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# AN ENERGETIC VARIATIONAL APPROACH FOR NONLINEAR DIFFUSION EQUATIONS IN MOVING THIN DOMAINS

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ABSTRACT. This paper concerns the processes of nonlinear diffusion in a moving domain which lies on a moving closed surface. The nonlinear diffusion equations and corresponding energy identities are derived by regarding the moving surface as a thin width (thickness) limit of moving thin domains, for which suitable boundary conditions are imposed to insure that there is no exchange of mass between the thin domains and the environments. We also employ an energetic variational approach to derive these nonlinear diffusion equations. Most of all, we show that these nonlinear energetic variational procedures can commute with the passing to the zero width limits.

## 1. INTRODUCTION

In this paper, we are interested in deriving diffusion equations on a moving surface, by regarding it as a limit of the problem in a moving thin domain with its width tends to zero.

Let us begin with an equation of the conservation of mass  $\rho$  with velocity  $u$  in a moving domain  $\Omega(t)$  in  $\mathbb{R}^n$  of the form

$$(1.1) \quad \partial_t \rho + \operatorname{div}(\rho u) = 0,$$

which represents the local conservation of mass. Considering the situation that there is no exchange of mass on the boundary, i.e.

$$(1.2) \quad u \cdot \nu_\Omega = V_\Omega^N$$

on the boundary, where  $V_\Omega^N$  is the normal velocity of the boundary of the moving domain  $\Omega(t)$  in the direction of the outward normal vector field  $\nu_\Omega$  of the boundary. Similar conservation law of mass  $\eta$  with velocity  $v$  on a moving surface  $\Gamma(t)$  can be derived from the local conservation of mass. It turns out (see Section 3) that, when the normal component of  $v$  is equal to the outward normal velocity  $V_\Gamma^N$  of the moving surface  $\Gamma(t)$ , the resulting equation is of the form:

$$(1.3) \quad \partial^\circ \eta - V_\Gamma^N H \eta + \operatorname{div}_\Gamma(\eta v^T) = 0,$$

where  $\partial^\circ = \partial_t + V_\Gamma^N \nu_\Gamma$  is the normal time derivative,  $\nu_\Gamma$  is the outward normal vector field of  $\Gamma(t)$ ,  $H$  is the  $(n - 1)$  times mean curvature of  $\Gamma(t)$ ,  $\operatorname{div}_\Gamma$  is the surface divergence operator on  $\Gamma(t)$ , and  $v^T$  is a tangential vector field satisfying  $v = V_\Gamma^N \nu_\Gamma + v^T$ . Note that this equation is obtained as the zero width limit of the corresponding equation (1.1) in a moving thin domain  $\Omega_\varepsilon(t)$  defined as the set of all points in  $\mathbb{R}^n$  with distance less than  $\varepsilon$  from  $\Gamma(t)$  (see Remark 4.2).

The conventional diffusion equations, or even the porous-media equations, can be viewed as the combination of incompressible fluids with the damping in the form of Darcy's law. Take the usual Darcy's law for the velocity  $u$ :

$$(1.4) \quad -\rho u = \nabla p(\rho)$$

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in the moving thin domain  $\Omega_\varepsilon(t)$ , where  $p$  is the pressure, then we can prove (see Theorem 4.1) that the zero width limit of the diffusion equations (1.1) with (1.4) yields diffusion equations on the moving surface  $\Gamma(t)$ : (1.3) and Darcy's law

$$(1.5) \quad -\eta v^T = \nabla_\Gamma p(\eta).$$

Here  $v^T$  is the tangential component of the velocity  $v$  and  $\nabla_\Gamma$  is the tangential gradient on  $\Gamma(t)$ .

The diffusion equations (1.1), (1.4) and (1.3), (1.5) possess specific energy identities. It can be easily proven (see Section 5) that for  $\rho$  and  $u$  satisfying (1.1) and (1.4) the energy identity

$$(1.6) \quad \frac{d}{dt} \int_{\Omega(t)} \omega(\rho) dx = - \int_{\Omega(t)} \rho |u|^2 dx - \int_{\partial\Omega(t)} p(\rho) V_\Omega^N d\mathcal{H}^{n-1}$$

holds. Here  $\omega$  is a function satisfying  $p(\rho) = \omega'(\rho)\rho - \omega(\rho)$ . Similarly, for  $\eta$  and  $v$  satisfying (1.3) and (1.5) we have

$$(1.7) \quad \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} = - \int_{\Gamma(t)} \eta |v^T|^2 d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta) V_\Gamma^N H d\mathcal{H}^{n-1},$$

where  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure. Fortunately, the energy identity (1.7) on the moving surface can be derived as the zero width limit of the energy identity (1.6) in the moving thin domain (see Theorem 5.3).

With the results from this paper, we can also note that the passing of zero width limit commutes with an energetic variational approach originated from the works of Lord Rayleigh [23] and Onsager [15, 16] and developed by Liu and others [3, 11, 24] (see Section 6). In summary, we show that the diagram below is commutative.

$$\begin{array}{ccc}
 & \text{integration} & \\
 & \text{by parts} & \\
 \text{Diffusion equation} & \xrightarrow{\hspace{1cm}} & \text{Energy identity} \\
 \text{in } \Omega_\varepsilon(t) & \xleftarrow{\hspace{1cm}} & \text{in } \Omega_\varepsilon(t) \\
 & \text{energetic} & \\
 & \text{variation} & \\
 \varepsilon \rightarrow 0 & & \varepsilon \rightarrow 0 \\
 \downarrow & & \downarrow \\
 \text{Diffusion equation} & \xrightarrow{\hspace{1cm}} & \text{Energy identity} \\
 \text{in } \Gamma(t) & \xleftarrow{\hspace{1cm}} & \text{on } \Gamma(t) \\
 & \text{energetic} & \\
 & \text{variation} & 
 \end{array}$$

A standard approach for finding the limit of a thin domain problem is a rescaling argument: one transforms a partial differential equation in a thin domain into that in a fixed in width reference domain by the change of variables and then gets a limit equation by assuming that a rescaled solution is independent of variables in thin directions. In our case where the thin domain and the surface both move, one may transform the moving thin domain into a fixed in time and width reference domain. However, it yields tedious calculations because of the geometry of the limit moving surface and it is difficult to bring a limit equation obtained on a stationary reference surface back to an equation on the original moving surface. One other method is to rescale the width of the moving thin domain without fixing time, which is used in [14] to find the limit of the Neumann type problem of the heat equation (equations (1.1), (1.2), and (1.4) with  $p(\rho) = \rho$ ) in moving thin domains. However, it is still complicated and requires a questionable assumption that the boundary condition holds in a middle of the moving thin domain. It is also artificial in the sense that we have to make rescaled solutions constant in the thin direction at an ‘‘appropriate’’

point to derive the limit energy identity and if we take a wrong point then we get a wrong limit (see Remarks 5.5).

To derive a limit with more straightforward calculations, we use another method which comes from an idea for determining coefficients of polynomials of formal expansion with respect to the width parameter  $\varepsilon$ : if  $A_0, A_1, \dots, A_n$  are independent of  $\varepsilon$  and

$$A_0 + \varepsilon A_1 + \dots + \varepsilon^n A_n + O(\varepsilon^{n+1}) = 0$$

for all  $\varepsilon > 0$ , then  $A_0 = A_1 = \dots = A_n = 0$ . We follow this idea but do not rescale the width of the moving thin domain. Since the moving surface  $\Gamma(t)$  admits the normal coordinate system

$$x = \pi(x, t) + d(x, t)\nu_\Gamma(\pi(x, t), t)$$

for  $x \in \Omega_\varepsilon(t)$ , where  $\pi(x, t)$  is the projection onto  $\Gamma(t)$  and  $d(x, t)$  is the signed distance function from  $\Gamma(t)$  increasing in the direction of the normal vector field  $\nu_\Gamma$ , we expand the density  $\rho$  and the velocity  $u$  defined on  $\Omega_\varepsilon(t)$  in powers of the signed distance:

$$\begin{aligned} \rho(x, t) &= \eta(\pi(x, t), t) + d(x, t)\eta^1(\pi(x, t), t) + \dots, \\ u(x, t) &= v(\pi(x, t), t) + d(x, t)v^1(\pi(x, t), t) + \dots. \end{aligned}$$

We differentiate both sides of the above equations and substitute them for equations (1.1), (1.2), and (1.4). Then under suitable assumptions we obtain the limit equations (1.3) and (1.5) as the zeroth order terms of expansions in the signed distance (or  $\varepsilon$ ) of the original equations (1.1), (1.2), and (1.4) (see Section 4). The same idea is valid for derivation of the energy identity (1.7) on the moving surface from that in the moving thin domain (1.6) (see Section 5). To get the limit energy identity we also use integral transformation formulas from surface integrals over the level-set surfaces  $\{x \in \mathbb{R}^n \mid d(x, t) = r\}$  ( $-\varepsilon < r < \varepsilon$ ) into that over the zero level-set surface  $\Gamma(t)$  (see Lemma 5.4).

