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WHAT DOES THE PARTIAL AUTOCORRELATION FUNCTION LOOK LIKE FOR LARGE LAGS

AKIHIKO INOUE

Abstract. We prove a representation of the partial autocorrelation function $\alpha(\cdot)$ of a stationary process $\{X_n : n \in \mathbf{Z}\}$, in terms of the AR($\infty$) and MA($\infty$) coefficients. We apply it to show what $\alpha(n)$ looks like for large n, especially, when $\{X_n\}$ is a long-memory process. For example, if $\{X_n\}$ is a fractional ARIMA(p, d, q) process, then we have $\alpha(n) \sim d/n$ as $n \to \infty$.

1. Introduction and Results

Let $\{X_n : n \in \mathbf{Z}\}$ be a real, zero-mean, weakly stationary process, defined on a probability space $(\Omega, \mathcal{F}, P)$, which we shall simply call a *stationary process*. We write $\gamma(\cdot)$ for the autocovariance function of $\{X_n\}$:

$$\gamma(n) := E[X_n X_0] \qquad (n \in \mathbf{Z}).$$

For $\{X_n\}$, we have another important function $\alpha(\cdot)$ called the *partial autocorrelation function*. For $n = 2, 3, \ldots$, $\alpha(n)$ is the correlation coefficient of the two residuals obtained after regressing X_0 and X_n on the intermediate values $X_1, \ldots, X_{n-1}$. By the Durbin Levinson algorithm (cf. Brockwell Davis [BD, Proposition 5.2.1]), we can calculate the value of $\alpha(n)$ easily (at least numerically) from those of $\gamma(0)$, $\gamma(1)$, $\ldots$, $\gamma(n)$, and conversely the value of $\gamma(n)$ from those of $\gamma(0)$, $\alpha(1)$, $\ldots$, $\alpha(n)$. In this sense, $\alpha(\cdot)$ has the same information as $\gamma(\cdot)$.

Unlike the autocovariance function $\gamma(\cdot)$, no restriction such as nonnegative definiteness is imposed on the partial autocorrelation function $\alpha(\cdot)$ (cf. Ramsey [Ra]). This suggests a potential advantage of $\alpha(\cdot)$ over $\gamma(\cdot)$ in time series analysis. It would be therefore desirable to have a dictionary where properties of $\gamma(\cdot)$, such as the asymptotics, get translated to those

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of $\alpha(\cdot)$, and vice versa. Unfortunately, our knowledge of the relationship between $\gamma(\cdot)$ and $\alpha(\cdot)$ has not been so good. The Durbin Levinson algorithm, which is very useful in numerical calculation, is not helpful for such a purpose. The difficulty comes from the fact that the definition of partial autocorrelation function involves the prediction from a *finite* part of time. This setting makes the analysis of $\alpha(\cdot)$ particularly difficult.

In [I2], the author introduced a method of studying the asymptotic behaviour of $|\alpha(n)|$ as $n \to \infty$. It was based on a representation of mean-squared finite-past prediction error, in terms of the AR(∞) and MA(∞) coefficients of the stationary process $\{X_n\}$ ([I2, Theorems 4.5 and 4.6]). By this method, a surprising regularity in the asymptotic behaviour of $|\alpha(n)|$ was revealed ([I2, Theorem 2.1]). Subsequently, in [I3] and Inoue Kasahara [IK], this method was applied to the fractional ARIMA(p, d, q) processes, and it was shown that $|\alpha(n)| \sim |d|/n$, as $n \to \infty$, in this case. However, of course, the original object was not the asymptotic behaviour of $|\alpha(n)|$ but that of $\alpha(n)$ itself. Unfortunately, the method of [I2] was not good enough for us to reach the latter.

In this paper, we achieve the original aim. Thus we show what $\alpha(n)$ looks like for large n, especially, when $\{X_n\}$ is a long-memory process. For example, if $\{X_n\}$ is a fractional ARIMA(p, d, q) process, then we have $\alpha(n) \sim d/n$ as $n \to \infty$ (Theorem 1.7). The key development in the methods is an unexpected representation of $\alpha(\cdot)$ in terms of the AR(∞) coefficients a_k and the MA(∞) coefficients c_k, of $\{X_n\}$ (Theorems 1.1–1.3). This representation enables us to study $\alpha(\cdot)$ directly via a_k and c_k. In a sense, it links the theory of finite-past prediction, in which $\alpha(\cdot)$ lives, to the theory of infinite-past prediction, where a_k and c_k live. The advantage of this link is that the latter type of prediction theory has been well developed since the early work of Szegö, Kolmogorov, Wiener, Krein, Beurling, and others. Via the link, we can study our finite-past prediction problem using the theory of infinite-past prediction.

Notice that such a link between the two types of prediction theories has already been found in the representation theorem of finite-past prediction error stated above. In fact, as in the proof of the latter, we prove the representation of $\alpha(\cdot)$ using a discrete-time analogue of the Seghier Dym theorem ([I2, Theorem 3.1]; see Theorem 1.5 below). The original Seghier Dym theorem ([S, D]) concerns the intersection of past and future of a continuous-time stationary process, and has originated from the work of Levinson McKean [LM].

We denote by H the closed real linear hull of $\{X_k : k \in \mathbf{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$. Then H is a real Hilbert space with inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For $I \subset \mathbf{Z}$, denote by H_I the closed real linear hull of $\{X_k : k \in I\}$ in H. In particular, for $m \in \mathbf{Z}$ and $n \in \mathbf{Z}$ with $m \leq n$, we write $H_{(-\infty,m]}$, $H_{[m,\infty)}$, and $H_{[m,n]}$ for H_I with $I = \{k \in \mathbf{Z} : -\infty < k \leq m\}$, $\{k \in \mathbf{Z} : m \leq k < \infty\}$, and $\{k \in \mathbf{Z} : m \leq k \leq n\}$, respectively. For $I \subset \mathbf{Z}$, we denote by P_I the orthogonal projection operator of H onto H_I. We write $P_I^{\perp} := I_H - P_I$, where I_H is the identity map of H. So $P_I^{\perp}$ is the orthogonal projection operator of H onto the orthogonal complement $H_I^{\perp}$ of H_I in H. For $Y \in H$, we may think of $P_I Y$ as the best linear predictor of Y on the observations X_k $(k \in I)$, whence $P_I^{\perp} Y = Y - P_I Y$ as its prediction error.

Throughout this paper, we assume that the stationary process $\{X_n\}$ is *purely nondeterministic*, that is,

$$\bigcap_{n=-\infty}^{\infty} H_{(-\infty,n]} = \{0\}$$

or, equivalently, there exists a positive even and integrable function $\Delta(\cdot)$ on $(-\pi, \pi)$ such that

$$\gamma(n) = \int_{-\pi}^{\pi} e^{in\theta} \Delta(\theta) d\theta \quad (n \in \mathbf{Z}), \qquad \int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty$$

(see [BD, Sect. 5.7] and Rozanov [Ro, Chapter II]). We call $\Delta(\cdot)$ the *spectral density* of $\{X_n\}$.

We set

$$U_n := \begin{cases} (X_1, X_0) & (n = 1), \\ \left(P_{[1,n-1]}^{\perp} X_n, P_{[1,n-1]}^{\perp} X_0\right) & (n = 2, 3, \dots), \end{cases}$$

$$V_n := \begin{cases} \|X_1\|^2 & (n = 1), \\ \|P_{[1,n-1]}^{\perp} X_n\|^2 & (n = 2, 3, \dots). \end{cases}$$

The partial autocorrelation function $\alpha(\cdot)$ of $\{X_n\}$ is defined by

$$\alpha(n) := \frac{U_n}{V_n} \qquad (n = 1, 2, \dots)$$

(cf. [BD, Sect. 3.4 and 5.2]). Thus, for $n = 2, 3, \ldots$, $\alpha(n)$ is the correlation coefficient between $P^{\perp}_{[1,n-1]}X_n$ and $P^{\perp}_{[1,n-1]}X_0$. We may say that $\alpha(n)$ is the *pure* correlation coefficient between X_0 and X_n.

