Explicit formulas for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and their applications

Hidenori Katsurada
Muroran Institute of Technology

Abstract

We give an explicit formula for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift. As an application we prove a certain algebraicity result for the values of twisted Rankin-Selberg series at integers of half-integral weight modular forms.

1 Introduction

It is an interesting problem to give an explicit formula for the Koecher-Maaß series of a Siegel modular form F for the symplectic group $Sp_n(\mathbf{Z})$, and several results have been obtained (cf. Böcherer [B], Ibukiyama and Katsurada [I-K1], [I-K2], [I-K3]). Such explicit formulas are not only interesting in its own right but also have some important applications in the theory of modular forms. For example, we refer to [B-S], [D-I], [Miz]. Now we consider a twist of such a Koecher-Maaß series by a Dirichlet character χ. As for this, in view of Saito [Sai1] for example, we can naturally consider the following Dirichlet series:

$$L^*(s, F, \chi) = \sum_T \frac{\chi(2^{2[n/2]} \det T) c_F(T)}{e(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree n, $c_F(T)$ is the T-th Fourier coefficient of F and $e(T) = \#\{U \in SL_n(\mathbf{Z}); T[U] = T\}$. We will sometimes call $L^*(s, F, \chi)$ the twisted Koecher-Maaß series of the second kind.

On the other hand, Choie and Kohnen [C-K] introduced a different type of "twist". For a positive integer N, let $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}); U \equiv 1_n \bmod N\}$ and $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}); T[U] = T\}$. For a primitive Dirichlet character χ mod N, the Koecher-Maaß series $L(s, F, \chi)$ of F twisted by χ is defined to be

$$L(s, F, \chi) = \sum_T \frac{\chi(\mathrm{tr}(T)) c_F(T)}{e_N(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_{n,N}(\mathbf{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree n. In [C-K], Choie and Kohnen proved a meromorphy continuation of $L(s, F, \chi)$ to the whole s-plane and a functional equation (cf. Theorem 2.1). Moreover they got a result on the algebraicity of its special values (cf. Theorem 2.2.) We shall call $L(s, F, \chi)$ the twist of the first kind.

In this paper we give explicit formulas for the twisted Koecher-Maaß series of the first and second kinds associated with the Duke-Imamoglu-Ikeda lift and apply them to the study of the special values of the Rankin-Selberg series for half-integral weight modular forms. We explain our main results more precisely. Let k and n be positive even integers such that $n \geq 4$ and $2k-n \geq 12$. For a cuspidal Hecke eigenform h in the Kohnen plus subspace of weight $k-n/1+1/2$ for $\varGamma_0(4)$, let $I_n(h)$ be the Duke-Imamoglu-Ikeda lift of h to the space of cusp forms of weight k for $Sp_n(\mathbf{Z})$. Moreover let $S(h)$ be the normalized Hecke eigenform of weight $2k-n$ for $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence, and $E_{n/2+1/2}$ be Cohen's Eisenstein series of weight $n/2+1/2$ for $\varGamma_0(4)$. We then give explicit formulas for $L(s, I_n(h), \chi)$ and $L^*(s, I_n(h), \chi)$ in terms of the twisted Rankin-Selberg series $R(s, h, E_{n/2+1/2}, \eta)$ of h and $E_{n/2+1/2}$ and twisted Hecke's L-function $L(s, S(h), \eta')$of $S(h)$, where η and η' are Dirichlet characters related with χ. It is relatively easy to get an explicit form of $L^*(s, I_n(h), \chi)$. In fact, by using the same argument as in Ibukiyama and Katsurada [I-K2], we can easily obtain its explicit formula (cf. Theorem 4.1). On the other hand, it seems nontrivial to get that of $L(s, I_n(h), \chi)$ (cf. Theorem 6.1), and we need some explicit formula for a certain character sum associated with a Dirichlet character (cf. Theorem 5.6). Using Theorem 6.1 combined with the result of Choie-Kohnen, we prove certain algebraicity results on $R(s, h, E_{n/2+1/2}, \eta)$ at an integer $s = m$ (cf. Theorems 7.1 and 7.2), which were announced in [Ka]. We note that the algebraicity of the special values of such a Rankin-Selberg series at half-integers was investigated by Shimura [Sh3]. However there are few results on the algebracity of such values at integers. As an attempt, Mizuno and the author [K-M] proved linear dependency of Rankin-Selberg L-values of a cuspidal Hecke eigenform belonging to Kohnen plus subspace of half integral weight and the Zagier's Eisenstein series of weight $3/2$. Our present result can be regarded as a generalization of our previous result.

Notation We denote by $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for a complex number x. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset X mod $aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, and $SL_m(R) = \{A \in M_m(R) \mid \det A = 1\}$, where $\det A$ denotes the determinant of a square matrix A and R^* is the unit group of R. We denote by $S_n(R)$ the set of symmetric matrices of degree n with entries in R. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $SL_n(\mathbf{Z})$ acts on the set $S_n(\mathbf{R})$ in the following way:

$$SL_n(\mathbf{Z}) \times S_n(\mathbf{R}) \ni (g, A) \longrightarrow {}^t gAg \in S_n(\mathbf{R}).$$

Let G be a subgroup of $GL_n(\mathbf{Z})$. For a subset $\mathcal{B}$ of $S_n(\mathbf{R})$ stable under the action of G we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. Two symmetric matrices A and A' with entries in R are said to be equivalent with respect to G and write $A \sim_G A'$ if there is an element X of G such that $A' = A[X]$. Let $\mathcal{L}_n$ denote the set of half-integral matrices of degree n over $\mathbf{Z}$, that is, $\mathcal{L}_n$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to $\mathbf{Z}$ or $\frac{1}{2}\mathbf{Z}$ according as $i = j$ or not.

2 Twisted Koecher-Maaß series

Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n and O_n denotes the unit matrix and the zero matrix of degree n, respectively. Furthermore, put

$$Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let l be an integer or a half-integer, and N a positive integer. Let $\Gamma_0^{(n)}(N)$ be the congruence subgroup of $Sp_n(\mathbf{Z})$ consisting of matrices whose left lower $n \times n$ block are congruent to O_n mod N. Moreover let χ be a Dirichlet character mod N. We then denote by $\mathfrak{M}_l(\Gamma_0^{(n)}(N), \chi)$ the space of modular forms of weight l and character χ for $\Gamma_0^{(n)}(N), \chi)$, and by $\mathfrak{S}_l(\Gamma_0^{(n)}(N), \chi)$ the subspace of $\mathfrak{M}_l(\Gamma_0^{(n)}(N), \chi)$ consisting of cusp forms. If χ is the trivial character mod N, we simply write $\mathfrak{M}_l(\Gamma_0^{(n)}(N), \chi)$ and $\mathfrak{S}_l(\Gamma_0^{(n)}(N), \chi)$ as $\mathfrak{M}_l(\Gamma_0^{(n)}(N))$ and $\mathfrak{S}_l(\Gamma_0^{(n)}(N))$, respectively. Let k be a positive integer, and let $F(Z) \in \mathfrak{M}_k(Sp_n(\mathbf{Z}))$. Then $F(Z)$ has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{L}_{n \geq 0}} c_F(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where $\mathrm{tr}(X)$ denotes the trace of a matrix X. For $N \in \mathbf{Z}_{>0}$, put $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \bmod N\}$, and for $T \in \mathcal{L}_{n>0}$ put $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$. For a primitive Dirichlet character χ mod N Let

$$L(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_{n,N}(\mathbf{Z})} \frac{\chi(\mathrm{tr}(T))c_F(T)}{e_N(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of the first kind of F as in Section 1. The following two theorems are due to Choie and Kohnen [C-K].

Theorem 2.1. *Let $F \in \mathfrak{S}_k(Sp_n(\mathbf{Z}))$, and χ a primitive character of conductor N. Put*

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2}\Gamma(s - (i-1)/2),$$

and

$$\Lambda(s,F,\chi) = N^{2s}\tau(\chi)^{-1}\gamma_n(s)L(s,F,\chi) \quad (\mathrm{Re}(s) >> 0),$$

where $\tau(\chi)$ is the Gauss sum of χ. Then $\Lambda(s,F,\chi)$ has an analytic continuation to the whole s-plane and has the following functional equation:

$$\Lambda(k-s,F,\chi) = (-1)^{nk/2}\chi(-1)\Lambda(s,F,\overline{\chi}).$$

Theorem 2.2. *Let F and χ be as above. Then there exists a finite dimensional $\overline{\mathbf{Q}}$-vector space V_F in $\mathbf{C}$ such that*

$$L(m,F,\chi)\pi^{-nm} \in V_F$$

for any primitive character χ and any integer m such that $(n+1)/2 \le m \le k-(n+1)/2$.

Now let

$$L^*(s,F,\chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{\chi(2^{2[n/2]}\det T)c_F(T)}{e(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of the second kind of F as in Section 1. We will discuss a relation between these two Dirichlet series in Section 5.

3 Review on the algebraicity of L-values of elliptic modular forms of integral and half-integral weight

In this section, we review on the special values of L functions of elliptic modular forms of integral and half-integral weights. For a modular form $g(z)$ of integral or half-integral weight for a certain congruence subgroup Γ of $SL_2(\mathbf{Z})$, let $\mathbf{Q}(g)$ denote the field generated over $\mathbf{Q}$ by all the Fourier coefficients of g, and for a Dirichlet character η let $\mathbf{Q}(\eta)$ denote the field generated over $\mathbf{Q}$ by all the values of η.

First let

$$f(z) = \sum_{m=1}^{\infty} c_f(m)\mathbf{e}(mz)$$

be a normalized Hecke eigenform in $\mathfrak{S}_k(SL_2(\mathbf{Z}))$, and χ be a primitive Dirichlet character. Then let us define Hecke's L-function $L(s,f,\chi)$ of f twisted by χ as

$$L(s,f,\chi) = \sum_{m=1}^{\infty} c_f(m)\chi(m)m^{-s}.$$

Then we have the following result (cf. [Sh2]):

Proposition 3.1. *There exist complex numbers $u_{\pm}(f)$ uniquely determined up to $\mathbf{Q}(f)^{\times}$ multiple such that*

$$\frac{L(m,f,\chi)}{(2\pi\sqrt{-1})^m \tau(\chi) u_j(f)} \in \mathbf{Q}(f)\mathbf{Q}(\chi)$$

for any integer $0 < m \le k-1$ and a primitive character χ, where $\tau(\chi)$ is the Gauss sum of χ, and $j = +$ or $-$ according as $(-1)^m \chi(-1) = 1$ or -1.

Corollary. *Under the above notation and the assumption, we have*

$$L(m,f,\chi)\pi^{-m} \in \overline{\mathbf{Q}} u_j(f)$$

for any integer $0 < m \le k-1$ and a primitive character χ.

