ON THE ANDRIANOV TYPE IDENTITY FOR POWER SERIES ATTACHED TO JACOBI FORMS AND ITS APPLICATION

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1. Introduction

The theory of Jacobi forms, namely automorphic forms on the Jacobi group and its generalization to higher degree have been studied by several authors (cf. [6 27 18 19 8]). In particular, Shintani introduced the standard $L$-function attached to a Jacobi form of arbitrary degree, and afterward Murase derived in a series of his papers [18 19] its meromorphic continuation and functional equation by making use of its integral expression. Moreover, Murase and Sugano derived in [20] an expression of the standard $L$-function attached to a Jacobi form in terms of a power series generated by its eigenvalues of Hecke operators. In this paper, we derive a local expression of the standard $L$-function attached to a Jacobi form in terms of a power series related to its Fourier coefficients. This can be regarded as an analogue of Andrianov's identity in [1] for Siegel modular forms. As an application, we shall also show the rationality theorem for a formal power series related to a polynomial appearing in the theory of local densities of quadratic forms, which is very similar to the result obtained in [5] by Böcherer and Sato.

Let us describe our main results precisely. Let $p$ be an arbitrary rational prime. For any nonzero element $a$ of the field $\mathbb{Q}_p$ of $p$-adic numbers, we put

$$\chi_p(a) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is unramified}, \\ 0 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is ramified}. \end{cases}$$

Let $n$ be a positive even integer. For each non-degenerate half-integral symmetric matrix $B'$ of degree $n$ over the ring $\mathbb{Z}_p$ of $p$-adic integers, we define the local Siegel series with complex parameter $s$ by

$$b_p(B'; s) := \sum_{R \in \text{Sym}_n(\mathbb{Q}_p)/\text{Sym}_n(\mathbb{Z}_p)} e_p(\text{tr}(-B'R)) \mu_p(R)^{-s},$$

where $\mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$, and $e_p$ is the standard additive character of $\mathbb{Q}_p$. It is well-known that such a kind of singular series appears naturally within the framework of

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studying Fourier coefficients of the Siegel Eisenstein series of degree $n$ and there exists a unique polynomial $F_p(B'; X)$ in one variable $X$ such that

$$b_p(B'; s) = \frac{(1-p^{-s})^{n/2}}{1 - \xi_p(B') p^{n/2-s}} F_p(B'; p^{-s}),$$

where $\xi_p(B') = \chi_p((-1)^{n/2} \det(2B'))$ (cf. [16]). Let $B$ be a non-degenerate symmetric matrix of degree $n - 1$ over a subring $R$ of $\mathbb{Z}_p$ satisfying the condition

$$(1) \quad (B + t r_B B')/4 \text{ is a half-integral symmetric matrix over } R \text{ for some } r_B \in R^{n-1}.$$

Then we can associate such a $B$ with a non-degenerate half-integral symmetric matrix $B'^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ t r_B/2 & (B + t r_B B')/4 \end{pmatrix}$ of degree $n$ over $R$. Here we easily see that the vector $r_B$ is uniquely determined by $B$ modulo $2 \mathcal{R}^{n-1}$, and therefore $B'^{(1)}$ is uniquely determined by $B$ up to $\text{GL}_{n-1}(R)$-equivalence. Then for such a $B$ over $\mathbb{Z}_p$, we define a polynomial $F_{p}^{(1)}(B; X)$ in $X$ by

$$F_{p}^{(1)}(B; X) := F_p(B'^{(1)}; X)$$

and put

$$G_{p}^{(1)}(B; X) = \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}_p) \setminus \text{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D) F_{p}^{(1)}(B[D^{-1}]; X) (p^n X^2)^{\text{ord}_p(\det D)},$$

where $\pi_p(D)$ denotes the generalized local Möbius function, that is, $\pi_p(D) = (-1)^i p^{i(i-1)/2}$ or 0 according as $D \in \text{GL}_{n-1}(\mathbb{Z}_p) \left( \frac{1_{n-1-i}}{p l_i} \right) \text{GL}_{n-1}(\mathbb{Z}_p)$ for some $0 \leq i \leq n - 1$ or not. We note that these polynomials do not depend on the choice of $r_B$. In addition, we also define a polynomial $B_{p}^{(1)}(B; t)$ in one variable $t$ by

$$B_{p}^{(1)}(B; t) := \frac{(1 - \xi_p(B'^{(1)}) p^{-(n-1)/2} t \prod_{i=1}^{n/2-1} (1 - p^{-2i+1} t^2)}{G_{p}^{(1)}(B; p^{-n+1/2} t)}.$$

On the other hand, for any positive even integers $k$ and $n$, let $\phi$ be a Jacobi form of weight $k$ and of index 1 with respect to the Jacobi modular group $\Gamma_{n-1}^J$ of degree $n - 1$, and $\sigma(\phi)$ be a Siegel modular form of weight $k - 1/2$ with respect to the congruence subgroup $\Gamma_{0}^{(n-1)}(4)$ of the Siegel modular group of degree $n - 1$ corresponding to $\phi$ under the Eichler-Zagier-Ibukiyama correspondence $\sigma$ (cf. §2.3 and 2.4 below). Let $\mathcal{D}_{p}^{(n-1)}(\mathbb{Z})$ be the set of all $(n-1) \times (n-1)$ matrices with entries in $\mathbb{Z}$ whose determinant is a power of $p$. For each positive definite half-integral symmetric matrix $B$ of degree $n - 1$ over $\mathbb{Z}$, we define a power series $\widetilde{G}_{\phi, p}(B; t)$ in $t$ by

$$\widetilde{G}_{\phi, p}(B; t) := \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \mathcal{D}_{p}^{(n-1)}(\mathbb{Z})} \pi_p(D) C_{\sigma(\phi)}(B[D^{-1}]) (p^k t)^{\text{ord}_p(\det D)},$$

where $C_{\sigma(\phi)}(B)$ denotes the $B$-th Fourier coefficient of $\sigma(\phi)$. Then our first main result is the following:
Theorem 1.1 (cf. Theorem 3.1 below). If \( \phi \) is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators with Satake \( p \)-parameters \((\chi^{(1)}_{\phi}(p), \ldots, \chi^{(n-1)}_{\phi}(p))\), then for each positive definite half-integral symmetric matrix \( B \) of degree \( n - 1 \) over \( \mathbb{Z} \) satisfying the condition (1), we have

\[
\frac{B_p^{(1)}(B; p^{n-1/2}t) \tilde{G}_{\phi,p}(B; t)}{\prod_{i=1}^{n-1} (1 - \chi^{(i)}_{\phi}(p)p^{n-1/2}t)(1 - \chi^{(i)}_{\phi}(p)^{-1}p^{n-1/2}t)} = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \mathcal{D}_{p^{n-1}}(\mathbb{Z})} C_{\sigma(\phi)}(B[W]) p^{-(k-n-1) \text{ord}_p(\det W)} t^{\text{ord}_p(\det W)}.
\]

This can be regarded as an analogue of the so-called Andrianov identity, which was obtained within the framework of studying standard \( L \)-functions attached to Siegel modular forms of integral weight (cf. \cite{1}, see also \cite{4}). We also note that the above identity for \( p \neq 2 \) can be derived from a similar result for Siegel modular forms of half-integral weight due to Shimura and Zhuravlev (cf. Corollary 5.2 in \cite{23}, see also Theorem 1.1 in \cite{26}). However, we cannot use their results to prove the above identity for \( p = 2 \).

Next, we explain an application of the above result to the rationality of a certain formal power series related to the polynomial \( F_p^{(1)}(B; X) \). For each non-degenerate half-integral symmetric matrix \( B \) of degree \( n - 1 \) over \( \mathbb{Z}_p \) satisfying the condition (1), we define a Laurent polynomial \( \tilde{F}_p^{(1)}(B; X) \) in \( X \) by

\[
\tilde{F}_p^{(1)}(B; X) := X^{-\text{ord}_p((-1)^{n/2} \det(2B^{(1)}) \delta(B^{(1)})^{-1})/2} F_p^{(1)}(B; p^{-(n+1)/2}X),
\]

and put

\[
\tilde{G}_p^{(1)}(B; X, t) = \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}_p) \setminus \text{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D) \tilde{F}_p^{(1)}(B[D^{-1}]; X) t^{\text{ord}_p(\det D)},
\]

where \( \delta(B^{(1)}) \) is the discriminant of the quadratic extension \( \mathbb{Q}_p \bigl( \sqrt{(-1)^{n/2} \det(2B^{(1)})} \bigr) / \mathbb{Q}_p \). Then we have a functional equation \( \tilde{F}_p^{(1)}(B; X) = \tilde{F}_p^{(1)}(B; X^{-1}) \) (cf. \cite{9}). Thus \( \tilde{F}_p^{(1)}(B; X) \) is a polynomial in \( X + X^{-1} \), and then \( \tilde{G}_p^{(1)}(B; X, t) \) is a polynomial in \( X + X^{-1} \) and \( t \). Now we put

\[
R_p^{(1)}(B; X, t) = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}_p) \setminus \text{GL}_{n-1}(\mathbb{Q}_p)} \tilde{F}_p^{(1)}(B[W]; X) t^{\text{ord}_p(\det W)}.
\]

Then by applying Theorem 1.1 to the Jacobi Eisenstein series, we obtain the following:

Theorem 1.2 (cf. Theorem 3.4 below). Let \( n \) be a positive even integer. If \( B \) is a non-degenerate half-integral symmetric matrix of degree \( n - 1 \) over \( \mathbb{Z}_p \) satisfying the condition (1), then we have

\[
R_p^{(1)}(B; X, t) = \frac{B_p^{(1)}(B; p^{n/2-1}t) \tilde{G}_p^{(1)}(B; X, t)}{\prod_{j=1}^{n-1} (1 - p^{j-1}X t)(1 - p^{j-1}X^{-1}t)}.
\]
We note that Böcherer and Sato ([5]) obtained a similar identity for a half-integral symmetric matrix of degree $n$. The above identity will play an important role in proving a conjecture on the period of the Ikeda lift proposed in [10] by Ikeda (cf. [13, 14]).

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**Notation.** We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. We put $e(x) = \exp(2\pi \sqrt{-1} x)$ for any $x \in \mathbb{C}$. For each rational prime $p$, let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ be the field of $p$-adic rational numbers and the ring of $p$-adic integers, respectively. We denote by $\text{ord}_p$ the valuation of $\mathbb{Q}_p$, normalized as $\text{ord}_p(p) = 1$, and by $e_p$ the continuous additive character of $\mathbb{Q}_p$ such that $e_p(x) = e(x)$ for any $x \in \mathbb{Q}$, which will be called the standard additive character of $\mathbb{Q}_p$. Let $R$ be a commutative ring. We denote by $R^\times$ the the unit group of $R$. We denote by $M_{m,n}(R)$ the set of $m \times n$ matrices with entries in $R$. In particular, we write $M_n(R) = M_{n,n}(R)$ and $R^n = M_{1,n}(R)$. We denote by $1_n$, $0_n \in M_n(R)$ the unit matrix and the zero matrix of degree $n$, respectively. We put $\text{GL}_n(R) = \{ U \in M_n(R) \mid \det U \in R^\times \}$, where $\det U$ is the determinant of $U$. For two matrices $X \in M_{m,n}(R)$ and $A \in M_m(R)$, we write $A[X] = {}^tXAX \in M_n(R)$, where $^tX$ denotes the transpose of $X$. Let $\text{Sym}_n(R)$ be the set of symmetric matrices of degree $n$ with entries in $R$. If $R$ is an integral domain of characteristic different from 2, let $\text{Sym}_n^*(R)$ be the subset of $\text{Sym}_n(R)$ consisting of all half-integral symmetric matrices of degree $n$, that is,

$$
\text{Sym}_n^*(R) := \left\{ T = (t_{ij}) \in \text{Sym}_n(\text{Frac}(R)) \mid \begin{array}{l}
t_{ii} \in R \quad (1 \leq i \leq n), \\
2t_{ij} \in R \quad (1 \leq i \neq j \leq n)
\end{array} \right\},
$$

where $\text{Frac}(R)$ is the field of fractions of $R$. In addition, for any subset $S$ of $\text{Sym}_n(R)$, we denote by $S^\times$ the subset of $S$ consisting of all non-degenerate elements in $S$. In particular, if $R$ is a subring of $\mathbb{R}$, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of $S$ consisting of all positive definite (resp. semi-positive definite) matrices. For any commutative ring $R$, the group $\text{GL}_n(R)$ acts on the set $\text{Sym}_n(R)$ in the following way:

$$
\text{GL}_n(R) \times \text{Sym}_n(R) \ni (U, A) \longmapsto A[U] \in \text{Sym}_n(R).
$$

For a subgroup $G$ of $\text{GL}_n(R)$, and a subset $S$ of $\text{Sym}_n(R)$ stable under the action of $G$, we denote by $S/G$ the set of $G$-orbits in $S$. We define an equivalence relation on $\text{Sym}_n(R)$ over a subring $R'$ of $R$ as follows: for any $A_1, A_2 \in \text{Sym}_n(R),$

$$
A_1 \sim_{R'} A_2 \overset{\text{def}}{=} A_2 = A_1[U] \text{ for some } U \in \text{GL}_n(R').
$$

(2)

For two square matrices $X \in M_m(R)$ and $Y \in M_n(R)$, we write $X \perp Y = (X \, Y^t)$. In particular, we often write $x \perp Y$ instead of $(x) \perp Y$ for any $x \in R$. Then we can simply write the diagonal matrix with entries $x_1, \ldots, x_n$ in $R$ by $x_1 \perp \cdots \perp x_n$.
2. Preliminaries

