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<th>Title</th>
<th>Explicit formulas for the twisted Koecher-Maaβ series of the Duke-Imamoglu-Ikeda lift and their applications</th>
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<td>Author(s)</td>
<td>Katsurada, Hidenori</td>
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<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 1028, 1-30</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-1-18</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/84423</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/70237">http://hdl.handle.net/2115/70237</a></td>
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<td>Type</td>
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<td>File Information</td>
<td>pre1028.pdf</td>
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Table 1: A table with the above information.
Explicit formulas for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and their applications

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Abstract
We give an explicit formula for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift. As an application we prove a certain algebraicity result for the values of twisted Rankin-Selberg series at integers of half-integral weight modular forms.

1 Introduction
It is an interesting problem to give an explicit formula for the Koecher-Maaß series of a Siegel modular form $F$ for the symplectic group $Sp_n(\mathbb{Z})$, and several results have been obtained (cf. Böcherer [B], Ibukiyama and Katsurada [I-K1], [I-K2], [I-K3]). Such explicit formulas are not only interesting in its own right but also have some important applications in the theory of modular forms. For example, we refer to [B-S], [D-I], [Miz]. Now we consider a twist of such a Koecher-Maaß series by a Dirichlet character $\chi$. As for this, in view of Saito [Sai1] for example, we can naturally consider the following Dirichlet series:

$$L^*(s, F, \chi) = \sum_T \chi(2^{n/2} \det T) c_F(T) e^\frac{(T)(\det T)}{T^{s}},$$

where $T$ runs over a complete set of representatives of $SL_n(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree $n$, $c_F(T)$ is the $T$-th Fourier coefficient of $F$ and $e(T) = \# \{U \in SL_n(\mathbb{Z}) ; T[U] = T \}$. We will sometimes call $L^*(s, F, \chi)$ the twisted Koecher-Maaß series of the second kind.

On the other hand, Choie and Kohnen [C-K] introduced a different type of “twist”. For a positive integer $N$, let $SL_{n,N}(\mathbb{Z}) = \{ U \in SL_n(\mathbb{Z}) ; U \equiv 1_n \mod N \}$ and $e_N(T) = \# \{ U \in SL_{n,N}(\mathbb{Z}) ; T[U] = T \}$. For a primitive Dirichlet character $\chi$ mod $N$, the Koecher-Maaß series $L(s, F, \chi)$ of $F$ twisted by $\chi$ is defined to be

$$L(s, F, \chi) = \sum_T \frac{\chi(\text{tr}(T)) c_F(T)}{e_N(T)(\det T)^s}.$$
where $T$ runs over a complete set of representatives of $SL_{n,N}(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree $n$. In [C-K], Choie and Kohnen proved a meromorphy continuation of $L(s, F, \chi)$ to the whole $s$-plane and a functional equation (cf. Theorem 2.1). Moreover they got a result on the algebraicity of its special values (cf. Theorem 2.2.) We shall call $L(s, F, \chi)$ the twist of the first kind.

In this paper we give explicit formulas for the twisted Koecher-Maaß series of the first and second kinds associated with the Duke-Imamoglu-Ikeda lift and apply them to the study of the special values of the Rankin-Selberg series for half-integral weight modular forms. We explain our main results more precisely. Let $k$ and $n$ be positive even integers such that $n \geq 4$ and $2k-n \geq 12$. For a cuspidal Hecke eigenform $h$ in the Kohnen plus subspace of weight $k-n/1+1/2$ for $\Gamma_0(4)$, let $I_n(h)$ be the Duke-Imamoglu-Ikeda lift of $h$ to the space of cusp forms of weight $k$ for $Sp_n(\mathbb{Z})$. Moreover let $S(h)$ be the normalized Hecke eigenform of weight $2k-n$ for $SL_2(\mathbb{Z})$ corresponding to $h$ under the Shimura correspondence, and $E_{n/2+1/2}$ be Cohen’s Eisenstein series of weight $n/2 + 1/2$ for $\Gamma_0(4)$. We then give explicit formulas for $L(s, I_n(h), \chi)$ and $L^*(s, I_n(h), \chi)$ in terms of the twisted Rankin-Selberg series $R(s, h, E_{n/2+1/2}, \eta)$ of $h$ and $E_{n/2+1/2}$ and twisted Hecke’s $L$-function $L(s, S(h), \eta')$ of $S(h)$, where $\eta$ and $\eta'$ are Dirichlet characters related with $\chi$. It is relatively easy to get an explicit form of $L^*(s, I_n(h), \chi)$. In fact, by using the same argument as in Ibukiyama and Katsurada [I-K2], we can easily obtain its explicit formula (cf. Theorem 4.1). On the other hand, it seems nontrivial to get that of $L(s, I_n(h), \chi)$ (cf. Theorem 6.1), and we need some explicit formula for a certain character sum associated with a Dirichlet character (cf. Theorem 5.6). Using Theorem 6.1 combined with the result of Choie-Kohnen, we prove certain algebraicity results on $R(s, h, E_{n/2+1/2}, \eta)$ at an integer $s = m$ (cf. Theorems 7.1 and 7.2), which were announced in [Ka]. We note that the algebraicity of the special values of such a Rankin-Selberg series at half-integers was investigated by Shimura [Sh3]. However there are few results on the algebraicity of such values at integers. As an attempt, Mizuno and the author [K-M] proved linear dependency of Rankin-Selberg $L$-values of a cuspidal Hecke eigenform belonging to Kohnen plus subspace of half integral weight and the Zagier’s Eisenstein series of weight $3/2$. Our present result can be regarded as a generalization of our previous result.

**Notation** We denote by $e(x) = \exp(2\pi \sqrt{-1}x)$ for a complex number $x$. For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m, n)$-matrices with entries in $R$. For an $(m, n)$-matrix $X$ and an $(m, m)$-matrix $A$, we write $A[X] = ^tXAX$, where $^tX$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X$ mod $aM_{mn}(R)$. Put $GL_m(R) = \{ A \in M_m(R) \mid \det A \in R^* \}$, and $SL_m(R) = \{ A \in M_m(R) \mid \det A = 1 \}$, where $\det A$ denotes the determinant of a square matrix $A$ and $R^*$ is the unit group of $R$. We denote by $S_n(R)$ the set of symmetric matrices of degree $n$ with entries in $R$. In particular, if $S$ is a subset of $S_n(\mathbb{R})$ with $\mathbb{R}$ the field of real numbers, we denote by $S_{\geq 0}$ (resp. $S_{> 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite)
matrices. The group $SL_n(\mathbb{Z})$ acts on the set $S_n(\mathbb{R})$ in the following way:

$$SL_n(\mathbb{Z}) \times S_n(\mathbb{R}) \ni (g,A) \mapsto gAg \in S_n(\mathbb{R}).$$

Let $G$ be a subgroup of $GL_n(\mathbb{Z})$. For a subset $\mathcal{B}$ of $S_n(\mathbb{R})$ stable under the action of $G$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to $G$. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$.

Two symmetric matrices $A$ and $A'$ with entries in $\mathbb{R}$ are said to be equivalent with respect to $G$ and write $A \sim_G A'$ if there is an element $X$ of $G$ such that $A' = AX$. Let $L_n$ denote the set of half-integral matrices of degree $n$ over $\mathbb{Z}$, that is, $L_n$ is the set of symmetric matrices of degree $n$ whose $(i,j)$-component belongs to $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$ according as $i = j$ or not.

## 2 Twisted Koecher-Maaß series

Put $J_n = \left( \begin{array}{cc} O_n & -1_n \\ 1_n & O_n \end{array} \right)$, where $1_n$ and $O_n$ denotes the unit matrix and the zero matrix of degree $n$, respectively. Furthermore, put

$$Sp_n(\mathbb{Z}) = \{ M \in GL_{2n}(\mathbb{Z}) \mid J_n[M] = J_n \}.$$

Let $l$ be an integer or a half-integer, and $N$ a positive integer. Let $\Gamma_0^{(n)}(N)$ be the congruence subgroup of $Sp_n(\mathbb{Z})$ consisting of matrices whose lower $n \times n$ block are congruent to $O_n$ mod $N$. Moreover let $\chi$ be a Dirichlet character mod $N$. We then denote by $\mathfrak{M}_l(\Gamma_0^{(n)}(N),\chi)$ the space of modular forms of weight $l$ and character $\chi$ for $\Gamma_0^{(n)}(N)$, and by $\mathfrak{S}_l(\Gamma_0^{(n)}(N),\chi)$ the subspace of $\mathfrak{M}_l(\Gamma_0^{(n)}(N),\chi)$ consisting of cusp forms. If $\chi$ is the trivial character mod $N$, we simply write $\mathfrak{M}_l(\Gamma_0^{(n)}(N),\chi)$ and $\mathfrak{S}_l(\Gamma_0^{(n)}(N),\chi)$ as $\mathfrak{M}_l(\Gamma_0^{(n)}(N))$ and $\mathfrak{S}_l(\Gamma_0^{(n)}(N))$, respectively. Let $k$ be a positive integer, and let $F(Z) \in \mathfrak{M}_k(Sp_n(\mathbb{Z}))$. Then $F(Z)$ has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{L}_n \geq 0} c_F(T) e(\text{tr}(TZ)),$$

where $\text{tr}(X)$ denotes the trace of a matrix $X$. For $N \in \mathbb{Z}_{>0}$, put $SL_{n,N}(\mathbb{Z}) = \{ U \in SL_n(\mathbb{Z}) \mid U \equiv 1_n \text{ mod } N \}$, and for $T \in \mathcal{L}_{n \geq 0}$ put $e_N(T) = \#\{ U \in SL_{n,N}(\mathbb{Z}) \mid T[U] = T \}$. For a primitive Dirichlet character $\chi$ mod $N$ let

$$L(s, F; \chi) = \sum_{T \in \mathcal{L}_{n \geq 0}/SL_{n,N}(\mathbb{Z})} \frac{\chi(\text{tr}(T))c_F(T)}{e_N(T)(\text{det}T)^s}$$

be the twisted Koecher-Maaß series of the first kind of $F$ as in Section 1. The following two theorems are due to Choie and Kohnen [C-K].

