ON THE ANDRIANOV TYPE IDENTITY FOR POWER SERIES ATTACHED TO JACOBI FORMS AND ITS APPLICATION

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1. Introduction

The theory of Jacobi forms, namely automorphic forms on the Jacobi group and its generalization to higher degree have been studied by several authors (cf. [6, 27, 18, 19, 8]). In particular, Shintani introduced the standard L-function attached to a Jacobi form of arbitrary degree, and afterward Murase derived in a series of his papers [18, 19] its meromorphic continuation and functional equation by making use of its integral expression. Moreover, Murase and Sugano derived in [20] an expression of the standard L-function attached to a Jacobi form in terms of a power series generated by its eigenvalues of Hecke operators. In this paper, we derive a local expression of the standard L-function attached to a Jacobi form in terms of a power series related to its Fourier coefficients. This can be regarded as an analogue of Andrianov's identity in [1] for Siegel modular forms. As an application, we shall also show the rationality theorem for a formal power series related to a polynomial appearing in the theory of local densities of quadratic forms, which is very similar to the result obtained in [5] by Böcherer and Sato.

Let us describe our main results precisely. Let p be an arbitrary rational prime. For any nonzero element a of the field $\mathbb{Q}_p$ of p-adic numbers, we put

$$\chi_p(a) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is unramified}, \\ 0 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is ramified}. \end{cases}$$

Let n be a positive even integer. For each non-degenerate half-integral symmetric matrix B' of degree n over the ring $\mathbb{Z}_p$ of p-adic integers, we define the local Siegel series with complex parameter s by

$$b_p(B';\, s) := \sum_{R \in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(-B'R))\, \mu_p(R)^{-s},$$

where $\mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$, and $\mathbf{e}_p$ is the standard additive character of $\mathbb{Q}_p$. It is well-known that such a kind of singular series appears naturally within the framework of

Date: February 1, 2009.
This work was partly supported by the JSPS International Training Program (ITP).

studying Fourier coefficients of the Siegel Eisenstein series of degree n and there exists a unique polynomial $F_p(B';X)$ in one variable X such that

$$b_p(B';\,s) = \frac{(1-p^{-s})\displaystyle\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\xi_p(B')p^{n/2-s}}\,F_p(B';\,p^{-s}),$$

where $\xi_p(B') = \chi_p((-1)^{n/2}\det(2B'))$ (cf. [16]). Let B be a non-degenerate symmetric matrix of degree $n-1$ over a subring R of $\mathbb{Z}_p$ satisfying the condition

(1) $(B + {}^t r_B r_B)/4$ is a half-integral symmetric matrix over R for some $r_B \in R^{n-1}$.

Then we can associate such a B with a non-degenerate half-integral symmetric matrix

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix}$$

of degree n over R. Here we easily see that the vector r_B is uniquely determined by B modulo $2R^{n-1}$, and therefore $B^{(1)}$ is uniquely determined by B up to $\mathrm{GL}_{n-1}(R)$-equivalence. Then for such a B over $\mathbb{Z}_p$, we define a polynomial $F_p^{(1)}(B;\,X)$ in X by

$$F_p^{(1)}(B;\,X) := F_p(B^{(1)};\,X)$$

and put

$$G_p^{(1)}(B;\,X) = \sum_{D\in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D)\,F_p^{(1)}(B[D^{-1}];\,X)\,(p^n X^2)^{\mathrm{ord}_p(\det D)},$$

where $\pi_p(D)$ denotes the generalized local Möbius function, that is, $\pi_p(D) = (-1)^i p^{i(i-1)/2}$ or 0 according as $D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\left(\begin{array}{c|c} \mathbf{1}_{n-1-i} & \\ \hline & p\mathbf{1}_i \end{array}\right)\mathrm{GL}_{n-1}(\mathbb{Z}_p)$ for some $0 \le i \le n-1$ or not. We note that these polynomials do not depend on the choice of r_B. In addition, we also define a polynomial $\mathbf{B}_p^{(1)}(B;\,t)$ in one variable t by

$$\mathbf{B}_p^{(1)}(B;\,t) := \frac{(1-\xi_p(B^{(1)})\,p^{-(n-1)/2}\,t)\displaystyle\prod_{i=1}^{n/2-1}(1-p^{-2i+1}t^2)}{G_p^{(1)}(B;\,p^{-n+1/2}\,t)}.$$

On the other hand, for any positive even integers k and n, let ϕ be a Jacobi form of weight k and of index 1 with respect to the Jacobi modular group Γ_{n-1}^J of degree $n-1$, and $\sigma(\phi)$ be a Siegel modular form of weight $k-1/2$ with respect to the congruence subgroup $\Gamma_0^{(n-1)}(4)$ of the Siegel modular group of degree $n-1$ corresponding to ϕ under the Eichler-Zagier-Ibukiyama correspondence σ (cf. §2.3 and 2.4 below). Let $\mathbf{D}_p^{(n-1)}(\mathbb{Z})$ be the set of all $(n-1)\times(n-1)$ matrices with entries in $\mathbb{Z}$ whose determinant is a power of p. For each positive definite half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}$, we define a power series $\widetilde{G}_{\phi,p}(B;\,t)$ in t by

$$\widetilde{G}_{\phi,p}(B;\,t) := \sum_{D\in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} \pi_p(D)\,C_{\sigma(\phi)}(B[D^{-1}])\,(p^k\,t)^{\mathrm{ord}_p(\det D)},$$

where $C_{\sigma(\phi)}(B)$ denotes the B-th Fourier coefficient of $\sigma(\phi)$. Then our first main result is the following:

Theorem 1.1 (cf. Theorem 3.1 below)**.** *If ϕ is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators with Satake p-parameters $(\chi_\phi^{(1)}(p), \ldots, \chi_\phi^{(n-1)}(p))$, then for each positive definite half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}$ satisfying the condition* (1)*, we have*

$$\frac{\mathbf{B}_p^{(1)}(B;\, p^{n-1/2}t)\, \widetilde{G}_{\phi,\,p}(B;\, t)}{\prod_{i=1}^{n-1}(1-\chi_\phi^{(i)}(p)p^{n-1/2}t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)}$$
$$= \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W])\, p^{-(k-n-1)\,\mathrm{ord}_p(\det W)}\, t^{\mathrm{ord}_p(\det W)}.$$

This can be regarded as an analogue of the so-called Andrianov identity, which was obtained within the framework of studying standard L-functions attached to Siegel modular forms of integral weight (cf. [1], see also [4]). We also note that the above identity for $p \neq 2$ can be derived from a similar result for Siegel modular forms of half-integral weight due to Shimura and Zhuravlev (cf. Corollary 5.2 in [23], see also Theorem 1.1 in [26]). However, we cannot use their results to prove the above identity for $p = 2$.

Next, we explain an application of the above result to the rationality of a certain formal power series related to the polynomial $F_p^{(1)}(B;\, X)$. For each non-degenerate half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}_p$ satisfying the condition (1), we define a Laurent polynomial $\widetilde{F}_p^{(1)}(B;\, X)$ in X by

$$\widetilde{F}_p^{(1)}(B;\, X) := X^{-\mathrm{ord}_p((-1)^{n/2}\det(2B^{(1)})\mathfrak{d}(B^{(1)})^{-1})/2} F_p^{(1)}(B;\, p^{-(n+1)/2}X),$$

and put

$$\widetilde{G}_p^{(1)}(B;\, X,\, t) = \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D)\, \widetilde{F}_p^{(1)}(B[D^{-1}];\, X)\, t^{\mathrm{ord}_p(\det D)},$$

where $\mathfrak{d}(B^{(1)})$ is the discriminant of the quadratic extension $\mathbb{Q}_p\Big(\sqrt{(-1)^{n/2}\det(2B^{(1)})}\Big)/\mathbb{Q}_p$. Then we have a functional equation $\widetilde{F}_p^{(1)}(B;\, X) = \widetilde{F}_p^{(1)}(B;\, X^{-1})$ (cf. [9]). Thus $\widetilde{F}_p^{(1)}(B;\, X)$ is a polynomial in $X + X^{-1}$, and then $\widetilde{G}_p^{(1)}(B;\, X,\, t)$ is a polynomial in $X + X^{-1}$ and t. Now we put

$$R_p^{(1)}(B;\, X,\, t) = \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \widetilde{F}_p^{(1)}(B[W];\, X)\, t^{\mathrm{ord}_p(\det W)}.$$

Then by applying Theorem 1.1 to the Jacobi Eisenstein series, we obtain the following:

Theorem 1.2 (cf. Theorem 3.4 below)**.** *Let n be a positive even integer. If B is a non-degenerate half-integral symmetric matrix of degree $n-1$ over $\mathbb{Z}_p$ satisfying the condition* (1)*, then we have*

$$R_p^{(1)}(B;\, X,\, t) = \frac{\mathbf{B}_p^{(1)}(B;\, p^{n/2-1}t)\, \widetilde{G}_p^{(1)}(B;\, X,\, t)}{\prod_{j=1}^{n-1}(1-p^{j-1}X\, t)(1-p^{j-1}X^{-1}t)}.$$

We note that Böcherer and Sato ([5]) obtained a similar identity for a half-integral symmetric matrix of degree n. The above identity will play an important role in proving a conjecture on the period of the Ikeda lift proposed in [10] by Ikeda (cf. [13, 14]).

Acknowledgments. The authors would like to express their hearty thanks to Professor A. Murase and Professor T. Sugano for their valuable comments and suggestions. They also thank Professor S. Böcherer for his useful comments.

Notation. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}\,x)$ for any $x \in \mathbb{C}$. For each rational prime p, let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ be the field of p-adic rational numbers and the ring of p-adic integers, respectively. We denote by ord_p the valuation of $\mathbb{Q}_p$ normalized as $\mathrm{ord}_p(p) = 1$, and by $\mathbf{e}_p$ the continuous additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for any $x \in \mathbb{Q}$, which will be called the standard additive character of $\mathbb{Q}_p$. Let R be a commutative ring. We denote by $R^\times$ the the unit group of R. We denote by $\mathrm{M}_{m,n}(R)$ the set of $m \times n$ matrices with entries in R. In particular, we write $\mathrm{M}_n(R) = \mathrm{M}_{n,n}(R)$ and $R^n = \mathrm{M}_{1,n}(R)$. We denote by $\mathbf{1}_n$, $\mathbf{0}_n \in \mathrm{M}_n(R)$ the unit matrix and the zero matrix of degree n, respectively. We put $\mathrm{GL}_n(R) = \{\, U \in \mathrm{M}_n(R) \mid \det U \in R^\times \,\}$, where $\det U$ is the determinant of U. For two matrices $X \in \mathrm{M}_{m,n}(R)$ and $A \in \mathrm{M}_m(R)$, we write $A[X] = {}^tXAX \in \mathrm{M}_n(R)$, where ${}^tX$ denotes the transpose of X. Let $\mathrm{Sym}_n(R)$ be the set of symmetric matrices of degree n with entries in R. If R is an integral domain of characteristic different from 2, let $\mathrm{Sym}_n^*(R)$ be the subset of $\mathrm{Sym}_n(R)$ consisting of all half-integral symmetric matrices of degree n, that is,

$$\mathrm{Sym}_n^*(R) := \left\{ T = (t_{ij}) \in \mathrm{Sym}_n(\mathrm{Frac}(R)) \;\middle|\; \begin{array}{ll} t_{ii} \in R & (1 \le i \le n), \\ 2\,t_{ij} \in R & (1 \le i \ne j \le n) \end{array} \right\},$$

where $\mathrm{Frac}(R)$ is the field of fractions of R. In addition, for any subset $\mathcal{S}$ of $\mathrm{Sym}_n(R)$, we denote by $\mathcal{S}^\times$ the subset of $\mathcal{S}$ consisting of all non-degenerate elements in $\mathcal{S}$. In particular, if R is a subring of $\mathbb{R}$, we denote by $\mathcal{S}_{>0}$ (resp. $\mathcal{S}_{\ge 0}$) the subset of $\mathcal{S}$ consisting of all positive definite (resp. semi-positive definite) matrices. For any commutative ring R, the group $\mathrm{GL}_n(R)$ acts on the set $\mathrm{Sym}_n(R)$ in the following way:

$$\mathrm{GL}_n(R) \times \mathrm{Sym}_n(R) \ni (U, A) \longmapsto A[U] \in \mathrm{Sym}_n(R).$$

For a subgroup G of $\mathrm{GL}_n(R)$, and a subset $\mathcal{S}$ of $\mathrm{Sym}_n(R)$ stable under the action of G, we denote by $\mathcal{S}/G$ the set of G-orbits in $\mathcal{S}$. We define an equivalence relation on $\mathrm{Sym}_n(R)$ over a subring R' of R as follows: for any A_1, $A_2 \in \mathrm{Sym}_n(R)$,

$$A_1 \sim_{R'} A_2 \overset{\text{def}}{\Longleftrightarrow} A_2 = A_1[U] \text{ for some } U \in \mathrm{GL}_n(R'). \tag{2}$$

For two square matrices $X \in \mathrm{M}_m(R)$ and $Y \in \mathrm{M}_n(R)$, we write $X \perp Y = \begin{pmatrix} X & \\ & Y \end{pmatrix}$. In particular, we often write $x \perp Y$ instead of $(x) \perp Y$ for any $x \in R$. Then we can simply write the diagonal matrix with entries $x_1, \cdots, x_n$ in R by $x_1 \perp \cdots \perp x_n$.