2.1. Siegel modular forms of integral weight.

Let $G_n(\mathbb{R})$ be the real symplectic group of degree $n$, that is,

$$G_n(\mathbb{R}) := \text{Sp}_n(\mathbb{R}) = \{ M \in \text{GL}_{2n}(\mathbb{R}) \mid {}^tM J_n M = J_n \},$$

where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. For any $S \in \text{Sym}_n(\mathbb{R})$ and $A \in \text{GL}_n(\mathbb{R})$, we put $\mathbf{n}_n(S) = \begin{pmatrix} I_n & S \\ 0_n & I_n \end{pmatrix}$ and $\mathbf{d}_n(A) = \begin{pmatrix} A & 0_n \\ 0_n & A^{-1} \end{pmatrix}$, respectively. Then we easily see that these elements $\mathbf{n}_n(S)$, $\mathbf{d}_n(A)$ and $J_n$ generate $G_n(\mathbb{R})$. The discrete subgroup $\Gamma_n := \text{Sp}_n(\mathbb{Z}) = G_n(\mathbb{R}) \cap \text{M}_{2n}(\mathbb{Z})$ of $G_n(\mathbb{R})$ is called the Siegel modular group of degree $n$. For any $N \in \mathbb{Z}_{>0}$, we denote by $\Gamma_0^{(n)}(N)$ the congruence subgroup of $\Gamma_n$ defined by

$$\Gamma_0^{(n)}(N) := \{ (\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}) \in \Gamma_n \mid \mathbf{C} \equiv 0_n \ (\text{mod} \ N) \}. $$

We denote the Siegel upper-half space of degree $n$ by $\mathbf{H}_n$, that is,

$$\mathbf{H}_n := \{ Z = X + \sqrt{-1} Y \in \text{Sym}_n(\mathbb{C}) \mid Y > 0 \ (\text{positive definite}) \}. $$

For any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n(\mathbb{R})$ and $Z \in \mathbf{H}_n$, we easily see that $j(M, Z) := CZ + D \in \text{GL}_n(\mathbb{C})$ and then we put $M(Z) := (AZ + B)(CZ + D)^{-1}$. As is well-known, this defines a transitive action of $G_n(\mathbb{R})$ on $\mathbf{H}_n$.

For any $k \in \mathbb{Z}$, a $\mathbb{C}$-valued holomorphic function $F(Z)$ on $\mathbf{H}_n$ is called a (holomorphic) Siegel modular form of degree $n$ and weight $k$ if it satisfies the following two conditions:

(i) $F(M(Z)) = \det(j(M, Z))^k F(Z)$ for any $M \in \Gamma_n$;

(ii) $F$ possesses a Fourier expansion of the form

$$F(Z) = \sum_{B \in \text{Sym}_n(\mathbb{Z}) \geq 0} A_F(B) e(\text{tr}(BZ)),$$

where tr denotes the trace of a matrix. If $F$ satisfies the stronger condition $A_F(B) = 0$ unless $B > 0$ (positive definite), then it is called a cusp form.

We denote by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ the $\mathbb{C}$-vector spaces consisting of all (holomorphic) Siegel modular forms and Siegel cusp forms of degree $n$ and weight $k$, respectively. For further details on the facts of Siegel modular forms of integral weight set out above, see [1] or [7].

2.2. Review of the theory of Jacobi forms of higher degree.

In this paragraph, we introduce some basic facts on Jacobi forms of integral weight whose index is a scalar. For further details on generalities of Jacobi forms, see [6] [18] [19] [27].

2.2.1. Jacobi group and complex analytic Jacobi forms.

Let $G_n = \text{Sp}_n(\mathbb{Q}) = \{ M \in \text{GL}_{2n}(\mathbb{Q}) \mid {}^tM J_n M = J_n \}$, and we naturally identify $G_n$ with its image under the natural inclusion

$$G_n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto [M] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \in G_{n+1}. $$
We denote by $H_n$ the Heisenberg group, that is,

$$H_n = \left\{ ([\lambda, \mu], \kappa) := \begin{pmatrix} 1 & 0 & \kappa \\ 0 & 1_n & \mu \\ 1 & 0 & 1_n \end{pmatrix} \left| \begin{array}{c} \begin{pmatrix} 1 & \lambda \\ 0 & 1_n \end{pmatrix} \\ 1 & -\lambda \end{pmatrix} \kappa \in \mathbb{Q} \right. \right\} \cdot (\lambda, \mu) \in \mathbb{Q}^n \oplus \mathbb{Q}^n, \quad \kappa \in \mathbb{Q}.$$

Then $G_n^J := \{ ([\lambda, \mu], [\kappa]) \cdot [M] \in G_{n+1} \mid ([\lambda, \mu], [\kappa]) \in H_n, M \in G_n \}$ is a $\mathbb{Q}$-algebraic subgroup of $G_{n+1}$ and is called the Jacobi group of degree $n$. We note that the Jacobi group $G_n^J$ is a semi-direct product $G_n \ltimes H_n$ of $H_n$ and $G_n$, and forms a connected non-reductive $\mathbb{Q}$-algebraic group with the center $Z_n^J = \{ (0, 0), \kappa \mid \kappa \in \mathbb{Q} \}$.

Then we have the following:

**Lemma 2.1.** For each $[([\lambda, \mu], [\kappa], ([\lambda', \mu'], [\kappa']) \in H_n$, and $M = (A B C D) \in G_n$, we have

1. $[([\lambda, \mu], [\kappa]) \cdot ([\lambda', \mu'], [\kappa']) = ([\lambda + \lambda', \mu + \mu'], \kappa + \kappa' + 2\lambda' \mu'),$
2. $[([\lambda, \mu], [\kappa]) \cdot [M] = [M] \cdot ([\lambda A + \mu C, \lambda B + \mu D], \kappa + (\lambda A + \mu C)^t (\lambda B + \mu D) - \lambda' \mu].$

**Proof.** Since it is an easy calculation, we omit the proof. □

According to the action of $G_{n+1}(\mathbb{R}) = \text{Sp}_{n+1}(\mathbb{R})$ on the Siegel upper-half space $\mathfrak{H}_{n+1}$, the group $G_n^J(\mathbb{R})$ of real points of $G_n^J$ naturally acts on the space $\mathfrak{H}_n \times \mathbb{C}^n$ as follows. For each $g = ([\lambda, \mu], [\kappa]) \in G_n^J(\mathbb{R})$ with $M = (A B C D) \in G_n(\mathbb{R})$ and $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we put

$$g(\tau, z) := (M(\tau), z(C\tau + D)^{-1} + \lambda M(\tau) + \mu).$$

Here we easily see that this action is transitive and the stabilizer of the point $(\sqrt{-1} 1_n, 0) \in \mathfrak{H}_n \times \mathbb{C}^n$ in $G_n^J(\mathbb{R})$ coincides with $Z_n^J(\mathbb{R}) \cdot K\infty$, where $K\infty$ is the stabilizer of $\sqrt{-1} 1_n \in \mathfrak{H}_n$ in $G_n(\mathbb{R})$, that is,

$$K\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G_n(\mathbb{R}) \mid A + \sqrt{-1} B \text{ is unitary} \right\}.$$

The map $g \mapsto g(\sqrt{-1} 1_n, 0)$ induces a diffeomorphism of $G_n^J(\mathbb{R})/(Z_n^J(\mathbb{R}) \cdot K\infty)$ onto $\mathfrak{H}_n \times \mathbb{C}^n$.

Let $l$ and $m$ be non-negative integers. For any $\mathbb{C}$-valued function $\phi(\tau, z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$, we define the action of $g \in G_n^J(\mathbb{R})$ on $\phi$ by

$$(\phi|_{l, m} g)(\tau, z) := J_{l, m}(g, (\tau, z))^{-1} \phi(g(\tau, z)),
$$

where for $g = ([\lambda, \mu], [\kappa]) \cdot [M]$, we put

$$J_{l, m}(g, (\tau, z)) := \det((C\tau + D)^l)
\times e(-m\kappa - m\tau [\lambda] - 2m\lambda' \mu + m((C\tau + D)^{-1} C)^l(z + \lambda\tau + \mu)).$$

It is easy to see that for any $g_i \in G_n^J(\mathbb{R})$ ($i = 1, 2$),

$$(\phi|_{l, m} g_1)|_{l, m} g_2 = \phi|_{l, m} (g_1 g_2).$$
In particular, it follows from Lemma 2.1 that for any $M, M' \in G_n(\mathbb{R})$ and $[(\lambda, \mu), \kappa], [(\lambda', \mu'), \kappa'] \in H_n(\mathbb{R})$, we have
\[
\begin{aligned}
\phi|_{l, m} [M] |_{l, m} [M'] &= \phi|_{l, m} [MM'], \\
\phi|_{l, m} [(\lambda, \mu), \kappa] |_{l, m} [(\lambda', \mu'), \kappa'] &= \phi|_{l, m} [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + 2\lambda' \mu'], \\
\phi|_{l, m} [M] |_{l, m} [(\lambda, \mu) M, \kappa + (\lambda, \mu) M \begin{pmatrix} 0_n & 1_n \\ 0_n & 0_n \end{pmatrix} tM^t (\lambda, \mu) - \lambda' \mu'] &= \phi|_{l, m} [(\lambda, \mu), \kappa] [M].
\end{aligned}
\]
We also define a subgroup of $G_n^J(\mathbb{R})$ by $\Gamma_n^J := \Gamma_n \ltimes H_n(\mathbb{Z})$, where $H_n(\mathbb{Z})$ is a subgroup of $H_n(\mathbb{R})$ consisting of all elements with integral entries.

Let $l$ and $m$ be positive integers. A holomorphic function $\phi(\tau, z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$ is called a (holomorphic) Jacobi form of degree $n$, weight $l$ and index $m$ if it satisfies the following two conditions:

(i) $\phi|_{l, m} \gamma = \phi$ for any $\gamma \in \Gamma_n^J$;

(ii) $\phi$ possesses a Fourier expansion of the form
\[
\phi(\tau, z) = \sum_{T \in \text{Sym}^*_n(\mathbb{Z}), r \in \mathbb{Z}^n} c_\phi(T, r) e(\text{tr}(T\tau) + r^t z)
\]
with $c_\phi(T, r) = 0$ unless $4mT - 4rr \geq 0$. If $\phi$ satisfies the stronger condition $c_\phi(T, r) = 0$ unless $4mT - 4rr > 0$, then it is called cuspidal.

We denote by $J_{l, m}(\Gamma_n^J)$ and $J_{l, m}^\text{cusp}(\Gamma_n^J)$ the $\mathbb{C}$-vector spaces consisting of all (holomorphic) Jacobi forms and cuspidal Jacobi forms of degree $n$, weight $l$ and index $m$, respectively.

As an important example of Jacobi form, we consider Fourier-Jacobi coefficients of Siegel modular forms of arbitrary degree $n > 1$. For any $k \in \mathbb{Z}$, let $F \in M_k(\Gamma_n)$ possess a Fourier expansion
\[
F(Z) = \sum_{B' \in \text{Sym}^*_n(\mathbb{Z})_{\geq 0}} A_F(B') e(\text{tr}(B'Z)) \quad (Z \in \mathfrak{H}_n),
\]
and we put $Z = \begin{pmatrix} \tau' & z \\ t_z & \tau \end{pmatrix}$ with $\tau \in \mathfrak{H}_{n-1}$, $z \in \mathbb{C}^{n-1}$ and $\tau' \in \mathfrak{H}_1$. Then we have the so-called Fourier-Jacobi expansion
\[
F\left( \begin{pmatrix} \tau' & z \\ t_z & \tau \end{pmatrix} \right) = \sum_{m=0}^\infty \phi_m(\tau, z) e(m\tau'),
\]
where
\[
(5) \quad \phi_m(\tau, z) = \sum_{T \in \text{Sym}^*_{n-1}(\mathbb{Z}), r \in \mathbb{Z}^{n-1}, 4mT - 4rr \geq 0} A_F\left( \begin{pmatrix} m & r/2 \\ tr/2 & T \end{pmatrix} \right) e(\text{tr}(T\tau) + r^t z).
\]
We easily see that the $m$-th coefficient $\phi_m \in J_{k, m}(\Gamma_{n-1}^J)$ for each $m \in \mathbb{Z}_{>0}$. In particular, if $F \in S_k(\Gamma_n)$, then $\phi_m \in J_{k, m}^\text{cusp}(\Gamma_{n-1}^J)$.

As another example, if $k$ is an even integer such that $k > n + 1$, then for each $m \in \mathbb{Z}_{>0}$, we define the Jacobi Eisenstein series of degree $n - 1$, weight $k$ and index $m$ by
\[
E_{k, m}^{(n-1)}(\tau, z) := \sum_{\gamma \in \Gamma_{n-1}^J \cap \Gamma_{n-1}^J \Gamma_{n-1}^J} J_{k, m}(\gamma, (\tau, z)) \quad (\tau \in \mathfrak{H}_{n-1}, z \in \mathbb{C}^{n-1}),
\]
where
\[
P_{n-1}^J := \{ ((\lambda, \mu), \kappa) \in G_{n-1}^J \mid C = 0_{n-1}, \lambda = 0 \}.
\]

We easily see that the right-hand side of the above definition is absolutely convergent and \(E_{k,m}^{(n-1)} \in J_{k,m}(\Gamma_{n-1})\). Moreover, Böcherer [3] showed that for any \(m \in \mathbb{Z}_{>0}\), there exists a certain relation between \(E_{k,m}^{(n-1)}\) and the \(m\)-th coefficient \(c_{k,m}^{(n-1)}\) of the above Fourier-Jacobi expansion of the Siegel Eisenstein series \(E_k^{(n)} \in M_k(\Gamma_n)\). In particular, when \(m = 1\), we have \(E_{k,1}^{(n-1)} = E_{k,1}^{(n)}\).