**Theorem 2.1.** Let $F \in \mathfrak{S}_k(Sp_n(\mathbb{Z}))$, and $\chi$ a primitive character of conductor $N$. Put

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{-i-1/2} \Gamma(s - (i-1)/2),$$

$$\gamma(s, F; \chi) = \sum_{T \in \mathcal{L}_{n \geq 0}/SL_{n,N}(\mathbb{Z})} \frac{\chi(\text{tr}(T))c_F(T)}{e_N(T)(\text{det}T)^s},$$

$$(2\pi)^{s} \Gamma(s) \gamma_n(s) \gamma(s, F; \chi) \text{ is holomorphic on } \mathbb{C} \text{ and vanishes at } s \geq 1/2.$$
and
\[ \Lambda(s, F, \chi) = N^{2s} \tau(\chi)^{-1} \gamma_n(s) L(s, F, \chi) \quad (\text{Re}(s) \gg 0), \]
where \( \tau(\chi) \) is the Gauss sum of \( \chi \). Then \( \Lambda(s, F, \chi) \) has an analytic continuation to the whole \( s \)-plane and has the following functional equation:
\[ \Lambda(k - s, F, \chi) = (-1)^{nk/2} \chi(-1) \Lambda(s, F, \chi). \]

**Theorem 2.2.** Let \( F \) and \( \chi \) be as above. Then there exists a finite dimensional \( \overline{\mathbb{Q}} \)-vector space \( V_F \) in \( \mathbb{C} \) such that
\[ L(m, F, \chi) \pi^{-nm} \in V_F \]
for any primitive character \( \chi \) and any integer \( m \) such that \((n + 1)/2 \leq m \leq k - (n + 1)/2\).

Now let
\[ L^*(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/\Gamma \mathcal{L}_n(Z)} \frac{\chi(2^{n/2} \det T)c_F(T)}{e(T)(\det T)^s} \]
be the twisted Koehler-Maaß series of the second kind of \( F \) as in Section 1. We will discuss a relation between these two Dirichlet series in Section 5.

### 3 Review on the algebraicity of L-values of elliptic modular forms of integral and half-integral weight

In this section, we review on the special values of \( L \) functions of elliptic modular forms of integral and half-integral weights. For a modular form \( g(z) \) of integral or half-integral weight for a certain congruence subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \), let \( \mathbb{Q}(g) \) denote the field generated over \( \mathbb{Q} \) by all the Fourier coefficients of \( g \), and for a Dirichlet character \( \eta \) let \( \mathbb{Q}(\eta) \) denote the field generated over \( \mathbb{Q} \) by all the values of \( \eta \).

First let
\[ f(z) = \sum_{m=1}^{\infty} c_f(m)e(mz) \]
be a normalized Hecke eigenform in \( \mathcal{S}_k(SL_2(\mathbb{Z})) \), and \( \chi \) be a primitive Dirichlet character. Then let us define Hecke’s \( L \)-function \( L(s, f, \chi) \) of \( f \) twisted by \( \chi \) as
\[ L(s, f, \chi) = \sum_{m=1}^{\infty} c_f(m)\chi(m)m^{-s}. \]

Then we have the following result (cf. [Sh2]):
Proposition 3.1. There exist complex numbers $u_{\pm}(f)$ uniquely determined up to $Q(f)^{\pm}$ multiple such that

$$\frac{L(m, f, \chi)}{(2\pi \sqrt{-1})^m \tau(u_{\pm}(f))} \in Q(f)Q(\chi)$$

for any integer $0 < m \leq k - 1$ and a primitive character $\chi$, where $\tau(\chi)$ is the Gauss sum of $\chi$, and $j = +$ or $-$ according as $(-1)^m \chi(-1) = 1$ or $-1$.

Corollary. Under the above notation and the assumption, we have

$$L(m, f, \chi) \pi^{-m} \in Q(h_{\pm}(f))$$

for any integer $0 < m \leq k - 1$ and a primitive character $\chi$.

We remark that we have $L(m, f, \chi) \neq 0$ if $m \neq k/2$, and $L(k/2, f, \chi) \neq 0$ for infinitely many $\chi$.

Next let us consider the half-integral weight case. From now on we simply write $\Gamma_0(M)$ as $\Gamma_0(M)$.

Let

$$h_1(z) = \sum_{m=1}^{\infty} c_{h_1}(m)e(mz)$$

be a Hecke eigenform in $\mathfrak{S}_{k_1 + 1/2}(\Gamma_0(4))$, and

$$h_2(z) = \sum_{m=0}^{\infty} c_{h_2}(m)e(mz)$$

be an element of $\mathfrak{M}_{k_2 + 1/2}(\Gamma_0(4))$. For a fundamental discriminant $D$ let $\chi_D$ be the Kronecker character corresponding to $D$. Let $\chi$ be a primitive character mod $N$. Then we define

$$R(s, h_1, h_2, \chi) = L(2s - k_1 + k_2 + 1, \omega) \sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s},$$

where $\omega(d) = \chi_{-4}^{k_1+k_2}\chi^2(d)$. We also define $R(s, h_1, h_2, \chi)$ as

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s}.$$

Now let $S(h_1)$ be the normalized Hecke eigenform in $\mathfrak{S}_{2k_1}(SL_2(\mathbb{Z}))$ corresponding to $h_1$ under the Shimura correspondence. Then the following result is due to Shimura [Sh3].

Proposition 3.2. Assume that $k_1 > k_2$. Under the above notation we have

$$\frac{R(m + 1/2, h_1, h_2, \chi)}{u_-(S(h_1))\tau(\chi^2)\pi^{-k_2+1+2m\sqrt{-1}}} \in Q(h_1)Q(h_2)Q(\chi)$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character $\chi$. 5
Proof. Let $N$ be the conductor of $\chi$. Put

$$h_2(\chi)(z) = \sum_{m=0}^{\infty} c_{h_2}(m)\chi(m)e(mz).$$

Then $h_2(\chi) \in \mathfrak{M}_{k_2+1/2}(4N^2, \chi^2)$. We can regard $h_1$ as an element of $\mathfrak{M}_{k_1+1/2}(I_0(4N^2))$. Then the assertion follows from [[Sh3], Theorem 2].

\[\square\]

Corollary. Assume that $c_{h_1}(n), c_{h_2}(n) \in \mathbb{Q}$ for any $n \in \mathbb{Z}_{\geq 0}$. Then there exists a one-dimensional $\mathbb{Q}$-vector space $U_{h_1,h_2}$ in $\mathbb{C}$ such that

$$\bar{R}(m + 1/2, h_1, h_2, \chi)\pi^{-2m} \in U_{h_1,h_2}$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character $\chi$.

4 Explicit formulas for the twisted Koecher-Maaß series of the second kind of the Duke-Imamoglu-Ikeda lift

Throughout this section, we assume that $n$ and $k$ are even positive integers. Let $h$ be a Hecke eigenform of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$ belonging to the Kohnen plus space. Then $h$ has the following Fourier expansion:

$$h(z) = \sum_{e} c_h(e)e(ez),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \mod 4$. Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m)e(mz)$$

be the normalized Hecke eigenform of weight $2k - n$ for $\text{SL}_2(\mathbb{Z})$ corresponding to $h$ via the Shimura correspondence (cf. [Ko]). For a prime number $p$ let $\beta_p$ be a nonzero complex number such that $\beta_p + \beta_p^{-1} = p^{k+n/2+1/2}c_{S(h)}(p)$. For a non-negative integers $l$ and $m$, the Cohen function $H(l, m)$ is given by $H(l, m) = L_{-m}(1 - l)$. Here

$$L_D(s) = \begin{cases} \zeta(2s - 1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a)\chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D \neq 0, D \equiv 0, 1 \mod 4 \\ 0, & D \equiv 2, 3 \mod 4, \end{cases}$$

6
where the positive integer $f$ is defined by $D = D_K f^2$ with the discriminant $D_K$ of $K = \mathbb{Q}(\sqrt{D})$, $\mu$ is the Möbius function, and $\sigma_s(n) = \sum_{d \mid n} d^s$. Furthermore, for an even integer $l \geq 4$, we define the Cohen Eisenstein series $E_{l+1/2}(z)$ by

$$E_{l+1/2}(z) = \sum_{c=0}^{\infty} H(l, c)e(cez).$$

It is known that $E_{l+1/2}(z)$ is a modular form of weight $l+1/2$ for $\Gamma_0(4)$ belonging to the Kohnen plus space.

For a prime number $p$ let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ be the field of $p$-adic numbers, and the ring of $p$-adic integers, respectively. We denote by $\nu_p$ the additive valuation on $\mathbb{Q}_p$, normalized so that $\nu_p(p) = 1$, and by $e_p$ the continuous homomorphism from the additive group $\mathbb{Q}_p$ to $\mathbb{C}^\times$ such that $e_p(a) = e(a)$ for $a \in \mathbb{Q}$. For a $p$-adic number $c$ put

$$\tilde{\xi}_p(c) = 1, -1 \text{ or } 0$$

according as $\mathbb{Q}_p(\sqrt{c}) = \mathbb{Q}_p, \mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p$ is quadratic unramified, or $\mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p$ is quadratic ramified. We note that $\tilde{\xi}_p(D) = \chi_D(p)$ for a fundamental discriminant $D$. For a non-degenerate half-integral matrix $T$ over $\mathbb{Z}_p$, let

$$b_p(T, s) = \sum_{R \in \text{SL}_2(\mathbb{Q}_p)/\text{SL}_2(\mathbb{Z}_p)} e_p(\text{tr}(TR))p^{-\nu_p(\mu_p(R))^s}$$

be the local Siegel series, where $\mu_p(R) = [R\mathbb{Z}_p^n + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$. Then there exists a polynomial $F_p(T, X)$ in $X$ such that

$$b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \xi_p(T)p^{n/2-s})^{-1} \prod_{i=1}^{n/2}(1 - p^{2i-2s})$$

(cf. [Ki1],) where $\xi_p(T) = \tilde{\xi}_p((-1)^{n/2} \det T)$. For a positive definite half integral matrix $T$ of degree $n$ write $(-1)^{n/2} \det(2T)$ as $(-1)^{n/2} \det(2T) = \nu_T f_T^2$ with $\nu_T$ a fundamental discriminant and $f_T$ a positive integer. We then put

$$c_{I_n(h)}(T) = c_h([\nu_T]) \prod_p (p^{k-n/2-1/2} \beta_p^{-1/2})^{\nu_p(\beta_p)} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).$$

We note that $c_{I_n(h)}(T)$ does not depend on the choice of $\beta_p$. Define a Fourier series $I_n(h)(Z)$ by

$$I_n(h)(Z) = \sum_{T \in \mathcal{L}_{>0}} c_{I_n(h)}(T)e(\text{tr}(TZ)).$$

In [I] Ikeda showed that $I_n(h)(Z)$ is a Hecke eigenform in $\mathfrak{S}_h(Sp_n(\mathbb{Z}))$ and its standard $L$-function $L(s, I_n(h), St)$ is given by

$$L(s, I_n(h), St) = \zeta(s) \prod_{i=1}^{n} L(s+k-i, S(h)).$$

We call $I_n(h)$ the Duke-Imamoglu-Ikeda lift (D-I-I lift) of $h$. 

