ON THE ANDRIANOV TYPE IDENTITY FOR POWER SERIES ATTACHED TO JACOBI FORMS AND ITS APPLICATION

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1. Introduction

The theory of Jacobi forms, namely automorphic forms on the Jacobi group and its generalization to higher degree have been studied by several authors (cf. [6, 27, 18, 19, 8]). In particular, Shintani introduced the standard L-function attached to a Jacobi form of arbitrary degree, and afterward Murase derived in a series of his papers [18, 19] its meromorphic continuation and functional equation by making use of its integral expression. Moreover, Murase and Sugano derived in [20] an expression of the standard L-function attached to a Jacobi form in terms of a power series generated by its eigenvalues of Hecke operators. In this paper, we derive a local expression of the standard L-function attached to a Jacobi form in terms of a power series related to its Fourier coefficients. This can be regarded as an analogue of Andrianov's identity in [1] for Siegel modular forms. As an application, we shall also show the rationality theorem for a formal power series related to a polynomial appearing in the theory of local densities of quadratic forms, which is very similar to the result obtained in [5] by Böcherer and Sato.

Let us describe our main results precisely. Let p be an arbitrary rational prime. For any nonzero element a of the field $\mathbb{Q}_p$ of p-adic numbers, we put

$$\chi_p(a) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is unramified}, \\ 0 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is ramified}. \end{cases}$$

Let n be a positive even integer. For each non-degenerate half-integral symmetric matrix B' of degree n over the ring $\mathbb{Z}_p$ of p-adic integers, we define the local Siegel series with complex parameter s by

$$b_p(B';\, s) := \sum_{R \in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(-B'R))\, \mu_p(R)^{-s},$$

where $\mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$, and $\mathbf{e}_p$ is the standard additive character of $\mathbb{Q}_p$. It is well-known that such a kind of singular series appears naturally within the framework of

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studying Fourier coefficients of the Siegel Eisenstein series of degree n and there exists a unique polynomial $F_p(B';X)$ in one variable X such that

$$b_p(B';\, s) = \frac{(1-p^{-s})\displaystyle\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\xi_p(B')p^{n/2-s}}\, F_p(B';\, p^{-s}),$$

where $\xi_p(B') = \chi_p((-1)^{n/2}\det(2B'))$ (cf. [16]). Let B be a non-degenerate symmetric matrix of degree $n-1$ over a subring R of $\mathbb{Z}_p$ satisfying the condition

(1) $(B + {}^t r_B r_B)/4$ is a half-integral symmetric matrix over R for some $r_B \in R^{n-1}$.

Then we can associate such a B with a non-degenerate half-integral symmetric matrix

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix}$$

of degree n over R. Here we easily see that the vector r_B is uniquely determined by B modulo $2R^{n-1}$, and therefore $B^{(1)}$ is uniquely determined by B up to $\mathrm{GL}_{n-1}(R)$-equivalence. Then for such a B over $\mathbb{Z}_p$, we define a polynomial $F_p^{(1)}(B;\, X)$ in X by

$$F_p^{(1)}(B;\, X) := F_p(B^{(1)};\, X)$$

and put

$$G_p^{(1)}(B;\, X) = \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D)\, F_p^{(1)}(B[D^{-1}];\, X)\, (p^n X^2)^{\mathrm{ord}_p(\det D)},$$

where $\pi_p(D)$ denotes the generalized local Möbius function, that is, $\pi_p(D) = (-1)^i p^{i(i-1)/2}$ or 0 according as $D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\left(\begin{array}{c|c} \mathbf{1}_{n-1-i} & \\ \hline & p\mathbf{1}_i \end{array}\right)\mathrm{GL}_{n-1}(\mathbb{Z}_p)$ for some $0 \le i \le n-1$ or not. We note that these polynomials do not depend on the choice of r_B. In addition, we also define a polynomial $\mathbf{B}_p^{(1)}(B;\, t)$ in one variable t by

$$\mathbf{B}_p^{(1)}(B;\, t) := \frac{(1-\xi_p(B^{(1)})\, p^{-(n-1)/2}\, t)\displaystyle\prod_{i=1}^{n/2-1}(1-p^{-2i+1}t^2)}{G_p^{(1)}(B;\, p^{-n+1/2}\, t)}.$$

On the other hand, for any positive even integers k and n, let ϕ be a Jacobi form of weight k and of index 1 with respect to the Jacobi modular group Γ_{n-1}^J of degree $n-1$, and $\sigma(\phi)$ be a Siegel modular form of weight $k-1/2$ with respect to the congruence subgroup $\Gamma_0^{(n-1)}(4)$ of the Siegel modular group of degree $n-1$ corresponding to ϕ under the Eichler-Zagier-Ibukiyama correspondence σ (cf. §2.3 and 2.4 below). Let $\mathbf{D}_p^{(n-1)}(\mathbb{Z})$ be the set of all $(n-1)\times(n-1)$ matrices with entries in $\mathbb{Z}$ whose determinant is a power of p. For each positive definite half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}$, we define a power series $\widetilde{G}_{\phi,p}(B;\, t)$ in t by

$$\widetilde{G}_{\phi,p}(B;\, t) := \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} \pi_p(D)\, C_{\sigma(\phi)}(B[D^{-1}])\, (p^k\, t)^{\mathrm{ord}_p(\det D)},$$

where $C_{\sigma(\phi)}(B)$ denotes the B-th Fourier coefficient of $\sigma(\phi)$. Then our first main result is the following:

Theorem 1.1 (cf. Theorem 3.1 below)**.** *If ϕ is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators with Satake p-parameters $(\chi_\phi^{(1)}(p), \ldots, \chi_\phi^{(n-1)}(p))$, then for each positive definite half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}$ satisfying the condition* (1)*, we have*

$$\frac{\mathbf{B}_p^{(1)}(B;\, p^{n-1/2}t)\,\widetilde{G}_{\phi,\,p}(B;\, t)}{\prod_{i=1}^{n-1}(1-\chi_\phi^{(i)}(p)p^{n-1/2}t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)}$$
$$= \sum_{W\in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W])\, p^{-(k-n-1)\,\mathrm{ord}_p(\det W)}\, t^{\mathrm{ord}_p(\det W)}.$$

This can be regarded as an analogue of the so-called Andrianov identity, which was obtained within the framework of studying standard L-functions attached to Siegel modular forms of integral weight (cf. [1], see also [4]). We also note that the above identity for $p \neq 2$ can be derived from a similar result for Siegel modular forms of half-integral weight due to Shimura and Zhuravlev (cf. Corollary 5.2 in [23], see also Theorem 1.1 in [26]). However, we cannot use their results to prove the above identity for $p = 2$.

Next, we explain an application of the above result to the rationality of a certain formal power series related to the polynomial $F_p^{(1)}(B;\, X)$. For each non-degenerate half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}_p$ satisfying the condition (1), we define a Laurent polynomial $\widetilde{F}_p^{(1)}(B;\, X)$ in X by

$$\widetilde{F}_p^{(1)}(B;\, X) := X^{-\mathrm{ord}_p((-1)^{n/2}\det(2B^{(1)})\mathfrak{d}(B^{(1)})^{-1})/2} F_p^{(1)}(B;\, p^{-(n+1)/2}X),$$

and put

$$\widetilde{G}_p^{(1)}(B;\, X,\, t) = \sum_{D\in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D)\, \widetilde{F}_p^{(1)}(B[D^{-1}];\, X)\, t^{\mathrm{ord}_p(\det D)},$$

where $\mathfrak{d}(B^{(1)})$ is the discriminant of the quadratic extension $\mathbb{Q}_p\Big(\sqrt{(-1)^{n/2}\det(2B^{(1)})}\Big)/\mathbb{Q}_p$. Then we have a functional equation $\widetilde{F}_p^{(1)}(B;\, X) = \widetilde{F}_p^{(1)}(B;\, X^{-1})$ (cf. [9]). Thus $\widetilde{F}_p^{(1)}(B;\, X)$ is a polynomial in $X + X^{-1}$, and then $\widetilde{G}_p^{(1)}(B;\, X,\, t)$ is a polynomial in $X + X^{-1}$ and t. Now we put

$$R_p^{(1)}(B;\, X,\, t) = \sum_{W\in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \widetilde{F}_p^{(1)}(B[W];\, X)\, t^{\mathrm{ord}_p(\det W)}.$$

Then by applying Theorem 1.1 to the Jacobi Eisenstein series, we obtain the following:

Theorem 1.2 (cf. Theorem 3.4 below)**.** *Let n be a positive even integer. If B is a non-degenerate half-integral symmetric matrix of degree $n-1$ over $\mathbb{Z}_p$ satisfying the condition* (1)*, then we have*

$$R_p^{(1)}(B;\, X,\, t) = \frac{\mathbf{B}_p^{(1)}(B;\, p^{n/2-1}t)\,\widetilde{G}_p^{(1)}(B;\, X,\, t)}{\prod_{j=1}^{n-1}(1-p^{j-1}X\,t)(1-p^{j-1}X^{-1}t)}.$$

We note that Böcherer and Sato ([5]) obtained a similar identity for a half-integral symmetric matrix of degree n. The above identity will play an important role in proving a conjecture on the period of the Ikeda lift proposed in [10] by Ikeda (cf. [13, 14]).