There is a long history in the study of partial differential equations in thin domains, such as the pioneering work by Hale and Raugel [7, 8], where they investigated damped hyperbolic equations and reaction-diffusion equations in a flat stationary thin domain of the form

$$(1.8) \quad \Omega_\varepsilon = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \omega, 0 < x_n < \varepsilon g(x')\},$$

where  $\omega$  is an open set in  $\mathbb{R}^{n-1}$  and  $g$  is a function on  $\omega$ . There is also a large number of the literature on reaction-diffusion equations in various types of thin domains such as a thin L-shaped domain [9], a moving flat thin domain of the form (1.8) with  $g$  time-dependent [17], and flat and curved thin domains with holes [18, 19, 20] (here a curved thin domain is a thin domain degenerating into a lower dimensional manifold). A main subject in the above literature is to compare the dynamics of equations in thin domains with that of limit equations in their degenerate sets rather than to find the limit equations of the original equations in the thin domains, since their degenerate sets are stationary and thus the rescaling argument works well for finding the limit equations. The Navier–Stokes equations in thin domains has been also studied well [10, 12, 22, 25, 26] since fluid flows in thin domains often appear in natural sciences like the flow of water in a large lake, geophysical flows, etc. Researchers are especially interested in the relation between the smallness of the width of thin domains and the large time behavior of solutions to the Navier–Stokes equations in thin domains. We refer to [21] and therein for other types of thin domains degenerating into stationary sets and mathematical analysis of partial differential equations in such thin domains.

In the case where the degenerate set of a thin domain moves, derivation of the limit of a partial differential equation in the thin domain is more complicated since the geometry of the degenerate set changes as it moves. Such a problem was first considered in [14] where the author derived both formally and rigorously the limit equation of the Neumann type problem of the heat equation (equations (1.1), (1.2), and (1.4) with  $p(\rho) = \rho$ ) in a moving thin domain degenerating into a closed smooth moving surface. He also found that the normal velocity and the mean curvature of the degenerate moving surface affects the limit equation, which is not observed in the case where the degenerate set of a thin domain does not move.

The rest of this paper is organized as follows. In Section 2, we fix some notations on various quantities related to the moving surface. In Section 3, we briefly observe that the transport equations in the moving domain and the moving surface are equivalent to the local mass conservation. In Section 4, we derive the limit equations (1.3) and (1.5) on the moving surface from the diffusion equations (1.1), (1.2), and (1.4) on the moving thin domain by means of expansion in terms of the signed distance. In Section 5, we derive the energy identities (1.6) and (1.7) from corresponding diffusion equations and then show that the energy identity (1.7) on the moving surface is the zero width limit of the energy identity (1.6) on the moving thin domain. In Section 6, we apply an energetic variational approach to the energy identities (1.6) and (1.7) to obtain Darcy's laws (1.4) and (1.5).

## 2. QUANTITIES ON A MOVING SURFACE

We start with several notations for a moving surface. Let  $\Gamma(t)$ ,  $t \in [0, T]$  be a closed (that is, compact and without boundary), connected, oriented and smooth moving surface in  $\mathbb{R}^n$  with  $n \geq 2$ . We write  $\nu_\Gamma(\cdot, t)$  and  $V_\Gamma^N(\cdot, t)$  for the unit outward normal vector field and the scalar outward normal velocity of  $\Gamma(t)$ , respectively (note that to describe the evolution of a closed surface it is sufficient to give the normal velocity). Since the smooth closed surface  $\Gamma(t)$  varies smoothly in time, the principal curvatures  $\kappa_1(\cdot, t), \dots, \kappa_{n-1}(\cdot, t)$  of  $\Gamma(t)$  are bounded uniformly in  $t \in [0, T]$ . Then there exists a constant  $\delta > 0$  independent of  $t$  such that the tubular neighborhood

$$N(t) := \{x \in \mathbb{R}^n \mid \text{dist}(x, \Gamma(t)) < \delta\}$$

of  $\Gamma(t)$  admits the normal coordinate system

$$(2.1) \quad x = \pi(x, t) + d(x, t)\nu_\Gamma(\pi(x, t), t)$$

for each  $x \in N(t)$ , where  $\pi(\cdot, t)$  is the orthogonal projection onto  $\Gamma(t)$  and  $d(\cdot, t)$  is the signed distance function from  $\Gamma(t)$  (see [5, Section 14.6] for example). Here we suppose that  $d(\cdot, t)$  increases along the direction of  $\nu_\Gamma(\cdot, t)$ . Then we have

$$(2.2) \quad \begin{aligned} \nabla d(x, t) &= \nu_\Gamma(\pi(x, t), t), & x \in N(t), \\ \partial_t d(y, t) &= -V_\Gamma^N(y, t), & y \in \Gamma(t). \end{aligned}$$

Moreover, differentiating both sides of

$$d(x, t) = \{x - \pi(x, t)\} \nabla d(x, t), \quad d(\pi(x, t), t) = 0$$

with respect to  $t$  we easily obtain

$$(2.3) \quad \partial_t d(x, t) = \partial_t d(\pi(x, t), t) = -V_\Gamma^N(\pi(x, t), t), \quad x \in N(t).$$

We write  $P_\Gamma := I_n - \nu_\Gamma \otimes \nu_\Gamma$  for the orthogonal projection onto the tangent space of  $\Gamma(t)$ . Here  $I_n$  denotes the identity matrix of size  $n$  and  $a \otimes b = (a_i b_j)_{i,j}$  is the tensor product of two vectors  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  in  $\mathbb{R}^n$ . We define

the tangential gradient of a function  $f$  on  $\Gamma(t)$  and the surface divergence of a (not necessarily tangential) vector field  $F$  on  $\Gamma(t)$  as

$$\nabla_{\Gamma} f := P_{\Gamma} \nabla \bar{f}, \quad \operatorname{div}_{\Gamma} F := \operatorname{tr}[P_{\Gamma} \nabla \bar{F}].$$

Here  $\bar{f}$  and  $\bar{F}$  are the constant extension of  $f$  and  $F$  in the normal direction of  $\Gamma(t)$  given by  $\bar{f}(x) := f(\pi(x, t))$  and  $\bar{F}(x) := F(\pi(x, t))$  for  $x \in N(t)$ . Also,  $\operatorname{tr}[M]$  denotes the trace of a square matrix  $M$  and we use the notation

$$\nabla G = \begin{pmatrix} \partial_1 G_1 & \dots & \partial_1 G_n \\ \vdots & \ddots & \vdots \\ \partial_n G_1 & \dots & \partial_n G_n \end{pmatrix}$$

for the gradient matrix of a vector field  $G = (G_1, \dots, G_n)$ . Note that  $\nabla_{\Gamma} f \cdot \nu_{\Gamma} = 0$  for any function  $f$  on  $\Gamma(t)$ . We also define the  $(n-1)$  times mean curvature  $H$  of  $\Gamma(t)$  as

$$(2.4) \quad H(y, t) := -\operatorname{div}_{\Gamma} \nu_{\Gamma}(y, t)$$

for  $y \in \Gamma(t)$ ,  $t \in [0, T]$ . Note that the mean curvature defined as above is equal to the sum of the principal curvatures:

$$(2.5) \quad H(y, t) = \sum_{i=1}^{n-1} \kappa_i(y, t), \quad y \in \Gamma(t).$$

Finally, we write  $\partial^{\circ} = \partial_t + V_{\Gamma}^N \nu_{\Gamma} \cdot \nabla$  for the normal time derivative (the time derivative along the normal velocity). Note that the formula

$$(2.6) \quad \partial^{\circ} f(y, t) = \frac{d}{dt}(f(\pi(y, t), t)), \quad y \in \Gamma(t)$$

holds for a function  $f(\cdot, t)$  on  $\Gamma(t)$  (see [2, Section 3.4]).

### 3. TRANSPORT EQUATION IN A MOVING DOMAIN AND ON A MOVING SURFACE

In this section we give the transport equation for a scalar quantity in a moving domain and on a moving surface. We use some of the same terminology and techniques as in [13]. We first consider transportation of a scalar quantity in a bounded moving domain  $\Omega(t)$  in  $\mathbb{R}^n$ . Let  $\rho(x, t)$  and  $u(x, t)$  be the density and the velocity field of the scalar quantity at  $x \in \Omega(t)$ , respectively. Our starting point is the local mass conservation

$$(3.1) \quad \frac{d}{dt} \int_{U(t)} \rho dx = 0$$

for any portion  $U(t)$  (relatively open set) of  $\Omega(t)$  moving with velocity  $u(\cdot, t)$ . Since the left-hand side is equal to  $\int_{U(t)} \{\partial_t \rho + \operatorname{div}(\rho u)\} dx$  by the Reynolds transport theorem [6] and the divergence theorem, the condition (3.1) for any  $U(t)$  is equivalent to the transport equation

$$(3.2) \quad \partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \Omega(t), t \in (0, T).$$

To make the total mass  $\int_{\Omega(t)} \rho dx$  conserved, we impose the boundary condition

$$(3.3) \quad u \cdot \nu_{\Omega} = V_{\Omega}^N \quad \text{on } \partial\Omega(t), t \in (0, T),$$

where  $\nu_{\Omega}(\cdot, t)$  and  $V_{\Omega}^N(\cdot, t)$  are the unit outward normal vector field and the scalar outward normal velocity of  $\partial\Omega(t)$ , respectively. The boundary condition (3.3) physically means that the quantity in  $\Omega(t)$  moves along the boundary of  $\Omega(t)$  and it does not go into and out of  $\Omega(t)$ .