7
Theorem 4.1. Let \( \chi \) be a primitive Dirichlet character mod \( N \). Then we have

\[
L^*(s, F, \chi) = c_n R(s, h, E_{n/2+1/2}, \chi) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi^2) \\
+ d_n c_h(1) \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), \chi^2),
\]

where \( c_n \) and \( d_n \) are nonzero rational numbers depending only on \( n \).

To prove Theorem 4.1, we reduce the problem to local computations. For \( a, b \in \mathbb{Q}_p^\times \) let \((a, b)_p\) the Hilbert symbol on \( \mathbb{Q}_p \). Following Kitaoka [Ki2], we define the Hasse invariant \( \varepsilon(A) \) of \( A \in \text{Sym}_n(\mathbb{Q}_p)^\times \) by

\[
\varepsilon(A) = \prod_{1 \leq i \leq j \leq n} (a_i, a_j)_p
\]

if \( A \) is equivalent to \( a_1 \perp \cdots \perp a_n \) over \( \mathbb{Q}_p \) with some \( a_1, a_2, \ldots, a_n \in \mathbb{Q}_p^\times \). For \( T \in \text{Sym}_n(\mathbb{Z}_p) \), put \( T^{(0)} = 2^{-1} T, F_p^{(0)}(T, X) = F_p(T^{(0)}, X) \), and so on. Then for non-degenerate symmetric matrices \( A \) of degree \( n \) with entries in \( \mathbb{Z}_p \) we define the local density \( \alpha_p(A) = \alpha_p(A, A) \) representing \( A \) by \( A \) as

\[
\alpha_p(A) = 2^{-1} \lim_{a \to \infty} p^{a(-n^2 + n(n+1)/2)} \# A_n(A, A),
\]

where

\[
A_n(A, A) = \{ X \in M_n(\mathbb{Z}_p)/p^a M_n(\mathbb{Z}_p) \mid A[X] - B \in p^a S_n(\mathbb{Z}_p) \},
\]

Furthermore put

\[
M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{\varepsilon(A')}
\]

for a positive definite symmetric matrix \( A \) of degree \( n \) with entries in \( \mathbb{Z} \), where \( \mathcal{G}(A) \) denotes the set of \( SL_n(\mathbb{Z}) \)-equivalence classes belonging to the genus of \( A \). Then by Siegel’s main theorem on the quadratic forms, we obtain

\[
M(A) = \kappa_n 2^{2-n} \det A^{(n+1)/2} \prod_p \alpha_p(A)^{-1}
\]

where \( e_n = 1 \) or \( 2 \) according as \( n = 1 \) or not, and \( \kappa_n = \prod_{i=1}^{n/2} \Gamma_C(2i) \) (cf. Theorem 6.8.1 in [Ki2]). Put

\[
\mathcal{F}_p = \{ d_0 \in \mathbb{Z}_p \mid \nu_p(d_0) \leq 1 \}
\]

if \( p \) is an odd prime, and

\[
\mathcal{F}_2 = \{ d_0 \in \mathbb{Z}_2 \mid d_0 \equiv 1 \text{ mod } 4, \text{ or } d_0/4 \equiv -1 \text{ mod } 4, \text{ or } \nu_2(d_0) = 3 \}.
\]
For \( d \in \mathbb{Z}_p^\times \) put

\[
S_n(\mathbb{Z}_p, d) = \{ T \in S_n(\mathbb{Z}_p) \mid (-1)^{n/2} \det T = p^{2i}d \text{ mod } \mathbb{Z}_p^\times \text{ with some } i \in \mathbb{Z} \},
\]

and \( S_n(\mathbb{Z}_p, d) = S_n(\mathbb{Z}_p, d) \cap S_n(\mathbb{Z}_p)_x \) for \( x = e \) or \( o \). Put \( \mathcal{L}^{(0)}_{n,p} = S_n(\mathbb{Z}_p)_e \) and \( \mathcal{L}^{(0)}_{n,p}(d) = S_n(\mathbb{Z}_p, d) \cap \mathcal{L}^{(0)}_{n,p} \). Let \( \iota_{n,p} \) be the constant function on \( \mathcal{L}^{(0)}_{n,p} \) taking the value 1, and \( \varepsilon_{n,p} \) the function on \( \mathcal{L}^{(0)}_{n,p} \) assigning the Hasse invariant of \( A \) for \( A \in \mathcal{L}^{(0)}_{n,p} \). We sometimes drop the suffix and write \( \iota_{n,p} \) as \( \iota \) or \( \epsilon \) and the others if there is no fear of confusion. From now on we sometimes write \( \omega = \epsilon^{l} \) with \( l = 0 \) or 1 according as \( \omega = \epsilon \) or \( \epsilon \). For \( d_0 \in \mathcal{F} \) and \( \omega = \epsilon^l \) with \( l = 0, 1 \), we define a formal power series \( P^{(l)}_{n,p}(d_0, \omega, X, t) \) in \( t \) by

\[
P^{(l)}_{n,p}(d_0, \omega, X, t) = \kappa(d_0, n, l)^{-1} \sum_{B \in \mathcal{L}^{(0)}_{n,p}(d_0)} \frac{ar{F}^{(l)}_{n,p}(B, X)}{\alpha_p(B)} \omega(B) t^{\nu_p(\det B)},
\]

where

\[
\kappa(d_0, n, l) = \kappa(d_0, n, l)_p = \{(-1)^{n(n+2)/8}(-1)^{n/2}d_0\}^{1/2}.
\]

Let \( \mathcal{F} \) denote the set of fundamental discriminants, and for \( l = \pm 1 \), put

\[
\mathcal{F}^{(l)} = \{ d_0 \in \mathcal{F} \mid l d_0 > 0 \}.
\]

**Theorem 4.2.** Let the notation and the assumption be as above. Then for \( \text{Re}(s) \gg 0 \), we have

\[
L^*(s, \mathcal{I}_n(h)) = \kappa_n 2^{ns+1-n} \times \sum_{d_0 \in \mathcal{F}^{(-1)^{n/2}}} c_h([d_0]) d_0^{n/4-k/2+1/4} \prod_p P^{(l)}_{n,p}(d_0, \iota_p, \alpha_p, \rho^{s+k/2+n/4+1/4} \chi(p)) + (-1)^{n(n+2)/8} \times \sum_{d_0 \in \mathcal{F}^{(-1)^{n/2}}} \{(-1)^{n/2}d_0\} \chi_h([d_0]) d_0^{n/4-k/2+1/4} \prod_p P^{(l)}_{n,p}(d_0, \varepsilon_p, \alpha_p, \rho^{s+k/2+n/4+1/4} \chi(p)).
\]

**Proof.** Let \( T \in S_n(\mathbb{Z}_p)_{e>0} \). Then the \( T \)-th Fourier coefficient \( c_{\mathcal{I}_n(h)}(T) \) of \( \mathcal{I}_n(h) \) is uniquely determined by the genus to which \( T \) belongs, and, by definition, it can be expressed as

\[
c_{\mathcal{I}_n(h)}(T) = c_h([T]) \bar{F}^{(0)}(T, \alpha_p)
\]

We also note that

\[
\bar{F}^{(0)}(T, \alpha_p) = [T]^{-k/2-n/4+1/4} (\det T)^{k/2-n/4-1/4}
\]

9
for $T \in S_n(Z_p)_{\varepsilon > 0}$. Hence we have

$$\sum_{T' \in \mathcal{G}(T)} \frac{c_{L_0}(T')}{e(T')} = \det T^{k/2 + n/4 - 1/4} [e_0^{(0)}(T)^{k/2 - n/4 - 1/4} \prod_p \tilde{F}_p^{(0)}(T, \alpha_p)] \alpha_p(T).$$

Thus, similarly to [I-K1], Theorem 3.3, (1), and [I-K2], Theorem 3.2, we obtain

$$L(s, I_n(h)) = \kappa_n 2^{n+1-n} \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} c_h([d_0]) |d_0|^{n/4-k/2+1/4}$$

$$\times \left\{ \prod_p P_{n,p}^{(0)}(d_0, t_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)) \right. \left. + (-1)^{n(n+2)/8} (-1)^{n/2} \prod_p P_{n,p}^{(0)}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)) \right\}.$$ 

This proves the assertion.

Proposition 4.3. Let $d_0 \in \mathcal{F}_p$ and $\xi_0 = \hat{\xi}(d_0)$. Then

$$P_n^{(0)}(d_0, t, X, t) = \frac{(p^{-1} t)^{v_p(d_0)}}{\phi_{n/2-1}(p^{-2})(1 - p^{-n/2} \xi_0)}$$

$$\times \frac{(1 + t^2 p^{-n/2-3/2})(1 + t^2 p^{-n/2-5/2} \xi_0^2) - \xi_0^2 t^2 p^{-n/2-2}(X + X^{-1} + p^{1/2-n/2} + p^{-1/2+n/2})}{(1 - p^{-2} t^2)(1 - p^{-2} X^{-1} t^2) \prod_{i=1}^{n/2}(1 - t^2 p^{-2i} X)(1 - t^2 p^{-2i} X^{-1})},$$

and

$$P_n^{(0)}(d_0, \varepsilon, X, t) = \frac{1}{\phi_{n/2-1}(p^{-2})(1 - p^{-n/2} \xi_0) \prod_{i=1}^{n/2}(1 - t^2 p^{-2i} X)(1 - t^2 p^{-2i} X^{-1})} \frac{\xi_0^2}{\xi_0}.$$

Proof. Put $H_k = \left( \begin{array}{cc} O & 1_k \\ 1_k & O \end{array} \right)$, and for $d \in Z_p^*$ put

$$D = \{ x \in M_{2k,n}(\mathbb{Z}_p) \mid \det(H_k[x]) \in dp^*\mathbb{Z}_p^* \text{ with some } i \in \mathbb{Z}_{\geq 0} \}.$$

We then define $Z_{2k}(u, \varepsilon, d)$ as

$$Z_{2k}(u, \varepsilon, d) = \int_D \varepsilon(H_k[x]) |\det(H_k[x])|_{p}^{s-k} dx$$

with $u = \tilde{s}$, where $|\cdot|_p$ denotes the normalized valuation on $\mathbb{Q}_p$, and $dx$ is the measure on $M_{2k,n}(\mathbb{Q}_p)$ normalized so that the volume of $M_{2k,n}(\mathbb{Z}_p)$ is 1. Moreover put

$$Z_{2k,\varepsilon}(u, \varepsilon, d) = \frac{1}{2} (Z_{2k,n}(u, \varepsilon, d) + Z_{2k,n}(-u, \varepsilon, d)).$$
and
\[ Z_{2k,0}(u, \varepsilon^l, d) = \frac{1}{2} (Z_{2k,n}(u, \varepsilon^l, d) - Z_{2k,n}(-u, \varepsilon^l, d)). \]

