Framed surfaces in the Euclidean space

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Abstract

A framed surface is a smooth surface in the Euclidean space with a moving frame. The framed surfaces may have singularities. We treat smooth surfaces with singular points, that is, singular surfaces more directly. By using the moving frame, the basic invariants and curvatures of the framed surface are introduced. Then we show that the existence and uniqueness for the basic invariants of the framed surfaces. We give properties of framed surfaces and typical examples. Moreover, we construct framed surfaces as one-parameter families of Legendre curves along framed curves. We give a criteria for singularities of framed surfaces by using the curvature of Legendre curves and framed curves.

1 Introduction

The geometry of smooth surfaces in the Euclidean space is a classical object. Recently, smooth surfaces with singular points are more important for differential geometry, differential equations and physics (for instance, [1, 2, 4, 5, 6, 7, 12, 14, 15, 16, 18, 19, 20, 21, 23, 24, 25, 26, 28]). One of the idea to treat the smooth surfaces with singular points is that we consider the fronts or frontals as smooth surfaces with singular points (cf. [1, 2, 20, 21, 26, 28]).

In this paper, we give an other consideration of smooth surfaces with singular points. The idea is a generalisation of not only the Legendre curves [8] but also framed curves in the Euclidean space [11]. It is also related the Cartan's moving frame (cf. [17]).

A framed surface in the Euclidean space is a smooth surface with a moving frame. The framed surface is a generalisation of not only regular surfaces but also frontals at least locally. The framed surfaces may have singularities. We would like to treat the surfaces with singular points more directly. In fact, we introduce the basic invariants of the framed surface in §2. Then we give the existence and uniqueness theorem of the basic invariants for the framed surface in §3. We investigate properties of the framed surfaces. We give a curvature and a concomitant mapping of the framed surfaces in §4. These mappings are useful to recognize a Legendre immersion or a framed immersion. Moreover, we construct framed surfaces as one-parameter families of Legendre curves along framed curves in §5. As an application of the construction, we give a criterion that the framed surface is locally diffeomorphic to the cuspidal edge, swallowtail

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and cuspidal cross cap by using the curvatures of the Legendre curves and the framed curves. We give concrete examples in §6.

All mappings and manifolds considered here are differential of class C^∞.

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2 Definitions and notations

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the canonical basis on $\mathbb{R}^3$. Let U be a simply connected domain of $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | |\boldsymbol{a}| = 1\}$. We denote a 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ.

Definition 2.1 We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0, \boldsymbol{x}_v(u, v) \cdot \boldsymbol{n}(u, v) = 0$ for all $(u, v) \in U$, where $\boldsymbol{x}_u(u, v) = (\partial \boldsymbol{x}/\partial u)(u, v)$ and $\boldsymbol{x}_v(u, v) = (\partial \boldsymbol{x}/\partial v)(u, v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

By definition, the framed base surface is a frontal. The definition and properties of frontals see [1, 2]. On the other hand, the frontal is a framed base surface at least locally. In this paper, we consider framed base surfaces as singular surfaces. If we do not confuse in the sentence, we also say that $\boldsymbol{x}$ is a framed surface.

We denote $\boldsymbol{t}(u, v) = \boldsymbol{n}(u, v) \times \boldsymbol{s}(u, v)$. Then $\{\boldsymbol{n}(u, v), \boldsymbol{s}(u, v), \boldsymbol{t}(u, v)\}$ is a moving frame along $\boldsymbol{x}(u, v)$. Thus, we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2}$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions and we call the functions *basic invariants* of the framed surface. We denote the above matrices by $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that (u, v) is a singular point of $\boldsymbol{x}$ if and only if $\det \mathcal{G}(u, v) = 0$.

Since the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should be satisfied the following conditions:

$$\begin{cases} a_{1,v} - b_1 g_2 = a_{2,u} - b_2 g_1, \\ b_{1,v} - a_2 g_1 = b_{2,u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \tag{3}$$

$$\begin{cases} e_{1,v} - f_1 g_2 = e_{2,u} - f_2 g_1, \\ f_{1,v} - e_2 g_1 = f_{2,u} - e_1 g_2, \\ g_{1,v} - e_1 f_2 = g_{2,u} - e_2 f_1. \end{cases} \tag{4}$$

3 Properties of framed surfaces

We consider basic properties of framed surfaces. We give fundamental theorems for framed surfaces, that is, the existence and uniqueness theorem for the basic invariants of framed surfaces.

Definition 3.1 Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are *congruent as framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{x}}(u, v) = A(\boldsymbol{x}(u, v)) + \boldsymbol{a}, \widetilde{\boldsymbol{n}}(u, v) = A(\boldsymbol{n}(u, v)), \widetilde{\boldsymbol{s}}(u, v) = A(\boldsymbol{s}(u, v)),$$

for all $(u, v) \in U$.

The existence theorem of framed surfaces follows from the existence of solutions of partial differential equations.

Theorem 3.2 (The Existence Theorem) *Let U be a simply connected domain in $\mathbb{R}^2$ and let $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ be smooth functions with the integrability conditions* (3) *and* (4)*. Then there exists a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants is $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.*

Proof. Since the integrability condition (4), there exists an orthonormal frame $\{\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}\}$ such that the condition (2) holds. Moreover, by the integrability condition (3), there exists a smooth mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ such that the condition (1) holds. Therefore, there exists a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants is $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. $\square$

Theorem 3.3 (The Uniqueness Theorem) *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$, respectively. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as framed surfaces if and only if the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ coincides.*

In order to prove the uniqueness theorem, we prepare the following two lemmas.

Lemma 3.4 *If $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as framed surfaces, then $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2) = (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$.*

Proof. By Definition 3.1 and a direct calculation, we obtain the lemma. $\square$

Lemma 3.5 *If* $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)=(\widetilde{\mathcal{G}},\widetilde{\mathcal{F}}_1,\widetilde{\mathcal{F}}_2)$ *and* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})(u_0,v_0)=(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})(u_0,v_0)$ *for some point* $(u_0,v_0)\in U$, *then* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})$.

Proof. Firstly, we show $(\boldsymbol{n},\boldsymbol{s},\boldsymbol{t})=(\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}},\widetilde{\boldsymbol{t}})$, where $\boldsymbol{n}\times\boldsymbol{s}=\boldsymbol{t}$ and $\widetilde{\boldsymbol{n}}\times\widetilde{\boldsymbol{s}}=\widetilde{\boldsymbol{t}}$. We define a function $f:U\to\mathbb{R}$ by $f(u,v)=\boldsymbol{n}(u,v)\cdot\widetilde{\boldsymbol{n}}(u,v)+\boldsymbol{s}(u,v)\cdot\widetilde{\boldsymbol{s}}(u,v)+\boldsymbol{t}(u,v)\cdot\widetilde{\boldsymbol{t}}(u,v)$. By the definition of the basic invariants, we have

$$\begin{aligned} f_u &= (e_1-\widetilde{e_1})(\boldsymbol{s}\cdot\widetilde{\boldsymbol{n}})+(f_1-\widetilde{f_1})(\boldsymbol{t}\cdot\widetilde{\boldsymbol{n}})+(e_1-\widetilde{e_1})(\boldsymbol{n}\cdot\widetilde{\boldsymbol{s}}) \\ &\quad +(f_1-\widetilde{f_1})(\boldsymbol{n}\cdot\widetilde{\boldsymbol{t}})+(g_1-\widetilde{g_1})(\boldsymbol{t}\cdot\widetilde{\boldsymbol{s}})+(g_1-\widetilde{g_1})(\boldsymbol{s}\cdot\widetilde{\boldsymbol{t}}). \end{aligned}$$

By the assumption $\mathcal{F}_1=\widetilde{\mathcal{F}}_1$, we have $f_u(u,v)=0$ for all $(u,v)\in U$. Similarly, we also have $f_v(u,v)=0$ for all $(u,v)\in U$. Moreover, by the assumption $(\boldsymbol{n},\boldsymbol{s})(u_0,v_0)=(\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})(u_0,v_0)$, we have $f(u_0,v_0)=3$. It conclude that $f(u,v)=3$ for all $(u,v)\in U$. Hence, we have $\boldsymbol{n}\cdot\widetilde{\boldsymbol{n}}=\boldsymbol{s}\cdot\widetilde{\boldsymbol{s}}=\boldsymbol{t}\cdot\widetilde{\boldsymbol{t}}=1$. It follows that $\boldsymbol{n}=\widetilde{\boldsymbol{n}},\boldsymbol{s}=\widetilde{\boldsymbol{s}}$ and $\boldsymbol{t}=\widetilde{\boldsymbol{t}}$.

Next, we show $\boldsymbol{x}=\widetilde{\boldsymbol{x}}$. By the assumption $\mathcal{G}_1=\widetilde{\mathcal{G}}_1$, we have $\boldsymbol{x}_u=a_1\boldsymbol{s}+b_1\boldsymbol{t}=\widetilde{a_1}\widetilde{\boldsymbol{s}}+\widetilde{b_1}\widetilde{\boldsymbol{t}}=\widetilde{\boldsymbol{x}}_u$ and $\boldsymbol{x}_v=a_2\boldsymbol{s}+b_2\boldsymbol{t}=\widetilde{a_2}\widetilde{\boldsymbol{s}}+\widetilde{b_2}\widetilde{\boldsymbol{t}}=\widetilde{\boldsymbol{x}}_v$. Then, we have $(\boldsymbol{x}-\widetilde{\boldsymbol{x}})_u=(\boldsymbol{x}-\widetilde{\boldsymbol{x}})_v=0$. Since $\boldsymbol{x}(u_0,v_0)=\widetilde{\boldsymbol{x}}(u_0,v_0)$, we have $\boldsymbol{x}(u,v)=\widetilde{\boldsymbol{x}}(u,v)$ for all $(u,v)\in U$. Therefore, we have $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})$. $\Box$

Proof of Theorem 3.3. The necessary part of the theorem is Lemma 3.4.

We prove the sufficient part of the theorem. For fix a point $(u_0,v_0)\in U$, there exist $A\in SO(3)$ and $\boldsymbol{a}\in\mathbb{R}^3$ such that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})(u_0,v_0)=(A\widetilde{\boldsymbol{x}}+\boldsymbol{a},A\widetilde{\boldsymbol{n}},A\widetilde{\boldsymbol{s}})(u_0,v_0)$. By Lemmas 3.4 and 3.5, we have $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(A\widetilde{\boldsymbol{x}}+\boldsymbol{a},A\widetilde{\boldsymbol{n}},A\widetilde{\boldsymbol{s}})$, that is, $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})$ are congruent as framed surfaces. $\Box$

Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be a framed surface with the basic invariants $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$. For the moving frame $\{\boldsymbol{n},\boldsymbol{s},\boldsymbol{t}\}$ along $\boldsymbol{x}$, there is an ability. We consider rotations and reflections of the vectors $\boldsymbol{s},\boldsymbol{t}$. We denote

$$\begin{pmatrix}\boldsymbol{s}^\theta(u,v)\\ \boldsymbol{t}^\theta(u,v)\end{pmatrix}=\begin{pmatrix}\cos\theta(u,v) & -\sin\theta(u,v)\\ \sin\theta(u,v) & \cos\theta(u,v)\end{pmatrix}\begin{pmatrix}\boldsymbol{s}(u,v)\\ \boldsymbol{t}(u,v)\end{pmatrix},$$

where $\theta:U\to\mathbb{R}$ is a smooth function. Then $\boldsymbol{n}\times\boldsymbol{s}^\theta=\boldsymbol{t}^\theta$ and $\{\boldsymbol{n},\boldsymbol{s}^\theta,\boldsymbol{t}^\theta\}$ is also a moving frame along $\boldsymbol{x}$. It follows that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}^\theta)$ is a framed surface. We call the frame $\{\boldsymbol{n},\boldsymbol{s}^\theta,\boldsymbol{t}^\theta\}$ a *rotation frame* by θ of the framed surface $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$. We denote by $(\mathcal{G}^\theta,\mathcal{F}_1^\theta,\mathcal{F}_2^\theta)$ the basic invariants of $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}^\theta)$. Moreover, we consider a moving frame $\{\boldsymbol{n}^r,\boldsymbol{s}^r,\boldsymbol{t}^r\}=\{-\boldsymbol{n},\boldsymbol{t},\boldsymbol{s}\}$ along $\boldsymbol{x}$ and call it a *reflection frame* of the framed surface $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$. We denote by $(\mathcal{G}^r,\mathcal{F}_1^r,\mathcal{F}_2^r)$ the basic invariants of $(\boldsymbol{x},\boldsymbol{n}^r,\boldsymbol{s}^r)$.

By a direct calculation, we have the following.

