb{R}^3 \times \Delta$ be framed immersions with $m^2(t) + n^2(t) \neq 0$. Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are parametrically equivalent

via a change of parameter $s : \overline{I} \to I$, that is, $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)(t) = (\gamma, \nu_1, \nu_2)(s(t))$ for all $t \in \overline{I}$. Since $\overline{m}^2(t) + \overline{n}^2(t) \neq 0$ and

$$\begin{aligned} \overline{J} &= \{(a,b) \in \mathbb{R}^2 \mid \overline{\alpha}(t) - a\overline{m}(t) - b\overline{n}(t) = 0 \text{ for all } t \in \overline{I}\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid \alpha(s(t)) - am(s(t)) - bn(s(t)) = 0 \text{ for all } t \in \overline{I}\}, \end{aligned}$$

we have $\mathcal{FS}(\overline{\gamma})(t,a,b) = \mathcal{FS}(\gamma)(s(t),a,b)$. Hence, the focal surface of a framed immersion is independent on a parameterization.

Remark 3.9 Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $m^2(t) + n^2(t) \neq 0$. The focal surface of (γ, ν_1, ν_2) is $\mathcal{R}$-equivalent to the focal surface of $(\gamma, \overline{\nu}_1, \overline{\nu}_2)$, where $\{\overline{\nu}_1, \overline{\nu}_2, \boldsymbol{\mu}\}$ is a rotated frame by $\theta(t)$. In fact, if we take a diffeomorphism $\phi : I \times \overline{J} \to I \times J$ defined by

$$(t,c,d) \mapsto (t, c\cos\theta(t) + d\sin\theta(t), -c\sin\theta(t) + d\cos\theta(t)),$$

then $\mathcal{FS}(\gamma)(\phi(t,c,d)) = \overline{\mathcal{FS}}(\gamma)(t,c,d)$. Here $\overline{J} = \{(c,d) \in \mathbb{R}^2 \mid \alpha(t) - c\overline{m}(t) - d\overline{n}(t) = 0$ for all $t \in I\}$ and $\overline{\mathcal{FS}}(\gamma)(t,c,d) : I \times \overline{J} \to \mathbb{R}^3, \overline{\mathcal{FS}}(\gamma)(t,c,d) = \gamma(t) - c\overline{\nu}_1(t) - d\overline{\nu}_2(t)$.

On the other hand, the focal surface of (γ, ν_1, ν_2) is $\mathcal{R}$-equivalent to the focal surface of $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$, where $\{\widetilde{\nu}_1, \widetilde{\nu}_2, -\boldsymbol{\mu}\}$ is a reflected frame by $\theta(t)$. In fact, if we take a diffeomorphism $\widetilde{\phi} : I \times \widetilde{J} \to I \times J$ defined by

$$(t,c,d) \mapsto (t, c\cos\theta(t) - d\sin\theta(t), -c\sin\theta(t) - d\cos\theta(t)),$$

then $\mathcal{FS}(\gamma)(\widetilde{\phi}(t,c,d)) = \widetilde{\mathcal{FS}}(\gamma)(t,c,d)$. Here $\widetilde{J} = \{(c,d) \in \mathbb{R}^2 \mid -\alpha(t) - c\widetilde{m}(t) - d\widetilde{n}(t) = 0$ for all $t \in I\}$ and $\widetilde{\mathcal{FS}}(\gamma)(t,c,d) : I \times \widetilde{J} \to \mathbb{R}^3, \widetilde{\mathcal{FS}}(\gamma)(t,c,d) = \gamma(t) - c\widetilde{\nu}_1(t) - d\widetilde{\nu}_2(t)$.

For simplicity, we consider a framed immersion with the Frenet type frame. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with the Frenet type frame $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ and $f(t) = L(t)M^2(t) \neq 0$. By Definitions 3.1 and 3.6, we have $\mathcal{E}v(\gamma) : I \to \mathbb{R}^3$,

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\alpha(t)}{M(t)}\boldsymbol{n}_1(t) + \frac{\begin{vmatrix} \alpha(t) & M(t) \\ \dot{\alpha}(t) & \dot{M}(t) \end{vmatrix}}{L(t)M^2(t)}\boldsymbol{n}_2(t)$$

and $\mathcal{FS}(\gamma) : I \times \mathbb{R} \to \mathbb{R}^3$,

$$\mathcal{FS}(\gamma)(t,\lambda) = \gamma(t) - \frac{\alpha(t)}{M(t)}\boldsymbol{n}_1(t) - \lambda\boldsymbol{n}_2(t).$$

We show that the image of the set of singular points of the focal surface coincide with the image of the evolute. Moreover, we give relationships between singularities of the evolute and of the focal surface.

Theorem 3.10 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) \neq 0$. Then we have the following:*

(1) *The image of the set of singular points of $\mathcal{FS}(\gamma)$ coincide with the image of $\mathcal{E}v(\gamma)$,*

(2) *The focal surface $\mathcal{FS}(\gamma)$ is locally diffeomorphic to the cuspidal edge CE at (t_0, a_0, b_0) if and only if t_0 is a regular point of $\mathcal{E}v(\gamma)$,*

(3) *The focal surface $\mathcal{FS}(\gamma)$ is locally diffeomorphic to the swallowtail SW at (t_0, a_0, b_0) if and only if the evolute $\mathcal{E}v(\gamma)$ is locally diffeomorphic to the 3/2-cusp C at t_0.*

Here, $CE : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0), (u, v) \mapsto (u, v^2, v^3)$ is the cuspidal edge (Figure 1), $SW : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0), (u, v) \mapsto (3u^4 + u^2v, 4u^3 + 2uv, v)$ is the swallowtail (Figure 2), and $C : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0), t \mapsto (t^2, t^3, t^4)$ is the 3/2-cusp (Figure 3).

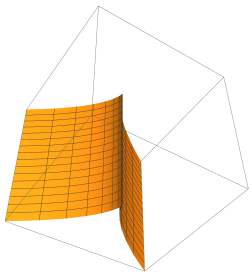


Figure 1: The cuspidal edge

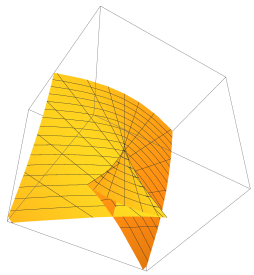


Figure 2: The swallowtail

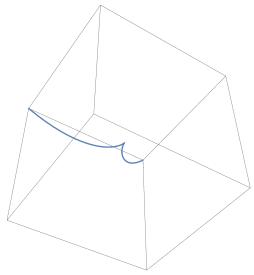


Figure 3: The 3/2-cusp

Proof. Since Remarks 3.4 and 3.9, it is enough to consider the Frenet type frame. By a direct calculation, we have

$$\frac{\partial \mathcal{FS}(\gamma)}{\partial t}(t, \lambda) \times \frac{\partial \mathcal{FS}(\gamma)}{\partial \lambda}(t, \lambda) = -\left(\frac{\begin{vmatrix} \alpha(t) & M(t) \\ \dot{\alpha}(t) & \dot{M}(t) \end{vmatrix}}{M^2(t)} + \lambda L(t) \right) \boldsymbol{\mu}(t).$$

Therefore, $(t_0, \lambda_0) \in I \times \mathbb{R}$ is a singular point of $\mathcal{FS}(\gamma)$ if and only if

$$\lambda_0 = -\frac{\begin{vmatrix} \alpha(t_0) & M(t_0) \\ \dot{\alpha}(t_0) & \dot{M}(t_0) \end{vmatrix}}{L(t_0)M^2(t_0)}.$$

This means that (1) holds.

