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# Large time behavior of derivatives of the vorticity for the two dimensional Navier-Stokes flow

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## Abstract

This paper studies the large time asymptotic behavior of derivatives of the vorticity solving the two-dimensional vorticity equations equivalent to the Navier-Stokes equations. It is well-known by now that the vorticity behaves asymptotically as the Oseen vortex provided that the initial vorticity is integrable. This paper shows that each derivative of the vorticity also behave asymptotically as that of the Oseen vortex. For the proof new spatial decay estimates for derivatives are established. These estimates control behavior at the space infinity. The convergence result follows from a rescaling and compactness argument.

## 1 Introduction

This paper is concerned with the large time behavior of an incompressible, viscous fluid in  $\mathbb{R}^2$ . The velocity of the fluid is described by the Navier-Stokes equations:

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{for } t > 0, \quad x \in \mathbb{R}^2, \\ \nabla \cdot u = 0 & \text{for } x \in \mathbb{R}^2, \end{cases} \quad (1)$$

where  $u = u(x, t) \in \mathbb{R}^2$  is the fluid velocity,  $p(x, t) \in \mathbb{R}$  is the pressure,  $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$ ,  $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$  and  $u_t = \partial_t u = \partial u/\partial t$ . The kinematic viscosity has been rescaled to be 1. We are interested in the large time behavior of the vorticity  $\omega = \text{rot } u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$  when initial vorticity is integrable. For this purpose, instead of (1), we consider an equation for the vorticity which is obtained by taking the curl of (1):

$$\omega_t - \Delta \omega + (u, \nabla)\omega = 0, \quad t > 0, \quad x \in \mathbb{R}^2. \quad (2)$$

The velocity  $u$  is obtained in terms of  $\omega$  via Biot-Savart law

$$u(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y, t) dy, \quad t > 0, \quad x \in \mathbb{R}^2, \quad (3)$$

where  $x^\perp = (-x_2, x_1)$ . The equation (2)-(3) are formally equivalent to (1).

The global well-posedness of the two dimensional vorticity equations in  $L^1(\mathbb{R}^2)$  is first obtained by Y. Giga, T. Miyakawa and H. Osada [10]. In fact they constructed a global solution even when initial data is a finite measure. This result is extended by various authors for example by M. Ben-Artzi [1], H. Brezis [2], and T. Kato [13]. Although the uniqueness of solution was known by [10] when the point mass part of the initial data is small, it is quite recent that the uniqueness is proved for a general measure by I. Gallagher and Th. Gallay [5]. As for the large time behavior of  $\omega$ , we know that it behaves like a constant multiple of the Gauss kernel  $g(x, t) = (4\pi t)^{-1} \exp(-|x|^2/4t)$ . Let us recall its precise form:

**Theorem 1.1** ([8], [4], [6]). *Assume that  $p \in [1, \infty]$ ,  $q \in (2, \infty]$ . Let  $\omega$  be the solution of (2)-(3) with initial data  $\omega_0 \in L^1(\mathbb{R}^2)$ . Let  $m = \int_{\mathbb{R}^2} \omega_0(x) dx$ , and  $g(x, t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$ . Then*

$$\lim_{t \rightarrow \infty} t^{1-\frac{1}{p}} |\omega(\cdot, t) - mg(\cdot, t)|_p = 0, \quad (4)$$

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{1}{q}} |u(\cdot, t) - mv^g(\cdot, t)|_q = 0. \quad (5)$$

Here  $v^g$  is the velocity field associated with  $g$  via Biot-Savart law (3), and  $|f|_p$  denotes the norm of  $f$  in  $L^p$ ; if  $f$  is a vector  $(f_1, f_2)$ , by  $|f|_p$  we mean  $(|f_1|^2 + |f_2|^2)^{\frac{1}{2}}|_p$ .

The above theorem shows that the solutions behave asymptotically as  $mg$  which is called the Oseen vortex. Note that the Gauss kernel is a solution of (2)-(3) with a Dirac mass as the initial data. The quantity  $m = \int_{\mathbb{R}^2} \omega_0(x) dx$  is called "total circulation" and it is preserved by the semi-flow defined by (2)-(3) in  $L^1(\mathbb{R}^2)$ ;

$$\int_{\mathbb{R}^2} \omega(x, t) dx = \int_{\mathbb{R}^2} \omega_0(x) dx, \quad t \geq 0. \quad (6)$$

Y. Giga and T. Kambe [8] first proved Theorem 1.1 when the Reynolds number  $\int_{\mathbb{R}^2} |\omega_0(x)| dx$  is sufficiently small by giving the delicate estimates of the bilinear form of the integral equation associated with (2). Later A. Carpio [4] proved Theorem 1.1 under the assumption that  $|m|$  is small by rescaling solutions:  $\omega_k(x, t) = k^2 \omega(kx, k^2 t)$ ,  $u_k(x, t) = ku(kx, k^2 t)$  for  $k > 0$ . Recently, Th. Gallay and C. E. Wayne [6] proved for a general initial data in  $L^1(\mathbb{R}^2)$  by introducing entropy-like Lyapunov function for a renormalized equation. In this paper, we study the large time behavior of derivatives  $\partial_t^b \partial_x^3 \omega$  of the vorticity  $\omega$

satisfying (4) and (5). Here  $\partial_x^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}$  for multi-index  $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ , where  $\partial_{x_i} = \partial/\partial x_i$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , the set of all nonnegative integers. Our main result is

**Theorem 1.2.** *Assume that  $p \in [1, \infty]$ ,  $q \in (2, \infty]$ ,  $b \in \mathbb{N}_0$  and  $\beta$  is a multi-index. Let  $\omega$  be the solution of (2)-(3) with initial data  $\omega_0 \in L^1(\mathbb{R}^2)$ ,  $m = \int_{\mathbb{R}^2} \omega_0(x) dx$ , and  $g(x, t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$ . Then*

$$\lim_{t \rightarrow \infty} t^{b + \frac{|\beta|}{2} + 1 - \frac{1}{p}} |\partial_t^b \partial_x^\beta \omega(\cdot, t) - \partial_t^b \partial_x^\beta m g(\cdot, t)|_p = 0. \quad (7)$$

$$\lim_{t \rightarrow \infty} t^{b + \frac{|\beta|}{2} + \frac{1}{2} - \frac{1}{q}} |\partial_t^b \partial_x^\beta u(\cdot, t) - \partial_t^b \partial_x^\beta m v^g(\cdot, t)|_q = 0. \quad (8)$$

This theorem shows that the derivatives of the solutions behave asymptotically as the derivatives of the Oseen vortex. To prove Theorem 1.2 next estimates for the solutions play important roles.