Using the spectral density $\Delta(\cdot)$ of $\{X_n\}$, we define the *outer function* $h(\cdot)$ of $\{X_n\}$ by

$$h(z) := \sqrt{2\pi}\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\Delta(\theta)d\theta\right\} \qquad (z \in \mathbf{C},\ |z| < 1)$$

(see [Ro] and Ibragimov Rozanov [IR] for background). The function $h(\cdot)$ is actually an outer function in the Hardy space H^{2+} of class 2 over the unit disk $|z| < 1$. Using $h(\cdot)$, we define the MA(∞) coefficients c_n by

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (|z| < 1),$$

and the AR(∞) coefficients a_n by

$$-\frac{1}{h(z)} = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1)$$

(see [I2, Sect. 4] for background). Both $\{c_n\}$ and $\{a_n\}$ are real sequences, and $\{c_n\}$ is in l^2. We define

$$b_j^m := \sum_{k=0}^{m} c_k a_{j+m-k} \qquad (m, j \in \mathbf{N} \cup \{0\}).$$

Notice that b_j^m here is equal to that in [IK] but it corresponds to b_{j-1}^{m+1} in [I2, I3].

Here is the representation of the partial autocorrelation function $\alpha(\cdot)$ in terms of the AR(∞) and MA(∞) coefficients.

Theorem 1.1. *Suppose that the AR(∞) coefficients a_n satisfy*

$$\sum_{k=0}^{\infty} |a_k| < \infty. \tag{1.1}$$

Then we have, for $n = 1, 2, \ldots$,

$$U_n = (c_0)^2 \sum_{k=1}^{\infty}\sum_{p=0}^{\infty} d_k(n,p) d_{k-1}(n,p), \tag{1.2}$$

$$V_n = (c_0)^2 \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} d_k(n,p)^2, \tag{1.3}$$

where, for $n \in \mathbf{N}$ *and* $p \in \mathbf{N} \cup \{0\}$, *we define* $d_0(n,p) := \delta_{p0}$,

$$d_1(n,p) := \sum_{m_1=0}^{\infty} a_{n+m_1+p} c_{m_1}, \tag{1.4}$$

and, for $k = 2, 3, \ldots$,

(1.5)

$$d_k(n,p) := \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{v=0}^{\infty} b_{n+p+v}^{m_1} c_v.$$

In particular, we have

$$\alpha(n) = \frac{\sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n,p) d_{k-1}(n,p)}{\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} d_k(n,p)^2} \qquad (n = 1, 2, \ldots). \tag{1.6}$$

We write $\sum^{\infty-}$ to indicate that the sum does not necessarily converge absolutely, i.e., $\sum_{k=m}^{\infty-} := \lim_{M\to\infty} \sum_{k=m}^{M}$. We need the next variant of Theorem 1.1 when we consider the fractional ARIMA(p,d,q) processes with $-1/2 < d < 0$.

Theorem 1.2. *The representations* (1.2) *and* (1.3), *hence* (1.6), *still hold if all the summations* $\sum^{\infty}$ *in* (1.4) *and* (1.5) *are replaced by* $\sum^{\infty-}$ *and if* (1.1) *is replaced by the following two conditions:*

$$\sum_{k=0}^{\infty} |c_k| < \infty, \tag{1.7}$$

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty. \tag{1.8}$$

In the applications, we need to rewrite (1.5). Notice that, under (1.1), the boundedness of the sequence $\{c_n\}$, which comes from the property $\{c_n\} \in l^2$, implies

$$\sum_{v=0}^{\infty} |c_v a_{n+v}| < \infty \qquad (n = 0, 1, \ldots).$$

Thus, we may consider the following condition under (1.1):

$$\sum_{v=0}^{\infty} |c_v a_{n+v}| = O(n^{-1}) \qquad (n \to \infty). \tag{1.9}$$

We shall see that this condition holds in many cases. Under (1.1), we define

$$\beta(n) := \sum_{v=0}^{\infty} c_v a_{v+n} \qquad (n = 0, 1, \dots). \tag{1.10}$$

Theorem 1.3. *We assume* (1.1) *and* (1.9). *Then, we have, for* $n \in \mathbf{N}$ *and* $p \in \mathbf{N} \cup \{0\}$,

$$d_1(n,p) = \beta(n+p), \tag{1.11}$$

$$d_2(n,p) = \sum_{m_1=0}^{\infty} \beta(m_1+n)\beta(m_1+n+p), \tag{1.12}$$

and, for $k = 3, 4, \dots,$

(1.13)

$$\begin{aligned} d_k(n,p) = &\sum_{m_{k-1}=0}^{\infty} \beta(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} \beta(m_{k-1}+m_{k-2}+n) \\ &\cdots \sum_{m_2=0}^{\infty} \beta(m_3+m_2+n) \sum_{m_1=0}^{\infty} \beta(m_2+m_1+n)\beta(m_1+n+p), \end{aligned}$$

the sums converging absolutely.

The key to the proofs of Theorems 1.1 and 1.2 is to extend [I2, Theorem 4.1] properly. For $n \in \mathbf{Z}$ and $k \in \mathbf{N}$, we define the projection operators P_n^k by

$$P_n^k := \begin{cases} P_{(-\infty,n-1]} & \text{if } k \text{ is odd}, \\ P_{[1,\infty)} & \text{if } k \text{ is even}. \end{cases}$$

Theorem 1.4. *Let* $Y_1, Y_2 \in H$. *Suppose that the spectral density* $\Delta(\cdot)$ *of* $\{X_n\}$ *satisfies*

$$\int_{-\pi}^{\pi} \frac{1}{\Delta(\theta)} d\theta < \infty. \tag{1.14}$$

(1) *We have*

$$\begin{aligned} (Y_1, Y_2) = &\left((P_1^1)^{\perp} Y_1, (P_1^1)^{\perp} Y_2\right) \\ &+ \sum_{k=1}^{\infty} \left((P_1^{k+1})^{\perp} P_1^k \cdots P_1^1 Y_1, (P_1^{k+1})^{\perp} P_1^k \cdots P_1^1 Y_2\right). \end{aligned} \tag{1.15}$$

(2) *We have, for $n = 2, 3, \ldots$,*

$$(1.16)\qquad \begin{aligned} \left(P^{\perp}_{[1,n-1]}Y_1, P^{\perp}_{[0,n-1]}Y_2\right) &= \left((P^1_n)^{\perp}Y_1, (P^1_n)^{\perp}Y_2\right) \\ &\quad + \sum_{k=1}^{\infty} \left((P^{k+1}_n)^{\perp}P^k_n \cdots P^1_n Y_1, (P^{k+1}_n)^{\perp}P^k_n \cdots P^1_n Y_2\right). \end{aligned}$$

If we put $Y_1 = Y_2$ in Theorem 1.4(2), then it reduces to [I2, Theorem 4.1]. As in the proof of the latter, the key to the proof of Theorem 1.4 is to use the following discrete analogue of the Seghier Dym theorem ([S, D]):

Theorem 1.5 ([I2]). *If the spectral density $\Delta(\cdot)$ of $\{X_n\}$ satisfies* (1.14)*, then, for every $n \in \mathbf{N} \cup \{0\}$, we have $H_{(-\infty,0]} \cap H_{[-n,\infty)} = H_{[-n,0]}$.*

We turn to the results on the asymptotic behaviour of $\alpha(n)$ as $n \to \infty$. We write $\mathcal{R}_0$ for the class of slowly varying functions at infinity: the class of positive, measurable ℓ, defined on some neighborhood $[A, \infty)$ of infinity, such that

$$\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1 \quad \text{for all } \lambda > 0$$

(see [BGT, Chapter 1]).

