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S. Izumiya and W.L. Marar

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THE EULER CHARACTERISTIC OF A GENERIC WAVE FRONT IN A 3-MANIFOLD

S.IZUMIYA AND W.L.MARAR

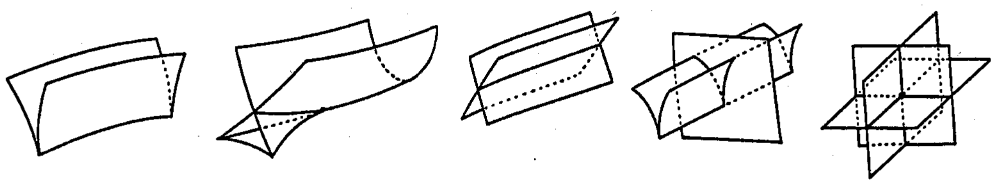
Abstract. We give a relation between Euler characteristics of a generic closed Legendrian surface and its wavefront.

INTRODUCTION

In this note we shall compute the Euler characteristic of a generic wave front in a 3-manifold.

Let N be a $(2n+1)$-dimensional smooth manifold and K be a contact structure on N (i.e. K is a nondegenerate tangent hyperplane field on N). An immersion $i : L \to N$ is said to be *Legendrian* if $\dim L = n$ and $di_x(T_x L) \subset K_x$ for any $x \in L$. We say that a smooth fibre bundle $\pi : E \to M$ is *Legendrian* if its total space E is furnished with a contact structure and its fibres are Legendrian submanifolds. For a Legendrian immersion $i : L \to E$, $\pi \circ i : L \to M$ is called *a Legendrian map* and the image of the Legendrian map $\pi \circ i$ is called *the wave front* of i. It is denoted by $W(i)$.

From now on, we only consider the case of $n = 2$. Then it is known that a generic wave front has (semi cubic) cuspidal edges (A_2), swallowtails (A_3) and points of transversal self intersection (A_1A_1, A_1A_2, $A_1A_1A_1$) as singularities ([1], see Fig. 1). We shall refer to the $A_1A_1A_1$-type point as *a triple point* of i.



A_2 A_3 A_1A_1 A_1A_2 $A_1A_1A_1$

Fig.1

If L is a closed surface, then the number of swallowtails and triple points are finite. Our main result is the following :

THEOREM. *Let $i : L \to E$ be a generic Legendrian immersion of a closed surface. Then we have*

$$\chi(W(i)) = \chi(L) + T(i) + \frac{S(i)}{2},$$

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where $\chi(X)$ is the Euler characteristic of X, $T(i)$ is the number of triple points on $W(i)$ and $S(i)$ is the number of swallowtails.

We remark that the corresponding result for a generic wave front in a 2-manifold is easily verified and that

$$\chi(W(i)) = -d(i),$$

where $d(i)$ denote the number of double points on $W(i)$.

On the other hand, we consider an equidistant surface of a closed surface in 3-dimensional Euclidian space. If the distance is sufficiently small, then the equidistant surface is diffeomorphic to the original surface. However, for some distances, singularities may be appeared in the equidistant surface and it may not be homeomorphic to the original surface. Hence, it is interesting to study the relation between topologies of the equidistant surface and of the original surface. These subjects are studied in the theory of Legendrian singularities. In fact, it is known that singularities of equidistant surfaces are locally diffeomorphic to singularities of wave fronts (see [1]). Then we show the general property of the Euler characteristic of global wave fronts.

In order to prove the theorem, we shall apply the method which has been introduced to compute the Euler characteristic of the image of a stable perturbation of an $\mathcal{A}$-finite map germ in [2].

All maps considered here are class C^∞ unless stated otherwise.

1.Proof of the theorem

In this section we shall give a proof of the theorem. Let $i : L \to E$ be a generic Legenendrian immersion of a closed surface. Since the Euler characteristic is a topological invariant, then we can ignore cuspidal edges. We now define the following sets:

$$D^2(i) = cl\{x \in S \mid \sharp(\pi \circ i)^{-1}\pi \circ i(x) \geq 2\},$$
$$D^3(i) = \{x \in D^2(i) \mid \sharp(\pi \circ i)^{-1}\pi \circ i(x) = 3\},$$
$$D^2(i,(2)) = \{x \in D^2(i) \mid \sharp(\pi \circ i)^{-1}\pi \circ i(x) = 1\},$$

where clX is the topological closure of X. Then we have the following diagram:

$$\begin{tikzcd} & D^3(i) \arrow[d, "h"] & \\ D^2(i,(2)) \arrow[r, "j"] & D^2(i) \arrow[d, "k"] & \\ & L \arrow[r, "\pi\circ i"] & W(i) \subset M, \end{tikzcd}$$

where h, j, k are inclusions.