2. Preliminaries

2.1. Siegel modular forms of integral weight.

Let $G_n(\mathbb{R})$ be the real symplectic group of degree n, that is,

$$G_n(\mathbb{R}) := \mathrm{Sp}_n(\mathbb{R}) = \{\, M \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^tMJ_nM = J_n \,\},$$

where $J_n = \left(\begin{smallmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{smallmatrix}\right)$. For any $S \in \mathrm{Sym}_n(\mathbb{R})$ and $A \in \mathrm{GL}_n(\mathbb{R})$, we put $\mathbf{n}_n(S) = \left(\begin{smallmatrix} \mathbf{1}_n & S \\ \mathbf{0}_n & \mathbf{1}_n \end{smallmatrix}\right)$ and $\mathbf{d}_n(A) = \left(\begin{smallmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{smallmatrix}\right)$, respectively. Then we easily see that these elements $\mathbf{n}_n(S)$, $\mathbf{d}_n(A)$ and J_n generate $G_n(\mathbb{R})$. The discrete subgroup $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z}) = G_n(\mathbb{R}) \cap \mathrm{M}_{2n}(\mathbb{Z})$ of $G_n(\mathbb{R})$ is called the *Siegel modular group* of degree n. For any $N \in \mathbb{Z}_{>0}$, we denote by $\Gamma_0^{(n)}(N)$ the congruence subgroup of Γ_n defined by

$$\Gamma_0^{(n)}(N) := \{\, \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid C \equiv \mathbf{0}_n \pmod{N} \,\}.$$

We denote the Siegel upper-half space of degree n by $\mathfrak{H}_n$, that is,

$$\mathfrak{H}_n := \{\, Z = X + \sqrt{-1}\, Y \in \mathrm{Sym}_n(\mathbb{C}) \mid Y > 0 \ \text{(positive definite)} \,\}.$$

For any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, we easily see that $j(M, Z) := CZ + D \in \mathrm{GL}_n(\mathbb{C})$ and then we put $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$. As is well-known, this defines a transitive action of $G_n(\mathbb{R})$ on $\mathfrak{H}_n$.

For any $k \in \mathbb{Z}$, a $\mathbb{C}$-valued holomorphic function $F(Z)$ on $\mathfrak{H}_n$ is called a (*holomorphic*) *Siegel modular form* of degree n and weight k if it satisfies the following two conditions:

(i) $F(M\langle Z\rangle) = \det(j(M, Z))^k F(Z)$ for any $M \in \Gamma_n$;

(ii) F possesses a Fourier expansion of the form

$$F(Z) = \sum_{B \in \mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} A_F(B)\, \mathbf{e}(\mathrm{tr}(BZ)),$$

where tr denotes the trace of a matrix. If F satisfies the stronger condition $A_F(B) = 0$ unless $B > 0$ (positive definite), then it is called a *cusp form*.

We denote by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ the $\mathbb{C}$-vector spaces consisting of all (holomorphic) Siegel modular forms and Siegel cusp forms of degree n and weight k, respectively. For further details on the facts of Siegel modular forms of integral weight set out above, see [1] or [7].

2.2. Review of the theory of Jacobi forms of higher degree.

In this paragraph, we introduce some basic facts on Jacobi forms of integral weight whose index is a scalar. For further details on generalities of Jacobi forms, see [6, 18, 19, 27].

2.2.1. *Jacobi group and complex analytic Jacobi forms.*

Let $G_n = \mathrm{Sp}_n(\mathbb{Q}) = \{\, M \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid {}^tMJ_nM = J_n \,\}$, and we naturally identify G_n with its image under the natural inclusion

$$G_n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto [M] := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{array}\right) \in G_{n+1}.$$

We denote by H_n the Heisenberg group, that is,

$$H_n = \left\{ [(\lambda,\mu),\kappa] := \left(\begin{array}{cc|cc} 1 & 0 & \kappa & \mu \\ 0 & \mathbf{1}_n & {}^t\mu & \mathbf{0}_n \\ \hline & & 1 & 0 \\ & & 0 & \mathbf{1}_n \end{array}\right) \left(\begin{array}{cc|cc} 1 & \lambda & & \\ 0 & \mathbf{1}_n & & \\ \hline & & 1 & 0 \\ & & -{}^t\lambda & \mathbf{1}_n \end{array}\right) \;\middle|\; \begin{array}{l} (\lambda,\mu) \in \mathbb{Q}^n \oplus \mathbb{Q}^n, \\ \kappa \in \mathbb{Q} \end{array} \right\}.$$

Then $G_n^J := \{\, [(\lambda,\mu),\,\kappa]\cdot[M] \in G_{n+1} \mid [(\lambda,\mu),\,\kappa] \in H_n,\, M \in G_n \,\}$ is a $\mathbb{Q}$-algebraic subgroup of G_{n+1} and is called the *Jacobi group* of degree n. We note that the Jacobi group G_n^J is a semi-direct product $G_n \ltimes H_n$ of H_n and G_n, and forms a connected non-reductive $\mathbb{Q}$-algebraic group with the center

$$Z_n^J = \{\, [(0,0),\kappa] \mid \kappa \in \mathbb{Q} \,\}.$$

Then we have the following:

Lemma 2.1. *For each* $[(\lambda,\mu),\,\kappa]$, $[(\lambda',\mu'),\,\kappa'] \in H_n$, *and* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n$, *we have*

$$[(\lambda,\mu),\,\kappa]\cdot[(\lambda',\mu'),\,\kappa'] = [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+2\lambda^t\mu'], \tag{3}$$

$$[(\lambda,\mu),\,\kappa]\cdot[M] = [M]\cdot[(\lambda A+\mu C,\lambda B+\mu D),\kappa+(\lambda A+\mu C)^t(\lambda B+\mu D)-\lambda^t\mu]. \tag{4}$$

Proof. Since it is an easy calculation, we omit the proof. □

According to the action of $G_{n+1}(\mathbb{R}) = \mathrm{Sp}_{n+1}(\mathbb{R})$ on the Siegel upper-half space $\mathfrak{H}_{n+1}$, the group $G_n^J(\mathbb{R})$ of real points of G_n^J naturally acts on the space $\mathfrak{H}_n \times \mathbb{C}^n$ as follows. For each $g = [(\lambda,\mu),\,\kappa]\cdot[M] \in G_n^J(\mathbb{R})$ with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$ and $(\tau,\,z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we put

$$g\langle\tau,\,z\rangle := (M\langle\tau\rangle,\, z(C\tau+D)^{-1} + \lambda\, M\langle\tau\rangle + \mu).$$

Here we easily see that this action is transitive and the stabilizer of the point $(\sqrt{-1}\,\mathbf{1}_n,\,0) \in \mathfrak{H}_n \times \mathbb{C}^n$ in $G_n^J(\mathbb{R})$ coincides with $Z_n^J(\mathbb{R})\cdot K_\infty$, where K_∞ is the stabilizer of $\sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$ in $G_n(\mathbb{R})$, that is,

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G_n(\mathbb{R}) \;\middle|\; A+\sqrt{-1}\,B \text{ is unitary} \right\}.$$

The map $g \mapsto g\langle\sqrt{-1}\,\mathbf{1}_n,\,0\rangle$ induces a diffeomorphism of $G_n^J(\mathbb{R})/(Z_n^J(\mathbb{R})\cdot K_\infty)$ onto $\mathfrak{H}_n \times \mathbb{C}^n$.

Let l and m be non-negative integers. For any $\mathbb{C}$-valued function $\phi(\tau,\,z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$, we define the action of $g \in G_n^J(\mathbb{R})$ on ϕ by

$$(\phi|_{l,m}\, g)(\tau,\,z) := J_{l,m}(g,\,(\tau,\,z))^{-1}\,\phi\,(g\langle\tau,\,z\rangle)\,,$$

where for $g = [(\lambda,\mu),\,\kappa]\cdot[M]$, we put

$$\begin{aligned} J_{l,m}(g,\,(\tau,\,z)) &:= \det(C\tau+D)^l \\ &\quad \times \mathbf{e}(-m\kappa - m\tau[{}^t\lambda] - 2m\lambda^t z - m\lambda\,{}^t\mu + m\{(C\tau+D)^{-1}C\}[{}^t(z+\lambda\tau+\mu)]). \end{aligned}$$

It is easy to see that for any $g_i \in G_n^J(\mathbb{R})$ $(i = 1,\,2)$,

$$(\phi|_{l,m}\, g_1)|_{l,m}\, g_2 = \phi|_{l,m}\,(g_1 g_2).$$

In particular, it follows from Lemma 2.1 that for any $M,\ M' \in G_n(\mathbb{R})$ and $[(\lambda,\mu),\ \kappa],\ [(\lambda',\mu'),\ \kappa'] \in H_n(\mathbb{R})$, we have

$$\begin{cases} \phi|_{l,m}[M]\,|_{l,m}[M'] = \phi|_{l,m}[MM'], \\ \phi|_{l,m}[(\lambda,\mu),\ \kappa]\,|_{l,m}[(\lambda',\mu'),\ \kappa'] = \phi|_{l,m}[(\lambda+\lambda',\mu+\mu'),\ \kappa+\kappa'+2\lambda{}^t\mu'], \\ \phi|_{l,m}[M]\,|_{l,m}[(\lambda,\mu)M,\ \kappa+(\lambda,\mu)M\left(\begin{smallmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{smallmatrix}\right){}^tM\,{}^t(\lambda,\mu)-\lambda{}^t\mu] = \phi|_{l,m}[(\lambda,\mu),\ \kappa]\cdot[M]. \end{cases}$$

We also define a subgroup of $G_n^J(\mathbb{R})$ by $\Gamma_n^J := \Gamma_n \ltimes H_n(\mathbb{Z})$, where $H_n(\mathbb{Z})$ is a subgroup of $H_n(\mathbb{R})$ consisting of all elements with integral entries.

Let l and m be positive integers. A holomorphic function $\phi(\tau,\ z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$ is called a (*holomorphic*) *Jacobi form* of degree n, weight l and index m if it satisfies the following two conditions:

(i) $\phi|_{l,m}\gamma = \phi$ for any $\gamma \in \Gamma_n^J$;
(ii) ϕ possesses a Fourier expansion of the form

$$\phi(\tau,\ z) = \sum_{T\in \mathrm{Sym}_n^*(\mathbb{Z}),\, r\in\mathbb{Z}^n} c_\phi(T,\ r)\,\mathbf{e}(\mathrm{tr}(T\tau)+r\,{}^tz)$$

with $c_\phi(T,\ r) = 0$ unless $4mT - {}^trr \geq 0$. If ϕ satisfies the stronger condition $c_\phi(T,\ r) = 0$ unless $4mT - {}^trr > 0$, then it is called *cuspidal*.

We denote by $J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$ the $\mathbb{C}$-vector spaces consisting of all (holomorphic) Jacobi forms and cuspidal Jacobi forms of degree n, weight l and index m, respectively.

As an important example of Jacobi form, we consider Fourier-Jacobi coefficients of Siegel modular forms of arbitrary degree $n > 1$. For any $k \in \mathbb{Z}$, let $F \in M_k(\Gamma_n)$ possess a Fourier expansion

$$F(Z) = \sum_{B'\in \mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} A_F(B')\,\mathbf{e}(\mathrm{tr}(B'Z)) \quad (Z \in \mathfrak{H}_n),$$

and we put $Z = \begin{pmatrix} \tau' & z \\ {}^tz & \tau \end{pmatrix}$ with $\tau \in \mathfrak{H}_{n-1}$, $z \in \mathbb{C}^{n-1}$ and $\tau' \in \mathfrak{H}_1$. Then we have the so-called Fourier-Jacobi expansion

$$F\left(\begin{pmatrix} \tau' & z \\ {}^tz & \tau \end{pmatrix}\right) = \sum_{m=0}^{\infty} \phi_m(\tau,\ z)\,\mathbf{e}(m\tau'),$$

where

$$\phi_m(\tau,\ z) = \sum_{\substack{T\in \mathrm{Sym}_{n-1}^*(\mathbb{Z}),\, r\in\mathbb{Z}^{n-1}, \\ 4mT-{}^trr\geq 0}} A_F\left(\begin{pmatrix} m & r/2 \\ {}^tr/2 & T \end{pmatrix}\right)\mathbf{e}(\mathrm{tr}(T\tau)+r\,{}^tz). \tag{5}$$

We easily see that the m-th coefficient $\phi_m \in J_{k,m}(\Gamma_{n-1}^J)$ for each $m \in \mathbb{Z}_{>0}$. In particular, if $F \in S_k(\Gamma_n)$, then $\phi_m \in J_{k,m}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$.

As another example, if k is an even integer such that $k > n+1$, then for each $m \in \mathbb{Z}_{>0}$, we define *the Jacobi Eisenstein series* of degree $n-1$, weight k and index m by

$$\mathfrak{E}_{k,m}^{(n-1)}(\tau,\ z) := \sum_{\gamma\in P_{n-1}^J\cap\Gamma_{n-1}^J\backslash\Gamma_{n-1}^J} J_{k,m}(\gamma,\ (\tau,\ z)) \quad (\tau \in \mathfrak{H}_{n-1},\ z \in \mathbb{C}^{n-1}),$$

where

$$P^J_{n-1} := \left\{ [(\lambda,\, \mu),\, \kappa] \cdot [\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)] \in G^J_{n-1} \mid C = \mathbf{0}_{n-1},\, \lambda = 0 \right\}.$$

We easily see that the right-hand side of the above definition is absolutely convergent and $\mathfrak{E}^{(n-1)}_{k,\,m} \in J_{k,\,m}(\Gamma^J_{n-1})$. Moreover, Böcherer ([3]) showed that for any $m \in \mathbb{Z}_{>0}$, there exists a certain relation between $\mathfrak{E}^{(n-1)}_{k,\,m}$ and the m-th coefficient $e^{(n-1)}_{k,\,m}$ of the above Fourier-Jacobi expansion of the Siegel Eisenstein series $E^{(n)}_k \in M_k(\Gamma_n)$. In particular, when $m = 1$, we have $\mathfrak{E}^{(n-1)}_{k,\,1} = e^{(n-1)}_{k,\,1}$.