First, we consider the following setting as a standard one for long-memory processes: for $\ell \in \mathcal{R}_0$ and $d \in (0, 1/2)$,

$$(1.17)\qquad c_n \sim n^{-(1-d)} \left\{\frac{\ell(n)}{B(d, 1-2d)}\right\}^{1/2} \qquad (n \to \infty),$$

$$(1.18)\qquad a_n \sim n^{-(1+d)} \left\{\frac{\ell(n)}{B(d, 1-2d)}\right\}^{-1/2} \frac{d\sin(\pi d)}{\pi} \qquad (n \to \infty).$$

Using [I1, Proposition 4.3] and the equality

$$\gamma(n) = \sum_0^{\infty} c_v c_{|n|+v} \qquad (n \in \mathbf{Z})$$

(cf. [I2, (5.3)]), we see that (1.17) implies

$$(1.19)\qquad \gamma(n) \sim n^{2d-1}\ell(n) \qquad (n \to \infty).$$

Since $-1 < 2d - 1 < 0$, this implies that the stationary process $\{X_n\}$ satisfying (1.17) with $0 < d < 1/2$ is a *long-memory process* (cf. [BD, Sect. 13.2]). The setting (1.17) with (1.18) may seem stringent at first glance, but, under the conditions (C1), (C2), and (A1) below, (1.17) (1.19) are equivalent (cf. [I2, Theorem 5.1]). Moreover, the fractional

ARIMA(p, d, q) processes with $0 < d < 1/2$, which are regarded as standard models of long-memory processes and we consider below, satisfy (1.17) and (1.18) for some constant functions ℓ (see Kokoszka and Taqqu [KT, Corollary 3.1]).

Theorem 1.6. *For $\ell \in \mathcal{R}_0$ and $d \in (0, 1/2)$, we assume (1.17) and (1.18). Then the partial autocorrelation function $\alpha(\cdot)$ of $\{X_n\}$ satisfies*

$$\alpha(n) \sim \frac{d}{n} \qquad (n \to \infty). \tag{1.20}$$

We can write (1.20) as $d = \lim_{n\to\infty} n\alpha(n)$. This suggests a method of estimation of the parameter d which is important in a long-memory process. See [IK, Sect. 5] for numerical calculation.

Next we consider the fractional ARIMA processes. For $d \in (-1/2, 1/2)$ and $p, q \in \mathbf{N} \cup \{0\}$, a stationary process $\{X_n\}$ is said to be a *fractional ARIMA(p, d, q) process* if it has a spectral density $\Delta(\cdot)$ of the form

$$\Delta(\theta) = \frac{1}{2\pi}\frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2}|1 - e^{i\theta}|^{-2d} \qquad (-\pi < \theta < \pi), \tag{1.21}$$

where $\Phi(z)$ and $\Theta(z)$ are polynomials with real coefficients of degrees p, q, respectively, satisfying the following condition:

$$\begin{aligned} &\Phi(z) \text{ and } \Theta(z) \text{ have no common zeros, and } \Phi(z) \neq 0 \text{ and} \\ &\Theta(z) \neq 0 \text{ for all } z \text{ in the closed unit disk } \{z \in \mathbf{C} : |z| \leq 1\}. \end{aligned} \tag{1.22}$$

The fractional ARIMA models were introduced independently by Granger and Joyeux [GJ] and Hosking [Ho], and have been widely used as useful parametric models describing long-memory processes. See Beran [Be, Sect. 2.5] and [BD, Sect. 13.2] for textbook treatments. If $d \in (-1/2, 1/2) \setminus \{0\}$, then the autocovariance function $\gamma(\cdot)$ of a fractional ARIMA(p, d, q) process $\{X_n\}$ satisfies

$$\gamma(n) \sim \left\{\frac{\Theta(1)}{\Phi(1)}\right\}^2 \frac{\Gamma(1 - 2d)\sin(\pi d)}{\pi} n^{2d-1} \qquad (n \to \infty) \tag{1.23}$$

(see, e.g., [I3]). In particular, if $0 < d < 1/2$, then $\{X_n\}$ is a long-memory process. On the other hand, if $d = 0$, then $\{X_n\}$ reduces to the ordinary ARMA(p, q) process (cf. [BD, Chapter 3]), and $\gamma(n)$ decays exponentially as $n \to \infty$.

If $\{X_n\}$ is a fractional ARIMA$(0, d, 0)$ process, then we have the following simple form of $\alpha(n)$:

$$\alpha(n) = \frac{d}{n-d} \qquad (n = 1, 2, \dots) \tag{1.24}$$

(see [Ho, Theorem 1(f)] as well as [BD, (13.2.10)]). From (1.24), we find that (1.20) always holds in this case whenever $d \in (-1/2, 1/2) \setminus \{0\}$. We can also ask what $\alpha(n)$ looks like for large n when $\{X_n\}$ is a general fractional ARIMA(p, d, q) process with $d \in (-1/2, 1/2) \setminus \{0\}$. The answer is again (1.20) as the next theorem shows.

Theorem 1.7. *Let $p, q \in \mathbf{N} \cup \{0\}$ and $d \in (-1/2, 1/2) \setminus \{0\}$, and let $\{X_n\}$ be a fractional ARIMA (p, d, q) process with partial autocorrelation function $\alpha(\cdot)$. Then we have* (1.20).

As we stated above, earlier work ([I3, IK]) asserts that $|\alpha(n)| \sim |d|/n$ as $n \to \infty$. One nice point of Theorem 1.7 is that the proof is much easier than that of the earlier result. All these advances in result and proof are due to the new representation of $\alpha(\cdot)$. In Section 6, we also consider the case $d = 0$ for completeness. Naturally, the result (Theorem 6.1) is that $\alpha(n)$ decays exponentially as $n \to \infty$ if $d = 0$.

Finally we consider the stationary process $\{X_n\}$ satisfying the following conditions (cf. [I2, Sect. 2]):

$$c_n \geq 0 \textit{ for all } n \geq 0; \tag{C1}$$

$$\{c_n\} \textit{ is eventually decreasing to zero;} \tag{C2}$$

$$\{a_n\} \textit{ is eventually decreasing to zero.} \tag{A1}$$

In [I2], the following extra condition is also required:

$$\{a_n - a_{n+1}\} \textit{ is eventually decreasing to zero,} \tag{A2}$$

but we do not need it here. Let $\ell \in \mathcal{R}_0$, and choose a positive constant B so large that $\ell(\cdot)$ is locally bounded on $[B, \infty)$ (see [BGT, Corollary 1.4.2]). When we say $\int^\infty \ell(s)ds/s = \infty$, it means that $\int_B^\infty \ell(s)ds/s = \infty$. If so, then we define another slowly varying function $\tilde{\ell}$ by

$$\tilde{\ell}(x) := \int_B^x \frac{\ell(s)}{s} ds \qquad (x \geq B) \tag{1.25}$$

(see [BGT, Sect. 1.5.6]). The asymptotic behaviour of $\tilde{\ell}(x)$ as $x \to \infty$ does not depend on the choice of B since we have assumed that $\int^{\infty} \ell(s)ds/s = \infty$.

Theorem 1.8. *Let* $-\infty < d < 1/2$ *and* $\ell \in \mathcal{R}_0$. *We assume* (C1), (C2), (A1), *and* (1.19).

(1) *If* $0 < d < 1/2$, *then* $\alpha(\cdot)$ *satisfies* (1.20);
(2) *if* $d = 0$ *and* $\int^{\infty} \ell(s)ds/s = \infty$, *then*

$$\alpha(n) \sim n^{-1} \frac{\ell(n)}{2\tilde{\ell}(n)} \qquad (n \to \infty); \tag{1.26}$$

(3) *if* $d = 0$ *with* $\int^{\infty} \ell(s)ds/s < \infty$ *or* $-\infty < d < 0$, *then*

$$\alpha(n) \sim \frac{n^{2d-1}\ell(n)}{\sum_{-\infty}^{\infty} \gamma(k)} \qquad (n \to \infty). \tag{1.27}$$

This theorem is an improvement of [I2, Theorem 2.1] in that the latter considers $|\alpha(n)|$ rather than $\alpha(n)$, under the stronger assumptions (C1)–(A2). By [I2, Theorem 7.3], this theorem applies to $\{X_n\}$ satisfying the following condition:

$$\begin{aligned} &\text{there exists a finite Borel measure } \sigma \text{ on } [0, 1) \\ &\text{such that } \gamma(n) = \int_0^1 t^{|n|}\sigma(dt) \quad (n \in \mathbf{Z}). \end{aligned} \tag{1.28}$$

As an example, we consider the case $\gamma(n) = (1 + |n|)^{2d-1}$ with $-\infty < d < 1/2$ in Section 5.