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Notation. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}\,x)$ for any $x \in \mathbb{C}$. For each rational prime p, let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ be the field of p-adic rational numbers and the ring of p-adic integers, respectively. We denote by ord_p the valuation of $\mathbb{Q}_p$ normalized as $\mathrm{ord}_p(p) = 1$, and by $\mathbf{e}_p$ the continuous additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for any $x \in \mathbb{Q}$, which will be called the standard additive character of $\mathbb{Q}_p$. Let R be a commutative ring. We denote by $R^\times$ the the unit group of R. We denote by $\mathrm{M}_{m,n}(R)$ the set of $m \times n$ matrices with entries in R. In particular, we write $\mathrm{M}_n(R) = \mathrm{M}_{n,n}(R)$ and $R^n = \mathrm{M}_{1,n}(R)$. We denote by $\mathbf{1}_n$, $\mathbf{0}_n \in \mathrm{M}_n(R)$ the unit matrix and the zero matrix of degree n, respectively. We put $\mathrm{GL}_n(R) = \{\, U \in \mathrm{M}_n(R) \mid \det U \in R^\times \,\}$, where $\det U$ is the determinant of U. For two matrices $X \in \mathrm{M}_{m,n}(R)$ and $A \in \mathrm{M}_m(R)$, we write $A[X] = {}^tXAX \in \mathrm{M}_n(R)$, where ${}^tX$ denotes the transpose of X. Let $\mathrm{Sym}_n(R)$ be the set of symmetric matrices of degree n with entries in R. If R is an integral domain of characteristic different from 2, let $\mathrm{Sym}_n^*(R)$ be the subset of $\mathrm{Sym}_n(R)$ consisting of all half-integral symmetric matrices of degree n, that is,

$$\mathrm{Sym}_n^*(R) := \left\{ T = (t_{ij}) \in \mathrm{Sym}_n(\mathrm{Frac}(R)) \;\middle|\; \begin{array}{ll} t_{ii} \in R & (1 \le i \le n), \\ 2\,t_{ij} \in R & (1 \le i \ne j \le n) \end{array} \right\},$$

where $\mathrm{Frac}(R)$ is the field of fractions of R. In addition, for any subset $\mathcal{S}$ of $\mathrm{Sym}_n(R)$, we denote by $\mathcal{S}^\times$ the subset of $\mathcal{S}$ consisting of all non-degenerate elements in $\mathcal{S}$. In particular, if R is a subring of $\mathbb{R}$, we denote by $\mathcal{S}_{>0}$ (resp. $\mathcal{S}_{\ge 0}$) the subset of $\mathcal{S}$ consisting of all positive definite (resp. semi-positive definite) matrices. For any commutative ring R, the group $\mathrm{GL}_n(R)$ acts on the set $\mathrm{Sym}_n(R)$ in the following way:

$$\mathrm{GL}_n(R) \times \mathrm{Sym}_n(R) \ni (U,\, A) \longmapsto A[U] \in \mathrm{Sym}_n(R).$$

For a subgroup G of $\mathrm{GL}_n(R)$, and a subset $\mathcal{S}$ of $\mathrm{Sym}_n(R)$ stable under the action of G, we denote by $\mathcal{S}/G$ the set of G-orbits in $\mathcal{S}$. We define an equivalence relation on $\mathrm{Sym}_n(R)$ over a subring R' of R as follows: for any $A_1,\, A_2 \in \mathrm{Sym}_n(R)$,

$$A_1 \sim_{R'} A_2 \overset{\mathrm{def}}{\Longleftrightarrow} A_2 = A_1[U] \text{ for some } U \in \mathrm{GL}_n(R'). \tag{2}$$

For two square matrices $X \in \mathrm{M}_m(R)$ and $Y \in \mathrm{M}_n(R)$, we write $X \perp Y = \left(\begin{smallmatrix} X & \\ & Y \end{smallmatrix}\right)$. In particular, we often write $x \perp Y$ instead of $(x) \perp Y$ for any $x \in R$. Then we can simply write the diagonal matrix with entries $x_1, \cdots, x_n$ in R by $x_1 \perp \cdots \perp x_n$.

2. Preliminaries

2.1. Siegel modular forms of integral weight.

Let $G_n(\mathbb{R})$ be the real symplectic group of degree n, that is,

$$G_n(\mathbb{R}) := \mathrm{Sp}_n(\mathbb{R}) = \{\, M \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^tMJ_nM = J_n \,\},$$

where $J_n = \left(\begin{smallmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{smallmatrix}\right)$. For any $S \in \mathrm{Sym}_n(\mathbb{R})$ and $A \in \mathrm{GL}_n(\mathbb{R})$, we put $\mathbf{n}_n(S) = \left(\begin{smallmatrix} \mathbf{1}_n & S \\ \mathbf{0}_n & \mathbf{1}_n \end{smallmatrix}\right)$ and $\mathbf{d}_n(A) = \left(\begin{smallmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{smallmatrix}\right)$, respectively. Then we easily see that these elements $\mathbf{n}_n(S)$, $\mathbf{d}_n(A)$ and J_n generate $G_n(\mathbb{R})$. The discrete subgroup $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z}) = G_n(\mathbb{R}) \cap \mathrm{M}_{2n}(\mathbb{Z})$ of $G_n(\mathbb{R})$ is called the *Siegel modular group* of degree n. For any $N \in \mathbb{Z}_{>0}$, we denote by $\Gamma_0^{(n)}(N)$ the congruence subgroup of Γ_n defined by

$$\Gamma_0^{(n)}(N) := \left\{\, \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid C \equiv \mathbf{0}_n \pmod{N} \,\right\}.$$

We denote the Siegel upper-half space of degree n by $\mathfrak{H}_n$, that is,

$$\mathfrak{H}_n := \{\, Z = X + \sqrt{-1}\, Y \in \mathrm{Sym}_n(\mathbb{C}) \mid Y > 0 \ \text{(positive definite)} \,\}.$$

For any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, we easily see that $j(M,\, Z) := CZ + D \in \mathrm{GL}_n(\mathbb{C})$ and then we put $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$. As is well-known, this defines a transitive action of $G_n(\mathbb{R})$ on $\mathfrak{H}_n$.

For any $k \in \mathbb{Z}$, a $\mathbb{C}$-valued holomorphic function $F(Z)$ on $\mathfrak{H}_n$ is called a (*holomorphic*) *Siegel modular form* of degree n and weight k if it satisfies the following two conditions:

(i) $F(M\langle Z\rangle) = \det(j(M,\, Z))^k F(Z)$ for any $M \in \Gamma_n$;

(ii) F possesses a Fourier expansion of the form

$$F(Z) = \sum_{B \in \mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} A_F(B)\, \mathbf{e}(\mathrm{tr}(BZ)),$$

where tr denotes the trace of a matrix. If F satisfies the stronger condition $A_F(B) = 0$ unless $B > 0$ (positive definite), then it is called a *cusp form*.

We denote by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ the $\mathbb{C}$-vector spaces consisting of all (holomorphic) Siegel modular forms and Siegel cusp forms of degree n and weight k, respectively. For further details on the facts of Siegel modular forms of integral weight set out above, see [1] or [7].

2.2. Review of the theory of Jacobi forms of higher degree.

In this paragraph, we introduce some basic facts on Jacobi forms of integral weight whose index is a scalar. For further details on generalities of Jacobi forms, see [6, 18, 19, 27].

2.2.1. *Jacobi group and complex analytic Jacobi forms.*

Let $G_n = \mathrm{Sp}_n(\mathbb{Q}) = \{\, M \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid {}^tMJ_nM = J_n \,\}$, and we naturally identify G_n with its image under the natural inclusion

$$G_n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto [M] := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{array}\right) \in G_{n+1}.$$

We denote by H_n the Heisenberg group, that is,

$$H_n = \left\{ [(\lambda,\mu),\kappa] := \left(\begin{array}{cc|cc} 1 & 0 & \kappa & \mu \\ 0 & \mathbf{1}_n & {}^t\mu & \mathbf{0}_n \\ \hline & & 1 & 0 \\ & & 0 & \mathbf{1}_n \end{array}\right) \left(\begin{array}{cc|cc} 1 & \lambda & & \\ 0 & \mathbf{1}_n & & \\ \hline & & 1 & 0 \\ & & -{}^t\lambda & \mathbf{1}_n \end{array}\right) \;\middle|\; \begin{array}{l} (\lambda,\mu) \in \mathbb{Q}^n \oplus \mathbb{Q}^n, \\ \kappa \in \mathbb{Q} \end{array} \right\}.$$

Then $G_n^J := \{ [(\lambda,\mu),\,\kappa]\cdot[M] \in G_{n+1} \mid [(\lambda,\mu),\,\kappa] \in H_n,\ M \in G_n \}$ is a $\mathbb{Q}$-algebraic subgroup of G_{n+1} and is called the *Jacobi group* of degree n. We note that the Jacobi group G_n^J is a semi-direct product $G_n \ltimes H_n$ of H_n and G_n, and forms a connected non-reductive $\mathbb{Q}$-algebraic group with the center

$$Z_n^J = \{ [(0,0),\kappa] \mid \kappa \in \mathbb{Q} \}.$$

Then we have the following:

Lemma 2.1. *For each* $[(\lambda,\mu),\,\kappa]$, $[(\lambda',\mu'),\,\kappa'] \in H_n$, *and* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n$, *we have*

$$[(\lambda,\mu),\,\kappa]\cdot[(\lambda',\mu'),\,\kappa'] = [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+2\lambda^t\mu'], \tag{3}$$

$$[(\lambda,\mu),\,\kappa]\cdot[M] = [M]\cdot[(\lambda A+\mu C,\lambda B+\mu D),\kappa+(\lambda A+\mu C)^t(\lambda B+\mu D)-\lambda^t\mu]. \tag{4}$$

Proof. Since it is an easy calculation, we omit the proof. □

According to the action of $G_{n+1}(\mathbb{R}) = \mathrm{Sp}_{n+1}(\mathbb{R})$ on the Siegel upper-half space $\mathfrak{H}_{n+1}$, the group $G_n^J(\mathbb{R})$ of real points of G_n^J naturally acts on the space $\mathfrak{H}_n \times \mathbb{C}^n$ as follows. For each $g = [(\lambda,\mu),\,\kappa]\cdot[M] \in G_n^J(\mathbb{R})$ with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$ and $(\tau,\,z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we put

$$g\langle\tau,\,z\rangle := (M\langle\tau\rangle,\,z(C\tau+D)^{-1}+\lambda\,M\langle\tau\rangle+\mu).$$

Here we easily see that this action is transitive and the stabilizer of the point $(\sqrt{-1}\,\mathbf{1}_n,\,0) \in \mathfrak{H}_n \times \mathbb{C}^n$ in $G_n^J(\mathbb{R})$ coincides with $Z_n^J(\mathbb{R})\cdot K_\infty$, where K_∞ is the stabilizer of $\sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$ in $G_n(\mathbb{R})$, that is,

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G_n(\mathbb{R}) \;\middle|\; A+\sqrt{-1}\,B \ \text{ is unitary} \right\}.$$

The map $g \mapsto g\langle\sqrt{-1}\,\mathbf{1}_n,\,0\rangle$ induces a diffeomorphism of $G_n^J(\mathbb{R})/(Z_n^J(\mathbb{R})\cdot K_\infty)$ onto $\mathfrak{H}_n \times \mathbb{C}^n$.