Next we give the transport equation for a scalar quantity on a moving surface. Let  $\Gamma(t)$  be a closed, connected, oriented moving surface in  $\mathbb{R}^n$ . As in Section 2, we

write  $\nu_\Gamma(\cdot, t)$  and  $V_\Gamma(\cdot, t)$  for the outward normal vector field and the scalar outward normal velocity of  $\Gamma(t)$ , respectively. Suppose that a scalar quantity on  $\Gamma(t)$  has the density  $\eta(y, t)$  for each  $y \in \Gamma(t)$  and moves with velocity

$$v(y, t) = V_\Gamma(y, t)\nu_\Gamma(y, t) + v^T(y, t), \quad y \in \Gamma(t),$$

where  $v^T(\cdot, t)$  is a given tangential velocity field on  $\Gamma(t)$ . Then its local mass conservation is expressed as

$$(3.4) \quad \int_{U(t)} \eta d\mathcal{H}^{n-1} = 0$$

for any portion  $U(t)$  (relatively open set) of  $\Gamma(t)$  moving with velocity  $v(\cdot, t)$ . The Leibniz formula [4, Lemma 2.2] yields

$$\frac{d}{dt} \int_{U(t)} \eta d\mathcal{H}^{n-1} = \int_{U(t)} \{\partial^\circ \eta - V_\Gamma^N H\eta + \operatorname{div}_\Gamma(\eta v^T)\} d\mathcal{H}^{n-1}.$$

From this formula, the condition (3.4) for any  $U(t)$  is equivalent to

$$(3.5) \quad \partial^\circ \eta - V_\Gamma^N H\eta + \operatorname{div}_\Gamma(\eta v^T) = 0 \quad \text{on } \Gamma(t), t \in (0, T).$$

This is the transport equation on the moving surface  $\Gamma(t)$ .

#### 4. ZERO WIDTH LIMIT FOR NONLINEAR DIFFUSION EQUATIONS

Let us consider nonlinear diffusion of a scalar quantity in  $\Omega(t)$  with density  $\rho$  and velocity  $u$ . Suppose that the diffusion process is described by the transport equation (3.2) and Darcy's law  $-\rho u = \nabla p(\rho)$ , where

$$(4.1) \quad p(\rho) := \omega'(\rho)\rho - \omega(\rho)$$

is the pressure with a given function  $\omega(\rho)$ ,  $\rho \in \mathbb{R}$ . We impose the boundary condition (3.3). Hence the nonlinear diffusion equations we deal with are

$$(4.2) \quad \partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \Omega(t), t \in (0, T),$$

$$(4.3) \quad -\rho u = \nabla p(\rho) \quad \text{in } \Omega(t), t \in (0, T),$$

$$(4.4) \quad u \cdot \nu_\Omega = V_\Omega^N \quad \text{on } \partial\Omega(t), t \in (0, T).$$

We consider these equations in a moving thin domain. For sufficiently small  $\varepsilon > 0$ , we define a moving thin domain  $\Omega_\varepsilon(t)$  as the set of all points in  $\mathbb{R}^n$  with distance less than  $\varepsilon$  from the moving surface  $\Gamma(t)$ :

$$(4.5) \quad \Omega_\varepsilon(t) := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Gamma(t)) < \varepsilon\}.$$

Our goal in this section is to find the limit equations of (4.2)–(4.4) in  $\Omega(t) = \Omega_\varepsilon(t)$  as  $\varepsilon$  goes to zero, that is, the moving thin domain  $\Omega_\varepsilon(t)$  degenerates into the moving surface  $\Gamma(t)$ . According to the normal coordinate system (2.1), we expand  $\rho$  and  $u$  in powers of the signed distance  $d(x, t)$  as

$$(4.6) \quad \rho(x, t) = \eta(\pi(x, t), t) + d(x, t)\eta^1(\pi(x, t), t) + R(d(x, t)^2)$$

and

$$(4.7) \quad u(x, t) = v(\pi(x, t), t) + d(x, t)v^1(\pi(x, t), t) + R(d(x, t)^2).$$

Here  $R(d(x, t)^k)$  ( $k \in \mathbb{N}$ ) is the terms of order equal to or higher than  $k$  with respect to small  $d(x, t)$ . In particular,  $R(f(x, t))$  for a function  $f(x, t)$  can be of the form

$$R(f(x, t)) = f(x, t)g(x, t)$$

with some (bounded) function  $g(x, t)$ . Note that we can differentiate  $R(d(x, t)^k)$  and its  $j$ -th order derivative is of the form  $R(d(x, t)^{k-j})$  for  $j \leq k$  although we cannot differentiate  $O(d(x, t)^k)$  since it only represents a quantity whose absolute

value is bounded above by  $|d(x, t)|^k$ . Also,  $R(d(x, t)^k) = O(\varepsilon^k)$  holds for  $x \in \Omega_\varepsilon(t)$  and  $k \in \mathbb{N}$  since  $d(x, t)$  is of order  $\varepsilon$  on  $\Omega_\varepsilon(t)$ .

Under the expansions (4.6) and (4.7), the limit equations of (4.2)–(4.4) in  $\Omega_\varepsilon(t)$  as  $\varepsilon$  goes to zero are given as equations  $\eta$  and  $v$  satisfy on  $\Gamma(t)$ .

**Theorem 4.1.** *Let  $\rho$  and  $u$  satisfy the equations (4.2)–(4.4) in the moving thin domain  $\Omega(t) = \Omega_\varepsilon(t)$  given by (4.5). Also, let  $\eta$  and  $v$  be the zeroth order terms in the expansions (4.6) and (4.7) of  $\rho$  and  $u$ , respectively. Then  $v$  is of the form*

$$(4.8) \quad v = V_\Gamma^N \nu_\Gamma + v^T \quad \text{on } \Gamma(t), t \in (0, T)$$

with some tangential velocity field  $v^T$  on  $\Gamma(t)$ , and  $\eta$  and  $v$  satisfy the equations

$$(4.9) \quad \partial^\circ \eta - V_\Gamma^N H \eta + \operatorname{div}_\Gamma(\eta v^T) = 0,$$

$$(4.10) \quad -\eta v^T = \nabla_\Gamma p(\eta)$$

on  $\Gamma(t)$ ,  $t \in (0, T)$ .

*Proof.* For the sake of simplicity, we use the abbreviations

$$(4.11) \quad f(\pi, t) = f(\pi(x, t), t), \quad R(d^k) = R(d(x, t)^k)$$

for a functions  $f(\cdot, t)$  on  $\Gamma(t)$  and  $k \in \mathbb{N}$ . We also abbreviate the product of several functions with the same argument like

$$(4.12) \quad [u_1 \cdot u_2](x, t) = u_1(x, t) \cdot u_2(x, t)$$

for vector fields  $u_1(\cdot, t)$  and  $u_2(\cdot, t)$  on  $\Omega_\varepsilon(t)$ . First we show that  $v$  is of the form (4.8). By the definition (4.5) of the moving thin domain  $\Omega_\varepsilon(t)$ , the unit outward normal vector and the outward normal velocity of its boundary are given by

$$(4.13) \quad \nu_\Omega(x, t) = \pm \nu_\Gamma(\pi, t), \quad V_\Omega^N(x, t) = \pm V_\Gamma^N(\pi, t)$$

for  $x \in \partial\Omega_\varepsilon(t)$  with  $d(x, t) = \pm\varepsilon$  (double-sign corresponds). Hence the boundary condition (4.4) reads

$$u(x, t) \cdot \nu_\Gamma(\pi, t) = V_\Gamma^N(\pi, t)$$

for  $x \in \partial\Omega_\varepsilon(t)$ . We substitute (4.7) for  $u$  in the above equality. Then

$$[v \cdot \nu](\pi, t) \pm \varepsilon[v^1 \cdot \nu](\pi, t) + O(\varepsilon^2) = V_\Gamma^N(\pi, t).$$

Since  $v$ ,  $v^1$ ,  $\nu_\Gamma$ , and  $V_\Gamma^N$  are independent of  $\varepsilon$ , it follows that

$$(4.14) \quad [v \cdot \nu](\pi, t) = V_\Gamma^N(\pi, t),$$

$$(4.15) \quad [v^1 \cdot \nu](\pi, t) = 0,$$

and thus  $v$  is of the form (4.8) with some tangential velocity field  $v^T$  on  $\Gamma(t)$ .

Let us derive the equations (4.9)–(4.10). We differentiate both sides of (4.6) with respect to  $t$  and apply (2.3) and (2.6) to get

$$(4.16) \quad \partial_t \rho(x, t) = \partial^\circ \eta(\pi, t) - [V_\Gamma^N \eta^1](\pi, t) + R(d).$$

Next we compute  $\operatorname{div}(\rho u) = \operatorname{tr}[\nabla(\rho u)]$ . Differentiating both sides of

$$\pi(x, t) = x - d(x, t)\nu_\Gamma(\pi, t)$$

with respect to  $x$  and applying (2.2) we have

$$(4.17) \quad \nabla \pi(x, t) = P_\Gamma(\pi, t) + R(d).$$

From the expansions (4.6) and (4.7),

$$(4.18) \quad [\rho u](x, t) = V(\pi, t) + d(x, t)V^1(\pi, t) + R(d^2),$$



where

$$(4.19) \quad V(\pi, t) := [\eta v](\pi, t),$$

$$(4.20) \quad V^1(\pi, t) := [\eta v^1](\pi, t) + [\eta^1 v](\pi, t).$$

We differentiate both sides of (4.18) with respect to  $x$ . Then by (2.2) and (4.17),

$$\begin{aligned} [\nabla(\rho u)](x, t) &= \nabla\pi(x, t)\nabla V(\pi, t) + \nabla d(x, t) \otimes V^1(\pi, t) + R(d) \\ &= [P_\Gamma \nabla V](\pi, t) + [\nu_\Gamma \otimes V^1](\pi, t) + R(d). \end{aligned}$$

From this formula and  $\text{tr}[\nu_\Gamma \otimes V^1] = \nu_\Gamma \cdot V^1$ , the divergence of  $\rho u$  is

$$[\text{div}(\rho u)](x, t) = \text{div}_\Gamma V(\pi, t) + [\nu_\Gamma \cdot V^1](\pi, t) + R(d).$$

