Explicit formulas for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and their applications

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Abstract
We give an explicit formula for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift. As an application we prove a certain algebraicity result for the values of twisted Rankin-Selberg series at integers of half-integral weight modular forms.

1 Introduction
It is an interesting problem to give an explicit formula for the Koecher-Maaß series of a Siegel modular form $F$ for the symplectic group $\text{Sp}_n(\mathbb{Z})$, and several results have been obtained (cf. Böcherer [B], Ibukiyama and Katsurada [I-K1], [I-K2], [I-K3]). Such explicit formulas are not only interesting in its own right but also have some important applications in the theory of modular forms. For example, we refer to [B-S], [D-I], [Miz]. Now we consider a twist of such a Koecher-Maaß series by a Dirichlet character $\chi$. As for this, in view of Saito [Sai1] for example, we can naturally consider the following Dirichlet series:

$$ L'(s, F, \chi) = \sum_T \frac{\chi(2\det T)c_F(T)e(T)\chi(T)}{\det T^s}, $$

where $T$ runs over a complete set of representatives of $\text{SL}_n(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree $n$, $c_F(T)$ is the $T$-th Fourier coefficient of $F$ and $e(T) = \#\{U \in \text{SL}_n(\mathbb{Z}); T[U] = T\}$. We will sometimes call $L'(s, F, \chi)$ the twisted Koecher-Maaß series of the second kind.

On the other hand, Choie and Kohnen [C-K] introduced a different type of “twist”. For a positive integer $N$, let $\text{SL}_{n,N}(\mathbb{Z}) = \{U \in \text{SL}_n(\mathbb{Z}); U \equiv 1_n \text{ mod } N\}$ and $e_N(T) = \#\{U \in \text{SL}_{n,N}(\mathbb{Z}); T[U] = T\}$. For a primitive Dirichlet character $\chi \text{ mod } N$, the Koecher-Maaß series $L(s, F, \chi)$ of $F$ twisted by $\chi$ is defined to be

$$ L(s, F, \chi) = \sum_T \frac{\chi(\text{tr}(T))c_F(T)}{e_N(T)\det T^s}, $$

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where $T$ runs over a complete set of representatives of $SL_{n,N}(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree $n$. In [C-K], Choie and Kohnen proved a meromorphy continuation of $L(s,F,\chi)$ to the whole $s$-plane and a functional equation (cf. Theorem 2.1). Moreover they got a result on the algebraicity of its special values (cf. Theorem 2.2.) We shall call $L(s,F,\chi)$ the twist of the first kind.

In this paper we give explicit formulas for the twisted Koecher-Maaß series of the first and second kinds associated with the Duke-Imamoglu-Ikeda lift and apply them to the study of the special values of the Rankin-Selberg series for half-integral weight modular forms. We explain our main results more precisely. Let $k$ and $n$ be positive even integers such that $n \geq 4$ and $2k-n \geq 12$. For a cuspidal Hecke eigenform $h$ in the Kohnen plus subspace of weight $k-n/1+1/2$ for $\Gamma_0(4)$, let $I_n(h)$ be the Duke-Imamoglu-Ikeda lift of $h$ to the space of cusp forms of weight $k$ for $Sp_n(\mathbb{Z})$. Moreover let $S(h)$ be the normalized Hecke eigenform of weight $2k-n$ for $SL_2(\mathbb{Z})$ corresponding to $h$ under the Shimura correspondence, and $E_{n/2+1/2}$ be Cohen’s Eisenstein series of weight $n/2+1/2$ for $\Gamma_0(4)$. We then give explicit formulas for $L(s,I_n(h),\chi)$ and $L^*(s,I_n(h),\chi)$ in terms of the twisted Rankin-Selberg series $R(s,h,E_{n/2+1/2},\eta)$ of $h$ and $E_{n/2+1/2}$ and twisted Hecke’s $L$-function $L(s,S(h),\eta')$ of $S(h)$, where $\eta$ and $\eta'$ are Dirichlet characters related with $\chi$. It is relatively easy to get an explicit form of $L^*(s,I_n(h),\chi)$. In fact, by using the same argument as in Ibukiyama and Katsurada [I-K2], we can easily obtain its explicit formula (cf. Theorem 4.1). On the other hand, it seems nontrivial to get that of $L(s,I_n(h),\chi)$ (cf. Theorem 6.1), and we need some explicit formula for a certain character sum associated with a Dirichlet character (cf. Theorem 5.6). Using Theorem 6.1 combined with the result of Choie-Kohnen, we prove certain algebraicity results on $R(s,h,E_{n/2+1/2},\eta)$ at an integer $s=m$ (cf. Theorems 7.1 and 7.2), which were announced in [Ka]. We note that the algebraicity of the special values of such a Rankin-Selberg series at half-integers was investigated by Shimura [Sh3]. However there are few results on the algebraicity of such values at integers. As an attempt, Mizuno and the author [K-M] proved linear dependency of Rankin-Selberg $L$-values of a cuspidal Hecke eigenform belonging to Kohnen plus subspace of half integral weight and the Zagier’s Eisenstein series of weight $3/2$. Our present result can be regarded as a generalization of our previous result.

**Notation** We denote by $e(x) = \exp(2\pi i x)$ for a complex number $x$. For a commutative ring $R$, we denote by $M_{m,n}(R)$ the set of $(m,n)$-matrices with entries in $R$. For an $(m,n)$-matrix $X$ and an $(m,m)$-matrix $A$, we write $A[X] = AXA^t$, where $A^t$ denotes the transpose of $A$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{m,n}(R)$ we often use the same symbol $X$ to denote the coset $X \mod aM_{m,n}(R)$. Put $GL_m(R) = \{ A \in M_m(R) \mid \det A \in R^* \}$, and $SL_m(R) = \{ A \in M_m(R) \mid \det A = 1 \}$, where $\det A$ denotes the determinant of a square matrix $A$ and $R^*$ is the unit group of $R$. We denote by $S_n(R)$ the set of symmetric matrices of degree $n$ with entries in $R$. In particular, if $S$ is a subset of $S_n(R)$ with $R$ the field of real numbers, we denote by $S_{\geq 0}$ (resp. $S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite)
Let \( \mathbf{SL}_n(\mathbb{Z}) \) acts on the set \( S_n(\mathbb{R}) \) in the following way:

\[
\mathbf{SL}_n(\mathbb{Z}) \times S_n(\mathbb{R}) \ni (g, A) \quad \rightarrow \quad gAg \in S_n(\mathbb{R}).
\]

Let \( G \) be a subgroup of \( \mathbf{GL}_n(\mathbb{Z}) \). For a subset \( \mathcal{B} \) of \( S_n(\mathbb{R}) \) stable under the action of \( G \) we denote by \( \mathcal{B}/G \) the set of equivalence classes of \( \mathcal{B} \) with respect to \( G \). We sometimes identify \( \mathcal{B}/G \) with a complete set of representatives of \( \mathcal{B}/G \). Two symmetric matrices \( A \) and \( A' \) with entries in \( R \) are said to be equivalent with respect to \( G \) and write \( A \sim_G A' \) if there is an element \( X \) of \( G \) such that \( A' = A[X] \). Let \( \mathcal{L}_n \) denote the set of half-integral matrices of degree \( n \) over \( \mathbb{Z} \), that is, \( \mathcal{L}_n \) is the set of symmetric matrices of degree \( n \) whose \((i,j)\)-component belongs to \( \mathbb{Z} \) or \( \frac{1}{2}\mathbb{Z} \) according as \( i = j \) or not.

\[2\text{ Twisted Koecher-Maaß series}\]

Put \( J_n = \left( \begin{array}{cc} O_n & -1_n \\ 1_n & O_n \end{array} \right) \), where \( 1_n \) and \( O_n \) denotes the unit matrix and the zero matrix of degree \( n \), respectively. Furthermore, put

\[S\mathbf{p}_n(\mathbb{Z}) = \{ M \in \mathbf{GL}_{2n}(\mathbb{Z}) \mid J_n[M] = J_n \}.
\]

Let \( l \) be an integer or a half-integer, and \( N \) a positive integer. Let \( \Gamma_0^{(n)}(N) \) be the congruence subgroup of \( S\mathbf{p}_n(\mathbb{Z}) \) consisting of matrices whose left lower \( n \times n \) block are congruent to \( O_n \) mod \( N \). Moreover let \( \chi \) be a Dirichlet character mod \( N \). We then denote by \( \mathfrak{M}_l(\Gamma_0^{(n)}(N), \chi) \) the space of modular forms of weight \( l \) and character \( \chi \) for \( \Gamma_0^{(n)}(N) \), and by \( \mathfrak{E}_l(\Gamma_0^{(n)}(N), \chi) \) the subspace of \( \mathfrak{M}_l(\Gamma_0^{(n)}(N), \chi) \) consisting of cusp forms. If \( \chi \) is the trivial character mod \( N \), we simply write \( \mathfrak{M}_l(\Gamma_0^{(n)}(N), \chi) \) and \( \mathfrak{E}_l(\Gamma_0^{(n)}(N), \chi) \) as \( \mathfrak{M}_l(\Gamma_0^{(n)}(N)) \) and \( \mathfrak{E}_l(\Gamma_0^{(n)}(N)) \), respectively. Let \( k \) be a positive integer, and let \( F(Z) \in \mathfrak{M}_k(S\mathbf{p}_n(\mathbb{Z})) \). Then \( F(Z) \) has the Fourier expansion:

\[
F(Z) = \sum_{T \in \mathcal{L}_{n \geq 0}} c_F(T) e^{i\text{tr}(TZ)},
\]

where \( \text{tr}(X) \) denotes the trace of a matrix \( X \). For \( N \in \mathbb{Z}_{>0} \), put \( S\mathbf{L}_{n,N}(\mathbb{Z}) = \{ U \in S\mathbf{L}_n(\mathbb{Z}) \mid U \equiv 1_n \mod N \} \), and for \( T \in \mathcal{L}_{n \geq 0} \) put \( e_N(T) = |\{ U \in S\mathbf{L}_{n,N}(\mathbb{Z}) \mid T[U] = T \} | \). For a primitive Dirichlet character \( \chi \) mod \( N \) Let

\[
L(s, F; \chi) = \sum_{T \in \mathcal{L}_{n \geq 0} / S\mathbf{L}_{n,N}(\mathbb{Z})} \frac{\chi(\text{tr}(T)) c_F(T)}{e_N(T)(\det T)^s},
\]

be the twisted Koecher-Maaß series of the first kind of \( F \) as in Section 1. The following two theorems are due to Choie and Kohnen \([C-K]\).