Then it is well known that
\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^n/2 p^{-\nu_p(d_0)} d_0) = \phi_n(p^{-1}) \sum_{T \in L_{n,p}^{(0)}} \frac{b_p(2^{-\delta_p T}, p^{-k})}{\alpha_p(T)} (p^k t)^{\nu_p(\det(T)}} \]
for \(d_0 \in \mathcal{F}_p\), where \(x(d_0) = e\) or \(o\) according as \(\nu_p(d_0)\) is even or odd. Recall that
\[ b_p(2^{-\delta_p T}, p^{-k}) = \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-k+2i})}{1 - \xi(2^{-\delta_p T} p^{-k+n/2})} F_p(0)^{(T, p^{-k})} \]
and
\[ F_p(0)^{(T, p^{-k})} = p^{-k/2+\left(n+1\right)/4} \nu_p(\det T) \nu_p(d_0) F_p(0)^{(T, p^{-k+n/2})}. \]

Hence we have
\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^n/2 p^{-\nu_p(d_0)} d_0) = \phi_n(p^{-1}) \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-k+2i})}{1 - \xi(2^{-\delta_p T} p^{-k+n/2})} \]
\[ \times p^{k/2\left(n+1\right)/4} \nu_p(d_0) P_n^{(0)}(d_0, \varepsilon^l, p^{-k+n/2}) T(d_0, \varepsilon^l, p^{-k+n/2+1/4}). \]

Let \(T(d_0, \omega, X, t)\) denote the right-hand side of the formula for \(\omega = \varepsilon^l\) \((l = 0, 1)\) in the proposition. Then, by [[Sai2, Theorem 3.4 (2)]], we have
\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^n/2 p^{-\nu_p(d_0)} d_0) = \phi_n(p^{-1}) \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-k+2i})}{1 - \xi(T) p^{-k+n/2}} \]
\[ \times p^{k/2\left(n+1\right)/4} \nu_p(d_0) T(d_0, \varepsilon^l, p^{-k+n/2+1/4}). \]
(Remark that there are misprints in [Sai2]; the \((q^{-1})_n\) on page 197, lines 9 and 15 should be \((q^{-1})_{n}r_\) Hence we have
\[ P_n^{(0)}(d_0, \varepsilon^l, p^{-k+n/2+1/4}) = T(d_0, \varepsilon^l, p^{-k+n/2+1/4}) \]
for infinitely many positive integers \(k\). Hence we have
\[ P_n^{(0)}(d_0, \varepsilon^l, X, t) = T(d_0, \varepsilon^l, X, t). \]

\[ \Box \]

**Proof of Theorem 4.1.**

Put \(\Omega = \{\omega_p\}\), and let \(d_0 \in \mathcal{F}((-1)^n/2)\). Put
\[ P(s, d_0, \Omega, \chi) = \prod_p P_n^{(0)}(d_0, \varepsilon^l, p^{-s+k/2+n/4+1/4}) \chi(p). \]
Then by Proposition 4.3, we have

\[ P(s, d_0, \{ t_p \}, \chi) \]

\[ = |d_0|^{-s+k/2+n/4-3/4} \chi(d_0) \prod_{i=1}^{n/2-1} \zeta(2i)L(n/2, \chi_{d_0}) \prod_{i=0}^{n/2} L(2s+2i-n, S(h), \chi^2) \]

\[ \times L(2s-n+1, S(h), \chi^2) \prod_p (1 + \pi^{-2s+k-1} \chi(p)^2)(1 + \chi_{d_0}(p)^2 \pi^{-2s+k-2} \chi(p)^2) \]

\[ - \chi_{d_0}(p)^3 \pi^{-2s+k-3/2} \chi(p)^2 \beta_p(1 + \pi^{1/2-n/2} \beta_p^{-1})(1 + \pi^{-1/2+n/2} \beta_p^{-1}) \}

We note that \( L(s, h) \) and \( L(s, E_{n/2+1}) \) can be expressed as

\[ L(s, h) = L(2s, S(h)) \sum_{d_0 \in F^{(-1)^{n/2}}} \zeta(2s)c(d_0)|d_0|^{-s} \prod_p (1 - \chi_{d_0}(p)^2 \pi^{-2s-k-1} \chi(p)^2), \]

and therefore, we easily see that \( L(s, h, E_{n/2+1/2}, \chi) \) can be expressed as

\[ L(s, h, E_{n/2+1/2}, \chi) = L(2s, S(h), \chi^2)L(2s-n+1, S(h), \chi^2) \]

\[ \times \sum_{d_0 \in F^{(-1)^{n/2}}} |d_0|^{-s} c(d_0) \chi(d_0) L(1-n/2, \chi_{d_0}) \]

\[ \times \prod_p ((1 + \pi^{-2s+k-1} \chi(p)^2)(1 + \chi_{d_0}(p)^2 \pi^{-2s+k-2} \chi(p)^2) \]

\[ - \chi_{d_0}(p)^3 \pi^{-2s+k-3/2} \chi(p)^2 \beta_p(1 + \pi^{1/2-n/2} \beta_p^{-1})(1 + \pi^{-1/2+n/2} \beta_p^{-1}) \}

(cf. [Sh1], Lemma 1.) Thus, by remarking the functional equation

\[ L(1-n/2, \chi_{d_0}) = 2^{1-n/2} \pi^{-n/2} \Gamma(n/2)|d_0|^{(n-1)/2} L(n/2, \chi_{d_0}), \]

we have

\[ \sum_{d_0 \in F^{(-1)^{n/2}}} c_h(|d_0|)|d_0|^{-s+k/2+n/4+1/4} P(s, d_0, \{ t_p \}, \chi) \]

\[ = \prod_{i=1}^{n/2-1} \zeta(2i) \frac{2^{n/2-1} \pi^{n/2}}{\Gamma(n/2)} L(s, h, E_{n/2+1/2}; \chi) \prod_{i=1}^{n/2-1} L(2s-2i+n, S(h), \chi^2). \]

On the other hand, if \( d_0 \neq 1 \), by Proposition 4.3, we have

\[ P(s, d_0, \{ t_p \}, \chi) = 0. \]
Thus if \( n \equiv 2 \mod 4 \), for any \( d_0 \in \mathcal{F}^{(-1)^{n/2}} \),
\[
P(s, d_0, \{\varepsilon_p\}, \chi) = 0.
\]
If \( n \equiv 0 \mod 4 \), by Proposition 4.3, we have
\[
P(s, 1, \{\varepsilon_p\}, \chi) = \zeta(n/2) \prod_{i=1}^{n/2-1} \zeta(2i) \prod_{i=1}^{n/2} L(2s - 2i + 1, S(h), \chi^2).
\]
Thus the assertion follows from Theorem 4.2.

5 Relation between twisted K-M series of the first and second kinds

Let \( N \) be a positive integer. Let \( g \) be a periodic function on \( \mathbb{Z} \) with a period \( N \) and \( \phi \) a polynomial in \( t_1, \ldots, t_r \). Then for an element \( u = (a_1 \mod N, \ldots, a_r \mod N) \in (\mathbb{Z}/N\mathbb{Z})^r \), the value \( g(\phi(a_1, \ldots, a_r)) \) does not depend on the choice of the representative of \( u \). Therefore we denote this value by \( g(\phi(u)) \). In particular we sometimes regard a Dirichlet character mod \( N \) as a function on \( \mathbb{Z}/N\mathbb{Z} \).

For a Dirichlet character \( \chi \mod N \) and \( A \in \mathcal{L}_{m>0} \), put
\[
h(A, \chi) = \sum_{U \in SL_m(\mathbb{Z}/N\mathbb{Z})} \chi(tr(A[U])).
\]
As was shown in [[K-M], Proposition 3.3], the twisted Koecher-Maaß series of the first kind of a Siegel modular form can be expressed in terms of \( h(A, \chi) \) as stated later. Therefore we shall compute \( h(A, \chi) \) in the case where \( A \) is an element of \( \mathcal{L}_{m>0} \). For \( A = (a_{ij})_{m \times m} \in \mathcal{S}_m(\mathbb{Z}/N\mathbb{Z}) \) and \( c \in \mathbb{Z}/N\mathbb{Z} \), put
\[
R_N(A, c) = \{X = (x_{ij})_{m \times m} \in M_n(\mathbb{Z}/N\mathbb{Z}) \mid \sum_{i=1}^m \sum_{\alpha, \beta = 1}^m a_{\alpha, \beta}x_{i\alpha}x_{i\beta} - c = 0
\]
\[
\text{and } \det X - 1 = 0 \}.
\]
Then we have
\[
h(A, \chi) = \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \chi(c) \#(R_N(A, c)).
\]
From now on let \( p \) be an odd prime number and \( F_p \) be the field with \( p \)-elements. For \( S \in \mathcal{S}_m(F_p) \) and \( T \in \mathcal{S}_r(F_p) \) put
\[
A(S, T) = \{Y = M_{r,m}(F_p) \mid YS \cdot Y = T \}.
\]
For an element \( S \in \mathcal{S}_m(F_p) \) with \( m \) even put \( \chi(S) = \left(\frac{(-1)^{m/2} \det S}{p}\right) \).
Lemma 5.1. Let $S \in S_m(F_p)^\times$.

1. Let $T \in S_r(F_p)$ with $m \geq r$.
   1.1. Let $r$ be even. Then
   \[
   \#A(S, T) = p^{r m - r (r+1)/2} (1 - \chi(S) p^{-m/2}) (1 + \chi(\langle -S \rangle \cap T) p^{(r-m)/2}) \prod_{m-r+1 \leq e \leq m-1} (1 - p^{-e})
   \]
   or
   \[
   \#A(S, T) = p^{r m - r (r+1)/2} \prod_{m-r+1 \leq e \leq m-1} (1 - p^{-e})
   \]
   according as $m$ is even or odd.
   1.2. Let $m$ be odd. Then
   \[
   \#A(S, T) = p^{r m - r (r+1)/2} (1 + \chi(\langle -S \rangle \cap T) p^{(r-m)/2}) \prod_{m-r+1 \leq e \leq m-1} (1 - p^{-e})
   \]
   or
   \[
   \#A(S, T) = p^{r m - r (r+1)/2} \prod_{m-r+1 \leq e \leq m-1} (1 - p^{-e})
   \]
   according as $m$ is even or odd. In particular, for $c \in F_p^\times$, we have
   \[
   \#A(S, c) = p^{m/2 - 1} (p^{m/2} - \left(\frac{-1}{p} \det S\right)^{\frac{m}{2}})
   \]
   or
   \[
   \#A(S, c) = p^{(m-1)/2} \left(\frac{-1}{p} \det S\right)^{\frac{(m-1)}{2}} + \left(\frac{-1}{p} \det S\right)^{\frac{(m-1)}{2}}
   \]
   according as $m$ is even or odd.