**Theorem 1.3.** *Assume that  $p \in [1, \infty]$ ,  $q \in (2, \infty]$ . Let  $\omega$  be the solution of (2)-(3) with initial data  $\omega_0 \in L^1(\mathbb{R}^2)$  and  $u$  be the velocity field associated with  $\omega$  via Biot-Savart law. Then, there exist positive constants  $W_i = W_i(b, \beta, p, |\omega_0|_1)$ ;  $i = 1, 2$ ,  $W = W(p, |\omega_0|_1)$  such that for all  $R \geq 1$  and  $t > 0$ ,*

( $L^p$  estimate)

$$|\partial_t^b \partial_x^\beta \omega(\cdot, t)|_p \leq \frac{W_1}{t^{b + \frac{|\beta|}{2} + 1 - \frac{1}{p}}} |\omega_0|_1, \quad (9)$$

$$|\partial_t^b \partial_x^\beta u(\cdot, t)|_q \leq \frac{W_1}{t^{b + \frac{|\beta|}{2} + \frac{1}{2} - \frac{1}{q}}} |\omega_0|_1, \quad (10)$$

(spatial decay estimate)

$$|\omega(\cdot, t)|_{p, 2R} \leq W \left( \frac{1}{R} \frac{|\omega_0|_1}{t^{\frac{1}{2} - \frac{1}{p}}} + \frac{|\omega_0|_{1, R}}{t^{1 - \frac{1}{p}}} \right), \quad (11)$$

(spatial decay estimate for the derivatives)

$$|\partial_t^b \partial_x^\beta \omega(\cdot, t)|_{p, 2R} \leq \frac{W_2}{t^{b + \frac{|\beta|}{2} - \frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1 - \frac{1}{p}}} \right) + \frac{1}{t^{\frac{3}{2} - \frac{1}{p}}} |\omega_0|_{1, R} \right\}, \quad (12)$$

where  $|\omega(\cdot, t)|_{p, R} := \left( \int_{|x| > R} |\omega(x, t)|^p dx \right)^{\frac{1}{p}}$ ,  $|\omega(\cdot, t)|_{\infty, R} := \text{ess. sup}_{|x| > R} |\omega(x, t)|$ .

The estimates (9), (10) were proved by T. Kato [13] for  $p \in (1, \infty)$  by using an interpolation method, and by Y. Giga and M.-H. Giga [9] for  $p \in [1, \infty]$  by a

Gronwall-type argument (see also Y. Giga [11], or Y. Giga and O. Sawada [12]). The spatial decay estimate similar to (11) is obtained by A. Carpio [4], but we give its proof for completeness in Section 2.1. The estimate (12) is the spatial decay estimate for derivatives of the solutions which is a main new contribution of this paper. In order to establish this estimate, we shall show three spatial decay estimates. First one is for  $\omega$  itself, that is, estimate (11). Second one is for the velocity  $u$ . Since  $u$  is represented by  $\omega$  via Biot-Savart law (3), we shall estimate the well-known Riesz potential; see Section 2.2.1. The last one is for the solution of the heat equation; see Section 2.2.2. Collecting these estimates, one can derive the estimate (12); see Section 2.2.3.

Now let us give the outline of the proof of Theorem 1.2 for  $\alpha = 0$ ,  $|\beta| = 1$ . First we consider the same rescaling as was used in A. Carpio [4]. We shall see that the convergence of  $\partial_x \omega(x, t)$  as time goes to infinity is equivalent to the convergence of the rescaled functions  $\partial_x \omega_k(x, 1)$  as  $k$  goes to infinity. Once we obtain Theorem 1.3, we can apply Ascoli-Arzelà type compactness theorem in  $L^p$  to the family of rescaled functions  $\{\partial_x \omega_k(x, 1)\}_{k \geq 1}$ . So every subsequence of  $\{\partial_x \omega_{k(l)}(x, 1)\}_{l=1}^{\infty}$  ( $k(l) \rightarrow \infty$  as  $l$  goes to infinity) has a convergent subsequence in  $L^p$ . Theorem 1.1 implies that the limit function is unique, so we obtain Theorem 1.2. By the induction we see that Theorem 1.2 also holds for higher derivatives of the solution. The details are given in Section 3.

After this work was completed, the author was informed of a recent work of I. Gallagher, Th. Gallay and P.-L. Lions [7] which give another proof for Theorem 1.1 using the rearrangement argument.

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## 2 Estimates near space infinity

We first show Theorem 1.3, which is the key theorem to show Theorem 1.2. The estimates (9) and (10) are already known (see [13], [9]). So we shall prove (11), (12).

### 2.1 Spatial decay estimate of solution

In this section we show the spatial decay estimate of the solution. We recall next pointwise estimate due to E. A. Carlen and M. Loss ([3, Theorem 3]).

For any  $\theta \in (0, 1)$ , there exists  $C_\theta > 0$  (depending only on  $\theta$  and  $|\omega_0|_1$ ) such that

$$|\omega(x, t)| \leq C_\theta \int_{\mathbb{R}^2} \frac{1}{t} e^{-\theta \frac{|x-y|^2}{4t}} |\omega_0(y)| dy, \quad x \in \mathbb{R}^2, \quad t > 0. \quad (13)$$

Remark that the constant  $C_\theta$  does not depend on  $u$ .

**Proof of Theorem 1.3(11).**

Fix any  $R > 0$ . We decompose the right side of the estimate (13) as  $|\omega(x, t)| \leq \omega_1(x, t) + \omega_2(x, t)$ ;

$$\omega_1(x, t) = C_\theta \int_{|y| \leq R} \frac{1}{t} e^{-\theta \frac{|x-y|^2}{4t}} |\omega_0(y)| dy ,$$

$$\omega_2(x, t) = C_\theta \int_{|y| \geq R} \frac{1}{t} e^{-\theta \frac{|x-y|^2}{4t}} |\omega_0(y)| dy .$$

It is easy to check the next inequality:

$$\frac{1}{t^l} e^{-\frac{A^2}{t}} \leq \frac{C_l}{A^{2l}}, \quad \forall t > 0 , \quad (14)$$

where  $C_l > 0$  is a positive constant depending only on  $l$ .

If  $|x| \geq 2R$ ,  $|y| \leq R$ , then  $|x - y| \geq |x| - |y| \geq \frac{|x|}{2}$ , so by (14) we observe that

$$\begin{aligned} \omega_1(x, t) &\leq C_\theta |\omega_0|_1 \frac{1}{t} e^{-\theta \frac{|x|^2}{16t}} \\ &\leq C(\theta) \frac{1}{t^{\frac{1}{2}} |x|} e^{-\theta \frac{|x|^2}{32t}} , \end{aligned}$$

where  $C(\theta)$  depends only on  $\theta, |\omega_0|_1$ . Hence

$$\begin{aligned} |\omega_1(\cdot, t)|_{p, 2R} &\leq C(\theta) \frac{1}{t^{\frac{1}{2}}} \left( \int_{|x| \geq 2R} \frac{1}{|x|^p} e^{-p\theta \frac{|x|^2}{32t}} dx \right)^{\frac{1}{p}} \\ &\leq C(\theta) \frac{1}{t^{\frac{1}{2}} R} \left( \int_{|x| \geq 2R} e^{-p\theta \frac{|x|^2}{32t}} dx \right)^{\frac{1}{p}} \\ &\leq C(\theta) \frac{1}{t^{\frac{1}{2}} R} \left( \int_{\mathbb{R}^2} e^{-p\theta \frac{|x|^2}{32t}} dx \right)^{\frac{1}{p}} \\ &\leq C(\theta, p) \frac{1}{R} t^{-\frac{1}{2} + \frac{1}{p}} \end{aligned}$$