**Theorem 2.1.** Let \( F \in \mathfrak{E}_k(S\mathbf{p}_n(\mathbb{Z})) \), and \( \chi \) a primitive character of conductor \( N \). Put

\[
\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2} \Gamma(s - (i-1)/2),
\]

then \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, F; \chi) \gamma_n(2s - 2z) ds \) converges absolutely and uniformly for \( c > \frac{1}{2} \), and for real \( s = c + it \) with \( c > \frac{1}{2} \) and \( |t| \rightarrow \infty \) the function \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, F; \chi) \gamma_n(2s - 2z) ds \) tends to some complex number depending on \( z \) as \( t \rightarrow \infty \).

**Theorem 2.2.** Let \( F \in \mathfrak{E}_k(S\mathbf{p}_n(\mathbb{Z})) \), and \( \chi \) a primitive character of conductor \( N \). Put

\[
\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2} \Gamma(s - (i-1)/2),
\]

then \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, F; \chi) \gamma_n(2s - 2z) ds \) converges absolutely and uniformly for \( c > \frac{1}{2} \), and for real \( s = c + it \) with \( c > \frac{1}{2} \) and \( |t| \rightarrow \infty \) the function \( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, F; \chi) \gamma_n(2s - 2z) ds \) tends to some complex number depending on \( z \) as \( t \rightarrow \infty \).
and

\[ \Lambda(s, F, \chi) = N^{2s} \tau(\chi)^{-1} \gamma_n(s) L(s, F, \chi) \quad (\text{Re}(s) \gg 0), \]

where \( \tau(\chi) \) is the Gauss sum of \( \chi \). Then \( \Lambda(s, F, \chi) \) has an analytic continuation to the whole \( s \)-plane and has the following functional equation:

\[ \Lambda(k - s, F, \chi) = (-1)^{nk/2} \chi(-1) \Lambda(s, F, \chi). \]

**Theorem 2.2.** Let \( F \) and \( \chi \) be as above. Then there exists a finite dimensional \( \bar{Q} \)-vector space \( V_F \) in \( \mathbb{C} \) such that

\[ L(m, F, \chi) \pi^{-nm} \in V_F \]

for any primitive character \( \chi \) and any integer \( m \) such that \( (n + 1)/2 \leq m \leq k - (n + 1)/2 \).

Now let

\[ L^*(s, F, \chi) = \sum_{T \in \mathcal{L} \cap \mathbb{Z}^2} \chi(\|T\|/2) e(T)(\det T)^s \]

be the twisted Koecher-Maaß series of the second kind of \( F \) as in Section 1. We will discuss a relation between these two Dirichlet series in Section 5.

### 3 Review on the algebraicity of L-values of elliptic modular forms of integral and half-integral weight

In this section, we review on the special values of L-functions of elliptic modular forms of integral and half-integral weights. For a modular form \( g(z) \) of integral or half-integral weight for a certain congruence subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \), let \( \mathbb{Q}(g) \) denote the field generated over \( \mathbb{Q} \) by all the Fourier coefficients of \( g \), and for a Dirichlet character \( \eta \) let \( \mathbb{Q}(\eta) \) denote the field generated over \( \mathbb{Q} \) by all the values of \( \eta \). First let

\[ f(z) = \sum_{m=1}^{\infty} c_f(m)e(mz) \]

be a normalized Hecke eigenform in \( \mathcal{S}_k(SL_2(\mathbb{Z})) \), and \( \chi \) be a primitive Dirichlet character. Then let us define Hecke’s L-function \( L(s, f, \chi) \) of \( f \) twisted by \( \chi \) as

\[ L(s, f, \chi) = \sum_{m=1}^{\infty} c_f(m) \chi(m)m^{-s}. \]

Then we have the following result (cf. [Sh2]):
Proposition 3.1. There exist complex numbers \( u_\pm(f) \) uniquely determined up to \( \mathbb{Q}(f) \) multiple such that

\[
\frac{L(m, f, \chi)}{(2\pi \sqrt{-1})^m \tau(\chi) u_j(f)} \in \mathbb{Q}(f) \mathbb{Q}(
\chi)
\]

for any integer \( 0 < m \leq k - 1 \) and a primitive character \( \chi \), where \( \tau(\chi) \) is the Gauss sum of \( \chi \), and \( j = + \) or \( - \) according as \( (-1)^m \chi(-1) = 1 \) or \( -1 \).

Corollary. Under the above notation and the assumption, we have

\[
L(m, f, \chi) \pi^{-m} \in \mathbb{Q} u_j(f)
\]

for any integer \( 0 < m \leq k - 1 \) and a primitive character \( \chi \).

We remark that we have \( L(m, f, \chi) \neq 0 \) if \( m \neq k/2 \), and \( L(k/2, f, \chi) \neq 0 \) for infinitely many \( \chi \).

Next let us consider the half-integral weight case. From now on we simply write \( \Gamma_0(M) \) as \( \Gamma_0(M) \).

Let \( \omega(d) = \chi_4^{-k_1} \chi_2^2(d) \). We also define \( R(s, h_1, h_2, \chi) \) as

\[
R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s},
\]

where \( \chi_4^{-k_1} \chi_2^2(d) \). Now let \( S(h_1) \) be the normalized Hecke eigenform in \( \mathcal{H}_{k_1+1/2}(\Gamma_0(4)) \) corresponding to \( h_1 \) under the Shimura correspondence. Then the following result is due to Shimura [Sh3].

Proposition 3.2. Assume that \( k_1 > k_2 \). Under the above notation we have

\[
\frac{\tilde{R}(m + 1/2, h_1, h_2, \chi)}{u_-(S(h_1)) \tau(\chi^2) \pi^{-k_2+1+2m} \sqrt{-1}} \in \mathbb{Q}(h_1) \mathbb{Q}(h_2) \mathbb{Q}(\chi)
\]

for any integer \( k_2 \leq m \leq k_1 - 1 \) and a primitive character \( \chi \).
Proof. Let $N$ be the conductor of $\chi$. Put

$$h_{2\chi}(z) = \sum_{m=0}^{\infty} c_{h_{2}\chi}(m)\chi(m)e(mz).$$

Then $h_{2\chi}(z) \in \mathbb{M}_{k_2+1/2}(4N^2, \chi^2)$. We can regard $h_1$ as an element of $\mathcal{E}_{k_1+1/2}(\Gamma_0(4N^2))$. Then the assertion follows from [[Sh3], Theorem 2].

Corollary. Assume that $c_{h_1}(n), c_{h_2}(n) \in \mathbb{Q}$ for any $n \in \mathbb{Z}_{\geq 0}$. Then there exists a one-dimensional $\mathbb{Q}$-vector space $U_{h_1, h_2}$ in $\mathbb{C}$ such that

$$\bar{R}(m+1/2, h_1, h_2, \chi)\pi^{-2m} \in U_{h_1, h_2}$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character $\chi$.

4 Explicit formulas for the twisted Koecher-Maaß series of the second kind of the Duke-Imamoglu-Ikeda lift

Throughout this section, we assume that $n$ and $k$ are even positive integers. Let $h$ be a Hecke eigenform of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$ belonging to the Kohnen plus space. Then $h$ has the following Fourier expansion:

$$h(z) = \sum_{e} \epsilon_h(e)e(\epsilon z),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \mod 4$. Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m)e(mz)$$

be the normalized Hecke eigenform of weight $2k - n$ for $SL_2(\mathbb{Z})$ corresponding to $h$ via the Shimura correspondence (cf. [Ko].) For a prime number $p$ let $\beta_p$ be a nonzero complex number such that $\beta_p + \beta_p^{-1} = p^{k+n/2+1/2}c_{S(h)}(p)$.

For a non-negative integers $l$ and $m$, the Cohen function $H(l, m)$ is given by $H(l, m) = L_{-m}(1-l)$. Here

$$L_D(s) = \begin{cases} 
\zeta(2s-1), & D = 0 \\
L(s, \chi_{D_K}) \sum_{a|f} \mu(a)\chi_{D_K}(a)a^{-s}\sigma_1 f(a), & D \neq 0, D \equiv 0, 1 \mod 4 \\
0, & D \equiv 2, 3 \mod 4,
\end{cases}$$

Here $\chi_{D_K}$ is the quadratic character modulo $D_K$. The function $L(s, \chi_{D_K})$ is the Dirichlet $L$-function associated with the character $\chi_{D_K}$.
where the positive integer \( f \) is defined by \( D = D_K f^2 \) with the discriminant \( D_K \) of \( K = \mathbb{Q}(\sqrt{D}) \), \( \mu \) is the Möbius function, and \( \sigma_s(n) = \sum_{d|n} d^s \). Furthermore, for an even integer \( l \geq 4 \), we define the Cohen Eisenstein series \( E_{l+1/2}(z) \) by

\[
E_{l+1/2}(z) = \sum_{\epsilon = 0}^{\infty} H(l, \epsilon) e(\epsilon z).
\]

It is known that \( E_{l+1/2}(z) \) is a modular form of weight \( l+1/2 \) for \( \Gamma_0(4) \) belonging to the Kohnen plus space.

For a prime number \( p \) let \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \) be the field of \( p \)-adic numbers, and the ring of \( p \)-adic integers, respectively. We denote by \( \nu_p \) the additive valuation on \( \mathbb{Q}_p \) normalized so that \( \nu_p(p) = 1 \), and by \( e_p \) the continuous homomorphism from the additive group \( \mathbb{Q}_p \) to \( \mathbb{C}^\times \) such that \( e_p(a) = e(a) \) for \( a \in \mathbb{Q} \). For a \( p \)-adic number \( c \) put

\[
\tilde{\xi}_p(c) = 1, -1 \text{ or } 0
\]

according as \( \mathbb{Q}_p(\sqrt{c}) = \mathbb{Q}_p \), \( \mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p \) is quadratic unramified, or \( \mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p \) is quadratic ramified. We note that \( \tilde{\xi}_p(D) = \chi_D(p) \) for a fundamental discriminant \( D \). For a non-degenerate half-integral matrix \( T \) over \( \mathbb{Z}_p \), let

\[
b_p(T, s) = \sum_{R \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} e_p(\text{tr}(TR)) p^{-\nu_p(\mu_p(R)) s}
\]

be the local Siegel series, where \( \mu_p(R) = [R \mathbb{Z}_p^n + \mathbb{Z}_p^n : \mathbb{Z}_p^n] \). Then there exists a polynomial \( F_p(T, X) \) in \( X \) such that

\[
b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \xi_p(T)p^{n/2-s})^{-1} \prod_{i=1}^{n/2}(1 - p^{2i-2s})
\]

(cf. [Ki1],) where \( \xi_p(T) = \tilde{\xi}_p((-1)^{n/2} \det T) \). For a positive definite half integral matrix \( T \) of degree \( n \) write \((-1)^{n/2} \det(2T)\) as \((-1)^{n/2} \det(2T) = \nu_T t_T^2 \) with \( \nu_T \) a fundamental discriminant and \( t_T \) a positive integer. We then put

\[
c_{I_n(h)}(T) = c_h([\nu_T]) \prod_p (p^{k-n/2-1/2} \beta_p)^{\nu_p(\beta_p)} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).
\]

We note that \( c_{I_n(h)}(T) \) does not depend on the choice of \( \beta_p \). Define a Fourier series \( I_n(h)(Z) \) by

\[
I_n(h)(Z) = \sum_{T \in \mathcal{L}_n > 0} c_{I_n(h)}(T) e(\text{tr}(TZ)).
\]

In [I] Ikeda showed that \( I_n(h)(Z) \) is a Hecke eigenform in \( \mathcal{S}_h(\text{Sp}_0(\mathbb{Z})) \) and its standard \( L \)-function \( L(s, I_n(h), St) \) is given by

\[
L(s, I_n(h), St) = \zeta(s) \prod_{i=1}^{n} L(s + k - i, S(h)).
\]

We call \( I_n(h) \) the Duke-Imamoglu-Ikeda lift (D-I-I lift) of \( h \).
Theorem 4.1. Let \( \chi \) be a primitive Dirichlet character mod \( N \). Then we have

\[
L^*(s, F, \chi) = c_n R(s, h, E_{n/2+1/2}, \chi) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \chi^2) + d_n c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1, S(h), \chi^2),
\]

where \( c_n \) and \( d_n \) are nonzero rational numbers depending only on \( n \).