We close this section with the remark that all the results in Theorems 1.6–1.8 are unified in

$$\alpha(n) \sim \frac{\gamma(n)}{\sum_{k=-n}^{n} \gamma(k)} \qquad (n \to \infty)$$

(we refer to [I2, Sect. 2 and 6] and [I3, Sect. 5] for the proof). This is mysterious. It seems likely that something essential is hidden behind this.

2. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.4. (1) We claim that

$$H_{(-\infty,0]} \cap H_{[1,\infty)} = \{0\}. \tag{2.1}$$

Indeed, by Theorem 1.5 and shift invariance, we have

$$H_{(-\infty,0]} \cap H_{[1,\infty)} \subset H_{(-\infty,0]} \cap H_{[0,\infty)} = H_{[0,0]},$$
$$H_{(-\infty,0]} \cap H_{[1,\infty)} \subset H_{(-\infty,1]} \cap H_{[1,\infty)} = H_{[1,1]},$$

whence

$$H_{(-\infty,0]} \cap H_{[1,\infty)} \subset H_{[0,0]} \cap H_{[1,1]}.$$

However, since $\{X_n\}$ is assumed to be purely nondeterministic, X_0 and X_1 are linearly independent in H, so that $H_{[0,0]} \cap H_{[1,1]} = \{0\}$. Thus (2.1) follows.

Now the orthogonal decompositions

$$H = H^{\perp}_{(-\infty,0]} \oplus H_{(-\infty,0]},$$
$$H = H^{\perp}_{[1,\infty)} \oplus H_{[1,\infty)}$$

of H imply the orthogonal decompositions

$$I_H = P^{\perp}_{(-\infty,0]} \oplus P_{(-\infty,0]}, \tag{2.2}$$
$$I_H = P^{\perp}_{[1,\infty)} \oplus P_{[1,\infty)} \tag{2.3}$$

of the identity map I_H, respectively. Repeated use of (2.2) and (2.3) yields, for $m = 2, 3, \dots$,

$$\begin{aligned}(Y_1, Y_2) &= \left((P_1^1)^{\perp} Y_1, (P_1^1)^{\perp} Y_2\right) \\ &+ \sum_{k=1}^{m-1} \left((P_1^{k+1})^{\perp} P_1^k \cdots P_1^1 Y_1, (P_1^{k+1})^{\perp} P_1^k \cdots P_1^1 Y_2\right) + R_1^m,\end{aligned}$$

where

$$R_1^m := \left(P_1^m \cdots P_1^1 Y_1, P_1^m \cdots P_1^1 Y_2\right).$$

As in the proof of [I2, Theorem 4.1], it follows from (2.1) and, e.g., Halmos [Ha, Problem 122] that

$$\operatorname*{s-lim}_{m\to\infty} P_1^m \cdots P_1^1 = 0,$$

whence $\lim_{m\to\infty} R_1^m = 0$. Thus (1.15) follows.
(2) For $n = 2, 3, \dots$, we have the orthogonal decompositions

$$H^{\perp}_{[1,n-1]} = H^{\perp}_{(-\infty,n-1]} \oplus \left(H^{\perp}_{[1,n-1]} \cap H_{(-\infty,n-1]}\right),$$
$$H^{\perp}_{[1,n-1]} = H^{\perp}_{[1,\infty)} \oplus \left(H^{\perp}_{[1,n-1]} \cap H_{[1,\infty)}\right)$$

of $H^{\perp}_{[1,n-1]}$, which in turn imply the orthogonal decompositions

$$P^{\perp}_{[1,n-1]} = P^{\perp}_{(-\infty,n-1]} \oplus P^{\perp}_{[1,n-1]}P_{(-\infty,n-1]}, \tag{2.4}$$

$$P^{\perp}_{[1,n-1]} = P^{\perp}_{[1,\infty)} \oplus P^{\perp}_{[1,n-1]}P_{[1,\infty)} \tag{2.5}$$

of $P_{[1,n-1]}$, respectively. Using (2.4) and (2.5) repeatedly, we find that, for $m = 2, 3, \ldots,$

$$\begin{aligned}\left(P^{\perp}_{[1,n-1]}Y_1, P^{\perp}_{[1,n-1]}Y_2\right) &= \left((P_n^1)^{\perp}Y_1, (P_n^1)^{\perp}Y_2\right) \\ &\quad + \sum_{k=1}^{m-1}\left((P_n^{k+1})^{\perp}P_n^k\cdots P_n^1Y_1, (P_n^{k+1})^{\perp}P_n^k\cdots P_n^1Y_2\right) + R_n^m,\end{aligned}$$

where

$$R_n^m := \left(P^{\perp}_{[1,n-1]}P_n^m\cdots P_n^1Y_1, P^{\perp}_{[1,n-1]}P_n^m\cdots P_n^1Y_2\right).$$

In the same way as above, Theorem 1.5 and [Ha, Problem 122] imply

$$\operatorname*{s-lim}_{m\to\infty} P_n^m\cdots P_n^1 = P_{[1,n-1]}.$$

Hence we have

$$\lim_{m\to\infty}\|P^{\perp}_{[1,n-1]}P_n^m\cdots P_n^1Y_i\| = \|P^{\perp}_{[1,n-1]}P_{[1,n-1]}Y_i\| = 0 \qquad (i = 1, 2),$$

so that $\lim_{m\to\infty} R_n^m = 0$. Thus (1.16) follows. □

Proof of Theorem 1.1. (Compare the proof of [I2, Theorem 4.5].) From (1.1), we have (1.14) (see [I2, Proposition 4.2]). Hence, it follows from Theorem 1.4 that, for $n = 1, 2, \ldots,$

$$\begin{aligned}U_n &= \left((P_n^2)^{\perp}P_n^1X_n, (P_n^2)^{\perp}X_0\right) \\ &\quad + \sum_{k=2}^{\infty}\left((P_n^{k+1})^{\perp}P_n^k\cdots P_n^1X_n, (P_n^{k+1})^{\perp}P_n^k\cdots P_n^2X_0\right),\end{aligned} \tag{2.6}$$

$$V_n = \|(P_n^1)^{\perp}X_n\|^2 + \sum_{k=1}^{\infty}\|(P_n^{k+1})^{\perp}P_n^k\cdots P_n^1X_n\|^2. \tag{2.7}$$

Let $n \in \mathbf{N}$. Suppose that k is even and ≥ 2. By [I2, Theorem 4.4], we have, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots,$

$$P_{(-\infty,n-1]} X_{m+n} = \sum_{j=0}^{\infty} b_{n+j}^{m} X_{-j} \qquad (\mathrm{mod}\ H_{[1,n-1]} \text{ if } n \geq 2),$$

$$P_{[1,\infty)} X_{-m} = \sum_{j=0}^{\infty} b_{n+j}^{m} X_{j+n} \qquad (\mathrm{mod}\ H_{[1,n-1]} \text{ if } n \geq 2),$$

whence

$$P_n^k \cdots P_n^1 X_n = c_0 \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{m_0=0}^{\infty} b_{n+m_0}^{m_1} X_{m_0+n} \quad (\mathrm{mod}\ H_{[1,n-1]} \text{ if } n \geq 2).$$