For the purpose of subsequent use, we give an explicit formula for the Fourier coefficients of $e^{(n-1)}_{k,\,1}$ in case n is even. Let k be a positive even integer such that $k > n+1$. Then the Siegel Eisenstein series $E^{(n)}_k$ of weight k with respect to Γ_n is defined by

$$E^{(n)}_k(Z) = \sum_{(C,\,D)} \det(CZ + D)^{-k} \quad (Z \in \mathfrak{H}_n)$$

where $(C,\, D)$ runs through a complete set of representatives of the equivalence classes of coprime symmetric pairs of size n. For each positive definite half-integral symmetric matrix B' of degree n, we denote by $\mathfrak{d}(B')$ the discriminant of the quadratic extension $\mathbb{Q}(\sqrt{(-1)^{n/2}\det(2B')})/\mathbb{Q}$ and put $\mathfrak{f}(B') = \sqrt{(-1)^{n/2}\det(2B')/\mathfrak{d}(B')}$. It is well-known that $\mathfrak{f}(B') \in \mathbb{Z}_{>0}$. Furthermore, we denote by $\chi_{B'}$ the Kronecker character corresponding to the above field extension. Then for each $B' \in \mathrm{Sym}^*_n(\mathbb{Z})_{>0}$, the B'-th Fourier coefficient $A^{(n)}_k(B')$ of $E^{(n)}_k$ is described as

$$A^{(n)}_k(B') = \xi(n,\, k) L(1 - k/2 + n/2, \chi_{B'})\, \mathfrak{f}(B')^{k-(n+1)/2} \prod_{p \,|\, \mathfrak{f}(B')} \widetilde{F}_p(B';\, p^{k-(n+1)/2}), \tag{6}$$

where $\xi(n,\, k) = 2^{n/2}\zeta(1-k)^{-1} \prod_{i=1}^{n/2} \zeta(1+2i-2k)^{-1}$, $L(s, \chi_{B'})$ denotes the Dirichlet L-function associated with $\chi_{B'}$, and

$$\widetilde{F}_p(B';\, X) = X^{-\mathrm{ord}_p(\mathfrak{f}(B'))} F_p(B';\, p^{-(n+1)/2}X).$$

We note that if $B \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})_{>0}$ satisfies the condition (1), then $\widetilde{F}^{(1)}_p(B;\, X) = \widetilde{F}_p(B^{(1)};\, X)$. Thus we have

Proposition 2.1. *Under the same assumption as above, let $e^{(n-1)}_{k,\,1}$ possess a Fourier expansion*

$$e^{(n-1)}_{k,\,1}(\tau,\, z) = \sum_{T \in \mathrm{Sym}^*_{n-1}(\mathbb{Z}),\, r \in \mathbb{Z}^{n-1}} c^{(n-1)}_{k,\,1}(T,\, r)\, \mathbf{e}(\mathrm{tr}(T\tau) + r\, {}^t z).$$

Then for each $T \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})$ such that $B_T = 4T - {}^t rr > 0$ with $r \in \mathbb{Z}^{n-1}$, we have

$$c^{(n-1)}_{k,\,1}(T,\, r) = \xi(n,\, k) L(1 - k + n/2, \chi_{B^{(1)}_T})\, \mathfrak{f}(B^{(1)}_T)^{k-(n+1)/2} \prod_{p \,|\, \mathfrak{f}(B^{(1)}_T)} \widetilde{F}^{(1)}_p(B_T;\, p^{k-(n+1)/2}),$$

where $B^{(1)}_T = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (B_T + {}^t rr)/4 \end{pmatrix} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix} \in \mathrm{Sym}^*_n(\mathbb{Z})_{>0}.$

Proof. Since
$$c_{k,1}^{(n-1)}(T,\,r)=A_k^{(n)}(B_T^{(1)}),$$
the assertion immediately follows form the equation (6). □

Returning to the general theory of Jacobi forms, now we consider the action of Hecke operators on Jacobi forms. Let $M\in \mathrm{Sp}_n(\mathbb{Q})$ and decompose the double coset $\Gamma_n^J M\Gamma_n^J$ into the disjoint right cosets:
$$\Gamma_n^J M\Gamma_n^J=\bigsqcup_{i=1}^{d}\Gamma_n^J\,g_i,$$
where we denote by d the number of right cosets, that is, $d=[\Gamma_n^J M\Gamma_n^J:\Gamma_n^J]$. Then for any $\phi\in J_{l,\,m}(\Gamma_n^J)$, we define the action of the double coset $\Gamma_n^J M\Gamma_n^J$ on ϕ by
$$\phi|_{l,\,m}\,\Gamma_n^J M\Gamma_n^J:=\sum_{i=1}^{d}\phi|_{l,\,m}\,g_i,$$
where the summation on the right hand side of the above is well-defined. We easily see that for any $\gamma\in\Gamma_n^J$,
$$(\phi|_{l,\,m}\,\Gamma_n^J M\Gamma_n^J)\,|_{l,\,m}\,\gamma=\phi|_{l,\,m}\,\Gamma_n^J M\Gamma_n^J,$$
that is, $\phi|_{l,\,m}\,\Gamma_n^J M\Gamma_n^J\in J_{l,\,m}(\Gamma_n^J)$. Moreover, if $\phi\in J_{l,\,m}^{\mathrm{cusp}}(\Gamma_n^J)$, then $\phi|_{l,\,m}\,\Gamma_n^J M\Gamma_n^J\in J_{l,\,m}^{\mathrm{cusp}}(\Gamma_n^J)$. Here we note that each of the double cosets $\Gamma_n^J M\Gamma_n^J$ with $M\in G_n(\mathbb{Q})$ contains a unique representative of the form
$$\mathbf{d}_n(\delta_1\perp\cdots\perp\delta_n)=(\delta_1\perp\cdots\perp\delta_n)\perp(\delta_1^{-1}\perp\cdots\perp\delta_n^{-1})$$
with $0<\delta_1\,|\cdots|\,\delta_n$. Moreover, let $D=\delta_1\perp\cdots\perp\delta_n$ and $D'=\delta_1'\perp\cdots\perp\delta_n'$ be two diagonal matrices with $0<\delta_1\,|\cdots|\,\delta_n$, $0<\delta_1'\,|\cdots|\,\delta_n'$. We easily see that if $(\delta_n,\,\delta_n')=1$, then for any $\phi\in J_{l,\,m}(\Gamma_n^J)$,
$$\phi|_{l,\,m}\,\Gamma_n^J\mathbf{d}_n(DD')\Gamma_n^J=\phi|_{l,\,m}\,\Gamma_n^J\mathbf{d}_n(D)\Gamma_n^J\,|_{l,\,m}\,\Gamma_n^J\mathbf{d}_n(D')\Gamma_n^J.$$

A Jacobi form $\phi\in J_{l,\,1}(\Gamma_n^J)$ is called a *Hecke eigenform* if it is a common eigenfunction of all actions of double cosets $\Gamma_n^J M\Gamma_n^J$ with $M\in G_n(\mathbb{Q})$, that is, for any $M\in G_n(\mathbb{Q})$, the equation
$$\phi|_{l,\,m}\,\Gamma_n^J M\Gamma_n^J=\lambda_\phi(M)\,\phi$$
holds for some $\lambda_\phi(M)\in\mathbb{C}$. We easily see from the above argument that ϕ is a Hecke eigenform if and only if it satisfies for any rational prime p and $D=p^{\alpha_1}\perp\cdots\perp p^{\alpha_n}\in\mathbf{D}_p^{(n)}(\mathbb{Z})$ with $0\le\alpha_1\le\cdots\le\alpha_n$,
$$\phi|_{l,\,m}\,\Gamma_n^J\mathbf{d}_n(D)\Gamma_n^J=\lambda_\phi(D)\,\phi$$
with $\lambda_\phi(D)\in\mathbb{C}$.

2.2.2. *Jacobi forms on the adele group.*

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and let $\Psi_{\mathbb{A}}$ be the character of $\mathbb{Q}\backslash\mathbb{A}$ such that $\Psi_{\mathbb{A}}(x_\infty) = \mathbf{e}(x_\infty)$ for any $x_\infty \in \mathbb{R}$. In addition, for each $m \in \mathbb{Z}$, we put $\Psi_{\mathbb{A}}^m(\kappa) = \Psi_{\mathbb{A}}(m\kappa)$ for any $\kappa \in \mathbb{A}$. We denote by $G_n^J(\mathbb{A})$ the adele group of the Jacobi group G_n^J defined in the previous paragraph. Then it follows from the strong approximation theorem for G_n^J that

$$G_n^J(\mathbb{A}) = G_n^J(\mathbb{Q})G_n^J(\mathbb{R})K_{\mathrm{fin}}^J,$$

where $K_{\mathrm{fin}}^J := \prod_{p<\infty} G_n^J(\mathbb{Z}_p)$.

Let l and m be positive integers. A $\mathbb{C}$-valued function f on $G_n^J(\mathbb{A})$ is called a *Jacobi form* of weight l and index m if it satisfies the following two conditions:

(i) The functional equation

$$f([(0,0),\kappa]\,\gamma\, g\, k_\infty\, k_{\mathrm{fin}}) = \det(j(k_\infty,\, \sqrt{-1}\,\mathbf{1}_n))^{-l}\,\Psi_{\mathbb{A}}^m(\kappa)f(g)$$

holds for any $\kappa \in \mathbb{A}$, $\gamma \in G_n^J(\mathbb{Q})$, $g \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\mathrm{fin}} \in K_{\mathrm{fin}}^J$;

(ii) For any $(\tau,\, z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we choose and fix an element $g_\infty \in G_n^J(\mathbb{R})$ such that $g_\infty\langle\sqrt{-1}\,\mathbf{1}_n,\, 0\rangle = (\tau,\, z)$ and put

$$\Phi_f(\tau,\, z) := J_{l,m}(g_\infty,\, (\sqrt{-1}\,\mathbf{1}_n,\, 0))f(g_\infty), \tag{7}$$

with the factor of automorphy $J_{l,m} : G_n^J(\mathbb{R}) \times (\mathfrak{H}_n \times \mathbb{C}^n) \to \mathbb{C}$ defined in §2.2.1. Here we easily see that the value Φ_f does not depend on the choice of g_∞. Then the function Φ_f is holomorphic on $\mathfrak{H}_n \times \mathbb{C}^n$. In particular, if it satisfies the further condition that

$$\left|\det(\mathrm{Im}(\tau))^{l/2}\exp(-2m\pi\,\mathrm{tr}\left(\mathrm{Im}(\tau)^{-1}[{}^t\,\mathrm{Im}(z)]\right))\Phi_f(\tau,\, z)\right| \text{ is bounded on } \mathfrak{H}_n \times \mathbb{C}^n,$$

then it is called *cuspidal*.

We denote by $J_{l,m}(G_n^J(\mathbb{A}))$ and $J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A}))$ the $\mathbb{C}$-vector spaces of the Jacobi forms and cuspidal Jacobi forms of weight l and index m on the group $G_n^J(\mathbb{A})$, respectively.

It is easy to see that for each $f \in J_{l,m}(G_n^J(\mathbb{A}))$, the associated function Φ_f is an element of $J_{l,m}(\Gamma_n^J)$. In particular, if $f \in J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A}))$, then $\Phi_f \in J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$. Furthermore we have

Lemma 2.2. *The map $J_{l,m}(G_n^J(\mathbb{A})) \ni f \mapsto \Phi_f \in J_{l,m}(\Gamma_n^J)$ induces $\mathbb{C}$-linear isomorphisms $J_{l,m}(G_n^J(\mathbb{A})) \cong J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A})) \cong J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$.*

Proof. Since it is straightforward, we omit the proof. □

2.3. **Standard L-functions attached to Jacobi forms.**

We study in this paragraph Shintani's standard L-functions attached to Jacobi forms. In particular, we derive an explicit formula for the standard L-function attached to the Jacobi Eisenstein series of index 1. It might be given in a classical way, but here we treat it adelically.

Let p be an arbitrary rational prime. For simplicity, we write G_p^J, G_p, K_p^J, K_p and Z_p^J instead of $G_n^J(\mathbb{Q}_p)$, $G_n(\mathbb{Q}_p)$, $G_n^J(\mathbb{Z}_p)$, $G_n(\mathbb{Z}_p)$ and $Z_n^J(\mathbb{Q}_p)$, respectively. We denote by Ψ_p and $|*|_p$ the restriction of $\Psi_{\mathbb{A}}$ to $\mathbb{Q}_p$ and the p-adic valuation of $\mathbb{Q}_p$ normalized as $|p|_p = p^{-1}$, respectively. Let $\mathscr{H}_p = \mathscr{H}(G_p^J,\, K_p^J;\, \Psi_p)$ be the $\mathbb{C}$-module consisting of $\mathbb{C}$-valued functions φ on G_p^J satisfying the following two conditions:

(i) The equation
$$\varphi([(0,0),\kappa]\, k\, g\, k') = \Psi_p(\kappa)\varphi(g)$$
holds for any $\kappa \in \mathbb{Q}_p$, k, $k' \in K_p^J$ and $g \in G_p^J$;

(ii) φ is compactly supported modulo Z_p^J.