By the above calculation, $\mathcal{FS}(\gamma) : I \times \mathbb{R} \to \mathbb{R}^3$ is a wave front (cf. [1, 9]). The assertions (2) and (3) can be proven by using the criterion for the cuspidal edge CE and swallowtail SW, see [9]. We consider a function $\Lambda : I \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Lambda(t, \lambda) = \frac{\begin{vmatrix} \alpha(t) & M(t) \\ \dot{\alpha}(t) & \dot{M}(t) \end{vmatrix}}{M^2(t)} + \lambda L(t).$$

Since $\Lambda_\lambda(t, \lambda) = L(t) \neq 0$, $\mathcal{FS}(\gamma)$ has only non-degenerate singularities. Moreover, we consider the singular curve $c : I \to I \times \mathbb{R} \subset \mathbb{R}^2$ of $\mathcal{FS}(\gamma)$ by

$$c(t) = \left(t, -\frac{\begin{vmatrix} \alpha(t) & M(t) \\ \dot{\alpha}(t) & \dot{M}(t) \end{vmatrix}}{L(t)M^2(t)} \right)$$

and a null vector field $\eta : I \to \mathbb{R}^2$ along $c(t)$ by $\eta(t) = (1, -\alpha(t)L(t)/M(t))$.

On the other hand, by using the Frenet-Serret type formula of the evolute, we have

$$\begin{aligned}
\dot{\mathcal{E}v}(\gamma)(t) &= \left(\frac{d}{dt} \left(\frac{\begin{vmatrix} \alpha(t) & M(t) \\ \dot{\alpha}(t) & \dot{M}(t) \end{vmatrix}}{L(t)M^2(t)} \right) - \frac{\alpha(t)L(t)}{M(t)} \right) \boldsymbol{n}_2(t) = \alpha_{\mathcal{E}v}(t)\boldsymbol{n}_2(t), \\
\ddot{\mathcal{E}v}(\gamma)(t) &= -\alpha_{\mathcal{E}v}(t)L(t)\boldsymbol{n}_1(t) + \dot{\alpha}_{\mathcal{E}v}(t)\boldsymbol{n}_2(t), \\
\dddot{\mathcal{E}v}(\gamma)(t) &= -\alpha_{\mathcal{E}v}(t)L(t)M(t)\boldsymbol{\mu}(t) + (-2\dot{\alpha}_{\mathcal{E}v}(t)L(t) - \alpha_{\mathcal{E}v}(t)\dot{L}(t))\boldsymbol{n}_1(t) \\
&\quad + (\ddot{\alpha}_{\mathcal{E}v}(t) - \alpha_{\mathcal{E}v}(t)L^2(t))\boldsymbol{n}_2(t).
\end{aligned}$$

By using the criterion in [9], the focal surface $\mathcal{FS}(\gamma)$ is locally diffeomorphic to the cuspidal edge CE at (t_0, λ_0) if and only if

$$\lambda_0 = -\frac{\begin{vmatrix} \alpha(t_0) & M(t_0) \\ \dot{\alpha}(t_0) & \dot{M}(t_0) \end{vmatrix}}{L(t_0)M^2(t_0)}$$

and

$$\det(\dot{c}(t_0), \eta(t_0)) = \frac{d}{dt} \left(\frac{\begin{vmatrix} \alpha(t_0) & M(t_0) \\ \dot{\alpha}(t_0) & \dot{M}(t_0) \end{vmatrix}}{L(t_0)M^2(t_0)} \right) - \frac{\alpha(t_0)L(t_0)}{M(t_0)} = \alpha_{\mathcal{E}v}(t_0) \neq 0.$$

This condition is equivalent to t_0 is a regular point of $\mathcal{E}v(\gamma)$, hence we have the assertion (2).

Moreover, the focal surface $\mathcal{FS}(\gamma)$ is locally diffeomorphic to the swallowtail SW at (t_0, λ_0) if and only if

$$\lambda_0 = -\frac{\begin{vmatrix} \alpha(t_0) & M(t_0) \\ \dot{\alpha}(t_0) & \dot{M}(t_0) \end{vmatrix}}{L(t_0)M^2(t_0)},$$

$\alpha_{\mathcal{E}v}(t_0) = 0$ and $\dot{\alpha}_{\mathcal{E}v}(t_0) \neq 0$. On the other hand, the evolute $\mathcal{E}v(\gamma)$ is locally diffeomorphic to the 3/2-cusp C at t_0 if and only if $\dot{\mathcal{E}v}(\gamma)(t_0) = 0$ and

$$\operatorname{rank}(\ddot{\mathcal{E}v}(\gamma)(t_0), \dddot{\mathcal{E}v}(\gamma)(t_0)) = 2,$$

that is, $\alpha_{\mathcal{E}v}(t_0) = 0$ and $\dot{\alpha}_{\mathcal{E}v}(t_0) \neq 0$. It follows that (3) holds. This completes the proof of Theorem. □

4 Properties of the evolutes of framed immersions

In this section, we consider properties of the evolutes of framed immersions.

4.1 Support functions

Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) = L(t)M^2(t) \neq 0$, where $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ is the Frenet type frame and $(L(t), M(t), 0, \alpha(t))$ is the curvature. We

define a family of functions $F_\mu : I \times \mathbb{R}^3 \to \mathbb{R}, F_\mu(t, \boldsymbol{x}) = (\gamma(t) - \boldsymbol{x}) \cdot \boldsymbol{\mu}(t)$. We call F_μ a *support function* of γ with respect to $\boldsymbol{\mu}$. We denote $F_\mu(\boldsymbol{x}_0)(t) = F_\mu(t, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$. Then we have the following proposition.

Proposition 4.1 *Under the above notations, we have the following:*

(1) $F_\mu(\boldsymbol{x}_0)(t_0) = 0$ *if and only if there exist* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that* $\gamma(t_0) - \boldsymbol{x}_0 = \lambda_1 \boldsymbol{n}_1(t_0) + \lambda_2 \boldsymbol{n}_2(t_0)$.

(2) $F_\mu(\boldsymbol{x}_0)(t_0) = \dot{F}_\mu(\boldsymbol{x}_0)(t_0) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that*

$$\gamma(t_0) - \boldsymbol{x}_0 = \frac{\alpha(t_0)}{M(t_0)} \boldsymbol{n}_1(t_0) + \lambda \boldsymbol{n}_2(t_0).$$

(3) $F_\mu(\boldsymbol{x}_0)(t_0) = \dot{F}_\mu(\boldsymbol{x}_0)(t_0) = \ddot{F}_\mu(\boldsymbol{x}_0)(t_0) = 0$ *if and only if*

$$\gamma(t_0) - \boldsymbol{x}_0 = \frac{\alpha(t_0)}{M(t_0)} \boldsymbol{n}_1(t_0) - \frac{\begin{vmatrix} \alpha(t_0) & M(t_0) \\ \dot{\alpha}(t_0) & \dot{M}(t_0) \end{vmatrix}}{L(t_0) M^2(t_0)} \boldsymbol{n}_2(t_0).$$

Proof. Since $F_\mu(\boldsymbol{x}_0)(t) = (\gamma(t) - \boldsymbol{x}_0) \cdot \boldsymbol{\mu}(t)$, we have the following calculations:

$$\begin{aligned} &(i)\ F_\mu(\boldsymbol{x}_0) = (\gamma - \boldsymbol{x}_0) \cdot \boldsymbol{\mu}, \\ &(ii)\ \dot{F}_\mu(\boldsymbol{x}_0) = \alpha + (\gamma - \boldsymbol{x}_0) \cdot (-M \boldsymbol{n}_1), \\ &(iii)\ \ddot{F}_\mu(\boldsymbol{x}_0) = \dot{\alpha} + (\gamma - \boldsymbol{x}_0) \cdot (-\dot{M} \boldsymbol{n}_1 - LM \boldsymbol{n}_2 - M^2 \boldsymbol{\mu}). \end{aligned}$$

By (i), the assertion (1) holds.

By (ii), $F_\mu(\boldsymbol{x}_0)(t_0) = \dot{F}_\mu(\boldsymbol{x}_0)(t_0) = 0$ if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\gamma(t_0) - \boldsymbol{x}_0 = \lambda_1 \boldsymbol{n}_1(t_0) + \lambda_2 \boldsymbol{n}_2(t_0)$ and $\alpha(t_0) - M(t_0)\lambda_1 = 0$. Since $M(t_0) \neq 0$, we have

$$\gamma(t_0) - \boldsymbol{x}_0 = \frac{\alpha(t_0)}{M(t_0)} \boldsymbol{n}_1(t_0) + \lambda \boldsymbol{n}_2(t_0).$$

Therefore the assertion (2) holds.