Let l and m be non-negative integers. For any $\mathbb{C}$-valued function $\phi(\tau,\,z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$, we define the action of $g \in G_n^J(\mathbb{R})$ on ϕ by

$$(\phi|_{l,\,m}\,g)(\tau,\,z) := J_{l,\,m}(g,\,(\tau,\,z))^{-1}\,\phi\,(g\langle\tau,\,z\rangle)\,,$$

where for $g = [(\lambda,\mu),\,\kappa]\cdot[M]$, we put

$$\begin{aligned} J_{l,\,m}(g,\,(\tau,\,z)) &:= \det(C\tau+D)^l \\ &\quad\times\mathbf{e}(-m\kappa-m\tau[{}^t\lambda]-2m\lambda^t z-m\lambda\,{}^t\mu+m\{(C\tau+D)^{-1}C\}[{}^t(z+\lambda\tau+\mu)]). \end{aligned}$$

It is easy to see that for any $g_i \in G_n^J(\mathbb{R})$ $(i=1,\,2)$,

$$(\phi|_{l,\,m}\,g_1)|_{l,\,m}\,g_2 = \phi|_{l,\,m}\,(g_1g_2).$$

In particular, it follows from Lemma 2.1 that for any $M,\ M' \in G_n(\mathbb{R})$ and $[(\lambda,\mu),\ \kappa],\ [(\lambda',\mu'),\ \kappa'] \in H_n(\mathbb{R})$, we have

$$\begin{cases} \phi|_{l,m}[M]\,|_{l,m}[M'] = \phi|_{l,m}[MM'], \\ \phi|_{l,m}[(\lambda,\mu),\ \kappa]\,|_{l,m}[(\lambda',\mu'),\ \kappa'] = \phi|_{l,m}[(\lambda+\lambda',\mu+\mu'),\ \kappa+\kappa'+2\lambda^t\mu'], \\ \phi|_{l,m}[M]\,|_{l,m}[(\lambda,\mu)M,\ \kappa+(\lambda,\mu)M\left(\begin{smallmatrix}\mathbf{0}_n & \mathbf{1}_n \\ \mathbf{0}_n & \mathbf{0}_n\end{smallmatrix}\right){}^tM\,{}^t(\lambda,\mu)-\lambda^t\mu] = \phi|_{l,m}[(\lambda,\mu),\ \kappa]\cdot[M]. \end{cases}$$

We also define a subgroup of $G_n^J(\mathbb{R})$ by $\Gamma_n^J := \Gamma_n \ltimes H_n(\mathbb{Z})$, where $H_n(\mathbb{Z})$ is a subgroup of $H_n(\mathbb{R})$ consisting of all elements with integral entries.

Let l and m be positive integers. A holomorphic function $\phi(\tau,\ z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$ is called a (*holomorphic*) *Jacobi form* of degree n, weight l and index m if it satisfies the following two conditions:

(i) $\phi|_{l,m}\gamma = \phi$ for any $\gamma \in \Gamma_n^J$;
(ii) ϕ possesses a Fourier expansion of the form

$$\phi(\tau,\ z) = \sum_{T\in \mathrm{Sym}_n^*(\mathbb{Z}),\, r\in\mathbb{Z}^n} c_\phi(T,\ r)\,\mathbf{e}(\mathrm{tr}(T\tau)+r\,{}^tz)$$

with $c_\phi(T,\ r)=0$ unless $4mT-{}^trr \geq 0$. If ϕ satisfies the stronger condition $c_\phi(T,\ r)=0$ unless $4mT-{}^trr>0$, then it is called *cuspidal*.

We denote by $J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$ the $\mathbb{C}$-vector spaces consisting of all (holomorphic) Jacobi forms and cuspidal Jacobi forms of degree n, weight l and index m, respectively.

As an important example of Jacobi form, we consider Fourier-Jacobi coefficients of Siegel modular forms of arbitrary degree $n>1$. For any $k\in\mathbb{Z}$, let $F\in M_k(\Gamma_n)$ possess a Fourier expansion

$$F(Z) = \sum_{B'\in\mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} A_F(B')\,\mathbf{e}(\mathrm{tr}(B'Z)) \quad (Z\in\mathfrak{H}_n),$$

and we put $Z = \begin{pmatrix} \tau' & z \\ {}^tz & \tau \end{pmatrix}$ with $\tau\in\mathfrak{H}_{n-1}$, $z\in\mathbb{C}^{n-1}$ and $\tau'\in\mathfrak{H}_1$. Then we have the so-called Fourier-Jacobi expansion

$$F\left(\begin{pmatrix} \tau' & z \\ {}^tz & \tau \end{pmatrix}\right) = \sum_{m=0}^{\infty}\phi_m(\tau,\ z)\,\mathbf{e}(m\tau'),$$

where

$$\phi_m(\tau,\ z) = \sum_{\substack{T\in\mathrm{Sym}_{n-1}^*(\mathbb{Z}),\, r\in\mathbb{Z}^{n-1}, \\ 4mT-{}^trr\geq 0}} A_F\left(\begin{pmatrix} m & r/2 \\ {}^tr/2 & T\end{pmatrix}\right)\mathbf{e}(\mathrm{tr}(T\tau)+r\,{}^tz). \tag{5}$$

We easily see that the m-th coefficient $\phi_m \in J_{k,m}(\Gamma_{n-1}^J)$ for each $m\in\mathbb{Z}_{>0}$. In particular, if $F\in S_k(\Gamma_n)$, then $\phi_m\in J_{k,m}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$.

As another example, if k is an even integer such that $k>n+1$, then for each $m\in\mathbb{Z}_{>0}$, we define *the Jacobi Eisenstein series* of degree $n-1$, weight k and index m by

$$\mathfrak{E}_{k,m}^{(n-1)}(\tau,\ z) := \sum_{\gamma\in P_{n-1}^J\cap\Gamma_{n-1}^J\backslash\Gamma_{n-1}^J} J_{k,m}(\gamma,\ (\tau,\ z)) \quad (\tau\in\mathfrak{H}_{n-1},\ z\in\mathbb{C}^{n-1}),$$

where

$$P_{n-1}^J := \left\{ [(\lambda,\, \mu),\, \kappa] \cdot [\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)] \in G_{n-1}^J \mid C = \mathbf{0}_{n-1},\, \lambda = 0 \right\}.$$

We easily see that the right-hand side of the above definition is absolutely convergent and $\mathfrak{E}_{k,\,m}^{(n-1)} \in J_{k,\,m}(\Gamma_{n-1}^J)$. Moreover, Böcherer ([3]) showed that for any $m \in \mathbb{Z}_{>0}$, there exists a certain relation between $\mathfrak{E}_{k,\,m}^{(n-1)}$ and the m-th coefficient $e_{k,\,m}^{(n-1)}$ of the above Fourier-Jacobi expansion of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$. In particular, when $m = 1$, we have $\mathfrak{E}_{k,\,1}^{(n-1)} = e_{k,\,1}^{(n-1)}$.

For the purpose of subsequent use, we give an explicit formula for the Fourier coefficients of $e_{k,\,1}^{(n-1)}$ in case n is even. Let k be a positive even integer such that $k > n+1$. Then the Siegel Eisenstein series $E_k^{(n)}$ of weight k with respect to Γ_n is defined by

$$E_k^{(n)}(Z) = \sum_{(C,\,D)} \det(CZ + D)^{-k} \quad (Z \in \mathfrak{H}_n)$$

where $(C,\, D)$ runs through a complete set of representatives of the equivalence classes of coprime symmetric pairs of size n. For each positive definite half-integral symmetric matrix B' of degree n, we denote by $\mathfrak{d}(B')$ the discriminant of the quadratic extension $\mathbb{Q}(\sqrt{(-1)^{n/2}\det(2B')})/\mathbb{Q}$ and put $\mathfrak{f}(B') = \sqrt{(-1)^{n/2}\det(2B')/\mathfrak{d}(B')}$. It is well-known that $\mathfrak{f}(B') \in \mathbb{Z}_{>0}$. Furthermore, we denote by $\chi_{B'}$ the Kronecker character corresponding to the above field extension. Then for each $B' \in \mathrm{Sym}_n^*(\mathbb{Z})_{>0}$, the B'-th Fourier coefficient $A_k^{(n)}(B')$ of $E_k^{(n)}$ is described as

$$A_k^{(n)}(B') = \xi(n,\, k) L(1 - k/2 + n/2, \chi_{B'})\, \mathfrak{f}(B')^{k-(n+1)/2} \prod_{p \,|\, \mathfrak{f}(B')} \widetilde{F}_p(B';\, p^{k-(n+1)/2}), \tag{6}$$

where $\xi(n,\, k) = 2^{n/2}\zeta(1-k)^{-1}\prod_{i=1}^{n/2}\zeta(1+2i-2k)^{-1}$, $L(s, \chi_{B'})$ denotes the Dirichlet L-function associated with $\chi_{B'}$, and

$$\widetilde{F}_p(B';\, X) = X^{-\mathrm{ord}_p(\mathfrak{f}(B'))} F_p(B';\, p^{-(n+1)/2}X).$$

We note that if $B \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_{>0}$ satisfies the condition (1), then $\widetilde{F}_p^{(1)}(B;\, X) = \widetilde{F}_p(B^{(1)};\, X)$. Thus we have

Proposition 2.1. *Under the same assumption as above, let $e_{k,\,1}^{(n-1)}$ possess a Fourier expansion*

$$e_{k,\,1}^{(n-1)}(\tau,\, z) = \sum_{T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z}),\, r \in \mathbb{Z}^{n-1}} c_{k,\,1}^{(n-1)}(T,\, r)\, \mathbf{e}(\mathrm{tr}(T\tau) + r\, {}^t z).$$

Then for each $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})$ such that $B_T = 4T - {}^t rr > 0$ with $r \in \mathbb{Z}^{n-1}$, we have

$$c_{k,\,1}^{(n-1)}(T,\, r) = \xi(n,\, k) L(1 - k + n/2, \chi_{B_T^{(1)}})\, \mathfrak{f}(B_T^{(1)})^{k-(n+1)/2} \prod_{p \,|\, \mathfrak{f}(B_T^{(1)})} \widetilde{F}_p^{(1)}(B_T;\, p^{k-(n+1)/2}),$$

where $B_T^{(1)} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (B_T + {}^t rr)/4 \end{pmatrix} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix} \in \mathrm{Sym}_n^*(\mathbb{Z})_{>0}$.