To prove Theorem 4.1, we reduce the problem to local computations. For \( a, b \in \mathbb{Q}_p^\times \) let \((a, b)_p\) the Hilbert symbol on \( \mathbb{Q}_p \). Following Kitaoka [Ki2], we define the Hasse invariant \( \varepsilon(A) \) of \( A \in S_m(\mathbb{Q}_p)^\times \) by

\[
\varepsilon(A) = \prod_{1 \leq i < j \leq n} (a_i, a_j)_p
\]

if \( A \) is equivalent to \( a_1 \perp \cdots \perp a_n \) over \( \mathbb{Q}_p \) with some \( a_1, a_2, \ldots, a_n \in \mathbb{Q}_p^\times \). For \( T \in S_n(\mathbb{Z}_p)_e \), put \( T^{(0)} = 2^{-1} T, F_p^{(0)}(T, X) = F_p(T^{(0)}, X) \), and so on. Then for non-degenerate symmetric matrices \( A \) of degree \( n \) with entries in \( \mathbb{Z}_p \) we define the local density \( \alpha_p(A) = \alpha_p(A, A) \) representing \( A \) by \( A \) as

\[
\alpha_p(A) = 2^{-1} \lim_{a \to \infty} p^a(-n^2 + n(n+1)/2) \# A_n(A, A),
\]

where

\[
A_n(A, A) = \{ X \in M_n(\mathbb{Z}_p)/p^a M_n(\mathbb{Z}_p) \mid A[X] - B \in p^a S_n(\mathbb{Z}_p)_e \}.
\]

Furthermore put

\[
M(A) = \frac{1}{\sum_{A' \in \mathcal{G}(A)} \varepsilon(A')}
\]

for a positive definite symmetric matrix \( A \) of degree \( n \) with entries in \( \mathbb{Z} \), where \( \mathcal{G}(A) \) denotes the set of \( SL_n(\mathbb{Z}) \)-equivalence classes belonging to the genus of \( A \). Then by Siegel's main theorem on the quadratic forms, we obtain

\[
M(A) = \kappa_n 2^{2-n} \det A^{(n+1)/2} \prod_p \alpha_p(A)^{-1}
\]

where \( e_n = 1 \) or \( 2 \) according as \( n = 1 \) or not, and \( \kappa_n = \prod_{i=1}^{n/2} \Gamma_C(2i) \) (cf. Theorem 6.8.1 in [Ki2]). Put

\[
\mathcal{F}_p = \{ d_0 \in \mathbb{Z}_p \mid \nu_p(d_0) \leq 1 \}
\]

if \( p \) is an odd prime, and

\[
\mathcal{F}_2 = \{ d_0 \in \mathbb{Z}_2 \mid d_0 \equiv 1 \mod 4, \text{ or } d_0/4 \equiv -1 \mod 4, \text{ or } \nu_2(d_0) = 3 \}.
\]
For \( d \in \mathbb{Z}_p^\times \) put

\[
S_n(z_p, d) = \{ T \in S_n(z_p) \mid (-1)^{n/2} \det T = p^2d \text{ mod } z_p^\Delta \text{ with some } i \in \mathbb{Z}, \}
\]

and \( S_n(z_p, d)_x = S_n(z_p, d) \cap S_n(z_p)_x \) for \( x = e \) or \( o \). Put \( L_n^{(0)}(z_p)^\times \) and \( L_n^{(0)}(z_p) \). Let \( t_{n,p} \) be the constant function on \( L_n^{(0)}(z_p) \) taking the value 1, and \( \varepsilon_{n,p} \) the function on \( L_n^{(0)}(z_p) \) assigning the Hasse invariant of \( A \) for \( A \in L_n^{(0)}(z_p) \). We sometimes drop the suffix and write \( t_{n,p} \) as \( t_p \) or \( t \) and the others if there is no fear of confusion. From now on we sometimes write \( \omega = e^l \) with \( l = 0 \) or 1 according as \( \omega = t \) or \( e \). For \( d_0 \in F \) and \( \omega = e^l \) with \( l = 0,1 \), we define a formal power series \( P_n^{(0)}(d_0, \omega, X, t) \) in \( t \) by

\[
P_n^{(0)}(d_0, \omega, X, t) = \kappa(d_0, n, l)^{-1} \sum_{B \in L_n^{(0)}(d_0)} \frac{\tilde{F}_p^{(0)}(B, X)}{\alpha_p(B)} \omega(B) t^{\nu_p(\det B)},
\]

where

\[
\kappa(d_0, n, l) = \kappa(d_0, n, l) = \{( -1)^{n(n+2)/8} ( -1)^{n/2} \},
\]

Let \( F \) denote the set of fundamental discriminants, and for \( l = \pm 1 \), put

\[
F^{(l)} = \{ d_0 \in F \mid ld_0 > 0 \}.
\]

**Theorem 4.2.** Let the notation and the assumption be as above. Then for \( \text{Re}(s) \gg 0 \), we have

\[
L^*(s, \Gamma_n(h)) = \kappa_n 2^{n^2 + 1 - n} \times \sum_{d_0 \in F^{(-1)^{n/2}}} c_h([d_0]) |d_0|^{n/4 - k/2 + 1/4} \prod_p P_{n,p}(d_0, t_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p))
\]

+ \( (-1)^{n(n+2)/8} \times \sum_{d_0 \in F^{(-1)^{n/2}}} ( -1)^{n/2} \omega_2(h_0, d_0) 2 c_h([d_0]) |d_0|^{n/4 - k/2 + 1/4} \prod_p P_{n,p}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)).\)

**Proof.** Let \( T \in S_n(z_p)_{e > 0}. \) Then the \( T \)-th Fourier coefficient \( c_{\Gamma_n(h)}(T) \) of \( \Gamma_n(h) \) is uniquely determined by the genus to which \( T \) belongs, and, by definition, it can be expressed as

\[
\chi_{\Gamma_n(h)}(T) = c_h([k_T^{(0)}]) (k_T^{(0)})^{k-n/2-1/2} \prod_p \tilde{F}_p^{(0)}(T, \alpha_p).
\]

We also note that

\[
(k_T^{(0)})^{k-n/2-1/2} = |k_T^{(0)}|^{-(k/2-n/4-1/4)} (\det T)^{(k/2-n/4-1/4)}
\]

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for $T \in S_n(Z_p)_{e > 0}$. Hence we have
\[
\sum_{T' \in \mathcal{G}(T)} \frac{c_{L(h)}(T')}{e(T')} = \det T^{k/2+n/4-1/4} |e_T^{(0)}|^{k/2-n/4-1/4} \prod_p \frac{\tilde{F}_p^{(0)}(T, \alpha_p)}{\alpha_p(T)}.
\]
Thus, similarly to [I-K1], Theorem 3.3, (1), and [I-K2], Theorem 3.2, we obtain
\[
L(s, I_n(h)) = \kappa_n 2^{s+1-n} \sum_{d_0 \in \mathcal{F}(-1)^{n/2}} c_h([d_0]) |d_0|^{n/4-k/2+1/4}
\]
\[
\times \left\{ \prod_p P_{n,p}^{(0)}(d_0, t_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)) \right\}
\]
\[
+ (-1)^{n(n+2)/8}((-1)^{n/2} d_0) \prod_p P_{n,p}^{(0)}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)) \right\}.
\]
This proves the assertion.

Proposition 4.3. Let $d_0 \in \mathcal{F}_p$ and $\xi_0 = \tilde{\xi}(d_0)$. Then
\[
P_n^{(0)}(d_0, t, X, t) = \frac{(p^{-1}t)^{\nu_p(d_0)}}{\phi_{n/2-1}(p^{-2})(1-\nu_p/2 \xi_0)}
\]
\[
\times \frac{(1 + t^2 p^{-n/2-3/2})(1 + t^2 p^{-n/2-5/2} \xi_0^2) - \xi_0 t^2 p^{-n/2-2}(X + X^{-1} + p^{1/2-n/2} + p^{-1/2+n/2})}{(1 - p^{-2}Xt^2)(1 - p^{-2}X^{-1}t^2) \prod_{i=1}^{n/2}(1 - t^2 p^{-2i-1}X)(1 - t^2 p^{-2i-1}X^{-1})},
\]
and
\[
P_n^{(0)}(d_0, \varepsilon, X, t) = \frac{\xi_0^2}{\phi_{n/2-1}(p^{-2})(1-\nu_p/2 \xi_0) \prod_{i=1}^{n/2}(1 - t^2 p^{-2i-1}X)(1 - t^2 p^{-2i-1}X^{-1})}.
\]

Proof. Put $H_k = \begin{pmatrix} O & 1_k \\ 1_k & O \end{pmatrix}$, and for $d \in Z_p^*$ put
\[
D = \{ x \in M_{2k,n}(Z_p) \mid \det(H_k(x)) \in dp^n Z_p^* \text{ with some } i \in Z_{\geq 0} \}.
\]
We then define $Z_{2k}(u, e^l, d)$ as
\[
Z_{2k}(u, e^l, d) = \int_D e^l(H_k[x]) |\det(H_k[x])|^{|s-k|} dx
\]
with $u = p^{-s}$, where $| \cdot |_p$ denotes the normalized valuation on $Q_p$, and $dx$ is the measure on $M_{2k,n}(Q_p)$ normalized so that the volume of $M_{2k,n}(Z_p)$ is 1. Moreover put
\[
Z_{2k,c}(u, e^l, d) = \frac{1}{2} (Z_{2k,n}(u, e^l, d) + Z_{2k,n}(-u, e^l, d)),
\]

and 

\[ Z_{2k,o}(u, \varepsilon^l, d) = \frac{1}{2} (Z_{2k,n}(u, \varepsilon^l, d) - Z_{2k,n}(-u, \varepsilon^l, d)). \]