2. We have
   \[
   \#A(S, 0) = p^{m/2 - 1} (p^{m/2} - \left(\frac{-1}{p} \det S\right)^{\frac{m}{2}}) + p^{m/2} \left(\frac{-1}{p} \det S\right)^{\frac{m}{2}}
   \]
   or
   \[
   \#A(S, 0) = p^{m-1}
   \]
   according as $m$ is even or odd.

Proof. The assertions (1) and (2) follow from [[Ki1], Theorem 1.3.2], and [[Ki1], Lemma 1.3.1], respectively.

Proposition 5.2. Let $A = a_1 \perp \cdots \perp a_m$ with $a_i \in F_p$. For $c \in F_p^\times$ put
\[
\mathcal{M}_p(A, c) = \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\]
\[
\gamma_{m,p} = p^{m^2-m(m+1)/2} (1-p^{-m/2}) \prod_{e=1}^{(m-2)/2} (1-p^{-2e})
\]

or

\[
\gamma_{m,p} = p^{m^2-m(m+1)/2} \prod_{e=1}^{(m-1)/2} (1-p^{-2e})
\]

depending as \( m \) is even or odd. Then we have

\[
\#\mathcal{R}_p(A, c) = \gamma_{m,p} \#\mathcal{M}_p(A, c).
\]

**Proof.** Let \( \Phi : GL_m(F_p) \rightarrow S_m(F_p) \cap GL_m(F_p) \) be the mapping defined by \( \Phi(X) = X^tX \). Then by Lemma 5.1, we have \( \#\Phi^{-1}(Z) = 2^\gamma_{m,p} \) for any \( Z \in S_m(F_p) \cap SL_m(F_p) \). We note that \( \det X = \pm 1 \) for any \( X \in \Phi^{-1}(Z) \). Hence we have \( \#(\Phi^{-1}(Z) \cap SL_m(F_p)) = \gamma_{m,p} \). Moreover we have

\[
\text{tr}(XAX) = \text{tr}(AX^tX),
\]

and hence \( X \in \mathcal{R}_p(A, c) \) if and only if \( \Phi(X) \in \mathcal{M}_p(A, c) \). This proves the assertion. \( \square \)

We rewrite \( \mathcal{M}_p(A, c) \) in more concise form. Let \( p \) be a prime number and \( l \) be a positive integer dividing \( p-1 \). Take an \( l \)-th root of unity \( \zeta_l \) and a prime ideal \( \mathfrak{p} \) of \( \mathbb{Q}(\zeta_l) \) lying above \( p \). Let \( a \) be an integer prime to \( p \). Then we have \( a^{(p-1)/l} \equiv \zeta_i^j \) mod \( \mathfrak{p} \) with some \( i \in \mathbb{Z} \). We then put \( \left( \frac{a}{p} \right)_l = \zeta_i^j \). We call \( \left( \frac{a}{p} \right)_l \) the \( l \)-th power residue symbol mod \( p \). In the case \( l = 2 \), this is the Legendre symbol, and we write it as \( \left( \frac{a}{p} \right) \) as usual. We note that this definition of the power residue symbol is different from the usual one, and depends on the choice of \( \zeta_l \) except the case \( l = 2 \). We denote by \( \left( \frac{\cdot}{N} \right) \) the Jacobi symbol for a positive odd integer. Let \( \chi \) be a primitive Dirichlet character of conductor \( N \). We assume that \( N \) is a square free odd integer, and write \( N = p_1 \cdot \cdots \cdot p_r \) with \( p_1, \cdots, p_r \) prime numbers. Put \( l_j = l_{m,p_j} = \text{GCD}(m, p_j - 1) \). For an \( r \)-tuple \( I = (i_1, i_2, \ldots, i_r) \) of integers put

\[
\chi_{(i_1, \ldots, i_r)} = \chi \prod_{j=1}^r \left( \frac{\cdot}{p_j} \right)_{l_j}^{i_j}.
\]

For two Dirichlet characters \( \chi \) and \( \eta \) mod \( N \) we define \( J_m(\chi, \eta) \) and \( I_m(\chi, \eta) \)

\[
J_m(\chi, \eta) = \sum_{Z \in S_m(Z/N)} \chi(\det Z)\eta(1 - \text{tr}(Z))
\]

and

\[
I_m(\chi, \eta) = \sum_{Z \in S_m(Z/N)} \chi(\det Z)\eta(\text{tr}(Z)).
\]
By definition, \( J_m(\chi, \eta) \) is an algebraic number. We note that \( J_1(\chi, \eta) \) is the Jacobi sum \( J(\chi, \eta) \) associated with \( \chi \) and \( \eta \). We also define \( J_m(\chi) \) as \( J_m(\chi) = J_m(\chi, \chi) \).

**Lemma 5.3.** Let \( \eta \) be a primitive character mod \( p \). Let \( c \in F_p \) and \( S \in \mathfrak{S}_l(F_p) \) of rank \( r \). Let \( S \sim S_0 \perp O_{l-r} \) with \( \det S_0 \neq 0 \). Put

\[
I_{\eta, S, c} = \sum_{w \in F_p^l} \eta(S[w] + c).
\]

Assume that \( r \) is odd, and that \( \eta^2 \neq 1 \). Then

\[
I_{\eta, S, c} = p^{l-(r+1)/2} J(\eta, \left( \frac{\star}{p} \right)) \left( \frac{-1}{p} \right)^{(r+1)/2} \frac{\det S_0}{p} \eta(c) \left( \frac{c}{p} \right).
\]

Assume that \( r \) is even, and that \( \eta \neq 1 \). Then

\[
I_{\eta, S, c} = p^{l-r/2} \left( \frac{-1}{p} \right)^{r/2} \frac{\det S_0}{p} \eta(c).
\]

Here we make the convention that \( \left( \frac{-1}{p} \right)^{r/2} \frac{\det S_0}{p} = 1 \) if \( r = 0 \).

**Proof.** We have

\[
I_{\eta, S, c} = p^l I_{\eta, S_0, c}.
\]

Hence we may assume that \( r = l \). Then

\[
I_{\eta, S, c} = \sum_{u \in F_p} \eta(u) \#A(S, u - c).
\]

Let \( l \) be odd. Then by Lemma 5.1,

\[
\#A(S, u - c) = p^{l-1/2} (p^{l-1/2} + \left( \frac{-1}{p} \right)^{(l-1)/2} \frac{\det S}{p}).
\]

Hence we have

\[
I_{\eta, S, c} = p^{l-1/2} \left( \frac{-1}{p} \right)^{(l+1)/2} \frac{\det S}{p} \sum_{u \in F_p} \eta(u) \left( \frac{u - c}{p} \right).
\]

Since \( \eta^2 \) is nontrivial, we have \( I_{\eta, S, c} = 0 \) if \( c = 0 \). If \( c \neq 0 \), then

\[
\sum_{u \in F_p} \eta(u) \left( \frac{u - c}{p} \right) = \left( \frac{-c}{p} \right) \sum_{u \in F_p} \eta(u) \left( \frac{1-c^{-1}u}{p} \right)
\]

\[
= \eta(c) \left( \frac{-c}{p} \right) \sum_{u \in F_p} \eta(u) \left( \frac{1-u}{p} \right) = \eta(c) \left( \frac{-c}{p} \right) J(\eta, \left( \frac{\star}{p} \right)).
\]

16
Let \( l \) be even. Then

\[
\#A(S, u - c) = (p^{l/2} - \left( \frac{(-1)^{l/2} \det S}{p} \right))p^{l/2 - 1} + p^{l/2} \left( \frac{(-1)^{l/2} \det S}{p} \right) a_0,
\]

where \( a_0 = 1 \) or \( 0 \) according as \( u = c \) or not. Hence

\[
I_{\eta, S, c} = p^{l/2} \left( \frac{(-1)^{l/2} \det S}{p} \right) \eta(c).
\]

\[\square\]

**Corollary.** Let \( d \in F_p^\times \). Then we have

\[
I_{\eta, S, cd} = \eta(d) \left( \frac{d}{p} \right) I_{\eta, S, c}.
\]

**Proposition 5.4.** Let \( \eta \) be a primitive character mod \( p \). For \( Z_1 \in S_{l-1}(F_p) \) and \( z_{l+1} \in F_p \), put

\[
I(Z_1, z_{l+1}) = \sum_{w \in M_{l-1}(F_p)} \eta \left( \begin{pmatrix} 1 & w \\ t & z_{l+1} \end{pmatrix} \right).
\]

(1) Assume that \( l \) is even, and that \( \eta^2 \neq 1 \). Then

\[
I(Z_1, z_{l+1}) = p^{(l-2)/2} J_{\eta, \left( \frac{\chi}{p} \right)} \left( \frac{(-1)^{l/2} \det Z_1}{p} \right) \eta(\det Z_1 z_{l+1}) \left( \frac{z_{l+1}}{p} \right).
\]

(2) Assume that \( l \) is odd, and that \( \eta^2 \neq 1 \). Then

\[
I(Z_1, z_{l+1}) = p^{(l-1)/2} \left( \frac{(-1)^{(l-1)/2} \det Z_1}{p} \right) \eta(\det Z_1 z_{l+1})
\]

**Proof.** We note that

\[
\det \begin{pmatrix} Z_1 & w \\ t & z_{l+1} \end{pmatrix} = -\text{Adj}(Z_1)[w] + \det Z_1 z_{l+1},
\]

where \( \text{Adj}(Z_1) \) is the \((l - 1) \times (l - 1)\) matrix whose \((i, j)\)-th component is the \((j, i)\)-th cofactor of \( Z_1 \). We also note that \( \det(-\text{Adj}(Z_1)) = (-1)^{l-1}(\det Z_1)^{l-2} \). Thus the assertion follows directly from Lemma 5.3 if \( \det Z_1 \neq 0 \). If \( \det Z_1 = 0 \), then \( \text{rank}_{F_p}(Z_1) \leq 1 \), the assertion follows also from Lemma 5.3. \( \square \)

**Theorem 5.5.** Let \( \chi \) be a primitive character mod \( p \). Let \( l = \text{GCD}(m, p - 1) \), and \( u_0 \) be a primitive \( l \)-th root of unity mod \( p \). Let \( A \in S_m(F_p) \).