For the purpose of subsequent use, we give an explicit formula for the Fourier coefficients of \(E_{k,1}^{(n-1)}\) in case \(n\) is even. Let \(k\) be a positive even integer such that \(k > n + 1\). Then the Siegel Eisenstein series \(E_k^{(n)}\) of weight \(k\) with respect to \(\Gamma_n\) is defined by
\[
E_k^{(n)}(Z) = \sum_{(C, D)} \det(CZ + D)^{-k} \quad (Z \in \mathfrak{H}_n)
\]
where \((C, D)\) runs through a complete set of representatives of the equivalence classes of coprime symmetric pairs of size \(n\). For each positive definite half-integral symmetric matrix \(B'\) of degree \(n\), we denote by \(\mathfrak{d}(B')\) the discriminant of the quadratic extension \(\mathbb{Q}(\sqrt{(-1)^{n/2}\det(2B')})/\mathbb{Q}\) and put \(f(B') = \sqrt{(-1)^{n/2}\det(2B')/\mathfrak{d}(B')}\). It is well-known that \(f(B') \in \mathbb{Z}_{>0}\). Furthermore, we denote by \(\chi_{B'}\) the Kronecker character corresponding to the above field extension. Then for each \(B' \in \text{Sym}_n^*(\mathbb{Z})\), the \(B'\)-th Fourier coefficient \(A_k^{(n)}(B')\) of \(E_k^{(n)}\) is described as
\[
A_k^{(n)}(B') = \xi(n, k)L(1 - k/2 + n/2, \chi_{B'}) f(B')^{-k(n+1)/2} \prod_{p|f(B')} \tilde{F}_p(B'; p^{-k(n+1)/2}),
\]
where \(\xi(n, k) = 2^{n/2}\zeta(1-k)^{-1} \prod_{i=1}^{n/2} \zeta(1+2i-2k)^{-1}\), \(L(s, \chi_{B'})\) denotes the Dirichlet \(L\)-function associated with \(\chi_{B'}\), and
\[
\tilde{F}_p(B'; X) = X^{-\text{ord}_p(f(B'))} F_p(B'; p^{-(n+1)/2}X).
\]
We note that if \(B \in \text{Sym}_n^*(\mathbb{Z})\) satisfies the condition (1), then \(\tilde{F}_p^{(1)}(B; X) = \tilde{F}_p(B^{(1)}; X)\). Thus we have

**Proposition 2.1.** Under the same assumption as above, let \(e_{k,1}^{(n-1)}\) possess a Fourier expansion
\[
e^{(n-1)}_{k,1}(\tau, z) = \sum_{T \in \text{Sym}_{n-1}^*(\mathbb{Z}), r \in \mathbb{Z}^{n-1}} c^{(n-1)}_{k,1}(T, r) \mathbf{e}(\text{tr}(T\tau) + r^t z).
\]
Then for each \(T \in \text{Sym}_{n-1}^*(\mathbb{Z})\) such that \(B_T = 4T - r\tau r > 0\) with \(r \in \mathbb{Z}^{n-1}\), we have
\[
c^{(n-1)}_{k,1}(T, r) = \xi(n, k)L(1 - k + n/2, \chi_{B_T^{(1)}}) f(B_T^{(1)})^{-k(n+1)/2} \prod_{p|f(B_T^{(1)})} \tilde{F}_p^{(1)}(B_T; p^{-k(n+1)/2}),
\]
where \(B_T^{(1)} = \left( \begin{array}{cc} 1 & r/2 \\ t\tau/2 & (B_T + t\tau r)/4 \end{array} \right) = \left( \begin{array}{cc} 1 & r/2 \\ t\tau/2 & T \end{array} \right) \in \text{Sym}_n^*(\mathbb{Z})\).
Proof. Since
\[ c_{k,1}^{(n-1)}(T, r) = A_k^{(n)}(B_T^{(1)}), \]
the assertion immediately follows form the equation (6). \qed

Returning to the general theory of Jacobi forms, now we consider the action of Hecke operators on Jacobi forms. Let \( M \in \text{Sp}_n(\mathbb{Q}) \) and decompose the double coset \( \Gamma_n^J M \Gamma_n^J \) into the disjoint right cosets:
\[ \Gamma_n^J M \Gamma_n^J = \bigsqcup_{i=1}^d \Gamma_n^J g_i, \]
where we denote by \( d \) the number of right cosets, that is, \( d = [\Gamma_n^J M \Gamma_n^J : \Gamma_n^J] \). Then for any \( \phi \in J_{l,m}(\Gamma_n^J) \), we define the action of the double coset \( \Gamma_n^J M \Gamma_n^J \) on \( \phi \) by
\[ \phi |_{l,m} \Gamma_n^J M \Gamma_n^J := \sum_{i=1}^d \phi |_{l,m} g_i, \]
where the summation on the right hand side of the above is well-defined. We easily see that for any \( \gamma \in \Gamma_n^J \),
\[ (\phi |_{l,m} \Gamma_n^J M \Gamma_n^J) |_{l,m} \gamma = \phi |_{l,m} \Gamma_n^J M \Gamma_n^J, \]
that is, \( \phi |_{l,m} \Gamma_n^J M \Gamma_n^J \in J_{l,m}(\Gamma_n^J) \). Moreover, if \( \phi \in J_{l,m}^{\text{cusp}}(\Gamma_n^J) \), then \( \phi |_{l,m} \Gamma_n^J M \Gamma_n^J \in J_{l,m}^{\text{cusp}}(\Gamma_n^J) \).

Here we note that each of the double cosets \( \Gamma_n^J M \Gamma_n^J \) with \( M \in G_n(\mathbb{Q}) \) contains a unique representative of the form
\[ d_n(\delta_1 \perp \cdots \perp \delta_n) = (\delta_1 \perp \cdots \perp \delta_n) \perp (\delta_1^{-1} \perp \cdots \perp \delta_n^{-1}) \]
with \( 0 < \delta_1 \cdots | \delta_n \). Moreover, let \( D = \delta_1 \perp \cdots \perp \delta_n \) and \( D' = \delta_1' \perp \cdots \perp \delta_n' \) be two diagonal matrices with \( 0 < \delta_1 \cdots | \delta_n \), \( 0 < \delta_1' \cdots | \delta_n' \). We easily see that if \( (\delta_n, \delta_n') = 1 \), then for any \( \phi \in J_{l,m}(\Gamma_n^J) \),
\[ \phi |_{l,m} \Gamma_n^J d_n(D D') \Gamma_n^J = \phi |_{l,m} \Gamma_n^J d_n(D) \Gamma_n^J |_{l,m} \Gamma_n^J d_n(D') \Gamma_n^J. \]

A Jacobi form \( \phi \in J_{1,1}(\Gamma_n^J) \) is called a \textit{Hecke eigenform} if it is a common eigenfunction of all actions of double cosets \( \Gamma_n^J M \Gamma_n^J \) with \( M \in G_n(\mathbb{Q}) \), that is, for any \( M \in G_n(\mathbb{Q}) \), the equation
\[ \phi |_{l,m} \Gamma_n^J M \Gamma_n^J = \lambda_\phi(M) \phi \]
holds for some \( \lambda_\phi(M) \in \mathbb{C} \). We easily see from the above argument that \( \phi \) is a Hecke eigenform if and only if it satisfies for any rational prime \( p \) and \( D = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in D_p^{(n)}(\mathbb{Z}) \) with \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_n \),
\[ \phi |_{l,m} \Gamma_n^J d_n(D) \Gamma_n^J = \lambda_\phi(D) \phi \]
with \( \lambda_\phi(D) \in \mathbb{C} \).
2.2.2. Jacobi forms on the adele group.

Let $A$ be the adele ring of $\mathbb{Q}$ and let $\Psi_A$ be the character of $\mathbb{Q}\backslash A$ such that $\Psi_A(x_\infty) = e(x_\infty)$ for any $x_\infty \in \mathbb{R}$. In addition, for each $m \in \mathbb{Z}$, we put $\Psi_A^m(\kappa) = \Psi_A(m\kappa)$ for any $\kappa \in A$. We denote by $G_n^J(A)$ the adele group of the Jacobi group $G_n^J$ defined in the previous paragraph. Then it follows from the strong approximation theorem for $G_n^J$ that

$$G_n^J(A) = G_n^J(\mathbb{Q})G_n^J(\mathbb{R})K_{\text{fin}},$$

where $K_{\text{fin}} := \prod_{p<\infty} G_n^J(\mathbb{Z}_p)$.

Let $l$ and $m$ be positive integers. A $\mathbb{C}$-valued function $f$ on $G_n^J(A)$ is called a Jacobi form of weight $l$ and index $m$ if it satisfies the following two conditions:

(i) The functional equation

$$f([([0,0],[\kappa]) \gamma \ g \ k_\infty \ k_{\text{fin}}]) = \det(j(k_\infty, \sqrt{-1} 1_n))^{-1} \Psi_A^m(\kappa)f(g)$$

holds for any $\kappa \in A$, $\gamma \in G_n^J(\mathbb{Q})$, $g \in G_n^J(A)$, $k_\infty \in K_\infty$ and $k_{\text{fin}} \in K_{\text{fin}}$;

(ii) For any $(\tau, z) \in \mathcal{H}_n \times \mathbb{C}^n$, we choose and fix an element $g_\infty \in G_n^J(\mathbb{R})$ such that $g_\infty(\sqrt{-1} 1_n, 0) = (\tau, z)$ and put

$$\Phi_f(\tau, z) := J_{l,m}(g_\infty, (\sqrt{-1} 1_n, 0))f(g_\infty),$$

with the factor of automorphy $J_{l,m} : G_n^J(\mathbb{R}) \times (\mathcal{H}_n \times \mathbb{C}^n) \to \mathbb{C}$ defined in §2.2.1. Here we easily see that the value $\Phi_f$ does not depend on the choice of $g_\infty$. Then the function $\Phi_f$ is holomorphic on $\mathcal{H}_n \times \mathbb{C}^n$. In particular, if it satisfies the further condition that

$$|\det(\text{Im}(\tau))^{l/2} \exp(-2m\pi \text{tr}(\text{Im}(\tau)^{-1}[t \text{Im}(z)]))\Phi_f(\tau, z)|$$

is bounded on $\mathcal{H}_n \times \mathbb{C}^n$, then it is called cuspidal.

We denote by $J_{l,m}(G_n^J(A))$ and $J_{l,m}^\text{cusp}(G_n^J(A))$ the $\mathbb{C}$-vector spaces of the Jacobi forms and cuspidal Jacobi forms of weight $l$ and index $m$ on the group $G_n^J(A)$, respectively.

It is easy to see that for each $f \in J_{l,m}(G_n^J(A))$, the associated function $\Phi_f$ is an element of $J_{l,m}(\Gamma_n^J)$. In particular, if $f \in J_{l,m}^\text{cusp}(G_n^J(A))$, then $\Phi_f \in J_{l,m}^\text{cusp}(\Gamma_n^J)$. Furthermore we have

Lemma 2.2. The map $J_{l,m}(G_n^J(A)) \ni f \mapsto \Phi_f \in J_{l,m}(\Gamma_n^J)$ induces $\mathbb{C}$-linear isomorphisms

$J_{l,m}(G_n^J(A)) \cong J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^\text{cusp}(G_n^J(A)) \cong J_{l,m}^\text{cusp}(\Gamma_n^J)$.

Proof. Since it is straightforward, we omit the proof. \qed

2.3. Standard $L$-functions attached to Jacobi forms.

We study in this paragraph Shintani’s standard $L$-functions attached to Jacobi forms. In particular, we derive an explicit formula for the standard $L$-function attached to the Jacobi Eisenstein series of index 1. It might be given in a classical way, but here we treat it adelically.
Let \( p \) be an arbitrary rational prime. For simplicity, we write \( G_p^J, G_p, K_p^J, K_p \) and \( Z_p^J \) instead of \( G_p^J(\mathbb{Q}_p), G_p(\mathbb{Q}_p), G_p^J(\mathbb{Z}_p), G_p(\mathbb{Z}_p) \) and \( Z_p^J(\mathbb{Q}_p) \), respectively. We denote by \( \Psi_p \) and \( |*|_p \) the restriction of \( \Psi_A \) to \( \mathbb{Q}_p \) and the \( p \)-adic valuation of \( \mathbb{Q}_p \) normalized as \( |p|_p = p^{-1} \), respectively. Let \( \mathscr{H}_p = \mathscr{H}(G_p^J, K_p^J; \Psi_p) \) be the \( \mathbb{C} \)-module consisting of \( \mathbb{C} \)-valued functions \( \varphi \) on \( G_p^J \) satisfying the following two conditions:

(i) The equation
\[
\varphi([(0, 0), \kappa] k k') = \Psi_p(\kappa) \varphi(g)
\]
holds for any \( \kappa \in \mathbb{Q}_p \), \( k, k' \in K_p^J \) and \( g \in G_p^J \);

(ii) \( \varphi \) is compactly supported modulo \( Z_p^J \).