Since  $v$  is of the form (4.8) and  $V$  is given by (4.19),

$$\begin{aligned} \text{div}_\Gamma V &= \text{div}_\Gamma[\eta(V_\Gamma^N \nu_\Gamma + v^T)] \\ &= \nabla_\Gamma(\eta V_\Gamma^N) \cdot \nu_\Gamma + \eta V_\Gamma^N \text{div}_\Gamma \nu_\Gamma + \text{div}_\Gamma(\eta v^T) \\ &= -\eta V_\Gamma^N H + \text{div}_\Gamma(\eta v^T) \end{aligned}$$

on  $\Gamma(t)$  by the definition of the tangential gradient and the mean curvature. Moreover, by (4.14), (4.15), and (4.20),

$$[\nu_\Gamma \cdot V^1](\pi, t) = [\eta^1 V_\Gamma^N](\pi, t).$$

Hence we have

$$(4.21) \quad [\text{div}(\rho u)](x, t) = -[V_\Gamma^N H \eta](\pi, t) + [\text{div}_\Gamma(\eta v^T)](\pi, t) + [\eta^1 V_\Gamma^N](\pi, t) + R(d).$$

Substituting (4.16) and (4.21) for (4.2), we obtain

$$\partial^\circ \eta(\pi, t) - [V_\Gamma^N H \eta](\pi, t) + [\text{div}_\Gamma(\eta v^T)](\pi, t) = R(d).$$

Since each term on the right-hand side is independent of  $d = d(x, t)$ , we conclude that  $\eta$  and  $v = V_\Gamma^N \nu_\Gamma + v^T$  satisfy (4.9).

Let us derive (4.10). We expand the pressure  $p(\rho)$  in  $d(x, t)$  as

$$(4.22) \quad p(\rho(x, t)) = p^0(\pi, t) + d(x, t)p^1(\pi, t) + R(d^2).$$

Then it follows from the expansions (4.6) and (4.22) that

$$(4.23) \quad p^0(\pi, t) = p(\eta(\pi, t)).$$

Moreover, we differentiate both sides of (4.22) with respect to  $x$  and apply (2.2) and (4.17) to get

$$\begin{aligned} \nabla p(\rho(x, t)) &= \nabla\pi(x, t)\nabla p^0(\pi, t) + p^1(\pi, t)\nabla d(x, t) + R(d) \\ &= \nabla_\Gamma p^0(\pi, t) + [p^1 \nu_\Gamma](\pi, t) + R(d). \end{aligned}$$

We substitute this for (4.3) and apply (4.8). Then we have

$$-[\eta v^T](\pi, t) - [\eta V_\Gamma^N \nu_\Gamma](\pi, t) + R(d) = \nabla_\Gamma p^0(\pi, t) + [p^1 \nu_\Gamma](\pi, t) + R(d).$$

Since all terms except of  $R(d)$  are independent of  $d = d(x, t)$  and the vectors  $v^T$  and  $\nabla_\Gamma p^0$  are tangential to  $\Gamma(t)$ , it follows that

$$(4.24) \quad -[\eta v^T](\pi, t) = \nabla_\Gamma p^0(\pi, t),$$

$$(4.25) \quad -[\eta V_\Gamma^N \nu_\Gamma](\pi, t) = p^1(\pi, t).$$

From (4.23) and (4.24) we obtain (4.10).  $\square$

**Remark 4.2.** By the proof of Theorem 4.1 we observe that the transport equation (3.5) on the moving surface  $\Gamma(t)$  can be derived as the limit of the transport equation (3.2) in the moving thin domain  $\Omega(t) = \Omega_\varepsilon(t)$  with the boundary condition (3.3) as  $\varepsilon$  goes to zero.

## 5. ENERGY LAW

The subject in this section is the energy law for nonlinear diffusion equations (4.2)–(4.4) and (4.9)–(4.10). As in Section 4, the pressure  $p(\rho)$  is given by (4.1) with a given function  $\omega(\rho)$ .

**Proposition 5.1.** *Assume that  $\rho$  and  $u$  satisfy (4.2)–(4.4). Then*

$$(5.1) \quad \frac{d}{dt} \int_{\Omega(t)} \omega(\rho) dx = - \int_{\Omega(t)} \rho |u|^2 dx - \int_{\partial\Omega(t)} p(\rho) V_\Omega^N d\mathcal{H}^{n-1}.$$

*Proof.* By the Reynolds transport theorem,

$$\frac{d}{dt} \int_{\Omega(t)} \omega(\rho) d\mathcal{H}^{n-1} = \int_{\Omega(t)} \partial_t \omega(\rho) dx + \int_{\partial\Omega(t)} \omega(\rho) V_\Omega^N d\mathcal{H}^{n-1}.$$

Since  $\partial_t \omega(\rho) = \omega'(\rho) \partial_t \rho$  and the transport equation (4.2) is satisfied,

$$\partial_t \omega(\rho) = -\omega'(\rho) \operatorname{div}(\rho u) = -\operatorname{div}(\omega'(\rho) \rho u) + \nabla \omega'(\rho) \cdot (\rho u).$$

Hence the divergence theorem and (4.4) yield

$$\int_{\Omega(t)} \partial_t \omega(\rho) dx = - \int_{\partial\Omega(t)} \omega'(\rho) \rho V_\Omega^N d\mathcal{H}^{n-1} + \int_{\Omega(t)} \nabla \omega'(\rho) \cdot (\rho u) dx.$$

Using this formula we get

$$\frac{d}{dt} \int_{\Omega(t)} \omega(\rho) dx = \int_{\Omega(t)} \nabla \omega'(\rho) \cdot (\rho u) dx - \int_{\partial\Omega(t)} \{\omega'(\rho) \rho - \omega(\rho)\} V_\Omega^N d\mathcal{H}^{n-1}.$$

The energy law (5.1) follows from this equality, (4.1), and

$$u = -\frac{\nabla p(\rho)}{\rho} = -\nabla \omega'(\rho)$$

by (4.1) and (4.3).  $\square$

**Proposition 5.2.** *Suppose that  $\eta$  and  $v$  of the form (4.8) satisfy (4.9) and (4.10). Then*

$$(5.2) \quad \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} = - \int_{\Gamma(t)} \eta |v^T|^2 d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta) V_\Gamma^N H d\mathcal{H}^{n-1}.$$

*Proof.* By the Leibniz formula [4, Lemma 2.2],

$$\frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} = I - \int_{\Gamma(t)} \omega(\eta) V_\Gamma^N H d\mathcal{H}^{n-1},$$

where

$$I = \int_{\Gamma(t)} \{\partial^\circ \omega(\eta) + \operatorname{div}_\Gamma(\omega(\eta) v^T)\} d\mathcal{H}^{n-1}.$$

By  $\partial^\circ \omega(\eta) = \omega'(\eta) \partial^\circ \eta$  and the transport equation (4.9),

$$\begin{aligned} \partial^\circ \omega(\eta) &= \omega'(\eta) \{V_\Gamma^N H \eta - \operatorname{div}_\Gamma(\eta v^T)\} \\ &= \omega'(\eta) V_\Gamma^N H \eta + \nabla_\Gamma \omega'(\eta) \cdot (\eta v^T) - \operatorname{div}_\Gamma(\omega'(\eta) \eta v^T). \end{aligned}$$

Hence

$$I = \int_{\Gamma(t)} \{\omega'(\eta)V_{\Gamma}^N H\eta + \nabla_{\Gamma}\omega'(\eta) \cdot (\eta v^T)\} d\mathcal{H}^{n-1} \\ + \int_{\Gamma(t)} \operatorname{div}_{\Gamma}[(\omega(\eta) - \omega'(\eta)\eta)v^T] d\mathcal{H}^{n-1}.$$

The second integral on the right-hand side vanishes by the Stokes formula and the fact that  $v^T$  is tangential and  $\Gamma(t)$  has no boundary. Therefore,

$$\frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} = \int_{\Gamma(t)} \nabla_{\Gamma}\omega'(\eta) \cdot (\eta v^T) d\mathcal{H}^{n-1} \\ + \int_{\Gamma(t)} \{\omega'(\eta)\eta - \omega(\eta)\}V_{\Gamma}^N H d\mathcal{H}^{n-1}.$$

Applying

$$v^T = -\frac{\nabla_{\Gamma}p(\eta)}{\eta} = -\nabla_{\Gamma}\omega'(\eta),$$

which follows from (4.1) and (4.10), to the first term on the right-hand side and (4.1) to the second term, we get the energy identity (5.2).  $\square$

Next we derive the energy law (5.2) as a limit of the energy law (5.1) with the moving thin domain  $\Omega(t) = \Omega_{\varepsilon}(t)$  when  $\varepsilon$  goes to zero. As in Section 4, we expand  $\rho$  and  $u$  in powers of the signed distance as (4.6) and (4.7) and determine the equation  $\eta$  and  $v$  satisfy.