Proposition 3.6 *Under the above notations, we have the relationships between the basic invariants* $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$ *and* $(\mathcal{G}^\theta,\mathcal{F}_1^\theta,\mathcal{F}_2^\theta)$, $(\mathcal{G}^r,\mathcal{F}_1^r,\mathcal{F}_2^r)$, *respectively.*

(1) *For any smooth function* $\theta:U\to\mathbb{R}$,

$$\mathcal{G}^\theta=\mathcal{G}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}a_1\cos\theta-b_1\sin\theta & a_1\sin\theta+b_1\cos\theta\\ a_2\cos\theta-b_2\sin\theta & a_2\sin\theta+b_2\cos\theta\end{pmatrix},$$

$$\mathcal{F}_1^\theta=\begin{pmatrix}0 & e_1\cos\theta-f_1\sin\theta & e_1\sin\theta+f_1\cos\theta\\ -e_1\cos\theta+f_1\sin\theta & 0 & g_1-\theta_u\\ -e_1\sin\theta-f_1\cos\theta & -g_1+\theta_u & 0\end{pmatrix},$$

$$\mathcal{F}_2^\theta = \begin{pmatrix} 0 & e_2\cos\theta - f_2\sin\theta & e_2\sin\theta + f_2\cos\theta \\ -e_2\cos\theta + f_2\sin\theta & 0 & g_2 - \theta_v \\ -e_2\sin\theta - f_2\cos\theta & -g_2 + \theta_v & 0 \end{pmatrix}.$$

(2)

$$\mathcal{G}^r = \mathcal{G}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 \\ b_2 & a_2 \end{pmatrix}, \mathcal{F}_1^r = \begin{pmatrix} 0 & -f_1 & -e_1 \\ f_1 & 0 & -g_1 \\ e_1 & g_1 & 0 \end{pmatrix}, \mathcal{F}_2^r = \begin{pmatrix} 0 & -f_2 & -e_2 \\ f_2 & 0 & -g_2 \\ e_2 & g_2 & 0 \end{pmatrix}.$$

Especially, we have

$$\begin{pmatrix} e_i^\theta \\ f_i^\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e_i \\ f_i \end{pmatrix}, \ i = 1, 2.$$

We consider the integrability conditions (3) and (4) of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ and $(\boldsymbol{x}, \boldsymbol{n}^r, \boldsymbol{s}^r)$, respectively. Since

$$\boldsymbol{x}_u = a_1\boldsymbol{s} + b_1\boldsymbol{t} = a_1^\theta\boldsymbol{s}^\theta + b_1^\theta\boldsymbol{t}^\theta = a_1^r\boldsymbol{s}^r + b_1^r\boldsymbol{t}^r, \boldsymbol{x}_v = a_2\boldsymbol{s} + b_2\boldsymbol{t} = a_2^\theta\boldsymbol{s}^\theta + b_2^\theta\boldsymbol{t}^\theta = a_2^r\boldsymbol{s}^r + b_2^r\boldsymbol{t}^r,$$

we also have

$$\begin{cases} a_{1,v}^\theta - b_1^\theta g_2^\theta = a_{2,u}^\theta - b_2^\theta g_1^\theta, \\ b_{1,v}^\theta - a_2^\theta g_1^\theta = b_{2,u}^\theta - a_1^\theta g_2^\theta, \\ a_1^\theta e_2^\theta + b_1^\theta f_2^\theta = a_2^\theta e_1^\theta + b_2^\theta f_1^\theta, \end{cases}$$

for any $\theta : U \to \mathbb{R}$, and

$$\begin{cases} a_{1,v}^r - b_1^r g_2^r = a_{2,u}^r - b_2^r g_1^r, \\ b_{1,v}^r - a_2^r g_1^r = b_{2,u}^r - a_1^r g_2^r, \\ a_1^r e_2^r + b_1^r f_2^r = a_2^r e_1^r + b_2^r f_1^r. \end{cases}$$

Proposition 3.7 *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surfaces with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Then the following are equivalent for any smooth function $\theta : U \to \mathbb{R}$.*
(1) $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$.
(2) $\mathcal{F}_{2,u}^\theta - \mathcal{F}_{1,v}^\theta = \mathcal{F}_1^\theta\mathcal{F}_2^\theta - \mathcal{F}_2^\theta\mathcal{F}_1^\theta$.
(3) $\mathcal{F}_{2,u}^r - \mathcal{F}_{1,v}^r = \mathcal{F}_1^r\mathcal{F}_2^r - \mathcal{F}_2^r\mathcal{F}_1^r$.

Proof. We prove that (1) is equivalent to (2). We define matrices $R(\theta)$ and Θ by

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \ \Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta \\ 0 & \theta & 0 \end{pmatrix}.$$

Then we have $\mathcal{F}_1^\theta = \Theta_u + R(\theta)\mathcal{F}_1R(-\theta)$ and $\mathcal{F}_2^\theta = \Theta_v + R(\theta)\mathcal{F}_2R(-\theta)$ by Proposition 3.6 (1). By a direct calculation, we have

$$\begin{aligned} \mathcal{F}_{2,u}^\theta - \mathcal{F}_{1,v}^\theta &= \Theta_{vu} + R(\theta)_u\mathcal{F}_2R(-\theta) + R(\theta)\mathcal{F}_{2,u}R(-\theta) + R(\theta)\mathcal{F}_2R(-\theta)_u \\ &\quad -\Theta_{uv} - R(\theta)_v\mathcal{F}_1R(-\theta) - R(\theta)\mathcal{F}_{1,v}R(-\theta) - R(\theta)\mathcal{F}_1R(-\theta)_v. \end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{F}_1^\theta\mathcal{F}_2^\theta-\mathcal{F}_2^\theta\mathcal{F}_1^\theta &= \Theta_u R(\theta)\mathcal{F}_2R(-\theta)+R(\theta)\mathcal{F}_1R(-\theta)\Theta_v-\Theta_vR(\theta)\mathcal{F}_1R(-\theta)\\ &\quad -R(\theta)\mathcal{F}_2R(-\theta)\Theta_u+R(\theta)(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_2\mathcal{F}_1)R(-\theta).\end{aligned}$$

By using the relations $\Theta_uR(\theta)=R(\theta)_u$, $R(-\theta)\Theta_u=R(-\theta)_u$, $\Theta_vR(\theta)=R(\theta)_v$ and $R(-\theta)\Theta_v=R(-\theta)_v$, we have $R(\theta)(\mathcal{F}_{2,u}-\mathcal{F}_{1,v})R(-\theta)=R(\theta)(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_2\mathcal{F}_1)R(-\theta)$. Since $R(\theta)$ and $R(-\theta)$ are invertible matrices, we conclude that (1) is equivalent to (2).

Next, we prove that (1) is equivalent to (3). We define a matrix R by

$$R=\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Then we have $\mathcal{F}_1^r=R\mathcal{F}_1R$ and $\mathcal{F}_2^r=R\mathcal{F}_2R$ by Proposition 3.6 (2). Thus, we have

$$\mathcal{F}_{2,u}^r-\mathcal{F}_{1,v}^r=R\mathcal{F}_{2,u}R-R\mathcal{F}_{1,v}R=R(\mathcal{F}_{2,u}-\mathcal{F}_{1,v})R.$$

On the other hand,

$$\mathcal{F}_1^r\mathcal{F}_2^r-\mathcal{F}_2^r\mathcal{F}_1^r=R\mathcal{F}_1RR\mathcal{F}_2R-R\mathcal{F}_2RR\mathcal{F}_1R=R(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_2\mathcal{F}_1)R.$$

Note that R^2 is equal to the unit matrix. Since R is an invertible matrix, we conclude that (1) is equivalent to (3). □

Next we consider a parameter change of the domain U and a diffeomorphism of the target space $\mathbb{R}^3$.

Proposition 3.8 *Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be a framed surface with the basic invariants $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$. Let $\phi:V\to U,(p,q)\mapsto\phi(p,q)=(u(p,q),v(p,q))$ be a parameter change, that is, a diffeomorphism of the domain. Then $(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})=(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})\circ\phi:V\to\mathbb{R}^3\times\Delta$ is a framed surface. Moreover, the basic invariants $(\widetilde{\mathcal{G}},\widetilde{\mathcal{F}}_1,\widetilde{\mathcal{F}}_2)$ of $(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{n}},\widetilde{\boldsymbol{s}})$ is given by*

$$\begin{aligned}\begin{pmatrix}\widetilde{a}_1&\widetilde{b}_1\\\widetilde{a}_2&\widetilde{b}_2\end{pmatrix}(p,q) &= \begin{pmatrix}u_p&v_p\\u_q&v_q\end{pmatrix}(p,q)\begin{pmatrix}a_1&b_1\\a_2&b_2\end{pmatrix}(\phi(p,q))\\ \begin{pmatrix}\widetilde{e}_1&\widetilde{f}_1&\widetilde{g}_1\\\widetilde{e}_2&\widetilde{f}_2&\widetilde{g}_2\end{pmatrix}(p,q) &= \begin{pmatrix}u_p&v_p\\u_q&v_q\end{pmatrix}(p,q)\begin{pmatrix}e_1&f_1&g_1\\e_2&f_2&g_2\end{pmatrix}(\phi(p,q)).\end{aligned}$$

Proof. By the chain rule, we have

$$\begin{aligned}\widetilde{\boldsymbol{x}}_p(p,q) &= \boldsymbol{x}_u(\phi(p,q))u_p(p,q)+\boldsymbol{x}_v(\phi(p,q))v_p(p,q)\\ &= \{a_1(\phi(p,q))\boldsymbol{s}(\phi(p,q))+b_1(\phi(p,q))\boldsymbol{t}(\phi(p,q))\}u_p(p,q)\\ &\quad+\{a_2(\phi(p,q))\boldsymbol{s}(\phi(p,q))+b_2(\phi(p,q))\boldsymbol{t}(\phi(p,q)))\}v_p(p,q)\\ &= \{a_1(\phi(p,q))u_p(p,q)+a_2(\phi(p,q))v_p(p,q)\}\widetilde{\boldsymbol{s}}(p,q)\\ &\quad+\{b_1(\phi(p,q))u_p(p,q)+b_2(\phi(p,q))v_p(p,q)\}\widetilde{\boldsymbol{t}}(p,q),\\ \widetilde{\boldsymbol{x}}_q(p,q) &= \boldsymbol{x}_u(\phi(p,q))u_q(p,q)+\boldsymbol{x}_v(\phi(p,q))v_q(p,q)\\ &= \{a_1(\phi(p,q))\boldsymbol{s}(\phi(p,q))+b_1(\phi(p,q))\boldsymbol{t}(\phi(p,q))\}u_q(p,q)\\ &\quad+\{a_2(\phi(p,q))\boldsymbol{s}(\phi(p,q))+b_2(\phi(p,q))\boldsymbol{t}(\phi(p,q)))\}v_q(p,q)\\ &= \{a_1(\phi(p,q))u_q(p,q)+a_2(\phi(p,q))v_q(p,q)\}\widetilde{\boldsymbol{s}}(p,q)\\ &\quad+\{b_1(\phi(p,q))u_q(p,q)+b_2(\phi(p,q))v_q(p,q)\}\widetilde{\boldsymbol{t}}(p,q).\end{aligned}$$

It follows that we have the first equation. The second equation in the proposition is proved similarly as the above by using the chain rule. $\square$

Proposition 3.9 *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Then there exists a smooth mapping $(\boldsymbol{n}^\Phi, \boldsymbol{s}^\Phi) : U \to \Delta$ such that $(\Phi \circ \boldsymbol{x}, \boldsymbol{n}^\Phi, \boldsymbol{s}^\Phi) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.*

Proof. We denote the Jacobi matrix of Φ at x by $D_\Phi(x)$, that is,

$$D_\Phi(x) = \begin{pmatrix} \dfrac{\partial \Phi_1}{\partial x_1}(x) & \dfrac{\partial \Phi_2}{\partial x_1}(x) & \dfrac{\partial \Phi_3}{\partial x_1}(x) \\ \\ \dfrac{\partial \Phi_1}{\partial x_2}(x) & \dfrac{\partial \Phi_2}{\partial x_2}(x) & \dfrac{\partial \Phi_3}{\partial x_2}(x) \\ \\ \dfrac{\partial \Phi_1}{\partial x_3}(x) & \dfrac{\partial \Phi_2}{\partial x_3}(x) & \dfrac{\partial \Phi_3}{\partial x_3}(x) \end{pmatrix}.$$

Since Φ is a diffeomorphism, $D_\Phi(x) \in GL(3, \mathbb{R})$. We define a mapping $(\boldsymbol{n}^\Phi, \boldsymbol{s}^\Phi) : U \to \Delta$ by

$$(\boldsymbol{n}^\Phi, \boldsymbol{s}^\Phi)(u, v) = \left(\frac{\boldsymbol{n}(u, v)\ {}^T(D_\Phi)^{-1}(\boldsymbol{x}(u, v))}{|\boldsymbol{n}(u, v)\ {}^T(D_\Phi)^{-1}(\boldsymbol{x}(u, v))|}, \frac{\boldsymbol{s}(u, v) D_\Phi(\boldsymbol{x}(u, v))}{|\boldsymbol{s}(u, v) D_\Phi(\boldsymbol{x}(u, v))|} \right),$$

where ${}^T A$ is the transpose of the matrix A. Then we show that $(\Phi \circ \boldsymbol{x}, \boldsymbol{n}^\Phi, \boldsymbol{s}^\Phi) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. In fact, since $(d/du)(\Phi \circ \boldsymbol{x})(u, v) = \boldsymbol{x}_u(u, v) D_\Phi \circ \boldsymbol{x}(u, v)$ and $(d/dv)(\Phi \circ \boldsymbol{x})(u, v) = \boldsymbol{x}_v(u, v) D_\Phi \circ \boldsymbol{x}(u, v)$, we have

$$\begin{aligned} \left(\frac{d}{du}(\Phi \circ \boldsymbol{x}) \right) \cdot \boldsymbol{n}^\Phi &= \frac{1}{|\boldsymbol{n}^T (D_\Phi)^{-1} \circ \boldsymbol{x}|} \boldsymbol{x}_u (D_\Phi \circ \boldsymbol{x})((D_\Phi)^{-1} \circ \boldsymbol{x})^T \boldsymbol{n} = \frac{1}{|\boldsymbol{n}^T (D_\Phi)^{-1} \circ \boldsymbol{x}|} \boldsymbol{x}_u{}^T \boldsymbol{n} = 0, \\ \left(\frac{d}{dv}(\Phi \circ \boldsymbol{x}) \right) \cdot \boldsymbol{n}^\Phi &= \frac{1}{|\boldsymbol{n}^T (D_\Phi)^{-1} \circ \boldsymbol{x}|} \boldsymbol{x}_v (D_\Phi \circ \boldsymbol{x})((D_\Phi)^{-1} \circ \boldsymbol{x})^T \boldsymbol{n} = \frac{1}{|\boldsymbol{n}^T (D_\Phi)^{-1} \circ \boldsymbol{x}|} \boldsymbol{x}_v{}^T \boldsymbol{n} = 0. \end{aligned}$$

Note that all vectors in this proof are row vectors. Moreover, we have

$$\begin{aligned} \boldsymbol{n}^\Phi \cdot \boldsymbol{s}^\Phi &= \frac{1}{|\boldsymbol{n}\ {}^T(D_\Phi)^{-1} \circ \boldsymbol{x}||\boldsymbol{s} D_\Phi \circ \boldsymbol{x}|} \boldsymbol{n}({}^T(D_\Phi)^{-1} \circ \boldsymbol{x})({}^T D_\Phi \circ \boldsymbol{x})^T \boldsymbol{s} \\ &= \frac{1}{|\boldsymbol{n}\ {}^T(D_\Phi)^{-1} \circ \boldsymbol{x}||\boldsymbol{s} D_\Phi \circ \boldsymbol{x}|}\ \boldsymbol{n}^T \boldsymbol{s} = 0. \end{aligned}$$

Therefore, $(\Phi \circ \boldsymbol{x}, \boldsymbol{n}^\Phi, \boldsymbol{s}^\Phi) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. $\square$

4 Curvatures of framed surfaces

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.