Then \( \mathscr{H}_p \) forms a \( \mathbb{C} \)-algebra via the convolution product
\[
(\varphi_1 \ast \varphi_2)(g) := \int_{Z_p^J \setminus G_p^J} \varphi_1(gx^{-1})\varphi_2(x)dx,
\]
where \( dx \) is a Haar measure on \( Z_p^J \setminus G_p^J \) normalized by \( \int_{Z_p^J \setminus Z_p^J \cap K_p^J} dx = 1 \). The algebra \( \mathscr{H}_p \) is called the Hecke algebra of \( (G_p^J, K_p^J) \) with respect to the additive character \( \Psi_p \).

We put
\[
N_p^J := \{ [(0, \mu), 0] \langle n \rangle (A) \in G_p^J \mid \mu \in \mathbb{Q}_p^n, A \in U_{n, p}, S \in \text{Sym}_n(\mathbb{Q}_p) \},
\]
\[
T_p = T(\mathbb{Q}_p) := \{ \langle n \rangle (t_1 \perp \cdots \perp t_n) \in G_p \mid t_i \in \mathbb{Q}_p^\times \}
\]
and \( T(\mathbb{Z}_p) := T_p \cap K_p \), where \( U_{n, p} \subset \text{GL}_n(\mathbb{Q}_p) \) is the group of upper unipotent matrices. We fix Haar measures \( d\langle n \rangle \) and \( dt \) on \( N_p^J \) and \( T_p \) respectively normalized by
\[
\int_{N_p^J \cap K_p^J} d\langle n \rangle = 1 \quad \text{and} \quad \int_{T(\mathbb{Z}_p)} dt = 1.
\]

We define the module \( \delta_{N_p^J}(t) \) of \( t \in T_p \) to be the ratio \( d(t \langle n \rangle t^{-1}) / d\langle n \rangle \). For any \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n \), we put
\[
\pi_{\alpha} = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \text{GL}_n(\mathbb{Q}_p),
\]
then we easily see that
\[
\delta_{N_p^J}(\pi_{\alpha}) = p^{-\sum_{i=1}^{n}(2n+3-2)\alpha_i}.
\]

Let \( X_0(T_p) \) be the group of unramified characters of \( T_p \), that is,
\[
X_0(T_p) := \{ \chi \in \text{Hom}(T_p, \mathbb{C}^\times) \mid \chi \text{ is trivial on } T(\mathbb{Z}_p) \}.
\]
In particular, if \( n = 1 \), then \( X_0(T_p) \) coincides with the group \( X_0(\mathbb{Q}_p^\times) \) consisting of all unramified characters of \( \mathbb{Q}_p^\times \). For any \( \chi \in X_0(T_p) \) and \( \varphi \in \mathscr{H}_p \), we define the zonal spherical function \( \widehat{\varphi}_\chi \) by
\[
\widehat{\varphi}_\chi(\varphi) := \sum_{\alpha \in \mathbb{Z}^n} \chi^{-1}(\langle n \rangle (\pi_{\alpha})) \overline{\varphi}(\langle n \rangle (\pi_{\alpha})),
\]
where
\[
\varphi(t) := \delta_{N_p^J}(t)^{-1/2} \int_{N_p^J} \varphi(nt) d\langle n \rangle \quad (t \in T_p).
\]
It is shown by Murase that the map $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ gives a $\mathbb{C}$-algebra homomorphism of $\mathcal{H}_p$ to $\mathbb{C}$ and that every $\mathbb{C}$-algebra homomorphism of $\mathcal{H}_p$ to $\mathbb{C}$ is given by $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ for some $\chi \in X_0(T_p)$ (cf. Proposition 4.10 and Theorem 4.15 in [8]).

On the other hand, for any $\chi \in X_0(T_p)$, let $\phi_\chi$ be a $\mathbb{C}$-valued function on $G^J_p$ defined by

$$\phi_\chi([(0, 0), \kappa] n t [(\lambda, 0), 0] k) = \Psi_p(\kappa)(\chi \delta^{-1/2}_{m_p}(t) \text{char}_{\mathbb{Z}_p}(\lambda))$$

for any $\kappa \in \mathbb{Q}_p$, $n \in N^J_p$, $t \in T_p$, $\lambda \in \mathbb{Q}_p^\times$ and $k \in K^J_p$, where we denote by $\text{char}_{\mathbb{Z}_p}$ the characteristic function of $\mathbb{Z}_p^n$. Here we note that each $\chi \in X_0(T_p)$ can be written in the form

$$\chi((d_n, \lambda, \tau_n)) = \chi^{(1)}(\tau_1) \cdots \chi^{(n)}(\tau_n),$$

with uniquely determined $n$ unramified characters $\chi^{(1)}, \ldots, \chi^{(n)} \in X_0(\mathbb{Q}_p^n)$. In this case, we simply write $\chi = (\chi^{(1)}, \ldots, \chi^{(n)})$. If $\chi = (\chi^{(1)}, \ldots, \chi^{(n)}) \in X_0(T_p)$, then it satisfies that

\begin{equation}
\phi_\chi([(0, 0), \kappa] n t [(\lambda, 0), 0] k) = \Psi_p(\kappa) \prod_{i=1}^n \chi^{(i)}(t_i) |t_i|^{(2n+3-2i)/2} \text{char}_{\mathbb{Z}_p}(\lambda)
\end{equation}

for any $\kappa \in \mathbb{Q}_p$, $n \in N^J_p$, $t = (d_n, \lambda, \tau_n) \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K^J_p$.

For each rational prime $p$, we define the action of Hecke algebra $\mathcal{H}_p$ on the space $J_{l, 1}(G^J_n(\mathbb{A}))$ by the following: for any $f \in J_{l, 1}(G^J_n(\mathbb{A}))$ and $\varphi \in \mathcal{H}_p$, $$(f \ast \varphi)(g) := \int_{\mathbb{Z}_p^\times \backslash G^J_p} f(gx^{-1}) \varphi(g^{-1}) dx \quad (g \in G^J_n(\mathbb{A})).$$

A Jacobi form $f \in J_{l, 1}(G^J_n(\mathbb{A}))$ is called a Hecke eigenform if it is a common eigenfunction of all elements of $\prod_p \mathcal{H}_p$, that is, for any rational prime $p$ and $\varphi \in \mathcal{H}_p$, the equation

$$f \ast \varphi = \lambda_f(\varphi) f$$

holds for some $\lambda_f(\varphi) \in \mathbb{C}$. Since for each $p$, the map $\lambda_f : \mathcal{H}_p \to \mathbb{C}$ gives a $\mathbb{C}$-algebra homomorphism of $\mathcal{H}_p$ to $\mathbb{C}$, it determines a $\chi_f = (\chi_f^{(1)}, \ldots, \chi_f^{(n)}) \in X_0(T_p)$ such that

$$\lambda_f(\varphi) = \widehat{\omega}_{\chi_f}(\varphi)$$

for any $\varphi \in \mathcal{H}_p$. We call either the collection $(\chi_f^{(1)}(p), \ldots, \chi_f^{(n)}(p))$ or $(\chi_f^{(1)}(p)^{-1}, \ldots, \chi_f^{(n)}(p)^{-1})$ the Satake $p$-parameters of $f$. Then for a Hecke eigenform $f \in J_{l, 1}(G^J_n(\mathbb{A}))$, we define the standard $L$-function attached to $\phi$ by

$$L(s, f, \text{St}) := \prod_{p < \infty} \prod_{i=1}^n \left\{ (1 - \chi_f^{(i)}(p) p^{-s}) (1 - \chi_f^{(i)}(p)^{-1} p^{-s}) \right\}^{-1},$$

which was introduced by Shintani in his unpublished paper, and afterward was studied by Murase (cf. [8], [9]).

By Lemma 2.2, for each element $f \in J_{l, 1}(G^J_n(\mathbb{A}))$ we obtain the associated element $\Phi_f \in J^\text{cusp}_{l, 1}(\mathbb{G}^J_n)$. Then we easily have the following relation between the action of the Hecke algebra $\mathcal{H}_p$ on $f$ and the operation $\Phi_f|_{l, 1}\Gamma_n^J M \Gamma_n^J$ for some $M \in G_n(\mathbb{Z}[p^{-1}])$:...
Lemma 2.3. Let \( f \in J_{l,1}(G_n^J(A)) \). For any \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \) with \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_n \), we have
\[
\Phi_{f*\varphi} = \Phi_f|_{l,1} \Gamma_n^J \mathfrak{d}_n(\pi_\alpha) \Gamma_n^J.
\]
Here \( \varphi_\alpha \) is an element of \( \mathcal{H}_p \) defined by
\[
\varphi_\alpha(g) = \begin{cases} 
\Psi_p(\kappa) & \text{if } g \in \mathbb{Z}_p^J \mathbb{K}_p^J \mathfrak{d}_n(\pi_\alpha) \mathbb{K}_p^J \text{ and } g = [(0,0), \kappa] k \mathfrak{d}_n(\pi_\alpha) k', \\
0 & \text{if } g \notin \mathbb{Z}_p^J \mathbb{K}_p^J \mathfrak{d}_n(\pi_\alpha) \mathbb{K}_p^J,
\end{cases}
\]
where \( \kappa \in \mathbb{Q}_p \) and \( k, k' \in \mathbb{K}_p^J \). In particular, if \( f \) is a Hecke eigenform, then \( \Phi_f \) is also a Hecke eigenform in the sense of \( \S 2.2.1 \).

Let \( \phi \in J_{l,1}(\Gamma_n^J) \) be a Hecke eigenform corresponding to a Hecke eigenform \( f \in J_{l,1}(G_n^J(A)) \) via the mapping defined in (7), that is, \( \phi = \Phi_f \). Then by Lemma 2.3, we naturally define the standard \( L \)-function attached to \( \phi \) as \( L(s, \phi, St) := L(s, f, St) \). Namely,
\[
L(s, \phi, St) := \prod_{\mathfrak{p} \in \infty} \prod_{i=1}^n \left( 1 - \frac{\chi^{(i)}_\phi(p) p^{-s}}{1 - \chi^{(i)}_\phi(p^{-1} p^{-s})} \right)^{-1},
\]
where we put \( \chi^{(i)}_\phi(p) = \chi^{(i)}_f(p) \) for \( i = 1, \ldots, n \).

If \( \phi \) is a cuspidal Hecke eigenform, then the following analytic properties of the standard \( L \)-function \( L(s, \phi, St) \) are shown by Murase ([19]):

Lemma 2.4 (cf. [19]). If \( \phi \in J_{l,1}^{\text{cusp}}(\Gamma_n^J) \) is a Hecke eigenform, then the standard \( L \)-function \( L(s, \phi, St) \) has a meromorphic continuation to the entire complex plane \( \mathbb{C} \). More precisely, put \( \Gamma_C(s) := 2(2\pi)^{-s} \Gamma(s) \), and the function
\[
L^*(s, \phi, St) = \prod_{i=1}^n \Gamma_C(s + l - 1/2 - i) L(s, \phi, St)
\]
is meromorphic on \( \mathbb{C} \), and satisfies the functional equation
\[
L^*(1 - s, \phi, St) = \varepsilon_n L^*(s, \phi, St),
\]
where
\[
\varepsilon_n = \begin{cases} 
-1 & \text{if } n \equiv 1, 2 \pmod{4}, \\
1 & \text{otherwise}.
\end{cases}
\]

Remark. Murase derived similar properties for the standard \( L \)-functions attached to more general cuspidal Jacobi forms whose index is a matrix.

On the other hand, we consider the standard \( L \)-function attached to the Jacobi Eisenstein series \( \mathfrak{c}_{l,1}^{(n)} \in J_{l,1}(G_n^J(A)) \) in the rest of this paragraph.

For any quasi-character \( \xi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C} \), we define a \( \mathbb{C} \)-valued function \( \widetilde{\phi}_\xi \) on \( G_n^J(A) \) by
\[
\widetilde{\phi}_\xi([(0, \mu), \kappa] g[(\lambda, 0), 0] k_\infty k_{\text{fin}}) = \xi(\det(A)) \varphi_0(\lambda) j(k_\infty, \sqrt{-1} 1_n)^{-1}
\]
for any \( \kappa \in \mathbb{Z} \), \( g = (A \ B \ C \ D) \in G_0^n(\mathbb{Z}) \), \( k_\infty \in \mathbb{K} \) and \( k_{\operatorname{fin}} \in K_{\operatorname{fin}}^J \), where \( \varphi_0 = \prod_v \varphi_{0,v} \),

\[
\varphi_{0,v}(\lambda) = \begin{cases} 
\operatorname{char}_{\mathbb{Z}_p}(\lambda) & \text{if } v = p < \infty, \\
\exp(-2\pi \lambda^v) & \text{if } v = \infty.
\end{cases}
\]

Then we define the Eisenstein series \( E_\xi \) on \( G_0^n(\mathbb{Z}) \) associated with \( \xi \) by

\[
E_\xi(g) := \sum_{\gamma \in \mathbb{D}_0^n(\mathbb{Q}) \setminus G_0^n(\mathbb{Q})} \tilde{\phi}_{\xi}(\gamma g) \quad (g \in G_0^n(\mathbb{Z})).
\]

In particular, we denote by \( E_{l,1}^{(n)} \) the Eisenstein series on \( G_0^n(\mathbb{Z}) \) associated with a special character \( \xi_l(x) = |x|_{\mathbb{A}}^l \ (x \in \mathbb{A}^\times) \). Then we easily see that \( E_{l,1}^{(n)} \) is an element of \( J_{l,1}(G_0^n(\mathbb{Z})) \) and corresponds to the Jacobi Eisenstein series \( E_{l,1} \in J_{l,1}(\Gamma_0^n) \) in the same manner as in Lemma 2.2. Therefore we also call \( E_{l,1} \) the Jacobi Eisenstein series of weight \( l \) and index 1.