We remark that we have $L(m,f,\chi) \ne 0$ if $m \ne k/2$, and $L(k/2,f,\chi) \ne 0$ for infinitely many χ.

Next let us consider the half-integral weight case. From now on we simply write $\Gamma_0^{(1)}(M)$ as $\Gamma_0(M)$. Let

$$h_1(z) = \sum_{m=1}^{\infty} c_{h_1}(m)\mathbf{e}(mz)$$

be a Hecke eigenform in $\mathfrak{S}_{k_1+1/2}(\Gamma_0(4))$, and

$$h_2(z) = \sum_{m=0}^{\infty} c_{h_2}(m)\mathbf{e}(mz)$$

be an element of $\mathfrak{M}_{k_2+1/2}(\Gamma_0(4))$. For a fundamental discriminant D let χ_D be the Kronecker character corresponding to D. Let χ be a primitive character mod N. Then we define

$$\widetilde{R}(s,h_1,h_2,\chi) = L(2s-k_1-k_2+1,\omega)\sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s},$$

where $\omega(d) = \chi_{-4}^{k_1-k_2}\chi^2(d)$. We also define $R(s,h_1,h_2,\chi)$ as

$$R(s,h_1,h_2,\chi) = L(2s-k_1-k_2+1,\chi^2)\sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s}.$$

Now let $S(h_1)$ be the normalized Hecke eigenform in $\mathfrak{S}_{2k_1}(SL_2(\mathbf{Z}))$ corresponding to h_1 under the Shimura correspondence. Then the following result is due to Shimura [Sh3].

Proposition 3.2. *Assume that $k_1 > k_2$. Under the above notation we have*

$$\frac{\widetilde{R}(m+1/2,h_1,h_2,\chi)}{u_-(S(h_1))\tau(\chi^2)\pi^{-k_2+1+2m}\sqrt{-1}} \in \mathbf{Q}(h_1)\mathbf{Q}(h_2)\mathbf{Q}(\chi)$$

for any integer $k_2 \le m \le k_1-1$ and a primitive character χ.

Proof. Let N be the conductor of χ. Put

$$h_{2\chi}(z) = \sum_{m=0}^{\infty} c_{h_2}(m)\chi(m)\mathbf{e}(mz).$$

Then $h_{2\chi}(z) \in \mathfrak{M}_{k_2+1/2}(4N^2, \chi^2)$. We can regard h_1 as an element of $\mathfrak{S}_{k_1+1/2}(\Gamma_0(4N^2))$. Then the assertion follows from [[Sh3], Theorem 2]. $\square$

Corollary. *Assume that* $c_{h_1}(n), c_{h_2}(n) \in \overline{\mathbf{Q}}$ *for any* $n \in \mathbf{Z}_{\geq 0}$. *Then there exists a one-dimensional* $\overline{\mathbf{Q}}$*-vector space* U_{h_1,h_2} *in* $\mathbf{C}$ *such that*

$$\widetilde{R}(m+1/2, h_1, h_2, \chi)\pi^{-2m} \in U_{h_1,h_2}$$

for any integer $k_2 \leq m \leq k_1 - 1$ *and a primitive character* χ.

4 Explicit formulas for the twisted Koecher-Maaß series of the second kind of the Duke-Imamoglu-Ikeda lift

Throughout this section, we assume that n and k are even positive integers. Let h be a Hecke eigenform of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$ belonging to the Kohnen plus space. Then h has the following Fourier expansion:

$$h(z) = \sum_{e} c_h(e)\mathbf{e}(ez),$$

where e runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \bmod 4$. Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m)\mathbf{e}(mz)$$

be the normalized Hecke eigenform of weight $2k - n$ for $SL_2(\mathbf{Z})$ corresponding to h via the Shimura correspondence (cf. [Ko].) For a prime number p let β_p be a nonzero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}c_{S(h)}(p)$. For a non-negative integers l and m, the Cohen function $H(l, m)$ is given by $H(l, m) = L_{-m}(1 - l)$. Here

$$L_D(s) = \begin{cases} \zeta(2s-1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a)\chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D \neq 0,\ D \equiv 0, 1 \bmod 4 \\ 0, & D \equiv 2, 3 \bmod 4, \end{cases}$$

where the positive integer f is defined by $D = D_K f^2$ with the discriminant D_K of $K = \mathbf{Q}(\sqrt{D})$, μ is the Möbius function, and $\sigma_s(n) = \sum_{d|n} d^s$. Furthermore, for an even integer $l \geq 4$, we define the Cohen Eisenstein series $E_{l+1/2}(z)$ by

$$E_{l+1/2}(z) = \sum_{e=0}^{\infty} H(l,e)\mathbf{e}(ez).$$

It is known that $E_{l+1/2}(z)$ is a modular form of weight $l+1/2$ for $\Gamma_0(4)$ belonging to the Kohnen plus space.

For a prime number p let $\mathbf{Q}_p$ and $\mathbf{Z}_p$ be the field of p-adic numbers, and the ring of p-adic integers, respectively. We denote by ν_p the additive valuation on $\mathbf{Q}_p$ normalized so that $\nu_p(p) = 1$, and by $\mathbf{e}_p$ the continuous homomorphism from the additive group $\mathbf{Q}_p$ to $\mathbf{C}^\times$ such that $\mathbf{e}_p(a) = \mathbf{e}(a)$ for $a \in \mathbf{Q}$. For a p-adic number c put

$$\widetilde{\xi}_p(c) = 1, -1 \text{ or } 0$$

according as $\mathbf{Q}_p(\sqrt{c}) = \mathbf{Q}_p$, $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic unramified, or $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic ramified. We note that $\tilde{\xi}_p(D) = \chi_D(p)$ for a fundamental discriminant D. For a non-degenerate half-integral matrix T over $\mathbf{Z}_p$, let

$$b_p(T,s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\operatorname{tr}(TR))p^{-\nu_p(\mu_p(R))s}$$

be the local Siegel series, where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. Then there exists a polynomial $F_p(T,X)$ in X such that

$$b_p(T,s) = F_p(T,p^{-s})(1-p^{-s})(1-\xi_p(T)p^{n/2-s})^{-1}\prod_{i=1}^{n/2}(1-p^{2i-2s})$$

(cf. [Ki1],) where $\xi_p(T) = \widetilde{\xi}_p((-1)^{n/2}\det T)$. For a positive definite half integral matrix T of degree n write $(-1)^{n/2}\det(2T)$ as $(-1)^{n/2}\det(2T) = \mathfrak{d}_T\mathfrak{f}_T^2$ with $\mathfrak{d}_T$ a fundamental discriminant and $\mathfrak{f}_T$ a positive integer. We then put

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T|)\prod_p (p^{k-n/2-1/2}\beta_p)^{\nu_p(\mathfrak{f}_T)}F_p(T,p^{-(n+1)/2}\beta_p^{-1}).$$

We note that $c_{I_n(h)}(T)$ does not depend on the choice of β_p. Define a Fourier series $I_n(h)(Z)$ by

$$I_n(h)(Z) = \sum_{T \in \mathcal{L}_{n>0}} c_{I_n(h)}(T)\mathbf{e}(\operatorname{tr}(TZ)).$$

In [I] Ikeda showed that $I_n(h)(Z)$ is a Hecke eigenform in $\mathfrak{S}_k(Sp_n(\mathbf{Z}))$ and its standard L-function $L(s, I_n(h), \mathrm{St})$ is given by

$$L(s, I_n(h), \mathrm{St}) = \zeta(s)\prod_{i=1}^{n} L(s+k-i, S(h)).$$

We call $I_n(h)$ the Duke-Imamoglu-Ikeda lift (D-I-I lift) of h.

Theorem 4.1. *Let χ be a primitive Dirichlet character mod N. Then we have*

$$L^*(s,F,\chi)=c_nR(s,h,E_{n/2+1/2},\chi)\prod_{j=1}^{n/2-1}L(2s-2j,S(h),\chi^2)$$

$$+d_nc_h(1)\prod_{j=1}^{n/2}L(2s-2j+1,S(h),\chi^2),$$

where c_n and d_n are nonzero rational numbers depending only on n.

To prove Theorem 4.1, we reduce the problem to local computations. For $a,b\in\mathbf{Q}_p^\times$ let $(a,b)_p$ the Hilbert symbol on $\mathbf{Q}_p$. Following Kitaoka [Ki2], we define the Hasse invariant $\varepsilon(A)$ of $A\in S_m(\mathbf{Q}_p)^\times$ by

$$\varepsilon(A)=\prod_{1\le i\le j\le n}(a_i,a_j)_p$$

if A is equivalent to $a_1\perp\cdots\perp a_n$ over $\mathbf{Q}_p$ with some $a_1,a_2,...,a_n\in\mathbf{Q}_p^\times$. For $T\in S_n(\mathbf{Z}_p)_e$, put $T^{(0)}=2^{-1}T, F_p^{(0)}(T,X)=F_p(T^{(0)},X)$, and so on. Then for non-degenerate symmetric matrices A of degree n with entries in $\mathbf{Z}_p$ we define the local density $\alpha_p(A)=\alpha_p(A,A)$ representing A by A as

$$\alpha_p(A)=2^{-1}\lim_{a\to\infty}p^{a(-n^2+n(n+1)/2)}\#\mathcal{A}_a(A,A),$$

where

$$\mathcal{A}_a(A,A)=\{X\in M_n(\mathbf{Z}_p)/p^aM_n(\mathbf{Z}_p)\mid A[X]-B\in p^aS_n(\mathbf{Z}_p)_e\},$$

Furthermore put

$$M(A)=\sum_{A'\in\mathcal{G}(A)}\frac{1}{e(A')}$$

for a positive definite symmetric matrix A of degree n with entries in $\mathbf{Z}$, where $\mathcal{G}(A)$ denotes the set of $SL_n(\mathbf{Z})$-equivalence classes belonging to the genus of A. Then by Siegel's main theorem on the quadratic forms, we obtain

$$M(A)=\kappa_n2^{2-n}\det A^{(n+1)/2}\prod_p\alpha_p(A)^{-1}$$

where $e_n=1$ or 2 according as $n=1$ or not, and $\kappa_n=\prod_{i=1}^{n/2}\Gamma_{\mathbf{C}}(2i)$ (cf. Theorem 6.8.1 in [Ki2]). Put

$$\mathcal{F}_p=\{d_0\in\mathbf{Z}_p\mid \nu_p(d_0)\le 1\}$$

if p is an odd prime, and

$$\mathcal{F}_2=\{d_0\in\mathbf{Z}_2\mid d_0\equiv 1 \bmod 4,\ \text{or}\ d_0/4\equiv -1 \bmod 4,\ \text{or}\ \nu_2(d_0)=3\}.$$