Since $\{X_m\}$ is purely nondeterministic, it permits the moving-average representation of the form

$$X_m = \sum_{j=-\infty}^{m} c_{m-j} \xi_j \qquad (m \in \mathbf{Z}),$$

where $\{\xi_j : j \in \mathbf{Z}\}$ is an orthonormal system of H such that

$$H_{(-\infty,m]} = H_{(-\infty,m]}(\xi) \qquad (m \in \mathbf{Z})$$

with $H_{(-\infty,m]}(\xi)$ being the closed subspace of H spanned by $\{\xi_j : -\infty < j \leq m\}$ (see [Ro, Chapter II]). Since

$$P_{(-\infty,n-1]}^{\perp} X_{m+n} = \sum_{j=0}^{m} c_{m-j} \xi_{j+n} \qquad (m = 0, 1, \ldots),$$

we have

$$(P_n^{k+1})^{\perp} P_n^k \cdots P_n^1 X_n = c_0 \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{m_0=0}^{\infty} b_{n+m_0}^{m_1} \sum_{j=0}^{m_0} c_{m_0-j} \xi_{j+n};$$

so that

$$\big((P_n^{k+1})^{\perp}P_n^k\cdots P_n^1X_n,\xi_{p+n}\big)=\begin{cases}c_0d_k(n,p) & (p=0,1,\dots),\\ 0 & (p=-1,-2,\dots).\end{cases}$$

Arguing similarly,

$$\big(\xi_{p+n},(P_n^{k+1})^{\perp}P_n^k\cdots P_n^2X_0\big)=\begin{cases}c_0d_{k-1}(n,p) & (p=0,1,\dots),\\ 0 & (p=-1,-2,\dots).\end{cases}$$

Thus, from the Parseval equality, we get

$$\begin{aligned}&\big((P_n^{k+1})^{\perp}P_n^k\cdots P_n^1X_n,(P_n^{k+1})^{\perp}P_n^k\cdots P_n^2X_0\big)\\ &\qquad\qquad\qquad=(c_0)^2\sum_{p=0}^{\infty}d_k(n,p)d_{k-1}(n,p),\end{aligned}\tag{2.8}$$

$$\|(P_n^{k+1})^{\perp}P_n^k\cdots P_n^1X_n\|^2=(c_0)^2\sum_{p=0}^{\infty}d_k(n,p)^2.\tag{2.9}$$

Similarly, we have (2.8) and (2.9) for k odd, and also

(2.10)

$$\big((P_n^2)^{\perp}P_n^1X_n,(P_n^2)^{\perp}X_0\big)=(c_0)^2d_1(n,0)=(c_0)^2\sum_{p=0}^{\infty}d_1(n,p)d_0(n,p),$$

$$\|(P_n^1)^{\perp}X_n\|^2=(c_0)^2=(c_0)^2\sum_{p=0}^{\infty}d_0(n,p)^2.\tag{2.11}$$

The assertions (1.2) and (1.3) now follow if we substitute (2.8) and (2.10) into (2.6), and (2.9) and (2.11) into (2.7). $\square$

Proof of Theorem 1.2. The condition (1.8) is equivalent to (1.14) (see [I2, Proposition 4.2]). Moreover, by [IK, Proposition 2.1], we have, for $n\in\mathbf{N}$ and $m\in\mathbf{N}\cup\{0\}$,

$$P_{(-\infty,n-1]}X_{m+n}=\sum_{j=0}^{\infty-}b_{n+j}^mX_{-j}\qquad(\text{mod } H_{[1,n-1]}\text{ if } n\ge 2),$$

$$P_{[1,\infty)}X_{-m}=\sum_{j=0}^{\infty-}b_{n+j}^mX_{j+n}\qquad(\text{mod } H_{[1,n-1]}\text{ if } n\ge 2).$$

Using these equalities, we can prove the theorem as in the proof of Theorem 1.1. We omit the details. $\square$

We turn to the proof of Theorem 1.3. For real sequences $\{C(n)\}_{n=0}^{\infty} \in l^{\infty}$ and $\{A(n)\}_{n=0}^{\infty} \in l^1$, we consider the condition

$$E(n) = O(n^{-1}) \qquad (n \to \infty), \tag{2.12}$$

where

$$E(n) := \sum_{v=0}^{\infty} |C(v)A(v+n)| \qquad (n = 0, 1, \dots). \tag{2.13}$$

The condition (2.12) implies that, for some positive constant K,

$$E(n) \leq \frac{K}{(n+1)} \qquad (n = 0, 1, \dots). \tag{2.14}$$

Lemma 2.1. *Let $\{C(n)\}_{n=0}^{\infty} \in l^{\infty}$ and $\{A(n)\}_{n=0}^{\infty} \in l^1$. We assume (2.12). Then, for $n, p \in \mathbf{N} \cup \{0\}$, we have*

$$\sum_{m_1=0}^{\infty} E(m_1+n)E(m_1+n+p) < \infty, \tag{2.15}$$

and, for $k = 3, 4, \dots,$

(2.16)

$$\sum_{m_{k-1}=0}^{\infty} E(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} E(m_{k-1}+m_{k-2}+n)$$
$$\cdots \sum_{m_2=0}^{\infty} E(m_3+m_2+n) \sum_{m_1=0}^{\infty} E(m_2+m_1+n)E(m_1+n+p) < \infty.$$

Proof. (Compare the proof of [IK, Lemma 3.1].) For $k \in \mathbf{N}$ and $n, p, m \in \mathbf{N} \cup \{0\}$, we define $E_k(n,p,m)$ inductively by

$$E_1(n,p,m) := E(n+p+m),$$
$$E_{k+1}(n,p,m) := \sum_{m_k=0}^{\infty} E(m+m_k+n)E_k(n,p,m_k) \qquad (k = 1, 2, \dots).$$

We claim that, for $k \in \mathbf{N}$,

$$\sum_{m=0}^{\infty} E_k(n,p,m)^2 < \infty \qquad (n, p \in \mathbf{N} \cup \{0\}). \tag{2.17}$$

Since the sums in (2.15) and (2.16) are $E_2(n,p,0)$ and $E_k(n,p,0)$, respectively, the lemma follows immediately from (2.17).

To prove (2.17), we use induction. By (2.14), we find that (2.17) holds for $k = 1$. We assume that (2.17) holds for $k = j$. Then by (2.14), we have

$$E_{j+1}(n,p,m) \le K \sum_{m_j=0}^{\infty} \frac{1}{m+m_j+1} E_j(n,p,m_j). \tag{2.18}$$

Since the operator T defined by

$$(Tu)_m := \sum_{i=0}^{\infty} \frac{u_i}{m+i+1} \qquad (u = (u_i) \in l^2)$$

is a bounded linear operator from l^2 to l^2 (see Hardy *et al.* [HLP, Chapter IX]), the inequality (2.18) implies (2.17) for $k = j+1$. Thus (2.17) follows by induction. □

Proof of Theorem 1.3. By Lemma 2.1, we can use the Fubini–Tonelli theorem to exchange the order of sums in (1.5), and we get (1.12) and (1.13) as in the proof of [I2, Theorem 4.6]. □

3. Proof of Theorem 1.6

Let $\ell \in \mathcal{R}_0$ and $q \in (0,1)$. For real sequences $\{A(n)\}_{n=0}^{\infty}$ and $\{C(n)\}_{n=0}^{\infty}$, we consider the following conditions:

$$C(n) \sim n^{-(1-q)}\ell(n) \qquad (n \to \infty). \tag{3.1}$$

$$A(n) \sim n^{-(1+q)} \frac{1}{\ell(n)} \cdot \frac{q \sin(\pi q)}{\pi} \qquad (n \to \infty). \tag{3.2}$$

We define

$$B(n) := \sum_{v=0}^{\infty} C(v) A(n+v) \qquad (n = 1, 2, \dots).$$

Proposition 3.1. *Let* $\ell \in \mathcal{R}_0$ *and* $q \in (0,1)$. *We assume* (3.1) *and* (3.2).