Proof. Since

$$c_{k,1}^{(n-1)}(T,\, r) = A_k^{(n)}(B_T^{(1)}),$$

the assertion immediately follows form the equation (6). □

Returning to the general theory of Jacobi forms, now we consider the action of Hecke operators on Jacobi forms. Let $M \in \mathrm{Sp}_n(\mathbb{Q})$ and decompose the double coset $\Gamma_n^J M \Gamma_n^J$ into the disjoint right cosets:

$$\Gamma_n^J M \Gamma_n^J = \bigsqcup_{i=1}^{d} \Gamma_n^J\, g_i,$$

where we denote by d the number of right cosets, that is, $d = [\Gamma_n^J M \Gamma_n^J : \Gamma_n^J]$. Then for any $\phi \in J_{l,\,m}(\Gamma_n^J)$, we define the action of the double coset $\Gamma_n^J M \Gamma_n^J$ on ϕ by

$$\phi|_{l,\,m}\, \Gamma_n^J M \Gamma_n^J := \sum_{i=1}^{d} \phi|_{l,\,m}\, g_i,$$

where the summation on the right hand side of the above is well-defined. We easily see that for any $\gamma \in \Gamma_n^J$,

$$(\phi|_{l,\,m}\, \Gamma_n^J M \Gamma_n^J)\,|_{l,\,m}\, \gamma = \phi|_{l,\,m}\, \Gamma_n^J M \Gamma_n^J,$$

that is, $\phi|_{l,\,m}\, \Gamma_n^J M \Gamma_n^J \in J_{l,\,m}(\Gamma_n^J)$. Moreover, if $\phi \in J_{l,\,m}^{\mathrm{cusp}}(\Gamma_n^J)$, then $\phi|_{l,\,m}\, \Gamma_n^J M \Gamma_n^J \in J_{l,\,m}^{\mathrm{cusp}}(\Gamma_n^J)$. Here we note that each of the double cosets $\Gamma_n^J M \Gamma_n^J$ with $M \in G_n(\mathbb{Q})$ contains a unique representative of the form

$$\mathbf{d}_n(\delta_1 \perp \cdots \perp \delta_n) = (\delta_1 \perp \cdots \perp \delta_n) \perp (\delta_1^{-1} \perp \cdots \perp \delta_n^{-1})$$

with $0 < \delta_1 \,|\, \cdots \,|\, \delta_n$. Moreover, let $D = \delta_1 \perp \cdots \perp \delta_n$ and $D' = \delta_1' \perp \cdots \perp \delta_n'$ be two diagonal matrices with $0 < \delta_1 \,|\, \cdots \,|\, \delta_n$, $0 < \delta_1' \,|\, \cdots \,|\, \delta_n'$. We easily see that if $(\delta_n,\, \delta_n') = 1$, then for any $\phi \in J_{l,\,m}(\Gamma_n^J)$,

$$\phi|_{l,\,m}\, \Gamma_n^J \mathbf{d}_n(DD')\Gamma_n^J = \phi|_{l,\,m}\, \Gamma_n^J \mathbf{d}_n(D)\Gamma_n^J\,|_{l,\,m}\, \Gamma_n^J \mathbf{d}_n(D')\Gamma_n^J.$$

A Jacobi form $\phi \in J_{l,\,1}(\Gamma_n^J)$ is called a *Hecke eigenform* if it is a common eigenfunction of all actions of double cosets $\Gamma_n^J M \Gamma_n^J$ with $M \in G_n(\mathbb{Q})$, that is, for any $M \in G_n(\mathbb{Q})$, the equation

$$\phi|_{l,\,m}\, \Gamma_n^J M \Gamma_n^J = \lambda_\phi(M)\, \phi$$

holds for some $\lambda_\phi(M) \in \mathbb{C}$. We easily see from the above argument that ϕ is a Hecke eigenform if and only if it satisfies for any rational prime p and $D = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \mathbf{D}_p^{(n)}(\mathbb{Z})$ with $0 \le \alpha_1 \le \cdots \le \alpha_n$,

$$\phi|_{l,\,m}\, \Gamma_n^J \mathbf{d}_n(D)\Gamma_n^J = \lambda_\phi(D)\, \phi$$

with $\lambda_\phi(D) \in \mathbb{C}$.

2.2.2. *Jacobi forms on the adele group.*

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and let $\Psi_{\mathbb{A}}$ be the character of $\mathbb{Q}\backslash\mathbb{A}$ such that $\Psi_{\mathbb{A}}(x_\infty) = \mathbf{e}(x_\infty)$ for any $x_\infty \in \mathbb{R}$. In addition, for each $m \in \mathbb{Z}$, we put $\Psi_{\mathbb{A}}^m(\kappa) = \Psi_{\mathbb{A}}(m\kappa)$ for any $\kappa \in \mathbb{A}$. We denote by $G_n^J(\mathbb{A})$ the adele group of the Jacobi group G_n^J defined in the previous paragraph. Then it follows from the strong approximation theorem for G_n^J that

$$G_n^J(\mathbb{A}) = G_n^J(\mathbb{Q})G_n^J(\mathbb{R})K_{\mathrm{fin}}^J,$$

where $K_{\mathrm{fin}}^J := \prod_{p<\infty} G_n^J(\mathbb{Z}_p)$.

Let l and m be positive integers. A $\mathbb{C}$-valued function f on $G_n^J(\mathbb{A})$ is called a *Jacobi form* of weight l and index m if it satisfies the following two conditions:

(i) The functional equation

$$f([(0,0),\kappa]\,\gamma\, g\, k_\infty\, k_{\mathrm{fin}}) = \det(j(k_\infty,\, \sqrt{-1}\,\mathbf{1}_n))^{-l}\, \Psi_{\mathbb{A}}^m(\kappa) f(g)$$

holds for any $\kappa \in \mathbb{A}$, $\gamma \in G_n^J(\mathbb{Q})$, $g \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\mathrm{fin}} \in K_{\mathrm{fin}}^J$;

(ii) For any $(\tau,\, z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we choose and fix an element $g_\infty \in G_n^J(\mathbb{R})$ such that $g_\infty\langle\sqrt{-1}\,\mathbf{1}_n,\, 0\rangle = (\tau,\, z)$ and put

$$\Phi_f(\tau,\, z) := J_{l,m}(g_\infty,\, (\sqrt{-1}\,\mathbf{1}_n,\, 0)) f(g_\infty), \tag{7}$$

with the factor of automorphy $J_{l,m} : G_n^J(\mathbb{R}) \times (\mathfrak{H}_n \times \mathbb{C}^n) \to \mathbb{C}$ defined in §2.2.1. Here we easily see that the value Φ_f does not depend on the choice of g_∞. Then the function Φ_f is holomorphic on $\mathfrak{H}_n \times \mathbb{C}^n$. In particular, if it satisfies the further condition that

$$\left|\det(\mathrm{Im}(\tau))^{l/2} \exp(-2m\pi \,\mathrm{tr}\left(\mathrm{Im}(\tau)^{-1}[{}^t\,\mathrm{Im}(z)]\right))\Phi_f(\tau,\, z)\right| \text{ is bounded on } \mathfrak{H}_n \times \mathbb{C}^n,$$

then it is called *cuspidal*.

We denote by $J_{l,m}(G_n^J(\mathbb{A}))$ and $J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A}))$ the $\mathbb{C}$-vector spaces of the Jacobi forms and cuspidal Jacobi forms of weight l and index m on the group $G_n^J(\mathbb{A})$, respectively.

It is easy to see that for each $f \in J_{l,m}(G_n^J(\mathbb{A}))$, the associated function Φ_f is an element of $J_{l,m}(\Gamma_n^J)$. In particular, if $f \in J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A}))$, then $\Phi_f \in J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$. Furthermore we have

Lemma 2.2. *The map $J_{l,m}(G_n^J(\mathbb{A})) \ni f \mapsto \Phi_f \in J_{l,m}(\Gamma_n^J)$ induces $\mathbb{C}$-linear isomorphisms $J_{l,m}(G_n^J(\mathbb{A})) \cong J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A})) \cong J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$.*

Proof. Since it is straightforward, we omit the proof. $\square$

2.3. **Standard L-functions attached to Jacobi forms.**

We study in this paragraph Shintani's standard L-functions attached to Jacobi forms. In particular, we derive an explicit formula for the standard L-function attached to the Jacobi Eisenstein series of index 1. It might be given in a classical way, but here we treat it adelically.

Let p be an arbitrary rational prime. For simplicity, we write G_p^J, G_p, K_p^J, K_p and Z_p^J instead of $G_n^J(\mathbb{Q}_p)$, $G_n(\mathbb{Q}_p)$, $G_n^J(\mathbb{Z}_p)$, $G_n(\mathbb{Z}_p)$ and $Z_n^J(\mathbb{Q}_p)$, respectively. We denote by Ψ_p and $|*|_p$ the restriction of $\Psi_{\mathbb{A}}$ to $\mathbb{Q}_p$ and the p-adic valuation of $\mathbb{Q}_p$ normalized as $|p|_p = p^{-1}$, respectively. Let $\mathscr{H}_p = \mathscr{H}(G_p^J, K_p^J; \Psi_p)$ be the $\mathbb{C}$-module consisting of $\mathbb{C}$-valued functions φ on G_p^J satisfying the following two conditions:

(i) The equation
$$\varphi([(0,0),\kappa]\, k\, g\, k') = \Psi_p(\kappa)\varphi(g)$$
holds for any $\kappa \in \mathbb{Q}_p$, k, $k' \in K_p^J$ and $g \in G_p^J$;

(ii) φ is compactly supported modulo Z_p^J.