For $d \in \mathbf{Z}_p^\times$ put

$$\begin{aligned} & S_n(\mathbf{Z}_p, d) \\ = \ & \{T \in S_n(\mathbf{Z}_p) \mid (-1)^{n/2} \det T = p^{2i} d \bmod \mathbf{Z}_p^{*\square} \text{ with some } \ i \in \mathbf{Z}\}, \end{aligned}$$

and $S_n(\mathbf{Z}_p,d)_x = S_n(\mathbf{Z}_p,d) \cap S_n(\mathbf{Z}_p)_x$ for $x = e$ or o. Put $\mathcal{L}_{n,p}^{(0)} = S_n(\mathbf{Z}_p)_e^\times$ and $\mathcal{L}_{n,p}^{(0)}(d) = S_n(\mathbf{Z}_p,d) \cap \mathcal{L}_{n,p}^{(0)}$. Let $\iota_{n,p}$ be the constant function on $\mathcal{L}_{n,p}^\times$ taking the value 1, and $\varepsilon_{n,p}$ the function on $\mathcal{L}_{n,p}^\times$ assigning the Hasse invariant of A for $A \in \mathcal{L}_{n,p}^\times$. We sometimes drop the suffix and write $\iota_{n,p}$ as ι_p or ι and the others if there is no fear of confusion. From now on we sometimes write $\omega = \varepsilon^l$ with $l = 0$ or 1 according as $\omega = \iota$ or ε. For $d_0 \in \mathcal{F}_p$ and $\omega = \varepsilon^l$ with $l = 0, 1$, we define a formal power series $P_{n,p}^{(0)}(d_0, \omega, X, t)$ in t by

$$P_{n,p}^{(0)}(d_0,\omega,X,t) = \kappa(d_0,n,l)^{-1} \sum_{B \in \mathcal{L}_{n,p}^{(0)}(d_0)} \frac{\widetilde{F}_p^{(0)}(B,X)}{\alpha_p(B)} \omega(B) t^{\nu_p(\det B)},$$

where

$$\kappa(d_0,n,l) = \kappa(d_0,n,l)_p = \{(-1)^{n(n+2)/8}((-1)^{n/2}2, d_0)_2\}^{l\delta_{2,p}}.$$

Let $\mathcal{F}$ denote the set of fundamental discriminants, and for $l = \pm 1$, put

$$\mathcal{F}^{(l)} = \{d_0 \in \mathcal{F} \mid ld_0 > 0\}.$$

Theorem 4.2. *Let the notation and the assumption be as above. Then for* $\mathrm{Re}(s) \gg 0$, *we have*

$$\begin{aligned} & L^*(s, I_n(h)) = \kappa_n 2^{ns+1-n} \\ & \times \{ \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} c_h(|d_0|)|d_0|^{n/4-k/2+1/4} \prod_p P_{n,p}^{(0)}(d_0, \iota_p, \alpha_p, p^{-s+k/2+n/4+1/4}\chi(p)) \\ & + \ (-1)^{n(n+2)/8} \\ & \times \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} ((-1)^{n/2}2, d_0)_2 c_h(|d_0|)|d_0|^{n/4-k/2+1/4} \prod_p P_{n,p}^{(0)}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4}\chi(p))\}. \end{aligned}$$

Proof. Let $T \in S_n(\mathbf{Z}_p)_{e>0}$. Then the T-th Fourier coefficient $c_{I_n(h)}(T)$ of $I_n(h)$ is uniquely determined by the genus to which T belongs, and, by definition, it can be expressed as

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T^{(0)}|)(\mathfrak{f}_T^{(0)})^{k-n/2-1/2} \prod_p \widetilde{F}^{(0)}(T, \alpha_p)$$

We also note that

$$(\mathfrak{f}_T^{(0)})^{k-n/2-1/2} = |\mathfrak{d}_T^{(0)}|^{-(k/2-n/4-1/4)} (\det T)^{(k/2-n/4-1/4)}$$

for $T \in S_n(\mathbf{Z}_p)_{e>0}$. Hence we have

$$\sum_{T' \in \mathcal{G}(T)} \frac{c_{I_n(h)}(T')}{e(T')} = \det T^{k/2+n/4-1/4} |\mathfrak{d}_T^{(0)}|^{k/2-n/4-1/4} \prod_p \frac{\widetilde{F}_p^{(0)}(T, \alpha_p)}{\alpha_p(T)}.$$

Thus, similarly to [I-K1], Theorem 3.3, (1), and [I-K2], Theorem 3.2, we obtain

$$L(s, I_n(h)) = \kappa_n 2^{ns+1-n} \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} c_h(|d_0|)|d_0|^{n/4-k/2+1/4}$$

$$\times\{\prod_p P_{n,p}^{(0)}(d_0, \iota_p, \alpha_p, p^{-s+k/2+n/4+1/4}\chi(p))$$

$$+(-1)^{n(n+2)/8}((-1)^{n/2}2, d_0)_2 \prod_p P_{n,p}^{(0)}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4}\chi(p))\}.$$

This proves the assertion.

□

Proposition 4.3. *Let $d_0 \in \mathcal{F}_p$ and $\xi_0 = \widetilde{\xi}(d_0)$. Then*

$$P_n^{(0)}(d_0, \iota, X, t) = \frac{(p^{-1}t)^{\nu_p(d_0)}}{\phi_{n/2-1}(p^{-2})(1-p^{-n/2}\xi_0)}$$

$$\times \frac{(1+t^2p^{-n/2-3/2})(1+t^2p^{-n/2-5/2}\xi_0^2) - \xi_0 t^2 p^{-n/2-2}(X+X^{-1}+p^{1/2-n/2}+p^{-1/2+n/2})}{(1-p^{-2}Xt^2)(1-p^{-2}X^{-1}t^2)\prod_{i=1}^{n/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})},$$

and

$$P_n^{(0)}(d_0, \varepsilon, X, t) = \frac{1}{\phi_{n/2-1}(p^{-2})(1-p^{-n/2}\xi_0)} \frac{\xi_0^2}{\prod_{i=1}^{n/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})}.$$

Proof. Put $H_k = \begin{pmatrix} O & 1_k \\ 1_k & O \end{pmatrix}$, and for $d \in \mathbf{Z}_p^*$ put

$$D = \{x \in M_{2k,n}(\mathbf{Z}_p) \mid \det(H_k[x]) \in dp^i \mathbf{Z}_p^{*\square} \text{ with some } i \in \mathbf{Z}_{\geq 0}\}.$$

We then define $Z_{2k}(u, \varepsilon^l, d)$ as

$$Z_{2k}(u, \varepsilon^l, d) = \int_D \varepsilon^l(H_k[x]) |\det(H_k[x])|_p^{s-k} dx$$

with $u = p^{-s}$, where $|*|_p$ denotes the normalized valuation on $\mathbf{Q}_p$, and dx is the measure on $M_{2k,n}(\mathbf{Q}_p)$ normalized so that the volume of $M_{2k,n}(\mathbf{Z}_p)$ is 1. Moreover put

$$Z_{2k,e}(u, \varepsilon^l, d) = \frac{1}{2}(Z_{2k,n}(u, \varepsilon^l, d) + Z_{2k,n}(-u, \varepsilon^l, d)),$$

and

$$Z_{2k,o}(u,\varepsilon^l,d)=\frac{1}{2}(Z_{2k,n}(u,\varepsilon^l,d)-Z_{2k,n}(-u,\varepsilon^l,d)).$$

Then it is well known that

$$Z_{2k,x(d_0)}(u,\varepsilon^l,(-1)^{n/2}p^{-\nu_p(d_0)}d_0)=\phi_n(p^{-1})\sum_{T\in\mathcal{L}_{n,p}^{(0)}(d_0)}\frac{b_p(2^{-\delta_{2,p}}T,p^{-k})}{\alpha_p(T)}(p^kt)^{\nu_p(\det(T)}$$

for $d_0\in\mathcal{F}_p$, where $x(d_0)=e$ or o according as $\nu_p(d_0)$ is even or odd. Recall that

$$b_p(2^{-\delta_{2,p}}T,p^{-k})=\frac{(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{-2k+2i})}{1-\xi(2^{-\delta_{2,p}}T)p^{-k+n/2}}F_p^{(0)}(T,p^{-k})$$

and

$$F_p^{(0)}(T,p^{-k})=p^{(-k/2+(n+1)/4)(\nu_p(\det T)-\nu_p(d_0))}\widetilde{F}_p^{(0)}(T,p^{-k+(n+1)/2}).$$

Hence we have

$$Z_{2k,x(d_0)}(u,\varepsilon^l,(-1)^{n/2}p^{-\nu_p(d_0)}d_0)=\phi_n(p^{-1})\frac{(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{-2k+2i})}{1-\xi(2^{-\delta_{2,p}}T)p^{-k+n/2}}$$

$$\times p^{(k/2-(n+1)/4)\nu_p(d_0)}P_n^{(0)}(d_0,\varepsilon^l,p^{-k+(n+1)/2},up^{-k/2+(n+1)/4}).$$

Let $T(d_0,\omega,X,t)$ denote the right-hand side of the formula for $\omega=\varepsilon^l$ $(l=0,1)$ in the proposition. Then, by [[Sai2], Theorem 3.4 (2)], we have

$$Z_{2k,x(d_0)}(u,\varepsilon^l,(-1)^{n/2}p^{-\nu_p(d_0)}d_0)=\phi_n(p^{-1})\frac{(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{-2k+2i})}{1-\xi(T)p^{-k+n/2}}$$

$$\times p^{(k/2-(n+1)/4)\nu_p(d_0)}T(d_0,\varepsilon^l,p^{-k+(n+1)/2},up^{-k/2+(n+1)/4}).$$

(Remark that there are misprints in [Sai2]; the $(q^{-1})_n$ on page 197, lines 9 and 15 should be $(q^{-1})_r$.) Hence we have

$$P_n^{(0)}(d_0,\varepsilon^l,p^{-k+(n+1)/2},up^{-k/2+(n+1)/4})=T(d_0,\varepsilon^l,p^{-k+(n+1)/2},up^{-k/2+(n+1)/4})$$

for infinitely many positive integers k. Hence we have

$$P_n^{(0)}(d_0,\varepsilon^l,X,t)=T(d_0,\varepsilon^l,X,t).$$

$\square$

Proof of Theorem 4.1.
Put $\Omega=\{\omega_p\}$, and let $d_0\in\mathcal{F}^{((-1)^{n/2})}$. Put

$$P(s,d_0,\Omega,\chi)=\prod_p P_{n,p}^{(0)}(d_0,\varepsilon_p,\alpha_p,p^{-s+k/2+n/4+1/4}\chi(p)).$$