(1) *We have*

$$B(n) \sim \frac{\sin(\pi q)}{\pi} n^{-1} \qquad (n \to \infty).$$

(2) *We have*

$$\sum_{v=0}^{\infty}|C(v)A(n+v)| \sim \frac{\sin(\pi q)}{\pi}n^{-1} \qquad (n\to\infty). \tag{3.3}$$

(3) *We have, for $s \geq 0$ and $u \geq 0$,*

$$B([ns]+[nu]+n) \sim \frac{\sin(\pi q)}{\pi(s+u+1)}n^{-1} \qquad (n\to\infty). \tag{3.4}$$

(4) *For every $r \in (1,\infty)$, there exists $N_1 \in \mathbf{N}$ such that*

(3.5)

$$|B([ns]+[nu]+n)| \leq \frac{r\sin(\pi q)}{\pi(s+u+1)}n^{-1} \qquad (s\geq 0,\ u\geq 0,\ n\geq N_1).$$

Proof. (1) follows from [I1, Proposition 4.3]. The conditions (3.1) and (3.2) imply that $C(n)$ and $A(n)$ are eventually positive, while, for $M \in \mathbf{N}$,

$$\sum_{v=0}^{M}|C(v)A(n+v)| = o(n^{-1}) \qquad (n\to\infty).$$

Thus (2) follows from (1). Since $[ns]+[nu]+n \sim n(s+u+1)$ as $n\to\infty$, (3) follows from (1). Let $r\in(1,\infty)$. Since $n/([ns]+[nu]+n)$ tends to $1/(s+u+1)$, as $n\to\infty$, uniformly in $s\geq 0$ and $u \geq 0$ (cf. [BGT, Theorem 1.5.2]), we may choose $N_2 \in \mathbf{N}$ so that

$$\frac{1}{([ns]+[nu]+n)} \leq \frac{r^{1/2}}{n(s+u+1)} \qquad (s\geq 0,\ u\geq 0,\ n\geq N_2).$$

On the other hand, by (1), we can choose $N_3 \in \mathbf{N}$ so that

$$|B(n)| \leq \frac{r^{1/2}\sin(\pi q)}{\pi n} \qquad (n \geq N_3).$$

If we put $N_1 := \max(N_2, N_3)$, then we have, for $s\geq 0$, $u\geq 0$, and $n\geq N_1$,

$$|B([ns]+[nu]+n)| \leq \frac{r^{1/2}\sin(\pi q)}{\pi([ns]+[nu]+n)} \leq \frac{r\sin(\pi q)}{\pi(s+u+1)}n^{-1}.$$

Thus (4) follows. □

Let $B(\cdot)$ be as above. For $n\in\mathbf{N}$ and $p\in\mathbf{N}\cup\{0\}$, we define

$$D_0(n,p) := \delta_{p0}, \quad D_1(n,p) := B(n+p),$$

$$D_2(n,p) := \sum_{m_1=0}^{\infty} B(m_1+n)B(m_1+n+p),$$

and, for $k = 3, 4, \ldots$,

$$D_k(n,p) := \sum_{m_{k-1}=0}^{\infty} B(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} B(m_{k-1}+m_{k-2}+n) \cdots \sum_{m_2=0}^{\infty} B(m_3+m_2+n) \sum_{m_1=0}^{\infty} B(m_2+m_1+n)B(m_1+n+p).$$

By (3.3) and Lemma 2.1, these sums converge absolutely, so that $D_k(n,p)$ are well-defined.

For $k = 1, 2, \ldots$ and $u \geq 0$, we define $f_k(u)$ by

$$f_1(u) := \frac{1}{\pi(1+u)}, \quad f_2(u) := \frac{1}{\pi^2}\int_0^\infty \frac{ds_1}{(s_1+1)(s_1+1+u)},$$

and, for $k = 3, 4, \ldots$,

$$f_k(u) := \frac{1}{\pi^k}\int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{1}{(s_{k-1}+1)} \times \left\{\prod_{m=1}^{k-2} \frac{1}{(s_{m+1}+s_m+1)}\right\} \times \frac{1}{(s_1+1+u)}.$$

We put

$$\sigma_k := \int_0^\infty f_k(u)^2 du \qquad (k = 1, 2, \ldots).$$

Then it follows from [I2, Lemma 6.5] that

$$\sum_{k=1}^{\infty} \sigma_k x^{2k} = \frac{1}{\pi^2}\arcsin^2 x \qquad (|x| < 1).$$

Proposition 3.2. *Let $\ell \in \mathcal{R}_0$ and $q \in (0,1)$. We assume (3.1) and (3.2).*

(1) *For $r \in (1, \infty)$ and $N_1 \in \mathbf{N}$ satisfying (3.5), we have*

$$|D_k(n,[nu])| \leq n^{-1}\{r\sin(\pi q)\}^k f_k(u) \qquad (u \geq 0,\ k \in \mathbf{N},\ n \geq N_1). \tag{3.6}$$

(2) *We have, for $k \in \mathbf{N}$ and $u \geq 0$,*

$$D_k(n,[nu]) \sim n^{-1}\{\sin(\pi q)\}^k f_k(u) \qquad (n \to \infty). \tag{3.7}$$

Proof. Suppose that $k \geq 3$ and write $D_k(n,[nu])$ in the form

$$\begin{aligned}
D_k(n,[nu]) &= \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 B([s_{k-1}]+n) \\
&\qquad \times \left\{ \prod_{m=1}^{k-2} B([s_{m+1}]+[s_m]+n) \right\} \times B([s_1]+n+[nu]) \\
&= n^{k-1} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 B([ns_{k-1}]+n) \\
&\qquad \times \left\{ \prod_{m=1}^{k-2} B([ns_{m+1}]+[ns_m]+n) \right\} \times B([ns_1]+n+[nu]).
\end{aligned}$$

Applying Proposition 3.1 and the dominated convergence theorem to this, we get (3.6) and (3.7). The proofs for the remaining cases $k = 1, 2$ are similar. □

With the fact $f_k \in L^2(0,\infty)$ in mind, we define $\tau_1 := \pi^{-1}$ and

$$\tau_k := \int_0^\infty f_k(u) f_{k-1}(u) du \qquad (k = 2, 3, \dots).$$

Proposition 3.3. *Let $\ell \in \mathcal{R}_0$ and $q \in (0,1)$. We assume* (3.1) *and* (3.2).

(1) *If* $0 < q < 1/2$, *then*

$$\sum_{k=1}^\infty \sum_{p=0}^\infty D_k(n,p) D_{k-1}(n,p) \sim \frac{q}{n} \qquad (n \to \infty). \tag{3.8}$$

(2) *If* $1/2 \leq q < 1$, *then*

$$\sum_{k=1}^\infty \sum_{p=0}^\infty D_k(n,p) D_{k-1}(n,p) \sim \frac{1-q}{n} \qquad (n \to \infty). \tag{3.9}$$

Proof. We have

$$\begin{aligned}
&n \sum_{k=1}^\infty \sum_{p=0}^\infty D_k(n,p) D_{k-1}(n,p) \\
&\quad = nD_1(n,0) + n \sum_{k=2}^\infty \int_0^\infty D_k(n,[u]) D_{k-1}(n,[u]) du \\
&\quad = nD_1(n,0) + \sum_{k=2}^\infty \int_0^\infty nD_k(n,[nu]) \cdot nD_{k-1}(n,[nu]) du.
\end{aligned}$$

To apply the dominated convergence theorem, we choose r so that

$$0 < r\sin(\pi q) < 1.$$

Using Schwarz's inequality, we see that

$$\begin{aligned}
&\sum_{k=2}^{\infty}\{r\sin(\pi q)\}^{2k-1}\int_0^{\infty} f_k(u)f_{k-1}(u)du \\
&\le \sum_{k=2}^{\infty}\{r\sin(\pi q)\}^{2k-1}(\sigma_k\sigma_{k-1})^{1/2} \\
&\le \left[\sum\nolimits_{k=2}^{\infty}\{r\sin(\pi q)\}^{2k-2}\sigma_{k-1}\right]^{1/2}\left[\sum\nolimits_{k=2}^{\infty}\{r\sin(\pi q)\}^{2k}\sigma_k\right]^{1/2} < \infty.
\end{aligned}$$

Hence Proposition 3.2 and the dominated convergence theorem yield

$$\lim_{n\to\infty} n\sum_{k=1}^{\infty}\sum_{p=0}^{\infty} D_k(n,p)D_{k-1}(n,p) = \sum_{k=1}^{\infty}\tau_k\sin^{2k-1}(\pi q). \tag{3.10}$$

By Lemma 3.4 below, the right-hand side is equal to q if $0 < q < 1/2$, and $1-q$ if $1/2 \le q < 1$. Thus (3.8) and (3.9) follow. □