Then we have

**Proposition 2.2.** The Jacobi Eisenstein series \( E_{l,1}^{(n)} \) is a Hecke eigenform, that is, for any \( \varphi \in \bigotimes_p \mathcal{H}_p \),

\[
E_{l,1}^{(n)} \ast \varphi = \lambda_\xi(\varphi) E_{l,1}^{(n)}
\]

with \( \lambda_\xi(\varphi) \in \mathbb{C} \). Moreover, the Satake p-parameters of \( E_{l,1}^{(n)} \) are taken of the form

\[
(p^{-(n+1)+i-1/2})_{1 \leq i \leq n}
\]

up to inversion.

**Proof.** For any quasi-character \( \xi \) of \( \mathbb{Q}^\times \setminus \mathbb{A}^\times \), we take a \( \chi = (\chi^{(1)}, \cdots, \chi^{(n)}) \in X_0(T_p) \) such that

\[
\chi^{(i)}(t_i) = \xi(t_i) |t_i|_{p}^{-2(n+3-2i)/2} \quad (t_i \in \mathbb{Q}_p^\times)
\]

for each \( 1 \leq i \leq n \). Then by the equation (9) and the definition of \( \tilde{\phi}_{\xi} \), we have \( \tilde{\phi}_{\xi} = \phi_\chi \).

Therefore it suffices to prove that for any \( \varphi \in \mathcal{H}_p \) and \( \lambda \in \mathbb{Q}_p^n \), the equation

\[
(\phi_\chi \ast \varphi)([(\lambda, 0), 0]) = c \cdot \operatorname{char}_{\mathbb{Z}_p}(\lambda)
\]

holds with some \( c \in \mathbb{C}^\times \). Indeed, if \( \lambda \notin \mathbb{Z}_p^n \), then there exists \( 0 \neq \mu \in \mathbb{Z}_p^n \) such that \( \Psi_p(\lambda' \mu) \neq 1 \). Thus we have

\[
(\phi_\chi \ast \varphi)([(\lambda, 0), 0]) = (\phi_\chi \ast \varphi)([(\lambda, 0), (0, \mu), 0]) = (\phi_\chi \ast \varphi)([(\lambda, \mu), (0, \lambda' \mu)]) = (\phi_\chi \ast \varphi)([(0, \mu), (\lambda, 0), 0]) = \Psi_p(\lambda' \mu)(\phi_\chi \ast \varphi)([(\lambda, 0), 0]),
\]

and \( (\phi_\chi \ast \varphi)([(\lambda, 0), 0]) = 0 \). Now we have proved that the Eisenstein series \( E_\xi \) is a Hecke eigenform. Moreover, it follows from the equation (10) that

\[
c = (\phi_\chi \ast \varphi)(1) = \int_{\mathbb{Z}_p^n \setminus G_0^n} \phi_\chi(g) \varphi(g^{-1}) dg
\]
and therefore the eigenvalue \( \lambda_E(\varphi) \) coincides with the zonal spherical function \( \hat{\omega}_\chi(\varphi) \). Therefore it follows from the equation (9) that
\[
\chi^{(i)}(t_i) = \xi_i(t_i) |t_i|^{-(2n+3-2i)/2} = |t_i|^{|-(2n+3-2i)/2|
\]
for each \( i \). By substituting \( t_i = p \), we obtain \( \chi^{(i)}(p) = p^{-l+(2n+3-2i)/2} \) and complete the proof.

By Proposition 2.2, we obtain the following conclusion:

**Corollary.** Let \( l \) be a positive even integer such that \( l > n + 2 \). Then we have
\[
L(s, E^{(n)}_{l,1}, St) = L(s, E_{l,1}^{(n)}, St) = \prod_{i=1}^{n} \zeta(s-l+1/2+i)\zeta(s-l-1/2-i).
\]
In particular, \( L(s, E_{l,1}^{(n)}, St) \) and \( L(s, E_{l,1}^{(n)}, St) \) converge absolutely for \( \text{Re}(s) > l - n - 1/2 \). In addition, they have meromorphic continuations to the entire complex plane \( \mathbb{C} \) and satisfy functional equations under \( s \mapsto 1 - s \).

**Remark.** Let \( k \) and \( n \) be positive even integers such that \( k > n + 1 \). As mentioned in §2.1, \( E^{(n-1)}_{k-1,1} \) coincides with the first Fourier-Jacobi coefficient \( e^{(n-1)}_{k,1} \) of the Siegel Eisenstein series \( E_k^{(n)} \in M_k(\Gamma_n) \) of degree \( n \) and weight \( k \). Thus it follows from Corollary of Proposition 2.2 that
\[
L(s, E^{(n)}_{l,1}, St) = \prod_{p \mid \text{p}} \prod_{i=1}^{n-1} \left( 1 - p^{-(k-(n+1)/2)}p^{-s+i-n/2} \right) \left( 1 - (p^{-(k-(n+1)/2)}p^{-s+i-n/2}) \right)^{-1}
\]
\[
= \prod_{i=1}^{n-1} L(s+k-1/2-i, E^{(1)}_{2k-n}),
\]
where \( E^{(1)}_{2k-n} \in M_{2k-n}(\Gamma_1) \). Moreover, replacing \( e^{(n-1)}_{k,1} \) by the first Fourier-Jacobi coefficient \( \phi_1 \in J^{\text{cusp}}_{k,1}(\Gamma_{n-1}) \) of a Siegel cusp form \( f \in S_k(\Gamma_n) \) which is connected to a normalized Hecke eigenform \( f \in S_{2k-n}(\Gamma_1) \) via a lifting procedure due to Ikeda (cf. [13]), then we also obtain a similar explicit formula for the standard \( L \)-function attached to \( \phi_1 \) (cf. [15]).

2.4. Eichler-Zagier-Ibukiyama correspondence between Jacobi forms and Siegel modular forms of half-integral weight.

For the purpose of the subsequent use, we review in this paragraph that there exists a natural \( \mathbb{C} \)-linear correspondence from the space of Jacobi forms of even integral weight and of index 1 into that of Siegel modular forms of half-integral weight.

For any \((\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^n\) and \((r_1, r_2) \in \mathbb{Q}^n \oplus \mathbb{Q}^n\), we define the \textit{theta series} of characteristic \((r_1, r_2)\) by
\[
\theta_{(r_1, r_2)}(\tau, z) = \theta^{(n)}_{(r_1, r_2)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^n} e \left( (\tau/2)[\lambda + r_1] + (\lambda + r_1)^t(z + r_2) \right).
\]
In particular, for any $r \in \mathbb{Z}^n$, we put $\theta_r(\tau, z) = \theta_r^{(n)}(\tau, z) := \theta^{(n)}_{(r/2, 0)}(2\tau, 2z)$. We note that the function $\theta_r(\tau, z)$ depends only on $r \mod 2\mathbb{Z}^n$. For a fixed $\tau \in \mathfrak{H}_n$, it is known that $(\theta_r(\tau, z))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}$ forms a basis of the $\mathbb{C}$-vector space $\Theta_r^{(n)}$ consisting of all $\mathbb{C}$-valued holomorphic functions $\theta(z)$ on $\mathbb{C}^n$ which satisfy that

$$\theta(z + \lambda \tau + \mu) = e(-\operatorname{tr}(\tau^T[\lambda] + 2^t\lambda z))\theta(z)$$

for any $\lambda, \mu \in \mathbb{C}^n$.

For any $\tau \in \mathfrak{H}_n$, we put

$$\theta(\tau) = \theta^{(n)}(\tau) := \theta^{(n)}_{(0, 0)}(2\tau, 0) = \sum_{\lambda \in \mathbb{Z}^n} e(\tau^T[\lambda]).$$

Then for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(4)$, we define the Shimura’s factor of automorphy by

$$J(M, \tau) = J^{(n)}(M, \tau) := \frac{\theta^{(n)}(M \langle \tau \rangle)}{\theta^{(n)}(\tau)}.$$

As is well-known, it follows that

$$J(M, \tau)^2 = (-1)^{(\det D - 1)/2} \det(C\tau + D).$$

For any $l \in \mathbb{Z}$, a holomorphic function $F(\tau)$ on $\mathfrak{H}_n$ is called a Siegel modular form of degree $n$ and weight $l - 1/2$ if it satisfies the following two conditions:

(i) $F(M \langle \tau \rangle) = J(M, \tau)^{2l-1} F(\tau)$ for any $M \in \Gamma_0^{(n)}(4)$;

(ii) For any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, the function $\det(C\tau + D)^{-l+1/2} F(M \langle \tau \rangle)$ possesses a Fourier expansion of the form

$$\det(C\tau + D)^{-l+1/2} F(M \langle \tau \rangle) = \sum_{B \in \operatorname{Sym}_n^+(\mathbb{Z})} C_{F, M}(B) e(\operatorname{tr}(B\tau)/4),$$

where $\det(C\tau + D)^{-l+1/2}$ is an appropriately defined single valued function of $\tau$. If $F$ satisfies the stronger condition $C_{F, M}(B) = 0$ unless $B > 0$ (positive definite), it is called a cusp form. We note that such a $F$ possesses a usual Fourier expansion

$$F(\tau) = \sum_{B \in \operatorname{Sym}_n^+(\mathbb{Z})} C_F(B) e(\operatorname{tr}(B\tau)).$$

We denote by $M_{l-1/2}(\Gamma_0^{(n)}(4))$ and $S_{l-1/2}(\Gamma_0^{(n)}(4))$ the $\mathbb{C}$-vector spaces of Siegel modular forms and Siegel cusp forms of degree $n$ and weight $l - 1/2$, respectively.

Furthermore, we introduce the generalized Kohnen plus space $M_{l-1/2}^{+}(\Gamma_0^{(n)}(4))$ consisting of all elements $F \in M_{l-1/2}(\Gamma_0^{(n)}(4))$ whose Fourier coefficients $C_F(B)$ satisfy the condition

$$C_F(B) = 0 \text{ unless } B \equiv (-1)^l \tau B r_B \mod 4 \operatorname{Sym}_n^+(\mathbb{Z}) \text{ for some } r_B \in \mathbb{Z}^{n-1},$$

and put $S_{l-1/2}^{+}(\Gamma_0^{(n)}(4)) := M_{l-1/2}^{+}(\Gamma_0^{(n)}(4)) \cap S_{l-1/2}(\Gamma_0^{(n)}(4))$. These spaces were introduced by Kohnen (\cite{17}) in case $n = 1$, and by Ibukiyama (\cite{8}) for general $n$. 


Now, we recall an important fact that if \( l \) is even, then there exists a \( \mathbb{C} \)-linear isomorphism between the space \( J_{l,1}(\Gamma_n^J) \) of Jacobi forms of index 1 and the generalized Kohnen plus space \( M_{l-1/2}^+(\Gamma_0^{(n)}(4)) \) as follows. Let \( \phi \in J_{l,1}(\Gamma_n^J) \) possess a Fourier expansion of the form
\[
\phi(\tau, z) = \sum_{T \in \text{Sym}_n^+(\mathbb{Z}), \ r \in \mathbb{Z}^n, \ 4T - tr_r \geq 0} c_\phi(T, r) \mathbf{e}(\text{tr}(T \tau) + r \cdot z).
\]
Since for each \( \tau \in \mathfrak{H}_n \), \( \phi(\tau, z) \) belongs in the space \( \Theta_{2, n}^{(l)} \) generated by \( (\theta_r(\tau, z))_{r \in 2\mathbb{Z}^n} \), we have that \( \phi \) can be expressed as a linear combination
\[
\phi(\tau, z) = \sum_{r \in 2\mathbb{Z}^n} h_r(\tau) \theta_r(\tau, z)
\]
with some \( 2^n \) holomorphic functions \( (h_r(\tau))_{r \in 2\mathbb{Z}^n} \) on \( \mathfrak{H}_n \) whose Fourier expansion is of the form
\[
h_r(\tau) = \sum_{T \in \text{Sym}_n^+(\mathbb{Z}), \ 4T - tr_r \geq 0} c_\phi(T, r) \mathbf{e}(\text{tr}((T - \frac{1}{4}rr/4) \tau)).
\]
Then we put
\[
\sigma(\phi)(\tau) = \sum_{r \in 2\mathbb{Z}^n} h_r(4\tau).
\]
The following statement is shown by Eichler and Zagier ([6]) in case \( n = 1 \) and by Ibukiyama for general \( n \):

**Proposition 2.3** (cf. Theorem 1, 2 in [3]). If \( l \) is even, then the map \( \phi \mapsto \sigma(\phi) \) gives a \( \mathbb{C} \)-linear isomorphism
\[
J_{l,1}(\Gamma_n^J) \cong M_{l-1/2}^+(\Gamma_0^{(n)}(4)),
\]
which is compatible with the actions of Hecke operators. Furthermore, its restriction to the space \( J_{l,1}^\text{cusp}(\Gamma_n^J) \) also induces a \( \mathbb{C} \)-linear isomorphism
\[
J_{l,1}^\text{cusp}(\Gamma_n^J) \cong S_{l-1/2}^+(\Gamma_0^{(n)}(4)).
\]
We call it the Eichler-Zagier-Ibukiyama correspondence.

**Remark.** When \( l \) is odd, the space \( J_{l,1}(\Gamma_n^J) \) is not isomorphic to the Kohnen plus space \( M_{l-1/2}^+(\Gamma_0^{(n)}(4)) \). However, a similar claim is also valid by introducing the space \( J_{l,1}^\text{skew}(\Gamma_n^J) \) of skew holomorphic Jacobi forms which was defined by Skoruppa ([24, 25]) in case \( n = 1 \) and by Arakawa ([2]) for general \( n \).