Then $\mathscr{H}_p$ forms a $\mathbb{C}$-algebra via the convolution product
$$(\varphi_1 * \varphi_2)(g) := \int_{Z_p^J\backslash G_p^J} \varphi_1(gx^{-1})\varphi_2(x)dx,$$
where dx is a Haar measure on $Z_p^J\backslash G_p^J$ normalized by $\displaystyle\int_{Z_p^J\backslash Z_p^J K_p^J} dx = 1$. The algebra $\mathscr{H}_p$ is called the *Hecke algebra* of (G_p^J, K_p^J) with respect to the additive character Ψ_p.

We put
$$N_p^J := \left\{ [(0,\mu),0]\, \mathbf{d}_n(A)\, \mathbf{n}_n(S) \in G_p^J \;\middle|\; \mu \in \mathbb{Q}_p^n,\ A \in U_{n,p},\ S \in \mathrm{Sym}_n(\mathbb{Q}_p) \right\},$$
$$T_p = T(\mathbb{Q}_p) := \{\, \mathbf{d}_n(t_1 \perp \cdots \perp t_n) \in G_p \mid t_i \in \mathbb{Q}_p^\times \,\}$$
and $T(\mathbb{Z}_p) := T_p \cap K_p$, where $U_{n,p} \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is the group of upper unipotent matrices. We fix Haar measures $d\mathbf{n}$ and dt on N_p^J and T_p respectively normalized by
$$\int_{N_p^J\cap K_p^J} d\mathbf{n} = 1 \ \text{ and } \ \int_{T(\mathbb{Z}_p)} dt = 1.$$
We define the module $\delta_{N_p^J}(t)$ of $t \in T_p$ to be the ratio $d(t\mathbf{n}t^{-1})/d\mathbf{n}$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, we put
$$\pi_\alpha = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \mathrm{GL}_n(\mathbb{Q}_p),$$
then we easily see that
$$\delta_{N_p^J}(\pi_\alpha) = p^{-\sum_{i=1}^n (2n+3-2i)\alpha_i}.$$

Let $X_0(T_p)$ be the group of unramified characters of T_p, that is,
$$X_0(T_p) := \{\, \chi \in \mathrm{Hom}(T_p, \mathbb{C}^\times) \mid \chi \text{ is trivial on } T(\mathbb{Z}_p) \,\}.$$
In particular, if $n = 1$, then $X_0(T_p)$ coincides with the group $X_0(\mathbb{Q}_p^\times)$ consisting of all unramified characters of $\mathbb{Q}_p^\times$. For any $\chi \in X_0(T_p)$ and $\varphi \in \mathscr{H}_p$, we define the *zonal spherical function* $\widehat{\omega}_\chi(\varphi)$ by
$$\widehat{\omega}_\chi(\varphi) := \sum_{\alpha \in \mathbb{Z}^n} \chi^{-1}(\mathbf{d}_n(\pi_\alpha))\widetilde{\varphi}(\mathbf{d}_n(\pi_\alpha)),$$
where
$$\widetilde{\varphi}(t) := \delta_{N,p}^J(t)^{-1/2} \int_{N_p^J} \varphi(\mathbf{n}t)d\mathbf{n} \quad (t \in T_p).$$

It is shown by Murase that the map $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ gives a $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$ and that every $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$ is given by $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ for some $\chi \in X_0(T_p)$ (cf. Proposition 4.10 and Theorem 4.15 in [18]).

On the other hand, for any $\chi \in X_0(T_p)$, let ϕ_χ be a $\mathbb{C}$-valued function on G_p^J defined by

$$\phi_\chi([(0,0),\kappa]\,\mathbf{n}\,t\,[(\lambda,0),0]\,k) = \Psi_p(\kappa)\,(\chi\,\delta_{N_p^J}^{-1/2})(t)\,\mathrm{char}_{\mathbb{Z}_p^n}(\lambda)$$

for any $\kappa \in \mathbb{Q}_p$, $\mathbf{n} \in N_p^J$, $t \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K_p^J$, where we denote by $\mathrm{char}_{\mathbb{Z}_p^n}$ the characteristic function of $\mathbb{Z}_p^n$. Here we note that each $\chi \in X_0(T_p)$ can be written in the form

$$\chi(\mathbf{d}_n(t_1 \perp \cdots \perp t_n)) = \chi^{(1)}(t_1)\cdots\chi^{(n)}(t_n),$$

with uniquely determined n unramified characters $\chi^{(1)}, \cdots, \chi^{(n)} \in X_0(\mathbb{Q}_p^\times)$. In this case, we simply write $\chi = (\chi^{(1)}, \cdots, \chi^{(n)})$. If $\chi = (\chi^{(1)}, \cdots, \chi^{(n)}) \in X_0(T_p)$, then it satisfies that

$$\phi_\chi([(0,0),\kappa]\,\mathbf{n}\,t\,[(\lambda,0),0]\,k) = \Psi_p(\kappa)\prod_{i=1}^{n}\chi^{(i)}(t_i)\,|t_i|_p^{(2n+3-2i)/2}\mathrm{char}_{\mathbb{Z}_p^n}(\lambda) \tag{8}$$

for any $\kappa \in \mathbb{Q}_p$, $\mathbf{n} \in N_p^J$, $t = \mathbf{d}_n(t_1 \perp \cdots \perp t_n) \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K_p^J$.

For each rational prime p, we define the action of Hecke algebra $\mathscr{H}_p$ on the space $J_{l,1}(G_n^J(\mathbb{A}))$ by the following: for any $f \in J_{l,1}(G_n^J(\mathbb{A}))$ and $\varphi \in \mathscr{H}_p$,

$$(f * \varphi)(g) := \int_{Z_p^J\backslash G_p^J} f(gx^{-1})\varphi(g^{-1})dx \quad (g \in G_n^J(\mathbb{A})).$$

A Jacobi form $f \in J_{l,1}(G_n^J(\mathbb{A}))$ is called a *Hecke eigenform* if it is a common eigenfunction of all elements of $\bigotimes_p \mathscr{H}_p$, that is, for any rational prime p and $\varphi \in \mathscr{H}_p$, the equation

$$f * \varphi = \lambda_f(\varphi)\,f$$

holds for some $\lambda_f(\varphi) \in \mathbb{C}$. Since for each p, the map $\lambda_f : \mathscr{H}_p \to \mathbb{C}$ gives a $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$, it determines a $\chi_f = (\chi_f^{(1)}, \cdots, \chi_f^{(n)}) \in X_0(T_p)$ such that

$$\lambda_f(\varphi) = \widehat{\omega}_{\chi_f}(\varphi)$$

for any $\varphi \in \mathscr{H}_p$. We call either the collection $(\chi_f^{(1)}(p), \cdots, \chi_f^{(n)}(p))$ or $(\chi_f^{(1)}(p)^{-1}, \cdots, \chi_f^{(n)}(p)^{-1})$ the *Satake p-parameters* of f. Then for a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$, we define the standard L-function attached to ϕ by

$$L(s,\,f,\,\mathrm{St}) := \prod_{p<\infty}\prod_{i=1}^{n}\left\{(1-\chi_f^{(i)}(p)\,p^{-s})(1-\chi_f^{(i)}(p)^{-1}p^{-s})\right\}^{-1},$$

which was introduced by Shintani in his unpublished paper, and afterward was studied by Murase (cf. [18, 19]).

By Lemma 2.2, for each element $f \in J_{l,1}(G_n^J(\mathbb{A}))$ we obtain the associated element $\Phi_f \in J_{l,1}^{\mathrm{cusp}}(\Gamma_n^J)$. Then we easily have the following relation between the action of the Hecke algebra $\mathscr{H}_p$ on f and the operation $\Phi_f|_{l,1}\Gamma_n^J M \Gamma_n^J$ for some $M \in G_n(\mathbb{Z}[p^{-1}])$:

Lemma 2.3. *Let $f \in J_{l,1}(G_n^J(\mathbb{A}))$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$ with $0 \le \alpha_1 \le \cdots \le \alpha_n$, we have*

$$\Phi_{f * \varphi_\alpha} = \Phi_f|_{l,1} \Gamma_n^J \mathbf{d}_n(\pi_\alpha)\Gamma_n^J.$$

Here φ_α is an element of $\mathscr{H}_p$ defined by

$$\varphi_\alpha(g) = \begin{cases} \Psi_p(\kappa) & \text{if } g \in Z_p^J K_p^J \mathbf{d}_n(\pi_\alpha) K_p^J \text{ and } g = [(0,0),\kappa]\, k\, \mathbf{d}_n(\pi_\alpha)\, k', \\ 0 & \text{if } g \notin Z_p^J K_p^J \mathbf{d}_n(\pi_\alpha) K_p^J, \end{cases}$$

where $\kappa \in \mathbb{Q}_p$ and $k, k' \in K_p^J$. In particular, if f is a Hecke eigenform, then Φ_f is also a Hecke eigenform in the sense of §2.2.1.

Let $\phi \in J_{l,1}(\Gamma_n^J)$ be a Hecke eigenform corresponding to a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$ via the mapping defined in (7), that is, $\phi = \Phi_f$. Then by Lemma 2.3, we naturally define the standard L-function attached to ϕ as $L(s, \phi, \mathrm{St}) := L(s, f, \mathrm{St})$. Namely,

$$L(s, \phi, \mathrm{St}) := \prod_{p<\infty} \prod_{i=1}^{n} \left\{(1 - \chi_\phi^{(i)}(p)\, p^{-s})(1 - \chi_\phi^{(i)}(p)^{-1} p^{-s})\right\}^{-1},$$

where we put $\chi_\phi^{(i)}(p) = \chi_f^{(i)}(p)$ for $i = 1, \cdots, n$.