where  $C(\theta, p)$  depends only on  $p, |\omega_0|_1, \theta$ . Next we estimate  $\omega_2$ . By Young's inequality,

$$\begin{aligned} |\omega_2(\cdot, t)|_{p, 2R} &\leq |\omega_2(\cdot, t)|_p \leq C_\theta \frac{1}{t} \left( \int_{\mathbb{R}^2} e^{-p\theta \frac{|y|^2}{4t}} dy \right)^{\frac{1}{p}} |\omega_0|_{1, R} \\ &\leq C_2 \frac{1}{t^{1 - \frac{1}{p}}} |\omega_0|_{1, R} \end{aligned}$$

where  $C_2$  depends only on  $\theta, p, |\omega_0|_1$ . Summing up, we get the estimate (11).

**Remark 2.1.1.** Fix any  $\delta \in (0, 1)$ . Since  $|x-y| \geq |x|-|y| \geq (1-\delta)|x|$  whenever  $|x| \geq 2R$  and  $|y| \leq 2\delta R$ , arguing as the same above, we observe that there exists  $W = W(p, |\omega_0|_1, \delta)$  such that

$$|\omega(\cdot, t)|_{p, 2R} \leq W \left( \frac{1}{Rt^{\frac{1}{2}-\frac{1}{p}}} |\omega_0|_1 + \frac{1}{t^{1-\frac{1}{p}}} |\omega_0|_{1, 2\delta R} \right). \quad (15)$$

**Remark 2.1.2.** The spatial decay estimate of solutions  $\omega$  like (11) has already obtained by A. Carpio [4] or by Y. Giga and M.-H. Giga [9, Chapter II]. Their methods are different from ours. They use a cut of function  $\phi_R(x)$  whose support is outside of the ball with center at origin and radius  $R$ , and consider the equation which  $\omega_R(x, t) := \omega(x, t)\phi_R(x)$  satisfies.

## 2.2 Spatial decay estimates for derivatives of solution

In this section, we prove the following estimate instead of Theorem 1.3 (12). Fix  $\delta \in (0, 1)$ . Then, for any  $R \geq 1$

$$|\partial_t^b \partial_x^\beta \omega(\cdot, t)|_{p, 2R} \leq \frac{W_{2, \delta}}{t^{b+\frac{|\beta|}{2}-\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1-\frac{1}{p}}} \right) + \frac{1}{t^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1, 2\delta^2+|\beta|+2bR} \right\}. \quad (16)$$

where  $W_{2, \delta}$  depends only on  $b, \beta, p, |\omega_0|_1$ , and  $\delta$ . Clearly, this implies that the estimate (12) holds. Note that by Remark 2.1.1, the estimate (16) holds for  $|\beta| = 0, b = 0$ . So We first prove (16) for the case  $b = 0$  by induction with respect to  $|\beta|$ , and next we shall prove for the case for arbitrary  $b \in \mathbb{N}_0$ . A solution  $\omega$  is a solution of the associated integral equation

$$\omega(x, t) = e^{t\Delta} f - \int_0^t e^{(t-s)\Delta} (u(s), \nabla) \omega(s) ds, \quad (17)$$

where  $e^{t\Delta}$  is the semigroup generated by the operator  $\Delta$ , whose representation is

$$e^{t\Delta} f = \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}} f(y) dy. \quad (18)$$

Since  $\nabla \cdot u = 0$ , (17) can be rewritten as

$$\omega(x, t) = e^{t\Delta} f - \int_0^t \nabla \cdot e^{(t-s)\Delta} u(s) \omega(s) ds, \quad (19)$$

and by a semigroup property of solution, (19) can be also expressed as

$$\omega(x, t) = e^{\frac{t}{2}\Delta} \omega\left(\frac{t}{2}\right) - \int_{\frac{t}{2}}^t \nabla \cdot e^{(t-s)\Delta} u(s) \omega(s) ds, \quad (20)$$

where  $\omega(\frac{t}{2}) = \omega(x, \frac{t}{2})$ . Differentiating both sides by  $x$ ,

$$\partial_x^\beta \omega(x, t) = \partial_x^\beta e^{\frac{t}{2}\Delta} \omega(\frac{t}{2}) - \int_{\frac{t}{2}}^t \nabla \cdot e^{(t-s)\Delta} \partial_x^\beta (u(s)\omega(s)) ds. \quad (21)$$

Using the representation (21), we shall show the decay estimate for the derivatives of the solution. In Section 2.2.1 we estimate the spatial decay of the velocity  $u(x, t)$ , which may be used in estimating nonlinear terms of (21). Moreover, in Section 2.2.2, we shall estimate the decay for the derivatives of functions convoluted with Gaussian. Summing up these results, we get (16) for  $b = 0$ .

### 2.2.1 Spatial decay estimate for the Riesz potential

In this section, we estimate the spatial decay of the velocity field  $u(x, t)$ . Since  $u(x, t)$  is represented by Biot-Savart law (3), it is sufficient to show the decay estimate for the Riesz potential. Here we may assume that the spatial dimension is  $n$ . The Riesz potential for  $\alpha \in (0, n)$  is given by

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) dy, \quad f \in C_0(\mathbb{R}^n), \quad (22)$$

where  $C_0(\mathbb{R}^n)$  denotes the space of all continuous functions with compact support in  $\mathbb{R}^n$ .

**Lemma 2.2.1.** (a) *Let  $p, r \in (1, \infty)$  satisfy  $\frac{1}{p} = \frac{1}{r} - \frac{\alpha}{n}$ ,  $n - \alpha - \frac{n}{p} > 0$ . Then there exist positive constants  $M_1, M_2$  (depending only on  $n, \alpha, p$ ) such that for all  $R > 0$ ,*

$$|I_\alpha(f)|_{p, 2R} \leq \frac{M_1}{R^{n-\alpha-\frac{n}{p}}} |f|_1 + M_2 |f|_{r, R}. \quad (23)$$

(b) *There exist positive constants  $M'_1, M'_2$  (depending only on  $n, \alpha$ ) such that for all  $R > 0$ ,*

$$|I_\alpha(f)|_{\infty, 2R} \leq \frac{M'_1}{R^{n-\alpha}} |f|_1 + M'_2 |f|_{1, R}^{\frac{\alpha}{n}} |f|_{\infty, R}^{1-\frac{\alpha}{n}}. \quad (24)$$

**Proof.** The idea of the proof is the same as the previous section. That is, we decompose as

$$\begin{aligned} |I_\alpha(f)| &\leq \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy = I_1 + I_2, \\ I_1 &= \int_{|y| \leq R} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy, \\ I_2 &= \int_{|y| \geq R} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy. \end{aligned}$$