Then $\mathscr{H}_p$ forms a $\mathbb{C}$-algebra via the convolution product
$$(\varphi_1 * \varphi_2)(g) := \int_{Z_p^J\backslash G_p^J} \varphi_1(gx^{-1})\varphi_2(x)dx,$$
where dx is a Haar measure on $Z_p^J\backslash G_p^J$ normalized by $\displaystyle\int_{Z_p^J\backslash Z_p^J K_p^J} dx = 1$. The algebra $\mathscr{H}_p$ is called the *Hecke algebra* of $(G_p^J,\, K_p^J)$ with respect to the additive character Ψ_p.

We put
$$N_p^J := \left\{\, [(0,\mu),0]\,\mathbf{d}_n(A)\,\mathbf{n}_n(S) \in G_p^J \;\middle|\; \mu \in \mathbb{Q}_p^n,\ A \in U_{n,p},\ S \in \mathrm{Sym}_n(\mathbb{Q}_p) \,\right\},$$
$$T_p = T(\mathbb{Q}_p) := \{\, \mathbf{d}_n(t_1 \perp \cdots \perp t_n) \in G_p \mid t_i \in \mathbb{Q}_p^\times \,\}$$
and $T(\mathbb{Z}_p) := T_p \cap K_p$, where $U_{n,p} \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is the group of upper unipotent matrices. We fix Haar measures $d\mathbf{n}$ and dt on N_p^J and T_p respectively normalized by
$$\int_{N_p^J\cap K_p^J} d\mathbf{n} = 1 \ \text{ and } \ \int_{T(\mathbb{Z}_p)} dt = 1.$$
We define the module $\delta_{N_p^J}(t)$ of $t \in T_p$ to be the ratio $d(t\mathbf{n}t^{-1})/d\mathbf{n}$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, we put
$$\pi_\alpha = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \mathrm{GL}_n(\mathbb{Q}_p),$$
then we easily see that
$$\delta_{N_p^J}(\pi_\alpha) = p^{-\sum_{i=1}^n (2n+3-2i)\alpha_i}.$$
Let $X_0(T_p)$ be the group of unramified characters of T_p, that is,
$$X_0(T_p) := \{\, \chi \in \mathrm{Hom}(T_p,\, \mathbb{C}^\times) \mid \chi \text{ is trivial on } T(\mathbb{Z}_p) \,\}.$$
In particular, if $n = 1$, then $X_0(T_p)$ coincides with the group $X_0(\mathbb{Q}_p^\times)$ consisting of all unramified characters of $\mathbb{Q}_p^\times$. For any $\chi \in X_0(T_p)$ and $\varphi \in \mathscr{H}_p$, we define the *zonal spherical function* $\widehat{\omega}_\chi(\varphi)$ by
$$\widehat{\omega}_\chi(\varphi) := \sum_{\alpha\in\mathbb{Z}^n} \chi^{-1}(\mathbf{d}_n(\pi_\alpha))\widetilde{\varphi}(\mathbf{d}_n(\pi_\alpha)),$$
where
$$\widetilde{\varphi}(t) := \delta_{N,p}^J(t)^{-1/2} \int_{N_p^J} \varphi(\mathbf{n}t)d\mathbf{n} \quad (t \in T_p).$$

It is shown by Murase that the map $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ gives a $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$ and that every $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$ is given by $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ for some $\chi \in X_0(T_p)$ (cf. Proposition 4.10 and Theorem 4.15 in [18]).

On the other hand, for any $\chi \in X_0(T_p)$, let ϕ_χ be a $\mathbb{C}$-valued function on G_p^J defined by

$$\phi_\chi([(0,0),\kappa]\,\mathbf{n}\,t\,[(\lambda,0),0]\,k) = \Psi_p(\kappa)\,(\chi\,\delta_{N_p^J}^{-1/2})(t)\,\mathrm{char}_{\mathbb{Z}_p^n}(\lambda)$$

for any $\kappa \in \mathbb{Q}_p$, $\mathbf{n} \in N_p^J$, $t \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K_p^J$, where we denote by $\mathrm{char}_{\mathbb{Z}_p^n}$ the characteristic function of $\mathbb{Z}_p^n$. Here we note that each $\chi \in X_0(T_p)$ can be written in the form

$$\chi(\mathbf{d}_n(t_1 \perp \cdots \perp t_n)) = \chi^{(1)}(t_1)\cdots\chi^{(n)}(t_n),$$

with uniquely determined n unramified characters $\chi^{(1)}, \cdots, \chi^{(n)} \in X_0(\mathbb{Q}_p^\times)$. In this case, we simply write $\chi = (\chi^{(1)}, \cdots, \chi^{(n)})$. If $\chi = (\chi^{(1)}, \cdots, \chi^{(n)}) \in X_0(T_p)$, then it satisfies that

$$\phi_\chi([(0,0),\kappa]\,\mathbf{n}\,t\,[(\lambda,0),0]\,k) = \Psi_p(\kappa)\prod_{i=1}^n \chi^{(i)}(t_i)\,|t_i|_p^{(2n+3-2i)/2}\mathrm{char}_{\mathbb{Z}_p^n}(\lambda) \tag{8}$$

for any $\kappa \in \mathbb{Q}_p$, $\mathbf{n} \in N_p^J$, $t = \mathbf{d}_n(t_1 \perp \cdots \perp t_n) \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K_p^J$.

For each rational prime p, we define the action of Hecke algebra $\mathscr{H}_p$ on the space $J_{l,1}(G_n^J(\mathbb{A}))$ by the following: for any $f \in J_{l,1}(G_n^J(\mathbb{A}))$ and $\varphi \in \mathscr{H}_p$,

$$(f * \varphi)(g) := \int_{Z_p^J\backslash G_p^J} f(gx^{-1})\varphi(g^{-1})dx \quad (g \in G_n^J(\mathbb{A})).$$

A Jacobi form $f \in J_{l,1}(G_n^J(\mathbb{A}))$ is called a *Hecke eigenform* if it is a common eigenfunction of all elements of $\bigotimes_p \mathscr{H}_p$, that is, for any rational prime p and $\varphi \in \mathscr{H}_p$, the equation

$$f * \varphi = \lambda_f(\varphi)\, f$$

holds for some $\lambda_f(\varphi) \in \mathbb{C}$. Since for each p, the map $\lambda_f : \mathscr{H}_p \to \mathbb{C}$ gives a $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$, it determines a $\chi_f = (\chi_f^{(1)}, \cdots, \chi_f^{(n)}) \in X_0(T_p)$ such that

$$\lambda_f(\varphi) = \widehat{\omega}_{\chi_f}(\varphi)$$

for any $\varphi \in \mathscr{H}_p$. We call either the collection $(\chi_f^{(1)}(p), \cdots, \chi_f^{(n)}(p))$ or $(\chi_f^{(1)}(p)^{-1}, \cdots, \chi_f^{(n)}(p)^{-1})$ the *Satake p-parameters* of f. Then for a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$, we define the standard L-function attached to ϕ by

$$L(s, f, \mathrm{St}) := \prod_{p<\infty}\prod_{i=1}^n \left\{(1-\chi_f^{(i)}(p)\,p^{-s})(1-\chi_f^{(i)}(p)^{-1}p^{-s})\right\}^{-1},$$

which was introduced by Shintani in his unpublished paper, and afterward was studied by Murase (cf. [18, 19]).

By Lemma 2.2, for each element $f \in J_{l,1}(G_n^J(\mathbb{A}))$ we obtain the associated element $\Phi_f \in J_{l,1}^{\mathrm{cusp}}(\Gamma_n^J)$. Then we easily have the following relation between the action of the Hecke algebra $\mathscr{H}_p$ on f and the operation $\Phi_f|_{l,1}\Gamma_n^J M \Gamma_n^J$ for some $M \in G_n(\mathbb{Z}[p^{-1}])$:

Lemma 2.3. *Let $f \in J_{l,1}(G_n^J(\mathbb{A}))$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$ with $0 \le \alpha_1 \le \cdots \le \alpha_n$, we have*

$$\Phi_{f * \varphi_\alpha} = \Phi_f|_{l,1}\, \Gamma_n^J \mathbf{d}_n(\pi_\alpha) \Gamma_n^J.$$

Here φ_α is an element of $\mathscr{H}_p$ defined by

$$\varphi_\alpha(g) = \begin{cases} \Psi_p(\kappa) & \text{if } g \in Z_p^J K_p^J \mathbf{d}_n(\pi_\alpha) K_p^J \text{ and } g = [(0,0),\kappa]\, k\, \mathbf{d}_n(\pi_\alpha)\, k', \\ 0 & \text{if } g \notin Z_p^J K_p^J \mathbf{d}_n(\pi_\alpha) K_p^J, \end{cases}$$

where $\kappa \in \mathbb{Q}_p$ and $k,\, k' \in K_p^J$. In particular, if f is a Hecke eigenform, then Φ_f is also a Hecke eigenform in the sense of §2.2.1.

Let $\phi \in J_{l,1}(\Gamma_n^J)$ be a Hecke eigenform corresponding to a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$ via the mapping defined in (7), that is, $\phi = \Phi_f$. Then by Lemma 2.3, we naturally define the standard L-function attached to ϕ as $L(s,\, \phi,\, \mathrm{St}) := L(s,\, f,\, \mathrm{St})$. Namely,

$$L(s,\, \phi,\, \mathrm{St}) := \prod_{p<\infty} \prod_{i=1}^{n} \left\{ (1 - \chi_\phi^{(i)}(p)\, p^{-s})(1 - \chi_\phi^{(i)}(p)^{-1} p^{-s}) \right\}^{-1},$$

where we put $\chi_\phi^{(i)}(p) = \chi_f^{(i)}(p)$ for $i = 1, \cdots, n$.

If ϕ is a cuspidal Hecke eigenform, then the following analytic properties of the standard L-function $L(s,\, \phi,\, \mathrm{St})$ are shown by Murase ([19]):

Lemma 2.4 (cf. [19])**.** *If $\phi \in J_{l,1}^{\mathrm{cusp}}(\Gamma_n^J)$ is a Hecke eigenform, then the standard L-function $L(s,\, \phi,\, \mathrm{St})$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$. More precisely, put $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, and the function*

$$L^*(s,\, \phi,\, \mathrm{St}) = \prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s + l - 1/2 - i)\, L(s,\, \phi,\, \mathrm{St})$$

is meromorphic on $\mathbb{C}$, and satisfies the functional equation

$$L^*(1-s,\, \phi,\, \mathrm{St}) = \varepsilon_n\, L^*(s,\, \phi,\, \mathrm{St}),$$

where

$$\varepsilon_n = \begin{cases} -1 & \text{if } n \equiv 1,\, 2 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Remark. Murase derived similar properties for the standard L-functions attached to more general cuspidal Jacobi forms whose index is a matrix.

On the other hand, we consider the standard L-function attached to the Jacobi Eisenstein series $\mathfrak{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$ in the rest of this paragraph.

For any quasi-character $\xi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}$, we define a $\mathbb{C}$-valued function $\widetilde{\phi_\xi}$ on $G_n^J(\mathbb{A})$ by

$$\widetilde{\phi_\xi}([(0,\mu),\kappa]\, g[(\lambda,0),0]\, k_\infty\, k_{\mathrm{fin}}) = \xi(\det(A))\varphi_0(\lambda) j(k_\infty, \sqrt{-1}\,\mathbf{1}_n)^{-l}$$

for any $\kappa \in \mathbb{A}$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\mathrm{fin}} \in K_{\mathrm{fin}}^J$, where $\varphi_0 = \prod_v \varphi_{0,v}$,

$$\varphi_{0,v}(\lambda) = \begin{cases} \mathrm{char}_{\mathbb{Z}_p^n}(\lambda) & \text{if } v = p < \infty, \\ \exp(-2\pi\lambda^t\lambda) & \text{if } v = \infty. \end{cases}$$

Then we define the Eisenstein series E_ξ on $G_n^J(\mathbb{A})$ associated with ξ by

$$E_\xi(g) := \sum_{\gamma \in P_n^J(\mathbb{Q})\backslash G_n^J(\mathbb{Q})} \widetilde{\phi}_\xi(\gamma g) \qquad (g \in G_n^J(\mathbb{A})).$$

In particular, we denote by $\mathcal{E}_{l,1}^{(n)}$ the Eisenstein series on $G_n^J(\mathbb{A})$ associated with a special character $\xi_l(x) = |x|_{\mathbb{A}}^l$ $(x \in \mathbb{A}^\times)$. Then we easily see that $\mathcal{E}_{l,1}^{(n)}$ is an element of $J_{l,1}(G_n^J(\mathbb{A}))$ and corresponds to the Jacobi Eisenstein series $\mathfrak{E}_{l,1}^{(n)} \in J_{l,1}(\Gamma_n^J)$ in the same manner as in Lemma 2.2. Therefore we also call $\mathcal{E}_{l,1}^{(n)}$ the Jacobi Eisenstein series of weight l and index 1. Then we have

Proposition 2.2. *The Jacobi Eisenstein series $\mathcal{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$ is a Hecke eigenform, that is, for any $\varphi \in \bigotimes_p \mathscr{H}_p$,*

$$\mathcal{E}_{l,1}^{(n)} * \varphi = \lambda_{\mathcal{E}}(\varphi)\, \mathcal{E}_{l,1}^{(n)}$$

with $\lambda_{\mathcal{E}}(\varphi) \in \mathbb{C}$. Moreover, the Satake p-parameters of $\mathcal{E}_{l,1}^{(n)}$ are taken of the form

$$\left(p^{l-(n+1)+i-1/2}\right)_{1 \le i \le n}$$

up to inversion.