Definition 4.1 We define a smooth mapping $C_F = (J_F, K_F, H_F) : U \to \mathbb{R}^3$ by

$$J_F = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, K_F = \det \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}, H_F = -\frac{1}{2} \left\{ \det \begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} - \det \begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix} \right\}.$$

We call $C_F = (J_F, K_F, H_F)$ a *curvature of the framed surface.*

Remark 4.2 By the integrability condition (4), we have $K_F = g_{1,v} - g_{2,u}$.

Suppose that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a regular surface. Then $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3 \times S^2$ is a Legendre immersion, where $\boldsymbol{n} = \boldsymbol{x}_u \times \boldsymbol{x}_v / |\boldsymbol{x}_u \times \boldsymbol{x}_v|$. Let $E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v$ be the coefficients of the first fundamental form and $L = -\boldsymbol{x}_u \cdot \boldsymbol{n}_u, M = -\boldsymbol{x}_u \cdot \boldsymbol{n}_v, N = -\boldsymbol{x}_v \cdot \boldsymbol{n}_v$ be the coefficients of the second fundamental form. There exists a smooth mapping $\boldsymbol{s} : U \to S^2$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface. Actually we may take $\boldsymbol{s} = \boldsymbol{x}_u / |\boldsymbol{x}_u|$ or $\boldsymbol{s} = \boldsymbol{x}_v / |\boldsymbol{x}_v|$. We denote the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ for the regular surface $\boldsymbol{x}$. The relationship between the first, second fundamental invariants and the basic invariant is as follows:

$$
\begin{aligned}
&E = a_1^2 + b_1^2, \quad F = a_1 b_1 + a_2 b_2, \quad G = a_2^2 + b_2^2 \\
&L = -a_1 e_1 - b_1 f_1, \quad M = -a_1 e_2 - b_1 f_2, \quad N = -a_2 e_2 - b_2 f_2.
\end{aligned}
$$

By the integrability condition (3), we have $M = -a_2 e_1 - b_2 f_1$. We denote the Gauss curvature and the mean curvature of the regular surface $\boldsymbol{x}$ by K and H. Then

$$
K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F)^2}.
$$

By a direct calculation, we give a relationship between the Gauss curvature, the mean curvature and the curvature of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ as follows.

Proposition 4.3 *Under the above notation, we have $K = K_F / J_F$ and $H = H_F / J_F$.*

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.

Note that a condition $H_F^2(u,v) - J_F(u,v) K_F(u,v) \geq 0$ holds for all $(u,v) \in U$.

We give a relationship between the curvature of the framed surface and the framed surfaces which given by a rotation frame and a reflection frame. We denote the curvatures $C_F^\theta = (J_F^\theta, K_F^\theta, H_F^\theta)$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ and $C_F^r = (J_F^r, K_F^r, H_F^r)$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}^r, \boldsymbol{s}^r)$, respectively.

Proposition 4.4 *Under the above notation, we have the following.*

(1) $(J_F^\theta, K_F^\theta, H_F^\theta) = (J_F, K_F, H_F)$ *for any smooth function* $\theta : U \to \mathbb{R}$.

(2) $(J_F^r, K_F^r, H_F^r) = (-J_F, -K_F, H_F)$.

Proof. (1) By Proposition 3.6 (1), we have

$$
\begin{aligned}
J_F^\theta &= \det \begin{pmatrix} a_1^\theta & b_1^\theta \\ a_2^\theta & b_2^\theta \end{pmatrix} = \det \left\{ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} = J_F, \\
K_F^\theta &= \det \begin{pmatrix} e_1^\theta & f_1^\theta \\ e_2^\theta & f_2^\theta \end{pmatrix} = \det \left\{ \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} = K_F.
\end{aligned}
$$

We show $H_F^\theta = H_F$. By Proposition 3.6 (1), we also have

$$
\begin{aligned}
\begin{pmatrix} a_1^\theta & f_1^\theta \\ a_2^\theta & f_2^\theta \end{pmatrix} &= \begin{pmatrix} a_1 \cos\theta - b_1 \sin\theta & e_1 \sin\theta + f_1 \cos\theta \\ a_2 \cos\theta - b_2 \sin\theta & e_2 \sin\theta + f_2 \cos\theta \end{pmatrix}, \\
\begin{pmatrix} b_1^\theta & e_1^\theta \\ b_2^\theta & e_2^\theta \end{pmatrix} &= \begin{pmatrix} a_1 \sin\theta + b_1 \cos\theta & e_1 \cos\theta - f_1 \sin\theta \\ a_2 \sin\theta + b_2 \cos\theta & e_2 \cos\theta - f_2 \sin\theta \end{pmatrix}.
\end{aligned}
$$

It follows that

$$\begin{aligned}
\det\begin{pmatrix} a_1^\theta & f_1^\theta \\ a_2^\theta & f_2^\theta \end{pmatrix} &= a_1e_2\cos\theta\sin\theta - b_1f_2\sin\theta\cos\theta + a_1f_2\cos^2\theta - b_1e_2\sin^2\theta \\
&\quad -e_1a_2\cos\theta\sin\theta + f_1b_2\cos\theta\sin\theta + e_1b_2\sin^2\theta - f_1a_2\cos^2\theta, \\
\det\begin{pmatrix} b_1^\theta & e_1^\theta \\ b_2^\theta & e_2^\theta \end{pmatrix} &= a_1e_2\cos\theta\sin\theta - b_1f_2\cos\theta\sin\theta - a_1f_2\sin^2\theta + b_1e_2\cos^2\theta \\
&\quad -e_1a_2\cos\theta\sin\theta + f_1b_2\sin\theta\cos\theta - e_1b_2\cos^2\theta + f_1a_2\sin^2\theta.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
H_F^\theta &= -\frac{1}{2}\left\{\det\begin{pmatrix} a_1^\theta & f_1^\theta \\ a_2^\theta & f_2^\theta \end{pmatrix} - \det\begin{pmatrix} b_1^\theta & e_1^\theta \\ b_2^\theta & e_2^\theta \end{pmatrix}\right\} \\
&= -\frac{1}{2}(a_1f_2\cos^2\theta - b_1e_2\sin^2\theta + e_1b_2\sin^2\theta - f_1a_2\cos^2\theta \\
&\quad +a_1f_2\sin^2\theta - b_1e_2\cos^2\theta + e_1b_2\cos^2\theta - f_1a_2\sin^2\theta) \\
&= -\frac{1}{2}\left(a_1f_2 - f_1a_2 - b_1e_2 + e_1b_2\right) = H_F.
\end{aligned}$$

(2) By Proposition 3.6 (2), we have

$$\begin{aligned}
J_F^r &= \det\begin{pmatrix} a_1^r & b_1^r \\ a_2^r & b_2^r \end{pmatrix} = \det\left\{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = -J_F, \\
K_F^r &= \det\begin{pmatrix} e_1^r & f_1^r \\ e_2^r & f_2^r \end{pmatrix} = \det\left\{\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = -K_F.
\end{aligned}$$

Moreover,

$$\begin{aligned}
H_F^r &= -\frac{1}{2}\left\{\det\begin{pmatrix} a_1^r & f_1^r \\ a_2^r & f_2^r \end{pmatrix} - \det\begin{pmatrix} b_1^r & e_1^r \\ b_2^r & e_2^r \end{pmatrix}\right\} \\
&= -\frac{1}{2}\left\{\det\begin{pmatrix} b_1 & -e_1 \\ b_2 & -e_2 \end{pmatrix} - \det\begin{pmatrix} a_1 & -f_1 \\ a_2 & -f_2 \end{pmatrix}\right\} = H_F.
\end{aligned}$$

□

Let $\phi : V \to U, (p,q) \mapsto \phi(p,q) = (u(p,q), v(p,q))$ be a parameter change. By Proposition 3.8, $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) = (\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ is a framed surface with the basic invariants $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$. We denote the curvature of the framed surface $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ by $(\widetilde{J}_F, \widetilde{K}_F, \widetilde{H}_F)$.

Proposition 4.5 *Under the above notation, the curvature $(\widetilde{J}_F, \widetilde{K}_F, \widetilde{H}_F) : V \to \mathbb{R}^3$ is given by*

$$(\widetilde{J}_F(p,q), \widetilde{K}_F(p,q), \widetilde{H}_F(p,q)) = (J_\phi(p,q)J_F(\phi(p,q)), J_\phi(p,q)K_F(\phi(p,q)), J_\phi(p,q)H_F(\phi(p,q))),$$

where J_ϕ is the Jacobian of the parameter change ϕ.

Proof. By Proposition 3.8, we have $\widetilde{J}_F(p,q) = J_\phi(p,q)J_F(\phi(p,q))$ and $\widetilde{K}_F(p,q) = J_\phi(p,q)K_F(\phi(p,q))$. Since

$$\begin{aligned}
\begin{pmatrix} \widetilde{a}_1 & \widetilde{f}_1 \\ \widetilde{a}_2 & \widetilde{f}_2 \end{pmatrix}(p,q) &= \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix}(\phi(p,q)), \\
\begin{pmatrix} \widetilde{b}_1 & \widetilde{e}_1 \\ \widetilde{b}_2 & \widetilde{e}_2 \end{pmatrix}(p,q) &= \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p,q)\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}(\phi(p,q)),
\end{aligned}$$

we have $\widetilde{H}_F(p,q)=J_\phi(p,q)H_F(\phi(p,q))$. $\Box$

The curvature is useful to recognize that the framed base surface is a front or not.

Proposition 4.6 *Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be a framed surface and $p\in U$. Then $(\boldsymbol{x},\boldsymbol{n}):U\to\mathbb{R}^3\times S^2$ is a Legendre immersion around p if and only if $C_F(p)\neq 0$.*

Proof. We show the necessarily part of the proposition, that is, if $C_F(p)=0$, then $(\boldsymbol{x},\boldsymbol{n}):U\to\mathbb{R}^3\times S^2$ is not a Legendre immersion at p. Since $J_F(p)=0$, there exist $k_1,k_2\in\mathbb{R}$ such that $k_1^2+k_2^2\neq 0$ and $k_1(a_1,a_2)+k_2(b_1,b_2)=0$ at p. Moreover, since $K_F(p)=0$, there exist $h_1,h_2\in\mathbb{R}$ such that $h_1^2+h_2^2\neq 0$ and $h_1(e_1,e_2)+h_2(f_1,f_2)=0$ at p. We divide into the following four cases: $k_1h_1\neq 0$, $k_2h_1\neq 0$, $k_1h_2\neq 0$ and $k_2h_2\neq 0$.

Suppose that $k_1h_1\neq 0$. In this case, we have $(a_1,a_2)=-(k_2/k_1)(b_1,b_2)$ and $(e_1,e_2)=-(h_2/h_1)(f_1,f_2)$ at p. Thus,

$$\begin{pmatrix}\boldsymbol{x}_u & \boldsymbol{n}_u\\ \boldsymbol{x}_v & \boldsymbol{n}_v\end{pmatrix}(p)=\begin{pmatrix}b_1\boldsymbol{w}_1 & f_1\boldsymbol{w}_2\\ b_2\boldsymbol{w}_1 & f_2\boldsymbol{w}_2\end{pmatrix}(p),$$

where $\boldsymbol{w}_1=-(k_2/k_1)\boldsymbol{s}+\boldsymbol{t}$ and $\boldsymbol{w}_2=-(h_2/h_1)\boldsymbol{s}+\boldsymbol{t}$. Since $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are non-zero vectors, $\operatorname{rank}\begin{pmatrix}\boldsymbol{x}_u & \boldsymbol{n}_u\\ \boldsymbol{x}_v & \boldsymbol{n}_v\end{pmatrix}(p)<2$ if and only if $\det\begin{pmatrix}b_1 & f_1\\ b_2 & f_2\end{pmatrix}(p)=0$.