Then it is well known that

\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^{n/2}p^{-\nu_p(d_0)}d_0) = \phi_n(p^{-1}) \sum_{T \in L_n(p)(d_0)} \frac{b_p(2^{-\delta_2,T}, p^{-k})}{\alpha_p(T)} (p^k \varepsilon)^{\nu_p(\det(T)} \]

for \( d_0 \in \mathcal{F}_p \), where \( x(d_0) = e \) or \( o \) according as \( \nu_p(d_0) \) is even or odd. Recall that

\[ b_p(2^{-\delta_2,T}, p^{-k}) = \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-2k+2i})}{1 - \xi(2^{-\delta_2,T})p^{-k+n/2}} F_p^{(0)}(T, p^{-k}) \]

and

\[ F_p^{(0)}(T, p^{-k}) = p^{(-k/2+(n+1)/4)(\nu_p(\det T)-\nu_p(d_0))} \tilde{F}_p^{(0)}(T, p^{-k+(n+1)/2}). \]

Hence we have

\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^{n/2}p^{-\nu_p(d_0)}d_0) = \phi_n(p^{-1}) \frac{1 - p^{-k}}{1 - \xi(2^{-\delta_2,T})p^{-k+n/2}} \]

\[ \times p^{(k/2-(n+1)/4)\nu_p(d_0)} T(p, \varepsilon^l, p^{-k+(n+1)/2}, u p^{-k/2+(n+1)/4}). \]

Let \( T(d_0, \omega, X, t) \) denote the right-hand side of the formula for \( \omega = \varepsilon^l \) \((l = 0, 1)\) in the proposition. Then, by [[Sai2], Theorem 3.4 (2)], we have

\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^{n/2}p^{-\nu_p(d_0)}d_0) = \phi_n(p^{-1}) \frac{1 - p^{-k}}{1 - \xi(T)p^{-k+n/2}} \]

\[ \times p^{(k/2-(n+1)/4)\nu_p(d_0)} T(d_0, \varepsilon^l, p^{-k+(n+1)/2}, u p^{-k/2+(n+1)/4}). \]

(Remark that there are misprints in [Sai2]; the \((q^{-1})_n\) on page 197, lines 9 and 15 should be \((q^{-1})_p\).) Hence we have

\[ P_n^{(0)}(d_0, \varepsilon^l, p^{-k+(n+1)/2}, u p^{-k/2+(n+1)/4}) = T(d_0, \varepsilon^l, p^{-k+(n+1)/2}, u p^{-k/2+(n+1)/4}) \]

for infinitely many positive integers \( k \). Hence we have

\[ P_n^{(0)}(d_0, \varepsilon^l, X, t) = T(d_0, \varepsilon^l, X, t). \]

\[ \square \]

**Proof of Theorem 4.1.**

Put \( \Omega = \{ \omega_p \} \), and let \( d_0 \in \mathcal{F}((-1)^{n/2}) \). Put

\[ P(s, d_0, \Omega, \chi) = \prod_p P_n^{(0)}(d_0, \varepsilon^l, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)). \]
Then by Proposition 4.3, we have

\[
P(s, d_0, \{t_p\}, \chi) = |d_0|^{-s+k/2+n/4-3/4} \chi(d_0) \prod_{i=1}^{n-1} \zeta(2i) L(n/2, \chi_{d_0}) \prod_{i=0}^{n/2} L(2s + 2i - n, S(h), \chi^2)
\]

\[\times \prod_{p} ((1 + p^{-2s+k-1} \chi(p)^2)(1 + \chi_{d_0}(p)^2 p^{-2s+k-2} \chi(p)^2)) - \chi_{d_0}(p)p^{-2s+k-3/2} \chi(p)^2 \beta_p(1 + p^{1/2-n/2} \beta_p^{-1})(1 + p^{-1/2+n/2} \beta_p^{-1})].
\]

We note that \(L(s, h)\) and \(L(s, E_{n/2+1})\) can be expressed as

\[
L(s, h) = L(2s, S(h)) \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} c(|d_0|)|d_0|^{-s} \prod_{p} (1 - \chi_{(-1)^{s-n/2}}(d_0)p^{k-n/2-1-2s}),
\]

and

\[
L(s, E_{n/2+1}) = \zeta(2s) \zeta(2s - n + 1) \times \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} L(1 - n/2, \chi_{d_0})|d_0|^{-s} \prod_{p} (1 - \chi_{d_0}(p)p^{n/2-1-2s}),
\]

and therefore, we easily see that \(L(s, h, E_{n/2+1/2}, \chi)\) can be expressed as

\[
L(s, h, E_{n/2+1/2}, \chi) = L(2s, S(h), \chi^2)L(2s - n + 1, S(h), \chi^2)
\]

\[\times \prod_{d_0 \in \mathcal{F}((-1)^{n/2})} |d_0|^{-s} c(|d_0|) \chi(d_0) L(1 - n/2, \chi_{d_0})
\]

\[\times \prod_{p} ((1 + p^{-2s+k-1} \chi(p)^2)(1 + \chi_{d_0}(p)^2 p^{-2s+k-2} \chi(p)^2)) - \chi_{d_0}(p)p^{-2s+k-3/2} \chi(p)^2 \beta_p(1 + p^{1/2-n/2} \beta_p^{-1})(1 + p^{-1/2+n/2} \beta_p^{-1})\]

(cf. [Sh1], Lemma 1.) Thus, by remarking the functional equation

\[
L(1 - n/2, \chi_{d_0}) = 2^{1-n/2} \pi^{n/2} \frac{\Gamma(n/2)}{|d_0|^{(n-1)/2}} L(n/2, \chi_{d_0}),
\]

we have

\[
\sum_{d_0 \in \mathcal{F}((-1)^{n/2})} c_h(|d_0|)|d_0|^{-s+k/2+n/4+1/4} P(s, d_0, \{t_p\}, \chi)
\]

\[= \prod_{i=1}^{n-1} \zeta(2i) \frac{2^{n/2-1} \pi^{n/2}}{\Gamma(n/2)} L(s, h, E_{n/2+1/2}; \chi) \prod_{i=1}^{n/2-1} L(2s - 2i + n, S(h), \chi^2).
\]

On the other hand, if \(d_0 \neq 1\), by Proposition 4.3, we have

\[
P(s, d_0, \{\varepsilon_p\}, \chi) = 0.
\]
Thus if \( n \equiv 2 \mod 4 \), for any \( d_0 \in \mathcal{F}((-1)^{n/2}) \),
\[ P(s, d_0, \{\varepsilon_p\}, \chi) = 0. \]

If \( n \equiv 0 \mod 4 \), by Proposition 4.3, we have
\[ P(s, 1, \{\varepsilon_p\}, \chi) = \zeta(n/2) \prod_{i=1}^{n/2} \zeta(2i) \prod_{i=1}^{n/2} L(2s - 2i + 1, S(h), \chi^2). \]

Thus the assertion follows from Theorem 4.2. \( \square \)

5 Relation between twisted K-M series of the first and second kinds

Let \( N \) be a positive integer. Let \( g \) be a periodic function on \( \mathbb{Z} \) with a period \( N \) and \( \phi \) a polynomial in \( t_1, \ldots, t_r \). Then for an element \( u = (a_1 \mod N, \ldots, a_r \mod N) \in (\mathbb{Z}/N\mathbb{Z})^r \), the value \( g(\phi(a_1, \ldots, a_r)) \) does not depend on the choice of the representative of \( u \). Therefore we denote this value by \( g(\phi(u)) \). In particular we sometimes regard a Dirichlet character mod \( N \) as a function on \( \mathbb{Z}/N\mathbb{Z} \).

For a Dirichlet character \( \chi \mod N \) and \( A \in \mathcal{L}_{m>0} \), put
\[ h(A, \chi) = \sum_{U \in SL_m(\mathbb{Z}/N\mathbb{Z})} \chi(\text{tr}(A[U])). \]

As was shown in [[K-M], Proposition 3.3], the twisted Koecher-Maaß series of the first kind of a Siegel modular form can be expressed in terms of \( h(A, \chi) \) as stated later. Therefore we shall compute \( h(A, \chi) \) in the case where \( A \) is an element of \( \mathcal{L}_{m>0} \). For \( A = (a_{ij})_{m \times m} \in S_m(\mathbb{Z}/N\mathbb{Z}) \) and \( c \in \mathbb{Z}/N\mathbb{Z} \), put
\[ R_N(A, c) = \{ X = (x_{ij})_{m \times m} \in M_n(\mathbb{Z}/N\mathbb{Z}) \mid \sum_{i=1}^{m} \sum_{\alpha, \beta=1}^{m} a_{\alpha, \beta} x_{i\alpha} x_{i\beta} = c \}
\[ \text{and } \det X - 1 = 0 \}. \]

Then we have
\[ h(A, \chi) = \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \chi(c) \#(R_N(A, c)). \]

From now on let \( p \) be an odd prime number and \( F_p \) be the field with \( p \)-elements. For \( S \in S_m(F_p) \) and \( T \in S_r(F_p) \) put
\[ A(S, T) = \{ Y = M_{r,m}(F_p) \mid YS^t Y = T \}. \]

For an element \( S \in S_m(F_p) \) with \( m \) even put \( \chi(S) = \left( \frac{(-1)^{m/2} \det S}{p} \right) \).
Lemma 5.1. Let \( S \in S_m(F_p)^\times \).

(1) Let \( T \in S_r(F_p) \) with \( m \geq r \).

(1.1) Let \( r \) be even. Then
\[
\#A(S, T) = p^r m - (r-1)/2 \left( 1 - \chi(S) p^{-m/2} \right) (1 + \chi((-S) \downarrow T) p^{(r-m)/2}) \prod_{m-r+1 \leq x \leq m-1} (1 - p^{-x})
\]
or
\[
\#A(S, T) = p^r m - (r-1)/2 \prod_{m-r+1 \leq x \leq m-1} (1 - p^{-x})
\]
according as \( m \) is even or odd.