Proof. For any quasi-character ξ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$, we take a $\chi = (\chi^{(1)}, \cdots, \chi^{(n)}) \in X_0(T_p)$ such that

$$\chi^{(i)}(t_i) = \xi(t_i)\, |t_i|_p^{-(2n+3-2i)/2} \qquad (t_i \in \mathbb{Q}_p^\times) \tag{9}$$

for each $1 \le i \le n$. Then by the equation (8) and the definition of $\widetilde{\phi}_\xi$, we have $\widetilde{\phi}_\xi = \phi_\chi$. Therefore it suffices to prove that for any $\varphi \in \mathscr{H}_p$ and $\lambda \in \mathbb{Q}_p^n$, the equation

$$(\phi_\chi * \varphi)([(\lambda, 0), 0]) = c \cdot \mathrm{char}_{\mathbb{Z}_p^n}(\lambda) \tag{10}$$

holds with some $c \in \mathbb{C}^\times$. Indeed, if $\lambda \notin \mathbb{Z}_p^n$, then there exists $0 \ne \mu \in \mathbb{Z}_p^n$ such that $\Psi_p(\lambda^t\mu) \ne 1$. Thus we have

$$\begin{aligned} (\phi_\chi * \varphi)([(\lambda, 0), 0]) &= (\phi_\chi * \varphi)([(\lambda, 0), 0] \cdot [(0, \mu), 0]) \\ &= (\phi_\chi * \varphi)([(\lambda, \mu), \lambda^t\mu]) \\ &= (\phi_\chi * \varphi)([(0, \mu), \lambda^t\mu] \cdot [(\lambda, 0), 0]) \\ &= \Psi_p(\lambda^t\mu)\, (\phi_\chi * \varphi)([(\lambda, 0), 0]), \end{aligned}$$

and $(\phi_\chi * \varphi)([(\lambda, 0), 0]) = 0$. Now we have proved that the Eisenstein series E_ξ is a Hecke eigenform. Moreover, it follows from the equation (10) that

$$c = (\phi_\chi * \varphi)(1) = \int_{Z_p^J\backslash G_p^J} \phi_\chi(g)\varphi(g^{-1})dg$$

and therefore the eigenvalue $\lambda_{\mathcal{E}}(\varphi)$ coincides with the zonal spherical function $\widehat{\omega}_{\chi}(\varphi)$. Therefore it follows from the equation (9) that

$$\chi^{(i)}(t_i) = \xi_l(t_i)\,|t_i|_p^{-(2n+3-2i)/2} = |t_i|_p^{l-(2n+3-2i)/2}$$

for each i. By substituting $t_i = p$, we obtain $\chi^{(i)}(p) = p^{-l+(2n+3-2i)/2}$ and complete the proof. □

By Proposition 2.2, we obtain the following conclusion:

Corollary. *Let l be a positive even integer such that $l > n+2$. Then we have*

$$L(s,\, \mathcal{E}_{l,1}^{(n)},\, \mathrm{St}) = L(s,\, \mathfrak{E}_{l,1}^{(n)},\, \mathrm{St}) = \prod_{i=1}^{n} \zeta(s-l+1/2+i)\zeta(s+l-1/2-i).$$

In particular, $L(s,\, \mathcal{E}_{l,1}^{(n)},\, \mathrm{St})$ and $L(s,\, \mathfrak{E}_{l,1}^{(n)},\, \mathrm{St})$ converge absolutely for $\mathrm{Re}(s) > l-n-1/2$. In addition, they have meromorphic continuations to the entire complex plane $\mathbb{C}$ and satisfy functional equations under $s \longmapsto 1-s$.

Remark. Let k and n be positive even integers such that $k > n+1$. As mentioned in §2.1, $\mathfrak{E}_{k,1}^{(n-1)}$ coincides with the first Fourier-Jacobi coefficient $e_{k,1}^{(n-1)}$ of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$ of degree n and weight k. Thus it follows from Corollary of Proposition 2.2 that

$$\begin{aligned} L(s,\, e_{l,1}^{(n)},\, \mathrm{St}) &= \prod_p \prod_{i=1}^{n-1} \{(1-p^{k-(n+1)/2}p^{-s+i-n/2})(1-(p^{k-(n+1)/2})^{-1}p^{-s+i-n/2})\}^{-1} \\ &= \prod_{i=1}^{n-1} L(s+k-1/2-i,\, E_{2k-n}^{(1)}), \end{aligned}$$

where $E_{2k-n}^{(1)} \in M_{2k-n}(\Gamma_1)$. Moreover, replacing $e_{k,1}^{(n-1)}$ by the first Fourier-Jacobi coefficient $\phi_1 \in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to a normalized Hecke eigenform $f \in S_{2k-n}(\Gamma_1)$ via a lifting procedure due to Ikeda (cf. [9]), then we also obtain a similar explicit formula for the standard L-function attached to ϕ_1 (cf. [15]).

2.4. **Eichler-Zagier-Ibukiyama correspondence between Jacobi forms and Siegel modular forms of half-integral weight.**

For the purpose of the subsequent use, we review in this paragraph that there exists a natural $\mathbb{C}$-linear correspondence from the space of Jacobi forms of even integral weight and of index 1 into that of Siegel modular forms of half-integral weight.

For any $(\tau,\, z) \in \mathfrak{H}_n \times \mathbb{C}^n$ and $(r_1,\, r_2) \in \mathbb{Q}^n \oplus \mathbb{Q}^n$, we define the *theta series* of characteristic $(r_1,\, r_2)$ by

$$\theta_{(r_1,\,r_2)}(\tau,\, z) = \theta_{(r_1,\,r_2)}^{(n)}(\tau,\, z) := \sum_{\lambda \in \mathbb{Z}^n} \mathbf{e}\left((\tau/2)[\,{}^t(\lambda+r_1)] + (\lambda+r_1)^t(z+r_2)\right).$$

In particular, for any $r \in \mathbb{Z}^n$, we put $\theta_r(\tau, z) = \theta_r^{(n)}(\tau, z) := \theta_{(r/2,0)}^{(n)}(2\tau, 2z)$. We note that the function $\theta_r(\tau, z)$ depends only on r mod $2\mathbb{Z}^n$. For a fixed $\tau \in \mathfrak{H}_n$, it is known that $(\theta_r(\tau, z))_{r\in\mathbb{Z}^n/2\mathbb{Z}^n}$ forms a basis of the $\mathbb{C}$-vector space $\Theta_\tau^{(n)}$ consisting of all $\mathbb{C}$-valued holomorphic functions $\theta(z)$ on $\mathbb{C}^n$ which satisfy that

$$\theta(z + \lambda\tau + \mu) = \mathbf{e}(-\operatorname{tr}(\tau[{}^t\lambda] + 2{}^t\lambda z))\theta(z)$$

for any $\lambda, \mu \in \mathbb{Z}^n$.

For any $\tau \in \mathfrak{H}_n$, we put

$$\theta(\tau) = \theta^{(n)}(\tau) := \theta_{(0,0)}^{(n)}(2\tau, 0) = \sum_{\lambda\in\mathbb{Z}^n} \mathbf{e}(\tau[{}^t\lambda]).$$

Then for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(4)$, we define the *Shimura's factor of automorphy* by

$$J(M, \tau) = J^{(n)}(M, \tau) := \frac{\theta^{(n)}(M\langle\tau\rangle)}{\theta^{(n)}(\tau)}.$$

As is well-known, it follows that

$$J(M, \tau)^2 = (-1)^{(\det D-1)/2} \det(C\tau + D).$$

For any $l \in \mathbb{Z}$, a holomorphic function $F(\tau)$ on $\mathfrak{H}_n$ is called a *Siegel modular form* of degree n and weight $l - 1/2$ if it satisfies the following two conditions:

(i) $F(M\langle\tau\rangle) = J(M, \tau)^{2l-1}F(\tau)$ for any $M \in \Gamma_0^{(n)}(4)$;

(ii) For any $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n$, the function $\det(C\tau+D)^{-l+1/2}F(M\langle\tau\rangle)$ possesses a Fourier expansion of the form

$$\det(C\tau + D)^{-l+1/2}F(M\langle\tau\rangle) = \sum_{B\in\mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} C_{F,M}(B)\,\mathbf{e}(\operatorname{tr}(B\tau)/4),$$

where $\det(C\tau + D)^{-l+1/2}$ is an appropriately defined single valued function of τ. If F satisfies the stronger condition $C_{F,M}(B) = 0$ unless $B > 0$ (positive definite), it is called a *cusp form*. We note that such a F possesses a usual Fourier expansion

$$F(\tau) = \sum_{B\in\mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} C_F(B)\,\mathbf{e}(\operatorname{tr}(B\tau)).$$

We denote by $M_{l-1/2}(\Gamma_0^{(n)}(4))$ and $S_{l-1/2}(\Gamma_0^{(n)}(4))$ the $\mathbb{C}$-vector spaces of Siegel modular forms and Siegel cusp forms of degree n and weight $l - 1/2$, respectively.

Furthermore, we introduce the *generalized Kohnen plus space* $M_{l-1/2}^+(\Gamma_0^{(n)}(4))$ consisting of all elements $F \in M_{l-1/2}(\Gamma_0^{(n)}(4))$ whose Fourier coefficients $C_F(B)$ satisfy the condition

$$C_F(B) = 0 \text{ unless } B \equiv (-1)^{l+1}\,{}^t r_B r_B \mod 4\,\mathrm{Sym}_n^*(\mathbb{Z}) \text{ for some } r_B \in \mathbb{Z}^{n-1},$$

and put $S_{l-1/2}^+(\Gamma_0^{(n)}(4)) := M_{l-1/2}^+(\Gamma_0^{(n)}(4)) \cap S_{l-1/2}(\Gamma_0^{(n)}(4))$. These spaces were introduced by Kohnen ([17]) in case $n = 1$, and by Ibukiyama ([8]) for general n.

Now, we recall an important fact that if l is even, then there exists a $\mathbb{C}$-linear isomorphism between the space $J_{l,1}(\Gamma_n^J)$ of Jacobi forms of index 1 and the generalized Kohnen plus space $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$ as follows. Let $\phi \in J_{l,1}(\Gamma_n^J)$ possess a Fourier expansion of the form

$$\phi(\tau,\, z) = \sum_{\substack{T \in \mathrm{Sym}_n^*(\mathbb{Z}),\, r \in \mathbb{Z}^n, \\ 4T - {}^t rr \geq 0}} c_\phi(T,\, r)\, \mathbf{e}(\mathrm{tr}(T\tau) + r\, {}^t z).$$

Since for each $\tau \in \mathfrak{H}_n$, $\phi(\tau,\, z)$ belongs in the space $\Theta_\tau^{(n)}$ generated by $(\theta_r(\tau,\, z))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}$, we have that ϕ can be expressed as a linear combination

$$\phi(\tau, z) = \sum_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} h_r(\tau)\, \theta_r(\tau, z)$$

with some 2^n holomorphic functions $(h_r(\tau))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}$ on $\mathfrak{H}_n$ whose Fourier expansion is of the form

$$h_r(\tau) = \sum_{\substack{T \in \mathrm{Sym}_n^*(\mathbb{Z}), \\ 4T - {}^t rr \geq 0}} c_\phi(T,\, r)\, \mathbf{e}(\mathrm{tr}\big((T - {}^t rr/4)\, \tau\big)).$$

Then we put

$$\sigma(\phi)(\tau) = \sum_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} h_r(4\tau).$$

The following statement is shown by Eichler and Zagier ([6]) in case $n = 1$ and by Ibukiyama for general n:

Proposition 2.3 (cf. Theorem 1, 2 in [8])**.** *If l is even, then the map $\phi \mapsto \sigma(\phi)$ gives a $\mathbb{C}$-linear isomorphism*

$$J_{l,1}(\Gamma_n^J) \cong M^+_{l-1/2}(\Gamma_0^{(n)}(4)),$$

which is compatible with the actions of Hecke operators. Furthermore, its restriction to the space $J^{\mathrm{cusp}}_{l,1}(\Gamma_n^J)$ also induces a $\mathbb{C}$-linear isomorphism

$$J^{\mathrm{cusp}}_{l,1}(\Gamma_n^J) \cong S^+_{l-1/2}(\Gamma_0^{(n)}(4)).$$

We call it the Eichler-Zagier-Ibukiyama correspondence.

Remark. When l is odd, the space $J_{l,1}(\Gamma_n^J)$ is not isomorphic to the Kohnen plus space $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$. However, a similar claim is also valid by introducing the space $J^{\mathrm{skew}}_{l,1}(\Gamma_n^J)$ of *skew holomorphic Jacobi forms* which was defined by Skoruppa ([24, 25]) in case $n = 1$ and by Arakawa ([2]) for general n.