We easily see by the definition that the Fourier expansion of \( \sigma(\phi) \) can be expressed in terms of Fourier coefficients of \( \phi \) as
\[
\sigma(\phi)(\tau) = \sum_{B \in \text{Sym}_n(\mathbb{Z})} c_\phi((B + \frac{1}{4}r_B r_B)/4, r_B) \mathbf{e}(\text{tr}(B \tau)),
\]
where \( r_B \) denotes an element of \( \mathbb{Z}^n \) such that \( B + \frac{1}{4}r_B r_B \in 4\text{Sym}_n^+(\mathbb{Z}) \). We note that \( r_B \) is uniquely determined by \( B \) modulo \( 2\mathbb{Z}^n \), and then \( c_\phi((B + \frac{1}{4}r_B r_B)/4, r_B) \) does not depend
on the choice of the representative of $r_B \mod 2\mathbb{Z}^n$. Moreover, if $\phi$ coincides with the first Fourier-Jacobi coefficient of a Siegel modular form $F \in M_l(\Gamma_{n+1})$, then we have

$$\sigma(\phi)(\tau) = \sum_{B \in \text{Sym}_n(\mathbb{Z}) \geq 0} A_F(B^{(1)}) e(\text{tr}(B\tau)), $$

where $B^{(1)} \in \text{Sym}_n^*(\mathbb{Z})$ denotes the matrix defined in §1, and $A_F(B^{(1)})$ is the $B^{(1)}$-th Fourier coefficient of $F$. In particular, let $n$ and $k$ be positive even integers such that $k > n + 1$ and we take $\phi = e_{k,1}^{(n-1)} \in J_{k,1}(\Gamma_{n-1})$, then we have the following explicit formula for the Fourier coefficients of the associated form $\sigma(e_{k,1}^{(n-1)}) \in M_{k-1/2}^+(\Gamma_{0}^{(n-1)}(4))$:

**Proposition 2.4.** Under the same assumption as in Proposition 2.1, let $\sigma(e_{k,1}^{(n-1)})$ possess a Fourier expansion

$$\sigma(e_{k,1}^{(n-1)})(\tau) = \sum_{B \in \text{Sym}_n(\mathbb{Z}) \geq 0} C_{k-1/2}^{(n-1)}(B) e(\text{tr}(B\tau)).$$

Then for each $B \in \text{Sym}_n^{*}(\mathbb{Z})_{>0}$ satisfying the condition (1), we have

$$C_{k-1/2}^{(n-1)}(B) = \xi(n, k)L(1-k+n/2, \chi_{B^{(1)}}) f(B^{(1)})^{k-(n+1)/2} \prod_{p | \langle B^{(1)} \rangle} \tilde{F}_p^{(1)}(B; p^{k-(n+1)/2}).$$

Proof. If $B = 4T - t r r$ with $T \in \text{Sym}_n^{*}(\mathbb{Z})$ and $r \in \mathbb{Z}^{n-1}$, then we have

$$C_{k-1/2}^{(n-1)}(B) = e_{k,1}^{(n-1)}(T, r).$$

Thus the assertion follows from Proposition 2.1.

\[\square\]

3. ANDRIANOVA type identity for power series attached to Jacobi forms

Throughout this paragraph, let $n$ and $k$ be positive even integers such that $k > n + 1$, and we fix a rational prime $p$. For a subring $R$ of $\mathbb{Z}_p$, we simply denote by $\text{Sym}_{n-1}(R)^{(1)}$ the subset of $\text{Sym}_{n-1}(R)^{\times}$ consisting of all elements which satisfy the condition (1) in §1, namely,

$$\text{Sym}_{n-1}(R)^{(1)} = \{ B \in \text{Sym}_{n-1}(R)^{\times} | B + tr_B r_B \in 4 \text{Sym}_{n-1}^*(R) \text{ for some } r_B \in R^{n-1} \}.$$ 

As mentioned in §1, for each element $B \in \text{Sym}_{n-1}(R)^{(1)}$, we can associate it with an element

$$B^{(1)} = \left( \begin{array}{cc} 1 & r_B/2 \\ tr_B/2 & (B + tr_Br_B)/4 \end{array} \right) \in \text{Sym}_n^*(R)^{\times}.$$ 

Then for such a $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we introduce a modified local Siegel series as follows. For each $R \in \text{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])$ and $r \in \mathbb{Z}_p^{n-1}$, if $R \in p^{-l}\text{Sym}_{n-1}(\mathbb{Z}_p)$ with $l \geq 0$, then we put

$$\omega(R; r) = p^{-(n-1)/2} \mu_p(R)^{1/2} \sum_{x \in \mathbb{Z}_p^{n-1}/p\mathbb{Z}_p^{n-1}} e_p(-R^{t} x) + r R^{t} r/2 + x R^{t} r/2,$$
where $\mu_p(R) = \mathbb{Z}_p^{-1} R + \mathbb{Z}_p^{-1}$, and we note that the right-hand side does not depend on the choice of $l$. Let $B \in \text{Sym}_{n-1}(\mathbb{Q}_p)$ possess $B = 4T - \langle r \rangle$ with $T \in \text{Sym}_{n-1}^*(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then we put

$$b_p^{(1)}(B; t) = \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}_p^{p-1})/\text{Sym}_{n-1}(\mathbb{Z}_p)} \omega(R; r) e_p(-\text{tr}(TR)) \tau_{\text{ord}_p}(\mu_p(R)),$$

We note that this series coincides with $\alpha_1(B, t)$ in [21] if $p \neq 2$ and $r = 0$. As will be shown later, the above definition does not depend on the choice of $T$ and $r$ (cf. Proposition 3.1 below).

On the other hand, if $m > 1$, then for each $S \in \text{Sym}_{m-1}^*(\mathbb{Z}_p)$, $T \in \text{Sym}_{n-1}(\mathbb{Q}_p)$, $r \in \mathbb{Z}_p^{n-1}$ and $e \in \mathbb{Z}_{>0}$, we put

$$\mathcal{A}_e(S, T, r) := \left\{ X \in M_{m,n-1}(\mathbb{Z}_p)/p^m M_{m,n-1}(\mathbb{Z}_p) \mid \begin{array}{c}
(-1 \perp S)[X] + t r x_1 / 2 \\
+t r x_1 / 2 - T \in p^e \text{Sym}_{n-1}^*(\mathbb{Z}_p) \end{array} \right\},$$

where $x_1 \in \mathbb{Z}_p^{n-1}$ denotes the first row of $X$. We easily check that it is well-defined. Furthermore, if both $S$ and \( \left( \begin{array}{cc} 1 & r / 2 \\ T & \end{array} \right) \) are non-degenerate, then $p^{e(-m(n-1)+n(n-1)/2)} \# \mathcal{A}_e(S, T, r)$ has the same value for any $e \geq \text{ord}_p(\det \left( \begin{array}{cc} 1 & r / 2 \\ T & \end{array} \right))$, which will be denoted by $\alpha_p^{(1)}(S, T, r)$. We note that $\alpha_p^{(1)}(S, T, r)$ coincides with the usual local density $\alpha_p(-1 \perp S, T)$ if $r = 0$. Then we obtain the following lemmas:

**Lemma 3.1.** Let $B \in \text{Sym}_{n-1}(\mathbb{Q}_p)^\times$ possess $B = 4T - \langle r \rangle$ with $T \in \text{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then we have

$$b_p^{(1)}(B; -k/2) = \alpha_p(H_{k-1}, T, r),$$

where $H_{k-1} = \underbrace{H \perp \cdots \perp H}_{k-1}$ with $H = \left( \begin{array}{cc} 0 & 1 / 2 \\ 1 / 2 & 0 \end{array} \right) \in \text{Sym}_{n}^*(\mathbb{Z}_p)$. In particular, $b_p^{(1)}(B; t) = 0$ unless $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$.

**Proof.** By Lemma 3.4 of [22], we have

$$b_p^{(1)}(B; -k/2)$$

$$= \sum_{R} \sum_{x \in \mathbb{Z}_p^{n-1}/p^e \mathbb{Z}_p^{n-1}} e_p(-R[l^t x] + r R^t x / 2 + x R^t r / 2) p^{-(k-1) \text{ord}_p(\mu_p(R))} p^{-(n-1)l} e_p(-\text{tr}(TR))$$

$$= \sum_{R} \sum_{x} e_p(-R[l^t x] + r R^t x / 2 + x R^t r / 2) p^{-(n-1)l} e_p(-\text{tr}(TR)) p^{-2l(k-1)n}$$

$$\times \sum_{Y \in M_{2k-2,n-1}(\mathbb{Z}_p)/p^e M_{2k-2,n-1}(\mathbb{Z}_p)} e_p(\text{tr}(H_{k-1}[Y] R))$$

$$= \sum_{R} \sum_{x} e_p(\text{tr}((-t^t x x + H_{k-1}[Y] + t^t r x / 2 + t^t r x / 2 - T) R)) p^{-l(2k-1)(n-1)}$$

$$= \# \mathcal{A}_e(H_{k-1}, T, r) p^{-l((2k-1)(n-1) - n(n-1)/2)}.$$

Thus the assertion holds. \(\square\)
Lemma 3.2. If $B \in \text{Sym}_{n-1}(\mathbb{Q}_p)^*$ possesses $B = 4T - 4rr$ with $T \in \text{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{-1}$, then we have

$$\alpha_p(H_k, B^{(1)}) = (1 - p^{-k})\alpha_p(H_{k-1}, T, r).$$

Proof. The proof is similar to that of Proposition 2.4 in [11], and here we give a sketch of the proof. For each $\xi = (\xi_i) \in \mathbb{Z}_p^{2k}$, we put

$$\mathcal{A}_e(H_k, B^{(1)}) = \{ X \in M_{2k,n}(\mathbb{Z}_p)/p^r M_{2k,n}(\mathbb{Z}_p) \mid H_k[X] - B^{(1)} \in p^r \text{Sym}^*_n(\mathbb{Z}_p) \}$$

and

$$\mathcal{A}_e(H_k, B^{(1)}; \xi) = \{ X = (x_{ij}) \in \mathcal{A}_e(H_k, B^{(1)}) \mid x_{i1} \equiv \xi_i \pmod{p} \text{ for } 1 \leq i \leq 2k \}.$$

We easily see that $\mathcal{A}_e(H_k, B^{(1)}; \xi) \neq \emptyset$ only if $\xi \in \mathcal{A}_e(H_k, 1)$. Now we fix such a $\xi_i$. Then we have $\xi_i \neq 0 \pmod{p \mathbb{Z}_p^{2k}}$. Thus by Lemma 2.3 in [11], we can take $U \in \text{GL}_{2k}(\mathbb{Z}_p)$ and $K \in L_{2k-2,p}$ such that

$$K \sim _{\mathbb{Z}_p} H_{2k-2}; \quad (i) K^{-1} H_{2k-2} K = Y; \quad (ii) U^{-1} \xi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

For each $X \in \mathcal{A}_e(H_k, B^{(1)}; \xi)$, we write $X$ as $X = (^t\xi | Y)$ with $Y \in M_{2k,n-1}(\mathbb{Z}_p)$, and write $Y$ as $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ with $y_1, y_2 \in \mathbb{Z}_p^{n-1}$ and $y_3 \in M_{2k-2,n-1}(\mathbb{Z}_p)$. Then by an easy calculation, we have

$$y_1 + y_2/2 - r/2 \in p^r \mathbb{Z}_p^{n-1}$$

and

$$-^t y_1 y_1 + K[y_3] + ^t y_1 y_2/2 + ^t y_2 y_1/2 - T \in p^r \text{Sym}^*_n(\mathbb{Z}_p).$$

Thus we have

$$-^t y_1 y_1 + K[y_3] + ^t r y_2/2 + ^t y_1 r/2 - T \in p^r \text{Sym}^*_n(\mathbb{Z}_p),$$

that is, $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{A}_e(H_{k-1}, T, r)$. Then the mapping $Y \longmapsto \begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$ induces a bijection between $\mathcal{A}_e(H_k, B^{(1)}; \xi)$ and $\mathcal{A}_e(H_{k-1}, T, r)$. Thus we have

$$p^e(2kn+n(n-1)/2) \# \mathcal{A}_e(H_k, B^{(1)})$$

$$= p^e(-2k+1) \# \mathcal{A}_e(H_k, 1) p^e(-(2k-1)(n-1)+n(n-1)/2) \# \mathcal{A}_e(H_{k-1}, T, r)$$

$$= \alpha_p(H_k, 1) \alpha_p(H_{k-1}, T, r)$$

$$= (1 - p^{-k}) \alpha_p(H_{k-1}, T, r).$$

Therefore the assertion holds. \hfill \Box

Now by combining Lemmas 3.1 and 3.2, we obtain the following:

Proposition 3.1. For each $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ and $s \in \mathbb{C}$, we have

$$b_p^{(1)}(B; p^{-s+1/2}) = (1 - p^{-s})^{-1} b_p(B^{(1)}; s).$$
Proof. It is well-known that for each \( B' \in \text{Sym}_n^*(\mathbb{Z}_p) \) with \( n < 2k \), the Siegel series \( b_p(B'; s) \) in §1 satisfies the equation

\[
b_p(B; k) = \alpha_p(H_k, B).
\]

Then by Lemmas 3.1 and 3.2, we have

\[
b_p^{(1)}(B; p^{-k+1/2}) = (1 - p^{-k})^{-1} b_p(B^{(1)}; k)
\]

for infinitely many \( k \), and therefore the assertion follows. \( \square \)

Remark. The definition of \( b_p^{(1)}(B; t) \) for \( B = 4T - t'rr \) with \( T \in \text{Sym}_{n-1}(\mathbb{Q}_p) \) and \( r \in \mathbb{Z}_p^{n-1} \) does not depend on the choice of \( T \) and \( r \). Indeed, if \( T \in \text{Sym}_{n-1}^*(\mathbb{Z}_p) \), then the vector \( r \) is uniquely determined by \( B \) modulo \( 2\mathbb{Z}_p^{n-1} \), and the matrix \(
\begin{pmatrix}
t_r/2 & T \\
1 & r
\end{pmatrix}
\)

is uniquely determined by \( B \) up to \( \text{GL}_{n-1}(\mathbb{Z}_p) \)-equivalence. Thus by Proposition 3.1, \( b_p^{(1)}(B; t) \) is uniquely determined by \( B \). If \( T \not\in \text{Sym}_{n-1}^*(\mathbb{Z}_p) \), then we have \( b_p^{(1)}(B; t) = 0 \). Furthermore, if \( B = 4T' - t'r'r' \) is another expression, then \( T' \) does not belong to \( \text{Sym}_{n-1}^*(\mathbb{Z}_p) \) either. This proves the well-definedness of \( b_p^{(1)}(B; t) \).