**Theorem 5.3.** *Let  $\rho$  and  $u$  satisfy the energy law (5.1) in the moving thin domain  $\Omega(t) = \Omega_{\varepsilon}(t)$  given by (4.5). Also, let  $\eta$  and  $v$  be the zeroth order terms in the expansions (4.6) and (4.7) of  $\rho$  and  $u$ , respectively. Assume that  $v$  is of the form (4.8) with some tangential velocity field  $v^T$  on  $\Gamma(t)$  and Darcy's law (4.3) holds in  $\Omega_{\varepsilon}(t)$ . Then  $\eta$  and  $v$  satisfy the energy law (5.2).*

We give change of variables formulas for integrals which we use in the proof of Theorem 5.3. For  $y \in \Gamma(t)$  and  $\rho \in [-\varepsilon, \varepsilon]$  we set

$$(5.3) \quad J(y, t, r) := \prod_{i=1}^{n-1} \{1 - r\kappa_i(y, t)\},$$

where  $\kappa_1(\cdot, t), \dots, \kappa_{n-1}(\cdot, t)$  are the principal curvatures of  $\Gamma(t)$ . It is the Jacobian that appears when we change variables of integrals over a tubular neighborhood  $\{x \in \mathbb{R}^n \mid -r < d(x, t) < r\}$  ( $r > 0$ ) of  $\Gamma(t)$  and a level-set surface  $\{x \in \mathbb{R}^n \mid d(x, t) = s\}$  ( $s \in \mathbb{R}$ ) in terms of the normal coordinate system around  $\Gamma(t)$  (see [5, Section 14.6] for example). The first formula in Lemma 5.4 is often called the co-area formula.

**Lemma 5.4.** *For a function  $f(x)$  on  $\Omega_{\varepsilon}(t)$ , the identity*

$$(5.4) \quad \int_{\Omega_{\varepsilon}(t)} f(x) dx = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} f(y + r\nu_{\Gamma}(y, t))J(y, t, r) dr d\mathcal{H}^{n-1}(y)$$

holds. Moreover,

$$(5.5) \quad \int_{\partial\Omega_{\varepsilon}(t)} f(x) d\mathcal{H}^{n-1}(x) = \int_{\Gamma(t)} f(y + \varepsilon\nu_{\Gamma}(y, t))J(y, t, \varepsilon) d\mathcal{H}^{n-1}(y) \\ + \int_{\Gamma(t)} f(y - \varepsilon\nu_{\Gamma}(y, t))J(y, t, -\varepsilon) d\mathcal{H}^{n-1}(y).$$

*Proof of Theorem 5.3.* As in the proof of Theorem 4.1, we use the abbreviations (4.11) and (4.12). Let us calculate each term of (5.1). We expand  $\omega(\rho)$  in powers of the signed distance  $d(x, t)$  as

$$\omega(\rho(x, t)) = \omega(\eta(\pi, t)) + d(x, t)\omega^1(\pi, t) + R(d^2).$$

Here the zeroth order term is  $\omega(\eta(\pi, t))$  since the zeroth order term of  $\rho(x, t)$  is  $\eta(\pi, t)$ . We divide the integral of  $\omega(\rho)$  over  $\Omega_\varepsilon(t)$  as

$$\int_{\Omega_\varepsilon(t)} \omega(\rho(x, t)) dx = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &:= \int_{\Omega_\varepsilon(t)} \omega(\eta(\pi, t)) dx, \\ I_2 &:= \int_{\Omega_\varepsilon(t)} d(x, t)\omega^1(\pi, t) dx, \\ I_3 &:= \int_{\Omega_\varepsilon(t)} R(d(x, t)^2) dx. \end{aligned}$$

By the co-area formula (5.4) and the fact that  $J(y, t, r)$  is a polynomial in  $r$  whose coefficients are polynomials in the principal curvatures and  $J(y, t, 0) = 1$ , we have

$$\begin{aligned} I_1 &= \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} \omega(\eta(y, t))J(y, t, r) dr d\mathcal{H}^{n-1}(y) \\ &= 2\varepsilon \int_{\Gamma(t)} \omega(\eta(y, t)) d\mathcal{H}^{n-1}(y) + \varepsilon^2 f_1(\varepsilon, t), \end{aligned}$$

where  $f_1(\varepsilon, t)$  is a polynomial in  $\varepsilon$  with time-dependent coefficients. Therefore,

$$(5.6) \quad \frac{dI_1}{dt} = 2\varepsilon \frac{d}{dt} \int_{\Gamma(t)} \omega(y, t) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).$$

Similarly we have

$$I_2 = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} r\omega^1(y, t)J(y, t, r) dr d\mathcal{H}^{n-1}(y) = \varepsilon^2 f_2(\varepsilon, t)$$

with a polynomial  $f_2(\varepsilon, t)$  in  $\varepsilon$  with time-dependent coefficients and thus

$$(5.7) \quad \frac{dI_2}{dt} = O(\varepsilon^2).$$

We apply the Reynolds transport theorem to the time derivative of  $I_3$ . Then, since the time derivative of  $R(d(x, t)^2)$  is  $R(d(x, t))$ , we have

$$\frac{dI_3}{dt} = \int_{\Omega_\varepsilon(t)} R(d(x, t)) dx + \int_{\partial\Omega_\varepsilon(t)} R(d(x, t)^2)V_\Omega^N(x, t) d\mathcal{H}^{n-1}(x).$$

Since  $J(y, t, r)$  is bounded independently of  $\varepsilon$ , the co-area formula (5.4) yields

$$\int_{\Omega_\varepsilon(t)} R(d(x, t)) dx = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} R(r)J(y, t, r) dr d\mathcal{H}^{n-1}(y) = O(\varepsilon^2).$$

Moreover, applying (4.13) and (5.5) to the integral over  $\partial\Omega_\varepsilon(t)$  and observing that  $R(d(x, t)^2) = R(\varepsilon^2)$  holds for  $x \in \partial\Omega_\varepsilon(t)$  we have

$$\int_{\partial\Omega_\varepsilon(t)} R(d(x, t)^2)V_\Omega^N(x, t) d\mathcal{H}^{n-1}(x) = O(\varepsilon^2).$$

Thus, we get the estimate

$$(5.8) \quad \frac{dI_3}{dt} = O(\varepsilon^2).$$

Since the integral of  $\omega(\rho)$  over  $\Omega_\varepsilon(t)$  is the sum of  $I_1$ ,  $I_2$ , and  $I_3$ , it follows from (5.6), (5.7), and (5.8) that

$$(5.9) \quad \frac{d}{dt} \int_{\Omega_\varepsilon(t)} \omega(\rho(x, t)) dx = 2\varepsilon \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta(y, t)) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).$$

Next we calculate the first term in the right-hand side of (5.1). From the expansions (4.6) and (4.7), the product  $\rho|u|^2$  is of the form

$$[\rho|u|^2](x, t) = [\eta|v|^2](\pi, t) + R(d).$$

Hence, by (5.4),

$$(5.10) \quad \begin{aligned} \int_{\Omega_\varepsilon(t)} [\rho|u|^2](x, t) dx &= \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} \{[\eta|v|^2](y, t) + R(r)\} J(y, t, r) dr d\mathcal{H}^{n-1}(y) \\ &= 2\varepsilon \int_{\Gamma(t)} [\eta|v|^2](y, t) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \end{aligned}$$

Let us compute the last term in the right-hand side of (5.1). We expand the pressure  $p(\rho)$  in  $d(x, t)$  as (4.22). Then, by the assumption that  $v$  is of the form (4.8) and Darcy's law (4.3) holds, we get (4.23) and (4.25) as in the proof of Theorem 4.1 and thus we can write

$$p(\rho(x, t)) = p(\eta(\pi, t)) - d(x, t)[\eta V_\Gamma^N](\pi, t) + R(d^2).$$

Therefore, by (4.13) and (5.5),

$$\int_{\partial\Omega_\varepsilon(t)} [p(\rho)V_\Omega^N](x, t) d\mathcal{H}^{n-1}(x) = J_1 + J_2 + O(\varepsilon^2),$$

where

$$\begin{aligned} J_1 &:= \int_{\Gamma(t)} [p(\eta)V_\Gamma^N](y, t) \{J(y, t, \varepsilon) - J(y, t, -\varepsilon)\} d\mathcal{H}^{n-1}(y), \\ J_2 &:= -\varepsilon \int_{\Gamma(t)} [\eta|V_\Gamma^N|^2](y, t) \{J(y, t, \varepsilon) + J(y, t, -\varepsilon)\} d\mathcal{H}^{n-1}(y). \end{aligned}$$

By (5.3) and (2.5) we have

$$\begin{aligned} J(y, t, \varepsilon) - J(y, t, -\varepsilon) &= -2\varepsilon H(y, t) + O(\varepsilon^2), \\ J(y, t, \varepsilon) + J(y, t, -\varepsilon) &= 2 + O(\varepsilon). \end{aligned}$$

Hence it follows that

$$\begin{aligned} J_1 &= -2\varepsilon \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y, t) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2), \\ J_2 &= -2\varepsilon \int_{\Gamma(t)} [\eta|V_\Gamma^N|^2](y, t) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \end{aligned}$$

Thus, the integral of  $p(\rho)V_\Omega^N$  over  $\partial\Omega_\varepsilon(t)$  becomes

$$(5.11) \quad \begin{aligned} \int_{\partial\Omega_\varepsilon(t)} [p(\rho)V_\Omega^N](x, t) d\mathcal{H}^{n-1}(x) \\ &= -2\varepsilon \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y, t) d\mathcal{H}^{n-1}(y) \\ &\quad - 2\varepsilon \int_{\Gamma(t)} [\eta|V_\Gamma^N|^2](y, t) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \end{aligned}$$

Finally, substituting (5.9), (5.10), and (5.11) for (5.1), dividing both sides by  $2\varepsilon$ , and observing that  $|v|^2 = |V_\Gamma^N|^2 + |v^T|^2$  we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta(y, t)) d\mathcal{H}^{n-1}(y) &= - \int_{\Gamma(t)} [\eta|v^T|^2](y, t) d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y, t) d\mathcal{H}^{n-1}(y) + O(\varepsilon). \end{aligned}$$

In the above equality all terms except  $O(\varepsilon)$  are independent of  $\varepsilon$ . Hence we conclude that  $\eta$  and  $v$  satisfy the energy law (5.2).  $\square$