We easily see by the definition that the Fourier expansion of $\sigma(\phi)$ can be expressed in terms of Fourier coefficients of ϕ as

$$\sigma(\phi)(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} c_\phi((B + {}^t r_B r_B)/4,\, r_B)\, \mathbf{e}(\mathrm{tr}(B\tau)),$$

where r_B denotes an element of $\mathbb{Z}^n$ such that $B + {}^t r_B r_B \in 4\mathrm{Sym}_n^*(\mathbb{Z})$. We note that r_B is uniquely determined by B modulo $2\mathbb{Z}^n$, and then $c_\phi((B + {}^t r_B r_B)/4,\, r_B)$ does not depend on the choice of the representative of r_B mod $2\mathbb{Z}^n$. Moreover, if ϕ coincides with the first Fourier-Jacobi coefficient of a Siegel modular form $F \in M_l(\Gamma_{n+1})$, then we have

$$\sigma(\phi)(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} A_F\big(B^{(1)}\big)\,\mathbf{e}(\mathrm{tr}(B\tau)),$$

where $B^{(1)} \in \mathrm{Sym}_n^*(\mathbb{Z})$ denotes the matrix defined in §1, and $A_F(B^{(1)})$ is the $B^{(1)}$-th Fourier coefficient of F. In particular, let n and k be positive even integers such that $k > n+1$ and we take $\phi = e_{k,1}^{(n-1)} \in J_{k,1}(\Gamma_{n-1}^J)$, then we have the following explicit formula for the Fourier coefficients of the associated form $\sigma(e_{k,1}^{(n-1)}) \in M_{k-1/2}^+(\Gamma_0^{(n-1)}(4))$:

Proposition 2.4. *Under the same assumption as in Proposition 2.1, let $\sigma(e_{k,1}^{(n-1)})$ possess a Fourier expansion*

$$\sigma(e_{k,1}^{(n-1)})(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} C_{k-1/2}^{(n-1)}(B)\,\mathbf{e}(\mathrm{tr}(B\tau)).$$

Then for each $B \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_{>0}$ satisfying the condition (1)*, we have*

$$C_{k-1/2}^{(n-1)}(B) = \xi(n,\,k)L(1-k+n/2,\,\chi_{B^{(1)}})\,\mathfrak{f}(B^{(1)})^{k-(n+1)/2} \prod_{p \mid \mathfrak{f}(B^{(1)})} \widetilde{F}_p^{(1)}(B;\,p^{k-(n+1)/2}).$$

Proof. If $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})$ and $r \in \mathbb{Z}^{n-1}$, then we have

$$C_{k-1/2}^{(n-1)}(B) = c_{k,1}^{(n-1)}(T,\,r).$$

Thus the assertion follows from Proposition 2.1. $\square$

3. Andrianov type identity for power series attached to Jacobi forms

Throughout this paragraph, let n and k be positive even integers such that $k > n+1$, and we fix a rational prime p. For a subring R of $\mathbb{Z}_p$, we simply denote by $\mathrm{Sym}_{n-1}(R)^{(1)}$ the subset of $\mathrm{Sym}_{n-1}(R)^\times$ consisting of all elements which satisfy the condition (1) in §1, namely,

$$\mathrm{Sym}_{n-1}(R)^{(1)} = \{\, B \in \mathrm{Sym}_{n-1}(R)^\times \mid B + {}^t r_B r_B \in 4\,\mathrm{Sym}_{n-1}^*(R) \text{ for some } r_B \in R^{n-1} \,\}.$$

As mentioned in §1, for each element $B \in \mathrm{Sym}_{n-1}(R)^{(1)}$, we can associate it with an element

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix} \in \mathrm{Sym}_n^*(R)^\times.$$

Then for such a $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we introduce a modified local Siegel series as follows. For each $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])$ and $r \in \mathbb{Z}_p^{n-1}$, if $R \in p^{-l}\mathrm{Sym}_{n-1}(\mathbb{Z}_p)$ with $l \geq 0$, then we put

$$\omega(R;\,r) = p^{-(n-1)l}\,\mu_p(R)^{1/2} \sum_{x \in \mathbb{Z}_p^{n-1}/p^l\mathbb{Z}_p^{n-1}} \mathbf{e}_p(-R[{}^t x] + rR\,{}^t x/2 + xR\,{}^t r/2),$$

where $\mu_p(R) = [\mathbb{Z}_p^{n-1}R+\mathbb{Z}_p^{n-1} : \mathbb{Z}_p^{n-1}]$, and we note that the right-hand side does not depend on the choice of l. Let $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ possess $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}^*_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then we put

$$b_p^{(1)}(B;\, t) = \sum_{R\in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z}_p)} \omega(R;\, r)\,\mathbf{e}_p(-\mathrm{tr}(TR))\, t^{\mathrm{ord}_p(\mu_p(R))}.$$

We note that this series coincides with $\alpha_1(B,\, t)$ in [21] if $p \neq 2$ and $r = 0$. As will be shown later, the above definition does not depend on the choice of T and r (cf. Proposition 3.1 below).

On the other hand, if $m > 1$, then for each $S \in \mathrm{Sym}^*_{m-1}(\mathbb{Z}_p)$, $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$, $r \in \mathbb{Z}_p^{n-1}$ and $e \in \mathbb{Z}_{>0}$, we put

$$\mathcal{A}_e(S,\, T,\, r) := \left\{ X \in \mathrm{M}_{m,n-1}(\mathbb{Z}_p)/p^e\mathrm{M}_{m,n-1}(\mathbb{Z}_p) \;\middle|\; \begin{array}{l} (-1\perp S)[X] + {}^t r\mathbf{x}_1/2 \\ \qquad + {}^t\mathbf{x}_1 r/2 - T \in p^e\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p) \end{array} \right\},$$

where $\mathbf{x}_1 \in \mathbb{Z}_p^{n-1}$ denotes the first row of X. We easily check that it is well-defined. Furthermore, if both S and $\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix}$ are non-degenerate, then $p^{e(-m(n-1)+n(n-1)/2)}\#\mathcal{A}_e(S,\, T,\, r)$ has the same value for any $e \geq \mathrm{ord}_p(\det \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix})$, which will be denoted by $\alpha_p^{(1)}(S,\, T,\, r)$. We note that $\alpha_p^{(1)}(S,\, T,\, r)$ coincides with the usual local density $\alpha_p(-1\perp S,\, T)$ if $r = 0$. Then we obtain the following lemmas:

Lemma 3.1. *Let $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)^\times$ possess $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then we have*

$$b_p^{(1)}(B;\, p^{-k+1/2}) = \alpha_p(H_{k-1},\, T,\, r),$$

where $H_{k-1} = \overbrace{H \perp \cdots \perp H}^{k-1}$ with $H = \left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right) \in \mathrm{Sym}^*_2(\mathbb{Z}_p)$. In particular, $b_p^{(1)}(B;\, t) = 0$ unless $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$.

Proof. By Lemma 3.4 of [22], we have

$$\begin{aligned}
& b_p^{(1)}(B;\, p^{-k+1/2}) \\
&= \sum_R \sum_{\mathbf{x}\in\mathbb{Z}_p^{n-1}/p^l\mathbb{Z}_p^{n-1}} \mathbf{e}_p(-R[{}^t\mathbf{x}] + rR\,{}^t\mathbf{x}/2 + \mathbf{x}R\,{}^t r/2)p^{-(k-1)\,\mathrm{ord}_p(\mu_p(R))}p^{-(n-1)l}\mathbf{e}_p(-\mathrm{tr}(TR)) \\
&= \sum_R \sum_{\mathbf{x}} \mathbf{e}_p(-R[{}^t\mathbf{x}] + rR\,{}^t\mathbf{x}/2 + \mathbf{x}R\,{}^t r/2)p^{-(n-1)l}\mathbf{e}_p(-\mathrm{tr}(TR))p^{-2l(k-1)n} \\
&\qquad \times \sum_{Y\in \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)/p^l\mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(H_{k-1}[Y]R)) \\
&= \sum_R \sum_x \sum_Y \mathbf{e}_p(\mathrm{tr}((-{}^t\mathbf{x}\mathbf{x} + H_{k-1}[Y] + {}^t r\mathbf{x}/2 + {}^t\mathbf{x}r/2 - T)R))\, p^{-l(2k-1)(n-1)} \\
&= \#\mathcal{A}_l(H_{k-1},\, T,\, r)\, p^{-l((2k-1)(n-1)-n(n-1)/2)}.
\end{aligned}$$

Thus the assertion holds. $\square$

Lemma 3.2. *If $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)^\times$ possesses $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$, then we have*

$$\alpha_p(H_k,\, B^{(1)}) = (1-p^{-k})\alpha_p(H_{k-1},\, T,\, r).$$

Proof. The proof is similar to that of Proposition 2.4 in [11], and here we give a sketch of the proof. For each $\xi = (\xi_i) \in \mathbb{Z}_p^{2k}$, we put

$$\mathcal{A}_e(H_k,\, B^{(1)}) = \{\, X \in \mathrm{M}_{2k,n}(\mathbb{Z}_p)/p^e\mathrm{M}_{2k,n}(\mathbb{Z}_p) \mid H_k[X] - B^{(1)} \in p^e\mathrm{Sym}_n^*(\mathbb{Z}_p)\,\}$$

and

$$\mathcal{A}_e(H_k,\, B^{(1)};\, \xi) = \{\, X = (x_{ij}) \in \mathcal{A}_e(H_k,\, B^{(1)}) \mid x_{i1} \equiv \xi_i \pmod{p^e} \text{ for } 1 \le i \le 2k\,\}.$$

We easily see that $\mathcal{A}_e(H_k, B^{(1)};\, \xi) \neq \phi$ only if $\xi \in \mathcal{A}_e(H_k,\, 1)$. Now we fix such a ξ. Then we have $\xi \not\equiv 0 \pmod{p\mathbb{Z}_p^{2k}}$. Thus by Lemma 2.3 in [11], we can take $U \in \mathrm{GL}_{2k}(\mathbb{Z}_p)$ and $K \in \mathcal{L}_{2k-2,p}$ such that

$$\text{(i)}\ \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \perp K = H_k[U]; \quad \text{(ii)}\ K \sim_{\mathbb{Z}_p} H_{2k-2}; \quad \text{(iii)}\ U^{-1}\xi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For each $X \in \mathcal{A}_e(H_k,\, B^{(1)};\, \xi)$, we write X as $X = ({}^t\xi \mid Y)$ with $Y \in \mathrm{M}_{2k,n-1}(\mathbb{Z}_p)$, and write Y as $Y = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ Y_3 \end{pmatrix}$ with $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{Z}_p^{n-1}$ and $Y_3 \in \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)$. Then by an easy calculation, we have

$$\mathbf{y}_1 + \mathbf{y}_2/2 - r/2 \in p^e\mathbb{Z}_p^{n-1}$$

and

$$-{}^t\mathbf{y}_1\mathbf{y}_1 + K[Y_3] + {}^t\mathbf{y}_1\mathbf{y}_2/2 + {}^t\mathbf{y}_2\mathbf{y}_1/2 - T \in p^e\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p).$$

Thus we have

$$-{}^t\mathbf{y}_1\mathbf{y}_1 + K[Y_3] + {}^t r\mathbf{y}_1/2 + {}^t\mathbf{y}_1 r/2 - T \in p^e\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p),$$

that is, $\begin{pmatrix} \mathbf{y}_1 \\ Y_3 \end{pmatrix} \in \mathcal{A}_e(H_{k-1},\, T,\, r)$. Then the mapping $Y \longmapsto \begin{pmatrix} \mathbf{y}_1 \\ Y_3 \end{pmatrix}$ induces a bijection between $\mathcal{A}_e(H_k,\, B^{(1)};\, \xi)$ and $\mathcal{A}_e(H_{k-1},\, T,\, r)$. Thus we have

$$\begin{aligned} & p^{e(-2kn+n(n+1)/2)}\#\mathcal{A}_e(H_k,\, B^{(1)}) \\ &= p^{e(-2k+1)}\#\mathcal{A}_e(H_k,\, 1)\, p^{e(-(2k-1)(n-1)+n(n-1)/2)}\#\mathcal{A}_e(H_{k-1},\, T,\, r) \\ &= \alpha_p(H_k,\, 1)\, \alpha_p(H_{k-1},\, T,\, r) \\ &= (1-p^{-k})\, \alpha_p(H_{k-1},\, T,\, r). \end{aligned}$$

Therefore the assertion holds. ☐

Now by combining Lemmas 3.1 and 3.2, we obtain the following:

Proposition 3.1. *For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ and $s \in \mathbb{C}$, we have*

$$b_p^{(1)}(B;\, p^{-s+1/2}) = (1-p^{-s})^{-1}\, b_p(B^{(1)};\, s).$$

Proof. It is well-known that for each $B' \in \mathrm{Sym}_n^*(\mathbb{Z}_p)^\times$ with $n < 2k$, the Siegel series $b_p(B';\, s)$ in §1 satisfies the equation

$$b_p(B;\, k) = \alpha_p(H_k,\, B).$$

Then by Lemmas 3.1 and 3.2, we have

$$b_p^{(1)}(B;\, p^{-k+1/2}) = (1-p^{-k})^{-1}\, b_p(B^{(1)};\, k)$$

for infinitely many k, and therefore the assertion follows. $\square$

Remark. The definition of $b_p^{(1)}(B;\, t)$ for $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$ does not depend on the choice of T and r. Indeed, if $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$, then the vector r is uniquely determined by B modulo $2\mathbb{Z}_p^{n-1}$, and the matrix $\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix}$ is uniquely determined by B up to $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$-equivalence. Thus by Proposition 3.1, $b_p^{(1)}(B;\, t)$ is uniquely determined by B. If $T \notin \mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$, then we have $b_p^{(1)}(B;\, t) = 0$. Furthermore, if $B = 4T' - {}^t r'r'$ is another expression, then T' does not belong to $\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$ either. This proves the well-definedness of $b_p^{(1)}(B;\, t)$.