By (iii) and a direct calculation, $F_\mu(\boldsymbol{x}_0)(t_0) = \dot{F}_\mu(\boldsymbol{x}_0)(t_0) = \ddot{F}_\mu(\boldsymbol{x}_0)(t_0) = 0$ if and only if

$$\gamma(t_0) - \boldsymbol{x}_0 = \frac{\alpha(t_0)}{M(t_0)} \boldsymbol{n}_1(t_0) - \frac{\begin{vmatrix} \alpha(t_0) & M(t_0) \\ \dot{\alpha}(t_0) & \dot{M}(t_0) \end{vmatrix}}{L(t_0) M^2(t_0)} \boldsymbol{n}_2(t_0).$$

Therefore the assertion (3) holds. □

For the support function of γ with respect to $\boldsymbol{\mu}$, the discriminant set of F_μ

$$\mathcal{D}_{F_\mu} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \text{there exists } t \in I \text{ such that } F_\mu = \frac{\partial F_\mu}{\partial t} = 0 \text{ at } (t, \boldsymbol{x})\}$$

coincide with the image of the focal surface $\mathcal{FS}(\gamma)$. Moreover, the secondary discriminant set of F_μ

$$\mathcal{D}^2_{F_\mu} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \text{there exists } t \in I \text{ such that } F_\mu = \frac{\partial F_\mu}{\partial t} = \frac{\partial^2 F_\mu}{\partial t^2} = 0 \text{ at } (t, \boldsymbol{x})\}$$

coincide with the image of the evolute $\mathcal{E}v(\gamma)$.

4.2 Parallel curves

We consider parallel curves of a framed curve and its evolute. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Let $\theta : I \to \mathbb{R}$ be a smooth function which satisfies $\dot{\theta}(t) = \ell(t)$ for all $t \in I$. The *parallel curve* $\gamma_{(a,b)} : I \to \mathbb{R}^3$ of the framed curve (γ, ν_1, ν_2) is defined by

$$\gamma_{(a,b)}(t) = \gamma(t) + (a\cos\theta(t) + b\sin\theta(t))\,\nu_1(t) + (-a\sin\theta(t) + b\cos\theta(t))\,\nu_2(t),$$

where $a, b \in \mathbb{R}$. Then we have the following results.

Proposition 4.2 *Under the above notations, the parallel curve $\gamma_{(a,b)}$ is also a framed base curve. More precisely, $(\gamma_{(a,b)}, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve with the curvature $(\ell(t), m(t), n(t), \alpha(t) + a(m(t)\cos\theta(t) - n(t)\sin\theta(t)) + b(m(t)\sin\theta(t) + n(t)\cos\theta(t)))$.*

Proof. Since

$$\begin{aligned}
\dot{\gamma}_{(a,b)}(t) = \; & \alpha(t)\boldsymbol{\mu}(t) + \Big(-a\dot{\theta}(t)\sin\theta(t) + b\dot{\theta}(t)\cos\theta(t)\Big)\,\nu_1(t) \\
& + (a\cos\theta(t) + b\sin\theta(t))\,(\ell(t)\nu_2(t) + m(t)\boldsymbol{\mu}(t)) \\
& + \Big(-a\dot{\theta}(t)\cos\theta(t) - b\dot{\theta}(t)\sin\theta(t)\Big)\,\nu_2(t) \\
& + (-a\sin\theta(t) + b\cos\theta(t))\,(-\ell(t)\nu_1(t) + n(t)\boldsymbol{\mu}(t)) \\
= \; & (\alpha(t) + a(m(t)\cos\theta(t) - n(t)\sin\theta(t)) + b(m(t)\sin\theta(t) + n(t)\cos\theta(t)))\,\boldsymbol{\mu}(t),
\end{aligned}$$

we have $\dot{\gamma}_{(a,b)}(t) \cdot \nu_1(t) = 0$ and $\dot{\gamma}_{(a,b)}(t) \cdot \nu_2(t) = 0$ for all $t \in I$. Hence $(\gamma_{(a,b)}, \nu_1, \nu_2)$ is a framed curve. It follows that $\gamma(t)$ and $\gamma_{(a,b)}(t)$ have the same frame $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$. Therefore, we have the curvature $(\ell(t), m(t), n(t), \alpha(t) + a(m(t)\cos\theta(t) - n(t)\sin\theta(t)) + b(m(t)\sin\theta(t) + n(t)\cos\theta(t)))$. □

Proposition 4.3 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) \neq 0$. Then $(\gamma_{(a,b)}, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed immersion with $f_{(a,b)}(t) \neq 0$ and $\mathcal{E}v(\gamma_{(a,b)})(t) = \mathcal{E}v(\gamma)(t)$ for any $a, b \in \mathbb{R}$.*

Proof. By Proposition 4.2, $(\gamma_{(a,b)}, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed curve and we have $f_{(a,b)}(t) = f(t) \neq 0$ for all $t \in I$. It follows that $(\gamma_{(a,b)}, \nu_1, \nu_2)$ is a framed immersion. By a direct calculation, we have $\mathcal{E}v(\gamma_{(a,b)})(t) = \mathcal{E}v(\gamma)(t)$ for any $a, b \in \mathbb{R}$. □

Proposition 4.4 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) \neq 0$. The image of the set of singular points of parallel curves $\gamma_{(a,b)}$ is contained in the focal surface $\mathcal{FS}(\gamma)$ for any $a, b \in \mathbb{R}$.*

Proof. We put $A_{(a,b)}(t) = a\cos\theta(t) + b\sin\theta(t)$ and $B_{(a,b)}(t) = -a\sin\theta(t) + b\cos\theta(t)$, where $\theta(t)$ satisfies $\dot{\theta}(t) = \ell(t)$ for all $t \in I$. In this case, $\gamma_{(a,b)}(t) = \gamma(t) + A_{(a,b)}(t)\nu_1(t) + B_{(a,b)}(t)\nu_2(t)$. If t_0 is a singular point of $\gamma_{(a,b)}$, then we have $\alpha(t_0) + m(t_0)A_{(a,b)}(t_0) + n(t_0)B_{(a,b)}(t_0) = 0$. This means that $\gamma_{(a,b)}(t_0) \in \mathcal{FS}(\gamma)(I \times J)$ for any $a, b \in \mathbb{R}$. □

Example 4.5 Let $(\gamma, \nu_1, \nu_2) : [0, 2\pi) \to \mathbb{R}^3$ be

$$\begin{aligned}
\gamma(t) &= (C_1 \cos^3 t, C_1 \sin^3 t, C_2 \cos 2t), \\
\nu_1(t) &= (-\sin t, -\cos t, 0), \\
\nu_2(t) &= \frac{1}{\sqrt{9C_1^2 + 16C_2^2}}(-4C_2 \cos t, 4C_2 \sin t, 3C_1),
\end{aligned}$$

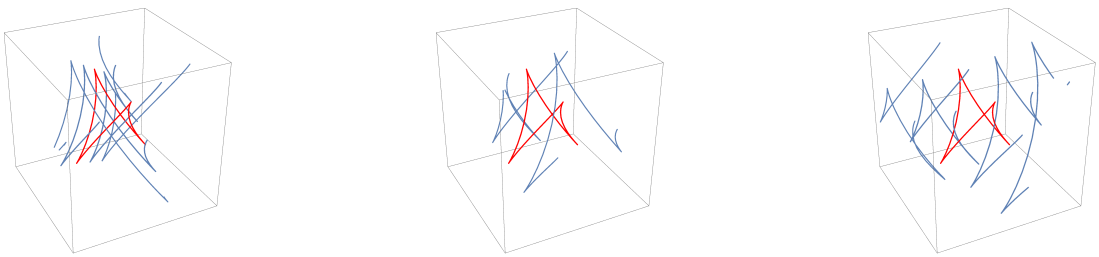
where C_1 and C_2 are non-zero constants. We call γ an *astroid*. Then (γ, ν_1, ν_2) is a framed immersion with the curvature

$$\ell(t) = \frac{4C_2}{\sqrt{9C_1^2 + 16C_2^2}}, \quad m(t) = \frac{3C_1}{\sqrt{9C_1^2 + 16C_2^2}}, \quad n(t) = 0, \quad \alpha(t) = \sqrt{9C_1^2 + 16C_2^2} \cos t \sin t.$$