(1.2) Let \( r \) be odd. Then
\[
\#A(S, T) = p^r m - (r-1)/2 \left( 1 - \chi(S) p^{-m/2} \right) \prod_{m-r+1 \leq x \leq m-1} (1 - p^{-x})
\]
or
\[
\#A(S, T) = p^r m - (r-1)/2 \left( 1 + \chi((-S) \downarrow T) p^{(r-m)/2} \right) \prod_{m-r+1 \leq x \leq m-1} (1 - p^{-x})
\]
according as \( m \) is even or odd. In particular, for \( c \in F_p^\times \), we have
\[
\#A(S, c) = p^r m/2 - 1 \left( \frac{(-1)^{m/2} \det S}{p} \right)
\]
or
\[
\#A(S, c) = p^{(m-1)/2} \left( \frac{(-1)^{m/2} \det S}{p} \right)
\]
according as \( m \) is even or odd.

(2) We have
\[
\#A(S, 0) = p^r m/2 - 1 \left( \frac{(-1)^{m/2} \det S}{p} \right) + p^r m/2 \left( \frac{(-1)^{m/2} \det S}{p} \right)
\]
or
\[
\#A(S, 0) = p^{m-1}
\]
according as \( m \) is even or odd.

Proof. The assertions (1) and (2) follow from [[Ki1], Theorem 1.3.2], and [[Ki1], Lemma 1.3.1], respectively.

Proposition 5.2. Let \( A = a_1 \cdots a_m \) with \( a_i \in F_p \). For \( c \in F_p^\times \) put
\[
\mathcal{M}_p(A, c) = \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\]

\[
\begin{align*}
\mathcal{M}_p(A, c) &= \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_p(A, c) &= \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_p(A, c) &= \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_p(A, c) &= \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\end{align*}
\]
and
\[ \gamma_{m,p} = p^{m^2-m(m+1)/2}(1-p^{-m/2}) \prod_{e=1}^{(m-2)/2} (1-p^{-2e}) \]
or
\[ \gamma_{m,p} = p^{m^2-m(m+1)/2} \prod_{e=1}^{(m-1)/2} (1-p^{-2e}) \]
according as \( m \) is even or odd. Then we have
\[ \#R_p(A,c) = \gamma_{m,p} \#M_p(A,c). \]

Proof. Let \( \Phi : GL_m(F_p) \to S_m(F_p) \cap GL_m(F_p) \) be the mapping defined by \( \Phi(X) = X^tX \). Then by Lemma 5.1, we have \( \#\Phi^{-1}(Z) = 2\gamma_{m,p} \) for any \( Z \in S_m(F_p) \cap SL_m(F_p) \). We note that \( \det X = \frac{1}{Z} \) for any \( X \in \Phi^{-1}(Z) \). Hence we have \( \#(\Phi^{-1}(Z) \cap SL_m(F_p)) = \gamma_{m,p} \). Moreover we have
\[ \text{tr}(XAX) = \text{tr}(AX^tX), \]
and hence \( X \in R_p(A,c) \) if and only if \( \Phi(X) \in M_p(A,c) \). This proves the assertion. \( \square \)

We rewrite \( M_p(A,c) \) in more concise form. Let \( p \) be a prime number and \( l \) be a positive integer dividing \( p-1 \). Take an \( l \)-th root of unity \( \zeta_l \) and a prime ideal \( \mathfrak{p} \) of \( \mathbb{Q}(\zeta_l) \) lying above \( p \). Let \( a \) be an integer prime to \( p \). Then we have
\[ a^{(p-1)/l} \equiv \zeta_l^i \pmod{\mathfrak{p}} \]
with some \( i \in \mathbb{Z} \). We then put \( \left( \frac{a}{p} \right)_l = \zeta_l^i \). We call \( \left( \frac{a}{p} \right)_l \) the \( l \)-th power residue symbol mod \( p \). In the case \( l = 2 \), this is the Legendre symbol, and we write it as \( \left( \frac{a}{p} \right) \) as usual. We note that this definition of the power residue symbol is different from the usual one, and depends on the choice of \( \mathfrak{p} \) and \( \zeta_l \) except the case \( l = 2 \). We denote by \( \left( \frac{a}{N} \right) \) the Jacobi symbol for a positive odd integer. Let \( \chi \) be a primitive Dirichlet character of conductor \( N \). We assume that \( N \) is a square free odd integer, and write \( N = p_1 \cdots p_r \) with \( p_1, \cdots, p_r \) prime numbers. Put \( l_j = l_{m,p_j} = \text{GCD}(m,p_j-1) \). For an \( r \)-tuple \( I = (i_1, i_2, \cdots, i_r) \) of integers put
\[ \chi_{(i_1, \cdots, i_r)} = \chi \prod_{j=1}^{r} \left( \frac{a}{p_j} \right)_{l_j}^{i_j}. \]

For two Dirichlet characters \( \chi \) and \( \eta \) mod \( N \) we define \( J_m(\chi,\eta) \) and \( I_m(\chi,\eta) \)
\[ J_m(\chi,\eta) = \sum_{Z \in S_m(Z/N)} \chi(\det Z)\eta(1-\text{tr}(Z)) \]
and
\[ I_m(\chi,\eta) = \sum_{Z \in S_m(Z/N)} \chi(\det Z)\eta(\text{tr}(Z)). \]
By definition, $J_m(\chi, \eta)$ is an algebraic number. We note that $J_1(\chi, \eta)$ is the Jacobi sum $J(\chi, \eta)$ associated with $\chi$ and $\eta$. We also define $J_m(\chi)$ as $J_m(\chi) = J_m(\chi, \chi)$.

**Lemma 5.3.** Let $\eta$ be a primitive character mod $p$. Let $c \in F_p$ and $S \in S_l(F_p)$ of rank $r$. Let $S \sim S_0 \perp O_{1-r}$ with $\det S_0 \neq 0$. Put

$$I_{\eta, S, c} = \sum_{\omega \in F_p} \eta(S[w] + c).$$

Assume that $r$ is odd, and that $\eta^2 \neq 1$. Then

$$I_{\eta, S, c} = p^{l-(r+1)/2} J(\eta, \left(\frac{\cdot}{p}\right)) \left(\frac{-1}{p}\right)^{(r+1)/2} \det S_0 \frac{\eta(c)}{p}.$$ 

Assume that $r$ is even, and that $\eta \neq 1$. Then

$$I_{\eta, S, c} = p^{l-r/2} \left(\frac{-1}{p}\right)^{r/2} \det S_0 \eta(c).$$

Here we make the convention that $\left(\frac{-1}{p}\right)^{r/2} \det S_0 = 1$ if $r = 0$.

**Proof.** We have

$$I_{\eta, S, c} = p^{l-r} I_{\eta, S_0, c}.$$ 

Hence we may assume that $r = l$. Then

$$I_{\eta, S, c} = \sum_{u \in F_p} \eta(u) \#A(S, u - c).$$

Let $l$ be odd. Then by Lemma 5.1,

$$\#A(S, u - c) = p^{l-1/2} \left(p^{l-1/2} + \left(\frac{-1}{p}\right)^{(l-1)/2} \det S \right).$$

Hence we have

$$I_{\eta, S, c} = p^{l-1/2} \left(\frac{-1}{p}\right)^{(l+1)/2} \det S \sum_{u \in F_p} \eta(u) \left(\frac{u - c}{p}\right).$$

Since $\eta^2$ is nontrivial, we have $I_{\eta, S, c} = 0$ if $c = 0$. If $c \neq 0$, then

$$\sum_{u \in F_p} \eta(u) \left(\frac{u - c}{p}\right) = \left(\frac{-c}{p}\right) \sum_{u \in F_p} \eta(u) \left(\frac{1 - c^{-1}u}{p}\right) = \eta(c) \left(\frac{-c}{p}\right) J(\eta, \left(\frac{\cdot}{p}\right)).$$
Let $l$ be even. Then
\[
#A(S, u-c) = \left(\frac{p^{l/2} - \left(\frac{(-1)^{l/2} \det S}{p}\right)}{p^{l/2-1}} + \frac{p^{l/2} \left(\frac{(-1)^{l/2} \det S}{p}\right)}{a_0}\right)
\] where $a_0 = 1$ or $0$ according as $u = c$ or not. Hence
\[
I_{\eta, S, c} = p^{l/2} \left(\frac{(-1)^{l/2} \det S}{p}\right) \eta(c).
\]

\[\square\]

**Corollary.** Let $d \in \mathbb{F}_p^\times$. Then we have
\[
I_{\eta, S, cd} = \eta(d) \left(\frac{d}{p}\right) I_{\eta, S, c}.
\]

**Proposition 5.4.** Let $\eta$ be a primitive character mod $p$. For $Z_1 \in S_{l-1}(\mathbb{F}_p)$ and $z_{ll} \in \mathbb{F}_p$, put
\[
I(Z_1, z_{ll}) = \sum_{w \in M_{l-1}(\mathbb{F}_p)} \eta \left(\frac{Z_1 w}{t_w z_{ll}}\right).
\]

(1) Assume that $l$ is even, and that $\eta^2 \neq 1$. Then
\[
I(Z_1, z_{ll}) = p^{(l-2)/2} f(\eta, \left(\frac{Z_1}{p}\right)) \left(\frac{(-1)^{l/2} \det Z_1}{p}\right) \eta(\det Z_1 z_{ll}) \left(\frac{z_{ll}}{p}\right).
\]

(2) Assume that $l$ is odd, and that $\eta^2 \neq 1$. Then
\[
I(Z_1, z_{ll}) = p^{(l-1)/2} \left(\frac{(-1)^{(l-1)/2} \det Z_1}{p}\right) \eta(\det Z_1 z_{ll}).
\]

**Proof.** We note that
\[
\det \left(\frac{Z_1 w}{t_w z_{ll}}\right) = -\text{Adj}(Z_1)[w] + \det Z_1 z_{ll},
\]
where $\text{Adj}(Z_1)$ is the $(l-1) \times (l-1)$ matrix whose $(i, j)$-th component is the $(j, i)$-th cofactor of $Z_1$. We also note that $\det(-\text{Adj}(Z_1)) = (-1)^{l-1}(\det Z_1)^{l-2}$. Thus the assertion follows directly from Lemma 5.3 if $\det Z_1 \neq 0$. If $\det Z_1 = 0$, then $\text{rank}_{\mathbb{F}_p}(Z_1) \leq 1$, the assertion follows also from Lemma 5.3. \[\square\]

**Theorem 5.5.** Let $\chi$ be a primitive character mod $p$. Let $l = \text{GCD}(m, p-1)$, and $u_0$ be a primitive $l$-th root of unity mod $p$. Let $A \in S_m(\mathbb{F}_p)$. 
(1) If $\chi(u_0) \neq 1$, then we have $h(A, \chi) = 0$.
(2) Assume that $\chi(u_0) = 1$. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^m = \chi$.
(2.1) Let $m$ be even. Then
\[
h(A, \chi) = \gamma_{m, p} \sum_{i=0}^{l-1} A_{m, i, p} \tilde{\chi}(i)(\det A) J_{m-1}(\bar{\chi}(i)),
\]

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where $A_{m,i,p} = p^{(m-2)/2}(-1)^{(m(p-1))/4}J(\chi(i),\left(\frac{z}{p}\right))$.