Now we put

$$\widetilde{b}_p^{(1)}(B;\, t) := \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z}_p)} \pi_p(D)\, b_p^{(1)}(B[D^{-1}];\, t)\, (p^{n-1}\, t^2)^{\mathrm{ord}_p(\det D)}.$$

Then by Proposition 3.1, we obtain the following rationality theorem for the polynomial $\mathbf{B}_p^{(1)}(B;\, t)$ defined in §1:

Proposition 3.2. *For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$\mathbf{B}_p^{(1)}(B;\, p^{n-1/2}\, t)\, \widetilde{b}_p^{(1)}(B;\, p^{1/2}\, t) = \prod_{i=1}^{n-1}(1-p^{2i}t^2).$$

Next, we study the standard L-function attached to a Hecke eigenform and some power series related to it. For a Hecke eigenform $\phi \in J_{k,\,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$, and $D \in \mathbf{D}_p^{(n-1)}(\mathbb{Z})$, let

$$\phi|_{k,\,1}\Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J = \lambda_\phi(D)\phi$$

with $\lambda_\phi(D) \in \mathbf{C}$. Then we define a power series $Z_p(t,\, \phi)$ by

$$Z_p(t,\, \phi) := \sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \lambda_\phi(D)\, t^{\mathrm{ord}_p(\det D)},$$

where $\mathbf{ED}_p^{(n-1)}(\mathbb{Z})$ denotes the set of all elementary divisors of the form $p^{\alpha_1} \perp \cdots \perp p^{\alpha_{n-1}}$ with $0 \le \alpha_1 \le \cdots \le \alpha_{n-1}$. The following statement is shown by Murase and Sugano:

Proposition 3.3 (cf. Lemma 6.5 in [20], see also Theorem 5.5 in [2])**.** *Let $\phi \in J_{k,\,1}(\Gamma_{n-1}^J)$ be a Hecke eigenform with Satake p-parameters $(\chi_\phi^{(1)}(p),\, \cdots,\, \chi_\phi^{(n-1)}(p)) \in \mathbb{C}^{n-1}$. Then we have*

$$Z_p(t,\, \phi) = \prod_{i=1}^{n-1} \frac{(1-p^{2i}t^2)}{(1-\chi_\phi^{(i)}(p)p^{n-1/2}\, t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}\, t)}.$$

Let

$$\mathscr{X}_p^{(n-1)} := \left\{ \begin{pmatrix} V \\ W \end{pmatrix} \in \mathrm{M}_{2n-2,n-1}(\mathbb{Z}) \;\middle|\; V, W \in \mathbf{D}_p^{(n-1)}(\mathbb{Z}),\ \gcd(V,W)=1 \right\},$$

where $\gcd(V,W)$ denotes the greatest common divisor of all entries of V and W. For each $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right) \in \mathscr{X}_p^{(n-1)}$, $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])$ and $(\lambda_1, \lambda_2) \in \mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}$, we put

$$M_{V,W,R} := \left(\begin{array}{c|c} {}^tW^{-1}{}^tV & {}^tW^{-1}RV^{-1} \\ \hline \mathbf{0}_{n-1} & WV^{-1} \end{array}\right) \in G_{n-1}(\mathbb{Z}[p^{-1}])$$

and

$$[\lambda_1, \lambda_2] := [(\lambda_1, \lambda_2), \lambda_1{}^t\lambda_2] = \left(\begin{array}{cc|cc} 1 & \lambda_1 & 0 & \lambda_2 \\ 0 & \mathbf{1}_{n-1} & {}^t\lambda_2 & \mathbf{0}_{n-1} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \mathbf{0}_{n-1} & -{}^t\lambda_1 & \mathbf{1}_{n-1} \end{array}\right) \in H_{n-1}(\mathbb{Z}).$$

Then by combining Lemma 2.1 and some easy calculation (cf. [4]), we obtain the following:

Lemma 3.3. *We have*

$$\Gamma_{n-1}^J G_{n-1}(\mathbb{Z}[p^{-1}])\Gamma_{n-1}^J = \bigcup_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J$$

$$= \bigsqcup_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} \bigsqcup_{R} \bigsqcup_{(\lambda_1,\lambda_2)} \Gamma_{n-1}^J [M_{V,W,R}] \cdot [\lambda_1, \lambda_2],$$

where $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)$, *$R$ and (λ_1, λ_2) run over all representatives of* $(\mathbf{1}_{n-1} \perp \mathrm{GL}_{n-1}(\mathbb{Z}))\backslash \mathscr{X}_p^{(n-1)}/\mathrm{GL}_{n-1}(\mathbb{Z})$, $\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W$, *and* $(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})+(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}$, *respectively. Furthermore, if* $M_{V,W,R} \in \Gamma_{n-1}^J\mathbf{d}_{n-1}(D)\Gamma_{n-1}^J$ *with* $D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})$, *then we have* $\mathrm{ord}_p(\det D) = \mathrm{ord}_p(\det V \det W\, \mu_p(R))$.

Therefore, we get the following explicit formula for the actions of Hecke operators:

Corollary. *For each $\phi \in J_{k,1}(\Gamma_{n-1}^J)$, we have*

$$\begin{aligned}\sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \left(\phi\,|_{k,1}\Gamma_{n-1}^J\mathbf{d}_{n-1}(D)\Gamma_{n-1}^J\right)(\tau, z) &= \sum_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} \sum_{R} p^{(-2n+3)\delta_{V,W,R}} \det V^{k-1} \det W^{-k} \\ &\times \sum_{(\lambda_1,\lambda_2) \in (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1}{}^tV \oplus \mathbb{Z}^{n-1})} \mathbf{e}(\tau[{}^t\lambda_1] + 2\lambda_1{}^tz) \\ &\times \phi(\tau[VW^{-1}] + R[W^{-1}],\ (z + \lambda_1\tau + \lambda_2)VW^{-1}),\end{aligned}$$

where $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)$ *and R run over the sets stated above, and* $\delta_{V,W,R} = \mathrm{ord}_p(\det V \det W \mu_p(R))$.

Proof. For each $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right) \in \mathscr{X}_p^{(n-1)}$ and $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])$, we have

$$\Gamma_{n-1}^J M_{V,W,R}\Gamma_{n-1}^J = \Gamma_{n-1}^J\mathbf{d}_{n-1}(D)\Gamma_{n-1}^J$$

for some $D = p^{\alpha_1}\perp\cdots\perp p^{\alpha_{n-1}} \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})$. Then we have

$$\begin{aligned}
&(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}\\
\simeq\ &(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})\mathbf{d}_{n-1}(D)/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})\mathbf{d}_{n-1}(D)\\
\simeq\ &\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D.
\end{aligned}$$

It follows from Lemma 3.3 that $\#(\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D) = p^{\delta_{V,W,R}}$ and $e_1,...,e_r \le \delta_{V,W,R}$. Thus we have a natural surjection π from $(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1\,t}V\oplus\mathbb{Z}^{n-1})$ to $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D$, and we have $\#\mathrm{ker}(\pi) = p^{2(n-3)\delta_{V,W,R}}\det V$. Thus the assertion holds. □

By the above corollary, we obtain the following conclusion:

Proposition 3.4. *Let $\phi\in J_{k,1}(\Gamma^J_{n-1})$ be a Hecke eigenform. If the associated form $\sigma(\phi)\in M^+_{k-1/2}(\Gamma_0^{(n-1)}(4))$ under the Eichler-Zagier-Ibukiyama correspondence possesses a Fourier expansion*

$$\sigma(\phi)(\tau) = \sum_{B\in\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p)_{\ge 0}} C_{\sigma(\phi)}(B)\,\mathbf{e}(\mathrm{tr}(B\tau)),$$

then for each $B\in\mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, we have

$$\begin{aligned}
&\prod_{i=1}^{n-1}\frac{1-p^{2i}t^2}{(1-\chi_\phi^{(i)}(p)p^{n-1/2}\,t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}\,t)}\,C_{\sigma(\phi)}(B)\\
&= \sum_{\left(\begin{smallmatrix}V\\W\end{smallmatrix}\right)} b_p^{(1)}(B[{}^tV^{-1}];\,t)\,C_{\sigma(\phi)}(B[{}^tV^{-1}][W])\,p^{-(k-n-1)}p^{k\,\mathrm{ord}_p(\det V)}t^{\mathrm{ord}_p(\det V\det W)},
\end{aligned}$$

where $\left(\begin{smallmatrix}V\\W\end{smallmatrix}\right)$ runs over the set stated in Lemma 3.3.

Proof. We put

$$\Lambda_p(t) = \sum_{D\in\mathbf{ED}_p^{(n-1)}(\mathbb{Z})}\Gamma^J_{n-1}\mathbf{d}_{n-1}(D)\Gamma^J_{n-1}\,t^{\mathrm{ord}_p(\det D)}.$$

Then by Corollary of Lemma 3.3, we have

$$\begin{aligned}
&(\phi\,|_{k,1}\,\Lambda_p(t))(\tau,z) = \sum_T\sum_r c_\phi(T,\,r)\\
&\times\sum_{\left(\begin{smallmatrix}V\\W\end{smallmatrix}\right)\in(\mathbf{1}_{n-1}\perp\mathrm{GL}_{n-1}(\mathbb{Z}))\backslash\mathscr{X}_p^{(n-1)}/\mathrm{GL}_{n-1}(\mathbb{Z})} p^{(k-1)\,\mathrm{ord}_p(\det V)-k\,\mathrm{ord}_p(\det W)}t^{\mathrm{ord}_p(\det V\det W)}\\
&\times\mathbf{e}(\mathrm{tr}(T[{}^tW^{-1\,t}V]\tau + {}^t(r\,{}^tW^{-1\,t}V)z))\\
&\times\sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W}\mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\,t^{\mathrm{ord}_p(\mu_p(R))}\\
&\times\sum_{\lambda_1\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\cdot{}^tV} p^{-(2n-3)\delta_{V,W,R}}\mathbf{e}(\mathrm{tr}(2\,{}^t\lambda_1 z + {}^t(r\,{}^tW^{-1\,t}V+\lambda_1)\lambda_1\tau))\\
&\times\sum_{\lambda_2\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}}\mathbf{e}(\mathrm{tr}({}^t(r\,{}^tW^{-1\,t}V+\lambda_1)\lambda_2)).
\end{aligned}$$

Since

$$\sum_{\lambda_2 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} \mathbf{e}(\mathrm{tr}({}^t(r\,{}^tW^{-1}\,{}^tV+\lambda_1)\lambda_2)) = \begin{cases} p^{(n-1)\delta_{V,W,R}} & \text{if } r\,{}^tW^{-1} \in \mathbb{Z}^{n-1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W} \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\, t^{\mathrm{ord}_p(\mu_p(R))}$$

$$= \begin{cases} (\det W)^n \displaystyle\sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\, t^{\mathrm{ord}_p(\mu_p(R))} & \text{if } T[{}^tW^{-1}] \in \mathrm{Sym}^*_{n-1}(\mathbb{Z}), \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} (\phi\,|_{k,1}\,\Lambda_p(t))(\tau,z) &= \sum_T \sum_r \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)+(-k+n+1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V \det W)} \\ &\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(TR))\, (pt)^{\mathrm{ord}_p(\mu_p(R))} \\ &\times \sum_{\lambda_1 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}.{}^tV} p^{-(n-1)\delta_{V,W,R}} c_\phi(T[{}^tW],\, r\,{}^tW) \\ &\times \mathbf{e}(\mathrm{tr}({}^t(r\,{}^tV+2\lambda_1)z))\, \mathbf{e}(\mathrm{tr}((T[{}^tV]+{}^t(r\,{}^tV+\lambda_1)\lambda_1)\tau)). \end{aligned}$$

For a fixed $r_0 \in \mathbb{Z}^{n-1}$, we put

$$\mathcal{S}_1(r_0) = \{\, \lambda_1 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\,{}^tV \mid 2\lambda_1 \equiv r_0 \bmod \mathbb{Z}^{n-1}\,{}^tV \,\},$$

and

$$\mathcal{S}_2(r_0) = \{\, r \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \mid r\,{}^tV \equiv r_0 \bmod 2\mathbb{Z}^{n-1} \,\}.$$

For each $\lambda_1 \in \mathcal{S}_1(r_0)$, the map $\lambda_1 \mapsto (2\lambda_1 - r_0){}^tV^{-1}$ induces a bijection between $\mathcal{S}_1(r_0)$ and $\mathcal{S}_2(r_0)$. Thus we have

$$\begin{aligned} (\phi|_{k,1}\,\Lambda_p(t))(\tau,\, z) &= \sum_T \sum_{r_0} \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)+(-k+n+1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V \det W)} \\ &\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(TR))\, (pt)^{\mathrm{ord}_p(\mu_p(R))}\, p^{-(n-1)\delta_{V,W,R}} \\ &\times \sum_{r \in \mathcal{S}_2(r_0)} c_\phi(T[{}^tW],\, r\,{}^tW)\, \mathbf{e}(\mathrm{tr}({}^tr_0 z))\, \mathbf{e}(\mathrm{tr}(T[{}^tV] + ({}^tr_0r_0 - {}^t(r\,{}^tV)(r\,{}^tV))/4)\tau)) \\ &= \sum_{T_0} \sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^t r_0 z)) \sum_{\binom{V}{W}} \sum_{r \in \mathcal{S}_2(r_0)} p^{k\,\mathrm{ord}_p(\det V)+(-k+n+1)\,\mathrm{ord}_p(\det W)} p^{-(n-1)\delta_{V,W,R}} \\ &\times c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^trr/4)[{}^tW],\, r\,{}^tW) \\ &\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}] + {}^t rr/4)R))\, (pt)^{\mathrm{ord}_p(\mu_p(R))}. \end{aligned}$$

Then for a fixed $r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}$, the map

$$(r + 2\mathbb{Z}^{n-1}) + 2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}/2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \ni r + 2u \mapsto u \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}$$

is a bijection, and we have

$$c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t(r+2u)(r+2u)/4)[{}^tW],\, (r+2u){}^tW)$$
$$= c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t rr/4)[{}^tW],\, r{}^tW).$$