Then by Proposition 4.3, we have

$$P(s,d_0,\{\iota_p\},\chi)$$

$$=|d_0|^{-s+k/2+n/4-3/4}\chi(d_0)\prod_{i=1}^{n/2-1}\zeta(2i)L(n/2,\chi_{d_0})\prod_{i=0}^{n/2}L(2s+2i-n,S(h),\chi^2)$$

$$\times L(2s-n+1,S(h),\chi^2)\prod_p\{(1+p^{-2s+k-1}\chi(p)^2)(1+\chi_{d_0}(p)^2p^{-2s+2k-2}\chi(p)^2)$$

$$-\chi_{d_0}(p)p^{-2s+k-3/2}\chi(p)^2\beta_p(1+p^{1/2-n/2}\beta_p^{-1})(1+p^{-1/2+n/2}\beta_p^{-1})\}.$$

We note that $L(s,h)$ and $L(s,E_{n/2+1})$ can be expressed as

$$L(s,h)=L(2s,S(h))\sum_{d_0\in\mathcal{F}((-1)^{n/2})}c(|d_0|)|d_0|^{-s}\prod_p(1-\chi_{(-1)^{k-n/2}d_0}(p)p^{k-n/2-1-2s}),$$

and

$$L(s,E_{n/2+1})=\zeta(2s)\zeta(2s-n+1)$$

$$\times\sum_{d_0\in\mathcal{F}((-1)^{n/2})}L(1-n/2,\chi_{d_0})|d_0|^{-s}\prod_p(1-\chi_{d_0}(p)p^{n/2-1-2s}),$$

and therefore, we easily see that $L(s,h,E_{n/2+1/2},\chi)$ can be expressed as

$$L(s,h,E_{n/2+1/2},\chi)=L(2s,S(h),\chi^2)L(2s-n+1,S(h),\chi^2)$$

$$\times\sum_{d_0\in\mathcal{F}((-1)^{n/2})}|d_0|^{-s}c(|d_0|)\chi(d_0)L(1-n/2,\chi_{d_0})$$

$$\times\prod_p\{(1+p^{-2s+k-1}\chi(p)^2)(1+\chi_{d_0}(p)^2p^{-2s+k-2}\chi(p)^2)$$

$$-\chi_{d_0}(p)p^{-2s+k-3/2}\chi(p)^2\beta_p(1+p^{1/2-n/2}\beta_p^{-1})(1+p^{-1/2+n/2}\beta_p^{-1})\}$$

(cf. [Sh1], Lemma 1.) Thus, by remarking the functional equation

$$L(1-n/2,\chi_{d_0})=2^{1-n/2}\pi^{-n/2}\Gamma(n/2)|d_0|^{(n-1)/2}L(n/2,\chi_{(d_0}),$$

we have

$$\sum_{d_0\in\mathcal{F}((-1)^{n/2})}c_h(|d_0|)|d_0|^{-s+k/2+n/4+1/4}P(s,d_0,\{\iota_p\},\chi)$$

$$=\prod_{i=1}^{n/2-1}\zeta(2i)\frac{2^{n/2-1}\pi^{n/2}}{\Gamma(n/2)}L(s,h,E_{n/2+1/2};\chi)\prod_{i=1}^{n/2-1}L(2s-2i+n,S(h),\chi^2).$$

On the other hand, if $d_0\neq 1$, by Proposition 4.3, we have

$$P(s,d_0,\{\varepsilon_p\},\chi)=0.$$

Thus if $n \equiv 2 \bmod 4$, for any $d_0 \in \mathcal{F}^{((-1)^{n/2})}$,

$$P(s, d_0, \{\varepsilon_p\}, \chi) = 0.$$

If $n \equiv 0 \bmod 4$, by Proposition 4.3, we have

$$P(s, 1, \{\varepsilon_p\}, \chi) = \zeta(n/2) \prod_{i=1}^{n/2-1} \zeta(2i) \prod_{i=1}^{n/2} L(2s - 2i + 1, S(h), \chi^2).$$

Thus the assertion follows from Theorem 4.2. $\square$

5 Relation between twisted K-M series of the first and second kinds

Let N be a positive integer. Let g be a periodic function on $\mathbf{Z}$ with a period N and ϕ a polynomial in $t_1, ..., t_r$. Then for an element $u = (a_1 \bmod N, ..., a_r \bmod N) \in (\mathbf{Z}/N\mathbf{Z})^r$, the value $g(\phi(a_1, ..., a_r))$ does not depend on the choice of the representative of u. Therefore we denote this value by $g(\phi(u))$. In particular we sometimes regard a Dirichlet character mod N as a function on $\mathbf{Z}/N\mathbf{Z}$.

For a Dirichlet character χ mod N and $A \in \mathcal{L}_{m>0}$, put

$$h(A, \chi) = \sum_{U \in SL_m(\mathbf{Z}/N\mathbf{Z})} \chi(\mathrm{tr}(A[U])).$$

As was shown in [[K-M], Proposition 3.3], the twisted Koecher-Maaß series of the first kind of a Siegel modular form can be expressed in terms of $h(A, \chi)$ as stated later. Therefore we shall compute $h(A, \chi)$ in the case where A is an element of $\mathcal{L}_{m>0}$. For $A = (a_{ij})_{m \times m} \in S_m(\mathbf{Z}/N\mathbf{Z})$ and $c \in \mathbf{Z}/N\mathbf{Z}$, put

$$\mathcal{R}_N(A, c) = \{X = (x_{ij})_{m \times m} \in M_n(\mathbf{Z}/N\mathbf{Z}) \mid \sum_{i=1}^{m} \sum_{\alpha,\beta=1}^{m} a_{\alpha,\beta} x_{i\alpha} x_{i\beta} - c = 0$$
$$\text{and } \det X - 1 = 0\}.$$

Then we have

$$h(A, \chi) = \sum_{c \in \mathbf{Z}/N\mathbf{Z}} \chi(c) \#(R_N(A, c)).$$

From now on let p be an odd prime number and F_p be the field with p-elements. For $S \in S_m(F_p)$ and $T \in S_r(F_p)$ put

$$\mathrm{A}(S, T) = \{Y = M_{r,m}(F_p) \mid YS\,{}^tY = T\}.$$

For an element $S \in S_m(F_p)$ with m even put $\chi(S) = \left(\frac{(-1)^{m/2} \det S}{p}\right)$.

Lemma 5.1. *Let $S \in S_m(F_p)^\times$.*
(1) *Let $T \in S_r(F_p)$ with $m \geq r$.*
(1.1) *Let r be even. Then*

$$\#\mathrm{A}(S,T) = p^{rm-r(r+1)/2}(1-\chi(S)p^{-m/2})(1+\chi((-S)\perp T)p^{(r-m)/2}) \prod_{\substack{m-r+1\leq e\leq m-1 \\ e \text{ even}}} (1-p^{-e})$$

or

$$\#\mathrm{A}(S,T) = p^{rm-r(r+1)/2} \prod_{\substack{m-r+1\leq e\leq m-1 \\ e \text{ even}}} (1-p^{-e})$$

according as m is even or odd.
(1.2) *Let r be odd. Then*

$$\#\mathrm{A}(S,T) = p^{rm-r(r+1)/2}(1-\chi(S)p^{-m/2}) \prod_{\substack{m-r+1\leq e\leq m-1 \\ e \text{ even}}} (1-p^{-e})$$

or

$$\#\mathrm{A}(S,T) = p^{rm-r(r+1)/2}(1+\chi((-S)\perp T)p^{(r-m)/2}) \prod_{\substack{m-r+1\leq e\leq m-1 \\ e \text{ even}}} (1-p^{-e})$$

according as m is even or odd. In particular, for $c \in F_p^\times$, we have

$$\#\mathrm{A}(S,c) = p^{m/2-1}(p^{m/2} - \left(\frac{(-1)^{m/2}\det S}{p}\right))$$

or

$$\#\mathrm{A}(S,c) = p^{(m-1)/2}(p^{(m-1)/2} + \left(\frac{(-1)^{(m+1)/2}c\det S}{p}\right))$$

according as m is even or odd.
(2) *We have*

$$\#\mathrm{A}(S,0) = p^{m/2-1}(p^{m/2} - \left(\frac{(-1)^{m/2}\det S}{p}\right)) + p^{m/2}\left(\frac{(-1)^{m/2}\det S}{p}\right)$$

or

$$\#\mathrm{A}(S,0) = p^{m-1}$$

according as m is even or odd.

Proof. The assertions (1) and (2) follow from [[Ki1], Theorem 1.3.2], and [[Ki1], Lemma 1.3.1], respectively. □

Proposition 5.2. *Let $A = a_1\perp\cdots\perp a_m$ with $a_i \in F_p$. For $c \in F_p^\times$ put*

$$\mathcal{M}_p(A,c) = \{Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c - \sum_{i=1}^m a_i z_{ii} = 0\},$$

and

$$\gamma_{m,p} = p^{m^2-m(m+1)/2}(1-p^{-m/2})\prod_{e=1}^{(m-2)/2}(1-p^{-2e})$$

or

$$\gamma_{m,p} = p^{m^2-m(m+1)/2}\prod_{e=1}^{(m-1)/2}(1-p^{-2e})$$

according as m is even or odd. Then we have

$$\#\mathcal{R}_p(A,c) = \gamma_{m,p}\#\mathcal{M}_p(A,c).$$

Proof. Let $\Phi : GL_m(F_p) \longrightarrow S_m(F_p) \cap GL_m(F_p)$ be the mapping defined by $\Phi(X) = X{}^tX$. Then by Lemma 5.1, we have $\#\Phi^{-1}(Z) = 2\gamma_{m,p}$ for any $Z \in S_m(F_p) \cap SL_m(F_p)$. We note that $\det X = \pm 1$ for any $X \in \Phi^{-1}(Z)$. Hence we have $\#(\Phi^{-1}(Z) \cap SL_m(F_p)) = \gamma_{m,p}$. Moreover we have

$$\mathrm{tr}({}^tXAX) = \mathrm{tr}(AX{}^tX),$$

and hence $X \in \mathcal{R}_p(A,c)$ if and only if $\Phi(X) \in \mathcal{M}_p(A,c)$. This proves the assertion. □