If we take $\theta : [0, 2\pi) \to \mathbb{R}, \theta(t) = (4C_2/\sqrt{9C_1^2 + 16C_2^2})t$, then $\theta(t)$ satisfies the condition $\dot{\theta}(t) = \ell(t)$. In the case of $C_1 = C_2 = 1$, the parallel curve $\gamma_{(a,b)}$ is given by

$$\begin{aligned}
\gamma_{(a,b)}(t) = \Bigg(&\cos^3 t + a\left(-\sin t \cos \frac{4}{5}t + \frac{4}{5}\cos t \sin \frac{4}{5}t\right) + b\left(-\sin t \sin \frac{4}{5}t - \frac{4}{5}\cos t \cos \frac{4}{5}t\right), \\
&\sin^3 t + a\left(-\cos t \cos \frac{4}{5}t - \frac{4}{5}\sin t \sin \frac{4}{5}t\right) + b\left(-\cos t \sin \frac{4}{5}t + \frac{4}{5}\sin t \cos \frac{4}{5}t\right), \\
&\cos 2t - \frac{3}{5}a \sin \frac{4}{5}t + \frac{3}{5}b \cos \frac{4}{5}t\Bigg),
\end{aligned}$$

see Figure 4. The evolute $\mathcal{E}v(\gamma)$ and focal surface $\mathcal{FS}(\gamma)$ of (γ, ν_1, ν_2) see Example 6.1 below.



In the case of $a = 0$. In the case of $a = b$. In the case of $b = 0$.

Figure 4: γ and $\gamma_{(a,b)}$ of 4.5 for $C_1 = C_2 = 1$.

4.3 Evolutes of framed immersions and spheres

For a regular space curve $\gamma : I \to \mathbb{R}^3$ with the Frenet frame $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ and $\tau(t) \neq 0$, we have $(d/dt)E_v(\gamma)(t) = 0$ for all $t \in I$ if and only if there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a positive real number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$ for all $t \in I$, where $S^2(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x} - \boldsymbol{c}| = r\}$ (cf. [3, 7]). We consider relationships between the evolutes of framed immersions and spheres. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) \neq 0$.

Proposition 4.6 *If $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$, then there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a non-negative real number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$ for all $t \in I$.*

Proof. Since $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$, we put $\boldsymbol{c} = \mathcal{E}v(\gamma)(t)$. By equation (2), we have

$$\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t)}{f(t)}\right) = -\left(\frac{\begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t)}{f(t)}\right)\ell(t),$$

$$\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t)}{f(t)}\right) = \left(\frac{\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t)}{f(t)}\right)\ell(t).$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}|\gamma(t) - \boldsymbol{c}|^2 &= 2\left(\frac{\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t)}{f(t)}\right)\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t)}{f(t)}\right) \\ &\quad + 2\left(\frac{\begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t)}{f(t)}\right)\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t)}{f(t)}\right) \\ &= 0\end{aligned}$$

for all $t \in I$. Hence there exists a non-negative real number $r = |\gamma(t) - \boldsymbol{c}|$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$ for all $t \in I$. □

Lemma 4.7 *$(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$ if and only if there exist functions $u, v : I \to \mathbb{R}$ and a constant $\boldsymbol{c} \in \mathbb{R}^3$ such that $\gamma(t) - \boldsymbol{c} = u(t)\nu_1(t) + v(t)\nu_2(t)$ for all $t \in I$.*

Proof. If $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$, then we can write $\gamma(t) - \boldsymbol{c} = u(t)\nu_1(t) + v(t)\nu_2(t)$, where $\boldsymbol{c} = \mathcal{E}v(\gamma)(t)$,

$$u(t) = \frac{\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t)}{f(t)}, \quad v(t) = -\frac{\begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t)}{f(t)}.$$

Conversely, there exist functions $u, v : I \to \mathbb{R}$ and a constant $\boldsymbol{c} \in \mathbb{R}^3$ such that $\gamma(t) - \boldsymbol{c} = u(t)\nu_1(t) + v(t)\nu_2(t)$ for all $t \in I$. Taking the derivative of the both side, we have

$$\alpha(t)\boldsymbol{\mu}(t) = (\dot{u}(t) - v(t)\ell(t))\nu_1(t) + (\dot{v}(t) + u(t)\ell(t))\nu_2(t) + (u(t)m(t) + v(t)n(t))\boldsymbol{\mu}(t).$$

Since $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ is an orthonormal frame, we have $\dot{u}(t) - v(t)\ell(t) = 0$, $\dot{v}(t) + u(t)\ell(t) = 0$, $\alpha(t) = u(t)m(t) + v(t)n(t)$ for all $t \in I$. Thus $\dot{\alpha}(t) = \dot{u}(t)m(t) + u(t)\dot{m}(t) + \dot{v}(t)n(t) + v(t)\dot{n}(t)$. By a direct calculation, we have

$$\begin{vmatrix} \alpha(t) & n(t) \\ \dot{\alpha}(t) & \dot{n}(t) \end{vmatrix} + \alpha(t)\ell(t)m(t) = u(t)f(t), \quad \begin{vmatrix} \alpha(t) & m(t) \\ \dot{\alpha}(t) & \dot{m}(t) \end{vmatrix} - \alpha(t)\ell(t)n(t) = -v(t)f(t).$$

Hence

$$\alpha_{\mathcal{E}v}(t) = \frac{1}{\sqrt{m^2(t)+n^2(t)}}\left(n(t)(\dot{u}(t)-v(t)\ell(t)) - m(t)(\dot{v}(t)+u(t)\ell(t))\right) = 0$$

for all $t \in I$. It follows that $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$. □

Proposition 4.8 *Suppose that the set of regular points of γ is dense in I. Then $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$ if and only if there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a positive real number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$ for all $t \in I$.*

Proof. Suppose that $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$. By Proposition 4.6, there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a non-negative real number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$. If $r = 0$, then $\gamma(t) = \boldsymbol{c}$ and hence $\dot{\gamma}(t) = 0$ for all $t \in I$. Since the set of regular points of γ is dense in I, the case does not occur. It follows that r is positive.

Conversely, suppose that there exist a constant vector $\boldsymbol{c} \in \mathbb{R}^3$ and a positive real number $r \in \mathbb{R}$ such that $\gamma(t) \in S^2(\boldsymbol{c}, r)$. By the assumption, we have $|\gamma(t) - \boldsymbol{c}|^2 = r^2$. Taking the derivative of the both side, we have $\alpha(t)\boldsymbol{\mu}(t) \cdot (\gamma(t) - \boldsymbol{c}) = 0$. Since the set of regular points of γ is dense in I, we have $\boldsymbol{\mu}(t) \cdot (\gamma(t) - \boldsymbol{c}) = 0$ for all $t \in I$. Then there exist functions $u, v : I \to \mathbb{R}$ and a constant $\boldsymbol{c} \in \mathbb{R}^3$ such that $\gamma(t) - \boldsymbol{c} = u(t)\nu_1(t) + v(t)\nu_2(t)$ for all $t \in I$. By Lemma 4.7, we have $(d/dt)\mathcal{E}v(\gamma)(t) = 0$ for all $t \in I$. □

4.4 Contact between framed immersions and evolutes

We consider contact between framed immersions and its evolutes. We recall the notion of the contact between framed curves, see [8]. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta; t \mapsto (\gamma(t), \nu_1(t), \nu_2(t))$ and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : \widetilde{I} \to \mathbb{R}^3 \times \Delta; u \mapsto (\widetilde{\gamma}(u), \widetilde{\nu}_1(u), \widetilde{\nu}_2(u))$ be framed curves with the curvatures $(\ell(t), m(t), n(t), \alpha(t))$ and $(\widetilde{\ell}(u), \widetilde{m}(u), \widetilde{n}(u), \widetilde{\alpha}(u))$, respectively. Let k be a natural number. We denote $\mathcal{F}(t) = (\ell(t), m(t), n(t), \alpha(t))$ and $\widetilde{\mathcal{F}}(u) = (\widetilde{\ell}(u), \widetilde{m}(u), \widetilde{n}(u), \widetilde{\alpha}(u))$ for convenience. We say that (γ, ν_1, ν_2) *and* $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *have k-th order contact at* $t = t_0$, $u = u_0$ if

$$\frac{d^i}{dt^i}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d^i}{du^i}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0), \ \frac{d^k}{dt^k}(\gamma, \nu_1, \nu_2)(t_0) \neq \frac{d^k}{du^k}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0),$$

for $i = 0, 1, ..., k-1$. Moreover, we say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least k-th order contact at $t = t_0$, $u = u_0$ if

$$\frac{d^i}{dt^i}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d^i}{du^i}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0),$$

for $i = 0, 1, ..., k-1$.