Now suppose that $\det\begin{pmatrix}b_1 & f_1\\ b_2 & f_2\end{pmatrix}(p)\neq 0$. By the assumption $H_F(p)=0$, we have

$$0 \;=\; \det\begin{pmatrix}a_1 & f_1\\ a_2 & f_2\end{pmatrix}(p)-\det\begin{pmatrix}b_1 & e_1\\ b_2 & e_2\end{pmatrix}(p)=\left(-\frac{k_2}{k_1}+\frac{h_2}{h_1}\right)\det\begin{pmatrix}b_1 & f_1\\ b_2 & f_2\end{pmatrix}(p).$$

It follows that

$$-\frac{k_2}{k_1}+\frac{h_2}{h_1}=0. \tag{5}$$

On the other hand, by the integrability condition (4),

$$0=\det\begin{pmatrix}a_1 & e_1\\ a_2 & e_2\end{pmatrix}(p)+\det\begin{pmatrix}b_1 & f_1\\ b_2 & f_2\end{pmatrix}(p)=\left(\frac{h_2k_2}{h_1k_1}+1\right)\det\begin{pmatrix}b_1 & f_1\\ b_2 & f_2\end{pmatrix}(p).$$

Hence, we have

$$\frac{h_2k_2}{h_1k_1}+1=0. \tag{6}$$

By the equations (5) and (6), we have $h_2^2/h_1^2+1=0$, and this is a contradiction. Therefore, we conclude $\det\begin{pmatrix}b_1 & f_1\\ b_2 & f_2\end{pmatrix}(p)=0$. It follows that $(\boldsymbol{x},\boldsymbol{n})$ is not an immersion at p. The other cases are also proved similarly.

Conversely, if $\operatorname{rank}\begin{pmatrix}\boldsymbol{x}_u & \boldsymbol{n}_u\\ \boldsymbol{x}_v & \boldsymbol{n}_v\end{pmatrix}(p)<2$, then there exist $k_1,k_2\in\mathbb{R}$ such that $k_1^2+k_2^2\neq 0$ and $k_1(a_1,b_1,e_1,f_1)+k_2(a_2,b_2,e_2,f_2)=0$ at p. By substituting this relations into C_F, we have $C_F(p)=0$. $\Box$

Remark 4.7 By Propositions 4.3 and 4.6, if $(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre immersion around $p \in U$ and p is a singular point of $\boldsymbol{x}$, then the Gauss curvature K or the mean curvature H must be divergence at the point p.

By Proposition 4.6, if $C_F(p) = 0$, then $\boldsymbol{x}$ is not a front but a frontal at the point, that is, $(\boldsymbol{x}, \boldsymbol{n})$ is not an immersion. How about the condition that the framed surface is an immersion or not? Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We define a smooth mapping $I_F : U \to \mathbb{R}^8$ by

$$I_F = \left(C_F, \det \begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix} \right).$$

We call the mapping $I_F : U \to \mathbb{R}^8$ a *concomitant mapping* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed immersion* if $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is an immersion.

Proposition 4.8 *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and $p \in U$. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed immersion around p if and only if $I_F(p) \neq 0$.*

Proof. We show the necessarily part of the proposition, that is, if $I_F(p) = 0$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is not a framed immersion at p. It is enough to show that

$$\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix} (p) < 2.$$

The above condition is equivalent to the following conditions,

$$\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{n}_u \\ \boldsymbol{x}_v & \boldsymbol{n}_v \end{pmatrix} (p), \ \operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{s}_v \end{pmatrix} (p), \ \operatorname{rank} \begin{pmatrix} \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix} (p) < 2.$$

By the assumption $C_F(p) = 0$ and Proposition 4.6, $\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{n}_u \\ \boldsymbol{x}_v & \boldsymbol{n}_v \end{pmatrix} (p) < 2$.

We show $\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{s}_v \end{pmatrix} (p) < 2$. By the definition of the basic invariants, we have

$$\begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{s}_v \end{pmatrix} = \begin{pmatrix} a_1 \boldsymbol{s} + b_1 \boldsymbol{t} & -e_1 \boldsymbol{s} + g_1 \boldsymbol{t} \\ a_2 \boldsymbol{s} + b_2 \boldsymbol{t} & -e_2 \boldsymbol{s} + g_2 \boldsymbol{t} \end{pmatrix}.$$

Since $J_F(p) = 0$ and $\det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix} (p) = 0$, there exist $k_1, k_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$ and $k_1(a_1, a_2) + k_2(b_1, b_2) = 0$ at p. Moreover, there exist $h_1, h_2 \in \mathbb{R}$ such that $h_1^2 + h_2^2 \neq 0$ and $h_1(e_1, e_2) + h_2(g_1, g_2) = 0$ at p. We divide into the following four cases: $k_1 h_1 \neq 0$, $k_2 h_1 \neq 0$, $k_1 h_2 \neq 0$ and $k_2 h_2 \neq 0$.

Suppose that $k_1 h_1 \neq 0$. In this case, we have $(a_1, a_2) = -(k_2/k_1)(b_1, b_2)$ and $(e_1, e_2) = -(h_2/h_1)(g_1, g_2)$ at p. Thus,

$$\begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{s}_v \end{pmatrix} (p) = \begin{pmatrix} b_1 \boldsymbol{w}_1 & g_1 \boldsymbol{w}_1 \\ b_2 \boldsymbol{w}_2 & g_2 \boldsymbol{w}_2 \end{pmatrix} (p),$$

where $\boldsymbol{w}_1 = -(k_2/k_1)\boldsymbol{s} + \boldsymbol{t}$ and $\boldsymbol{w}_2 = (h_2/h_1)\boldsymbol{s} + \boldsymbol{t}$. Since $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are non-zero vectors, $\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{s}_v \end{pmatrix} (p) < 2$ if and only if $\det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix} (p) = 0$. By the assumption $I_F(p) = 0$, we have $\det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}(p) = 0$. Therefore, $\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{s}_v \end{pmatrix}(p) < 2$. The other cases are also proved similarly.

Next, we show $\operatorname{rank} \begin{pmatrix} \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix}(p) < 2$. By the definition of the basic invariants, we have

$$\begin{pmatrix} \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix}(p) = \begin{pmatrix} e_1\boldsymbol{s} + f_1\boldsymbol{t} & -e_1\boldsymbol{s} + g_1\boldsymbol{t} \\ e_2\boldsymbol{s} + f_2\boldsymbol{t} & -e_2\boldsymbol{s} + g_2\boldsymbol{t} \end{pmatrix}(p).$$

Since we assume $K_F(p) = 0$ and $\det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix}(p) = 0$, there exist $k_1, k_2, h_1, h_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0, h_1^2 + h_2^2 \neq 0$, $k_1(e_1, e_2) + k_2(f_1, f_2) = 0$ and $h_1(e_1, e_2) + h_2(g_1, g_2) = 0$ at p. We divide into the following four cases: $k_1h_1 \neq 0$, $k_2h_1 \neq 0$, $k_1h_2 \neq 0$ and $k_2h_2 \neq 0$.

Suppose that $k_1h_1 \neq 0$. In this case, we have $(e_1, e_2) = -(k_2/k_1)(f_1, f_2)$ and $(e_1, e_2) = -(h_2/h_1)(g_1, g_2)$ at p. Thus,

$$\begin{pmatrix} \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix}(p) = \begin{pmatrix} f_1\boldsymbol{w}_1 & g_1\boldsymbol{w}_1 \\ f_2\boldsymbol{w}_2 & g_2\boldsymbol{w}_2 \end{pmatrix}(p),$$

where $\boldsymbol{w}_1 = -(k_2/k_1)\boldsymbol{s} + \boldsymbol{t}$ and $\boldsymbol{w}_2 = (h_2/h_1)\boldsymbol{s} + \boldsymbol{t}$. Since $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are non-zero vectors, $\operatorname{rank} \begin{pmatrix} \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix}(p) < 2$ if and only if $\det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}(p) = 0$. By the assumption $I_F(p) = 0$, we have $\det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}(p) = 0$. Therefore, $\operatorname{rank} \begin{pmatrix} \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix}(p) < 2$. The other cases are also proved similarly. Therefore, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is not an immersion at p.

Conversely, if $\operatorname{rank} \begin{pmatrix} \boldsymbol{x}_u & \boldsymbol{n}_u & \boldsymbol{s}_u \\ \boldsymbol{x}_v & \boldsymbol{n}_v & \boldsymbol{s}_v \end{pmatrix}(p) < 2$, then there exist $k_1, k_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$ and $k_1(a_1, b_1, e_1, f_1, g_1) + k_2(a_2, b_2, e_2, f_2, g_2) = 0$ at p. By substituting this relations into I_F, we have $I_F(p) = 0$. $\Box$

As a summary, we have the following result.

Corollary 4.9 *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and $p \in U$.*

(1) *$\boldsymbol{x}$ is an immersion (a regular surface) around p if and only if $J_F(p) \neq 0$.*

(2) *$(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre immersion around p if and only if $C_F(p) \neq 0$.*

(3) *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed immersion around p if and only if $I_F(p) \neq 0$.*

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with I_F. We denote $I_F = (I_{F,1}, \cdots, I_{F,8})$ and $C_F = (J_F, K_F, H_F) = (I_{F,1}, I_{F,2}, I_{F,3})$. Let $\phi : V \to U, (p, q) \mapsto \phi(p, q) = (u(p, q), v(p, q))$ be a parameter change of the domain. We denote the concomitant mapping of the framed surface $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) = (\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ by $\widetilde{I_F}$. By Proposition 3.8, we have the following proposition.

Proposition 4.10 *Under the above notation, the concomitant mapping $\widetilde{I}_F : V \to \mathbb{R}^8$ is given by*

$$(\widetilde{I}_{F,1}(p, q), \cdots, \widetilde{I}_{F,8}(p, q)) = (J_\phi(p, q)I_{F,1}(\phi(p, q)), \cdots, J_\phi(p, q)I_{F,8}(\phi(p, q))).$$

Remark 4.11 *We denote the concomitant mapping of the framed surfaces which given by a rotation frame (respectively, a reflection frame) by I_F^θ (respectively I_F^r). By Proposition 3.6* (1) *and (2), we have the following.*

$$
\begin{aligned}
I^\theta_{F,4} &= \det\begin{pmatrix} a^\theta_1 & g^\theta_1 \\ a^\theta_2 & g^\theta_2 \end{pmatrix} = I_{F,4}\cos\theta - I_{F,5}\sin\theta - \det\begin{pmatrix} a_1 & \theta_u \\ a_2 & \theta_v \end{pmatrix}\cos\theta + \det\begin{pmatrix} b_1 & \theta_u \\ b_2 & \theta_v \end{pmatrix}\sin\theta, \\
I^\theta_{F,5} &= \det\begin{pmatrix} b^\theta_1 & g^\theta_1 \\ b^\theta_2 & g^\theta_2 \end{pmatrix} = I_{F,4}\sin\theta + I_{F,5}\cos\theta - \det\begin{pmatrix} a_1 & \theta_u \\ a_2 & \theta_v \end{pmatrix}\sin\theta - \det\begin{pmatrix} b_1 & \theta_u \\ b_2 & \theta_v \end{pmatrix}\cos\theta, \\
I^\theta_{F,6} &= \det\begin{pmatrix} e^\theta_1 & g^\theta_1 \\ e^\theta_2 & g^\theta_2 \end{pmatrix} = I_{F,6}\cos\theta - I_{F,7}\sin\theta - \det\begin{pmatrix} e_1 & \theta_u \\ e_2 & \theta_v \end{pmatrix}\cos\theta + \det\begin{pmatrix} f_1 & \theta_u \\ f_2 & \theta_v \end{pmatrix}\sin\theta, \\
I^\theta_{F,7} &= \det\begin{pmatrix} f^\theta_1 & g^\theta_1 \\ f^\theta_2 & g^\theta_2 \end{pmatrix} = I_{F,6}\sin\theta + I_{F,7}\cos\theta - \det\begin{pmatrix} e_1 & \theta_u \\ e_2 & \theta_v \end{pmatrix}\sin\theta - \det\begin{pmatrix} f_1 & \theta_u \\ f_2 & \theta_v \end{pmatrix}\cos\theta, \\
I^\theta_{F,8} &= \det\begin{pmatrix} a^\theta_1 & e^\theta_1 \\ a^\theta_2 & e^\theta_2 \end{pmatrix} = (\cos^2\theta - \sin^2\theta)I_{F,8} - \cos\theta\sin\theta\left\{\det\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} + \det\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
I^r_{F,4} &= \det\begin{pmatrix} a^r_1 & g^r_1 \\ a^r_2 & g^r_2 \end{pmatrix} = \det\begin{pmatrix} b_1 & -g_1 \\ b_2 & -g_2 \end{pmatrix} = -\det\begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}, \\
I^r_{F,5} &= \det\begin{pmatrix} b^r_1 & g^r_1 \\ b^r_2 & g^r_2 \end{pmatrix} = \det\begin{pmatrix} a_1 & -g_1 \\ a_2 & -g_2 \end{pmatrix} = -\det\begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix}, \\
I^r_{F,6} &= \det\begin{pmatrix} e^r_1 & g^r_1 \\ e^r_2 & g^r_2 \end{pmatrix} = \det\begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \\
I^r_{F,7} &= \det\begin{pmatrix} f^r_1 & g^r_1 \\ f^r_2 & g^r_2 \end{pmatrix} = \det\begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix}, \\
I^r_{F,8} &= \det\begin{pmatrix} a^r_1 & e^r_1 \\ a^r_2 & e^r_2 \end{pmatrix} = \det\begin{pmatrix} b_1 & -f_1 \\ b_2 & -f_2 \end{pmatrix} = -\det\begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix} = \det\begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix},
\end{aligned}
$$

that is, $I^r_F = (-J_F, -K_F, H_F, -I_{F,5}, -I_{F,4}, I_{F,7}, I_{F,6}, I_{F,8})$.