We rewrite $\mathcal{M}_p(A,c)$ in more concise form. Let p be a prime number and l be a positive integer dividing $p-1$. Take an l-th root of unity ζ_l and a prime ideal $\mathfrak{p}$ of $\mathbf{Q}(\zeta_l)$ lying above p. Let a be an integer prime to p. Then we have $a^{(p-1)/l} \equiv \zeta_l^i \bmod \mathfrak{p}$ with some $i \in \mathbf{Z}$. We then put $\left(\dfrac{a}{p}\right)_l = \zeta_l^i$. We call $\left(\dfrac{*}{p}\right)_l$ the l-th power residue symbol mod p. In the case $l = 2$, this is the Legendre symbol, and we write it as $\left(\dfrac{*}{p}\right)$ as usual. We note that this definition of the power residue symbol is different from the usual one, and depends on the choice of $\mathfrak{p}$ and ζ_l except the case $l = 2$. We denote by $\left(\dfrac{*}{N}\right)$ the Jacobi symbol for a positive odd integer. Let χ be a primitive Dirichlet character of conductor N. We assume that N is a square free odd integer, and write $N = p_1 \cdots p_r$ with $p_1, \cdots, p_r$ prime numbers. Put $l_j = l_{m,p_j} = \mathrm{GCD}(m, p_j - 1)$. For an r-tuple $I = (i_1, i_2, \cdots, i_r)$ of integers put

$$\chi_{(i_1,\cdots,i_r)} = \chi\prod_{j=1}^{r}\left(\frac{*}{p_j}\right)_{l_j}^{i_j}.$$

For two Dirichlet characters χ and η mod N we define $J_m(\chi,\eta)$ and $I_m(\chi,\eta)$

$$J_m(\chi,\eta) = \sum_{Z \in S_m(\mathbf{Z}/N\mathbf{Z})}\chi(\det Z)\eta(1-\mathrm{tr}(Z))$$

and

$$I_m(\chi,\eta) = \sum_{Z \in S_m(\mathbf{Z}/N\mathbf{Z})}\chi(\det Z)\eta(\mathrm{tr}(Z)).$$

By definition, $J_m(\chi,\eta)$ is an algebraic number. We note that $J_1(\chi,\eta)$ is the Jacobi sum $J(\chi,\eta)$ associated with χ and η. We also define $J_m(\chi)$ as $J_m(\chi) = J_m(\chi,\chi)$.

Lemma 5.3. *Let η be a primitive character mod p. Let $c \in F_p$ and $S \in S_l(F_p)$ of rank r. Let $S \sim S_0 \perp O_{l-r}$ with $\det S_0 \neq 0$. Put*

$$I_{\eta,S,c} = \sum_{\mathbf{w} \in F_p^l} \eta(S[{}^t\mathbf{w}] + c).$$

Assume that r is odd, and that $\eta^2 \neq 1$. Then

$$I_{\eta,S,c} = p^{l-(r+1)/2} J(\eta, (\frac{*}{p})) \left(\frac{(-1)^{(r+1)/2} \det S_0}{p} \right) \eta(c) \left(\frac{c}{p} \right).$$

Assume that r is even, and that $\eta \neq 1$. Then

$$I_{\eta,S,c} = p^{l-r/2} \left(\frac{(-1)^{r/2} \det S_0}{p} \right) \eta(c).$$

Here we make the convention that $\left(\frac{(-1)^{r/2} \det S_0}{p} \right) = 1$ if $r = 0$.

Proof. We have

$$I_{\eta,S,c} = p^{l-r} I_{\eta,S_0,c}.$$

Hence we may assume that $r = l$. Then

$$I_{\eta,S,c} = \sum_{u \in F_p} \eta(u) \# A(S, u - c).$$

Let l be odd. Then by Lemma 5.1,

$$\# A(S, u - c) = p^{(l-1)/2} (p^{(l-1)/2} + \left(\frac{(-1)^{(l-1)/2} (u - c) \det S}{p} \right)).$$

Hence we have

$$I_{\eta,S,c} = p^{(l-1)/2} \left(\frac{(-1)^{(l+1)/2} \det S}{p} \right) \sum_{u \in F_p} \eta(u) \left(\frac{u-c}{p} \right).$$

Since η^2 is nontrivial, we have $I_{\eta,S,c} = 0$ if $c = 0$. If $c \neq 0$, then

$$\sum_{u \in F_p} \eta(u) \left(\frac{u-c}{p} \right) = \left(\frac{-c}{p} \right) \sum_{u \in F_p} \eta(u) \left(\frac{1 - c^{-1} u}{p} \right)$$

$$= \eta(c) \left(\frac{-c}{p} \right) \sum_{u \in F_p} \eta(v) \left(\frac{1-v}{p} \right) = \eta(c) \left(\frac{-c}{p} \right) J(\eta, \left(\frac{*}{p} \right)).$$

Let l be even. Then

$$\#A(S, u-c) = (p^{l/2} - \left(\frac{(-1)^{l/2}\det S}{p}\right))p^{l/2-1} + p^{l/2}\left(\frac{(-1)^{l/2}\det S}{p}\right)a_0,$$

where $a_0 = 1$ or 0 according as $u = c$ or not. Hence

$$I_{\eta,S,c} = p^{l/2}\left(\frac{(-1)^{l/2}\det S}{p}\right)\eta(c).$$

□

Corollary. *Let $d \in F_p^{\times}$. Then we have*

$$I_{\eta,S,cd} = \eta(d)\left(\frac{d}{p}\right)^r I_{\eta,S,c}.$$

Proposition 5.4. *Let η be a primitive character mod p. For $Z_1 \in S_{l-1}(F_p)$ and $z_{ll} \in F_p$, put*

$$I(Z_1, z_{ll}) = \sum_{w \in M_{l-1,1}(F_p)} \eta(\left(\begin{smallmatrix} Z_1 & w \\ {}^t w & z_{ll} \end{smallmatrix}\right)).$$

(1) *Assume that l is even, and that $\eta^2 \neq 1$. Then*

$$I(Z_1, z_{ll}) = p^{(l-2)/2} J(\eta, (\frac{*}{p}))\left(\frac{(-1)^{l/2}\det Z_1}{p}\right)\eta(\det Z_1 z_{ll})\left(\frac{z_{ll}}{p}\right).$$

(2) *Assume that l is odd, and that $\eta^2 \neq 1$. Then*

$$I(Z_1, z_{ll}) = p^{(l-1)/2}\left(\frac{(-1)^{(l-1)/2}\det Z_1}{p}\right)\eta(\det Z_1 z_{ll}).$$

Proof. We note that

$$\det\begin{pmatrix} Z_1 & w \\ {}^t w & z_{ll} \end{pmatrix} = -\mathrm{Adj}(Z_1)[w] + \det Z_1 z_{ll},$$

where $\mathrm{Adj}(Z_1)$ is the $(l-1)\times(l-1)$ matrix whose (i,j)-th component is the (j,i)-th cofactor of Z_1. We also note that $\det(-\mathrm{Adj}(Z_1)) = (-1)^{l-1}(\det Z_1)^{l-2}$. Thus the assertion follows directly from Lemma 5.3 if $\det Z_1 \neq 0$. If $\det Z_1 = 0$, then $\mathrm{rank}_{F_p}(Z_1) \leq 1$, the assertion follows also from Lemma 5.3. □

Theorem 5.5. *Let χ be a primitive character mod p. Let $l = \mathrm{GCD}(m, p-1)$, and u_0 be a primitive l-th root of unity mod p. Let $A \in S_m(F_p)$.*
(1) *If $\chi(u_0) \neq 1$, then we have $h(A,\chi) = 0$.*
(2) *Assume that $\chi(u_0) = 1$. Fix a character $\widetilde{\chi}$ such that $\widetilde{\chi}^m = \chi$.*
(2.1) *Let m be even. Then*

$$h(A,\chi) = \gamma_{m,p}\sum_{i=0}^{l-1} A_{m,i,p}\widetilde{\chi}_{(i)}(\det A)J_{m-1}(\overline{\widetilde{\chi}_{(i)}}),$$

where $A_{m,i,p} = p^{(m-2)/2}(-1)^{m(p-1)/4}J(\widetilde{\chi}_{(i)}, \left(\frac{*}{p}\right))$.
(2.2) *Let* m *be odd and assume that* $\chi^2 \neq 1$. *Then*

$$h(A,\chi) = \gamma_{m,p}\sum_{i=0}^{l-1} A_{m,i,p}\widetilde{\chi}_{(i)}(\det A)J_{m-1}(\overline{\widetilde{\chi}_{(i)}}),$$

where $p^{(m-1)/2}(-1)^{(m-1)(p-1)/4}$.

Proof. If $A = O_m$ then we have $h(A,\chi) = 0$. Hence we assume that $A \neq O_m$. Then we may assume that $A = a_1 \perp \cdots \perp a_{m-1} \perp d$ with $d \neq 0$. Put

$$\widetilde{\mathcal{M}}_p(A,c)$$

$$= \{(Z_1, w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p) \mid \det \begin{pmatrix} Z_1 & w \\ {}^t w & d^{-1}(1 - \sum_{i=1}^{m-1} a_i z_{ii}) \end{pmatrix} c^m = 1\}.$$

Write $Z \in S_m(F_p)$ as $Z = \begin{pmatrix} Z_1 & w \\ {}^t w & z_m \end{pmatrix}$ with $Z_1 \in S_{m-1}(F_p), w \in M_{m-1,1}(F_p), z \in F_p$. Then the mapping $S_m(F_p) \ni Z \mapsto (c^{-1}Z_1, c^{-1}w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p)$ induces a bijection from $\mathcal{M}_p(A,c)$ to $\widetilde{\mathcal{M}}_p(A,c)$, and hence $\#\widetilde{\mathcal{M}}_p(A,c) = \#\mathcal{M}_p(A,c)$. Put

$$K(A) = \sum_c \#\widetilde{\mathcal{M}}_p(A,c)\chi(c).$$

Assume that $\chi(u_0) \neq 1$. Then we have

$$K(A) = \sum_{c \in F_p} \chi(cu_0)\#\widetilde{\mathcal{M}}_p(A, cu_0).$$

We note that $\widetilde{\mathcal{M}}_p(A,cu_0) = \widetilde{\mathcal{M}}_p(A,c)$. Hence we have

$$K(A) = \chi(u_0)K(A).$$

Hence we have $K(A) = 0$.