In general, we may assume that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least first order contact at any point $t = t_0$, $u = u_0$, up to congruence as framed curves.

Theorem 4.9 ([8]) *If (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$ then*

$$\frac{d^i}{dt^i}\mathcal{F}(t_0) = \frac{d^i}{du^i}\widetilde{\mathcal{F}}(u_0), \tag{3}$$

for $i = 0, 1, ..., k-1$. Conversely, if conditions (3) hold, then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$, up to congruence as framed curves.

Proposition 4.10 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta; t \mapsto (\gamma(t), \nu_1(t), \nu_2(t))$ and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : \widetilde{I} \to \mathbb{R}^3 \times \Delta; u \mapsto (\widetilde{\gamma}(u), \widetilde{\nu}_1(u), \widetilde{\nu}_2(u))$ be framed immersions with $f(t) \neq 0$ and $\widetilde{f}(u) \neq 0$, respectively. If (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k+2)$-th order contact at $t = t_0$, $u = u_0$ for $k \in \mathbb{N}$, then $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ and $(\mathcal{E}v(\widetilde{\gamma}), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)$ have at least k-th order contact at $t = t_0$, $u = u_0$.*

Proof. We give a proof by using the induction on k. First, we suppose that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least third order contact at $t = t_0$, $u = u_0$. By Theorem 4.9, we have

$$(\gamma, \nu_1, \nu_2)(t_0) = (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0),\ \frac{d}{dt}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0),$$

$$\frac{d^2}{dt^2}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d^2}{du^2}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0),\ \mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0),\ \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\widetilde{\mathcal{F}}(u_0).$$

Therefore we have $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)(t_0) = (\mathcal{E}v(\widetilde{\gamma}), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)(u_0)$.

Suppose that the assumption holds for the case $k+2$. We assume that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k+3)$-th order contact at $t = t_0$, $u = u_0$ and $(d^i/dt^i)(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)(t_0) = (d^i/du^i)(\mathcal{E}v(\widetilde{\gamma}), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)(u_0)$ for $i = 0, 1, ..., k-1$. By Theorem 4.9, we have $(d^i/dt^i)\mathcal{F}(t_0) = (d^i/du^i)\widetilde{\mathcal{F}}(u_0)$ for $i = 0, 1, ..., k+1$. We put

$$(\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{v}_3(t)) = (\boldsymbol{\mu}(t), \boldsymbol{n}_1(t), \boldsymbol{n}_2(t)),\ (\widetilde{\boldsymbol{v}}_1(u), \widetilde{\boldsymbol{v}}_2(u), \widetilde{\boldsymbol{v}}_3(u)) = (\widetilde{\boldsymbol{\mu}}(u), \widetilde{\boldsymbol{n}}_1(u), \widetilde{\boldsymbol{n}}_2(u))$$

and $\mathcal{F}_{\mathcal{E}v}(t) = (L(t), M(t), 0, \alpha_{\mathcal{E}v}(t))$, $\widetilde{\mathcal{F}}_{\mathcal{E}v}(u) = (\widetilde{L}(u), \widetilde{M}(u), 0, \widetilde{\alpha}_{\mathcal{E}v}(u))$ for convenience. It follows that

$$\begin{aligned}
\frac{d^k}{dt^k}\mathcal{E}v(\gamma)(t) &= \left(\frac{d^{k-1}}{dt^{k-1}}\alpha_{\mathcal{E}v}(t)\right)\boldsymbol{v}_3(t) + \sum_{i=1}^{3} f_i\left(\mathcal{F}_{\mathcal{E}v}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}_{\mathcal{E}v}(t)\right)\boldsymbol{v}_i(t),\\
\frac{d^k}{dt^k}\boldsymbol{v}_1(t) &= -\left(\frac{d^{k-1}}{dt^{k-1}}M(t)\right)\boldsymbol{v}_2(t) + \sum_{i=1}^{3} g_i\left(\mathcal{F}_{\mathcal{E}v}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}_{\mathcal{E}v}(t)\right)\boldsymbol{v}_i(t),\\
\frac{d^k}{dt^k}\boldsymbol{v}_2(t) &= \left(\frac{d^{k-1}}{dt^{k-1}}M(t)\right)\boldsymbol{v}_1(t) + \left(\frac{d^{k-1}}{dt^{k-1}}L(t)\right)\boldsymbol{v}_3(t)\\
&\quad + \sum_{i=1}^{3} h_i\left(\mathcal{F}_{\mathcal{E}v}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}_{\mathcal{E}v}(t)\right)\boldsymbol{v}_i(t)
\end{aligned}$$

for some smooth functions f_i, g_i and h_i $(i = 1, 2, 3)$. Moreover,

$$\begin{aligned}
\frac{d^j}{dt^j}M(t) &= \sigma_j\left((m,n)(t), \ldots, \frac{d^j}{dt^j}(m,n)(t)\right),\\
\frac{d^j}{dt^j}L(t) &= \phi_j\left(\mathcal{F}(t), \ldots, \frac{d^j}{dt^j}\mathcal{F}(t), \frac{d^{j+1}}{dt^{j+1}}(m,n)(t)\right),\\
\frac{d^j}{dt^j}\alpha_{\mathcal{E}v}(t) &= \psi_j\left(\mathcal{F}(t), \ldots, \frac{d^{j+1}}{dt^{j+1}}\mathcal{F}(t), \frac{d^{j+2}}{dt^{j+2}}(m,n,\alpha)(t)\right)
\end{aligned}$$

for some smooth functions σ_j, ϕ_j, and ψ_j for $j = 0, 1, \ldots, k-1$. By the same calculations, we

have

$$\frac{d^k}{du^k}\mathcal{E}v(\widetilde{\gamma})(u) = \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{\alpha}_{\mathcal{E}v}(u)\right)\widetilde{\boldsymbol{v}}_3(u) + \sum_{i=1}^{3} f_i\left(\widetilde{\mathcal{F}}_{\mathcal{E}v}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}_{\mathcal{E}v}(u)\right)\widetilde{\boldsymbol{v}}_i(u),$$

$$\frac{d^k}{du^k}\widetilde{\boldsymbol{v}}_1(u) = -\left(\frac{d^{k-1}}{du^{k-1}}\widetilde{M}(u)\right)\widetilde{\boldsymbol{v}}_2(u) + \sum_{i=1}^{3} g_i\left(\widetilde{\mathcal{F}}_{\mathcal{E}v}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}_{\mathcal{E}v}(u)\right)\widetilde{\boldsymbol{v}}_i(u),$$

$$\begin{aligned}\frac{d^k}{du^k}\widetilde{\boldsymbol{v}}_2(u) &= \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{M}(u)\right)\widetilde{\boldsymbol{v}}_1(u) + \left(\frac{d^{k-1}}{du^{k-1}}\widetilde{L}(u)\right)\widetilde{\boldsymbol{v}}_3(u)\\ &\quad + \sum_{i=1}^{3} h_i\left(\widetilde{\mathcal{F}}_{\mathcal{E}v}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}_{\mathcal{E}v}(u)\right)\widetilde{\boldsymbol{v}}_i(u)\end{aligned}$$

and

$$\begin{aligned}\frac{d^j}{du^j}\widetilde{M}(u) &= \sigma_j\left((\widetilde{m},\widetilde{n})(u), \dots, \frac{d^j}{du^j}(\widetilde{m},\widetilde{n})(u)\right),\\ \frac{d^j}{du^j}\widetilde{L}(u) &= \phi_j\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^j}{du^j}\widetilde{\mathcal{F}}(u), \frac{d^{j+1}}{du^{j+1}}(\widetilde{m},\widetilde{n})(u)\right),\\ \frac{d^j}{du^j}\widetilde{\alpha}_{\mathcal{E}v}(u) &= \psi_j\left(\widetilde{\mathcal{F}}(u), \dots, \frac{d^{j+1}}{du^{j+1}}\widetilde{\mathcal{F}}(u), \frac{d^{j+2}}{du^{j+2}}(\widetilde{m},\widetilde{n},\widetilde{\alpha})(u)\right).\end{aligned}$$

It follows that $(d^k/dt^k)(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)(t_0) = (d^k/du^k)(\mathcal{E}v(\widetilde{\gamma}), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)(u_0)$.