(2.2) Let $m$ be odd and assume that $\chi^2 \neq 1$. Then

$$h(A, \chi) = \gamma_m \sum_{i=0}^{l-1} A_{m,i,p} \lambda(\chi(i)) (\det A) J_{m-1}(\lambda(\chi(i)))$$

where $p^{(m-1)/2}(-1)^{(m-1)(p-1)/4}$.

Proof. If $A = O_m$ then we have $h(A, \chi) = 0$. Hence we assume that $A \neq O_m$. Then we may assume that $A = a_1 \cdots a_{m-1} \cdot d$ with $d \neq 0$. Put

$$\mathcal{M}_p(A, c)$$

$$= \{(Z_1, w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p) \mid \det \begin{pmatrix} Z_1 & w \\ t w & z_m \end{pmatrix} d^{-1}(1 - \sum_{i=1}^{m-1} a_i z_{ii}) e^m = 1\}.$$ Write $Z \in S_m(F_p)$ as $Z = \begin{pmatrix} Z_1 & w \\ t w & z_m \end{pmatrix}$ with $Z_1 \in S_{m-1}(F_p), w \in M_{m-1,1}(F_p), z \in F_p$. Then the mapping $S_m(F_p) \ni Z \mapsto (c^{-1} Z_1, c^{-1} w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p)$ induces a bijection from $\mathcal{M}_p(A, c)$ to $\mathcal{M}_p(A, c)$, and hence $\# \mathcal{M}_p(A, c) = \# \mathcal{M}_p(A, c)$. Put

$$K(A) = \sum_c \# \mathcal{M}_p(A, c) \chi(c).$$

Assume that $\chi(u_0) \neq 1$. Then we have

$$K(A) = \sum_{c \in F_p} \chi(c u_0) \# \mathcal{M}_p(A, cu_0).$$

We note that $\mathcal{M}_p(A, cu_0) = \mathcal{M}_p(A, c)$. Hence we have

$$K(A) = \chi(u_0) K(A).$$

Hence we have $K(A) = 0$.

Assume that $\chi(u_0) = 1$. Then we can take a Dirichlet character $\tilde{\chi}$ such that $\tilde{\chi}^m = \chi$. First assume that $\det A = 0$. Then we may assume that we have $A = A_0 \cdot 0$ with $A_0 \in S_{m-1}(F_p)$. Let $P_{m-1,m}$ be the set of $(m-1) \times m$ matrices with entries in $F_p$ of rank $m-1$. Then for each $X_1 \in P_{m-1,m}$ there exist exactly $p^{m-1}$ elements $X_2 \in M_{1,m}(F_p)$ such that $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in SL_m(F_p)$. Hence we have

$$h(A, \chi) = \sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1]).$$

Let $m$ be even. Then we can take an element $\alpha \in F_p^\times$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that $\alpha U_0 U_0^\prime = \alpha 1_m$ in view of (1.1) of Lemma 5.1. Hence

$$h(A, \chi) = p^{m-1} \sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1 U_0]) = \chi(\alpha) h(A, \chi).$$
Hence we have $h(A, \chi) = 0$. Let $m$ be odd and assume that $\chi^2 \neq 1$. Then we can take an element $\alpha \in (F_p^\times)^2$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that $U_0 U_0^t = \alpha I_m$ in view of (1.2) of Lemma 5.1. Thus by the same argument as above we have $h(A, \chi) = 0$. This proves the assertion.

Next assume that $\det A \neq 0$. We may assume that $A = 1_{m-1} \perp d$ with $d = \det A$. Then we have

$$K(A) = \sum_c \# \bar{\mathcal{M}_p}(A, c) \chi(\text{cyclic genus}).$$

Hence we have

$$K(A) = \sum_{(Z, w)} \bar{\chi}(\det \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right)), \quad (Z, w) \text{ runs over elements of } S_{m-1}(F_p) \times M_{m-1,1}(F_p) \text{ such that}$$

$$(*) \quad \det \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right) = u^m \quad \text{with some } u \in F_p^\times, \quad \text{for such a matrix} \quad \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right), \quad \text{there exist exactly } l \text{ elements } u \text{ of } F_p \text{ satisfying } (*) \quad \text{We have}$$

$$\sum_{i=0}^{l-1} \left( \frac{v}{p} \right)_i^l = l \text{ or } 0 \quad \text{according as } v = u^m \text{ with some } u \in F_p^\times \text{ or not. Hence we have}$$

$$K(A) = \sum_{i=0}^{l-1} \bar{\chi}(\det \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right)) \times \left( \frac{\det \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right)}{p} \right)_i^l$$

$$= \sum_{i=0}^{l-1} \bar{\chi}(i)(\det \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right))$$

Put

$$K(A)_i = \sum_{i=0}^{l-1} \bar{\chi}(i)(\det \left( \begin{array}{cc} Z & w \\ t_w & d^{-1}(1 - \text{tr}(Z)) \end{array} \right))$$

We note that $\bar{\chi}^2(i) \neq 1$ for any $i$. Hence by Proposition 5.4 we have

$$K(A)_i = A_{m_i, p} \sum_{Z \in S_{m-1}(F_p)} \bar{\chi}(Z)(\det A) \bar{\chi}(Z)(1 - \text{tr}(Z)).$$
where $\tilde{\chi}_i^* = \tilde{\chi}_i \left( \frac{2}{p} \right)^{m-1}$. This proves the assertion if $m$ is odd. Assume that $m$ is even. Then it is easily seen that the set $\{ \tilde{\chi}_i \left( \frac{2}{p} \right) \}_{i=0}^{l-1}$ of Dirichlet characters coincides with $\{ \tilde{\chi}_i \}_{i=0}^{l-1}$. Moreover $\tilde{\chi}_i^2 \neq 1$ for any $i$. This proves the assertion.

**Theorem 5.6.** Let $N = p_1 \cdots p_r$. Let $\chi$ be a primitive Dirichlet character mod $N$. Let $u_{0,i}$ be a primitive $l_i$-th root of unity mod $p_i$. Let $A \in S_m(F_p)$.

1. If $\chi^*(u_{0,i}) \neq 1$ for some $i$. Then we have $h(A, \chi) = 0$.
2. Assume that $\chi^*(u_{0,i}) = 1$ for any $i$. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^m = \chi$.

(2.1) Let $m$ be even. Then we have

\[
h(A, \chi) = \prod_{i=1}^{r} (-1)^{m(p_i-1)/2} p_i^{(m-2)/2} \gamma_{m,p_i}
\]
\[
\times \sum_{i=0}^{l_i-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(1, \ldots, i_r) (\text{det } A) \tilde{J}(\tilde{\chi}(1, \ldots, i_r), \left( \frac{\cdot}{N} \right)) J_{m-1}(\tilde{\chi}(1, \ldots, i_r)).
\]

(2.2) Let $m$ be odd, and assume that $\chi^2$ is primitive. Then we have

\[
h(A, \chi) = \prod_{i=1}^{r} (-1)^{(m-1)(p_i-1)/2} p_i^{(m-1)/2} \gamma_{m,p_i}
\]
\[
\times \sum_{i=0}^{l_i-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(1, \ldots, i_r) (\text{det } A) J_{m-1}(\tilde{\chi}(1, \ldots, i_r)).
\]

**Proof.** We note that $J_m(\eta_1, \eta_2) = \prod_{i=1}^{r} J_m(\eta_i^{(p_i)}, \eta_i^{(p_i)})$ for primitive characters $\eta_1$ and $\eta_2$ mod $N$. Moreover $\eta_2$ is primitive if and only if $\eta_i^{(p_i)}$ is primitive if and only if $\eta_i^{(p_i)} \neq 1$ for any $1 \leq i \leq r$. Thus the assertion follows from Theorem 5.5 and [[K-M], Lemma 3.2].

Now we give explicit formulas for $J_m(\chi, \eta)$ and $I_m(\chi, \eta)$.

**Proposition 5.7.** Let $\chi$ and $\eta$ be primitive characters mod $p$. Assume that $\chi^2 \neq 1$. Put $c_m(\chi, \eta) = 1$ or 0 according as $\chi^m \eta = 1$ or not.

(1) **Assume that $m$ is odd.** Then

\[
I_m(\chi, \eta) = c_m(\chi, \eta) \left( -1 \right)^{(m-1)/2} p^{(m-1)/2} (p-1) J_{m-1}(\chi \left( \frac{\cdot}{p} \right), \eta).
\]

(2) **Assume that $m$ is even.** Then

\[
I_m(\chi, \eta) = c_m(\chi, \eta) \left( -1 \right)^{m/2} p^{(m-2)/2} (p-1) J_{m-1}(\chi \left( \frac{\cdot}{p} \right), \eta).
\]
Proof. By Proposition 5.4, we have

\[ I_m(\chi, \eta) = I'_m \times \begin{cases} \frac{p^{(m-1)/2}}{p} \left( \frac{(-1)^{(m-1)/2}}{p} \right) & \text{if } m \text{ is odd} \\ \frac{p^{(m-2)/2}}{p} \left( \frac{(-1)^{(m-2)/2}}{p} \right) J(\chi, \left( \frac{z}{p} \right)) & \text{if } m \text{ is even} \end{cases} \]

where

\[ I'_m = \sum_{z_{mm} \in F_p^\times, z_{1} \in S_{m-1}(F_p)^\times} \chi(z_{mm}) \chi(\det Z_1) \left( \frac{\det Z_1}{p} \right) \eta(z_{mm} + \text{tr}(Z_1)) \left( \frac{z_{mm}}{p} \right)^{m-1}. \]

Then we have

\[ I'_m = \sum_{z_{mm} \in F_p^\times, z_{1} \in S_{m-1}(F_p)^\times} \chi(z_{mm}) \eta(z_{mm}) \chi(\det Z_1) \left( \frac{\det Z_1}{p} \right) \eta(1 + z_{mm} \text{tr}(Z_1)) \left( \frac{z_{mm}}{p} \right)^{m-1}. \]

Put \( Y_1 = -z_{mm}^{-1} Z_1. \) Then \( \det Y_1 = (-1)^{m-1} z_{mm}^{-1} \det Z_1. \) Hence we have

\[ I'_m = \chi((-1)^{m-1}) \left( \frac{(-1)^{m-1}}{p} \right) \]

\[ \times \sum_{z_{mm} \in F_p^\times} \chi(z_{mm}) \eta(z_{mm}) \sum_{Y_1 \in S_{m-1}(F_p)^\times} \chi(\det Y_1) \left( \frac{\det Y_1}{p} \right) \eta(1 - \text{tr}(Y_1)). \]