Since  $|x - y| \geq \frac{|x|}{2}$  whenever  $|x| \geq 2R$  and  $|y| \leq R$ , we have

$$I_1 \leq \frac{4}{|x|^{n-\alpha}} |f|_1 ,$$

hence

$$\begin{aligned} |I_1|_{p,2R} &\leq 4|f|_1 \left( \int_{|x| \geq 2R} \frac{1}{|x|^{p(n-\alpha)}} dx \right)^{\frac{1}{p}} \\ &= M_1 \frac{|f|_1}{R^{n-\alpha-\frac{n}{p}}} , \end{aligned}$$

where  $M_1$  depends only on  $n, p, \alpha$ . Clearly, this holds also for  $p = \infty$ . Recalling Hardy-Littlewood-Sobolev's inequality (see for instance [9, Chepter VI]), for  $1 < p, r < \infty$  with  $\frac{1}{p} = \frac{1}{r} - \frac{\alpha}{n}$ , we have

$$|I_2|_{p,2R} \leq |I_2|_p \leq C(\alpha, p) |f|_{r,R} ,$$

this proves (a).

Let us estimate  $|I_2|_{\infty,R}$ . For  $L > 0$  we decompose  $I_2 = I_{2,1} + I_{2,2}$ :

$$\begin{aligned} I_{2,1} &= \int_{|y| \geq R, |x-y| \leq L} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy , \\ I_{2,2} &= \int_{|y| \geq R, |x-y| \geq L} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy . \end{aligned}$$

Using Young's inequality, we have

$$|I_{2,1}|_{\infty,R} \leq \int_{|y| \leq L} \frac{1}{|y|^{n-\alpha}} dy |f|_{\infty,R} = C_{2,1} L^\alpha |f|_{\infty,R} ,$$

where  $C_{2,1}$  depends only on  $n, \alpha$ . Using Young's inequality again, we have

$$|I_{2,2}|_{\infty,R} \leq \frac{1}{L^{n-\alpha}} |f|_{1,R} .$$

Taking  $L = \left( \frac{|f|_{1,R}}{|f|_{\infty,R}} \right)^{\frac{1}{n}}$ , we get

$$|I_2|_{\infty,R} \leq |I_{2,1}|_{\infty,R} + |I_{2,2}|_{\infty,R} \leq M'_2 |f|_{1,R}^{\frac{\alpha}{n}} |f|_{\infty,R}^{1-\frac{\alpha}{n}} ,$$

where  $M'_2$  depends only on  $n, \alpha$ . This completes the proof.

**Remark 2.2.1.** Fix  $\delta \in (0, 1)$ . Arguing as in Remark 2.1.1, we have, instead of (23) and (24),

$$|I_\alpha(f)|_{p,2R} \leq \frac{M_1}{R^{n-\alpha-\frac{n}{p}}} |f|_1 + M_2 |f|_{r,2\delta R} , \quad (25)$$

where  $M_1$  depends only on  $n, \alpha, p$  and  $\delta$ , and,  $M_2$  depends only on  $n, \alpha, p$ .

$$|I_\alpha(f)|_{\infty,2R} \leq \frac{M'_1}{R^{n-\alpha}} |f|_1 + M'_2 |f|_{1,2\delta R}^{\frac{\alpha}{n}} |f|_{\infty,2\delta R}^{1-\frac{\alpha}{n}} , \quad (26)$$

where  $M'_1$  depends only on  $n, \alpha, p$  and  $\delta$ , and,  $M'_2$  depends only on  $n, \alpha, p$ .

### 2.2.2 Decay estimate for the derivatives of the convolution with Gauss kernel

In this section we estimate the spatial decay property for the derivatives of the semigroup, or the derivatives of the convolution with the Gauss kernel

$$\partial_x^\beta e^{\Delta t} f = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \partial_x^\beta e^{-\frac{|x-y|^2}{4t}} f(y) dy . \quad (27)$$

**Lemma 2.2.2.** *Let  $p \in [1, \infty]$ ,  $|\beta| = k$ . Then, there exist positive constants  $M_{3,k}$  (depending only on  $n, p, k$ ) and  $M_{4,k}$  (depending only on  $n, k$ ) such that for all  $R > 0$ ,*

$$|\partial_x^\beta e^{\Delta t} f|_{p, 2R} \leq \frac{M_{3,k}}{R^{n+1-\frac{n}{p}} t^{\frac{k-1}{2}}} |f|_1 + \frac{M_{4,k}}{t^{\frac{k}{2}}} |f|_{p,R} \quad \text{for all } f \in C_0(\mathbb{R}^n) . \quad (28)$$

**Proof.** Note that

$$|\partial_x^\beta e^{\Delta t} f(x, t)| \leq \sum_{\substack{i,j \geq 0 \\ 2j-i=k}} C_{i,j} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy ,$$

where  $C_{i,j}$  depends only on  $i$  and  $j$ . We decompose each term as

$$\begin{aligned} & \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|y| \leq R} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy + \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|y| \geq R} \frac{|x-y|^i}{t^j} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy \\ &= J_{i,j,1} + J_{i,j,2} . \end{aligned}$$

We now take  $l = \frac{n}{2} + \frac{i+1}{2}$  in (14) and apply it to  $J_{i,j,1}$  to get

$$J_{i,j,1} \leq \frac{C}{\pi^{\frac{n}{2}} t^{\frac{k-1}{2}}} \int_{|y| \leq R} \frac{1}{|x-y|^{n+1}} |f(y)| dy .$$

Since  $|x-y| \geq \frac{|x|}{2}$  whenever  $|x| \geq 2R$  and  $|y| \leq R$ , we have

$$J_{i,j,1} \leq \frac{C}{|x|^{n+1} t^{\frac{k-1}{2}}} \int_{|y| \leq R} |f(y)| dy \leq \frac{C_{n,i}}{|x|^{n+1} t^{\frac{k-1}{2}}} |f|_1 ,$$

where  $C_{n,i}$  depends only on  $n$  and  $i$ . Hence

$$|J_{i,j,1}|_{p, 2R} \leq \frac{C_{n,i}}{t^{\frac{k-1}{2}}} |f|_1 \left( \int_{|x| \geq 2R} \frac{1}{|x|^{p(n+1)}} dx \right)^{\frac{1}{p}} \leq M_{i,3} \frac{|f|_1}{R^{n+1-\frac{n}{p}} t^{\frac{k-1}{2}}} ,$$

where  $M_{i,3}$  depends only on  $n, i, p$ .