If ϕ is a cuspidal Hecke eigenform, then the following analytic properties of the standard L-function $L(s, \phi, \mathrm{St})$ are shown by Murase ([19]):

Lemma 2.4 (cf. [19])**.** *If $\phi \in J_{l,1}^{\mathrm{cusp}}(\Gamma_n^J)$ is a Hecke eigenform, then the standard L-function $L(s, \phi, \mathrm{St})$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$. More precisely, put $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, and the function*

$$L^*(s, \phi, \mathrm{St}) = \prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s + l - 1/2 - i)\, L(s, \phi, \mathrm{St})$$

is meromorphic on $\mathbb{C}$, and satisfies the functional equation

$$L^*(1 - s, \phi, \mathrm{St}) = \varepsilon_n\, L^*(s, \phi, \mathrm{St}),$$

where

$$\varepsilon_n = \begin{cases} -1 & \text{if } n \equiv 1,\, 2 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Remark. Murase derived similar properties for the standard L-functions attached to more general cuspidal Jacobi forms whose index is a matrix.

On the other hand, we consider the standard L-function attached to the Jacobi Eisenstein series $\mathfrak{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$ in the rest of this paragraph.

For any quasi-character $\xi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}$, we define a $\mathbb{C}$-valued function $\widetilde{\phi_\xi}$ on $G_n^J(\mathbb{A})$ by

$$\widetilde{\phi_\xi}([(0,\mu),\kappa]\, g[(\lambda,0),0]\, k_\infty\, k_{\mathrm{fin}}) = \xi(\det(A))\varphi_0(\lambda) j(k_\infty, \sqrt{-1}\,\mathbf{1}_n)^{-l}$$

for any $\kappa \in \mathbb{A}$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\mathrm{fin}} \in K_{\mathrm{fin}}^J$, where $\varphi_0 = \prod_v \varphi_{0,v}$,

$$\varphi_{0,v}(\lambda) = \begin{cases} \mathrm{char}_{\mathbb{Z}_p^n}(\lambda) & \text{if } v = p < \infty, \\ \exp(-2\pi\lambda^t\lambda) & \text{if } v = \infty. \end{cases}$$

Then we define the Eisenstein series E_ξ on $G_n^J(\mathbb{A})$ associated with ξ by

$$E_\xi(g) := \sum_{\gamma \in P_n^J(\mathbb{Q}) \backslash G_n^J(\mathbb{Q})} \widetilde{\phi}_\xi(\gamma g) \qquad (g \in G_n^J(\mathbb{A})).$$

In particular, we denote by $\mathcal{E}_{l,1}^{(n)}$ the Eisenstein series on $G_n^J(\mathbb{A})$ associated with a special character $\xi_l(x) = |x|_{\mathbb{A}}^l$ $(x \in \mathbb{A}^\times)$. Then we easily see that $\mathcal{E}_{l,1}^{(n)}$ is an element of $J_{l,1}(G_n^J(\mathbb{A}))$ and corresponds to the Jacobi Eisenstein series $\mathfrak{E}_{l,1}^{(n)} \in J_{l,1}(\Gamma_n^J)$ in the same manner as in Lemma 2.2. Therefore we also call $\mathcal{E}_{l,1}^{(n)}$ the Jacobi Eisenstein series of weight l and index 1. Then we have

Proposition 2.2. *The Jacobi Eisenstein series $\mathcal{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$ is a Hecke eigenform, that is, for any $\varphi \in \bigotimes_p \mathscr{H}_p$,*

$$\mathcal{E}_{l,1}^{(n)} * \varphi = \lambda_{\mathcal{E}}(\varphi)\, \mathcal{E}_{l,1}^{(n)}$$

with $\lambda_{\mathcal{E}}(\varphi) \in \mathbb{C}$. Moreover, the Satake p-parameters of $\mathcal{E}_{l,1}^{(n)}$ are taken of the form

$$\left(p^{l-(n+1)+i-1/2}\right)_{1 \le i \le n}$$

up to inversion.

Proof. For any quasi-character ξ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, we take a $\chi = (\chi^{(1)}, \cdots, \chi^{(n)}) \in X_0(T_p)$ such that

$$\chi^{(i)}(t_i) = \xi(t_i)\, |t_i|_p^{-(2n+3-2i)/2} \qquad (t_i \in \mathbb{Q}_p^\times) \tag{9}$$

for each $1 \le i \le n$. Then by the equation (8) and the definition of $\widetilde{\phi}_\xi$, we have $\widetilde{\phi}_\xi = \phi_\chi$. Therefore it suffices to prove that for any $\varphi \in \mathscr{H}_p$ and $\lambda \in \mathbb{Q}_p^n$, the equation

$$(\phi_\chi * \varphi)([(\lambda, 0), 0]) = c \cdot \mathrm{char}_{\mathbb{Z}_p^n}(\lambda) \tag{10}$$

holds with some $c \in \mathbb{C}^\times$. Indeed, if $\lambda \notin \mathbb{Z}_p^n$, then there exists $0 \ne \mu \in \mathbb{Z}_p^n$ such that $\Psi_p(\lambda^t\mu) \ne 1$. Thus we have

$$\begin{aligned} (\phi_\chi * \varphi)([(\lambda, 0), 0]) &= (\phi_\chi * \varphi)([(\lambda, 0), 0] \cdot [(0, \mu), 0]) \\ &= (\phi_\chi * \varphi)([(\lambda, \mu), \lambda^t\mu]) \\ &= (\phi_\chi * \varphi)([(0, \mu), \lambda^t\mu] \cdot [(\lambda, 0), 0]) \\ &= \Psi_p(\lambda^t\mu)\, (\phi_\chi * \varphi)([(\lambda, 0), 0]), \end{aligned}$$

and $(\phi_\chi * \varphi)([(\lambda, 0), 0]) = 0$. Now we have proved that the Eisenstein series E_ξ is a Hecke eigenform. Moreover, it follows from the equation (10) that

$$c = (\phi_\chi * \varphi)(1) = \int_{Z_p^J \backslash G_p^J} \phi_\chi(g)\varphi(g^{-1})dg$$

and therefore the eigenvalue $\lambda_{\mathcal{E}}(\varphi)$ coincides with the zonal spherical function $\widehat{\omega}_{\chi}(\varphi)$. Therefore it follows from the equation (9) that

$$\chi^{(i)}(t_i) = \xi_l(t_i)\,|t_i|_p^{-(2n+3-2i)/2} = |t_i|_p^{l-(2n+3-2i)/2}$$

for each i. By substituting $t_i = p$, we obtain $\chi^{(i)}(p) = p^{-l+(2n+3-2i)/2}$ and complete the proof. □

By Proposition 2.2, we obtain the following conclusion:

Corollary. *Let l be a positive even integer such that $l > n+2$. Then we have*

$$L(s,\,\mathcal{E}_{l,1}^{(n)},\,\mathrm{St}) = L(s,\,\mathfrak{E}_{l,1}^{(n)},\,\mathrm{St}) = \prod_{i=1}^{n}\zeta(s-l+1/2+i)\zeta(s+l-1/2-i).$$

In particular, $L(s,\,\mathcal{E}_{l,1}^{(n)},\,\mathrm{St})$ and $L(s,\,\mathfrak{E}_{l,1}^{(n)},\,\mathrm{St})$ converge absolutely for $\mathrm{Re}(s) > l-n-1/2$. In addition, they have meromorphic continuations to the entire complex plane $\mathbb{C}$ and satisfy functional equations under $s \longmapsto 1-s$.

Remark. Let k and n be positive even integers such that $k > n+1$. As mentioned in §2.1, $\mathfrak{E}_{k,1}^{(n-1)}$ coincides with the first Fourier-Jacobi coefficient $e_{k,1}^{(n-1)}$ of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$ of degree n and weight k. Thus it follows from Corollary of Proposition 2.2 that

$$\begin{aligned} L(s,\,e_{l,1}^{(n)},\,\mathrm{St}) &= \prod_p\prod_{i=1}^{n-1}\{(1-p^{k-(n+1)/2}p^{-s+i-n/2})(1-(p^{k-(n+1)/2})^{-1}p^{-s+i-n/2})\}^{-1} \\ &= \prod_{i=1}^{n-1}L(s+k-1/2-i,\,E_{2k-n}^{(1)}), \end{aligned}$$

where $E_{2k-n}^{(1)} \in M_{2k-n}(\Gamma_1)$. Moreover, replacing $e_{k,1}^{(n-1)}$ by the first Fourier-Jacobi coefficient $\phi_1 \in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to a normalized Hecke eigenform $f \in S_{2k-n}(\Gamma_1)$ via a lifting procedure due to Ikeda (cf. [9]), then we also obtain a similar explicit formula for the standard L-function attached to ϕ_1 (cf. [15]).

2.4. **Eichler-Zagier-Ibukiyama correspondence between Jacobi forms and Siegel modular forms of half-integral weight.**

For the purpose of the subsequent use, we review in this paragraph that there exists a natural $\mathbb{C}$-linear correspondence from the space of Jacobi forms of even integral weight and of index 1 into that of Siegel modular forms of half-integral weight.

For any $(\tau,\,z) \in \mathfrak{H}_n \times \mathbb{C}^n$ and $(r_1,\,r_2) \in \mathbb{Q}^n \oplus \mathbb{Q}^n$, we define the *theta series* of characteristic $(r_1,\,r_2)$ by

$$\theta_{(r_1,r_2)}(\tau,\,z) = \theta_{(r_1,r_2)}^{(n)}(\tau,\,z) := \sum_{\lambda\in\mathbb{Z}^n}\mathbf{e}\left((\tau/2)[\,{}^t(\lambda+r_1)] + (\lambda+r_1)^t(z+r_2)\right).$$

In particular, for any $r \in \mathbb{Z}^n$, we put $\theta_r(\tau, z) = \theta_r^{(n)}(\tau, z) := \theta_{(r/2,0)}^{(n)}(2\tau, 2z)$. We note that the function $\theta_r(\tau, z)$ depends only on $r \bmod 2\mathbb{Z}^n$. For a fixed $\tau \in \mathfrak{H}_n$, it is known that $(\theta_r(\tau, z))_{r\in\mathbb{Z}^n/2\mathbb{Z}^n}$ forms a basis of the $\mathbb{C}$-vector space $\Theta_\tau^{(n)}$ consisting of all $\mathbb{C}$-valued holomorphic functions $\theta(z)$ on $\mathbb{C}^n$ which satisfy that

$$\theta(z + \lambda\tau + \mu) = \mathbf{e}(-\operatorname{tr}(\tau[{}^t\lambda] + 2{}^t\lambda z))\theta(z)$$

for any $\lambda,\ \mu \in \mathbb{Z}^n$.