Therefore, $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ and $(\mathcal{E}v(\widetilde{\gamma}), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)$ have at least $(k+1)$-th order contact at $t = t_0$, $u = u_0$. By the induction, we have the result. □

The following result is a different property between evolutes of framed immersions and evolutes of Legendre immersions (cf. [6]).

Proposition 4.11 *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) = L(t)M^2(t) \neq 0$, where $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ is the Frenet type frame. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ are not congruent as framed curves.*

Proof. If $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ are congruent as framed curves, then $(L, M, 0, \alpha) = (-M, 0, L, \alpha_{\mathcal{E}v})$ for all $t \in I$ since Theorem 2.5. It follows that $M(t) = L(t) = 0$ for all $t \in I$. By the assumption $f(t) \neq 0$, the case does not occur. □

Proposition 4.12 *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) = L(t)M^2(t) \neq 0$, where $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ is the Frenet type frame. Then we have the following:*

(1) *There exists a rotated frame $\{\overline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{n}}_1(t), \boldsymbol{n}_2(t)\}$ along $\mathcal{E}v(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v(\gamma), \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{n}}_1)$ are congruent as framed immersions if and only if $L(t) = -M(t)$ and $\alpha(t) = \alpha_{\mathcal{E}v}(t)$ for all $t \in I$.*

(2) *There exists a reflected frame $\{\widetilde{\boldsymbol{\mu}}(t), \widetilde{\boldsymbol{n}}_1(t), -\boldsymbol{n}_2(t)\}$ along $\mathcal{E}v(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v(\gamma), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)$ are congruent as framed immersions if and only if $L(t) = M(t)$ and $\alpha(t) = -\alpha_{\mathcal{E}v}(t)$ for all $t \in I$.*

Proof. Suppose that there exists a rotated frame $\{\overline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{n}}_1(t), \boldsymbol{n}_2(t)\}$ along $\mathcal{E}v(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v(\gamma), \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{n}}_1)$ are congruent as framed curves. By Theorem 2.5 and Proposition 3.5, we have

$$(L(t), M(t), 0, \alpha(t)) = \Big(-M(t) - \dot{\theta}(t), -L(t)\sin\theta(t), L(t)\cos\theta(t), \alpha_{\mathcal{E}v}(t)\Big)$$

for all $t \in I$. Since $L(t)\cos\theta(t) = 0$ and $L(t) \neq 0$ for all $t \in I$, we have $\cos\theta(t) = 0$ and $\dot{\theta}(t) = 0$. If $\sin\theta(t) = -1$, then $L(t) = -M(t)$ and $M(t) = L(t)$. It follows that $L(t) = M(t) = 0$. Since $f(t) \neq 0$, the case does not occur. It follows that $\sin\theta(t) = 1$. Hence, we have $L(t) = -M(t)$ and $\alpha(t) = \alpha_{\mathcal{E}v}(t)$ for all $t \in I$.

Conversely, suppose that $L(t) = -M(t)$ and $\alpha(t) = \alpha_{\mathcal{E}v}(t)$ for all $t \in I$. We take a rotated frame $\{\overline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{n}}_1(t), \boldsymbol{n}_2(t)\}$ along $\mathcal{E}v(\gamma)(t)$ by $\theta(t) = \pi/2$. Then, the curvature of $(\mathcal{E}v(\gamma), \overline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{n}}_1(t))$ is given by $(-M(t), -L(t), 0, \alpha_{\mathcal{E}v}(t))$. By Theorem 2.5, $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v(\gamma), \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{n}}_1)$ are congruent as framed curves. Therefore the assertion (1) holds.

By the similar arguments to the above, we have the assertion (2). □

5 The k-th evolutes of framed immersions

In this section, we consider repeated evolutes of framed immersions. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with the curvature (ℓ, m, n, α) and $f(t) \neq 0$. By Proposition 3.5, we have $f_{\mathcal{E}v}(t) = -L^2(t)M(t) \neq 0$, where $M(t) = \sqrt{m^2(t) + n^2(t)}$, $L(t) = f(t)/(m^2(t) + n^2(t))$. Then the evolute $\mathcal{E}v(\gamma)$ has the evolute $\mathcal{E}v(\mathcal{E}v(\gamma))(t)$. We denote $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$, $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$,

$$(\ell_1(t), m_1(t), n_1(t), \alpha_1(t)) = (-M(t), 0, L(t), \alpha_{\mathcal{E}v}(t))$$

and

$$\{\nu_1^1(t), \nu_2^1(t), \boldsymbol{\mu}^1(t)\} = \{\boldsymbol{\mu}(t), \boldsymbol{n}_1(t), \boldsymbol{n}_2(t)\}$$

for convenience. We give the form of the k-th evolute of the framed immersion, where k is a natural number greater than or equal to 2. We define $\mathcal{E}v^k(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{k-1}(\gamma))(t)$ and

$$\nu_1^k(t) = \begin{cases} \nu_1^1(t) & (k : \text{odd}) \\ \boldsymbol{\mu}^1(t) & (k : \text{even}) \end{cases}, \quad \nu_2^k(t) = \begin{cases} \nu_2^1(t) & (k : \text{odd}) \\ -\nu_2^1(t) & (k : \text{even}) \end{cases}, \quad \boldsymbol{\mu}^k(t) = \nu_1^k(t) \times \nu_2^k(t),$$

$$\ell_k(t) = \begin{cases} \ell_1(t) & (k : \text{odd}) \\ n_1(t) & (k : \text{even}) \end{cases}, \quad m_k(t) = 0, \quad n_k(t) = \begin{cases} n_1(t) & (k : \text{odd}) \\ \ell_1(t) & (k : \text{even}) \end{cases},$$

$$\alpha_k(t) = -\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha_{k-1}(t) & n_{k-1}(t) \\ \dot{\alpha}_{k-1}(t) & \dot{n}_{k-1}(t) \end{vmatrix}}{\ell_{k-1}(t)n_{k-1}^2(t)}\right) + \frac{\alpha_{k-1}(t)\ell_{k-1}(t)}{n_{k-1}(t)}$$

inductively.

Theorem 5.1 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) \neq 0$ and $k \geq 2$. Then $(\mathcal{E}v^k(\gamma), \nu_1^k, \nu_2^k) : I \to \mathbb{R}^3 \times \Delta$ is a framed immersion with the curvature $(\ell_k, m_k, n_k, \alpha_k)$, where the k-th evolute of the framed immersion is given by*

$$\mathcal{E}v^k(\gamma)(t) = \mathcal{E}v^{k-1}(\gamma)(t) - \frac{\begin{vmatrix} \alpha_{k-1}(t) & n_{k-1}(t) \\ \dot{\alpha}_{k-1}(t) & \dot{n}_{k-1}(t) \end{vmatrix}}{\ell_{k-1}(t)n_{k-1}^2(t)}\nu_1^{k-1}(t) - \frac{\alpha_{k-1}(t)}{n_{k-1}(t)}\nu_2^{k-1}(t).$$

Proof. We give a proof by using the induction on k. First we consider the case of $k=2$. By the definition of the evolute,

$$\mathcal{E}v^2(\gamma)(t) = \mathcal{E}v^1(\gamma)(t) - \frac{\begin{vmatrix} \alpha_1(t) & n_1(t) \\ \dot{\alpha}_1(t) & \dot{n}_1(t) \end{vmatrix}}{\ell_1(t)n_1^2(t)}\nu_1^1(t) - \frac{\alpha_1(t)}{n_1(t)}\nu_2^1(t).$$

Since

$$\frac{d}{dt}\mathcal{E}v^2(\gamma)(t) = \left(-\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha_1(t) & n_1(t) \\ \dot{\alpha}_1(t) & \dot{n}_1(t) \end{vmatrix}}{\ell_1(t)n_1^2(t)}\right) + \frac{\alpha_1(t)\ell_1(t)}{n_1(t)}\right)\nu_1^1(t),$$

we have $\dot{\mathcal{E}v}^2(\gamma)(t)\cdot\boldsymbol{\mu}^1(t) = 0$ and $\dot{\mathcal{E}v}^2(\gamma)(t)\cdot(-\nu_2^1(t)) = 0$. Hence, $(\mathcal{E}v^2(\gamma), \nu_1^2, \nu_2^2)$ is a framed immersion. Moreover, we have $\ell_2(t) = n_1(t)$, $m_2(t) = 0$, $n_2(t) = \ell_1(t)$ and

$$\alpha_2(t) = \left(-\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha_1(t) & n_1(t) \\ \dot{\alpha}_1(t) & \dot{n}_1(t) \end{vmatrix}}{\ell_1(t)n_1^2(t)}\right) + \frac{\alpha_1(t)\ell_1(t)}{n_1(t)}\right).$$

The assertion in the case of $k=2$ holds.