(1) If \( \chi(u_0) \neq 1 \), then we have \( h(A, \chi) = 0 \).

(2) Assume that \( \chi(u_0) = 1 \). Fix a character \( \bar{\chi} \) such that \( \bar{\chi}^m = \chi \).

(2.1) Let \( m \) be even. Then

\[
h(A, \chi) = \gamma_{m, p} \sum_{i=0}^{l-1} A_{m, i, p} \bar{\chi}(i)(\det A)J_{m-1}(\bar{\chi}(i)),
\]
where $A_{m,i,p} = p^{(m-2)/2}(-1)^{m(p-1)/4} J(\chi(i), \left( \begin{smallmatrix} z \\ p \end{smallmatrix} \right))$.

(2.2) Let $m$ be odd and assume that $\chi^2 \neq 1$. Then

$$h(A, \chi) = \gamma_{m,p} \sum_{i=0}^{l-1} A_{m,i,p} \tilde{\chi}(i) (\det A) J_{m-1}(\tilde{\chi}(i)),$$

where $p^{(m-1)/2}(-1)^{(m-1)(p-1)/4}$.

**Proof.** If $A = O_m$ then we have $h(A, \chi) = 0$. Hence we assume that $A \neq O_m$. Then we may assume that $A = a_1 \perp \cdots \perp a_{m-1} \perp d$ with $d \neq 0$. Put

$$\tilde{M}_p(A, c) = \{(Z_1, w) \in S_m(F_p) \times M_{m-1,1}(F_p) \mid \det \left( \begin{array}{cc} Z_1 & w \\ t & -a \end{array} \right) d^{-1}(1 - \sum_{i=1}^{m-1} a_i z_i) \} e_m = 1 \}. $$

Write $Z \in S_m(F_p)$ as $Z = \left( \begin{smallmatrix} Z_1 & W \\ t & z \end{smallmatrix} \right)$ with $Z_1 \in S_{m-1}(F_p), w \in M_{m-1,1}(F_p), z \in F_p$. Then the mapping $S_m(F_p) \ni Z \mapsto (c^{-1}Z_1, c^{-1}w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p)$ induces a bijection from $\tilde{M}_p(A, c)$ to $\tilde{M}_p(A, c)$, and hence $\#\tilde{M}_p(A, c) = \#M_p(A, c)$. Put

$$K(A) = \sum_{c} \#\tilde{M}_p(A, c) \chi(c).$$

Assume that $\chi(u_0) \neq 1$. Then we have

$$K(A) = \sum_{c \in F_p} \chi(c u_0) \#\tilde{M}_p(A, c u_0).$$

We note that $\tilde{M}_p(A, c u_0) = \tilde{M}_p(A, c)$. Hence we have

$$K(A) = \chi(u_0) K(A).$$

Hence we have $K(A) = 0$.

Assume that $\chi(u_0) = 1$. Then we can take a Dirichlet character $\tilde{\chi}$ such that $\tilde{\chi}^m = \chi$. First assume that $\det A = 0$. Then we may assume that we have $A = A_0 \perp 0$ with $A_0 \in S_{m-1}(F_p)$. Let $P_{m-1,m}$ be the set of $(m-1) \times m$ matrices with entries in $F_p$ of rank $m-1$. Then for each $X_1 \in P_{m-1,m}$ there exist exactly $p^{m-1}$ elements $X_2 \in M_{1,m}(F_p)$ such that $\left( \begin{array}{c} X_1 \\ X_2 \end{array} \right) \in SL_m(F_p)$. Hence we have

$$h(A, \chi) = p^{m-1} \sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1]).$$

Let $m$ be even. Then we can take an element $\alpha \in F_p^\times$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_{m}(F_p)$ such that $^tU_0U_0 = \alpha I_m$ in view of (1.1) of Lemma 5.1. Hence

$$h(A, \chi) = p^{m-1} \sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1U_0]) = \chi(\alpha) h(A, \chi).$$

18
Hence we have $h(A, \chi) = 0$. Let $m$ be odd and assume that $\chi^2 \neq 1$. Then we can take an element $\alpha \in (F_p^\times)^2$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that $U_0U_0 = \alpha 1_m$ in view of (1.2) of Lemma 5.1. Thus by the same argument as above we have $h(A, \chi) = 0$. This proves the assertion.

Next assume that $\det A \neq 0$. We may assume that

$$A = 1_{m-1} \downarrow d$$

with $d = \det A$. Then we have

$$K(A) = \sum_c \# M_p(A, c) \chi(c^m).$$

Hence we have

$$K(A) = \sum_{(Z_1, w)} \chi(\det \left( Z_1 \begin{pmatrix} w & \cdot \\ 1 & \cdot \end{pmatrix} \right)),

$$

where $(Z_1, w)$ runs over elements of $S_{m-1}(F_p) \times M_{m-1, 1}(F_p)$ such that

$$\chi(z) = 1 \quad \text{for all } z \in F_p^\times.

$$

Hence by Proposition 5.4 we have

$$K(A)_{l} = \sum_{i=0}^{l-1} \chi_{(i)}(\det A) \chi^*_{(i)}(\det Z_1) \chi^*_{(i)}(1 - \tr Z_1),$$

We note that $\chi^2_{(i)} \neq 1$ for any $i$. Hence by Proposition 5.4 we have

$$K(A)_{l} = A_{m, i, p} \sum_{Z_1 \in S_{m-1}(F_p)} \chi^*_{(i)}(\det A) \chi^*_{(i)}(\det Z_1) \chi^*_{(i)}(1 - \tr Z_1),$$

$19$
where \( \widehat{\chi}_{i}^{\ast} = \widehat{\chi}(i) \left( \frac{2}{p} \right)^{m-1} \). This proves the assertion if \( m \) is odd. Assume that \( m \) is even. Then it is easily seen that the set \( \{ \widehat{\chi}(i) \}_{i=0}^{l-1} \) of Dirichlet characters coincides with \( \{ \widehat{\chi}(i) \}_{i=0}^{l-1} \). Moreover \( \widehat{\chi}^{2} \neq 1 \) for any \( i \). This proves the assertion.

**Theorem 5.6.** Let \( N = p_{1} \cdots p_{r} \). Let \( \chi \) be a primitive Dirichlet character mod \( N \). Let \( u_{0,i} \) be a primitive \( l_{i} \)-th root of unity mod \( p_{i} \). Let \( A \in S_{m}(F_{p}) \).

1. If \( \chi^{(p_{i})}(u_{0,i}) \neq 1 \) for some \( i \). Then we have \( h(A, \chi) = 0 \).
2. Assume that \( \chi^{(p_{i})}(u_{0,i}) = 1 \) for any \( i \). Fix a character \( \tilde{\chi} \) such that \( \tilde{\chi}^{m} = \chi \).

(2.1) Let \( m \) be even. Then we have

\[
h(A, \chi) = \prod_{i=1}^{r} (-1)^{m(p_{i}-1)/2} p_{i}^{(m-2)/2} \gamma_{m, p_{i}}
\]

\[
\times \sum_{i_{1}=0}^{l_{1}} \cdots \sum_{i_{r}=0}^{l_{r}} \tilde{\chi}(i_{1}, i_{2}, \ldots, i_{r}) (\det A) J(\tilde{\chi}(i_{1}, i_{2}, \ldots, i_{r})) \left( \frac{\ast}{N} \right) J_{m-1}(\tilde{\chi}(i_{1}, i_{2}, \ldots, i_{r})).
\]

(2.2) Let \( m \) be odd, and assume that \( \chi^{2} \) is primitive. Then we have

\[
h(A, \chi) = \prod_{i=1}^{r} (-1)^{(m-1)(p_{i}-1)/2} p_{i}^{(m-1)/2} \gamma_{m, p_{i}}
\]

\[
\times \sum_{i_{1}=0}^{l_{1}} \cdots \sum_{i_{r}=0}^{l_{r}} \tilde{\chi}(i_{1}, i_{2}, \ldots, i_{r}) (\det A) J_{m-1}(\tilde{\chi}(i_{1}, i_{2}, \ldots, i_{r})).
\]

**Proof.** We note that \( J_{m}(\eta_{1}, \eta_{2}) = \prod_{i=1}^{r} J_{m}(\eta_{1}^{(p_{i})}, \eta_{2}^{(p_{i})}) \) for primitive characters \( \eta_{1} \) and \( \eta_{2} \) mod \( N \). Moreover \( \eta_{2}^{2} \) is primitive if and only if \( \eta_{2}^{2} \neq 1 \) for any \( 1 \leq i \leq r \). Thus the assertion follows from Theorem 5.5 and [[K-M], Lemma 3.2].

Now we give explicit formulas for \( J_{m}(\chi, \eta) \) and \( I_{m}(\chi, \eta) \).

**Proposition 5.7.** Let \( \chi \) and \( \eta \) be primitive characters mod \( p \). Assume that \( \chi^{2} \neq 1 \). Put \( c_{m}(\chi, \eta) = 1 \) or \( 0 \) according as \( \chi^{m} \eta = 1 \) or not.