For any $\tau \in \mathfrak{H}_n$, we put

$$\theta(\tau) = \theta^{(n)}(\tau) := \theta_{(0,0)}^{(n)}(2\tau, 0) = \sum_{\lambda\in\mathbb{Z}^n} \mathbf{e}(\tau[{}^t\lambda]).$$

Then for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(4)$, we define the *Shimura's factor of automorphy* by

$$J(M, \tau) = J^{(n)}(M, \tau) := \frac{\theta^{(n)}(M\langle\tau\rangle)}{\theta^{(n)}(\tau)}.$$

As is well-known, it follows that

$$J(M, \tau)^2 = (-1)^{(\det D-1)/2} \det(C\tau + D).$$

For any $l \in \mathbb{Z}$, a holomorphic function $F(\tau)$ on $\mathfrak{H}_n$ is called a *Siegel modular form* of degree n and weight $l - 1/2$ if it satisfies the following two conditions:

(i) $F(M\langle\tau\rangle) = J(M, \tau)^{2l-1}F(\tau)$ for any $M \in \Gamma_0^{(n)}(4)$;

(ii) For any $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n$, the function $\det(C\tau+D)^{-l+1/2}F(M\langle\tau\rangle)$ possesses a Fourier expansion of the form

$$\det(C\tau + D)^{-l+1/2}F(M\langle\tau\rangle) = \sum_{B\in\mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} C_{F,M}(B)\,\mathbf{e}(\operatorname{tr}(B\tau)/4),$$

where $\det(C\tau + D)^{-l+1/2}$ is an appropriately defined single valued function of τ. If F satisfies the stronger condition $C_{F,M}(B) = 0$ unless $B > 0$ (positive definite), it is called a *cusp form*. We note that such a F possesses a usual Fourier expansion

$$F(\tau) = \sum_{B\in\mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} C_F(B)\,\mathbf{e}(\operatorname{tr}(B\tau)).$$

We denote by $M_{l-1/2}(\Gamma_0^{(n)}(4))$ and $S_{l-1/2}(\Gamma_0^{(n)}(4))$ the $\mathbb{C}$-vector spaces of Siegel modular forms and Siegel cusp forms of degree n and weight $l - 1/2$, respectively.

Furthermore, we introduce the *generalized Kohnen plus space* $M_{l-1/2}^+(\Gamma_0^{(n)}(4))$ consisting of all elements $F \in M_{l-1/2}(\Gamma_0^{(n)}(4))$ whose Fourier coefficients $C_F(B)$ satisfy the condition

$$C_F(B) = 0 \text{ unless } B \equiv (-1)^{l+1}\,{}^t r_B r_B \mod 4\,\mathrm{Sym}_n^*(\mathbb{Z}) \text{ for some } r_B \in \mathbb{Z}^{n-1},$$

and put $S_{l-1/2}^+(\Gamma_0^{(n)}(4)) := M_{l-1/2}^+(\Gamma_0^{(n)}(4)) \cap S_{l-1/2}(\Gamma_0^{(n)}(4))$. These spaces were introduced by Kohnen ([17]) in case $n = 1$, and by Ibukiyama ([8]) for general n.

Now, we recall an important fact that if l is even, then there exists a $\mathbb{C}$-linear isomorphism between the space $J_{l,1}(\Gamma_n^J)$ of Jacobi forms of index 1 and the generalized Kohnen plus space $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$ as follows. Let $\phi \in J_{l,1}(\Gamma_n^J)$ possess a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{T\in \mathrm{Sym}_n^*(\mathbb{Z}),\, r\in\mathbb{Z}^n,\\ 4T-{}^t rr\geq 0}} c_\phi(T, r)\,\mathbf{e}(\mathrm{tr}(T\tau) + r\,{}^t z).$$

Since for each $\tau \in \mathfrak{H}_n$, $\phi(\tau, z)$ belongs in the space $\Theta_\tau^{(n)}$ generated by $(\theta_r(\tau, z))_{r\in\mathbb{Z}^n/2\mathbb{Z}^n}$, we have that ϕ can be expressed as a linear combination

$$\phi(\tau, z) = \sum_{r\in\mathbb{Z}^n/2\mathbb{Z}^n} h_r(\tau)\,\theta_r(\tau, z)$$

with some 2^n holomorphic functions $(h_r(\tau))_{r\in\mathbb{Z}^n/2\mathbb{Z}^n}$ on $\mathfrak{H}_n$ whose Fourier expansion is of the form

$$h_r(\tau) = \sum_{\substack{T\in \mathrm{Sym}_n^*(\mathbb{Z}),\\ 4T-{}^t rr\geq 0}} c_\phi(T, r)\,\mathbf{e}(\mathrm{tr}\big((T - {}^t rr/4)\,\tau\big)).$$

Then we put

$$\sigma(\phi)(\tau) = \sum_{r\in\mathbb{Z}^n/2\mathbb{Z}^n} h_r(4\tau).$$

The following statement is shown by Eichler and Zagier ([6]) in case $n = 1$ and by Ibukiyama for general n:

Proposition 2.3 (cf. Theorem 1, 2 in [8]). *If l is even, then the map $\phi \mapsto \sigma(\phi)$ gives a $\mathbb{C}$-linear isomorphism*

$$J_{l,1}(\Gamma_n^J) \cong M^+_{l-1/2}(\Gamma_0^{(n)}(4)),$$

which is compatible with the actions of Hecke operators. Furthermore, its restriction to the space $J^{\mathrm{cusp}}_{l,1}(\Gamma_n^J)$ also induces a $\mathbb{C}$-linear isomorphism

$$J^{\mathrm{cusp}}_{l,1}(\Gamma_n^J) \cong S^+_{l-1/2}(\Gamma_0^{(n)}(4)).$$

We call it the Eichler-Zagier-Ibukiyama correspondence.

Remark. When l is odd, the space $J_{l,1}(\Gamma_n^J)$ is not isomorphic to the Kohnen plus space $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$. However, a similar claim is also valid by introducing the space $J^{\mathrm{skew}}_{l,1}(\Gamma_n^J)$ of *skew holomorphic Jacobi forms* which was defined by Skoruppa ([24, 25]) in case $n = 1$ and by Arakawa ([2]) for general n.

We easily see by the definition that the Fourier expansion of $\sigma(\phi)$ can be expressed in terms of Fourier coefficients of ϕ as

$$\sigma(\phi)(\tau) = \sum_{B\in\mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} c_\phi((B + {}^t r_B r_B)/4,\, r_B)\,\mathbf{e}(\mathrm{tr}(B\tau)),$$

where r_B denotes an element of $\mathbb{Z}^n$ such that $B + {}^t r_B r_B \in 4\mathrm{Sym}_n^*(\mathbb{Z})$. We note that r_B is uniquely determined by B modulo $2\mathbb{Z}^n$, and then $c_\phi((B + {}^t r_B r_B)/4,\, r_B)$ does not depend on the choice of the representative of r_B mod $2\mathbb{Z}^n$. Moreover, if ϕ coincides with the first Fourier-Jacobi coefficient of a Siegel modular form $F \in M_l(\Gamma_{n+1})$, then we have

$$\sigma(\phi)(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} A_F\bigl(B^{(1)}\bigr)\, \mathbf{e}(\mathrm{tr}(B\tau)),$$

where $B^{(1)} \in \mathrm{Sym}_n^*(\mathbb{Z})$ denotes the matrix defined in §1, and $A_F(B^{(1)})$ is the $B^{(1)}$-th Fourier coefficient of F. In particular, let n and k be positive even integers such that $k > n+1$ and we take $\phi = e_{k,1}^{(n-1)} \in J_{k,1}(\Gamma_{n-1}^J)$, then we have the following explicit formula for the Fourier coefficients of the associated form $\sigma(e_{k,1}^{(n-1)}) \in M_{k-1/2}^+(\Gamma_0^{(n-1)}(4))$:

Proposition 2.4. *Under the same assumption as in Proposition 2.1, let $\sigma(e_{k,1}^{(n-1)})$ possess a Fourier expansion*

$$\sigma(e_{k,1}^{(n-1)})(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} C_{k-1/2}^{(n-1)}(B)\, \mathbf{e}(\mathrm{tr}(B\tau)).$$

Then for each $B \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_{>0}$ satisfying the condition (1)*, we have*

$$C_{k-1/2}^{(n-1)}(B) = \xi(n,\, k)L(1-k+n/2,\, \chi_{B^{(1)}})\, \mathfrak{f}(B^{(1)})^{k-(n+1)/2} \prod_{p \,|\, \mathfrak{f}(B^{(1)})} \widetilde{F}_p^{(1)}(B;\, p^{k-(n+1)/2}).$$

Proof. If $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})$ and $r \in \mathbb{Z}^{n-1}$, then we have

$$C_{k-1/2}^{(n-1)}(B) = c_{k,1}^{(n-1)}(T,\, r).$$

Thus the assertion follows from Proposition 2.1. $\square$

3. Andrianov type identity for power series attached to Jacobi forms

Throughout this paragraph, let n and k be positive even integers such that $k > n+1$, and we fix a rational prime p. For a subring R of $\mathbb{Z}_p$, we simply denote by $\mathrm{Sym}_{n-1}(R)^{(1)}$ the subset of $\mathrm{Sym}_{n-1}(R)^\times$ consisting of all elements which satisfy the condition (1) in §1, namely,

$$\mathrm{Sym}_{n-1}(R)^{(1)} = \{\, B \in \mathrm{Sym}_{n-1}(R)^\times \mid B + {}^t r_B r_B \in 4\,\mathrm{Sym}_{n-1}^*(R) \text{ for some } r_B \in R^{n-1} \,\}.$$