Next we estimate  $J_{i,j,2}$ . Using Young's inequality, we have

$$\begin{aligned} |J_{i,j,2}|_{p,2R} &\leq |J_{i,j,2}|_p \leq \frac{1}{(4\pi t)^{\frac{n}{2}} t^j} \int_{\mathbb{R}^n} |y|^i e^{-\frac{|y|^2}{4t}} dy |f|_{p,R} \\ &= \frac{2^i}{\pi^{\frac{n}{2}} t^{j-\frac{i}{2}}} \int_{\mathbb{R}^n} |z|^i e^{-|z|^2} dz |f|_{p,R} \\ &= \frac{M_{i,4}}{t^{\frac{k}{2}}} |f|_{p,R}, \end{aligned}$$

where  $M_{i,4}$  depends only on  $n, i$ . Thus

$$\begin{aligned} |\partial_x^\beta e^{t\Delta} f|_{p,2R} &\leq \sum_{\substack{i,j \geq 0 \\ 2j-i=k}} C_{i,j} (|J_{i,j,1}|_{p,2R} + |J_{i,j,2}|_{p,2R}) \\ &\leq \sum_{\substack{i,j \geq 0 \\ 2j-i=k}} C_{i,j} \left( M_{i,3} \frac{|f|_1}{R^{n+1-\frac{n}{p}} t^{\frac{k-1}{2}}} + \frac{M_{i,4}}{t^{\frac{k}{2}}} |f|_{p,R} \right) \\ &\leq M_{3,k} \frac{|f|_1}{R^{n+1-\frac{n}{p}} t^{\frac{k-1}{2}}} + \frac{M_{4,k}}{t^{\frac{k}{2}}} |f|_{p,R}, \end{aligned}$$

where  $M_{3,k}$  depends only on  $n, k$  and  $p$ , and,  $M_{4,k}$  depends only on  $n, k$ . The proof is now complete.

**Remark 2.2.2.** Fix  $\delta \in (0, 1)$ . Arguing as in Remark 2.1.1, we have, instead of (28),

$$|\partial_x^\beta e^{t\Delta} f|_{p,2R} \leq \frac{M_{3,k}}{R^{n+1-\frac{n}{p}} t^{\frac{k-1}{2}}} |f|_1 + \frac{M_{4,k}}{t^{\frac{k}{2}}} |f|_{p,2\delta R}, \quad (29)$$

where  $M_{3,k}$  depends only on  $n, p, k$  and  $\delta$ , and,  $M_{4,k}$  depends only on  $n, k$ .

**Remark 2.2.3.** The spatial decay estimates of functions convolved with the heat kernel like (28) are already obtained by Y. Giga and M.-H. Giga [9, Chapter I] for the case  $|\beta| = 0$ .

### 2.2.3 Proof of the spatial decay estimates for derivatives

We are now in position to prove the estimate (16) for  $b = 0$  by induction. The constants in this section may depend on  $\delta$ , but we do not refer to it for simplicity of the notation.

**Proof of the estimate (16) for  $b = 0$ .**

Assume that the estimate (16) holds for multi-index  $\beta$  with  $|\beta| = k - 1 \geq 0$ . We have to prove (16) for  $|\beta| = k$ . By the representation (21) we observe that

$$|\partial_x^\beta \omega(\cdot, t)|_{p, 2R} \leq |\partial_x^\beta e^{\frac{t}{2}\Delta} \omega(\frac{t}{2})|_{p, 2R} + \int_{\frac{t}{2}}^t |\nabla \cdot e^{(t-s)\Delta} \partial_x^\beta h(s)|_{p, 2R} ds =: K_1 + K_2, \quad (30)$$

where  $h(s) := u(s)\omega(s)$ . Let us estimate  $K_1$ . It follows from Remark 2.2.2 that

$$K_1 \leq \frac{M_{3,k}}{R^{3-\frac{2}{p}} t^{\frac{k-1}{2}}} |\omega(\frac{t}{2})|_1 + \frac{M_{4,k}}{t^{\frac{k}{2}}} |\omega(\frac{t}{2})|_{p, 2\delta R},$$

and by (9) for  $p = 1$ , and (15) (note that the constant  $W$  below may be different from  $W$  in (15) because of  $\delta$ ) we have

$$\begin{aligned} K_1 &\leq \frac{M_{3,k} W_1}{R^{3-\frac{2}{p}} t^{\frac{k-1}{2}}} |\omega_0|_1 + \frac{M_{4,k} W}{t^{\frac{k}{2}}} \left( \frac{|\omega_0|_1}{R(\frac{t}{2})^{\frac{1}{2}-\frac{1}{p}}} + \frac{|\omega_0|_{1, 2\delta^2 R}}{(\frac{t}{2})^{1-\frac{1}{p}}} \right) \\ &\leq \frac{B_{1,k}}{t^{\frac{k}{2}-\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1-\frac{1}{p}}} \right) + \frac{|\omega_0|_{1, 2\delta^2 R}}{t^{\frac{3}{2}-\frac{1}{p}}} \right\}, \end{aligned} \quad (31)$$

where  $B_{1,k}$  depends only on  $k, p, |\omega_0|_1$ .

Next we estimate  $K_2$ . Again, from Remark 2.2.2 it follows that

$$|\nabla \cdot e^{(t-s)\Delta} \partial_x^\beta h(s)|_{p, 2R} \leq \frac{M_{3,1}}{R^{3-\frac{2}{p}}} |\partial_x^\beta h(s)|_1 + \frac{M_{4,1}}{(t-s)^{\frac{1}{2}}} |\partial_x^\beta h(s)|_{p, 2\delta R} =: K_{2,1} + K_{2,2}.$$

By (9) and (10) for  $q = \infty$  we have

$$|\partial_x^\beta h(s)|_1 \leq \sum_{\alpha \leq \beta} |\partial_x^\alpha u(s)|_\infty |\partial_x^{\beta-\alpha} \omega(s)|_1 \leq \frac{C_k}{s^{\frac{k+1}{2}}}, \quad (32)$$

where  $C_k$  depends only on  $k$  and  $|\omega_0|_1$ , so since  $R \geq 1$ , we have

$$K_{2,1} \leq \frac{M_{3,1} C_k}{R s^{\frac{k+1}{2}}}. \quad (33)$$

Now we estimate  $|\partial_x^\beta h(s)|_{p, 2\delta R}$ . By Young's inequality,

$$\begin{aligned} |\partial_x^\beta h(s)|_{p, 2\delta R} &\leq |u(s)|_{\infty, 2\delta R} |\partial_x^\beta \omega(s)|_p + |\partial_x^\beta u(s)|_\infty |\omega(s)|_{p, 2\delta R} \\ &\quad + \sum_{\substack{\alpha \neq 0 \\ \alpha \neq \beta}} |\partial_x^\alpha u(s)|_\infty |\partial_x^{\beta-\alpha} \omega(s)|_{p, 2\delta R} \\ &=: K_{2,2,1} + K_{2,2,2} + K_{2,2,3}. \end{aligned}$$

By Remark 2.2.1 and (9) we observe that

$$K_{2,2,1} \leq \frac{W_1 |\omega_0|_1}{s^{1+\frac{k}{2}-\frac{1}{p}}} \left( \frac{M'_1}{R} |\omega(s)|_1 + M'_2 |\omega(s)|_{1, 2\delta^2 R}^{\frac{1}{2}} |\omega(s)|_{\infty, 2\delta^2 R}^{\frac{1}{2}} \right).$$