Thus we have

$$(\phi|_{k,1}\Lambda_p(t))(\tau,\, z)$$
$$= \sum_{T_0}\sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^t r_0 z)) \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V\det W)}$$
$$\times \sum_{R\in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} (pt)^{\mathrm{ord}_p(\mu_p(R))}$$
$$\times \sum_{\substack{r\in\mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1},\\ r{}^tV\equiv r_0 \bmod 2\mathbb{Z}^{n-1}}} c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t rr/4)[{}^tW],\, r{}^tW)$$
$$\times \sum_{u\in\mathbb{Z}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} p^{-(n-1)\delta_{V,W,R}} \mathbf{e}(\mathrm{tr}((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}] + {}^t rr/4 + {}^t uu + {}^t ur/2 + {}^t ru/2)R)).$$

We easily see for an element $r\in\mathbb{Z}^{n-1}$ that the summation

$$\sum_{R\in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} (pt)^{\mathrm{ord}_p(\nu(R))} \sum_{u\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} p^{-(n-1)\delta_{V,W,R}}$$
$$\times \mathbf{e}(\mathrm{tr}(((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}] + {}^t rr/4 + {}^t uu + {}^t ur/2 + {}^t ru/2)R))$$

equals $b_p^{(1)}((4T_0 - {}^t r_0 r_0)[{}^tV^{-1}];\, t)$ or 0 according as $(T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}] + {}^t rr/4 \in \mathrm{Sym}^*_{n-1}(\mathbb{Z}_p)$ or not, namely, according as $(4T_0 - {}^t r_0 r_0)[{}^tV^{-1}]\in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ or not. In the former case, the vector r is uniquely determined by T_0, r_0, and V, which will be denoted by $r_1 = r_1(T_0, r_0, V)$. Furthermore we have

$$(4T_0 - {}^t r_0 r_0)[{}^tV^{-1}] + {}^t rr)[{}^tV] = (4T_0 - {}^t r_0 r_0) + {}^t(r{}^tV)r{}^tV \in 4\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p),$$

and we have $r{}^tV\equiv r_0 \bmod 2\mathbb{Z}^{n-1}$. Thus we have

$$(\phi|_{k,1}\Lambda_p(t))(\tau,\, z) = \sum_{T_0}\sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^t r_0 z)) \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V\det W)}$$
$$\times b_p^{(1)}((4T_0 - {}^t r_0 r_0)[{}^tV^{-1}];\, t)\, c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t r_1 r_1/4)[{}^tW],\, r_1{}^tW).$$

Now we take an element $B\in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ so that $B = 4T_0 - {}^t r_0 r_0$ with $T_0\in \mathrm{Sym}^*_{n-1}(\mathbb{Z}_p)$ and $r_0\in\mathbb{Z}^{n-1}$. Then we have

$$c_\phi(T_0, r_0) = C_{\sigma(\phi)}(B),\; c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t r_1 r_1/4)[{}^tW],\, r_1^t W) = C_{\sigma(\phi)}(B[{}^tV^{-1}][{}^tW]),$$

and

$$b_p^{(1)}((4T_0 - {}^t r_0 r_0)[{}^tV^{-1}];\, t) = b_p^{(1)}(B[{}^tV^{-1}];\, t).$$

Since $\phi|_{k,1}\Lambda_p(t) = Z_p(t,\, \phi)\,\phi$, the assertion follows immediately from Proposition 3.3. □

For each $B\in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, let $\widetilde{G}_{\phi,p}(B;\, t)$ be the polynomial in t defined in §1. Then by making use of the same argument as in [4] combined with Propositions 3.2 and 3.4, we obtain the following:

Theorem 3.1. *Let n and k be positive even integers such that $k > n+1$, and let $\phi \in J_{k,1}(\Gamma^J_{n-1})$ be a Hecke eigenform with Satake p-parameters $(\chi^{(1)}_\phi(p), \cdots, \chi^{(n-1)}_\phi(p)) \in \mathbb{C}^{n-1}$. Then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, we have*

$$\frac{\mathbf{B}^{(1)}_p(B;\, p^{n-1/2}\, t)\, \widetilde{G}_{\phi,\, p}(B;\, t)}{\prod_{i=1}^{n-1}(1-\chi^{(i)}_\phi(p)p^{n-1/2}\, t)(1-\chi^{(i)}_\phi(p)^{-1}p^{n-1/2}\, t)} = \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}^{(n-1)}_p(\mathbb{Z})} C_{\sigma(\phi)}(B[W])\, p^{-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det W)}.$$

For each $D \in \mathrm{M}_{n-1}(\mathbb{Z}) \cap \mathrm{GL}_{n-1}(\mathbb{Q})$, we define the generalized global Möbius function $\pi(D)$ as $\prod_p \pi_p(D)$, where π_p is the local Möbius function defined in §1. We easily see that this is a finite product of $\pi_p(D)$. Then for each $B \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})^{(1)}_{>0}$, we put

$$\widetilde{H}_\phi(B;\, s) = \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathrm{M}_{n-1}(\mathbb{Z})\cap \mathrm{GL}_{n-1}(\mathbb{Q})} \pi(D)\, C_{\sigma(\phi)}(B[D^{-1}])\, \det D^{-s+k} \quad (s \in \mathbb{C}),$$

which is a finite sum, and we have $\widetilde{H}_\phi(B;\, s) = \prod_p \widetilde{G}_{\phi,\, p}(B;\, p^{-s})$. In addition, we also put $\mathbf{B}^{(1)}(B;\, s) = \prod_p \mathbf{B}^{(1)}_p(B;\, p^{-s})$. Then Theorem 3.1 can be restated globally as follows:

Theorem 3.2. *Under the same situation as above, we have*

$$\begin{aligned}&\mathbf{B}^{(1)}(B;\, s)\, L(s,\, \phi,\, \mathrm{St})\, \widetilde{H}_\phi(B;\, s+n-1/2) \\ &= \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathrm{M}_{n-1}(\mathbb{Z})\cap \mathrm{GL}_{n-1}(\mathbb{Q})} C_{\sigma(\phi)}(B[W])\, (\det W)^{-s-k+3/2}.\end{aligned}$$

Moreover, by applying Theorem 3.1 to the Jacobi Eisenstein series $\mathfrak{E}^{(n-1)}_{k,\,1} = e^{(n-1)}_{k,\,1} \in J_{k,\,1}(\Gamma^J_{n-1})$, we obtain the following conclusion:

Theorem 3.3. *Let n and k be as above. Then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$\frac{\mathbf{B}^{(1)}_p(B;\, p^{n-1/2}t)\, \widetilde{G}^{(1)}_p(B;\, p^{k-(n+1)/2},\, p^{(n+1)/2}\, t)}{\prod_{i=1}^{n-1}(1-p^{j-1}\, p^{k-(n+1)/2}\, p^{(n+1)/2}\, t)(1-p^{j-1}\, p^{-k+(n+1)/2}\, p^{(n+1)/2}\, t)} = \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathbf{D}^{(n-1)}_p(\mathbb{Z}_p)} \widetilde{F}^{(1)}_p(B[W];\, p^{k-(n+1)/2})\, (p^{(n+1)/2}\, t)^{\mathrm{ord}_p(\det W)},$$

where $\widetilde{F}^{(1)}_p(B;\, X)$ and $\widetilde{G}^{(1)}_p(B;\, X,\, t)$ are polynomials defined in §1.

Proof. By Proposition 2.4, the B-th Fourier coefficient of $\sigma(e^{(n-1)}_{k,\,1}) \in M^+_{k-1/2}(\Gamma^{(n-1)}_0(4))$ is expressed as

$$\xi(n,\, k)L(1-k/2+n/2, \chi_{B^{(1)}})\, \mathfrak{f}(B^{(1)})^{k-(n+1)/2} \prod_{p \mid \mathfrak{f}(B^{(1)})} \widetilde{F}^{(1)}_p(B;\, p^{k-(n+1)/2}).$$

Thus the assertion follows from Theorem 3.1 and Corollary of Proposition 2.2. □

For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, let $R_p^{(1)}(B;\, X,\, t)$ be the formal power series in $X + X^{-1}$ and t, which is defined in §1. Then we obtain the rationality for $R_p^{(1)}(B;\, X,\, t)$ as follows:

Theorem 3.4. *Let n be a positive even integer. Then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$R_p^{(1)}(B;\, X,\, t) = \frac{\mathbf{B}_p^{(1)}(B;\, p^{n/2-1}t)\, \widetilde{G}_p^{(1)}(B;\, X,\, t)}{\prod_{j=1}^{n-1}(1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)}.$$

Proof. We write the both-hand sides of the above equation as power series in t as

$$R_p^{(1)}(B;\, X,\, t) = \sum_{i=1}^{\infty} A_i(X)t^i,$$

and

$$\frac{\mathbf{B}_p^{(1)}(B;\, p^{n/2-1}t)\, \widetilde{G}_p^{(1)}(B;\, X,\, t)}{\prod_{j=1}^{n-1}(1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)} = \sum_{i=1}^{\infty} B_i(X)t^i,$$

where for each i, $A_i(X)$ and $B_i(X)$ are polynomials in $X + X^{-1}$. Then by Theorem 3.3, we have

$$A_i(p^{k-(n+1)/2}) = B_i(p^{k-(n+1)/2})$$

for infinitely many k. Thus we have

$$A_i(X) = B_i(X)$$

for each i. Therefore we complete the proof. □

Remark. For a given pair of positive even integers n and k as in Theorem 3.1, let $f \in S_{2k-n}(\Gamma_1)$ be a Hecke eigenform, which possesses a Fourier expansion

$$f(z) = \sum_{N=1}^{\infty} a_f(N)\, \mathbf{e}(Nz) \quad (z \in \mathfrak{H}_1)$$

normalized by $a_f(1) = 1$. For each rational prime p, we denote by α_p the Satake p-parameter of f, that is, an algebraic number determined by $\alpha_p + \alpha_p^{-1} = a_f(p)\, p^{-k+(n+1)/2}$ uniquely up to inversion. Then by substituting $X = \alpha_p$ in the main identity of Theorem 3.4, we can also derive a similar identity to Theorem 3.3 for a power series related to the first Fourier-Jacobi coefficient of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to f under Ikeda's lifting procedure (cf. [9]). We note that it will play an important role in a proof of Ikeda's conjecture on the period of such a F, which was proposed in [10] (cf. [13, 14]).

References

[1] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundl. Math. Wiss., 286, Springer-Verlag, Berlin, 1987.

[2] T. Arakawa, *Jacobi Eisenstein series and a basis problem for Jacobi forms*, Comment. Math. Univ. St.Paul. **43** (1994), no. 2, 181–216.

[3] S. Böcherer, *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen*, Math. Z. **183** (1983), 21–46.

[4] __________, *Eine Rationalitätsatz für formale Heckereihen zur Siegelschen Modulgruppe*, Abh. Math. Sem. Univ. Hamburg **56** (1986), 35–47.

[5] S. Böcherer and F. Sato, *Rationality of certain formal power series related to local densities*, Comment. Math. Univ. St.Paul. **36** (1987), 53–86.

[6] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math., vol. 55, Birkhäuser Boston Inc., Boston, Mass., 1985.

[7] E. Freitag, *Siegelsche Modulfunktionen*, Grundl. Math. Wiss., 254, Springer-Verlag, Berlin, 1983.

[8] T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights*, Comment. Math. Univ. St.Paul. **41** (1992), no. 2, 109–124.

[9] T. Ikeda, *On the lifting of elliptic modular forms to Siegel cusp forms of degree* $2n$, Ann. of Math. **154** (2001), no. 3, 641–681.

[10] ________, *Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture*, Duke Math. J. **131** (2006), no. 3, 469–497.

[11] H. Katsurada, *An explicit formula for Siegel series*, Amer. J. Math. **121** (1999), 415–452.

[12] H. Katsurada and H. Kawamura, *A certain Dirichlet series of Rankin-Selberg type associated with the Ikeda lifting*, J. Number Theory **128** (2008), 2025–2052.

[13] ____________________________, *On Ikeda's conjecture on the period of the Ikeda lift and its application*, to appear in RIMS Kôukyûroku Bessatsu.

[14] ____________________________, *Ikeda's conjecture on the period of the Ikeda lift*, preprint (2008).

[15] S. Hayashida, H. Kawamura and N.-P. Skoruppa, *Linear correspondence between Jacobi forms of integral weight and Siegel modular forms of half-integral weight*, preprint (2008).

[16] Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J. **95** (1984), 73–84.

[17] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249–266.

[18] A. Murase, *L-functions attached to Jacobi forms of degree n. Part I. The basic identity*, J. Reine Angew. Math. **401** (1989), 122–156.

[19] _________, *L-functions attached to Jacobi forms of degree n. Part II. Functional equation*, Math. Ann. **290** (1991), 247–276.

[20] A. Murase and T. Sugano, *Whittaker-Shintani functions on the symplectic group of Fourier-Jacobi type*, Composit. Math. **79** (1991), 321–349.

[21] G. Shimura, *On Eisenstein series*, Duke Math. **50**(1983), 417–476.

[22] __________, *Euler products and Fourier coefficients of automorphic forms on symplectic groups*, Invent. Math. **116** (1994) 531–576.

[23] __________, *Zeta functions and Eisenstein series on mataplectic groups*, Invent. Math. **121** (1995), 21-60.

[24] N.-P. Skoruppa, *Explicit formulas for the Fourier coefficients of Jacobi and elliptic forms*, Invent. Math. **102** (1990), no. 3, 501–520.

[25] _____________, *Developments in the theory of Jacobi forms*, Automorphic functions and their applications (Khabarovsk, 1988), 167–185, Acad. Sci. USSR, Inst. Appl. Math., Khabarovsk, 1990.

[26] V. G. Zhuravlëv, *Euler expansions of theta-transformations of Siegel modular forms of half integer weight and their analytic properties*, Math. USSR Sbornik **51** (1985), 169–190.

[27] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.