Now we put

\[
\bar{b}_p^{(1)}(B; t) := \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}_p) \setminus D(n-1)(\mathbb{Z}_p)} \pi_p(D) b_p^{(1)}(B[D^{-1}]; t) (p^{n-1} t^2)^{\text{ord}_p(\det D)}.
\]

Then by Proposition 3.1, we obtain the following rationality theorem for the polynomial \( B_p^{(1)}(B; t) \) defined in §1:

**Proposition 3.2.** For each \( B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)} \), we have

\[
B_p^{(1)}(B; p^{n-1/2} t) \bar{b}_p^{(1)}(B; p^{1/2} t) = \prod_{i=1}^{n-1} (1 - p^{2i} t^2).
\]

Next, we study the standard \( L \)-function attached to a Hecke eigenform and some power series related to it. For a Hecke eigenform \( \phi \in J_{k,1}^\text{cusp} (\Gamma_{n-1}^J) \), and \( D \in D(n-1)(\mathbb{Z}) \), let

\[
\phi|_{k,1} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J = \lambda_{\phi}(D) \phi
\]

with \( \lambda_{\phi}(D) \in \mathbb{C} \). Then we define a power series \( Z_p(t, \phi) \) by

\[
Z_p(t, \phi) := \sum_{D \in \mathbb{E} D^{(n-1)}(\mathbb{Z})} \lambda_{\phi}(D) t^{\text{ord}_p(\det D)},
\]

where \( \mathbb{E} D^{(n-1)}(\mathbb{Z}) \) denotes the set of all elementary divisors of the form \( p^{\alpha_1} \perp \cdots \perp p^{\alpha_{n-1}} \) with \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1} \). The following statement is shown by Murase and Sugano:

**Proposition 3.3** (cf. Lemma 6.5 in [20], see also Theorem 5.5 in [2]). Let \( \phi \in J_{k,1}(\Gamma_{n-1}^J) \) be a Hecke eigenform with Satake \( p \)-parameters \( (\chi_{\phi}^{(1)}(p), \cdots, \chi_{\phi}^{(n-1)}(p)) \in \mathbb{C}^{n-1} \). Then we have

\[
Z_p(t, \phi) = \prod_{i=1}^{n-1} \frac{(1 - p^{2i} t^2)}{(1 - \chi_{\phi}^{(i)}(p)) p^{n-1/2} t (1 - \chi_{\phi}^{(i)}(p)^{-1} p^{n-1/2} t)}.
\]
Let
\[
\mathcal{Z}_p^{(n-1)} := \left\{ \begin{pmatrix} V & \emptyset \\ W & \emptyset \end{pmatrix} \in M_{2n-2,n-1}(\mathbb{Z}) \mid V, W \in D_p^{(n-1)}(\mathbb{Z}), \ \gcd(V, W) = 1 \right\},
\]
where \(\gcd(V, W)\) denotes the greatest common divisor of all entries of \(V\) and \(W\). For each \((\frac{V}{W}) \in \mathcal{Z}_p^{(n-1)}\), \(R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])\) and \((\lambda_1, \lambda_2) \in \mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}\), we put
\[
M_{V, W, R} := \begin{pmatrix} tW^{-1}V & tW^{-1}WV^{-1} \\ 0_{n-1} & 1 \end{pmatrix} \in G_{n-1}(\mathbb{Z}[p^{-1}])
\]
and
\[
[\lambda_1, \lambda_2] := [(\lambda_1, \lambda_2), \lambda_1^t \lambda_2] = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} 0 & \lambda_2 \\ 1_{n-1} & 0 \end{pmatrix} \in H_{n-1}(\mathbb{Z}).
\]
Then by combining Lemma 2.1 and some easy calculation (cf. \[\text{(3)}\]), we obtain the following:

**Lemma 3.3.** We have
\[
\Gamma_{n-1}^J G_{n-1}(\mathbb{Z}[p^{-1}]) \Gamma_{n-1}^J = \bigcup_{D \in \mathcal{Z}_p^{(n-1)}(\mathbb{Z})} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J
\]
\[
= \bigcup_{(\frac{V}{W}) \in \mathcal{Z}_p^{(n-1)}} \bigcup_{R \in \text{Sym}_{n-1}(Z)} \Gamma_{n-1}^J [M_{V, W, R}] \cdot [\lambda_1, \lambda_2],
\]
where \((\frac{V}{W})\), \(R\) and \((\lambda_1, \lambda_2)\) run over all representatives of \((1_{n-1} \bot \text{GL}_{n-1}(\mathbb{Z})) \setminus \mathcal{Z}_p^{(n-1)} / \text{GL}_{n-1}(\mathbb{Z})\), \(\text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\text{WSym}_{n-1}(\mathbb{Z})\), and \((\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) \oplus (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) M_{V, W, R} / (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) M_{V, W, R}\), respectively. Furthermore, if \(M_{V, W, R} \in \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J\) with \(D \in \mathcal{Z}_p^{(n-1)}(\mathbb{Z})\), then we have \(\text{ord}_p(\det D) = \text{ord}_p(\det V \det W \mu_p(R))\).

Therefore, we get the following explicit formula for the actions of Hecke operators:

**Corollary.** For each \(\phi \in J_{k, 1}(\Gamma_{n-1}^J)\), we have
\[
\sum_{D \in \mathcal{Z}_p^{(n-1)}(\mathbb{Z})} \left( \phi \mid_{k, 1} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J \right)(\tau, z) = \sum_{(\frac{V}{W})} \sum_{R} p^{(-2n+3)\delta_{V, W, R}} \det V^{-1} \det W^{-k} \times \sum_{(\lambda_1, \lambda_2) \in (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1})/p^{V, W, R} (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1})} e(\tau^t \lambda_1 + 2 \lambda_1^t z) \times \phi(\tau [VW^{-1}] + R[W^{-1}], (z + \lambda_1 \tau + \lambda_2)VW^{-1}),
\]
where \((\frac{V}{W})\) and \(R\) run over the sets stated above, and \(\delta_{V, W, R} = \text{ord}_p(\det V \det W \mu_p(R))\).

**Proof.** For each \((\frac{V}{W}) \in \mathcal{Z}_p^{(n-1)}\) and \(R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])\), we have
\[
\Gamma_{n-1}^J M_{V, W, R} \Gamma_{n-1}^J = \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J
\]
for some \( D = p^{n_1} \perp \cdots \perp p^{n_m} \in \mathbf{ED}^{(n-1)}(\mathbb{Z}) \). Then we have
\[
(Z^{n-1} \oplus Z^{n-1}) + (Z^{n-1} \oplus Z^{n-1}) M_{V,W,R} / (Z^{n-1} \oplus Z^{n-1}) M_{V,W,R}
\]
\[
\simeq (Z^{n-1} \oplus Z^{n-1}) + (Z^{n-1} \oplus Z^{n-1}) d_{n-1}(D) / (Z^{n-1} \oplus Z^{n-1}) d_{n-1}(D)
\]
\[
\simeq Z^{n-1} / Z^{n-1} D.
\]

It follows from Lemma 3.3 that \( \#(Z^{n-1} / Z^{n-1} D) = p^{\delta_{V,W,R}} \) and \( e_1, ..., e_r \leq \delta_{V,W,R} \). Thus we have a natural surjection \( \pi \) from \( (Z^{n-1} \oplus Z^{n-1}) / p^{\delta_{V,W,R}} (Z^{n-1} V \oplus Z^{n-1}) \) to \( Z^{n-1} / Z^{n-1} D \), and we have \( \# \ker(\pi) = p^{2(n-3)\delta_{V,W,R}} \det V \). Thus the assertion holds. \( \square \)

By the above corollary, we obtain the following conclusion:

**Proposition 3.4.** Let \( \phi \in J_{k,1}(\Gamma^J_{n-1}) \) be a Hecke eigenform. If the associated form \( \sigma(\phi) \in M_{k-1/2}^+(\Gamma_0(4)) \) under the Eichler-Zagier-Ibukiyama correspondence possesses a Fourier expansion
\[
\sigma(\phi)(\tau) = \sum_{B \in \text{Sym}_{n-1}^+(\mathbb{Z}_p)} C_{\sigma(\phi)}(B) e(\text{tr}(B \tau)),
\]
then for each \( B \in \text{Sym}_{n-1}(\mathbb{Z})^1 \), we have
\[
\prod_{i=1}^{n-1} \frac{1 - p^{2i}t^2}{(1 - \chi^{(i)}_\phi(p))p^{n-1/2}t(1 - \chi^{(i)}_\phi(p))^{-1}p^{n-1/2}t} C_{\sigma(\phi)}(B)
\]
\[
= \sum_{(\frac{V}{W})} b_p^{(1)}(B[[V^{-1}]]; t) C_{\sigma(\phi)}(B[[V^{-1}][W]]) p^{-(k-n-1)} p^{k \ord_p(\det V)} p^{\ord_p(\det V \det W)},
\]
where \( (\frac{V}{W}) \) runs over the set stated in Lemma 3.3.

**Proof.** We put
\[
\Lambda_p(t) = \sum_{D \in \mathbf{ED}^{(n-1)}(\mathbb{Z})} \Gamma^J_{n-1}(D) \Gamma^J_{n-1} t^{\ord_p(\det D)}.
\]
Then by Corollary of Lemma 3.3, we have
\[
(\phi|_{k,1} \Lambda_p(t))(\tau, z) = \sum_T \sum_r c_\phi(T, r)
\]
\[
\times \sum_{(\frac{V}{W}) \in (1_n \perp \cdot \cdot \cdot \perp 1_n) \setminus \mathbf{ED}^{(n-1)}(\mathbb{Z}) / \text{GL}_{n-1}(\mathbb{Z})} p^{(k-1) \ord_p(\det V) - k \ord_p(\det W)} p^{\ord_p(\det V \det W)}
\]
\[
\times e(\text{tr}(T[[V^{-1}]] \tau + t([V^{-1} V]z))
\]
\[
\times \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}) \setminus (\mathbb{Z}^{p-1})^{\text{Sym}_{n-1}(\mathbb{Z}) W}} e(\text{tr}(T[[W^{-1}]] R)) t^{\ord_p(\mu_p(R))}
\]
\[
\times \sum_{\lambda_1 \in (\mathbb{Z})^{p-1} / p^{\delta_{V,W,R}} Z^{n-1} V} p^{-(2n-3)\delta_{V,W,R}} e(\text{tr}(2\lambda_1 z + t([V^{-1} V + \lambda_1] \lambda_1 \tau))
\]
\[
\times \sum_{\lambda_2 \in (\mathbb{Z})^{p-1} / p^{\delta_{V,W,R}} Z^{n-1} V} e(\text{tr}(t([V^{-1} V + \lambda_1] \lambda_2))).
\]
Since
\[ \sum_{\lambda_2 \in \mathbb{Z}^{n-1}/p^j \delta_{V,W,R} \mathbb{Z}^{n-1}} e(\text{tr}(s(r^t W^{-1} V + \lambda_1) \lambda_2)) = \begin{cases} p^{(n-1) \delta_{V,W,R}} & \text{if } r^t W^{-1} \in \mathbb{Z}^{n-1}, \\ 0 & \text{otherwise}, \end{cases} \]
and
\[ \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} e(\text{tr}(T[r^t W^{-1}] R)) \lambda_{\text{ord}_p(\mu_p(R))} \]
\[ \in \begin{cases} (\det W)^n & \text{if } T[r^t W^{-1}] \in \text{Sym}_{n-1}^*(\mathbb{Z}), \\ 0 & \text{otherwise}, \end{cases} \]
we have
\[ (\phi | k, 1 \Lambda_p(t))(\tau, z) = \sum_T \sum_r \sum_{\binom{V}{W}} p^{k \text{ord}_p(\det V) + (-k+n+1) \text{ord}_p(\det W)} \lambda_{\text{ord}_p(\det V \det W)} \]
\[ \times \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} e(\text{tr}(TR)) (p t)^{\text{ord}_p(\mu_p(R))} \]
\[ \times \sum_{\lambda_1 \in \mathbb{Z}^{n-1}/p^j \delta_{V,W,R} \mathbb{Z}^{n-1}} p^{-(n-1) \delta_{V,W,R}} e_\phi(T[r^t W], r^t W) \]
\[ \times e(\text{tr}(s(r^t V + 2\lambda_1) z)) e(\text{tr}((T[r^t V] + t(r^t V + \lambda_1) \lambda_1) \tau)). \]