Here, by (15), we have

$$\begin{aligned} |\omega(s)|_{1,2\delta^2 R} |\omega(s)|_{\infty,2\delta^2 R} &\leq W^2 \left( \frac{1}{R} |\omega_0|_1 s^{\frac{1}{2}} + |\omega_0|_{1,2\delta^3 R} \right) \left( \frac{1}{R} \frac{|\omega_0|_1}{s^{\frac{1}{2}}} + \frac{|\omega_0|_{1,2\delta^3 R}}{s} \right) \\ &= W^2 \left( \frac{|\omega_0|_1}{R} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}} \right)^2, \end{aligned}$$

so

$$|\omega(s)|_{1,2\delta^2 R}^{\frac{1}{2}} |\omega(s)|_{\infty,2\delta^2 R}^{\frac{1}{2}} \leq W \left( \frac{|\omega_0|_1}{R} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}} \right).$$

Thus since  $|\omega(s)|_1 \leq |\omega_0|_1$ , we have

$$\begin{aligned} K_{2,2,1} &\leq \frac{W_1 |\omega_0|_1}{s^{1+\frac{k}{2}-\frac{1}{p}}} \left\{ \frac{M'_1}{R} |\omega_0|_1 + M'_2 W \left( \frac{|\omega_0|_1}{R} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}} \right) \right\} \\ &\leq \frac{B_{2,2,1}}{s^{1+\frac{k}{2}-\frac{1}{p}}} \left( \frac{1}{R} + \frac{|\omega_0|_{1,2\delta^3 R}}{s^{\frac{1}{2}}} \right), \end{aligned} \quad (34)$$

where  $B_{2,2,1}$  depends only on  $k, p, |\omega_0|_1$ .

Using (10) and (15), we have

$$\begin{aligned} K_{2,2,2} &\leq \frac{W W_3 |\omega_0|_1}{s^{\frac{k+1}{2}}} \left( \frac{1}{R} \frac{|\omega_0|_1}{s^{\frac{1}{2}-\frac{1}{p}}} + \frac{|\omega_0|_{1,2\delta^2 R}}{s^{1-\frac{1}{p}}} \right) \\ &\leq \frac{B_{2,2,2}}{s^{1+\frac{k}{2}-\frac{1}{p}}} \left( \frac{1}{R} + \frac{|\omega_0|_{1,2\delta^2 R}}{s^{\frac{1}{2}}} \right), \end{aligned} \quad (35)$$

where  $B_{2,2,2}$  depends only on  $k, p, |\omega_0|_1$ .

Next we estimate  $K_{2,2,3}$ . By the assumption of the induction we have

$$|\partial_x^{\beta-\alpha} \omega(s)|_{p,2\delta R} \leq \frac{B_{\alpha,\beta}}{s^{b+\frac{|\beta-\alpha|}{2}-\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{s^{1-\frac{1}{p}}} \right) + \frac{1}{s^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{3+|\beta-\alpha|} R} \right\},$$

where  $B_{\alpha,\beta}$  depends only on  $|\beta-\alpha|, p, |\omega_0|_1$ . Note that  $|\beta-\alpha| = |\beta| - |\alpha|$ . Then,

$$\begin{aligned} K_{2,2,3} &\leq \sum_{\substack{\alpha \neq 0 \\ \alpha \neq \beta}} \frac{C_\alpha}{s^{\frac{|\alpha|}{2}+\frac{1}{2}}} |\omega_0|_1 \frac{B_{\alpha,\beta}}{s^{\frac{|\beta-\alpha|}{2}-\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{s^{1-\frac{1}{p}}} \right) + \frac{1}{s^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{3+|\beta-\alpha|} R} \right\} \\ &\leq \frac{\max\{C_\alpha B_{\alpha,\beta}\}}{s^{\frac{k}{2}}} \sum_{\substack{\alpha \neq 0 \\ \alpha \neq \beta}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{s^{1-\frac{1}{p}}} \right) + \frac{1}{s^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{3+|\beta-\alpha|} R} \right\} \\ &\leq \frac{B_{2,2,3}}{s^{\frac{k}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{s^{1-\frac{1}{p}}} \right) + \frac{1}{s^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{2+k} R} \right\}, \end{aligned} \quad (36)$$

where  $B_{2,2,3}$  depends only on  $k, p, |\omega_0|_1$ . Collecting estimates (33)-(36), we have

$$\begin{aligned} &|\nabla \cdot e^{(t-s)\Delta} \partial_x^\beta h(s)|_{p,2R} \\ &\leq \frac{B_{2,2,4}}{s^{\frac{k}{2}}} \left[ \frac{1}{R} \left\{ \frac{1}{s^{\frac{1}{2}}} + \frac{1}{(t-s)^{\frac{1}{2}}} \left( 1 + \frac{1}{s^{1-\frac{1}{p}}} \right) \right\} + \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{2+k} R} \right], \end{aligned} \quad (37)$$

where  $B_{2,2,4}$  depends only on  $k, p, |\omega_0|_1$ . Hence

$$\begin{aligned} K_2 &\leq B_{2,2,4} \int_{\frac{t}{2}}^t \frac{1}{s^{\frac{k}{2}}} \left[ \frac{1}{R} \left\{ \frac{1}{s^{\frac{1}{2}}} + \frac{1}{(t-s)^{\frac{1}{2}}} \left( 1 + \frac{1}{s^{1-\frac{1}{p}}} \right) \right\} + \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{2+k}R} \right] ds \\ &\leq \frac{B_{2,k}}{t^{\frac{k}{2}-\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1-\frac{1}{p}}} \right) + \frac{1}{t^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{2+k}R} \right\}, \end{aligned} \quad (38)$$

where  $B_{2,k}$  depends only on  $k, p, |\omega_0|_1$ . By (31) and (38), the estimate (16) holds for  $b = 0$ .

**Proof of the estimate (16) for arbitrary  $\beta, b$ .**

Let us show that (16) holds for arbitrary multi-index  $\beta$  and  $b \in \mathbb{N}_0$ . First we consider the case  $b = 1$ . Since  $\omega$  is a solution of (2), i.e.,

$$\partial_t \omega = \Delta \omega - (u, \nabla) \omega,$$

we have

$$\partial_t \partial_x^\beta \omega = \partial_x^\beta \Delta \omega - \partial_x^\beta (u, \nabla) \omega.$$

Thus

$$\begin{aligned} |\partial_t \partial_x^\beta \omega(\cdot, t)|_{p,2R} &\leq |\partial_x^\beta \Delta \omega(\cdot, t)|_{p,2R} + |\partial_x^\beta (u, \nabla) \omega(\cdot, t)|_{p,2R} \\ &\leq |\partial_x^\beta \Delta \omega(\cdot, t)|_{p,2R} + \sum_{\alpha \leq \beta} |\partial_x^\alpha u \partial_x^{\beta-\alpha} \nabla \omega(\cdot, t)|_{p,2R} \\ &=: A_1 + A_2. \end{aligned}$$