Assume that $\chi(u_0) = 1$. Then we can take a Dirichlet character $\widetilde{\chi}$ such that $\widetilde{\chi}^m = \chi$. First assume that $\det A = 0$. Then we may assume that we have $A = A_0 \perp 0$ with $A_0 \in S_{m-1}(F_p)$. Let $P_{m-1,m}$ be the set of $(m-1)\times m$ matrices with entries in F_p of rank $m-1$. Then for each $X_1 \in P_{m-1,m}$ there exist exactly p^{m-1} elements $X_2 \in M_{1,m}(F_p)$ such that $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in SL_m(F_p)$. Hence we have

$$h(A,\chi) = p^{m-1}\sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1]).$$

Let m be even. Then we can take an element $\alpha \in F_p^{\times}$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that ${}^tU_0U_0 = \alpha 1_m$ in view of (1.1) of Lemma 5.1. Hence

$$h(A,\chi) = p^{m-1}\sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1U_0]) = \chi(\alpha)h(A,\chi).$$

Hence we have $h(A,\chi)=0$. Let m be odd and assume that $\chi^2 \neq 1$. Then we can take an element $\alpha \in (F_p^\times)^\square$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that ${}^tU_0U_0 = \alpha 1_m$ in view of (1.2) of Lemma 5.1. Thus by the same argument as above we have $h(A,\chi)=0$. This proves the assertion. Next assume that $\det A \neq 0$. We may assume that

$$A = 1_{m-1} \perp d$$

with $d = \det A$. Then we have

$$K(A) = \sum_c \#\widetilde{\mathcal{M}}_p(A,c)\widetilde{\chi}(c^m).$$

Hence we have

$$K(A) = \sum_{(Z_1,w)} \overline{\widetilde{\chi}(\det \begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix})},$$

where (Z_1,w) runs over elements of $S_{m-1}(F_p) \times M_{m-1,1}(F_p)$ such that

$$(*) \qquad \det \begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix} = u^m$$

with some $u \in F_p^\times$, and for such a matrix $\begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix}$, there exist exactly l elements u of F_p satisfying (*). We have

$$\sum_{i=0}^{l-1} \left(\frac{v}{p}\right)_l^i = l \text{ or } 0$$

according as $v = u^m$ with some $u \in F_p^\times$ or not. Hence we have

$$\begin{aligned} K(A) &= \sum_{i=0}^{l-1} \overline{\widetilde{\chi}(\det \begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix})} \\ &\quad \times \overline{\left(\frac{\det \begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix}}{p}\right)_l^i} \\ &= \sum_{i=0}^{l-1} \overline{\widetilde{\chi}_{(i)}(\det \begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix})} \end{aligned}$$

Put

$$K(A)_i = \sum_{i=0}^{l-1} \overline{\widetilde{\chi}_{(i)}(\det \begin{pmatrix} Z_1 & w \\ {}^tw & d^{-1}(1-\mathrm{tr}(Z_1)) \end{pmatrix})}$$

We note that $\widetilde{\chi}_{(i)}^2 \neq 1$ for any i. Hence by Proposition 5.4 we have

$$K(A)_i = A_{m,i,p} \sum_{Z_1 \in S_{m-1}(F_p)} \widetilde{\chi}^*_{(i)}(\det A)\overline{\widetilde{\chi}^*_{(i)}(\det Z_1)\widetilde{\chi}^*_{(i)}(1-\mathrm{tr}(Z_1))},$$

where $\widetilde{\chi}^*_{(i)} = \widetilde{\chi}_{(i)}\left(\frac{*}{p}\right)^{m-1}$. This proves the assertion if m is odd. Assume that m is even. Then it is easily seen that the set $\{\widetilde{\chi}_{(i)}\left(\frac{*}{p}\right)\}_{i=0}^{l-1}$ of Dirichlet characters coincides with $\{\widetilde{\chi}_{(i)}\}_{i=0}^{l-1}$. Moreover $\widetilde{\chi}^2_{(i)} \neq 1$ for any i. This proves the assertion. $\square$

Theorem 5.6. *Let $N = p_1 \cdots p_r$. Let χ be a primitive Dirichlet character mod N. Let $u_{0,i}$ be a primitive l_i-th root of unity mod p_i. Let $A \in S_m(F_p)$.*
(1) *If $\chi^{(p_i)}(u_{0,i}) \neq 1$ for some i. Then we have $h(A, \chi) = 0$.*
(2) *Assume that $\chi^{(p_i)}(u_{0,i}) = 1$ for any i. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^m = \chi$.*
(2.1) *Let m be even. Then we have*

$$h(A, \chi) = \prod_{i=1}^{r}(-1)^{m(p_i-1)/4} p_i^{(m-2)/2}\gamma_{m,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \widetilde{\chi}_{(i_1,i_2,\cdots,i_r)}(\det A) J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right)) J_{m-1}(\bar{\widetilde{\chi}}_{(i_1,i_2,\cdots,i_r)}).$$

(2.2) *Let m be odd, and assume that χ^2 is primitive. Then we have*

$$h(A, \chi) = \prod_{i=1}^{r}(-1)^{(m-1)(p_i-1)/4} p_i^{(m-1)/2}\gamma_{m,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \widetilde{\chi}_{(i_1,i_2,\cdots,i_r)}(\det A) J_{m-1}(\bar{\widetilde{\chi}}_{(i_1,i_2,\cdots,i_r)}).$$

Proof. We note that $J_m(\eta_1, \eta_2) = \prod_{i=1}^r J_m(\eta_1^{(p_i)}, \eta_2^{(p_i)})$ for primitive characters η_1 and η_2 mod N. Moreover η_j^2 is primitive if and only if ${\eta_j^{(p_i)}}^2 \neq 1$ for any $1 \leq i \leq r$. Thus the assertion follows from Theorem 5.5 and [[K-M], Lemma 3.2]. $\square$

Now we give explicit formulas for $J_m(\chi, \eta)$ and $I_m(\chi, \eta)$.

Proposition 5.7. *Let χ and η be primitive characters mod p. Assume that $\chi^2 \neq 1$. Put $c_m(\chi, \eta) = 1$ or 0 according as $\chi^m\eta = 1$ or not.*
(1) *Assume that m is odd. Then*

$$I_m(\chi, \eta) = c_m(\chi, \eta)\left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2}(p-1)J_{m-1}(\chi\left(\frac{*}{p}\right), \eta).$$

(2) *Assume that m is even. Then*

$$I_m(\chi, \eta) = c_m(\chi, \eta)\left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}(p-1)\chi(-1)J(\chi, \left(\frac{*}{p}\right))J_{m-1}(\chi\left(\frac{*}{p}\right), \eta).$$

Proof. By Proposition 5.4, we have

$$I_m(\chi,\eta) = I'_m \times \begin{cases} p^{(m-1)/2}\left(\frac{(-1)^{(m-1)/2}}{p}\right) & \text{if } m \text{ is odd} \\ p^{(m-2)/2}\left(\frac{(-1)^{(m-2)/2}}{p}\right) J(\chi,\left(\frac{*}{p}\right)) & \text{if } m \text{ is even,} \end{cases}$$

where

$$I'_m = \sum_{\substack{z_{mm}\in F_p \\ Z_1\in S_{m-1}(F_p)^\times}} \chi(z_{mm})\chi(\det Z_1)\left(\frac{\det Z_1}{p}\right)\eta(z_{mm}+\mathrm{tr}(Z_1))\left(\frac{z_{mm}}{p}\right)^{m-1}.$$

Then we have

$$I'_m = \sum_{\substack{z_{mm}\in F_p^\times \\ Z_1\in S_{m-1}(F_p)^\times}} \chi(z_{mm})\eta(z_{mm})\chi(\det Z_1)\left(\frac{\det Z_1}{p}\right)\eta(1+z_{mm}^{-1}\mathrm{tr}(Z_1))\left(\frac{z_{mm}}{p}\right)^{m-1}.$$

Put $Y_1 = -z_{mm}^{-1}Z_1$. Then $\det Y_1 = (-1)^{m-1}z_{mm}^{1-m}\det Z_1$. Hence we have

$$I'_m = \chi((-1)^{m-1})\left(\frac{(-1)^{m-1}}{p}\right)$$

$$\times \sum_{z_{mm}\in F_p^\times} \chi(z_{mm})^m\eta(z_{mm}) \sum_{Y_1\in S_{m-1}(F_p)^\times} \chi(\det Y_1)\left(\frac{\det Y_1}{p}\right)\eta(1-\mathrm{tr}(Y_1)).$$

We have

$$\sum_{z_{mm}\in F_p^\times} \chi(z_{mm})^m\eta(z_{mm}) = p-1 \text{ or } 0$$

according as $\chi^m\eta$ is trivial or not. This proves the assertion. □

Proposition 5.8. *Let χ and η be as in Proposition 5.7.*
(1) *Assume that m is odd. Then*

$$J_m(\chi,\eta) = \left(\frac{(-1)^{(m-1)/2}}{p}\right)p^{(m-1)/2}$$

$$\times\{J(\chi,\chi^{m-1}\eta)J_{m-1}(\chi\left(\frac{*}{p}\right),\eta)+\eta(-1)I_{m-1}(\chi\left(\frac{*}{p}\right),\eta)\}.$$

(2) *Assume that m is even. Then*

$$J_m(\chi,\eta) = \left(\frac{-1}{p}\right)^{m/2}p^{(m-2)/2}J(\chi,\left(\frac{*}{p}\right))$$

$$\times\{J(\chi\left(\frac{*}{p}\right),\chi^{m-1}\left(\frac{*}{p}\right)\eta)J_{m-1}(\chi\left(\frac{*}{p}\right),\eta)+\eta(-1)I_{m-1}(\chi\left(\frac{*}{p}\right),\eta)\}.$$

Proof. By Proposition 5.4, we have

$$J_m(\chi,\eta) = (J'_m + J''_m) \times \begin{cases} p^{(m-1)/2} \left(\frac{(-1)^{(m-1)/2}}{p}\right) & \text{if } m \text{ is odd} \\ p^{(m-2)/2} \left(\frac{(-1)^{(m-2)/2}}{p}\right) J(\chi, \left(\frac{*}{p}\right)) & \text{if } m \text{ is even,} \end{cases}$$

where

$$J'_m = \sum_{\substack{z_{mm}\in F_p, z_{mm}\neq 1 \\ Z_1\in S_{m-1}(F_p)^\times}} \left(\frac{\det Z_1}{p}\right)\left(\frac{z_{mm}}{p}\right)^{m-1} \chi(z_{mm})\chi(\det Z_1)\eta(1-z_{mm}-\mathrm{tr}(Z_1)),$$

and

$$J''_m = \sum_{Z_1\in S_{m-1}(F_p)^\times} \left(\frac{\det Z_1}{p}\right) \chi(\det Z_1)\eta(-\mathrm{tr}(Z_1)).$$

Then we have $J''_m = \eta(-1)I_{m-1}(\chi\left(\frac{*}{p}\right), \eta)$. Moreover

$$J'_m = \sum_{\substack{z_{mm}\in F_p, z_{mm}\neq 1 \\ Z_1\in S_{m-1}(F_p)^\times}} \chi(z_{mm})\left(\frac{\det Z_1}{p}\right)\left(\frac{z_{mm}}{p}\right)^{m-1} \chi(\det Z_1)$$

$$\times\eta(1-z_{mm})\eta(1-(1-z_{mm})^{-1}\mathrm{tr}(Z_1)).$$

Put $Y_1 = (1-z_{mm})^{-1}Z_1$. Then $\det Y_1 = (1-z_{mm})^{1-m}\det Z_1$. Hence we have

$$J'_m = \sum_{z_{mm}\in F_p} \chi(z_{mm})\left(\frac{z_{mm}}{p}\right)^{m-1}\left(\frac{1-z_{mm}}{p}\right)^{m-1} \chi(1-z_{mm})^{m-1}\eta(1-z_{mm})$$

$$\times \sum_{Y_1\in S_{m-1}(F_p)^\times} \left(\frac{\det Y_1}{p}\right)\chi(\det Y_1)\eta(1-\mathrm{tr}(Y_1)).$$