By the characterization of generic wave fronts (see [ ]), $D^2(i)$ is a union of curves on L with self-intersection and circles, $D^3(i)$ is the inverse image of triple points and $D^2(i,(2))$ is the set of swallowtails of $\pi \circ i$. It follows that these are immersed submanifolds of L with dim $D^2(i) = 1$ and dim $D^3(i) =$ dim $D^2(i,(2)) = 0$.

In order to prove the theorem, we need the following formula.

LEMMA 1.1. $\chi(W(i)) = \chi(L) - \frac{1}{2}\chi(D^2(i)) + \frac{1}{2}\chi(D^2(i,(2)) - \frac{1}{6}\chi(D^3(i))$.

PROOF: Consider the equation

$$(*) \qquad \chi(W(i)) = \alpha\,\chi(L) + \beta\,\chi(D^2(i)) + \gamma\,\chi(D^2(i,(2))) + \delta\,\chi(D^3(i)),$$

where α, β, γ and δ are unknown variables. We solve this by a purely combinatorial method.

We now construct a triangulation K_i of the stratified set $W(i)$ as follows: We start to triangulate $W(i)$ by including the image of $D^2(i,(2))$ and the image od $D^3(i)$ among the vertices of K_i. After this, we build up the one-skeleton $K_i^{(1)}$ of K_i so that the image of $D^2(i)$ is a subcomplex of $K_i^{(1)}$. We complete our procedure by constructing the two-skeleton $K_i^{(2)}$.

Since $\pi \circ i$ and its restrictions to $D^2(i)$, $D^2(i,(2))$ and $D^3(i)$ are proper and finite-to-one mappings, then we can pull back K_i to obtain a triangulation for L, $D^2(i)$, $D^2(i,(2))$ and $D^3(i)$. Let C_j^X be the number of j-cells in X, where $X = W(i)$, L, $D^2(i)$, $D^2(i,(2))$ or $D^3(i)$. Then the equation (*) can be written by

$$\sum_j (-1)^j C_j^{W(i)} = \alpha \sum_j (-1)^j C_j^L + \beta \sum_j (-1)^j C_j^{D^2(i)}$$
$$+\gamma \sum_j (-1)^j C_j^{D^2(i,(2))} + \delta \sum_j (-1)^j C_j^{D^3(i)},$$

where $C_j^X = 0$ if $i > dim X$. So, if we can find real numbers α, β, γ and δ such that

$$(**) \qquad C_j^{W(i)} = \alpha\, C_j^L + \beta\, C_j^{D^2(i)} + \gamma\, C_j^{D^2(i,(2))} + \delta\, C_j^{D^3(i)},$$

for any j, then we have solutions of the equation (*). By the construction of the triangulation, we may concentrate on solving (**) in the case when $j = 0$. We remark that $\pi \circ i$ is 3 to 1 over the points in the image of $D^3(i)$, 1 to 1 over the points in the image of $D^2(i,(2))$, 2 to 1 over the points in the image of $D^2(i) - (D^2(i,(2)) \cup D^3(i))$, and 1 to 1 over the points in the image of $S - D^2(f)$. It follows that the equation

$$C_0^{W(i)} = \alpha\, C_0^L + \beta\, C_0^{D^2(i)} + \gamma\, C_0^{D^2(i,(2))} + \delta\, C_0^{D^3(i)}$$

is equivalent to the system of linear equations :

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

We can easily solve this equation, so that $\alpha = 1$, $\beta = -1/2$, $\gamma = 1/2$ and $\delta = -1/6$. This completes the proof.

Then we can prove the theorem.

PROOF OF THE THEOREM: By the definition we have $\chi(D^2(i,(2))) = S(i)$ and $\chi(D^3(i)) = 3T(i)$. Since $D^2(i)$ is a union of closed curves on the surface L with $3T(i)$ crossings, then we can triangulate it with $3T(i)+n$ 0-cells and $6T(i)+n$ 1-cells, where n is the number of circles in $D^2(i)$. It follows that $\chi(D^2(i)) = -3T(i)$. If we substitute these on the formula in Lemma 1.1, then we have

$$\chi(W(i)) = \chi(L) + T(i) + \frac{1}{2}S(i).$$

This completes the proof of the theorem.

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(S. Izumiya) Department of Mathematics, Faculty of Science, Hokkaido Univeristy, Sapporo 060, Japan
(W. L. Marar) Instituto de Ciências Matemáticas de São Carlos, Universidade de São Paulo, Caixa Postal 668, 13560-São Carlos (SP)-Brazil