The estimate (16) for  $b = 0$  yields

$$A_1 \leq \frac{W_2}{t^{\frac{|\beta|}{2}+\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1-\frac{1}{p}}} \right) + \frac{1}{t^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{4+|\beta|}R} \right\}. \quad (39)$$

Moreover, it follows from (10) and (12) for  $b = 0$ ,

$$\begin{aligned} A_2 &\leq \sum_{\alpha \leq \beta} |\partial_x^\alpha u|_\infty |\partial_x^{\beta-\alpha} \nabla \omega(\cdot, t)|_{p,2R} \\ &\leq \sum_{\alpha \leq \beta} \frac{W_{3,\alpha}}{t^{\frac{|\alpha|}{2}+\frac{1}{2}}} |\omega_0|_1 \frac{W_{2,\beta-\alpha}}{t^{\frac{|\beta-\alpha|}{2}+\frac{1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1-\frac{1}{p}}} \right) + \frac{1}{t^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{3+|\beta-\alpha|}R} \right\} \\ &\leq \frac{W'}{t^{\frac{|\beta|+1}{2}}} \left\{ \frac{1}{R} \left( 1 + \frac{1}{t^{1-\frac{1}{p}}} \right) + \frac{1}{t^{\frac{3}{2}-\frac{1}{p}}} |\omega_0|_{1,2\delta^{3+|\beta|}R} \right\}, \end{aligned} \quad (40)$$

where  $W'$  depends only on  $\beta, p, |\omega_0|_1$ . By (39) and (40), the estimate (16) holds for  $b = 1, \beta$ . The case  $b \geq 2$  can be shown as the same above, we omit it. This completes the proof of Theorem 1.3.

### 3 Large time behavior of derivatives

#### 3.1 Rescaling

We are now in position to prove the main theorem. We rescale the pair  $(\omega, u)$  by

$$\begin{aligned}\omega_k(x, t) &= k^2 \omega(kx, k^2 t), \quad k > 0 \\ u_k(x, t) &= ku(kx, k^2 t), \quad k > 0\end{aligned}$$

where  $(\omega, u)$  is the solution of (2)-(3) with initial vorticity  $\omega_0 \in L^1(\mathbb{R}^2)$ . Then the rescaled pair  $(\omega_k, u_k)$  satisfies vorticity equation (2) with initial vorticity  $\omega_k(0) = k^2 \omega_0(kx)$ , and  $u_k$  is the velocity field associated with  $\omega_k$  via Biot-Savart law. Remark that  $|\omega_k(0)|_1 = |\omega_0|_1$  and the Gauss kernel  $g$  is invariant with respect to this rescaling, i.e., for all  $k > 0$ ,  $g_k(x, t) = g(x, t)$ ,  $\forall x \in \mathbb{R}^2, t > 0$ . We can easily check the following lemma.

**Lemma 3.1.1.** *Assume that  $p \in [1, \infty]$ ,  $b \in \mathbb{N}_0$  and  $\beta$  is a multi-index. Let  $\omega$  be the solution of (2)-(3) with initial vorticity  $\omega_0 \in L^1(\mathbb{R}^2)$ ,  $m = \int_{\mathbb{R}^2} \omega_0(x) dx$ , and  $g(x, t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$ . Then, the following two formulae are equivalent.*

$$\lim_{t \rightarrow \infty} t^{b + \frac{|\beta|}{2} + 1 - \frac{1}{p}} |\partial_t^b \partial_x^\beta \omega(\cdot, t) - \partial_t^b \partial_x^\beta mg(\cdot, t)|_p = 0, \quad (41)$$

$$\lim_{k \rightarrow \infty} |\partial_t^b \partial_x^\beta \omega_k(\cdot, 1) - \partial_t^b \partial_x^\beta mg(\cdot, 1)|_p = 0. \quad (42)$$

**Proof.** Note that

$$\partial_t^b \partial_x^\beta (\omega_k - mg)(x, t) = k^{2+2b+|\beta|} \partial_t^b \partial_x^\beta (\omega - mg)(kx, k^2 t),$$

so that

$$\begin{aligned}|\partial_t^b \partial_x^\beta \omega_k(\cdot, 1) - \partial_t^b \partial_x^\beta mg(\cdot, 1)|_p &= k^{2+2b+|\beta|} |\partial_t^b \partial_x^\beta \omega(k\cdot, k^2) - \partial_t^b \partial_x^\beta mg(k\cdot, k^2)|_p \\ &= k^{2+2b+|\beta| - \frac{2}{p}} |\partial_t^b \partial_x^\beta \omega(\cdot, k^2) - \partial_t^b \partial_x^\beta mg(\cdot, k^2)|_p.\end{aligned}$$

Taking  $k^2 = t$ , we observe that the statement holds.

#### 3.2 Compactness of rescaled functions

Lemma 3.1.1 implies that we should study the compactness of the family of the rescaled functions  $\{\omega_k(x, 1)\}_{k \geq 1}$ . Theorem 1.3 yields compactness for rescaled functions.

**Lemma 3.2.1.** *Assume that  $p \in [1, \infty]$ ,  $b \in \mathbb{N}_0$  and  $\beta$  is a multi-index. Let  $\{\omega_k(x, 1)\}_{k \geq 1}$  be a family of the rescaled functions as above. Then, the following statements hold.*

(a) There exists a positive constants  $C = C(\beta, b, p, |\omega_0|_1)$  such that for all  $k \geq 1$ ,

$$|\partial_t^b \partial_x^\beta \omega_k(\cdot, 1)|_p \leq C \quad (43)$$

holds.

(b) For any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $k \geq 1$  and  $|y| < \delta$ ,

$$|\partial_t^b \partial_x^\beta \omega_k(\cdot - y, 1) - \partial_t^b \partial_x^\beta \omega_k(\cdot, 1)|_p \leq \epsilon \quad (44)$$

holds.

(c) For any  $\epsilon > 0$ , there exists  $R \geq 1$  such that for all  $k \geq 1$ ,

$$|\partial_t^b \partial_x^\beta \omega_k(\cdot, 1)|_{p,R} \leq \epsilon \quad (45)$$

holds.

**Proof.** Note that  $|\omega_k(0)|_1 = |\omega_0|_1$ ,  $|\omega_k(0)|_{1,R} \leq |\omega_0|_{1,R}$  for all  $k \geq 1$ . Hence (a), (c) clearly follows from by Theorem 1.3 (9), (12), respectively. The statement (b) follows from the estimate (11). The details will be omitted.