**Remark 5.5** (A failure of a simple rescaling argument for a moving surface). We can also get the limit energy identity by the rescaling argument but it is somewhat misleading. Let  $\rho$  and  $u$  satisfy the energy identity (5.1) in the moving thin domain  $\Omega(t) = \Omega_\varepsilon(t)$ . For  $y \in \Gamma(t)$ ,  $t \in (0, T)$ , and  $r \in (-1, 1)$ , we set

$$\eta(y, t, r) := \rho(y + \varepsilon r \nu_\Gamma(y, t), t), \quad v(y, t, r) := u(y + \varepsilon r \nu_\Gamma(y, t), t).$$

Then by (4.13) and the integral transformation formulas (5.4) and (5.5) we can write (5.1) in terms of  $\eta$  and  $v$  as

$$\begin{aligned} (5.12) \quad \varepsilon \frac{d}{dt} \int_{\Gamma(t)} \int_{-1}^1 \omega(\eta(y, r)) J(y, \varepsilon r) dr d\mathcal{H}^{n-1}(y) \\ = -\varepsilon \int_{\Gamma(t)} \int_{-1}^1 [\eta|v|^2](y, r) J(y, \varepsilon r) dr d\mathcal{H}^{n-1}(y) \\ - \int_{\Gamma(t)} p(\eta(y, 1)) V_\Gamma^N(y) J(y, \varepsilon) d\mathcal{H}^{n-1}(y) \\ + \int_{\Gamma(t)} p(\eta(y, -1)) V_\Gamma^N(y) J(y, -\varepsilon) d\mathcal{H}^{n-1}(y). \end{aligned}$$

Here we used the abbreviation (4.12) and suppressed the argument  $t$  of functions. If we assume that  $\eta$  and  $v$  are independent of the variable  $r$  in (5.12), then since

$$J(y, \varepsilon) - J(y, -\varepsilon) = -2\varepsilon H(y) + O(\varepsilon^2)$$

by (5.3) and (2.5), it follows that

$$\begin{aligned} 2\varepsilon \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta(y)) d\mathcal{H}^{n-1}(y) &= -2\varepsilon \int_{\Gamma(t)} [\eta|v|^2](y) d\mathcal{H}^{n-1}(y) \\ &\quad + 2\varepsilon \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2) \end{aligned}$$

and by dividing both sides by  $2\varepsilon$  and taking the principal term we obtain

$$\frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} = - \int_{\Gamma(t)} \eta|v|^2 d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta)V_\Gamma^N H d\mathcal{H}^{n-1}.$$

In this equality  $v$  should be of the form  $v = V_\Gamma^N \nu_\Gamma + v^T$  with some tangential velocity field  $v^T$ , since it is the velocity of a substance on the moving surface  $\Gamma(t)$  with the normal velocity  $V_\Gamma^N$ . Hence we get

$$\frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} = - \int_{\Gamma(t)} \eta(|V_\Gamma^N|^2 + |v^T|^2) d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta)V_\Gamma^N H d\mathcal{H}^{n-1},$$

which includes an additional term  $\int_{\Gamma(t)} \eta|V_\Gamma^N|^2 d\mathcal{H}^{n-1}$  compared to the limit energy identity (5.2). This improper term appears because we ignore the difference between  $p(\eta(y, t, 1))$  and  $p(\eta(y, t, -1))$  in (5.12). Of course it vanishes if the shape of the surface does not change, i.e.  $V_\Gamma^N = 0$ . This is the reason why this simple rescaling

argument is popular to derive a thin width limit problem in a formal level when the degenerate set of a thin domain does not change its shape.

**Remark 5.6** (Corrected rescaling argument). To obtain the correct limit (5.2) we should rewrite the sum of the last two terms in the right-hand side of (5.12) into the sum of

$$\begin{aligned} I_1 &= - \int_{\Gamma(t)} \{p(\eta(y, 1)) - p(\eta(y, -1))\} V_\Gamma^N(y) d\mathcal{H}^{n-1}(y), \\ I_2 &= \varepsilon \int_{\Gamma(t)} \{p(\eta(y, 1)) + p(\eta(y, -1))\} [V_\Gamma^N H](y) d\mathcal{H}^{n-1}(y), \end{aligned}$$

and a residual term  $O(\varepsilon^2)$  and calculate them properly (here we again suppressed the argument  $t$  of functions). For  $I_2$  we merely assume that  $\eta$  is independent of  $r$  to get

$$(5.13) \quad I_2 = 2\varepsilon \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y) d\mathcal{H}^{n-1}(y).$$

For a proper calculation of  $I_1$  we need to impose Darcy's law (4.3) in  $\Omega_\varepsilon(t)$  and describe it in terms of the rescaled functions. By the definition of  $\eta$ ,

$$p(\rho(x)) = p(\eta(\pi(x), \varepsilon^{-1}d(x)))$$

for  $x \in \Omega_\varepsilon(t)$ . We differentiate both sides in  $x$  and use (2.2) and (4.17). Then

$$\nabla p(\rho(x)) = \nabla_\Gamma p(\eta(\pi, \varepsilon^{-1}d)) + \varepsilon^{-1} \partial_r p(\eta(\pi, \varepsilon^{-1}d)) \nu_\Gamma(\pi) + O(\varepsilon),$$

where we abbreviate  $\pi(x)$  and  $d(x)$  to  $\pi$  and  $d$  in the right-hand side. Substituting this for (4.3) and taking the normal component of the resulting equation we obtain

$$(5.14) \quad \partial_r p(\eta(y, r)) = -\varepsilon [\eta v](y, r) \cdot \nu_\Gamma(y) + O(\varepsilon^2)$$

for  $y \in \Gamma(t)$  and  $r \in (-1, 1)$ . We apply the mean value theorem and (5.14) to the difference between  $p(\eta(y, 1))$  and  $p(\eta(y, -1))$ . Then

$$p(\eta(y, 1)) - p(\eta(y, -1)) = 2\partial_r p(\eta(y, \theta)) = -2\varepsilon [\eta v](y, \theta) \cdot \nu_\Gamma(y) + O(\varepsilon^2)$$

with some  $\theta = \theta(y, t) \in (-1, 1)$ . Hence  $I_1$  is expressed as

$$I_1 = 2\varepsilon \int_{\Gamma(t)} [\eta v](y, \theta) \cdot \nu_\Gamma(y) V_\Gamma^N(y) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2)$$

and by assuming that  $\eta$  and  $v$  are independent of the third argument we get

$$(5.15) \quad I_1 = 2\varepsilon \int_\Gamma [\eta(v \cdot \nu_\Gamma) V_\Gamma^N](y) d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).$$

We substitute (5.13) and (5.15) for (5.12), assume that the rescaled functions are constant in the variable  $r$  for the left-hand side and the first term on the right-hand side, and divide both sides by  $2\varepsilon$  after calculations. Then the principal term on the resulting equation is

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1} &= - \int_{\Gamma(t)} \eta |v|^2 d\mathcal{H}^{n-1} + \int_{\Gamma(t)} \eta (v \cdot \nu_\Gamma) V_\Gamma^N d\mathcal{H}^{n-1} \\ &\quad + \int_{\Gamma(t)} p(\eta) V_\Gamma^N H d\mathcal{H}^{n-1}. \end{aligned}$$

Finally we suppose that  $v$  is of the form  $v = V_\Gamma^N \nu_\Gamma + v^T$  with some tangential velocity  $v^T$ , which is natural since it is the velocity of a substance on the moving surface  $\Gamma(t)$  with the normal velocity  $V_\Gamma^N$  as we mentioned. Then we obtain the proper limit energy identity (5.2) from the above equality.

## 6. ENERGETIC VARIATION FOR DERIVATION OF DARCY'S LAW

In this section we discuss the energetic variational approach [3, 11, 24] for non-linear diffusion equations in a moving domain and on a moving surface. For a general non-equilibrium thermodynamic system, if the system is isothermal, then the combination of the first and second laws of thermodynamics yields

$$\frac{d}{dt}E^{\text{total}} = \dot{W} - \Delta,$$

where  $E^{\text{total}} = \mathcal{K} + \mathcal{F}$  is the sum of the kinetic energy  $\mathcal{K}$  and the Helmholtz free energy  $\mathcal{F}$ ,  $\Delta$  is the entropy production, and  $\dot{W}$  is the rate of change of work by the external environment. If the system is closed, i.e.,  $\dot{W} = 0$ , we further get the energy dissipation law

$$\frac{d}{dt}E^{\text{total}} = -2\mathcal{D},$$

where  $\mathcal{D} = \Delta/2$  is sometimes called the energy dissipation. For a conservative system ( $\Delta = 0$ ), the principle of least action (LAP) [1] states that the variation of the kinetic and the free energy with respect to the flow map in Lagrangian coordinates. Formally it can be written as

$$\begin{aligned} \delta \left( \int_0^T \mathcal{K} dt \right) &= \int_0^T \int (F_i \cdot \delta x) dx dt, \\ \delta \left( \int_0^T \mathcal{F} dt \right) &= \int_0^T \int (F_c \cdot \delta x) dx dt, \end{aligned}$$

where  $\delta$  represents the procedure of variation. Sometimes such a calculation is also referred to as the principle of virtual work. The LAP gives the inertial force  $F_i$  and the conservative force  $F_c$ , respectively, and the equation of motion is described by balance of forces:

$$F_i = F_c.$$

For a dissipative system, we use the maximum dissipation principle (MDP) [15, 16] to the dissipative force  $F_d$ : by taking the variation of the dissipation with respect to the velocity in Eulerian coordinates, we have

$$\delta \mathcal{D} = F_d \cdot \delta u.$$

When all forces are derived, the equation of motion for a dissipative system is formulated as balance of forces (Newton's third law):

$$F_i = F_c + F_d.$$

Let us apply the above energetic variational framework to the energy laws (5.1) and (5.2). For (5.1),

$$\begin{aligned} \mathcal{K} &= 0, \quad \mathcal{F} = \int_{\Omega(t)} \omega(\rho) dx, \\ \mathcal{D} &= \frac{1}{2} \int_{\Omega(t)} \rho |u|^2 dx, \quad \dot{W} = - \int_{\partial\Omega(t)} p(\rho) V_\Omega^N d\mathcal{H}^{n-1}. \end{aligned}$$

Let  $x(\cdot, t): \Omega(0) \rightarrow \Omega(t)$  be the flow map of the velocity field  $u$ , i.e.,

$$x(X, 0) = X, \quad \frac{d}{dt}x(X, t) = u(x(X, t), t), \quad X \in \Omega(0).$$

We write  $F$  for the deformation matrix of  $x$ :

$$F(X, t) = \frac{\partial x}{\partial X}(X, t).$$



The MDP gives the dissipative force

$$(6.1) \quad \frac{\delta \mathcal{D}}{\delta u} = \rho u.$$

On the other hand, the LAP shows that the conservative force is given by the gradient of the pressure.