This proves the assertion. □

Theorem 5.9. *Let χ be a primitive character mod p.*
(1) *Let m be odd, and assume that $\chi^2 \neq 1$.*
(1.1) *Assume that $\chi^m \neq 1$. Then*

$$J_m(\chi\left(\frac{*}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2} J(\chi\left(\frac{*}{p}\right)^i, \chi^m) J_{m-1}(\chi\left(\frac{*}{p}\right)^{i+1}, \chi).$$

(1.2) *Assume that $\chi^m = 1$. Then*

$$J_m(\chi\left(\frac{*}{p}\right)^i, \chi) = p^{m-1}\left(\frac{-1}{p}\right)^{i+1} J(\chi\left(\frac{*}{p}\right)^{i+1}, \left(\frac{*}{p}\right)) J_{m-2}(\chi\left(\frac{*}{p}\right)^i, \chi).$$

(2) *Let m be even.*

(2.1) *Assume that* $\chi^m \left(\frac{*}{p}\right)^{i+1} \neq 1$. *Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-2)/2} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J(\chi \left(\frac{*}{p}\right)^{i+1}, \chi^m \left(\frac{*}{p}\right)^{i+1}) J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi).$$

(2.2) *Assume that* $\chi^m \left(\frac{*}{p}\right)^{i+1} = 1$. *Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \chi(-1) p^{m-1} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

Proof. Let m be odd. Then, by (1) of Proposition 5.8, we have

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2}$$

$$\times \{J(\chi \left(\frac{*}{p}\right)^i, \chi^m) J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi) + \chi(-1) I_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi)\}.$$

Thus the assertion holds if $\chi^m \neq 1$. Assume that $\chi^m = 1$. Then by (2) of Proposition 5.8 and (2) of Proposition 5.7 we have

$$J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-3)/2} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right))$$

$$\times J(\chi \left(\frac{*}{p}\right)^i, \chi^{m-1} \left(\frac{*}{p}\right)^i) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

and

$$I_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi) = \left(\frac{-1}{p}\right)^{(m-3)/2} p^{(m-3)/2} (p-1) \chi(-1) \left(\frac{-1}{p}\right)^{i+1}$$

$$\times J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

We note that $J(\chi \left(\frac{*}{p}\right)^i, \chi^m) = -1, \chi(-1) = 1$ and

$$J(\chi \left(\frac{*}{p}\right)^i, \chi^{m-1} \left(\frac{*}{p}\right)^i) = J(\chi \left(\frac{*}{p}\right)^i, \overline{\chi \left(\frac{*}{p}\right)^i}) = \chi(-1) \left(\frac{-1}{p}\right)^i = \left(\frac{-1}{p}\right)^i.$$

This proves the assertion.

Let m be even. Then, by (2) of Proposition 5.8, we have

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-2)/2} p^{(m-2)/2} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right))$$

$$\times\{J(\chi\left(\frac{*}{p}\right)^{i},\chi^{m}\left(\frac{*}{p}\right)^{i+1})J_{m-1}(\chi\left(\frac{*}{p}\right)^{i+1},\chi)+\chi(-1)I_{m-1}(\chi\left(\frac{*}{p}\right)^{i+1},\chi)\}.$$

Thus the assertion holds if $\chi^{m}\left(\frac{*}{p}\right)^{i+1}\neq 1$. Assume that $\chi^{m}\left(\frac{*}{p}\right)^{i+1}=1$. Then by (1) of Proposition 5.7 and (1) of Proposition 5.8, we have

$$J_{m-1}(\chi\left(\frac{*}{p}\right)^{i+1},\chi)=\left(\frac{-1}{p}\right)^{(m-2)/2}p^{(m-2)/2}$$
$$\times J(\chi\left(\frac{*}{p}\right)^{i+1},\chi^{m-1})J_{m-2}(\chi\left(\frac{*}{p}\right)^{i},\chi),$$

and

$$I_{m-1}(\chi\left(\frac{*}{p}\right)^{i+1},\chi)=\left(\frac{-1}{p}\right)^{(m-2)/2}p^{(m-2)/2}J_{m-2}(\chi\left(\frac{*}{p}\right)^{i},\chi).$$

We note that $J(\chi\left(\frac{*}{p}\right)^{i},\chi^{m}\left(\frac{*}{p}\right)^{i+1})=-1, \left(\frac{-1}{p}\right)^{i+1}=1$ and

$$J(\chi\left(\frac{*}{p}\right)^{i+1},\chi^{m-1})=J(\chi\left(\frac{*}{p}\right)^{i+1},\overline{\chi\left(\frac{*}{p}\right)^{i+1}})=\chi(-1)\left(\frac{-1}{p}\right)^{i+1}=\chi(-1).$$

This proves the assertion.

□

Corollary. *Let χ be a primitive character with an odd square free conductor N. Assume that χ^2 is primitive. Then the value $J_m(\chi)$ is nonzero.*

Proof. The assertion follows directly from the above theorem if N is an odd prime. In general case, the assertion can also be proved by remarking that $J_m(\chi)=\prod_{p|N}J_m(\chi^{(p)})$ and that ${\chi^{(p)}}^2\neq 1$ for any $p|N$. □

To compare our present result with the result in [K-M], we give the following:

Proposition 5.10. *Let χ be a primitive Dirichlet character mod p. Assume that $\chi^2\neq 1$. Then we have*

$$J(\chi,\left(\frac{*}{p}\right))J(\chi\left(\frac{*}{p}\right),\chi\left(\frac{*}{p}\right))=\left(\frac{-1}{p}\right)\bar{\chi}(4)p.$$

Proof. Put

$$I=\sum_{(z,w)\in F_p^2}\chi(z(1-z)-w^2).$$

Then by using the same argument as in the proof of Theorem 5.5, we have

$$I=J(\chi,\left(\frac{*}{p}\right))\sum_{z\in F_p}\chi(z(1-z))\left(\frac{z(1-z)}{p}\right)$$

$$= J(\chi, \left(\frac{*}{p}\right)) J(\chi \left(\frac{*}{p}\right), \chi \left(\frac{*}{p}\right)).$$

On the other hand, we have

$$I = \sum_{(y,w) \in F_p^2} \chi(-y^2 - w^2 + 1/4).$$

Hence by Lemma 5.3 we have

$$I = p \left(\frac{-1}{p}\right) \bar{\chi}(4).$$

This proves the assertion. □

By virtue of the above proposition, we see that Theorem 5.6 coincides with [K-M], Proposition 3.7 in case $m = 2$.

Now let

$$F(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} c_F(A) \mathbf{e}(\mathrm{tr}(AZ))$$

be an element of $\mathfrak{M}_k(Sp_n(\mathbf{Z}))$ and let χ be a Dirichlet character mod N. Assume $N \neq 2$. Then by [[K-M], Proposition 3.1], we have

$$L(s, F, \chi) = \sum_{A \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{c_F(A) h(A, \chi)}{e(A)(\det A)^s}.$$

Thus by Theorem 5.6 we easily obtain:

Theorem 5.11. *Let $N, p_i, l_i, u_{0,i}$ $(i = 1, \cdots, r)$ and χ be as in Theorem 5.6, and let F be an element of $\mathfrak{M}_k(Sp_n(\mathbf{Z}))$.*
(1). If $\chi^{(p_i)}(u_{0,i}) \neq 1$ for some i. Then we have $L(s, F, \chi) = 0$.
(2). Assume that $\chi^{(p_i)}(u_{0,i}) = 1$ for any i. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^n = \chi$.
(2.1) Let n be even. Then we have

$$L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-2)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}}_{(i_1,\cdots,i_r)}(2^n) \overline{J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))} \, \overline{J_{n-1}(\widetilde{\chi}_{(i_1,\cdots,i_r)})} L^*(s, F, \widetilde{\chi}_{(i_1,i_2,\cdots,i_r)}).$$

(2.2) Let n be odd, and assume that $\chi^2 \neq 1$. Then we have

$$L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}}_{(i_1,\cdots,i_r)}(2^{n-1}) \overline{J_{n-1}(\widetilde{\chi}_{(i_1,i_2,\cdots,i_r)})} L^*(s, F, \widetilde{\chi}_{(i_1,i_2,\cdots,i_r)}).$$

6 Twisted Koecher-Maaß series of the first kind of the DII lift

By Theorems 4.1 and 5.11, we obtain the following.

Theorem 6.1. *Let k and n be positive even integers such that $n \geq 4,\ 2k-n \geq 12$. Let $h(z)$ and $E_{n/2+1/2}$ be as in Section 4. Let N be a square free odd integer, and $N = p_1 \cdots p_r$ be the prime decomposition of N. For each $i = 1, \cdots, r$ let $l_i = \mathrm{GCD}(n, p_i - 1)$ and $u_{0,i} \in \mathbf{Z}$ be a primitive l_i-th root of unity mod p_i.*
(1) Assume $\chi^{(p_i)}(u_i) \neq 1$ for some i. Then $L(s, I_n(h), \chi) = 0$.
(2) Assume $\chi^{(p_i)}(u_i) = 1$ for any i. Then

$$L(s, I_n(h), \chi) = 2^{ns}\overline{\widetilde{\chi}(2^n)} \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))}\,\overline{J_{n-1}(\widetilde{\chi}_{(i_1,\cdots,i_r)})}$$

$$\times\{c_{n,N} R(s, h, E_{n/2+1/2}, \widetilde{\chi}_{(i_1,\cdots,i_r)}) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \widetilde{\chi}^2_{(i_1,\cdots,i_r)})$$

$$+ d_{n,N} c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1, S(h), \widetilde{\chi}^2_{(i_1,\cdots,i_r)})\},$$

where $c_{n,N}$ and $d_{n,N}$ are nonzero rational numbers depending only on n and N, and $\widetilde{\chi}$ is a character s.t. $\widetilde{\chi}^n = \chi$.

Remark. In the case $n = 2$, an explicit formula for $L(s, I_2(h), \chi)$ was given by Katsurada-Mizuno [K-M].