Applying the Riesz criterion ([14, Theorem XIII.66]) to the above lemma, we know that the family of the rescaled functions  $\{\omega_k(x, 1)\}_{k \geq 1}$  is relatively compact with valued in Sobolev space  $W^{|\beta|, p}(\mathbb{R}^2)$  for  $p \in [1, \infty)$  and all multi-index  $\beta$ . We observe that  $\{\omega_k(x, 1)\}_{k \geq 1}$  is also relatively compact with valued in Sobolev space  $W^{|\beta|, \infty}(\mathbb{R}^2)$  by using Ascoli-Arzelà theorem. Let  $\{\omega_{k(l)}(x, 1)\}_{l=1}^{l=\infty}$  ( $k(l)$  goes to  $\infty$  as  $l$  goes to  $\infty$ ) be a convergent subsequence of  $\{\omega_k(x, 1)\}_{k \geq 1}$  in  $W^{|\beta|, p}(\mathbb{R}^2)$ . We have already known that  $\omega_k(x, 1) \rightarrow mg(x, 1)$  in  $L^p(\mathbb{R}^2)$  as  $k$  goes to  $\infty$  by Theorem 1.1 and Lemma 3.1.1. So  $\omega_{k(l)}(x, 1) \rightarrow mg(x, 1)$  in  $W^{|\beta|, p}(\mathbb{R}^2)$ . This implies  $\omega_k(x, 1) \rightarrow mg(x, 1)$  in  $W^{|\beta|, p}(\mathbb{R}^2)$  as  $k$  goes to  $\infty$ . So Lemma 3.1.1(42) holds for  $b = 0$ .

Next we consider the case  $b = 1$ . Since  $\partial_t \omega_k = \Delta \omega_k - (u_k, \nabla) \omega_k$ , we have

$$\partial_t \partial_x^\beta \omega_k = \partial_x^\beta \Delta \omega_k - \partial_x^\beta (u_k, \nabla) \omega_k .$$

Thus if we note the fact  $\partial_t g = \Delta g = \Delta g - (v^g, \nabla) g$ , where  $v^g$  is a function associated with the Gauss kernel  $g$  by Biot-Savart law (3), we observe that

$$\begin{aligned} & |\partial_t \partial_x^\beta \omega_k(\cdot, 1) - \partial_t \partial_x^\beta mg(\cdot, 1)|_p \\ & \leq |\partial_x^\beta \Delta \omega_k(\cdot, 1) - \partial_x^\beta \Delta mg(\cdot, 1)|_p + |\partial_x^\beta (u_k, \nabla) \omega_k(\cdot, 1) - \partial_x^\beta (v^g, \nabla) mg(\cdot, 1)|_p \\ & \leq |\partial_x^\beta \Delta \omega_k(\cdot, 1) - \partial_x^\beta \Delta mg(\cdot, 1)|_p + |\partial_x^\beta (u_k - mv^g, \nabla) \omega_k(\cdot, 1)|_p \\ & \quad + |\partial_x^\beta (mv^g, \nabla) (\omega_k - mg)(\cdot, 1)|_p \\ & =: G_1 + G_2 + G_3 . \end{aligned}$$

From the result of the case  $b = 0$ , the first term  $G_1$  goes to zero. By Biot-Savart law (3) we observe that

$$\partial_x^\alpha (u_k - v^g)(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \partial_x^\alpha (\omega_k - mg)(y, t) dy ,$$



and by Hardy-Littlewood-Sobolev's inequality, if  $1 < q, r < \infty$ ,  $\frac{1}{q} = \frac{1}{r} - \frac{1}{2}$ , then

$$|\partial_x^\alpha(u_k - v^g)(\cdot, 1)|_q \leq C(q)|\partial_x^\alpha(\omega_k - mg)(\cdot, 1)|_r . \quad (46)$$

Moreover, by Gagliardo-Nirenberg's inequality (see for instance [9, Chepter VI]), for  $2 < q < \infty$  we have

$$|\partial_x^\alpha(u_k - v^g)(\cdot, 1)|_\infty \leq C'|\partial_x^\alpha(u_k - v^g)(\cdot, 1)|_q^{1-\frac{2}{q}}|\nabla\partial_x^\alpha(u_k - v^g)(\cdot, 1)|_q^{\frac{2}{q}} . \quad (47)$$

Combining (46) and (47) with the estimate (9) and the results (7) for  $b = 0$ , we observe that  $G_2$  goes to zero as  $k \rightarrow \infty$ . Finally, since  $|\partial_x^\alpha v^g(\cdot, 1)|_\infty \leq C(\alpha)$ ,  $G_3$  goes to zero. So (7) also holds for  $b = 1$ . By induction, arguing as the same above, we also know that (7) holds for any  $\beta$ ,  $b$ . Once we obtain (7), then (8) is obvious if we use inequalities such as (46), (47). We omit the details. The proof of Theorem 1.2 is now complete.

## References

- [1] Ben-Artzi, M. Global solutions of two-dimensional Navier-Stokes and Euler equations. Arch. Rational Mech. Anal. **1994**, 128, 329-358.
- [2] Brezis, H. Remarks on the preceeding paper by M. Ben-Artzi: "Grobal solutions of two-dimensional Navier-Stokes and Euler equations". Arch. Rational Mech. Anal. **1994**, 128, 359-360.
- [3] Carlen, E. A.; Loss, M. Optimal smoothing and decay estimates for visously damped conservation laws, with applications to the 2-D Navier-Stokes equation. Duke Math. J. **1996**, 81, 135-157.
- [4] Carpio, A. Asymptotic behavior for the vorticity equations in dimensions two and three. Commun. in Patial Differential Equations **1994**, 19, 827-872.
- [5] Gallagher, I.; Gallay, Th. Uniqueness for two-dimensional Navier-Stokes equation with a measure as initial vorticity., to appear.
- [6] Gallay, Th.; Wayne, C. E. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. Comm. Math. Phys., to appear.
- [7] Gallagher, I.; Gallay, Th.; Lions, P.-L. On the uniqueness of the solution of the two-dimensional Navier-Stokes equation with a Dirac mass as initial vorticity., preprint.
- [8] Giga, Y.; Kambe, T. Large time behavior of the vorticity of two dimensional viscous flow and its application to vortex formation. Comm. Math. Phys. **1988**, 117, 549-568.
- [9] Giga, Y.; Giga, M.-H. *Nonlinear Partial Differential Equation, Self-similar solutions and asymptotic behavior*; Kyoritsu: 1999 (in Japanese), English version to be published by Birkhäuser.

- [10] Giga, Y.; Miyakawa, T.; Osada, H. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.* **1988**, 104, 223-250.
- [11] Giga, Y. On the Two-Dimensional Nonstationary Vorticity Equations. *Tosio Kato's method and principle for evolution equations in mathematical physics*; University of Tokyo Press Production Center: Tokyo, 2002; 86-96.
- [12] Giga, Y.; Sawada, O. On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem. *Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday*; Kluwer Acad. Publ.: Dordrecht, 2003; Vol. 1, 2, 549-562.
- [13] Kato, T. The Navier-Stokes equation for an incompressible fluid in  $\mathbb{R}^2$  with a measure as the initial vorticity. *Differential Integral Equations* **1994**, 7, 949-966.
- [14] Reed, M.; Simon, B. *Methods of modern mathematical physics. IV. Analysis of operators*; Academic Press: New York, 1978.