We have

\[ \sum_{z_{mm} \in F_p^\times} \chi(z_{mm})^m \eta(z_{mm}) = p - 1 \text{ or } 0 \]

according as \( \chi^m \eta \) is trivial or not. This proves the assertion. \( \square \)

**Proposition 5.8.** Let \( \chi \) and \( \eta \) be as in Proposition 5.7.

1. Assume that \( m \) is odd. Then

\[ J_m(\chi, \eta) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} \]

\[ \times \{ J(\chi, \chi^{m-1} \eta) J_{m-1}(\chi \left( \frac{z}{p} \right), \eta) + \eta(-1) I_{m-1}(\chi \left( \frac{z}{p} \right), \eta) \}. \]

2. Assume that \( m \) is even. Then

\[ J_m(\chi, \eta) = \left( \frac{-1}{p} \right)^{m/2} p^{(m-2)/2} J(\chi, \left( \frac{z}{p} \right)) \]

\[ \times \{ J(\chi \left( \frac{z}{p} \right), \chi^{m-1} \left( \frac{z}{p} \right) \eta) J_{m-1}(\chi \left( \frac{z}{p} \right), \eta) + \eta(-1) I_{m-1}(\chi \left( \frac{z}{p} \right), \eta) \}. \]
Proof. By Proposition 5.4, we have

\[ J_m(\chi, \eta) = (J'_m + J''_m) \times \begin{cases} 
  p^{(m-1)/2} \left( \frac{(-1)^{(m-1)/2}}{p} \right) J(\chi, \left( \frac{z}{p} \right)) & \text{if } m \text{ is odd} \\
  p^{(m-2)/2} \left( \frac{(-1)^{(m-2)/2}}{p} \right) J(\chi, \left( \frac{z}{p} \right)) & \text{if } m \text{ is even,}
\end{cases} \]

where

\[ J'_m = \sum_{z_{mm} \in F_p, \ z_{mm} \neq 1 \atop Z_1 \in S_{m-1}(F_p)^{\times}} \left( \frac{\det Z_1}{p} \right) \left( \frac{z_{mm}}{p} \right)^{m-1} \chi(z_{mm}) \chi(\det Z_1) \eta(1-z_{mm}^{-1} \tr(Z_1)), \]

and

\[ J''_m = \sum_{Z_1 \in S_{m-1}(F_p)^{\times}} \left( \frac{\det Z_1}{p} \right) \chi(\det Z_1) \eta(-\tr(Z_1)). \]

Then we have \( J''_m = \eta(-1)J_{m-1}(\chi(\left( \frac{z}{p} \right)), \eta). \) Moreover

\[ J'_m = \sum_{z_{mm} \in F_p, \ z_{mm} \neq 1 \atop Z_1 \in S_{m-1}(F_p)^{\times}} \chi(z_{mm}) \left( \frac{\det Z_1}{p} \right) \left( \frac{z_{mm}}{p} \right)^{m-1} \chi(\det Z_1) \times \eta(1-z_{mm}) \eta(1-(1-z_{mm}^{-1}) \tr(Z_1)). \]

Put \( Y_1 = (1-z_{mm})^{-1} Z_1. \) Then \( \det Y_1 = (1-z_{mm})^{-m} \det Z_1. \) Hence we have

\[ J'_m = \sum_{z_{mm} \in F_p} \chi(z_{mm}) \left( \frac{z_{mm}}{p} \right)^{m-1} \left( \frac{1-z_{mm}}{p} \right)^{m-1} \chi(1-z_{mm})^{m-1} \eta(1-z_{mm}) \times \sum_{Y_1 \in S_{m-1}(F_p)^{\times}} \left( \frac{\det Y_1}{p} \right) \chi(\det Y_1) \eta(1-\tr(Y_1)). \]

This proves the assertion. \( \square \)

**Theorem 5.9.** Let \( \chi \) be a primitive character mod \( p. \)

(1) Let \( m \) be odd, and assume that \( \chi^2 \neq 1. \)

(1.1) Assume that \( \chi^m \neq 1. \) Then

\[ J_m(\chi(\left( \frac{z}{p} \right)^i), \chi) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} J(\chi(\left( \frac{z}{p} \right)^i), \chi^m) J_{m-1}(\chi(\left( \frac{z}{p} \right)^{i+1}), \chi). \]

(1.2) Assume that \( \chi^m = 1. \) Then

\[ J_m(\chi(\left( \frac{z}{p} \right)^i), \chi) = p^{m-1} \left( \frac{-1}{p} \right)^{i+1} J(\chi(\left( \frac{z}{p} \right)^{i+1}), \chi^m) J_{m-2}(\chi(\left( \frac{z}{p} \right)^i), \chi). \]
(2) Let $m$ be even.

(2.1) Assume that $\chi^m \left( \frac{a}{p} \right)^{i+1} \neq 1$. Then

\[ J_m(\chi \left( \frac{a}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-2)/2} \frac{J(\chi \left( \frac{a}{p} \right)^i, \left( \frac{a}{p} \right)^{i+1})J(\chi \left( \frac{a}{p} \right)^{i+1}, \chi^m \left( \frac{a}{p} \right)^{i+1})J_{m-1}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi). \]

(2.2) Assume that $\chi^m \left( \frac{a}{p} \right)^{i+1} = 1$. Then

\[ J_m(\chi \left( \frac{a}{p} \right)^i, \chi) = \chi(-1)p^{m-1}J(\chi \left( \frac{a}{p} \right)^i, \left( \frac{a}{p} \right)^{i+1})J_{m-2}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi). \]

Proof. Let $m$ be odd. Then, by (1) of Proposition 5.8, we have

\[ J_m(\chi \left( \frac{a}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-1)/2}p^{(m-1)/2} \]

\[ \times \{J(\chi \left( \frac{a}{p} \right)^i, \chi^m)J_{m-1}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi) + \chi(-1)J_{m-1}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi)\}. \]

Thus the assertion holds if $\chi^m \neq 1$. Assume that $\chi^m = 1$. Then by (2) of Proposition 5.8 and (2) of Proposition 5.7 we have

\[ J_{m-1}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi) = \left( \frac{-1}{p} \right)^{(m-1)/2}p^{(m-3)/2}J(\chi \left( \frac{a}{p} \right)^i, \left( \frac{a}{p} \right)^{i+1}) \]

\[ \times J(\chi \left( \frac{a}{p} \right)^i, \chi^{m-1} \left( \frac{a}{p} \right)^i)J_{m-2}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi). \]

and

\[ I_{m-1}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi) = \left( \frac{-1}{p} \right)^{(m-3)/2}p^{(m-3)/2}(p-1)\chi(-1) \left( \frac{-1}{p} \right)^{i+1} \]

\[ \times J(\chi \left( \frac{a}{p} \right)^i, \left( \frac{a}{p} \right)^{i+1})J_{m-2}(\chi \left( \frac{a}{p} \right)^i, \chi). \]

We note that $J(\chi \left( \frac{a}{p} \right)^i, \chi^{m-1}) = -1, \chi(-1) = 1$ and

\[ J(\chi \left( \frac{a}{p} \right)^i, \chi^{m-1} \left( \frac{a}{p} \right)^i) = J(\chi \left( \frac{a}{p} \right)^i, \chi \left( \frac{a}{p} \right)^i) = \chi(-1) \left( \frac{-1}{p} \right)^i = \left( \frac{-1}{p} \right)^{i}. \]

This proves the assertion.

Let $m$ be even. Then, by (2) of Proposition 5.8, we have

\[ J_m(\chi \left( \frac{a}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-2)/2}p^{(m-2)/2}J(\chi \left( \frac{a}{p} \right)^i, \left( \frac{a}{p} \right)^{i+1}) \]
Thus the assertion holds if $\chi^m\left(\frac{\tau}{p}\right)^{i+1} \neq 1$. Assume that $\chi^m\left(\frac{\tau}{p}\right)^{i+1} = 1$. Then by (1) of Proposition 5.7 and (1) of Proposition 5.8, we have

$$J_{m-1}\left(\frac{\tau}{p}\right)^{i+1}, \chi = \left(-\frac{1}{p}\right)^{(m-2)/2} p^{(m-2)/2}$$

and

$$I_{m-1}\left(\frac{\tau}{p}\right)^{i+1}, \chi = \left(-\frac{1}{p}\right)^{(m-2)/2} p^{(m-2)/2} J_{m-2}\left(\frac{\tau}{p}\right)^i, \chi.$$}

We note that $J\left(\frac{\tau}{p}\right)^i, \chi^m\left(\frac{\tau}{p}\right)^{i+1} = -1, \left(\frac{-1}{p}\right)^{i+1} = 1$ and

$$J\left(\frac{\tau}{p}\right)^i, \chi^{m-1} = J\left(\frac{\tau}{p}\right)^{i+1}, \chi^{m-1} = \chi(-1) \left(-\frac{1}{p}\right)^{i+1} = \chi(-1).$$

This proves the assertion.

\[\square\]

**Corollary.** Let $\chi$ be a primitive character with an odd square free conductor $N$. Assume that $\chi^2$ is primitive. Then the value $J_m(\chi)$ is nonzero.

**Proof.** The assertion follows directly from the above theorem if $N$ is an odd prime. In general case, the assertion can also be proved by remarking that $J_m(\chi) = \prod_{p|N} J_m(\chi^{(p)})$ and that $\chi^{(p)} \neq 1$ for any $p|N$.

To compare our present result with the result in [K-M], we give the following:

**Proposition 5.10.** Let $\chi$ be a primitive Dirichlet character mod $p$. Assume that $\chi^2 \neq 1$. Then we have

$$J(\chi, \left(\frac{s}{p}\right)) J(\chi, \left(\frac{s}{p}\right), \chi) = \left(-\frac{1}{p}\right) \chi(4)p.$$

**Proof.** Put

$$I = \sum_{(z,w) \in F_p^2} \chi(z(1-z) - w^2).$$

Then by using the same argument as in the proof of Theorem 5.5, we have

$$I = J(\chi, \left(\frac{s}{p}\right)) \sum_{z \in F_p} \chi(z(1-z)) \left(\frac{z(1-z)}{p}\right)$$

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\[ J(\chi, \left( \frac{\alpha}{p} \right)) J(\chi, \left( \frac{\alpha}{p} \right), \chi(\frac{\alpha}{p})). \]

On the other hand, we have

\[ I = \sum_{(y, w) \in F_2} \chi(-y^2 - w^2 + 1/4). \]

Hence by Lemma 5.3 we have

\[ I = p \left( \frac{-1}{p} \right) \tilde{\chi}(4). \]

This proves the assertion. \(\square\)

By virtue of the above proposition, we see that Theorem 5.6 coincides with [K-M], Proposition 3.7 in case \(m = 2\).