For a fixed \( r_0 \in \mathbb{Z}^{n-1} \), we put
\[ S_1(r_0) = \{ \lambda_1 \in \mathbb{Z}^{n-1}/p^j \delta_{V,W,R} \mathbb{Z}^{n-1} | r^t V \equiv r_0 \text{ mod } \mathbb{Z}^{n-1} \}, \]
and
\[ S_2(r_0) = \{ r \in \mathbb{Z}^{n-1}/p^j \delta_{V,W,R} \mathbb{Z}^{n-1} | r^t V \equiv r_0 \text{ mod } 2\mathbb{Z}^{n-1} \}. \]

For each \( \lambda_1 \in S_1(r_0) \), the map \( \lambda_1 \mapsto (2\lambda_1 - r_0)^t V^{-1} \) induces a bijection between \( S_1(r_0) \) and \( S_2(r_0) \). Thus we have
\[ (\phi | k, 1 \Lambda_p(t))(\tau, z) = \sum_T \sum_{r_0} \sum_{\binom{V}{W}} p^{k \text{ord}_p(\det V) + (-k+n+1) \text{ord}_p(\det W)} \lambda_{\text{ord}_p(\det V \det W)} \]
\[ \times \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} e(\text{tr}(TR)) (p t)^{\text{ord}_p(\mu_p(R))} \]
\[ \times \sum_{\lambda_1 \in S_2(r_0)} p^{-(n-1) \delta_{V,W,R}} e_\phi(T[r^t W], r^t W) e(\text{tr}(s(r_0 z))) e(\text{tr}((T[r^t V] + t(r_0 r_0 - t(r^t V)(r^t V))/4) \tau)) \]
\[ = \sum_{T_0} \sum_{r_0} e(\text{tr}(T_0 \tau + t r_0 z)) \sum_{\binom{V}{W}} p^{k \text{ord}_p(\det V) + (-k+n+1) \text{ord}_p(\det W)} p^{-(n-1) \delta_{V,W,R}} \]
\[ \times e_\phi((T_0 - t r_0 r_0/4)[r^t V^{-1}] [r^t W] + (t r r/4)[r^t V], r^t W) \]
\[ \times \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} e(\text{tr}((T_0 - t r_0 r_0/4)[r^t V^{-1}] + t r r/4) R)) (p t)^{\text{ord}_p(\mu_p(R))}. \]

Then for a fixed \( r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1} \), the map
\[ (r + 2\mathbb{Z}^{n-1}) + 2p^j \delta_{V,W,R} \mathbb{Z}^{n-1}/2p^j \delta_{V,W,R} \mathbb{Z}^{n-1} \ni r + 2u \mapsto u \in \mathbb{Z}^{n-1}/p^j \delta_{V,W,R} \mathbb{Z}^{n-1} \]
is a bijection, and we have
\[ c_\phi((T_0 - \gamma_0 r_0/4)[tV^{-1}][W] + (\gamma + 2u)(r + 2u)/4)[W], (r + 2u)^tW) \]
\[ = c_\phi((T_0 - \gamma_0 r_0/4)[tV^{-1}][W] + (\gamma r/4)[tV], r^tW). \]

Thus we have
\[ (\phi|_{k, \Lambda_p}(t))(\tau, z) \]
\[ = \sum_{T_0} \sum_{r_0} \sum_{t} e(\text{tr}(T_0 \tau + t r_0 z)) \sum_{W} p^{k\text{ord}_p(\det V) - (k-n-1)\text{ord}_p(\det W)} \text{ord}_p(\det V) \text{det W} \]
\[ \times \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} (pt)^{\text{ord}_p(\mu_p(R))} \]
\[ \times \sum_{r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}, r \equiv r_0 \mod 2\mathbb{Z}^{n-1}} c_\phi((T_0 - \gamma_0 r_0/4)[tV^{-1}][W] + (\gamma r/4)[tV], r^tW) \]
\[ \times \sum_{u \in \mathbb{Z}[p^{tV,R \mathbb{Z}}^{n-1}]} p^{-(n-1)\delta_{V,R}} e(\text{tr}((T_0 - \gamma_0 r_0/4)[tV^{-1}] + \gamma r/4 + t uu + t ur/2 + t ru/2)R)). \]

We easily see for an element \( r \in \mathbb{Z}^{n-1} \) that the summation
\[ \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} (pt)^{\text{ord}_p(\mu(R))} \sum_{u \in \mathbb{Z}^{n-1}/p^{tV,R \mathbb{Z}}^{n-1}} p^{-(n-1)\delta_{V,R}} e(\text{tr}((T_0 - \gamma_0 r_0/4)[tV^{-1}] + \gamma r/4 + t uu + t ur/2 + t ru/2)R)) \]
equals \( b^{(1)}_p((4T_0 - \gamma_0 r_0)[tV^{-1}]; t) \) or 0 according as \( (T_0 - \gamma_0 r_0/4)[tV^{-1}] + \gamma r/4 \in \text{Sym}_{n-1}(\mathbb{Z}_p) \) or not, namely, according as \( (4T_0 - \gamma_0 r_0/4)[tV^{-1}] \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)} \) or not. In the former case, the vector \( r \) is uniquely determined by \( T_0, r_0, \) and \( V \), which will be denoted by \( r_1 = r_1(T_0, r_0, V) \).

Furthermore we have
\[ (4T_0 - \gamma_0 r_0/4)[tV^{-1}] + \gamma r/4)\text{tr}V) \in \text{Sym}_{n-1}(\mathbb{Z}_p), \]
and we have \( r^tV \equiv r_0 \mod 2\mathbb{Z}^{n-1} \). Thus we have
\[ (\phi|_{k, \Lambda_p}(t))(\tau, z) = \sum_{T_0} \sum_{r_0} \sum_{t} e(\text{tr}(T_0 \tau + t r_0 z)) \sum_{W} p^{k\text{ord}_p(\det V) - (k-n-1)\text{ord}_p(\det W)} \text{ord}_p(\det V) \text{det W} \]
\[ \times b^{(1)}_p((4T_0 - \gamma_0 r_0)[tV^{-1}]; t) c_\phi((T_0 - \gamma_0 r_0/4)[tV^{-1}][W] + (\gamma r_1/4)[tV], r_1^tW). \]

Now we take an element \( B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)} \) so that \( B = 4T_0 - \gamma_0 r_0 \) with \( T_0 \in \text{Sym}_{n-1}(\mathbb{Z}_p) \) and \( r_0 \in \mathbb{Z}^{n-1} \). Then we have
\[ c_\phi(T_0, r_0) = C_{\sigma(\phi)}(B), c_\phi((T_0 - \gamma_0 r_0/4)[tV^{-1}][W] + (\gamma r_1/4)[tV], r_1^tW) = C_{\phi}(B)[tV^{-1}][tW], \]
and
\[ b^{(1)}_p((4T_0 - \gamma_0 r_0)[tV^{-1}]; t) = b^{(1)}_p(B[tV^{-1}]; t). \]

Since \( \phi|_{k, \Lambda_p}(t) = Z_p(t, \phi) \), the assertion follows immediately from Proposition 3.3. \( \square \)

For each \( B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}, \) let \( G_{\phi, p}(B, t) \) be the polynomial in \( t \) defined in §1. Then by making use of the same argument as in [4] combined with Propositions 3.2 and 3.4, we obtain the following:
Theorem 3.1. Let $n$ and $k$ be positive even integers such that $k > n + 1$, and let $\phi \in J_{k,1}(\Gamma_{n-1})$ be a Hecke eigenform with Satake $p$-parameters $(\chi^{(1)}_{\phi}(p), \cdots, \chi^{(n-1)}_{\phi}(p)) \in \mathbb{C}^{n-1}$. Then for each $B \in \text{Sym}_{n-1}(\mathbb{Z})^{(1)}$, we have

$$\prod_{i=1}^{n-1} (1 - \chi_{\phi}^{(i)}(p)p^{n-1/2} t)(1 - \chi_{\phi}^{(i)}(p)^{-1}p^{n-1/2} t) = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \text{D}_p^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W]) p^{-(k-n-1)\text{ord}_p(\det W)}.$$ 

For each $D \in M_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})$, we define the generalized global Möbius function $\pi(D)$ as $\prod_p \pi_p(D)$, where $\pi_p$ is the local Möbius function defined in §1. We easily see that this is a finite product of $\pi_p(D)$. Then for each $B \in \text{Sym}_{n-1}^*(\mathbb{Z})^{(1)}$, we put

$$\widetilde{H}_\phi(B; s) = \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \text{M}_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})} \pi(D) C_{\sigma(\phi)}(B[D^{-1}]) \det D^{-s+k} \quad (s \in \mathbb{C}),$$

which is a finite sum, and we have $\widetilde{H}_\phi(B; s) = \prod_p \tilde{G}_{\phi, p}(B; p^{-s})$. In addition, we also put $B^{(1)}(B; s) = \prod_p B_p^{(1)}(B; p^{-s})$. Then Theorem 3.1 can be restated globally as follows:

Theorem 3.2. Under the same situation as above, we have

$$B^{(1)}(B; s) L(s, \phi, \text{St}) \tilde{H}_\phi(B; s + n - 1/2) = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \text{M}_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})} C_{\sigma(\phi)}(B[W]) (\det W)^{-s-k+3/2}.$$ 

Moreover, by applying Theorem 3.1 to the Jacobi Eisenstein series $\xi^{(n-1)}_{k,1} = e^{(n-1)}_{k,1} \in J_{k,1}(\Gamma_{n-1})$, we obtain the following conclusion:

Theorem 3.3. Let $n$ and $k$ be as above. Then for each $B \in \text{Sym}_{n-1}(\mathbb{Z})^{(1)}$, we have

$$\prod_{i=1}^{n-1} (1 - p^{i-1} p^{-k-(n+1)/2} p^{(n+1)/2} t)(1 - p^{-i} p^{-k-(n+1)/2} p^{(n+1)/2} t) = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \text{D}_p^{(n-1)}(\mathbb{Z})} \tilde{F}_p^{(1)}(B[W]; p^{-k-(n+1)/2} (p^{(n+1)/2} t)^{\text{ord}_p(\det W)}, \tilde{G}_p^{(1)}(B; X, t))$$

where $\tilde{F}_p^{(1)}(B; X)$ and $\tilde{G}_p^{(1)}(B; X, t)$ are polynomials defined in §1.

Proof. By Proposition 2.4, the $B$-th Fourier coefficient of $\sigma(e^{(n-1)}_{k,1}) \in M_{k-1}^+(\Gamma_0^{(n-1)}(4))$ is expressed as

$$\xi(n, k) L(1 - k/2 + n/2, \chi_{\text{St}}) f(B^{(1)}(1))^{-k-(n+1)/2} \prod_{p \mid B^{(1)}} \tilde{F}_p^{(1)}(B; p^{-k-(n+1)/2}).$$

Thus the assertion follows from Theorem 3.1 and Corollary of Proposition 2.2. \qed
For each $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, let $R_p^{(1)}(B; X, t)$ be the formal power series in $X + X^{-1}$ and $t$, which is defined in §1. Then we obtain the rationality for $R_p^{(1)}(B; X, t)$ as follows:

**Theorem 3.4.** Let $n$ be a positive even integer. Then for each $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have

$$R_p^{(1)}(B; X, t) = \frac{B_p^{(1)}(B; p^{n/2-1}t) \tilde{G}_p^{(1)}(B; X, t)}{\prod_{j=1}^{n-1} (1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)}.$$ 

**Proof.** We write the both-hand sides of the above equation as power series in $t$ as

$$R_p^{(1)}(B; X, t) = \sum_{i=1}^{\infty} A_i(X)t^i,$$

and

$$\frac{B_p^{(1)}(B; p^{n/2-1}t) \tilde{G}_p^{(1)}(B; X, t)}{\prod_{j=1}^{n-1} (1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)} = \sum_{i=1}^{\infty} B_i(X)t^i,$$

where for each $i$, $A_i(X)$ and $B_i(X)$ are polynomials in $X + X^{-1}$. Then by Theorem 3.3, we have

$$A_i(p^{k-(n+1)/2}) = B_i(p^{k-(n+1)/2})$$

for infinitely many $k$. Thus we have

$$A_i(X) = B_i(X)$$

for each $i$. Therefore we complete the proof. \(\square\)

**Remark.** For a given pair of positive even integers $n$ and $k$ as in Theorem 3.1, let $f \in S_{2k-n}(\Gamma_1)$ be a Hecke eigenform, which possesses a Fourier expansion

$$f(z) = \sum_{N=1}^{\infty} a_f(N)e(Nz) \quad (z \in \mathcal{H}_1)$$

normalized by $a_f(1) = 1$. For each rational prime $p$, we denote by $\alpha_p$ the Satake $p$-parameter of $f$, that is, an algebraic number determined by $\alpha_p + \alpha_p^{-1} = a_f(p)p^{k-(n+1)/2}$ uniquely up to inversion. Then by substituting $X = \alpha_p$ in the main identity of Theorem 3.4, we can also derive a similar identity to Theorem 3.3 for a power series related to the first Fourier-Jacobi coefficient of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to $f$ under Ikeda’s lifting procedure (cf. [9]). We note that it will play an important role in a proof of Ikeda’s conjecture on the period of such a $F$, which was proposed in [10] (cf. [13, 14]).
References