Lemma 3.4. *For $|x| < 1$, we have $\sum_{k=1}^{\infty}\tau_k x^{2k-1} = \pi^{-1}\arcsin x$.*

Proof. (Compare the proof of [I2, Lemma 6.5].) For $0 < d < 1/2$, let $\{Y_n : n \in \mathbf{Z}\}$ be a fractional ARIMA$(0,d,0)$ process such that $E[(Y_0)^2] = \Gamma(1-2d)/\Gamma^2(1-d)$. We denote by $\{c'_n\}$, $\{a'_n\}$, and $\alpha'(\cdot)$ the sequence of MA(∞) coefficients, sequence of AR(∞) coefficients, and partial autocorrelation function, respectively, of $\{Y_n\}$. Then we have, for $n = 0, 1, \ldots,$

$$c'_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\Gamma(d)}, \qquad a'_n = \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}$$

(see, e.g., [BD, Sect. 13.2]). Let $d_k(n,p)$ be as in Theorem 1.1. Recall U_n and V_n from Section 1. Then, since

$$\begin{aligned}
c'_n &\sim n^{-(1-d)}\frac{1}{\Gamma(d)} \qquad (n\to\infty), \\
a'_n &\sim n^{-(1+d)}\Gamma(d)\frac{d\sin(\pi d)}{\pi} \qquad (n\to\infty),
\end{aligned}$$

it follows from Theorems 1.1 and 1.3 and (3.10) that

$$\begin{aligned}\lim_{n\to\infty} nU_n &= (c_0)^2 \lim_{n\to\infty} n \sum_{k=1}^{\infty}\sum_{p=0}^{\infty} d_k(n,p)d_{k-1}(n,p)\\ &= (c_0)^2 \sum_{k=1}^{\infty} \tau_k \sin^{2k-1}(\pi d).\end{aligned}$$

However, we have

$$V_n \to (c_0)^2 \qquad (n\to\infty) \tag{3.11}$$

(see [I2, Sect. 4]), hence it follows from (1.24) that

$$d = \lim_{n\to\infty} n\alpha(n) = \lim_{n\to\infty} nU_n/V_n = (c_0)^{-2}\lim_{n\to\infty} nU_n.$$

Combining, we obtain $\sum_1^\infty \tau_k \sin^{2k-1}(\pi d) = d$. The lemma follows if we substitute $\pi^{-1}\arcsin x$ with $0 < x < 1$ into d and use analytic continuation. □

Remark. From Lemma 3.4, it follows that

$$\tau_k = \frac{1}{\pi}\cdot\frac{(2k-2)!}{2^{2k-2}((k-1)!)^2(2k-1)} \qquad (k = 1, 2, \ldots).$$

Proof of Theorem 1.6. The condition (1.18) implies (1.1). Therefore, using Theorems 1.1 and 1.3, and Proposition 3.3(1), we obtain (1.20). □

4. Proof of Theorem 1.7

We prove Theorem 1.7. Let $\{X_n\}$ be a fractional ARIMA(p,d,q) process with spectral density (1.21). If $d \in (-1/2, 1/2)\setminus\{0\}$, then we have

$$c_n \sim n^{-(1-d)}\frac{K_1}{\Gamma(d)} \qquad (n\to\infty), \tag{4.1}$$

$$a_n \sim n^{-(1+d)}\frac{\Gamma(d)}{K_1}\cdot\frac{d\sin(\pi d)}{\pi} \qquad (n\to\infty), \tag{4.2}$$

where

$$K_1 := \Theta(1)/\Phi(1)$$

(see [KT, Corollary 3.1]; see also [I3, Sect. 3]). Therefore, if $0 < d < 1/2$, then (1.20) follows immediately from Theorem 1.6.

We now assume that $-1/2 < d < 0$. Then (4.1) and (4.2) imply (1.7) and (1.8), respectively, so that we can use Theorem 1.2. Let $d_k(n,p)$ be as in Theorem 1.2. We define

$$\phi_n := \begin{cases} -a_0 & (n=0), \\ a_{n-1} - a_n & (n=1,2,\dots). \end{cases}$$

and

$$\psi_n := -\sum_{k=n+1}^{\infty} c_k \qquad (n=0,1,\dots).$$

Notice that ϕ_n here corresponds to $-\phi_n$ in [IK]. We put $q := 1+d$. Then, by [IK, Lemma 4.1] and [KT, Corollary 3.1] (see also [I3, Lemma 2.1]), we have

$$\psi_n \sim n^{-(1-q)} \frac{K_1}{\Gamma(q)} \qquad (n \to \infty),$$

$$\phi_n \sim n^{-(1+q)} \frac{\Gamma(q)}{K_1} \cdot \frac{q \sin(\pi q)}{\pi} \qquad (n \to \infty).$$

We define

$$\beta_-(n) := \sum_{v=0}^{\infty} \psi_v \phi_{v+n+1} \qquad (n=0,1,\dots).$$

Notice that $\beta_-(n)$ corresponds to $-\beta(n)$ in [IK]. By [IK, Theorem 3.3], we have, for $n=1,2,\dots$ and $p \in \mathbf{N} \cup \{0\}$,

$$d_1(n,p) = \beta_-(n+p), \tag{4.3}$$

$$d_2(n,p) = \sum_{m_1=0}^{\infty} \beta_-(m_1+n)\beta_-(m_1+n+p), \tag{4.4}$$

and, for $k=3,4,\dots$,

(4.5)

$$d_k(n,p) = \sum_{m_{k-1}=0}^{\infty} \beta_-(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} \beta_-(m_{k-1}+m_{k-2}+n)$$
$$\cdots \sum_{m_2=0}^{\infty} \beta_-(m_3+m_2+n) \sum_{m_1=0}^{\infty} \beta_-(m_2+m_1+n)\beta_-(m_1+n+p).$$

Notice that (4.3)–(4.5), together with Theorem 1.2, give a representation of $\alpha(\cdot)$ in terms of absolutely convergent sums involving the AR(∞) and MA(∞) coefficients, for the fractional ARIMA(p,d,q) processes with

$-1/2 < d < 0$. Notice also that, in [IK, Theorem 3.3], (4.3)–(4.5) are stated for $n \geq 2$ but we can prove the case $n = 1$ in the same way.

Since $1/2 < q < 1$, it follows from Theorem 1.2, Proposition 3.3(2), and (4.3)–(4.5) that

$$(c_0)^{-2} U_n = \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n,p) d_{k-1}(n,p) \sim \frac{1-q}{n} = \frac{d}{n} \qquad (n \to \infty),$$

which, together with (3.11), gives (1.20). This completes the proof of Theorem 1.7.

5. Proof of Theorem 1.8

In this section, we assume (C1), (C2), and (A1). Notice that (C1) and (A1) imply (1.1) (see [I2, Proposition 4.3]), whence we may define $\beta(\cdot)$ by (1.10).

The next lemma follows immediately from [BGT, Proposition 1.5.9a].

Lemma 5.1. *Let $\ell \in \mathcal{R}_0$. If $\int^{\infty} \ell(s) ds/s = \infty$, then $\ell(n)/\tilde{\ell}(n)$ tends to 0 as $n \to \infty$. If $\int^{\infty} \ell(s) ds/s < \infty$, then $\ell(n)$ tends to 0 as $n \to \infty$.*

Proposition 5.2. *Let $\ell \in \mathcal{R}_0$ and $-\infty < d \leq 0$. We assume* (1.19).