Suppose that the assumption holds for the case of k-th evolute. By the definition,

$$\mathcal{E}v^{k+1}(\gamma)(t) = \mathcal{E}v^k(\gamma)(t) - \frac{\begin{vmatrix} \alpha_k(t) & n_k(t) \\ \dot{\alpha}_k(t) & \dot{n}_k(t) \end{vmatrix}}{\ell_k(t)n_k^2(t)}\nu_1^k(t) - \frac{\alpha_k(t)}{n_k(t)}\nu_2^k(t).$$

Since

$$\frac{d}{dt}\mathcal{E}v^{k+1}(\gamma)(t) = \left(-\frac{d}{dt}\left(\frac{\begin{vmatrix} \alpha_k(t) & n_k(t) \\ \dot{\alpha}_k(t) & \dot{n}_k(t) \end{vmatrix}}{\ell_k(t)n_k^2(t)}\right) + \frac{\alpha_k(t)\ell_k(t)}{n_k(t)}\right)\nu_1^k(t),$$

we have $\dot{\mathcal{E}v}^{k+1}(\gamma)(t)\cdot\boldsymbol{\mu}^k(t) = 0$ and $\dot{\mathcal{E}v}^{k+1}(\gamma)(t)\cdot\big(-\nu_2^k(t)\big) = 0$. Hence $(\mathcal{E}v^{k+1}(\gamma), \nu_1^{k+1}, \nu_2^{k+1})$ is a framed immersion. Moreover, by a direct calculation, the curvature is given by $\ell_{k+1}(t) = n_k(t)$, $m_{k+1}(t) = 0$, $n_{k+1}(t) = \ell_k(t)$ and $\alpha_{k+1}(t)$. □

Proposition 5.2 *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) = L(t)M^2(t) \neq 0$, where $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ is the Frenet type frame. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \nu_1^k, \nu_2^k)$ are not congruent as framed curves.*

Proof. In the case when k is an odd number, if $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \boldsymbol{\mu}, \boldsymbol{n}_1)$ are congruent as framed curves, then we have $L(t) = -M(t)$ and $M(t) = 0$ for all $t \in I$, that is, $L(t) = M(t) = 0$ for all $t \in I$. By the assumption $f(t) \neq 0$, the case does not occur. On the other hand, in the case when k is an even number, if $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \boldsymbol{n}_2, -\boldsymbol{n}_1)$ are congruent as framed curves, then we also have $M(t) = 0$ for all $t \in I$. By the assumption $f(t) \neq 0$, the case does not occur. □

Proposition 5.3 *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed immersion with $f(t) = L(t)M^2(t) \neq 0$, where $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ is the Frenet type frame.*

(1) *In case when k is an odd number, then we have the following:*

(a) *There exists a rotated frame $\{\overline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{n}}_1(t), \boldsymbol{n}_2(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{n}}_1)$ are congruent as framed immersions if and only if $L(t) = -M(t)$ and $\alpha(t) = \alpha_k(t)$ for all $t \in I$.*

(b) *There exists a reflected frame $\{\widetilde{\boldsymbol{\mu}}(t), \widetilde{\boldsymbol{n}}_1(t), -\boldsymbol{n}_2(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{n}}_1)$ are congruent as framed immersions if and only if $L(t) = M(t)$ and $\alpha(t) = -\alpha_k(t)$ for all $t \in I$.*

(2) *In case when k is an even number, then we have the following:*

(c) *There exists a rotated frame $\{\overline{\boldsymbol{n}}_2(t), -\overline{\boldsymbol{n}}_1(t), \boldsymbol{\mu}(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \overline{\boldsymbol{n}}_2, -\overline{\boldsymbol{n}}_1)$ are congruent as framed immersions if and only if $\alpha(t) = \alpha_k(t)$ for all $t \in I$,*

(d) *There does not exist a reflected frame $\{\widetilde{\boldsymbol{n}}_2(t), -\widetilde{\boldsymbol{n}}_1(t), -\boldsymbol{\mu}(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \widetilde{\boldsymbol{n}}_2, -\widetilde{\boldsymbol{n}}_1)$ are congruent as framed immersions.*

Proof. By the similar argument to proof of Proposition 4.12, we have the assertion (1). Here we prove the assertion (2).

(c) Suppose that there exists a rotated frame $\{\overline{\boldsymbol{n}}_2(t), -\overline{\boldsymbol{n}}_1(t), \boldsymbol{\mu}(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \overline{\boldsymbol{n}}_2, -\overline{\boldsymbol{n}}_1)$ are congruent as framed curves. By Theorem 2.5,

$$(L(t), M(t), 0, \alpha(t)) = \Big(L(t) - \dot{\theta}(t), M(t)\sin\theta(t), -M(t)\cos\theta(t), \alpha_k(t)\Big)$$

for all $t \in I$. Hence, we have $\theta(t) = \pi/2$ and $\alpha(t) = \alpha_k(t)$ for all $t \in I$. Conversely, suppose that $\alpha(t) = \alpha_k(t)$ for all $t \in I$. If we take a rotated frame $\{\overline{\boldsymbol{n}}_2(t), -\overline{\boldsymbol{n}}_1(t), \boldsymbol{\mu}(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t) = \pi/2$, then the curvature of $(\mathcal{E}v^k(\gamma), \overline{\boldsymbol{n}}_2, -\overline{\boldsymbol{n}}_1)$ is given by $(L(t), M(t), 0, \alpha_k(t))$. Therefore the assertion (c) holds.

(d) If there exists a reflected frame $\{\widetilde{\boldsymbol{n}}_2(t), -\widetilde{\boldsymbol{n}}_1(t), -\boldsymbol{\mu}(t)\}$ along $\mathcal{E}v^k(\gamma)(t)$ by $\theta(t)$ such that $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\mathcal{E}v^k(\gamma), \widetilde{\boldsymbol{n}}_2, -\widetilde{\boldsymbol{n}}_1)$ are congruent as framed curves, then we have

$$(L(t), M(t), 0, \alpha(t)) = \Big(-L(t) + \dot{\theta}(t), -M(t)\sin\theta(t), -M(t)\cos\theta(t), \alpha_k(t)\Big)$$

for all $t \in I$. Hence, we have $\theta(t) = 3\pi/2$ and $L(t) = -L(t)$ for all $t \in I$. It follows that $L(t) = 0$. By the assumption $f(t) \neq 0$, the case does not occur. □

6 Examples

We give examples to understand the phenomena for evolutes and focal surfaces of framed immersions.