**Lemma 6.1.** *Suppose that  $\rho$  and  $u$  satisfy the transport equation (3.2). Then*

$$(6.2) \quad \frac{\delta \mathcal{F}}{\delta x} = \nabla p(\rho),$$

where  $p(\rho)$  is given by (4.1).

*Proof.* Throughout the proof we use the notation

$$f^\sharp(X, t) = f(x(X, t), t), \quad X \in \Omega(0)$$

for a function  $f(\cdot, t)$  on  $\Omega(t)$ . Since the transport equation (3.2) is satisfied, the density  $\rho$  is given by

$$(6.3) \quad \rho(x(X, t), t) = \frac{\rho_0(X)}{\det F(X, t)}, \quad X \in \Omega(0)$$

with the initial density  $\rho_0$  and thus

$$(6.4) \quad \mathcal{F}(x) = \int_{\Omega(0)} \omega \left( \frac{\rho_0(X)}{\det F(X, t)} \right) \det F(X, t) dX.$$

Let  $\{x^\varepsilon(\cdot, t)\}_\varepsilon$  be a family of flow maps and  $u^\varepsilon = dx^\varepsilon/dt$  satisfying

$$x^\varepsilon(\cdot, 0) = x(\cdot, 0), \quad x^\varepsilon(\cdot, T) = x(\cdot, T) \quad \text{for all } \varepsilon,$$

$$x^\varepsilon(\cdot, t)|_{\varepsilon=0} = x(\cdot, t), \quad u^\varepsilon(\cdot, t)|_{\varepsilon=0} = u(\cdot, t), \quad \left. \frac{d}{d\varepsilon} x^\varepsilon(\cdot, t) \right|_{\varepsilon=0} = w(x(\cdot, t), t)$$

with any given vector field  $w(\cdot, t)$  on  $\Omega(t)$ . We write  $F^\varepsilon$  for the deformation matrix of  $x^\varepsilon$ . Suppose that density  $\rho^\varepsilon$  and  $u^\varepsilon$  satisfy the transport equation (3.2) with the same initial density  $\rho_0$ . Then the relation (6.3) with  $\eta$ ,  $x$ , and  $F$  replaced by  $\rho^\varepsilon$ ,  $x^\varepsilon$ , and  $F^\varepsilon$  holds and by (6.4) the free energy  $\mathcal{F}$  with respect to the perturbed flow map  $x^\varepsilon$  is given by

$$(6.5) \quad \mathcal{F}(x^\varepsilon) = \int_{\Omega(0)} \omega \left( \frac{\rho_0}{\det F^\varepsilon} \right) \det F^\varepsilon dX.$$

We differentiate  $\int_0^T \mathcal{F}(x^\varepsilon) dt$  with respect to  $\varepsilon$  at  $\varepsilon = 0$ . Since  $F^\varepsilon|_{\varepsilon=0} = F$  and

$$\left. \frac{dF^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = \left. \frac{\partial}{\partial X} \frac{dx^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = \left. \frac{\partial w^\sharp}{\partial X} \right|_{\varepsilon=0},$$

the derivative of the determinant of  $F^\varepsilon$  with respect to  $\varepsilon$  at  $\varepsilon = 0$  is

$$(6.6) \quad \begin{aligned} \left. \frac{d}{d\varepsilon} \det F^\varepsilon \right|_{\varepsilon=0} &= \text{tr} \left( (F^\varepsilon)^{-1} \frac{dF^\varepsilon}{d\varepsilon} \right) \det F^\varepsilon \Big|_{\varepsilon=0} \\ &= \text{tr} \left( F^{-1} \frac{\partial w^\sharp}{\partial X} \right) \det F = (\text{div } w)^\sharp \det F, \end{aligned}$$

where  $F^{-1}$  and  $(F^\varepsilon)^{-1}$  are the inverse matrix of  $F$  and  $F^\varepsilon$ . We differentiate the integrand of (6.5) at  $\varepsilon = 0$  and apply (6.3), (6.6), and  $F^\varepsilon|_{\varepsilon=0} = F$  to obtain

$$\left. \frac{d}{d\varepsilon} \left( \omega \left( \frac{\rho_0}{\det F^\varepsilon} \right) \det F^\varepsilon \right) \right|_{\varepsilon=0} = \{-\omega'(\rho^\sharp)\rho^\sharp + \omega(\rho^\sharp)\}(\text{div } w)^\sharp \det F.$$

Therefore,

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \int_0^T \mathcal{F}(x^\varepsilon) dt \right|_{\varepsilon=0} &= \int_0^T \int_{\Omega(0)} \{-\omega'(\rho^\#)\rho^\# + \omega(\rho^\#)\} (\operatorname{div} w)^\# \det F dX dt \\ &= \int_0^T \int_{\Omega(t)} \{-\omega'(\rho)\rho + \omega(\rho)\} \operatorname{div} w dx dt \\ &= \int_0^T \int_{\Omega(t)} \nabla[\omega'(\rho)\rho - \omega(\rho)] \cdot w dx dt \end{aligned}$$

and thus (6.2) follows.  $\square$

By (6.1), (6.2), and  $\mathcal{K} = 0$ , the balance of forces

$$\frac{\delta \mathcal{K}}{\delta x} = \frac{\delta \mathcal{F}}{\delta x} + \frac{\delta \mathcal{D}}{\delta u}$$

is of the form

$$0 = \nabla p(\rho) + \rho u, \quad \text{i.e.,} \quad -\rho u = \nabla p(\rho),$$

which is exactly Darcy's law in a moving domain. Combining this with the transport equation (3.2), we obtain the nonlinear diffusion equations

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad -\rho u = \nabla p(\rho)$$

in the moving domain  $\Omega(t)$ , where  $p(\rho)$  is given by (4.1).

From the above discussion, we expect that the energetic variational approach for (5.2) yields Darcy' law (4.10) on a moving surface. For (5.2), we have

$$\begin{aligned} \mathcal{K} &= 0, \quad \mathcal{F} = \int_{\Gamma(t)} \omega(\eta) d\mathcal{H}^{n-1}, \\ \mathcal{D} &= \frac{1}{2} \int_{\Gamma(t)} \eta |v^T|^2 d\mathcal{H}^{n-1}, \quad \dot{\mathcal{W}} = \int_{\Gamma(t)} p(\eta) H V_\Gamma^N d\mathcal{H}^{n-1}. \end{aligned}$$

The variation of  $\mathcal{D}$  with respect to the total velocity  $v = V_\Gamma^N \nu + v^T$  gives

$$(6.7) \quad \frac{\delta \mathcal{D}}{\delta v} = \eta v^T,$$

since  $v^T = P_\Gamma v$ . Let us apply the LAP to the free energy  $\mathcal{F}$ .

**Lemma 6.2.** *Suppose that  $\eta$  and  $v$  of the form (4.8) satisfy the transport equation (3.5). Then*

$$(6.8) \quad \frac{\delta \mathcal{F}}{\delta y} = \nabla_\Gamma p(\eta),$$

where  $p(\eta)$  is given by (4.1).

We localize the integral over  $\Gamma(t)$  with a partition of unity of  $\Gamma(t)$  as in [13, Section 2.4] and take the variation of  $\mathcal{F}$  with respect to a flow map in ‘‘local Lagrangian coordinates.’’ Let  $y(\cdot, t): U \rightarrow \Gamma(t)$  be a mapping defined on an open set  $U$  in  $\mathbb{R}^{n-1}$  such that

$$(6.9) \quad y(Y, 0) \in \Gamma(0), \quad \frac{d}{dt} y(Y, t) = v(y(Y, t), t), \quad Y \in U.$$

We consider a localized surface integral

$$(6.10) \quad \mathcal{F}(y) = \int_{y(U, t)} \omega(\eta) d\mathcal{H}^{n-1}$$

and take its variation with respect to  $y$ . Let  $\{y^\varepsilon(\cdot, t)\}_\varepsilon$  be a family of flow maps on  $\Gamma(t)$  in local Lagrangian coordinates, i.e.  $y^\varepsilon(\cdot, t): U \rightarrow \Gamma(t)$  for each  $\varepsilon$ , and  $dy^\varepsilon/dt = v^\varepsilon$  such that

$$(6.11) \quad \begin{aligned} y^\varepsilon(\cdot, 0) &= y(\cdot, 0), \quad y^\varepsilon(\cdot, T) = y(\cdot, T) \quad \text{for all } \varepsilon, \\ y^\varepsilon(\cdot, t)|_{\varepsilon=0} &= y(\cdot, t), \quad v^\varepsilon(\cdot, t)|_{\varepsilon=0} = v(\cdot, t), \quad \left. \frac{d}{d\varepsilon} y^\varepsilon(\cdot, t) \right|_{\varepsilon=0} = w(y(\cdot, t), t) \end{aligned}$$

with any given tangential vector field  $w(\cdot, t)$  on  $\Gamma(t)$ . We use the notation

$$(6.12) \quad f^\sharp(Y, t) = f(y(Y, t), t), \quad Y \in U$$

for a function  $f(\cdot, t)$  on  $\Gamma(t)$ .