Proposition 4.12 *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *be a framed surface with the basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.

(1) *Suppose that* $(g_1, g_2) \neq (0,0)$ *at* $p \in U$. *If*

$$\det\begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix} = \det\begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix} = \det\begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix} = \det\begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} = 0$$

at p, *then* $I_F(p) = 0$.

(2) *Suppose that* $(g_1, g_2) = (0,0)$ *at* $p \in U$. *If* $C_F(p) = 0$, *then* $I_F(p) = 0$.

Proof. (1) By the assumptions, there exist $k_i \in \mathbb{R}, i = 1, \dots, 4$ such that

$$(a_1, a_2) = k_1(g_1, g_2), (b_1, b_2) = k_2(g_1, g_2), (e_1, e_2) = k_3(g_1, g_2), (f_1, f_2) = k_4(g_1, g_2)$$

at $p \in U$. It follows that $I_F(p) = 0$.

(2) Since $C_F(p) = 0$ and Proposition 4.6, $(\boldsymbol{x}, \boldsymbol{n})$ is not an immersion at $p \in U$. It follows that $\det\begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix} = 0$. Hence we have $I_F(p) = 0$. □

Next, we consider parallel surfaces of framed surfaces. For a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$, we define a parallel surface $\boldsymbol{x}^\lambda : U \to \mathbb{R}^3$ of the framed surface by $\boldsymbol{x}^\lambda(u,v) = \boldsymbol{x}(u,v) + \lambda\boldsymbol{n}(u,v)$, where $\lambda \in \mathbb{R}$.

Proposition 4.13 *Under the above notations, $\boldsymbol{x}^\lambda$ is a framed base surface. Indeed, $(\boldsymbol{x}^\lambda, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.*

Proof. By definition,

$$\begin{aligned}
\boldsymbol{x}^\lambda_u &= \boldsymbol{x}_u + \lambda \boldsymbol{n}_u = (a_1 + \lambda e_1)\boldsymbol{s} + (b_1 + \lambda f_1)\boldsymbol{t}, \\
\boldsymbol{x}^\lambda_v &= \boldsymbol{x}_v + \lambda \boldsymbol{n}_v = (a_2 + \lambda e_2)\boldsymbol{s} + (b_2 + \lambda f_2)\boldsymbol{t}.
\end{aligned}$$

Thus, $\boldsymbol{x}^\lambda_u \cdot \boldsymbol{n} = \boldsymbol{x}^\lambda_v \cdot \boldsymbol{n} = 0$. Since $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface, we have $\boldsymbol{n} \cdot \boldsymbol{s} = 0$. Therefore, $(\boldsymbol{x}^\lambda, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface. □

By a direct calculation, we have the following proposition.

Proposition 4.14 *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with the basic invariant $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and the concomitant mapping I_F. Then, the basic invariant $(\mathcal{G}^\lambda, \mathcal{F}_1^\lambda, \mathcal{F}_2^\lambda)$ and the concomitant mapping I^λ_F of the parallel surface $(\boldsymbol{x}^\lambda, \boldsymbol{n}, \boldsymbol{s})$ are given by*

$$\begin{aligned}
\mathcal{G}^\lambda &= \mathcal{G} + \lambda \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}, \ \mathcal{F}_1^\lambda = \mathcal{F}_1, \ \mathcal{F}_2^\lambda = \mathcal{F}_2, \\
J^\lambda_F &= J_F - 2H_F\lambda + K_F\lambda^2, \ K^\lambda_F = K_F, \ H^\lambda_F = H_F - K_F\lambda, \\
I^\lambda_{F,4} &= I_{F,4} + \lambda I_{F,6}, \ I^\lambda_{F,5} = I_{F,5} + \lambda I_{F,7}, \ I^\lambda_{F,6} = I_{F,6}, \ I^\lambda_{F,7} = I_{F,7}, \ I^\lambda_{F,8} = I_{F,8}.
\end{aligned}$$

5 Framed surfaces as one-parameter families of Legendre curves along framed curves

We consider a framed curve in the Euclidean space ([11]) and a one-parameter family of Legendre curves ([8, 27]). We construct framed surfaces as one-parameter families of Legendre curves along the framed curves. The idea is a cut off the surface by a plane of a special direction along a space curve.

Let $I, J \subset \mathbb{R}$ be intervals with parameters u, v, respectively. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, we denote the orthonormal plane of $\boldsymbol{a}$ through $\boldsymbol{b}$ by $\langle \boldsymbol{a} \rangle^\perp_{\boldsymbol{b}}$, that is,

$$\langle \boldsymbol{a} \rangle^\perp_{\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^3 | \boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{b}) = 0\}.$$

If $\boldsymbol{b}$ is the origin, then we denote $\langle \boldsymbol{a} \rangle^\perp_0$ by $\langle \boldsymbol{a} \rangle^\perp$ briefly.

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α), see Appendix A (cf. [11]). We denote $\boldsymbol{\mu}(u) = \nu_1(u) \times \nu_2(u)$. For each $u \in I$, we consider a Legendre curve $(\boldsymbol{x}(u, \cdot), \nu^L(u, \cdot)) : J \to \langle \boldsymbol{\mu}(u) \rangle^\perp_{\gamma(u)} \times (S^2 \cap \langle \boldsymbol{\mu}(u) \rangle^\perp)$, that is, $\boldsymbol{x}_v(u, v) \cdot \nu^L(u, v) = 0$ for all $(u, v) \in I \times J$. We identify the Euclidean plane $\mathbb{R}^2$ and the plane $\langle \boldsymbol{\mu}(u) \rangle^\perp_{\gamma(u)}$ via $(a_1, a_2) \mapsto \gamma(u) + a_1\nu_1(u) + a_2\nu_2(u)$, and S^1 and $S^2 \cap \langle \boldsymbol{\mu}(u) \rangle^\perp$ via $(b_1, b_2) \mapsto b_1\nu_1(u) + b_2\nu_2(u)$. We consider induced innor product on $\langle \boldsymbol{\mu}(u) \rangle^\perp$ by $(a_1\nu_1(u) + a_2\nu_2(u)) \cdot (b_1\nu_1(u) + b_2\nu_2(u)) = a_1b_1 + a_2b_2$. Under the identification, $(\boldsymbol{x}(u, \cdot), \nu^L(u, \cdot))$ is a Legendre curve in the sense of Appendix B (cf. [8]). The curvature of the Legendre curve $(\boldsymbol{x}(u, \cdot), \nu^L(u, \cdot))$ is denoted by $(\ell^L(u, \cdot), \beta^L(u, \cdot))$. By definition, there exist functions $x_1, x_2 : I \times J \to \mathbb{R}$ such that $\boldsymbol{x} : I \times J \to \mathbb{R}^3$ is given by $\boldsymbol{x}(u, v) = \gamma(u) + x_1(u, v)\nu_1(u) + x_2(u, v)\nu_2(u)$. We assume that x_1 and x_2 are smooth functions, namely, $\boldsymbol{x}$ is a smooth surface. We denote $\nu^L(u, v) = \nu^L_1(u, v)\nu_1(u) + \nu^L_2(u, v)\nu_2(u)$

and $\boldsymbol{\mu}^L(u,v) = -\nu_2^L(u,v)\nu_1(u) + \nu_1^L(u,v)\nu_2(u)$. We also assume that ν_1^L and ν_2^L are smooth functions. It follows that the curvature of the Legendre curve $(\ell^L, \beta^L) : I \times J \to \mathbb{R}^2$ is a smooth mapping. Furthermore, we suppose that there exists a smooth function $\theta : I \times J \to \mathbb{R}$ such that $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in I \times J$, where $\boldsymbol{n}(u,v) = \cos\theta(u,v)\nu^L(u,v) + \sin\theta(u,v)\boldsymbol{\mu}(u)$. We define $\boldsymbol{s} : I \times J \to S^2$ by $\boldsymbol{s}(u,v) = -\boldsymbol{\mu}^L(u,v)$.

Theorem 5.1 *Under the above notations, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : I \times J \to \mathbb{R}^3 \times \Delta$ is a framed surface with the basic invariants,*

$$
\begin{aligned}
a_1(u,v) &= (x_{1u}(u,v) - x_2(u,v)\ell(u))\nu_2^L(u,v) - (x_{2u}(u,v) + x_1(u,v)\ell(u))\nu_1^L(u,v),\\
b_1(u,v) &= \sin\theta(u,v)\big((x_{1u}(u,v) - x_2(u,v)\ell(u))\nu_1^L(u,v) + (x_{2u}(u,v) + x_1(u,v)\ell(u))\nu_2^L(u,v)\big)\\
&\quad - \cos\theta(u,v)(\alpha(u) + x_1(u,v)m(u) + x_2(u,v)n(u)),\\
a_2(u,v) &= -\beta^L(u,v),\\
b_2(u,v) &= 0,\\
e_1(u,v) &= \sin\theta(u,v)(n(u)\nu_1^L(u,v) - m(u)\nu_2^L(u,v))\\
&\quad + \cos\theta(u,v)(\nu_{1u}^L(u,v)\nu_2^L(u,v) - \nu_{2u}^L(u,v)\nu_1^L(u,v) - \ell(u)),\\
f_1(u,v) &= -\theta_u(u,v) - m(u)\nu_1^L(u,v) - n(u)\nu_2^L(u,v),\\
g_1(u,v) &= \sin\theta(u,v)(\nu_{2u}^L(u,v)\nu_1^L(u,v) - \nu_{1u}^L(u,v)\nu_2^L(u,v) + \ell(u))\\
&\quad + \cos\theta(u,v)(n(u)\nu_1^L(u,v) - m(u)\nu_2^L(u,v)),\\
e_2(u,v) &= -\cos\theta(u,v)\ell^L(u,v),\\
f_2(u,v) &= -\theta_v(u,v),\\
g_2(u,v) &= \sin\theta(u,v)\ell^L(u,v).
\end{aligned}
$$

Proof. By definition, we have $\boldsymbol{n}(u,v) \cdot \boldsymbol{s}(u,v) = 0$ for all $(u,v) \in I \times J$. It follows that $(\boldsymbol{n}, \boldsymbol{s}) \in \Delta$. By the assumption, we have $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in I \times J$. Since $\boldsymbol{x}_v(u,v) \cdot \nu^L(u,v) = 0$, we have

$$
\begin{aligned}
\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) &= (x_{1v}(u,v)\nu_1(u) + x_{2v}\nu_2(u)) \cdot (\cos\theta(u,v)\nu^L(u,v) + \sin\theta(u,v)\boldsymbol{\mu}(u))\\
&= \cos\theta(u,v)(x_{1v}(u,v)\nu_1^L(u,v) + x_{2v}(u,v)\nu_2^L(u,v)) = 0
\end{aligned}
$$

for all $(u,v) \in I \times J$. Hence $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : I \times J \to \mathbb{R}^3 \times \Delta$ is a framed surface. We omit (u,v) and u below. By a direct calculation, we have

$$
\begin{aligned}
\boldsymbol{x}_u &= (x_{1u} - x_2\ell)\nu_1 + (x_{2u} + x_1\ell)\nu_2 + (\alpha + x_1 m + x_2 n)\boldsymbol{\mu},\\
\boldsymbol{x}_v &= x_{1v}\nu_1 + x_{2v}\nu_2,\\
\boldsymbol{n} &= \cos\theta\nu_1^L\nu_1 + \cos\theta\nu_2^L\nu_2 + \sin\theta\boldsymbol{\mu},\\
\boldsymbol{s} &= \nu_2^L\nu_1 - \nu_1^L\nu_2,\\
\boldsymbol{t} &= \boldsymbol{n} \times \boldsymbol{s} = \sin\theta\nu_1^L\nu_1 + \sin\theta\nu_2^L\nu_2 - \cos\theta\boldsymbol{\mu},\\
\boldsymbol{n}_u &= (-\theta_u\sin\theta\nu_1^L + \cos\theta\nu_{1u}^L - \cos\theta\nu_2^L\ell - \sin\theta m)\nu_1\\
&\quad + (-\theta_u\sin\theta\nu_2^L + \cos\theta\nu_1^L\ell + \cos\theta\nu_{2u}^L - \sin\theta n)\nu_2\\
&\quad + \cos\theta(\nu_1^L m + \nu_2^L n + \theta_u)\boldsymbol{\mu},\\
\boldsymbol{s}_u &= (\nu_{2u}^L + \nu_1^L\ell)\nu_1 + (-\nu_{1u}^L + \nu_2^L\ell)\nu_2 + (\nu_2^L m - \nu_1^L n)\boldsymbol{\mu},\\
\boldsymbol{n}_v &= -\theta_v\sin\theta\nu^L + \cos\theta\nu_v^L + \theta_v\cos\theta\boldsymbol{\mu},\\
\boldsymbol{s}_v &= \ell^L\nu^L.
\end{aligned}
$$

It follows that we have the basic invariants as the above. $\Box$

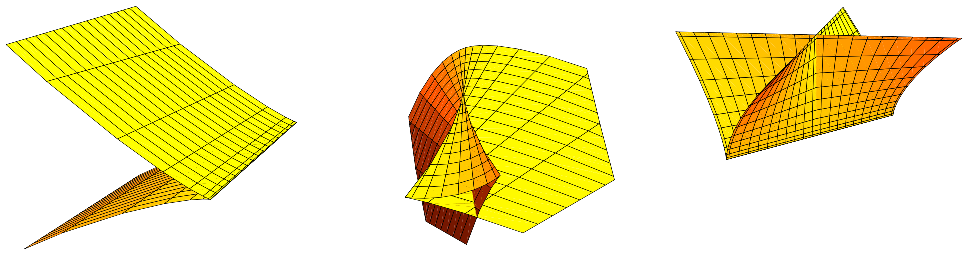
By a direct calculation, we have the following condition:

$$\begin{aligned}\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v) &= (x_{1u}(u,v)-x_2(u,v)\ell(u))\cos\theta(u,v)\nu_1^L(u,v)\\ &\quad +(x_{2u}(u,v)+x_1(u,v)\ell(u))\cos\theta(u,v)\nu_2^L(u,v)\\ &\quad +(\alpha(u)+x_1(u,v)m(u)+x_2(u,v)n(u))\sin\theta(u,v)\\ &= 0\end{aligned}$$

for all $(u,v)\in I\times J$.