(1) *If $d = 0$ and $\int^{\infty} \ell(s) ds/s = \infty$, then*

$$c_n \sim n^{-1} \ell(n) \{2\tilde{\ell}(n)\}^{-1/2} \qquad (n \to \infty), \tag{5.1}$$

$$a_n \sim n^{-1} \ell(n) \{2\tilde{\ell}(n)\}^{-3/2} \qquad (n \to \infty), \tag{5.2}$$

$$\beta(n) \sim n^{-1} \ell(n) \{2\tilde{\ell}(n)\}^{-1} \qquad (n \to \infty). \tag{5.3}$$

(2) *If $d = 0$ with $\int^{\infty} \ell(s) ds/s < \infty$ or $-\infty < d < 0$, then*

$$c_n \sim n^{2d-1} \ell(n) \left\{ \sum\nolimits_{-\infty}^{\infty} \gamma(k) \right\}^{-1/2} \qquad (n \to \infty), \tag{5.4}$$

$$a_n \sim n^{2d-1} \ell(n) \left\{ \sum\nolimits_{-\infty}^{\infty} \gamma(k) \right\}^{-3/2} \qquad (n \to \infty), \tag{5.5}$$

$$\beta(n) \sim n^{2d-1} \ell(n) \left\{ \sum\nolimits_{-\infty}^{\infty} \gamma(k) \right\}^{-1} \qquad (n \to \infty). \tag{5.6}$$

Proof. The assertions (5.1) and (5.2) follow from [I2, Theorem 5.2]. Using them, we obtain (5.3) (see [I2, (6.19)]). The assertions (5.4) and (5.5) follow from [I2, Theorem 5.3]. From them, we get (5.6) (see the proof of [I2, Theorem 6.7]). □

Proposition 5.3. *Let $\ell \in \mathcal{R}_0$ and $-\infty < d \le 0$. We assume* (1.19).

(1) *For every $R \in (1, \infty)$, there exists $N \in \mathbf{N}$ such that*

$$\left|\frac{\beta([ns]+[nu]+n)}{\beta(n)}\right| \le \frac{R}{(s+u+1)} \qquad (s \ge 0,\ u \ge 0,\ n \ge N). \tag{5.7}$$

(2) *For every $r \in (0, 1)$, there exists $N \in \mathbf{N}$ such that*

$$|\beta([ns]+[nu]+n)| \le \frac{r}{\pi(s+u+1)} n^{-1} \qquad (s \ge 0,\ u \ge 0,\ n \ge N). \tag{5.8}$$

Proof. By Proposition 5.2 and [BGT, Theorem 1.5.2], we have

$$\beta([ns]+[nu]+n)/\beta(n) \to (s+u+1)^{2d-1} \qquad (n \to \infty)$$

uniformly in $s \ge 0$ and $u \ge 0$. Since

$$(s+u+1)^{2d-1} \le (s+u+1)^{-1},$$

(1) follows. By Lemma 5.1 and Proposition 5.2, we have

$$\lim_{n\to\infty} n\beta(n) = 0. \tag{5.9}$$

This and (1) show (2). □

Proposition 5.4. *Let $\ell \in \mathcal{R}_0$ and $-\infty < d \le 0$. We assume* (1.19).

(1) *For $r \in (0, 1)$, $R \in (1, \infty)$, and $N \in \mathbf{N}$ satisfying both* (5.7) *and* (5.8), *we have*

$$\left|\frac{d_k(n, [nu])}{\beta(n)}\right| \le \pi R r^{k-1} f_k(u) \qquad (u \ge 0,\ k \in \mathbf{N},\ n \ge N).$$

(2) *For $r \in (0, 1)$, there exists $N \in \mathbf{N}$ such that*

$$|nd_k(n, [nu])| \le r^{k-1} f_k(u) \qquad (u \ge 0,\ k \in \mathbf{N},\ n \ge N).$$

(3) *We have, for $k \in \mathbf{N}$ and $u \ge 0$,*

$$d_k(n, [nu]) = O(\beta(n)) \qquad (n \to \infty).$$

(4) *We have, for $k \in \mathbf{N}$ and $u \ge 0$,*

$$d_k(n, [nu]) = o(n^{-1}) \qquad (n \to \infty).$$

Proof. Suppose that $k \geq 3$. By (C1), (A1), and (5.9), we see that (1.9) holds. Hence, using Theorem 1.3, we write $d_k(n,[nu])$ in the form

$$\begin{aligned} d_k(n,[nu]) = &\int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \beta([ns_{k-1}]+n) \\ &\times \left\{ \prod_{m=1}^{k-2} n\beta([ns_{m+1}]+[ns_m]+n) \right\} \times n\beta([ns_1]+n+[nu]). \end{aligned}$$

Using this, we get (1) from (5.7) and (5.8). The cases $k = 1, 2$ can be proved similarly. Now (2) follows from (1) and (5.9), and (3) from (1). Finally, (4) follows from (3) and (5.9). □

Proof of Theorem 1.8. Assume that $0 < d < 1/2$. Then, from [I2, Theorem 5.1], we see that (1.17) and (1.18) hold. Hence (1.20) follows immediately from Theorem 1.6. Next assume that $-\infty < d \leq 0$. For n so large that $\beta(n) > 0$, we write, using Theorem 1.1,

$$\frac{U_n}{(c_0)^2\beta(n)} = 1 + \sum_{k=2}^{\infty} \int_0^\infty \frac{d_k(n,[nu])}{\beta(n)} \cdot n d_{k-1}(n,[nu]) du.$$

As in the proof of Proposition 3.3, we use Proposition 5.4 and the dominated convergence theorem to get

$$\lim_{n\to\infty} \sum_{k=2}^{\infty} \int_0^\infty \frac{d_k(n,[nu])}{\beta(n)} \cdot n d_{k-1}(n,[nu]) du = 0.$$

From this and (3.11), we see that

$$\alpha(n) = \frac{U_n}{V_n} \sim \beta(n) \qquad (n \to \infty).$$

Thus (2) and (3) follow from (5.3) and (5.4), respectively. □

Example. Let $-\infty < d < 1/2$, and let $\{X_n\}$ be a stationary process with autocovariance function of the form $\gamma(n) = (1+|n|)^{-(1-2d)}$. Then $\{X_n\}$ satisfies (1.28) (cf. [I2, Example in Sect. 7]). Let $\alpha(\cdot)$ be the partial autocorrelation function of $\{X_n\}$. Applying Theorem 1.7 to $\{X_n\}$, we get the following result:

(1) if $0 < d < 1/2$, then

$$\alpha(n) \sim \frac{d}{n} \qquad (n \to \infty);$$

(2) if $d = 0$, then

$$\alpha(n) \sim \frac{1}{2n \log n} \qquad (n \to \infty);$$

(3) if $-\infty < d < 0$, then

$$\alpha(n) \sim \frac{n^{2d-1}}{\{2\zeta(1-2d) - 1\}} \qquad (n \to \infty).$$

Here $\zeta(s)$ is the Riemann zeta function.

6. ARMA PROCESSES

In this section, we consider the fractional ARIMA$(p, 0, q)$ processes, that is, the ARMA(p, q) processes. Let $p, q \in \mathbf{N} \cup \{0\}$, and let $\Phi(z)$ and $\Theta(z)$ be polynomials with real coefficients of degrees p, q, respectively, satisfying (1.22). Let $\{X_n\}$ be an ARMA(p, q) process with spectral density

$$\Delta(\theta) = \frac{1}{2\pi} \frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2} \qquad (-\pi < \theta < \pi).$$

We put

$$R := \begin{cases} \max(1/|u_1|, \ldots, 1/|u_q|) & \text{if } q \geq 1, \\ 0 & \text{if } q = 0, \end{cases}$$

where $u_1, \ldots, u_q$ are the (complex) zeros of $\Theta(z)$:

$$\Theta(z) = \text{const.} \times (z - u_1) \cdots (z - u_q).$$

From the assumption (1.22), we see that $|u_k| > 1$ for $k = 1, \ldots, q$, whence $R \in [0, 1)$. Let $\alpha(\cdot)$ be the partial autocorrelation function of $\{X_n\}$. The next theorem implies that $\alpha(n)$ decays exponentially as $n \to \infty$.

Theorem 6.1. *For every $r > R$, we have*

$$\alpha(n) = O(r^n) \qquad (n \to \infty). \tag{6.1}$$

Proof. The outer function $h(\cdot)$ of $\{X_n\}$ is given by $h(z) = \Theta(z)/\Phi(z)$ for $|z| < 1$. Hence we see that, for every $r > R$,

$$a_n = O(r^n) \qquad (n \to \infty).$$

Moreover, we see that c_n also decays exponentially as $n \to \infty$. Therefore, (6.1) follows easily from Theorems 1.1 and 1.3. □

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