1. Assume that \( m \) is odd. Then

\[
I_{m}(\chi, \eta) = c_{m}(\chi, \eta) \left( \frac{-1}{p} \right)^{m-1/2} p^{m-1/2} (p-1) J_{m-1}(\chi \left( \frac{\ast}{p} \right), \eta).
\]

2. Assume that \( m \) is even. Then

\[
I_{m}(\chi, \eta) = c_{m}(\chi, \eta) \left( \frac{-1}{p} \right)^{m/2} p^{m-2/2} (p-1) \chi(-1) J(\chi \left( \frac{\ast}{p} \right)) J_{m-1}(\chi \left( \frac{\ast}{p} \right), \eta).
\]

20
Proof. By Proposition 5.4, we have

\[ I_m(\chi, \eta) = I'_m \times \begin{cases} p^{(m-1)/2} \left( -1 \right)^{(m-1)/2} \frac{\chi(m)}{p} \frac{\eta(m)}{p} \left( \frac{m}{p} \right) & \text{if } m \text{ is odd} \\ p^{m-2} \left( -1 \right)^{(m-2)/2} \frac{\chi(m)}{p} \frac{\eta(m)}{p} \left( \frac{m}{p} \right) & \text{if } m \text{ is even} \end{cases} \]

where

\[ I'_m = \sum_{z_{mm} \in \mathbb{F}_p} \chi(z_{mm}) \chi(\det Z_1) \left( \frac{\det Z_1}{p} \right) \eta(z_{mm} + \text{tr}(Z_1)) \left( \frac{z_{mm}}{p} \right)^{m-1}. \]

Then we have

\[ I'_m = \sum_{z_{mm} \in \mathbb{F}_p} \chi(z_{mm}) \eta(z_{mm}) \chi(\det Z_1) \left( \frac{\det Z_1}{p} \right) \eta(1 + z_{mm} \text{tr}(Z_1)) \left( \frac{z_{mm}}{p} \right)^{m-1}. \]

Put \( Y_1 = -z_{mm}^p Z_1 \). Then \( \det Y_1 = (-1)^{m-1} - z_{mm}^p \det Z_1 \). Hence we have

\[ I'_m = \chi((-1)^{m-1}) \left( \frac{(-1)^{m-1}}{p} \right) \times \sum_{z_{mm} \in \mathbb{F}_p^\times} \chi(z_{mm}) \eta(z_{mm}) \sum_{Y_1 \in S_{m-1}(\mathbb{F}_p)\times} \chi(\det Y_1) \left( \frac{\det Y_1}{p} \right) \eta(1 - \text{tr}(Y_1)). \]

We have

\[ \sum_{z_{mm} \in \mathbb{F}_p^\times} \chi(z_{mm})^m \eta(z_{mm}) = p - 1 \text{ or } 0 \]

according as \( \chi^m \eta \) is trivial or not. This proves the assertion.

\[ \square \]

**Proposition 5.8.** Let \( \chi \) and \( \eta \) be as in Proposition 5.7.

1. Assume that \( m \) is odd. Then

\[ J_m(\chi, \eta) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} \times \{ J(\chi, \chi^{m-1} \eta)J_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) + \eta(-1)I_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) \}. \]

2. Assume that \( m \) is even. Then

\[ J_m(\chi, \eta) = \left( \frac{-1}{p} \right)^{m/2} p^{m/2} J(\chi \left( \frac{\chi}{p} \right)) \times \{ J(\chi \left( \frac{\chi}{p} \right), \chi^{m-1} \left( \frac{\chi}{p} \right) \eta)J_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) + \eta(-1)I_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) \}. \]
Proof. By Proposition 5.4, we have

\[ J_m(\chi, \eta) = (J'_m + J''_m) = \begin{cases} p^{(m-1)/2} \left( \frac{(-1)^{(m-1)/2}}{p} \right) J(\chi, \left( \frac{x}{p} \right)) & \text{if } m \text{ is odd} \\
p^{(m-2)/2} \left( \frac{(-1)^{(m-2)/2}}{p} \right) J(\chi, \left( \frac{x}{p} \right)) & \text{if } m \text{ is even} \end{cases} \]

where

\[ J'_m = \sum_{z \in \mathbb{F}_p, z \neq 0} \chi(z)^\sigma \left( \frac{z}{p} \right) \eta(1-zz^{-1}) \]

and

\[ J''_m = \sum_{z \in \mathbb{F}_p, z \neq 0} \chi(z)^\sigma \left( \frac{z}{p} \right) \eta(1-zz^{-1}). \]

Then we have

\[ J''_m = \eta(-1)J_{m-1}(\chi, \left( \frac{x}{p} \right), \eta). \]

Moreover

\[ J'_m = \sum_{z \in \mathbb{F}_p, z \neq 0} \chi(z)^\sigma \left( \frac{z}{p} \right) \eta(1-zz^{-1}) \]

Put \( Y_1 = (1-zz^{-1})^{-1}Z_1 \). Then \( \det Y_1 = (1-zz^{-1})^{-m} \det Z_1 \). Hence we have

\[ J'_m = \sum_{z \in \mathbb{F}_p, z \neq 0} \chi(z)^\sigma \left( \frac{z}{p} \right) \eta(1-zz^{-1}) \]

This proves the assertion. \( \Box \)

Theorem 5.9. Let \( \chi \) be a primitive character mod \( p \).

1. Let \( m \) be odd, and assume that \( \chi^2 \neq 1 \).
   1.1 Assume that \( \chi^m \neq 1 \). Then
   \[ J_m(\chi, \left( \frac{\ast}{p} \right), \chi) = \left( -\frac{1}{p} \right)^{(m-1)/2} p^{(m-1)/2} J(\chi, \left( \frac{\ast}{p} \right), \chi^m) J_{m-1}(\chi, \left( \frac{\ast}{p} \right)^{i+1}, \chi). \]
   1.2 Assume that \( \chi^m = 1 \). Then
   \[ J_m(\chi, \left( \frac{\ast}{p} \right), \chi) = p^{m-1} \left( -\frac{1}{p} \right)^{i+1} J(\chi, \left( \frac{\ast}{p} \right)^{i+1}, \chi) J_{m-2}(\chi, \left( \frac{\ast}{p} \right)^{i}, \chi). \]

22
(2) Let $m$ be even.

(2.1) Assume that $\chi^m \left(\frac{s}{p}\right)^{i+1} \neq 1$. Then

$$J_m(\chi \left(\frac{s}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-2)/2} J(\chi \left(\frac{s}{p}\right)^i, \chi) J(\chi \left(\frac{s}{p}\right)^{i+1}, \chi^m \left(\frac{s}{p}\right)^{i+1}) J_{m-1}(\chi \left(\frac{s}{p}\right)^i, \chi).$$

(2.2) Assume that $\chi^m \left(\frac{s}{p}\right)^{i+1} = 1$. Then

$$J_m(\chi \left(\frac{s}{p}\right)^i, \chi) = (\chi(-1))^{p^{m-1}} J(\chi \left(\frac{s}{p}\right)^i, \chi) J_{m-2}(\chi \left(\frac{s}{p}\right)^i, \chi).$$

\textbf{Proof.} Let $m$ be odd. Then, by (1) of Proposition 5.8, we have

$$J_m(\chi \left(\frac{s}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2}$$

$$\times \{ J(\chi \left(\frac{s}{p}\right)^i, \chi^m) J_{m-1}(\chi \left(\frac{s}{p}\right)^{i+1}, \chi) + \chi(-1) J_{m-1}(\chi \left(\frac{s}{p}\right)^{i+1}, \chi) \}.$$
Thus the assertion holds if \( \chi_m \left( \frac{\ast}{p} \right) \) \( \neq \) 1. Assume that \( \chi_m \left( \frac{\ast}{p} \right) \) \( = \) 1. Then by (1) of Proposition 5.7 and (1) of Proposition 5.8, we have

\[
J_{m-1} \left( \frac{\ast}{p} \right) \left[ \begin{array}{c}
\chi \left( \frac{\ast}{p} \right) \\
\chi \left( \frac{\ast}{p} \right) \chi_m \left( \frac{\ast}{p} \right) \chi_m ^{-1}
\end{array} \right]
\]

\[
\times J \left( \frac{\ast}{p} \right) \chi_m ^{-1} \chi_m ^{-1} \chi \left( \frac{\ast}{p} \right) J \left( \frac{\ast}{p} \right) \left( \frac{\ast}{p} \right)
\]

and

\[
I_{m-1} \left( \frac{\ast}{p} \right) \chi \left( \frac{\ast}{p} \right) \chi \left( \frac{\ast}{p} \right) \chi_m ^{-1} = \chi \left( \frac{\ast}{p} \right) \chi_m ^{-1}.
\]

This proves the assertion.

\[\square\]

**Corollary.** Let \( \chi \) be a primitive character with an odd square free conductor \( N \). Assume that \( \chi^2 \) is primitive. Then the value \( J_m (\chi) \) is nonzero.

**Proof.** The assertion follows directly from the above theorem if \( N \) is an odd prime. In general case, the assertion can also be proved by remarking that \( J_m (\chi) \) \( = \prod_{p \mid N} J_m (\chi (p)) \) and that \( \chi (p) \) \( = 1 \) for any \( p \mid N \).

To compare our present result with the result in [K-M], we give the following:

**Proposition 5.10.** Let \( \chi \) be a primitive Dirichlet character mod \( p \). Assume that \( \chi^2 \neq 1 \). Then we have

\[
J (\chi, \left( \frac{\ast}{p} \right)) \chi \left( \frac{\ast}{p} \right) \chi \left( \frac{\ast}{p} \right) = \left( \frac{-1}{p} \right) \chi (4) p.
\]

**Proof.** Put

\[
I = \sum_{(z,w) \in \mathbb{F}_p^2} \chi \left( z(1-z) - w^2 \right).
\]

Then by using the same argument as in the proof of Theorem 5.5, we have

\[
I = J (\chi, \left( \frac{\ast}{p} \right)) \sum_{z \in \mathbb{F}_p} \chi (z(1-z)) \left( \frac{z(1-z)}{p} \right)
\]

24
\[ = J(\chi; \left( \frac{a}{p} \right)) J(\chi; \left( \frac{a}{p} \right), \chi(\frac{a}{p})). \]

On the other hand, we have

\[ I = \sum_{(y, w) \in F_2^p} \chi(-y^2 - w^2 + 1/4). \]

Hence by Lemma 5.3 we have

\[ I = \rho \left( \frac{-1}{p} \right) \tilde{\chi}(4). \]

This proves the assertion. \(\square\)

By virtue of the above proposition, we see that Theorem 5.6 coincides with [K-M], Proposition 3.7 in case \(m = 2\).

Now let

\[ F(Z) = \sum_{A \in L_{n \geq 0}} c_F(A) e(\text{tr}(AZ)) \]

be an element of \(\mathfrak{M}_k(Sp_n(Z))\) and let \(\chi\) be a Dirichlet character mod \(N\). Assume \(N \neq 2\). Then by [[K-M], Proposition 3.1], we have

\[ L(s, F, \chi) = \sum_{A \in L_{n > 0}/SL_n(Z)} \frac{c_F(A) b(A, \chi)}{e(A)(\text{det} A)^s}. \]

Thus by Theorem 5.6 we easily obtain:

**Theorem 5.11.** Let \(N, p_i, l_i, u_{0,i}, \ldots, r \) and \(\chi\) be as in Theorem 5.6, and let \(F\) be an element of \(\mathfrak{M}_k(Sp_n(Z))\).

(1) If \(\chi^{(p_i)}(u_{0,i}) \neq 1\) for some \(i\). Then we have \(L(s, F, \chi) = 0\).

(2) Assume that \(\chi^{(p_i)}(u_{0,i}) = 1\) for any \(i\). Fix a character \(\tilde{\chi}\) such that \(\tilde{\chi}^n = \chi\).