By the above construction, we say that the framed surface $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is *a one-parameter family of Legendre curves along a framed curve*.

As an application of Theorem 5.1, we give a condition that the surface $\boldsymbol{x}$ is diffeomorphic to the cuspidal edge, the swallowtail and the cuspidal cross cap, see Figure 1 and Examples 6.1, 6.2 and 6.3.



cuspidal edge swallowtail cuspidal cross cap
Figure 1.

We recall the criteria for singularities of frontals stated in [5, 18] (see also, [15]). Let $\boldsymbol{x}:U\to\mathbb{R}^3$ be the frontal of a Legendre surface $(\boldsymbol{x},\boldsymbol{n})$. We define a function $\lambda:U\to\mathbb{R}$ by $\lambda(u,v)=\det(\boldsymbol{x}_u,\boldsymbol{x}_v,\boldsymbol{n})(u,v)$ where (u,v) is a coordinate system on U. We call λ a *discriminant function* (or, a *signed area density function*). When a singular point p of $\boldsymbol{x}$ is non-degenerate, that is, $d\lambda(p)\neq 0$, there exists a smooth parametrization $\delta(t):(-\varepsilon,\varepsilon)\to U$, $\delta(0)=p$ of the singular set $S(\boldsymbol{x})$. We call the curve $\delta(t)$ the singular curve of $\boldsymbol{x}$. Moreover, there exists a smooth vector field $\eta(t)$ along δ satisfying that $\eta(t)$ generates $\ker d\boldsymbol{x}_{\delta(t)}$. Now we define a function $\phi_{\boldsymbol{x}}(t)$ on $(-\varepsilon,\varepsilon)$ by $\phi_{\boldsymbol{x}}(t)=\det((\boldsymbol{x}\circ\delta)',\boldsymbol{n}\circ\delta,d\boldsymbol{n}(\eta))(t)$. By using these notations, we have the following theorem.

Theorem 5.2 ([5, 18]) *Let $(\boldsymbol{x},\boldsymbol{n}):U\to\mathbb{R}^3\times S^2$ be a Legendre surface and $p\in U$ be a non-degenerate singular point of $\boldsymbol{x}$. Then the following assertions hold.*

(1) *If $\eta\lambda(p)\neq 0$, then $\boldsymbol{x}$ to be a front near p if and only if $\phi_{\boldsymbol{x}}(0)\neq 0$ holds.*

(2) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\boldsymbol{x}$ to be a front near p and $\eta\lambda(p)\neq 0$ hold.*

(3) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $\boldsymbol{x}$ to be a front near p and $\eta\lambda(p)=0$ and $\eta\eta\lambda(p)\neq 0$ hold.*

(4) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\eta\lambda(p)\neq 0$, $\phi_{\boldsymbol{x}}(0)=0$ and $\phi'_{\boldsymbol{x}}(0)\neq 0$ hold.*

Here, $\eta\lambda:U\to\mathbb{R}$ means the directional derivative of λ by the vector field $\tilde{\eta}$, where $\tilde{\eta}$ is an extended vector field of η to U.

In this paper, if there is no confusion, we denote $\tilde{\eta}$ by η. By using the above theorem, we give criterion of singular points of the framed base surface which is given by a one-parameter family of Legendre curves along a framed curve.

Theorem 5.3 *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : I \times J \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of Legendre curves along a framed curve. Suppose that $\boldsymbol{x}(u, 0) = \gamma(u)$, the set of singular points of γ is dense in I and $(0, 0)$ is a non-degenerate singular point of $\boldsymbol{x}$. Then we have the following two cases.*

(A) Suppose that $\beta^L(0, 0) = 0$ and $\alpha(0) \neq 0$.

(1) $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\beta^L_v(0, 0) \neq 0$ and $\ell^L(0, 0) \neq 0$.

(2) $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $\beta^L_v(0, 0) = 0, \beta^L_{vv}(0, 0) \neq 0, \beta^L_u(0, 0) \neq 0$ and $\ell^L(0, 0) \neq 0$.

(3) $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\beta^L_v(0, 0) \neq 0, \ell^L(0, 0) = 0$ and $(\ell^L \circ \delta)'(0) \neq 0$.

(B) Suppose that $\beta^L(0, 0) \neq 0$ and $\alpha(0) = 0$.

(1) $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\alpha'(0) \neq 0$ and $\nu^L_1(0, 0)m(0) + \nu^L_2(0, 0)n(0) \neq 0$.

(2) $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $\alpha'(0) = 0, \alpha''(0) \neq 0, \nu^L_2(0, 0)m(0) - \nu^L_1(0, 0)n(0) \neq 0$ and $\nu^L_1(0, 0)m(0) + \nu^L_2(0, 0)n(0) \neq 0$.

(3) $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\alpha'(0) \neq 0, \nu^L_1(0, 0)m(0) + \nu^L_2(0, 0)n(0) = 0$ and $((\beta^L(\nu^L_1 m + \nu^L_2 n + \theta_u) + a_1\theta_v) \circ \delta)'(0) \neq 0$.

Here δ is a singular curve of $\boldsymbol{x}$.

Proof. Let $\boldsymbol{x}(u, v) = \gamma(u) + x_1(u, v)\nu_1(u) + x_2(u, v)\nu_2(u)$. By the assumption $\gamma(u) = \boldsymbol{x}(u, 0)$, we have $x_1(u, 0) = x_2(u, 0) = 0$ for all $u \in I$. Moreover, since the set of singular points of γ is dense in I and $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0$, we have $\sin\theta(u, 0) = 0$ and hence $\cos\theta(u, 0) = \pm 1$. By $b_2(u, v) = 0$ in Theorem 5.1, we have $\lambda(u, v) = -b_1(u, v)a_2(u, v) = \beta^L(u, v)b_1(u, v)$. Since $(0, 0)$ is a non-degenerate singular point of $\boldsymbol{x}$, we divide two cases: (A) $\beta^L(0, 0) = 0$ and $b_1(0, 0) \neq 0$, (B) $\beta^L(0, 0) \neq 0$ and $b_1(0, 0) = 0$. Moreover, we have $\lambda_u(0, 0) \neq 0$ or $\lambda_v(0, 0) \neq 0$. By the integrability condition of $a_1e_2 + b_1f_2 = a_2e_1 + b_2f_1$, we have $\alpha\theta_v = -\beta^L(\nu^L_{1u}\nu^L_2 - \nu^L_{2u}\nu^L_1 - \ell)$ at $(0, 0)$. The other integrability conditions automatically hold at $(0, 0)$.

First we consider the of type (A). By Theorem 5.1, $b_1(0, 0) \neq 0$ if and only if $\alpha(0) \neq 0$. Moreover, $b_1(u, 0) = \pm\alpha(u) \neq 0$ around $0 \in I$. Therefore, γ is a regular curve around $0 \in I$. In this case, (u, v) is a singular point of $\boldsymbol{x}$ if and only if $\beta^L(u, v) = 0$. Since $d\boldsymbol{x} = \boldsymbol{x}_u du + \boldsymbol{x}_v dv = (a_1\boldsymbol{s} + b_1\boldsymbol{t})du + a_2\boldsymbol{s}dv$ and $a_2(u, v) = -\beta^L(u, v)$, the null vector field η is given by $\partial/\partial v$. Therefore, the condition $\eta\lambda(0, 0) \neq 0$ is equivalent to $\beta^L_v(0, 0) \neq 0$, and the conditions $\eta\lambda(0, 0) = 0$ and $\eta\eta\lambda(0, 0) \neq 0$ are equivalent to $\beta^L_v(0, 0) = 0$ and $\beta^L_{vv}(0, 0) \neq 0$. Since $(0, 0)$ is a non-degenerate singular point of $\boldsymbol{x}$, we have $\beta^L_u(0, 0) \neq 0$ or $\beta^L_v(0, 0) \neq 0$. By the integrability condition, we have $\theta_v(0, 0) = 0$. By a direct calculation, we have $K_F = -\ell^L(\nu^L_1 m + \nu^L_2 n)$ and $H_F = \alpha\ell^L$ at $(0, 0)$. It follows that $\boldsymbol{x}$ is a front around $(0, 0)$ if and only if $\ell^L(0, 0) \neq 0$ by Proposition 4.6. Therefore, by Theorem 5.2, $\boldsymbol{x}$ at $(0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal edge (respectively, the swallowtail) if and only if $\beta^L_v(0, 0) \neq 0$ and $\ell^L(0, 0) \neq 0$ (respectively, $\beta^L_v(0, 0) = 0, \beta^L_{vv}(0, 0) \neq 0, \beta^L_u(0, 0) \neq 0$ and $\ell^L(0, 0) \neq 0$).

We now consider the condition for the cuspidal cross cap. Since $\eta\lambda(0, 0) = \beta^L_v(0, 0) \neq 0$, the singular curve δ is given by the form $\delta(t) = (t, v(t))$, where v is a smooth function with

$v(0)=0$. By a direct calculation,

$$\begin{aligned}
(\boldsymbol{x}\circ\delta)' &= (\alpha+x_1m+x_2n)\boldsymbol{\mu}+(x_{1u}-\beta^L\nu_2^Lv'-x_2\ell)\nu_1+(x_{2u}+\beta^L\nu_1^Lv'+x_1\ell)\nu_2\\
\boldsymbol{n}\circ\delta &= \cos\theta(\nu_1^L\nu_1+\nu_2^L\nu_2)+\sin\theta\boldsymbol{\mu}\\
d\boldsymbol{n}(\eta) &= (-\theta_v\sin\theta\nu_1^L-\cos\theta\ell^L\nu_2^L)\nu_1+(-\theta_v\sin\theta\nu_2^L+\cos\theta\ell^L\nu_1^L)\nu_2+\theta_v\cos\theta\boldsymbol{\mu}.
\end{aligned}$$

By straightforward calculations, we have

$$\begin{aligned}
\phi_{\boldsymbol{x}} &= \det((\boldsymbol{x}\circ\delta)',\boldsymbol{n}\circ\delta,d\boldsymbol{n}(\eta))\\
&= (\alpha+x_1m+x_2n)\ell^L+(x_{1u}-\beta^L\nu_2^Lv'-x_2\ell)(\theta_v\nu_2^L-\sin\theta\cos\theta\ell^L\nu_1^L)\\
&\quad+(x_{2u}+\beta^L\nu_1^Lv'+x_1\ell)(-\theta_v\nu_1^L-\sin\theta\cos\theta\ell^L\nu_2^L).
\end{aligned}$$

It follows that $\phi_{\boldsymbol{x}}(0)=\alpha(0)\ell^L(0,0)$ and $\phi'_{\boldsymbol{x}}(0)=\alpha(0)(\ell^L\circ\delta)'(0)$ under the condition $\phi_{\boldsymbol{x}}(0)=0$. Therefore, by Theorem 5.3, $\boldsymbol{x}$ at $(0,0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\beta_v^L(0,0)\neq 0,\ell^L(0,0)=0$ and $(\ell^L\circ\delta)'(0)\neq 0$.

Second we consider the of type (B). Since $b_1(0,0)=\mp\alpha(0)=0$, 0 is a singular point of γ. In this case, (u,v) is a singular point of $\boldsymbol{x}$ if and only if $b_1(u,v)=0$. Since $d\boldsymbol{x}=\boldsymbol{x}_udu+\boldsymbol{x}_vdv=(a_1\boldsymbol{s}+b_1\boldsymbol{t})du+a_2\boldsymbol{s}dv=a_1\boldsymbol{s}du-\beta^L\boldsymbol{s}dv$ on the singular set of $\boldsymbol{x}$, the null vector field η is given by $\beta^L(u,v)\partial/\partial u+a_1(u,v)\partial/\partial v$. Note that we have $a_1(u,0)=0$ for all $u\in I$. Therefore, the condition $\eta\lambda(0,0)\neq 0$ is equivalent to $\alpha'(0)\neq 0$, and the conditions $\eta\lambda(0,0)=0$ and $\eta\eta\lambda(0,0)\neq 0$ are equivalent to $\alpha'(0)=0$ and $\alpha''(0)\neq 0$. Since $(0,0)$ is a non-degenerate singular point of $\boldsymbol{x}$, we have $b_{1u}(0,0)\neq 0$ or $b_{1v}(0,0)\neq 0$, that is, $\alpha'(0)\neq 0$ or $\nu_2^L(0,0)m(0)-\nu_1^L(0,0)n(0)\neq 0$. By a direct calculation and the integrability condition, we have $K_F=-\ell^L(\nu_1^Lm+\nu_2^Ln)$ and $H_F=(1/2)\beta^L(\nu_1^Lm+\nu_2^Ln)$ at $(0,0)$. It follows that $\boldsymbol{x}$ is a front around $(0,0)$ if and only if $\nu_1^L(0,0)m(0)+\nu_2^L(0,0)n(0)\neq 0$ by Proposition 4.6. Therefore, by Theorem 5.2, $\boldsymbol{x}$ at $(0,0)$ is $\mathcal{A}$-equivalent to the cuspidal edge (respectively, the swallowtail) if and only if $\alpha'(0)\neq 0$ and $\nu_1^L(0,0)m(0)+\nu_2^L(0,0)n(0)\neq 0$ (respectively, $\alpha'(0)=0,\alpha''(0)\neq 0,\nu_2^L(0,0)m(0)-\nu_1^L(0,0)n(0)\neq 0$ and $\nu_1^L(0,0)m(0)+\nu_2^L(0,0)n(0)\neq 0$).