**Lemma 6.3.** *Let  $g = (g_{ij})_{i,j}$  be a matrix given by*

$$(6.13) \quad g_{ij} = \frac{\partial y}{\partial Y_i} \cdot \frac{\partial y}{\partial Y_j}, \quad i, j = 1, \dots, n-1$$

and  $g^\varepsilon = (g_{ij}^\varepsilon)_{i,j}$  be a matrix given as above with  $y$  replaced by  $y^\varepsilon$ . Then

$$(6.14) \quad \left. \frac{d}{d\varepsilon} \sqrt{\det g^\varepsilon} \right|_{\varepsilon=0} = (\operatorname{div}_\Gamma w)^\sharp \sqrt{\det g}.$$

*Proof.* Since  $g^\varepsilon|_{\varepsilon=0} = g$  and

$$\frac{d}{d\varepsilon} \det g^\varepsilon = \operatorname{tr} \left( (g^\varepsilon)^{-1} \frac{dg^\varepsilon}{d\varepsilon} \right) \det g^\varepsilon,$$

where  $(g^\varepsilon)^{-1}$  is the inverse matrix of  $g^\varepsilon$ , we have

$$(6.15) \quad \left. \frac{d}{d\varepsilon} \sqrt{\det g^\varepsilon} \right|_{\varepsilon=0} = \frac{1}{2} \operatorname{tr} \left( g^{-1} \left. \frac{dg^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} \right) \sqrt{\det g},$$

where  $g^{-1} = (g^{ij})_{i,j}$  is the inverse matrix of  $g$ . Moreover, since

$$\begin{aligned} \left. \frac{dg_{ij}^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} &= \left( \frac{\partial}{\partial Y_i} \frac{dy^\varepsilon}{d\varepsilon} \cdot \frac{\partial y^\varepsilon}{\partial Y_j} + \frac{\partial y^\varepsilon}{\partial Y_j} \cdot \frac{\partial}{\partial Y_i} \frac{dy^\varepsilon}{d\varepsilon} \right) \Big|_{\varepsilon=0} \\ &= \frac{\partial w^\sharp}{\partial Y_i} \cdot \frac{\partial y}{\partial Y_j} + \frac{\partial y}{\partial Y_i} \cdot \frac{\partial w^\sharp}{\partial Y_j} \end{aligned}$$

for each  $i, j = 1, \dots, n-1$ , where we used the notation (6.12), and  $g^{-1}$  is symmetric,

$$\begin{aligned} \operatorname{tr} \left( g^{-1} \left. \frac{dg^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} \right) &= \sum_{i,j=1}^{n-1} g^{ij} \left( \frac{\partial w^\sharp}{\partial Y_i} \cdot \frac{\partial y}{\partial Y_j} + \frac{\partial y}{\partial Y_i} \cdot \frac{\partial w^\sharp}{\partial Y_j} \right) \\ &= 2 \sum_{i,j=1}^{n-1} g^{ij} \frac{\partial w^\sharp}{\partial Y_i} \cdot \frac{\partial y}{\partial Y_j} = 2(\operatorname{div}_\Gamma w)^\sharp. \end{aligned}$$

Substituting this for (6.15), we get (6.14).  $\square$

*Proof of Lemma 6.2.* We first express the free energy  $\mathcal{F}$  in ‘‘local Lagrangian coordinates.’’ Let  $U$  be an open set in  $\mathbb{R}^{n-1}$  and  $y(\cdot, t): U \rightarrow \Gamma(t)$  be a flow map satisfying (6.9). For every open subset  $U'$  of  $U$ , the integral

$$\int_{U'(t)} \eta(y, t) d\mathcal{H}^{n-1}(y) = \int_{U'} \eta(y(Y, t), t) \sqrt{\det g(Y, t)} dY,$$

where  $U'(t) = y(U', t)$ , is constant in  $t$ , since  $\eta$  and  $v$  satisfy the transport equation (3.5). Hence

$$(6.16) \quad \eta(y(Y, t), t) \sqrt{\det g(Y, t)} = \eta(y(Y, 0), 0) \sqrt{\det g(Y, 0)}, \quad Y \in U$$

and the localized surface integral (6.10) is expressed as

$$(6.17) \quad \begin{aligned} \mathcal{F}(y) &= \int_U \omega(\eta(y(Y, t), t)) \sqrt{\det g(Y, t)} dY \\ &= \int_U \omega \left( \frac{\eta_0(Y)}{\sqrt{\det g(Y, t)}} \right) \sqrt{\det g(Y, t)} dY, \end{aligned}$$

where  $\eta_0(Y)$  is given by the right-hand side of (6.16), where  $g$  is the function of  $y = y(Y, t)$  as (6.13).

Next we take a variation of  $\mathcal{F}$  with respect to the flow map  $y$ . Let  $\{y^\varepsilon(\cdot, t)\}_\varepsilon$  be a family of flow maps on  $\Gamma(t)$  in local Lagrangian coordinates satisfies (6.11). Suppose that density  $\eta^\varepsilon$  and  $v^\varepsilon = dy^\varepsilon/dt$  satisfy the transport equation (3.5) and  $\eta^\varepsilon|_{t=0} = \eta|_{t=0}$  holds on  $y^\varepsilon(U, 0) = y(U, 0)$ . Then the relation (6.16) with  $\eta^\varepsilon$ ,  $y^\varepsilon$ , and  $g^\varepsilon$  holds and by (6.17) the free energy  $\mathcal{F}$  with respect to the perturbed flow map  $y^\varepsilon$  is given by

$$\mathcal{F}(y^\varepsilon) = \int_U \omega \left( \frac{\eta_0}{\sqrt{\det g^\varepsilon}} \right) \sqrt{\det g^\varepsilon} dY.$$

Note that the right-hand side of (6.16) with  $\eta^\varepsilon$ ,  $y^\varepsilon$ , and  $g^\varepsilon$  is equal to  $\eta_0(Y)$  since  $\eta^\varepsilon|_{t=0} = \eta|_{t=0}$ ,  $y^\varepsilon|_{t=0} = y|_{t=0}$ , and  $g^\varepsilon|_{t=0} = g|_{t=0}$ . We differentiate the integrand of the right-hand side with respect to  $\varepsilon$  at  $\varepsilon = 0$ . Then by (6.14), (6.17), and  $g^\varepsilon|_{\varepsilon=0} = g$  we get

$$\frac{d}{d\varepsilon} \left( \omega \left( \frac{\eta_0}{\sqrt{\det g^\varepsilon}} \right) \sqrt{\det g^\varepsilon} \right) \Big|_{\varepsilon=0} = \{-\omega'(\eta^\sharp)\eta^\sharp + \omega(\eta^\sharp)\}(\operatorname{div}_\Gamma w)^\sharp \sqrt{\det g}.$$

Here we used the notation (6.12). Hence

$$\begin{aligned} \frac{d}{d\varepsilon} \int_0^T \mathcal{F}(y^\varepsilon) dt \Big|_{\varepsilon=0} &= \int_0^T \int_U \{-\omega'(\eta^\sharp)\eta^\sharp + \omega(\eta^\sharp)\}(\operatorname{div}_\Gamma w)^\sharp \sqrt{\det g} dY dt \\ &= \int_0^T \int_{U(t)} \{-\omega'(\eta)\eta + \omega(\eta)\} \operatorname{div}_\Gamma w d\mathcal{H}^{n-1} dt \\ &= \int_0^T \int_{U(t)} \nabla_\Gamma[\omega'(\eta)\eta - \omega(\eta)] \cdot w d\mathcal{H}^{n-1} dt, \end{aligned}$$

where  $U(t) = y(U, t)$ . (Note that, since we localize the surface integral by using a partition of unity of  $\Gamma(t)$ , we may assume the function  $\omega'(\eta)\eta - \omega(\eta)$  is compactly supported in  $U(t)$ .) Since  $w$  is an arbitrary tangential vector field on  $\Gamma(t)$ , we conclude from the above equality that (6.8) holds.  $\square$

By (6.7), (6.8), and  $\mathcal{K} = 0$ , the balance of forces

$$\frac{\delta \mathcal{K}}{\delta y} = \frac{\delta \mathcal{F}}{\delta y} + \frac{\delta \mathcal{D}}{\delta v}$$

is of the form

$$0 = \nabla_\Gamma p(\eta) + \eta v^T, \quad \text{i.e.,} \quad -\eta v^T = \nabla_\Gamma p(\eta),$$

which is Darcy's law on a moving surface as we expected. Finally, combining this with the transport equation (3.5) we obtain the nonlinear equation

$$\begin{aligned} \partial^\circ \eta - V_\Gamma^N H \eta + \operatorname{div}_\Gamma(\eta v^T) &= 0, \\ -\eta v^T &= \nabla_\Gamma p(\eta) \end{aligned}$$

on the moving surface  $\Gamma(t)$ , where  $p(\eta)$  is given by (4.1).

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