(2.1) Let \(n\) be even. Then we have

\[ L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-2)(p_i-1)/4} \gamma_{n,p_i} \]

\[ \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(i_1, \ldots, i_r) (2^n J(\tilde{\chi}(i_1, \ldots, i_r); \frac{n}{N})) J_{n-1}(\tilde{\chi}(i_1, \ldots, i_r)) L^*(s, F, \tilde{\chi}(i_1, \ldots, i_r)). \]

(2.2) Let \(n\) be odd, and assume that \(\chi^2 \neq 1\). Then we have

\[ L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i} \]

\[ \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(i_1, \ldots, i_r) (2^{n-1} J_{n-1}(\tilde{\chi}(i_1, i_2, \ldots, i_r)) L^*(s, F, \tilde{\chi}(i_1, i_2, \ldots, i_r)). \]
6 Twisted Koehler-Maaß series of the first kind of the DII lift

By Theorems 4.1 and 5.11, we obtain the following.

**Theorem 6.1.** Let $k$ and $n$ be positive even integers such that $n \geq 4$, $2k - n \geq 12$. Let $h(z)$ and $E_{n/2 + 1/2}$ be as in Section 4. Let $N$ be a square free odd integer, and $N = p_1 \cdots p_r$ be the prime decomposition of $N$. For each $i = 1, \cdots, r$ let $l_i = \text{GCD}(n, p_i - 1)$ and $u_{i,i} \in \mathbb{Z}$ be a primitive $l_i$-th root of unity mod $p_i$.

1. Assume $\chi(p_i)(u_i) \neq 1$ for some $i$. Then $L(s, I_n(h), \chi) = 0$.
2. Assume $\chi(p_i)(u_i) = 1$ for any $i$. Then

\[
L(s, I_n(h), \chi) = 2^{n\alpha} \chi(2^n) \sum_{i_1 = 0}^{l_1 - 1} \cdots \sum_{i_r = 0}^{l_r - 1} J(\tilde{\chi}_{(i_1, \cdots, i_r)} , \left(\frac{N}{N}\right)) J_n - 1(\tilde{\chi}_{(i_1, \cdots, i_r)}) \\
\times \{c_{n,N}R(s, h, E_{n/2 + 1/2}, \tilde{\chi}_{(i_1, \cdots, i_r)}) \prod_{j=1}^{n/2 - 1} L(2s - 2j, S(h), \tilde{\chi}^2_{(i_1, \cdots, i_r)}) \} \\
+ d_{n,N}c_b(1) \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), \tilde{\chi}^2_{(i_1, \cdots, i_r)})
\]

where $c_{n,N}$ and $d_{n,N}$ are nonzero rational numbers depending only on $n$ and $N$, and $\tilde{\chi}$ is a character s.t. $\tilde{\chi}^n = \chi$.

**Remark.** In the case $n = 2$, an explicit formula for $L(s, I_2(h), \chi)$ was given by Katsurada-Mizuno [K-M].

7 Applications

Let $h_1$ and $h_2$ be modular forms of weight $k_1 + 1/2$ and $k_2 + 1/2$, respectively, and $\chi$ be a Dirichlet character. In Section 2, we reviewed on the algebraicity of the values $R(m, h_1, h_2, \chi)$ at half integers. We then naturally ask the following question:

**Question.** What can one say about the algebraicity of $R(m, h_1, h_2, \chi)$ with $m$ an integer?

As an application of Theorem 6.1, we give a partial answer to this question. We note that

\[
R(s, h_1, h_2, \chi) = (1 - 2^{-2s+k_1+k_2-1}\chi^2(2))^{-1} \tilde{R}(s, h_1, h_2, \chi)
\]

if the conductor of $\chi$ is odd. Hence it suffices to consider the above question for $R(m, h_1, h_2, \chi)$ with integer $m$ if $k_1 + k_2$ is even.
Let \( k \) and \( n \) be positive even integers such that \( n \geq 4, 2k - n \geq 12. \) Let \( h(z) \) and \( E_{n/2+1/2} \) be as in Section 4. For a Dirichlet character \( \chi \) of odd square free conductor \( N = p_1 \cdots p_r, \) we define

\[
R^{(x)}(s, h, E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} J(\chi_{(i_1, \ldots, i_r)}, \left( \binom{\ast}{N} \right))^* J_{n-1}(\chi_{(i_1, \ldots, i_r)})
\]

\[
\times R(s, h, E_{n/2+1/2}, \chi_{(i_1, \ldots, i_r)}) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi_{(i_1, \ldots, i_r)}^2),
\]

where \( l_i = \gcd(n, p_i - 1) \) as in Theorem 6.1.

**Theorem 7.1.** There exists a finite dimensional \( \mathcal{Q} \)-vector space \( W_{h,E_{n/2+1/2}} \) in \( C \) such that

\[
\frac{R^{(x)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h,E_{n/2+1/2}}
\]

for any integer \( n/2 + 1 \leq m \leq k - n/2 - 1 \) and a character \( \chi \) of odd square free conductor such that \( \chi^n \) is primitive.

**Proof.** Put

\[
M^{(x)}(s, S(h)) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} J(\chi_{(i_1, \ldots, i_r)}, \left( \binom{\ast}{N} \right))^* J_{n-1}(\chi_{(i_1, \ldots, i_r)})
\]

\[
\times \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), (\chi_{(i_1, \ldots, i_r)})^2).
\]

Then by Corollary to Proposition 3.1, we have

\[
\frac{M^{(x)}(m, S(h))}{\pi^{mn}} \in \mathcal{Q}u_{-(S(h))^{n/2}}\pi^{-n^2/4}.
\]

By Theorem 6.1, we have

\[
L(m, I_n(h), \chi^n) = 2^{mn}\chi(2^n)\left\{ c_{n,N}R^{(x)}(m, h, E_{n/2+1/2}) + d_{n,N}c_h(1)M^{(x)}(m, S(h)) \right\}.
\]

Hence by Theorem 2.2, we have

\[
\frac{R^{(x)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \mathcal{Q}u_1 \otimes \mathcal{Q} V_{I_n(h)} + \mathcal{Q}u_2
\]

with some complex numbers \( u_1 \) and \( u_2, \) where \( V_{I_n(h)} \) is the \( \mathcal{Q} \)-vector space associated with \( I_n(h) \) in Theorem 2.2. This proves the assertion. \( \square \)

By the above theorem, we immediately obtain the following:
Theorem 7.2. Let \( d > \dim \mathbb{Q} \mathcal{W}_h, E_{n/2+1/2} \). Let \( m_1, m_2, \ldots, m_d \) be integers such that \( n/2+1 \leq m_1, m_2, \ldots, m_d \leq k - n/2 - 1 \) and \( \chi_1, \chi_2, \ldots, \chi_d \) be Dirichlet characters of odd square free conductors \( N_1, N_2, \ldots, N_d \), respectively such that \( \chi_i^n \) is primitive for any \( i = 1, 2, \ldots, d \). Then the values \( \frac{R(\chi_i)(m_i, h, E_{n/2+1/2})}{\pi^{m_i n}} \), \( i = 1, 2, \ldots, d \), are linearly dependent over \( \mathbb{Q} \).

Corollary. In addition to the notation and the assumption as above, assume that \( n \equiv 2 \mod 4 \). Write \( N_i = \prod_{i=1}^r p_{ij} \) with \( p_{ij} \) an odd prime number, and let \( l_{ij} = \text{GCD}(p_{ij} - 1, n) \). Then the values \( \left\{ \frac{R(m_i, h, E_{n/2+1/2}, \chi_{i(a_{i1}, \ldots, a_{ir})})}{\pi^{2m_i n}} \right\}_{1 \leq i \leq r, 0 \leq a_{ir} \leq l_{ir} - 1} \) are linearly dependent over \( \mathbb{Q} \). In particular, if \( \chi_1, \chi_2, \ldots, \chi_d \) are Dirichlet characters of odd prime conductors \( p_1, p_2, \ldots, p_d \), respectively such that \( \chi_i^n \) is primitive for any \( i = 1, 2, \ldots, d \), then the values \( \left\{ R(m_i, h, E_{n/2+1/2}, \chi_i) \left( \frac{\pi}{r} \right)^{a_i} \right\}_{1 \leq i \leq r, 0 \leq a_i \leq l_i - 1} \) are linearly dependent over \( \mathbb{Q} \), where \( l_i = \text{GCD}(n, p_i - 1) \) for \( i = 1, \ldots, d \).

Proof. By Theorem 1.1, the value \( \frac{L_r(n, S(h), \chi_{i(a_{i1}, \ldots, a_{ir})})}{\pi^{n(n-1)/2}} \) belongs to \( \mathbb{Q}[S(h)]^{n/2-1} \cdot \pi^{-n^2/4+n/2} \), and in particular if \( n \equiv 2 \mod 4 \), then it is nonzero for any \( \chi_i \). Moreover, by Corollary to Theorem 5.10, \( J(\chi_{i1}, \ldots, \chi_{ir}), (\frac{\pi}{r})^{a_i} J_{n-1}(\chi_{i1}, \ldots, \chi_{ir}) \) is zero and belongs to \( \mathbb{Q} \). Thus the assertion holds.

As another application of Theorem 7.1, we also have a functional equation for \( R(\chi)(s, h, E_{n/2+1/2}) \). Namely, by Theorem 3.1 we obtain:

Theorem 7.3. Let \( h \) be as above. Let \( \chi \) be a primitive character of odd square free conductor \( N \). Assume that \( n \equiv 2 \mod 4 \), and that \( \chi^n \) is primitive. Put

\[
R(\chi)(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^n)^{-1} \gamma_n(s) R(\chi)(s, h, E_{n/2+1/2}),
\]

where \( \tau(\chi^n) \) is the Gauss sum, and

\[
\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^n \Gamma(s - (i - 1)/2).
\]

Then \( R(\chi)(s, h, E_{n/2+1/2}) \) has an analytic continuation to the whole \( s \)-plane, and has the following functional equation:

\[
R(\chi)(k - s, h, E_{n/2+1/2}) = R(\chi)(s, h, E_{n/2+1/2}).
\]

Remark. (1) As functions of \( s \), the Dirichlet series \( \{ R(s, h, E_{n/2+1/2}, \chi_{ij}) \}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1} \) are linearly independent over \( \mathbb{C} \).

(2) In the case of \( n = 2 \), this type of result was given for \( R(m, h, E_{3/2}) \) with \( E_{3/2} \) Zagier’s Eisenstein series of weight 3/2 by [K-M].

(3) The meromorphy of this type of series was derived in [Sh3] by using so called the Rankin-Selberg integral expression in more general setting, but we don’t know whether the functional equation of the above type can be directly proved without using the above method.
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