We now consider the condition for the cuspidal cross cap. Since $\eta\lambda(0,0)\neq 0$ is equivalent to $\alpha'(0)\neq 0$, the singular curve δ is given by the form $\delta(t)=(u(t),t)$, where u is a smooth function with $u(0)=0$. By a direct calculation and $b_1(u(t),t)=0$,

$$\begin{aligned}
(\boldsymbol{x}\circ\delta)' &= (\alpha+x_1m+x_2n)u'\boldsymbol{\mu}+(x_{1u}u'-\beta^L\nu_2^L-x_2\ell u')\nu_1+(x_{2u}u'+\beta^L\nu_1^L+x_1\ell u')\nu_2\\
&= \tan\theta((x_{1u}-x_2\ell)\nu_1^L+(x_{2u}+x_1\ell)\nu_2^L)u'\boldsymbol{\mu}\\
&\quad+(x_{1u}u'-\beta^L\nu_2^L-x_2\ell u')\nu_1+(x_{2u}u'+\beta^L\nu_1^L+x_1\ell u')\nu_2\\
\boldsymbol{n}\circ\delta &= \cos\theta(\nu_1^L\nu_1+\nu_2^L\nu_2)+\sin\theta\boldsymbol{\mu}\\
d\boldsymbol{n}(\eta) &= (\sin\theta(-\theta_u\beta^L\nu_1^L-\beta^Lm-\theta_va_1\nu_1^L)+\cos\theta(\beta^L\nu_{1u}^L-\beta^L\nu_2\ell-a_1\ell^L\nu_2^L))\nu_1\\
&\quad+(\sin\theta(-\theta_u\beta^L\nu_2^L-\beta^Ln-\theta_va_1\nu_2^L)+\cos\theta(\beta^L\nu_{2u}^L+\beta^L\nu_1\ell+a_1\ell^L\nu_1^L))\nu_2\\
&\quad+\cos\theta(\beta^L(\nu_1^Lm+\nu_2n+\theta_u)+a_1\theta_v)\boldsymbol{\mu}.
\end{aligned}$$

By straightforward calculations, we have

$$
\begin{aligned}
\phi_{\boldsymbol{x}} &= \det((\boldsymbol{x}\circ\delta)', \boldsymbol{n}\circ\delta, d\boldsymbol{n}(\eta)) \\
&= \sin\theta\left((x_{1u}-x_2\ell)\nu_1^L+(x_{2u}+x_1\ell)\nu_2^L\right)u' \\
&\quad\times\Big(\sin\theta\beta^L(-\nu_1^L n+\nu_2^L m)+\cos\theta(\beta^L\nu_1^L\nu_{2u}^L-\beta^L\nu_2^L\nu_{1u}^L+\beta^L\ell+a_1\ell^L)\Big) \\
&\quad+(x_{1u}u'-\beta^L\nu_2^L-x_2\ell u')\Big(\cos^2\theta\nu_2^L(\beta^L(\nu_1^L m+\nu_2^L n+\theta_u)+a_1\theta_v) \\
&\quad-\sin\theta(\sin\theta(-\theta_u\beta^L\nu_2^L-\beta^L n-\theta_v a_1\nu_2^L)+\cos\theta(\beta^L\nu_{2u}^L+\beta^L\nu_1^L\ell+a_1\ell^L\nu_1^L))\Big) \\
&\quad+(x_{2u}u'+\beta^L\nu_1^L+x_1\ell u')\Big(-\cos^2\theta\nu_1^L(\beta^L(\nu_1^L m+\nu_2^L n+\theta_u)+a_1\theta_v) \\
&\quad+\sin\theta(\sin\theta(-\theta_u\beta^L\nu_1^L-\beta^L m-\theta_v a_1\nu_1^L)+\cos\theta(\beta^L\nu_{1u}^L-\beta^L\nu_2^L\ell-a_1\ell^L\nu_2^L))\Big).
\end{aligned}
$$

It follows that $\phi_{\boldsymbol{x}}(0)=-(\beta^L(0,0))^2(\nu_1^L(0,0)m(0)+\nu_2^L(0,0)n(0))$, and $\phi'_{\boldsymbol{x}}(0)=(\beta^L(\nu_1^L m+\nu_2^L n+\theta_u)+a_1\theta_v)\circ\delta)'(0)$ under the condition $\phi_{\boldsymbol{x}}(0)=0$. Therefore, by Theorem 5.3, $\boldsymbol{x}$ at $(0,0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\alpha'(0)\neq 0, \nu_1^L(0,0)m(0)+\nu_2^L(0,0)n(0)=0$ and $(\beta^L(\nu_1^L m+\nu_2^L n+\theta_u)+a_1\theta_v)\circ\delta)'(0)\neq 0$. This complete the proof of the Theorem. $\square$

Remark 5.4 Under the same assumptions in Theorem 5.3, if $\gamma(u)$ is the image of the singular curve of $\boldsymbol{x}$, then the singular set $S(\boldsymbol{x})=\{(u,0)|u\in I\}$ and of type (A). Since the null vector field η and the singular direction δ' are linearly independent at $(0,0)$, the singular point $(0,0)$ can not be the swallowtail.

Remark 5.5 The condition $\nu_2^L(0,0)m(0)-\nu_1^L(0,0)n(0)\neq 0, \nu_1^L(0,0)m(0)+\nu_2^L(0,0)n(0)\neq 0$ in Theorme 5.3 (B) (2) is equivalent to the condition $(m(0),n(0))\neq(0,0)$.

Corollary 5.6 *Under the above notations as in Theorem 5.1, suppose that $\gamma: I\to\mathbb{R}^3$ is a regular curve, $\boldsymbol{x}(u,\cdot): J\to\langle\boldsymbol{\mu}(u)\rangle^{\perp}_{\gamma(u)}$ is diffeomorphic to the 3/2-cusp at $0\in J$ and $\boldsymbol{x}(u,0)=\gamma(u)$ for all $u\in I$. Then $\boldsymbol{x}: I\times J\to\mathbb{R}^3$ is a front around $(u,0)$. More precisely, $(\boldsymbol{x},\boldsymbol{n}): I\times J\to\mathbb{R}^3\times S^2$ is a Legendre immersion around $(u,0)$. Moreover, $\boldsymbol{x}$ is diffeomorphic to the cuspidal edge at $(u,0)$.*

Proof. Since γ is a regular curve, we have $\alpha(u)\neq 0$ for all $u\in I$. Moreover, $\boldsymbol{x}(u,\cdot)$ is diffeomorphic to the 3/2-cusp at $0\in J$ if and only if $\boldsymbol{x}_v(u,0)=0$ and $\det(\boldsymbol{x}_{vv}(u,0),\boldsymbol{x}_{vvv}(u,0))\neq 0$, for all $u\in I$ (cf. [4, 9, 13]). By the definition of the curvature $(\ell^L(u,v),\beta^L(u,v))$ of the Legendre curve $(\boldsymbol{x}(u,\cdot),\nu^L(u,\cdot))$, we have

$$
\begin{aligned}
\boldsymbol{x}_v(u,v) &= \beta^L(u,v)\boldsymbol{\mu}^L(u,v), \\
\boldsymbol{x}_{vv}(u,v) &= \beta_v^L(u,v)\boldsymbol{\mu}^L(u,v)-\beta^L(u,v)\ell^L(u,v)\nu^L(u,v) \\
\boldsymbol{x}_{vvv}(u,v) &= (\beta_{vv}^L(u,v)-\beta^L(u,v)(\ell^L(u,v))^2)\boldsymbol{\mu}^L(u,v)-2\beta_v^L(u,v)\ell^L(u,v)\nu^L(u,v).
\end{aligned}
$$

It follows that $\beta^L(u,0)=0, \beta_v^L(u,0)\neq 0$ and $\ell^L(u,0)\neq 0$ for all $u\in I$. Since $\boldsymbol{x}(u,0)=\gamma(u)$, we have $x_1(u,0)=x_2(u,0)=0$ for all $u\in I$. Therefore $x_{1u}(u,0)=x_{2u}(u,0)=0$. Moreover, by the condition $\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v)=0$ for all $(u,v)\in I\times J$, we have $\alpha(u)\sin\theta(u,0)=0$ and hence $\sin\theta(u,0)=0$. Then $a_1(u,0)=0, b_1(u,0)=-\cos\theta(u,0)\alpha(u), a_2(u,0)=-\beta^L(u,0), b_2(u,0)=0, e_2(u,0)=-\cos\theta(u,0)\ell^L(u,0), f_2(u,0)=-\theta_v(u,0), g_2(u,0)=0$. It follows that $H_F(u,0)=(1/2)\cos^2\theta(u,0)\alpha(u)\ell^L(u,0)\neq 0$ for all $u\in I$. By Proposition 4.6, $(\boldsymbol{x},\boldsymbol{n})$ is a Legendre immersion around $(u,0)$. Hence, $\boldsymbol{x}$ is a front around $(u,0)$. Moreover, by Theorem 5.3 (A) (1), $\boldsymbol{x}$ is diffeomorphic to the cuspidal edge at $(u,0)$. $\square$

We also have the following result.

Theorem 5.7 *Suppose that $\boldsymbol{x} : U \to \mathbb{R}^3$ is diffeomorphic to the cuspidal edge at $0 \in U$. Then there exist a parameter change $\phi : I \times J \to U$ around 0 and a smooth mapping $(\boldsymbol{n}, \boldsymbol{s}) : I \times J \to \Delta$ such that the framed surface $(\boldsymbol{x} \circ \phi, \boldsymbol{n}, \boldsymbol{s}) : I \times J \to \mathbb{R}^3 \times \Delta$ is given by a one-parameter family of $3/2$-cusp at $0 \in J$ along a regular curve $\gamma : I \to \mathbb{R}^3$ around $0 \in I$.*

Proof. The normal form of cuspidal edge by using coordinate transformations on the source and isometries on the target is given by [20]. Since the property of one-parameter family of Legendre curves along a framed curve are invariant as isometries on the target, there exists a parameter change $\phi : I \times J \to U$ around 0 such that $\widetilde{\boldsymbol{x}} = \boldsymbol{x} \circ \phi$ is given by the following form around $(0,0) \in I \times J$:

$$\widetilde{\boldsymbol{x}}(u,v) = \left(u, a(u) + \frac{v^2}{2}, b(u) + v^2 b_2(u) + v^3 b_3(u,v)\right),$$

where $a(0) = \dot{a}(0) = b(0) = \dot{b}(0) = b_2(0) = 0$ and $b_3(0,0) \neq 0$, by the proof of Theorem 3.1 in [20]. Here we relabelled the coefficient functions.

We define a regular curve $\gamma : I \to \mathbb{R}^3, \gamma(u) = (u,0,0)$. If we take $(\nu_1, \nu_2) : I \to \Delta$ by $\nu_1(u) = (0,1,0), \nu_2(u) = (0,0,1)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve. By $\widetilde{\boldsymbol{x}}_v(u,v) = (0, v, 2vb_2(u) + 3v^2 b_3(u,v) + v^3 b_{3v}(u,v))$, we have $\nu^L(u,v) = \nu_1^L(u,v)\nu_1(u) + \nu_2^L(u,v)\nu_2(u)$ and $\boldsymbol{\mu}^L(u,v) = -\nu_2^L(u,v)\nu_1(u) + \nu_1^L(u,v)\nu_2(u)$, where

$$\begin{aligned}
\nu_1^L(u,v) &= -\frac{2b_2(u) + 3vb_3(u,v) + v^2 b_{3v}(u,v)}{\sqrt{(2b_2(u) + 3vb_3(u,v) + v^2 b_{3v}(u,v))^2 + 1}}, \\
\nu_2^L(u,v) &= \frac{1}{\sqrt{(2b_2(u) + 3vb_3(u,v) + v^2 b_{3v}(u,v))^2 + 1}}.
\end{aligned}$$

It follows that the curvature of the Legendre curve $(\widetilde{\boldsymbol{x}}(u,\cdot), \nu^L(u,\cdot))$ is given by

$$\begin{aligned}
\ell^L(u,v) &= \frac{3b_3(u,v) + 5vb_{3v}(u,v) + v^2 b_{3vv}(u,v)}{(2b_2(u) + 3vb_3(u,v) + v^2 b_{3v}(u,v))^2 + 1}, \\
\beta^L(u,v) &= -v\sqrt{(2b_2(u) + 3vb_3(u,v) + v^2 b_{3v}(u,v))^2 + 1}.
\end{aligned}$$

We denote

$$\varphi(u,v) = \frac{a'(u)(2b^2 + 3vb_3(u,v) + v^2 b_{3v}(u,v)) + b'(u) + v^2 b_2'(u) + v^3 b_{3u}(u,v)}{\sqrt{(2b_2(u) + 3vb_3(u,v) + v^2 b_{3v}(u,v))^2 + 1}}.$$