7 Applications

Let h_1 and h_2 be modular forms of weight $k_1 + 1/2$ and $k_2 + 1/2$, respectively, and χ be a Dirichlet character . In Section 2, we reviewed on the algebraicity of the values $\widetilde{R}(m, h_1, h_2, \chi)$ at half integers. We then naturally ask the following question:
Question. What can one say about the algebraicity of $\widetilde{R}(m, h_1, h_2, \chi)$ with m an integer?

As an application of Theorem 6.1, we give a partial answer to this question. We note that

$$R(s, h_1, h_2, \chi) = (1 - 2^{-2s+k_1+k_2-1}\chi^2(2))^{-1} \widetilde{R}(s, h_1, h_2, \chi)$$

if the conductor of χ is odd. Hence it suffices to consider the above question for $R(m, h_1, h_2, \chi)$ with integer m if $k_1 + k_2$ is even.

Let k and n be positive even integers such that $n \geq 4,\ 2k-n \geq 12$. Let $h(z)$ and $E_{n/2+1/2}$ be as in Section 4. For a Dirichlet character χ of odd square free conductor $N = p_1 \cdots p_r$, we define

$$R^{(\chi)}(s,h,E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{J(\chi_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))}\, \overline{J_{n-1}(\chi_{(i_1,\cdots,i_r)})}$$

$$\times R(s,h,E_{n/2+1/2},\chi_{(i_1,\ldots,i_r)}) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \chi^2_{(i_1,\cdots,i_r)}),$$

where $l_i = \mathrm{GCD}(n, p_i - 1)$ as in Theorem 6.1.

Theorem 7.1. *There exists a finite dimensional $\overline{\mathbf{Q}}$-vector space $W_{h,E_{n/2+1/2}}$ in $\mathbf{C}$ such that*

$$\frac{R^{(\chi)}(m,h,E_{n/2+1/2})}{\pi^{mn}} \in W_{h,E_{n/2+1/2}}$$

for any integer $n/2+1 \leq m \leq k-n/2-1$ and a character χ of odd square free conductor such that χ^n is primitive.

Proof. Put

$$\mathbf{M}^{(\chi)}(s,S(h)) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{J(\widetilde{\chi}_{(i_1,\cdots,i_r)}, \left(\frac{*}{N}\right))}\, \overline{J_{n-1}(\chi_{(i_1,\cdots,i_r)})}$$

$$\times \prod_{j=1}^{n/2} L(2s-2j+1, S(h), (\chi_{(i_1,\cdots,i_r)})^2).$$

Then by Corollary to Proposition 3.1, we have

$$\frac{\mathbf{M}^{(\chi)}(m,S(h))}{\pi^{mn}} \in \overline{\mathbf{Q}} u_-(S(h))^{n/2} \pi^{-n^2/4}.$$

By Theorem 6.1, we have

$$L(m, I_n(h), \chi^n)$$
$$= 2^{nm}\overline{\chi(2^n)}\{c_{n,N} R^{(\chi)}(m,h,E_{n/2+1/2}) + d_{n,N} c_h(1) \mathbf{M}^{(\chi)}(m,S(h))\}.$$

Hence by Theorem 2.2, we have

$$\frac{R^{(\chi)}(m,h,E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)} + \overline{\mathbf{Q}} u_2$$

with some complex numbers u_1 and u_2, where $V_{I_n(h)}$ is the $\overline{\mathbf{Q}}$-vector space associated with $I_n(h)$ in Theorem 2.2. This proves the assertion. □

By the above theorem, we immediately obtain the following:

Theorem 7.2. *Let $d > \dim_{\overline{\mathbf{Q}}} W_{h,E_{n/2+1/2}}$. Let $m_1, m_2, \cdots, m_d$ be integers such that $n/2+1 \leq m_1, m_2, \cdots, m_d \leq k-n/2-1$ and $\chi_1, \chi_2, \cdots, \chi_d$ be Dirichlet characters of odd square free conductors $N_1, N_2, \cdots, N_d$, respectively such that χ_i^n is primitive for any $i = 1, 2, \cdots d$. Then the values $\dfrac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \cdots, \dfrac{R^{(\chi_d)}(m_d, h, E_{n/2+1/2})}{\pi^{m_d n}}$ are linearly dependent over $\overline{\mathbf{Q}}$.*

.

Corollary. *In addition to the notation and the assumption as above, assume that $n \equiv 2 \bmod 4$. Write N_i as $N_i = \prod_{j=1}^{r_i} p_{ij}$ with p_{ij} an odd prime number, and let $l_{ij} = \mathrm{GCD}(p_{ij}-1, n)$. Then the values $\left\{ \dfrac{R(m_i, h, E_{n/2+1/2}, \chi_{i(a_{i1},\cdots,a_{ir_i})})}{\pi^{2m_i}} \right\}_{1 \leq i \leq d, 0 \leq a_{i1} \leq l_{i1}-1, \cdots, 0 \leq a_{ir_i} \leq l_{ir_i}}$ are linearly dependent over $\overline{\mathbf{Q}}$. In particular, if $\chi_1, \chi_2, \cdots, \chi_d$ are Dirichlet characters of odd prime conductors $p_1, p_2, \cdots, p_d$, respectively such that χ_i^n is primitive for any $i = 1, 2, \cdots d$, then the values $\{R(m_i, h, E_{n/2+1/2}, \chi \left(\frac{*}{p_i}\right)_{l_i}^{a_i})\pi^{-2m_i}\}_{1 \leq i \leq d, 0 \leq a_i \leq l_i - 1}$ are linearly dependent over $\overline{\mathbf{Q}}$, where $l_i = \mathrm{GCD}(n, p_i - 1)$ for $i = 1, \cdots, d$.*

Proof. By Theorem 1.1, the value $\frac{\mathbf{L}_n(m, S(h), \chi^2_{(i_1,\cdots,i_r)})}{\pi^{m(n-2)}}$ belongs to $\overline{\mathbf{Q}} u_+(S(h))^{n/2-1} \pi^{-n^2/4+n/2}$, and in particular if $n \equiv 2 \bmod 4$, then it is nonzero for any χ. Moreover, by Corollary to Theorem 5.10, $J(\chi_{(i_1,\cdots,i_r)}, (\frac{*}{N})) J_{n-1}(\chi_{(i_1,\cdots,i_r)})$ is non-zero and belongs to $\overline{\mathbf{Q}}$. Thus the assertion holds. □

As another application of Theorem 7.1, we also have a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$. Namely, by Theorem 3.1 we obtain:

Theorem 7.3. *Let h be as above. Let χ be a primitive character of odd square free conductor N. Assume that $n \equiv 2 \bmod 4$, and that χ^n is primitive. Put*

$$\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^n)^{-1} \gamma_n(s) R^{(\chi)}(s, h, E_{n/2+1/2}),$$

where $\tau(\chi^n)$ is the Gauss sum, and

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2} \Gamma(s - (i-1)/2).$$

Then $\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2})$ has an analytic continuation to the whole s-plane, and has the following functional equation:

$$\mathcal{R}^{(\chi)}(k - s, h, E_{n/2+1/2}) = \mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}).$$

Remark. (1) As functions of s, the Dirichlet series $\{R(s, h, E_{n/2+1/2}, \chi_{i(j)})\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$ are linearly independent over $\mathbf{C}$.

(2) In the case of $n = 2$, this type of result was given for $R(m, h, E_{3/2})$ with $E_{3/2}$ Zagier's Eisenstein series of weight 3/2 by [K-M].

(3) The meromorphy of this type of series was derived in [Sh3] by using so called the Rankin-Selberg integral expression in more general setting, but we don't know whether the functional equation of the above type can be directly proved without using the above method.

References

[B] S. Böcherer, Bemerkungen über die Dirichletreichen von Koecher und Maaß, Math. Gottingensis des Schrift. des SFB. Geometry and Analysis Heft **68** (1986).

[B-S] S. Böcherer and R. Schulze-Pillot, The Dirichlet series of Koecher-Maaß and modular forms of weight 3/2, Math. Z. 209(1992), 273-287.

[C-K] Y. Choie and W. Kohnen, Special values of Koecher-Maaß series of Siegel cusp forms, Pacific J. Math. **198** (2001), 373-383.

[D-I] W. Duke, Ö. Imamoḡlu, A converse theorem and the Saito-Kurokawa lift. Internat. Math. Res. Notices 1996, no. 7, 347–355.

[I] T. Ikeda, On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$, Ann. of Math. 154(2001), 641-681.

[I-K1] T. Ibukiyama and H. Katsurada, An explicit formula for Koecher-Maass Dirichlet series for Eisenstein series of Klingen type. J. Number Theory 102(2003), 223-256

[I-K2] T. Ibukiyama and H. Katsurada, An explicit formula for the Koecher-Maass Dirichlet series for the Ikeda lifting. Abh. Math. Sem. Univ. Hamburg 74 (2004), 101-121

[I-K3] T. Ibukiyama, H. Katsurada, Koecher-Maass series for real analytic Siegel Eisenstein series, in Automorphic Forms and Zeta Functions, Proceedings of the conference in memory of Tsuneo Arakawa, 170–197, World Scientific (2006)

[Ka] H. Katsurada, On the special values of certain L-series related to half-integral weight modular forms, to appear in Springer Proceedings in Math. and Stat.

[K-M] H. Katsurada and Y. Mizuno, Linear dependence of certain L-values of half-integral weight modular forms, J. London Math. 85(2012), 455-471

[Ki1] Y. Kitaoka, Dirichlet series in the theory of Siegel modular forms, Nagoya Math. J. 95(1984), 73-84.

[Ki2] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge Tracts in Mathematics, 106. Cambridge University Press, Cambridge, 1993

[Ko] W. Kohnen, Modular forms of half-integral weight on $\Gamma_0(4)$, Math. Ann. 48(1980), 249-266.

[Miz] Y. Mizuno, The Rankin-Selberg convolution for Cohen's Eisenstein series of half-integral weights, Abh. Math. Sem. Hamburg 75(2005),1-20.

[Sai1] H. Saito, On L-functions associated with the vector space of binary quadratic forms. Nagoya Math. J. **130** (1993), 149–176.

[Sai2] H. Saito, *Explicit formula of orbital p-adic Zeta functions associated to symmetric and Hermitian matrices*, Comment. Math. Univ. St. Paul. **46**(1997), 175-216.

[Sh1] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. pure appl. Math. 29(1976), 65-76.

[Sh2] G. Shimura, On the periods of modular forms, Math. Ann. 229(1977), 211-221.

[Sh3] G. Shimura, The critical values of certain zeta functions associated with modular forms of half-integral weight, J. Math. Soc. Japan 33(1981), 649-672.

Hidenori Katsurada
Muroran Institute of Technology,
27-1 Mizumoto, Muroran, 050-8585, Japan
e-mail: hidenori@mmm. muroran-it.ac.jp