Now let

\[ F(Z) = \sum_{A \in L_{n \geq 0}} c_F(A)\mathrm{e}(\mathrm{tr}(AZ)) \]

be an element of \(\mathbb{R}_k(Sp_n(\mathbb{Z}))\) and let \(\chi\) be a Dirichlet character mod \(N\). Assume \(N \neq 2\). Then by [[K-M], Proposition 3.1], we have

\[ L(s, F, \chi) = \sum_{A \in L_{n \geq 0}/SL_n(\mathbb{Z})} \frac{c_F(A)h(A, \chi)}{\mathrm{e}(\det(A))}. \]

Thus by Theorem 5.6 we easily obtain:

**Theorem 5.11.** Let \(N, p, l, u_i, (i = 1, \ldots, r)\) and \(\chi\) be as in Theorem 5.6, and let \(F\) be an element of \(\mathbb{R}_k(Sp_n(\mathbb{Z}))\).

1. If \(\chi^{(p_i)}(u_{0,i}) \neq 1\) for some \(i\). Then we have \(L(s, F, \chi) = 0\).
2. Assume that \(\chi^{(p_i)}(u_{0,i}) = 1\) for any \(i\). Fix a character \(\tilde{\chi}\) such that \(\tilde{\chi}^n = \chi\).

2.1 Let \(n\) be even. Then we have

\[ L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-2)(p_i-1)}/4 \gamma_{n, p_i} \]

\[ \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(\alpha_{1, \ldots, r}) (2^n J(\tilde{\chi}(\alpha_{1, \ldots, r})) (\frac{n}{N})) J_{n-1}(\tilde{\chi}(\alpha_{1, \ldots, r})) L^*(s, F, \tilde{\chi}(\alpha_{1, \ldots, r})). \]

2.2 Let \(n\) be odd, and assume that \(\chi^2 \neq 1\). Then we have

\[ L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-1)(p_i-1)}/4 \gamma_{n, p_i} \]

\[ \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(\alpha_{1, \ldots, r}) (2^{n-1} J_{n-1}(\tilde{\chi}(\alpha_{1, \ldots, r})) L^*(s, F, \tilde{\chi}(\alpha_{1, \ldots, r})). \]
6 Twisted Koecher-Maaß series of the first kind of the DII lift

By Theorems 4.1 and 5.11, we obtain the following.

**Theorem 6.1.** Let \( k \) and \( n \) be positive even integers such that \( n \geq 4, \) \( 2k - n \geq 12. \) Let \( h(z) \) and \( E_{n/2+1/2} \) be as in Section 4. Let \( N \) be a square free odd integer, and \( N = p_1 \cdots p_r \) be the prime decomposition of \( N. \) For each \( i = 1, \cdots, r \) let \( l_i = \text{GCD}(n, p_i - 1) \) and \( u_{i, 1} \in \mathbb{Z} \) be a primitive \( l_i \)-th root of unity mod \( p_i. \)

1. Assume \( \chi^{(p_i)}(u_i) \neq 1 \) for some \( i. \) Then \( L(s, I_n(h), \chi) = 0. \)
2. Assume \( \chi^{(p_i)}(u_i) = 1 \) for any \( i. \) Then

\[
L(s, I_n(h), \chi) = 2^{n/2} \prod_{i=0}^{l_i-1} J(h, h, \chi) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \tilde{\chi}^2(h, h, \chi))
\]

\[
\times \{ c_{n, N} R(s, h, E_{n/2+1/2}, \tilde{\chi}(h, h, \chi)) \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), \tilde{\chi}^2(h, h, \chi)) \}
\]

where \( c_{n, N} \) and \( d_{n, N} \) are nonzero rational numbers depending only on \( n \) and \( N, \) and \( \tilde{\chi} \) is a character s.t. \( \tilde{\chi}^n = \chi. \)

**Remark.** In the case \( n = 2, \) an explicit formula for \( L(s, I_2(h), \chi) \) was given by Katsurada-Mizuno [K-M].

7 Applications

Let \( h_1 \) and \( h_2 \) be modular forms of weight \( k_1 + 1/2 \) and \( k_2 + 1/2, \) respectively, and \( \chi \) be a Dirichlet character. In Section 2, we reviewed on the algebraicity of the values \( R(m, h_1, h_2, \chi) \) at half integers. We then naturally ask the following question:

**Question.** What can one say about the algebraicity of \( R(m, h_1, h_2, \chi) \) with \( m \) an integer?

As an application of Theorem 6.1, we give a partial answer to this question. We note that

\[
R(s, h_1, h_2, \chi) = (1 - 2^{-2s+k_1+k_2-1} \chi^2(2))^{-1} \tilde{R}(s, h_1, h_2, \chi)
\]

if the conductor of \( \chi \) is odd. Hence it suffices to consider the above question for \( R(m, h_1, h_2, \chi) \) with integer \( m \) if \( k_1 + k_2 \) is even.
Let $k$ and $n$ be positive even integers such that $n \geq 4$, $2k - n \geq 12$. Let $h(z)$ and $E_{n/2+1/2}$ be as in Section 4. For a Dirichlet character $\chi$ of odd square free conductor $N = p_1 \cdots p_r$, we define

$$R(x)(s, h, E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} J(\chi(i_1, \ldots, i_r), (\frac{\star}{N}) J_{n-1}(\chi(i_1, \ldots, i_r)),$$

$$\times R(s, h, E_{n/2+1/2}, \chi(i_1, \ldots, i_r)) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi_{i_1, \ldots, i_r}^2),$$

where $l_i = \text{GCD}(n, p_i - 1)$ as in Theorem 6.1.

**Theorem 7.1.** There exists a finite dimensional $\mathbb{Q}$-vector space $W_{h, E_{n/2+1/2}}$ in $\mathbb{C}$ such that

$$\frac{R(x)(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h, E_{n/2+1/2}},$$

for any integer $n/2 + 1 \leq m \leq k - n/2 - 1$ and a character $\chi$ of odd square free conductor such that $\chi^n$ is primitive.

**Proof.** Put

$$M(x)(s, S(h)) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} J(\chi(i_1, \ldots, i_r), (\frac{\star}{N}) J_{n-1}(\chi(i_1, \ldots, i_r))$$

$$\times \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), (\chi(i_1, \ldots, i_r))^2).$$

Then by Corollary to Proposition 3.1, we have

$$\frac{M(x)(m, S(h))}{\pi^{mn}} \in \mathbb{Q}u_1(S(h))^{n/2} \mathbb{Q}^{-n^2/4}.$$

By Theorem 6.1, we have

$$L(m, I_n(h), \chi^n) = 2^{nm} \chi(2n) \{c_n, N R(x)(m, h, E_{n/2+1/2}) + d_n, N c_\chi(1) M(x)(m, S(h))\}.$$

Hence by Theorem 2.2, we have

$$\frac{R(x)(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \mathbb{Q}u_1 \otimes \mathbb{Q} V_{I_n(h)} + \mathbb{Q}u_2$$

with some complex numbers $u_1$ and $u_2$, where $V_{I_n(h)}$ is the $\mathbb{Q}$-vector space associated with $I_n(h)$ in Theorem 2.2. This proves the assertion. □

By the above theorem, we immediately obtain the following:
Let $l_1 = \gcd(p_{ij} - 1, n)$. Then the values $\frac{R^{(\chi_i)}(m_i, h, E_{n/2+1/2})}{\pi^{m_i n}}$ are linearly dependent over $\mathbb{Q}$.

**Corollary.** In addition to the notation and the assumption as above, assume that $n \equiv 2 \mod 4$. Write $N_i = \prod_{j=1}^{r_i} p_{ij}$ with $p_{ij}$ an odd prime number, and let $l_{ij} = \gcd(p_{ij} - 1, n)$. Then the values $\frac{R(m_i, h, E_{n/2+1/2}, \chi_{i(a_{1j}, \ldots, a_{r_ij})})}{\pi^{2m_i}}$ are linearly dependent over $\mathbb{Q}$. In particular, if $\chi_1, \chi_2, \ldots, \chi_d$ are Dirichlet characters of odd prime conductors $p_1, p_2, \ldots, p_d$, respectively such that $\chi_i^n$ is primitive for any $i = 1, 2, \ldots, d$, then the values $\frac{R(m_i, h, E_{n/2+1/2}, \chi_{i(a_{1j}, \ldots, a_{r_ij})})}{\pi^{2m_i}}$ are linearly dependent over $\mathbb{Q}$, where $l_i = \gcd(n, p_i - 1)$ for $i = 1, \ldots, d$.

**Proof.** By Theorem 1.1, the value $\frac{L_{n,6}(n, S(h), \chi_{i(a_{1j}, \ldots, a_{r_ij})})}{\pi^{(n-2)/2}}$ belongs to $\mathbb{Q}[R_{+}(S(h))^{n/2-1} \pi^{-n^2/4+n/2}]$ and in particular if $n \equiv 2 \mod 4$, then it is nonzero for any $\chi$. Moreover, by Corollary to Theorem 5.10, $J(\chi_{i(a_{1j}, \ldots, a_{r_ij})})$ is non-zero and belongs to $\mathbb{Q}$. Thus the assertion holds.

As another application of Theorem 7.1, we also have a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$. Namely, by Theorem 3.1 we obtain:

**Theorem 7.3.** Let $h$ be as above. Let $\chi$ be a primitive character of odd square free conductor $N$. Assume that $n \equiv 2 \mod 4$, and that $\chi^n$ is primitive. Put

$$
R^{(\chi)}(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^n)^{-1} \gamma_n(s) R^{(\chi)}(s, h, E_{n/2+1/2}),
$$

where $\tau(\chi^n)$ is the Gauss sum, and

$$
\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \sin((i-1)/2) \Gamma(s - (i-1)/2).
$$

Then $R^{(\chi)}(s, h, E_{n/2+1/2})$ has an analytic continuation to the whole $s$-plane, and has the following functional equation:

$$
R^{(\chi)}(k - s, h, E_{n/2+1/2}) = R^{(\chi)}(s, h, E_{n/2+1/2}).
$$

**Remark.** (1) As functions of $s$, the Dirichlet series

$$
\{R(s, h, E_{n/2+1/2}, \chi_{i(j)}))\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}
$$

are linearly independent over $\mathbb{C}$.

(2) In the case of $n = 2$, this type of result was given for $R(m, h, E_{3/2})$ with $E_{3/2}$ Zagier’s Eisenstein series of weight $3/2$ by [K-M].

(3) The meromorphy of this type of series was derived in [Sh3] by using so called the Rankin-Selberg integral expression in more general setting, but we don’t know whether the functional equation of the above type can be directly proved without using the above method.
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