Example 6.1 Let $(\gamma, \nu_1, \nu_2) : [0, 2\pi) \to \mathbb{R}^3$ be

$$\begin{aligned}
\gamma(t) &= (C_1 \cos^3 t, C_1 \sin^3 t, C_2 \cos 2t), \\
\nu_1(t) &= (-\sin t, -\cos t, 0), \\
\nu_2(t) &= \frac{1}{\sqrt{9C_1^2 + 16C_2^2}}(-4C_2 \cos t, 4C_2 \sin t, 3C_1),
\end{aligned}$$

where C_1 and C_2 are non-zero constants. By Example 4.5, we have

$$f(t) = \frac{36C_1^2 C_2}{(9C_1^2 + 16C_2^2)^{\frac{3}{2}}} \neq 0.$$

The evolute and the second evolute of the astroid are given by

$$\mathcal{E}v(\gamma)(t) = \left(\frac{1}{3C_1} \left(12C_1^2 + 16C_2^2\right) \cos^3 t, \frac{1}{3C_1} \left(12C_1^2 + 16C_2^2\right) \sin^3 t, -\frac{1}{4C_2} \left(9C_1^2 + 12C_2^2\right) \cos 2t \right)$$

and

$$\begin{aligned}
\mathcal{E}v^2(\gamma)(t) = \Bigg(& \frac{C_1}{12C_1^2 + 16C_2^2} \left(\frac{4}{3C_1^2} \left(12C_1^2 + 16C_2^2\right)^2 + \frac{1}{C_2^2} \left(9C_1^2 + 12C_2^2\right)^2 \right) \cos^3 t, \\
& \frac{C_1}{12C_1^2 + 16C_2^2} \left(\frac{4}{3C_1^2} \left(12C_1^2 + 16C_2^2\right)^2 + \frac{1}{C_2^2} \left(9C_1^2 + 12C_2^2\right)^2 \right) \sin^3 t, \\
& \frac{C_1}{9C_1^2 + 12C_2^2} \left(\frac{1}{C_1^2} \left(12C_1^2 + 16C_2^2\right)^2 + \frac{3}{4C_2^2} \left(9C_1^2 + 12C_2^2\right)^2 \right) \cos 2t \Bigg).
\end{aligned}$$

By the above calculation, we can be confirmed that the evolute of an astroid is also an astroid. In the case of $C_1 = C_2 = 1$, the astroid (blue curve) and the evolute (red curve), see Figure 5. Moreover, the focal surface of the astroid is given by

$$\mathcal{FS}(\gamma)(t, \lambda) = \left(\cos t \left(\frac{4\lambda}{5} + \cos^2 t + \frac{25}{3} \sin^2 t \right), \frac{\sin t}{15} \left(70 - 12\lambda + 55 \cos 2t\right), -\frac{3}{5}\lambda + \cos 2t \right),$$

see Figure 6.

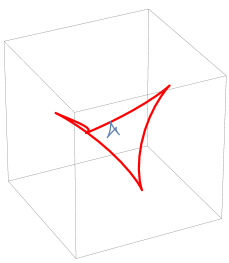


Figure 5: γ and $\mathcal{E}v(\gamma)$

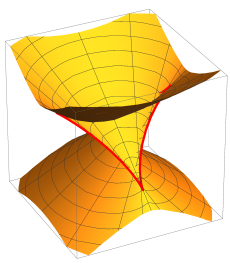


Figure 6: $\mathcal{E}v(\gamma)$ and $\mathcal{FS}(\gamma)$

Example 6.2 Let $(\gamma, \nu_1, \nu_2) : [0, 2\pi) \to S^2 \times \Delta \subset \mathbb{R}^3 \times \Delta$ be

$$\gamma(t) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{\sqrt{3}}{2}\cos t \right),$$
$$\nu_1(t) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{\sqrt{3}}{2}\cos t \right),$$
$$\nu_2(t) = \left(\frac{3}{4}\sin t + \frac{1}{4}\sin 3t, -\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, -\frac{\sqrt{3}}{2}\sin t \right).$$

Then (γ, ν_1, ν_2) is a framed curve (cf. [11]). We call γ a spherical nephroid. By definition, $\boldsymbol{\mu} : [0, 2\pi) \to S^2$ is given by $\boldsymbol{\mu}(t) = ((\sqrt{3}/2)\cos 2t, (\sqrt{3}/2)\sin 2t, -1/2)$. By a direct calculation, the curvature of (γ, ν_1, ν_2) is given by $\ell(t) = 0$, $m(t) = \sqrt{3}\sin t$, $n(t) = \sqrt{3}\cos t$, $\alpha(t) = \sqrt{3}\sin t$. It follows that $f(t) = 3 \neq 0$. The evolute of the spherical nephroid is a constant point $(0, 0, 0)$, see Proposition 4.8. Moreover, k-th evolutes ($k \geq 2$) are also the constant point $(0, 0, 0)$. The spherical nephroid and its evolute, see Figure 7. Moreover, the focal surface of spherical nephroid is given by

$$\mathcal{FS}(\gamma)(t, \lambda) = \left(\frac{\cos 2t}{2}(\lambda + \cos t), \frac{\sin 2t}{2}(\lambda + \cos t), \frac{\sqrt{3}}{2}(\lambda + \cos t) \right),$$

see Figure 8.

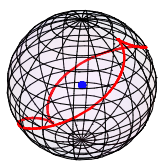


Figure 7: γ and $\mathcal{E}v(\gamma)$

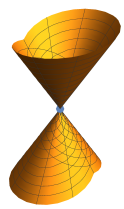


Figure 8: $\mathcal{E}v(\gamma)$ and $\mathcal{FS}(\gamma)$

Example 6.3 Let $(\gamma, \nu_1, \nu_2) : \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be

$$\gamma(t) = \left((t+1)\sin t + \cos t, -(t+1)\cos t + \sin t, \frac{1}{2}t^2 + t \right),$$
$$\nu_1(t) = (\sin t, -\cos t, 0),$$
$$\nu_2(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, -1).$$

Since $\dot{\gamma}(t) = (t+1)(\cos t, \sin t, 1)$, we have $\dot{\gamma}(t) \cdot \nu_1(t) = 0$ and $\dot{\gamma}(t) \cdot \nu_2(t) = 0$ for all $t \in I$. Therefore, (γ, ν_1, ν_2) is a framed curve. By definition, $\boldsymbol{\mu} : I \to S^2$ is given by $\boldsymbol{\mu}(t) = (1/\sqrt{2})(\cos t, \sin t, 1)$. By a direct calculation, the curvature of (γ, ν_1, ν_2) is given by $\ell(t) =$

$1/\sqrt{2}$, $m(t) = 1/\sqrt{2}$, $n(t) = 0$, $\alpha(t) = \sqrt{2}t + \sqrt{2}$. Hence (γ, ν_1, ν_2) is a framed immersion with $f(t) = \sqrt{2}/4 \neq 0$. The evolute is given by

$$\mathcal{E}v(\gamma)(t) = \left(-(t+1)\sin t - \cos t, (t+1)\cos t - \sin t, \frac{1}{2}t^2 + t + 2\right).$$

Moreover, $(\mathcal{E}v(\gamma), \boldsymbol{\mu}, \nu_1)$ is a framed immersion with the curvature $(-1/\sqrt{2}, 0, 1/\sqrt{2}, -\sqrt{2}t - \sqrt{2})$. By Proposition 4.12, there exists a reflected frame $\{-\nu_1(t), -\boldsymbol{\mu}(t), -\nu_2(t)\}$ along $\mathcal{E}v(\gamma)$ such that (γ, ν_1, ν_2) and $(\mathcal{E}v(\gamma), -\nu_1(t), -\boldsymbol{\mu}(t))$ are congruent as framed curves, see Figures 9 and 10. In fact, if we take

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3), \ \boldsymbol{a} = (0, 0, 2) \in \mathbb{R}^3,$$

then we have $\mathcal{E}v(\gamma)(t) = A(\gamma(t)) + \boldsymbol{a}$, $-\nu_1(t) = A(\nu_1(t))$, $-\boldsymbol{\mu}(t) = A(\nu_2(t))$ for all $t \in I$.

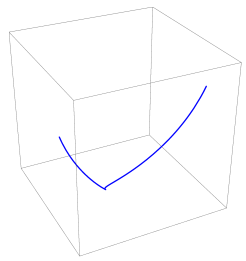


Figure 9: The original curve γ

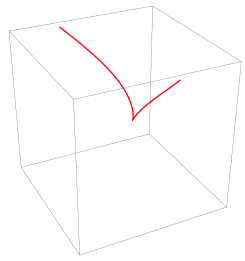


Figure 10: The evolute $\mathcal{E}v(\gamma)$

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