Then we define a smooth mapping $(\boldsymbol{n}, \boldsymbol{s}) : I \times J \to \Delta$ by

$$\boldsymbol{n}(u,v) = \frac{1}{\sqrt{1 + \varphi^2(u,v)}}\nu^L(u,v) - \frac{\varphi(u,v)}{\sqrt{1 + \varphi^2(u,v)}}\boldsymbol{\mu}(u), \boldsymbol{s}(u,v) = -\boldsymbol{\mu}^L(u,v).$$

Since $\widetilde{\boldsymbol{x}}_u(u,v) = (1, a'(u), b'(u) + v^2 b_2'(u) + v^3 b_{3u}(u,v))$, we have $\widetilde{\boldsymbol{x}}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in I \times J$. It follows from Theorem 5.1 that $(\widetilde{\boldsymbol{x}}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface. Moreover, since

$x_1(u,v) = a(u) + v^2/2$ and $x_2(u,v) = b(u) + v^2 b_2(u,v) + v^3 b_3(u,v)$, we have

$$\begin{aligned}
(x_1,x_2)_v(u,v) &= (v, 2vb_2(u) + 3v^2 b_3(u,v) + v^3 b_{3v}(u,v)),\\
(x_1,x_2)_{vv}(u,v) &= (1, 2b_2(u) + 6vb_3(u,v) + 6v^2 b_{3v}(u,v) + v^3 b_{3vv}(u,v)),\\
(x_1,x_2)_{vvv}(u,v) &= (0, 6b_3(u,v) + 18vb_{3v}(u,v) + 9v^2 b_{3vv}(u,v) + v^3 b_{3vvv}(u,v)).
\end{aligned}$$

It follows that $(x_1,x_2)_v(u,0) = 0$ and $\det((x_1,x_2)_{vv}(u,0), (x_1,x_2)_{vvv}(u,0)) = 6b_3(u,0) \neq 0$ around $(0,0) \in I \times J$. Therefore, $(u,0)$ is a 3/2-cusp of $\widetilde{\boldsymbol{x}}(u,\cdot)$ around $0 \in I$. $\Box$

The singularities of the swallowtail and of the cuspidal cross cap are more complicated (cf. [6, 22, 25]). The corresponding results for Corollary 5.6 and Theorem 5.7 of the the swallowtail and the cuspidal cross cap (and other singularities) are future problems (cf. [10]).

6 Examples

We give typical examples of singularities of smooth surfaces. We detect the basic invariants and curvatures of framed surfaces.

Example 6.1 (cuspidal edge) A singular point $p \in U$ of a mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ is called a *cuspidal edge* if the map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent (right-left equivalent) to the $(u,v) \mapsto (u, v^2, v^3)$ at 0. Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = (u, v^2, v^3)$. If we take $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$, $\boldsymbol{n}(u,v) = (1/\sqrt{9v^2+4})(0, -3v, 2)$, $\boldsymbol{s}(u,v) = (1,0,0)$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. Since $\boldsymbol{t}(u,v) = (1/\sqrt{9v^2+4})(0, 2, 3v)$, we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & v\sqrt{9v^2+4} \end{pmatrix}, \quad \begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6/(9v^2+4) & 0 \end{pmatrix}.$$

It follows that the curvature C_F of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by

$$J_F(u,v) = v\sqrt{9v^2+4},\ K_F(u,v) = 0,\ H_F(u,v) = \frac{3}{9v^2+4}.$$

Example 6.2 (swallowtail) A singular point $p \in U$ of a mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ is called a *swallowtail* if the map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the $(u,v) \mapsto (3u^4 + u^2 v, -4u^3 - 2uv, v)$ at 0. Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = (3u^4 + u^2 v, -4u^3 - 2uv, v)$. If we take $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$, $\boldsymbol{n}(u,v) = (1/\sqrt{1+u^2+u^4})(1, u, u^2)$, $\boldsymbol{s}(u,v) = (1/\sqrt{1+u^2})(u, -1, 0)$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.

Since $\boldsymbol{t}(u,v) = (1/\sqrt{1+u^2+u^4}\sqrt{1+u^2})(u^2, u^3, -1-u^2)$, we have the following basic invariants.

$$\begin{aligned}
\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} &= \begin{pmatrix} (12u^2+2v)\sqrt{1+u^2} & 0 \\ \frac{u(2+u^2)}{\sqrt{1+u^2}} & -\frac{\sqrt{1+u^2+u^4}}{\sqrt{1+u^2}} \end{pmatrix},\\
\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\sqrt{1+u^2+u^4}\sqrt{1+u^2}} & -\frac{u(2+u^2)}{(1+u^2+u^4)\sqrt{1+u^2}} & \frac{u^2}{(1+u^2)\sqrt{1+u^2+u^4}} \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

It follows that the curvature C_F of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by

$$J_F(u,v) = 2(6u^2+v)\sqrt{1+u^2+u^4},\ K_F(u,v) = 0,\ H_F(u,v) = -\frac{1+5u^2+5u^4+u^6}{2(1+u^2+u^4)(1+u^2)}.$$

Example 6.3 (cuspidal cross cap) A singular point $p \in U$ of a mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ is called a *cuspidal cross cap* if the map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the $(u,v) \mapsto (u, v^2, uv^3)$ at 0. Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = (u, v^2, uv^3)$. If we take $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$,

$$\boldsymbol{n}(u,v) = \frac{1}{\sqrt{4v^6 + 9u^2v^2 + 4}}(-2v^3, -3uv, 2), \boldsymbol{s}(u,v) = \frac{1}{\sqrt{1+v^6}}(1, 0, v^3),$$

then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.

Since $\boldsymbol{t}(u,v) = (1/\sqrt{4v^6 + 9u^2v^2 + 4}\sqrt{1+v^6})(-3uv^4, 2v^6 + 2, 3uv)$, we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1+v^6} & 0 \\ \frac{3uv^5}{\sqrt{1+v^6}} & \frac{v\sqrt{4v^6+9u^2v^2+4}}{\sqrt{1+v^6}} \end{pmatrix},$$

$$\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{6v\sqrt{1+v^6}}{4v^6+9u^2v^2+4} & 0 \\ -\frac{6v^2}{\sqrt{4v^6+9u^2v^2+4}\sqrt{1+v^6}} & \frac{6u(2v^6-1)}{(4v^6+9u^2v^2+4)\sqrt{1+v^6}} & \frac{9uv^3}{\sqrt{4v^6+9u^2v^2+4}(1+v^6)} \end{pmatrix}.$$

It follows that the curvature C_F of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by

$$J_F(u,v) = v\sqrt{4v^6 + 9u^2v^2 + 4},\ K_F(u,v) = -\frac{36v^3}{(4v^6 + 9u^2v^2 + 4)^{3/2}},\ H_F(u,v) = -\frac{3u(5v^6-1)}{4v^6 + 9u^2v^2 + 4}.$$

Example 6.4 (cross cap) A singular point $p \in U$ of a mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ is called a *cross cap* if the map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the $(u,v) \mapsto (u, v^2, uv)$ at 0. Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = (u, v^2, uv)$. Then it is well-known that the cross cap is not a frontal. However, if we consider the polar coordinate $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2, (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, then $\boldsymbol{x} \circ \phi$ is a frontal and the images are same (cf. [7]). Note that ϕ is not diffeomorphic but surjective. We rewrite $\boldsymbol{x} \circ \phi$ as $\boldsymbol{x} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{x}(r,\theta) = (r\cos\theta, r^2\sin\theta, r^2\cos\theta\sin\theta)$. In this case, if we take $(\boldsymbol{n}, \boldsymbol{s}) : \mathbb{R} \times \mathbb{R} \to \Delta$,

$$\begin{aligned} \boldsymbol{n}(r,\theta) &= \frac{1}{\sqrt{4r^2\sin^4\theta + 3\sin^2\theta + 1}}(-2r\sin^2\theta, -\cos\theta, 2\sin\theta), \\ \boldsymbol{s}(r,\theta) &= \frac{1}{\sqrt{3\sin^2\theta + 1}}(0, 2\sin\theta, \cos\theta), \end{aligned}$$

then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ is a framed surface. Since

$$\boldsymbol{t}(r,\theta) = \frac{1}{\sqrt{(4r^2\sin^4\theta + 3\sin^2\theta + 1)(3\sin^2\theta + 1)}}(-(3\sin^2\theta + 1), 2r\sin^2\theta\cos\theta, -4r\sin^3\theta),$$

we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \frac{2r\sin\theta(\sin^2\theta+1)}{\sqrt{3\sin^2\theta+1}} & \frac{-\cos\theta\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}}{\sqrt{3\sin^2\theta+1}} \\ r^2\cos\theta\sqrt{3\sin^2\theta+1} & \frac{r\sin\theta\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}}{\sqrt{3\sin^2\theta+1}} \end{pmatrix},$$

$$\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \frac{2\sin^2\theta\sqrt{3\sin^2\theta+1}}{4r^2\sin^4\theta+3\sin^2\theta+1} & 0 \\ \frac{2}{\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}\sqrt{3\sin^2\theta+1}} & \frac{2r\sin\theta\cos\theta(3\sin^2\theta+2)}{(4r^2\sin^4\theta+3\sin^2\theta+1)\sqrt{3\sin^2\theta+1}} & \frac{4r\sin^2\theta}{\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}(3\sin^2\theta+1)} \end{pmatrix}$$

It follows that the curvature C_F of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by

$$
\begin{aligned}
J_F(r,\theta) &= \frac{r^2(2\sin\theta(\sin^2\theta+1)+\cos^2\theta+1)\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}}{3\sin^2\theta+1}, \\
K_F(r,\theta) &= -\frac{2\sin^2\theta}{(4r^2\sin^4\theta+3\sin^2\theta+1)^{2/3}}, \\
H_F(r,\theta) &= -\frac{2\cos\theta(-3r^2\sin^6\theta+8r^2\sin^4\theta+3r^2\sin\theta+3\sin^2\theta+2)}{(4r^2\sin^4\theta+3\sin^2\theta+1)(2\sin^2\theta+1)}.
\end{aligned}
$$

Especially, $C_F(r,\theta) \neq 0$ for any $(r,\theta) \in \mathbb{R} \times \mathbb{R}$, that is, $\boldsymbol{x}$ is a front by Proposition 4.6.

A Framed curves in the Euclidean space

We quickly review on the theory of framed curves in the Euclidean space, see detail [11].

A framed curve in the Euclidean space is a smooth curve with a moving frame. We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *framed curve* if $\dot\gamma(t) \cdot \nu_1(t) = 0$ and $\dot\gamma(t) \cdot \nu_2(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.

We put on $\boldsymbol{\mu}(t) = \nu_1(t) \times \nu_2(t)$. Then $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the framed base curve $\gamma(t)$ in $\mathbb{R}^3$ and we have the Frenet-Serret type formula,

$$
\begin{pmatrix} \dot\nu_1(t) \\ \dot\nu_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \alpha(t)\boldsymbol{\mu}(t)
$$

where $\ell(t) = \dot\nu_1(t) \cdot \nu_2(t)$, $m(t) = \dot\nu_1(t) \cdot \boldsymbol{\mu}(t), n(t) = \dot\nu_2(t) \cdot \boldsymbol{\mu}(t)$ and $\alpha(t) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ, m, n, α) *the curvature of the framed curve.* Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition A.1 Let (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2)$ are *congruent as framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{a}$, $\widetilde{\nu_1}(t) = A(\nu_1(t))$ and $\widetilde{\nu_2}(t) = A(\nu_2(t))$ for all $t \in I$.

We have the existence and uniqueness for framed curves similarly to the case of regular space curves.

Theorem A.2 (The Existence Theorem for framed curves, [11]) *Let $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ whose curvature of the framed curve is (ℓ, m, n, α).*

Theorem A.3 (The Uniqueness Theorem for framed curves, [11]) *Let (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvature (ℓ, m, n, α) and $(\widetilde\ell, \widetilde m, \widetilde n, \widetilde\alpha)$, respectively. Then (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2)$ are congruent as framed curves if and only if the curvatures (ℓ, m, n, α) and $(\widetilde\ell, \widetilde m, \widetilde n, \widetilde\alpha)$ coincides.*

B Legendre curves in the Euclidean plane

We quickly review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see detail [8].

We say that $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma,\nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal if there exists $\nu : I \to S^1$ such that (γ,ν) is a Legendre curve. Examples of Legendre curves see [13, 14]. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. Then $\{\nu(t), \boldsymbol{\mu}(t)\}$ is a moving frame of a frontal $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet type formula,

$$\begin{pmatrix} \dot\nu(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot\nu(t) \cdot \boldsymbol{\mu}(t)$ and $\beta(t) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ,β) *the curvature of the Legendre curve.*

Definition B.1 Let (γ,ν) and $(\widetilde\gamma,\widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a} \in \mathbb{R}^2$ such that $\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{a}$ and $\widetilde\nu(t) = A(\nu(t))$ for all $t \in I$.

Theorem B.2 (The Existence Theorem for Legendre curves, [8]) *Let $(\ell,\beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ whose curvature of the Legendre curve is (ℓ,β).*

Theorem B.3 (The Uniqueness Theorem for Legendre curves, [8]) *Let (γ,ν) and $(\widetilde\gamma,\widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ,β) and $(\widetilde\ell,\widetilde\beta)$, respectively. Then (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ are congruent as Legendre curves if and only if the curvatures (ℓ,β) and $(\widetilde\ell,\widetilde\beta)$ coincides.*

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