As mentioned in §1, for each element $B \in \mathrm{Sym}_{n-1}(R)^{(1)}$, we can associate it with an element

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix} \in \mathrm{Sym}_n^*(R)^\times.$$

Then for such a $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we introduce a modified local Siegel series as follows. For each $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])$ and $r \in \mathbb{Z}_p^{n-1}$, if $R \in p^{-l}\mathrm{Sym}_{n-1}(\mathbb{Z}_p)$ with $l \geq 0$, then we put

$$\omega(R;\, r) = p^{-(n-1)l}\, \mu_p(R)^{1/2} \sum_{x \in \mathbb{Z}_p^{n-1}/p^l\mathbb{Z}_p^{n-1}} \mathbf{e}_p(-R[{}^t x] + rR\,{}^t x/2 + xR\,{}^t r/2),$$

where $\mu_p(R) = [\mathbb{Z}_p^{n-1}R + \mathbb{Z}_p^{n-1} : \mathbb{Z}_p^{n-1}]$, and we note that the right-hand side does not depend on the choice of l. Let $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ possess $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}^*_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then we put

$$b_p^{(1)}(B;\, t) = \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z}_p)} \omega(R;\, r)\,\mathbf{e}_p(-\mathrm{tr}(TR))\, t^{\mathrm{ord}_p(\mu_p(R))}.$$

We note that this series coincides with $\alpha_1(B,\, t)$ in [21] if $p \neq 2$ and $r = 0$. As will be shown later, the above definition does not depend on the choice of T and r (cf. Proposition 3.1 below).

On the other hand, if $m > 1$, then for each $S \in \mathrm{Sym}^*_{m-1}(\mathbb{Z}_p)$, $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$, $r \in \mathbb{Z}_p^{n-1}$ and $e \in \mathbb{Z}_{>0}$, we put

$$\mathcal{A}_e(S,\, T,\, r) := \left\{ X \in \mathrm{M}_{m,n-1}(\mathbb{Z}_p)/p^e\mathrm{M}_{m,n-1}(\mathbb{Z}_p) \;\middle|\; \begin{array}{l} (-1\perp S)[X] + {}^t r\mathbf{x}_1/2 \\ \qquad + {}^t\mathbf{x}_1 r/2 - T \in p^e\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p) \end{array} \right\},$$

where $\mathbf{x}_1 \in \mathbb{Z}_p^{n-1}$ denotes the first row of X. We easily check that it is well-defined. Furthermore, if both S and $\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix}$ are non-degenerate, then $p^{e(-m(n-1)+n(n-1)/2)}\#\mathcal{A}_e(S,\, T,\, r)$ has the same value for any $e \geq \mathrm{ord}_p(\det \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix})$, which will be denoted by $\alpha_p^{(1)}(S,\, T,\, r)$. We note that $\alpha_p^{(1)}(S,\, T,\, r)$ coincides with the usual local density $\alpha_p(-1\perp S,\, T)$ if $r = 0$. Then we obtain the following lemmas:

Lemma 3.1. *Let $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)^\times$ possess $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then we have*

$$b_p^{(1)}(B;\, p^{-k+1/2}) = \alpha_p(H_{k-1},\, T,\, r),$$

where $H_{k-1} = \overbrace{H \perp \cdots \perp H}^{k-1}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \in \mathrm{Sym}^*_2(\mathbb{Z}_p)$. In particular, $b_p^{(1)}(B;\, t) = 0$ unless $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$.

Proof. By Lemma 3.4 of [22], we have

$$\begin{aligned}
&b_p^{(1)}(B;\, p^{-k+1/2}) \\
&= \sum_R \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}/p^l\mathbb{Z}_p^{n-1}} \mathbf{e}_p(-R[{}^t\mathbf{x}] + rR\,{}^t\mathbf{x}/2 + \mathbf{x}R\,{}^t r/2)p^{-(k-1)\,\mathrm{ord}_p(\mu_p(R))}p^{-(n-1)l}\mathbf{e}_p(-\mathrm{tr}(TR)) \\
&= \sum_R \sum_{\mathbf{x}} \mathbf{e}_p(-R[{}^t\mathbf{x}] + rR\,{}^t\mathbf{x}/2 + \mathbf{x}R\,{}^t r/2)p^{-(n-1)l}\mathbf{e}_p(-\mathrm{tr}(TR))p^{-2l(k-1)n} \\
&\qquad \times \sum_{Y \in \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)/p^l\mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(H_{k-1}[Y]R)) \\
&= \sum_R \sum_x \sum_Y \mathbf{e}_p(\mathrm{tr}((-{}^t\mathbf{x}\mathbf{x} + H_{k-1}[Y] + {}^t r\mathbf{x}/2 + {}^t\mathbf{x}r/2 - T)R))\, p^{-l(2k-1)(n-1)} \\
&= \#\mathcal{A}_l(H_{k-1},\, T,\, r)\, p^{-l((2k-1)(n-1)-n(n-1)/2)}.
\end{aligned}$$

Thus the assertion holds. ∎

Lemma 3.2. *If $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)^\times$ possesses $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$, then we have*

$$\alpha_p(H_k,\, B^{(1)}) = (1-p^{-k})\alpha_p(H_{k-1},\, T,\, r).$$

Proof. The proof is similar to that of Proposition 2.4 in [11], and here we give a sketch of the proof. For each $\xi = (\xi_i) \in \mathbb{Z}_p^{2k}$, we put

$$\mathcal{A}_e(H_k,\, B^{(1)}) = \{\, X \in \mathrm{M}_{2k,n}(\mathbb{Z}_p)/p^e\mathrm{M}_{2k,n}(\mathbb{Z}_p) \mid H_k[X] - B^{(1)} \in p^e\mathrm{Sym}_n^*(\mathbb{Z}_p)\,\}$$

and

$$\mathcal{A}_e(H_k,\, B^{(1)};\, \xi) = \{\, X = (x_{ij}) \in \mathcal{A}_e(H_k,\, B^{(1)}) \mid x_{i1} \equiv \xi_i \pmod{p^e} \text{ for } 1 \le i \le 2k\,\}.$$

We easily see that $\mathcal{A}_e(H_k, B^{(1)};\, \xi) \ne \phi$ only if $\xi \in \mathcal{A}_e(H_k,\, 1)$. Now we fix such a ξ. Then we have $\xi \not\equiv 0 \pmod{p\mathbb{Z}_p^{2k}}$. Thus by Lemma 2.3 in [11], we can take $U \in \mathrm{GL}_{2k}(\mathbb{Z}_p)$ and $K \in \mathcal{L}_{2k-2,p}$ such that

$$\text{(i)} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \perp K = H_k[U]; \quad \text{(ii) } K \sim_{\mathbb{Z}_p} H_{2k-2}; \quad \text{(iii) } U^{-1}\xi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For each $X \in \mathcal{A}_e(H_k,\, B^{(1)};\, \xi)$, we write X as $X = ({}^t\xi \mid Y)$ with $Y \in \mathrm{M}_{2k,n-1}(\mathbb{Z}_p)$, and write Y as $Y = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ Y_3 \end{pmatrix}$ with $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{Z}_p^{n-1}$ and $Y_3 \in \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)$. Then by an easy calculation, we have

$$\mathbf{y}_1 + \mathbf{y}_2/2 - r/2 \in p^e\mathbb{Z}_p^{n-1}$$

and

$$-{}^t\mathbf{y}_1\mathbf{y}_1 + K[Y_3] + {}^t\mathbf{y}_1\mathbf{y}_2/2 + {}^t\mathbf{y}_2\mathbf{y}_1/2 - T \in p^e\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p).$$

Thus we have

$$-{}^t\mathbf{y}_1\mathbf{y}_1 + K[Y_3] + {}^t r\mathbf{y}_1/2 + {}^t\mathbf{y}_1 r/2 - T \in p^e\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p),$$

that is, $\begin{pmatrix} \mathbf{y}_1 \\ Y_3 \end{pmatrix} \in \mathcal{A}_e(H_{k-1},\, T,\, r)$. Then the mapping $Y \longmapsto \begin{pmatrix} \mathbf{y}_1 \\ Y_3 \end{pmatrix}$ induces a bijection between $\mathcal{A}_e(H_k,\, B^{(1)};\, \xi)$ and $\mathcal{A}_e(H_{k-1},\, T,\, r)$. Thus we have

$$\begin{aligned} & p^{e(-2kn+n(n+1)/2)}\#\mathcal{A}_e(H_k,\, B^{(1)}) \\ &= p^{e(-2k+1)}\#\mathcal{A}_e(H_k,\, 1)\, p^{e(-(2k-1)(n-1)+n(n-1)/2)}\#\mathcal{A}_e(H_{k-1},\, T,\, r) \\ &= \alpha_p(H_k,\, 1)\,\alpha_p(H_{k-1},\, T,\, r) \\ &= (1-p^{-k})\,\alpha_p(H_{k-1},\, T,\, r). \end{aligned}$$

Therefore the assertion holds. □

Now by combining Lemmas 3.1 and 3.2, we obtain the following:

Proposition 3.1. *For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ and $s \in \mathbb{C}$, we have*

$$b_p^{(1)}(B;\, p^{-s+1/2}) = (1-p^{-s})^{-1}\, b_p(B^{(1)};\, s).$$

Proof. It is well-known that for each $B' \in \mathrm{Sym}_n^*(\mathbb{Z}_p)^\times$ with $n < 2k$, the Siegel series $b_p(B';\, s)$ in §1 satisfies the equation

$$b_p(B;\, k) = \alpha_p(H_k,\, B).$$

Then by Lemmas 3.1 and 3.2, we have

$$b_p^{(1)}(B;\, p^{-k+1/2}) = (1 - p^{-k})^{-1}\, b_p(B^{(1)};\, k)$$

for infinitely many k, and therefore the assertion follows. □

Remark. The definition of $b_p^{(1)}(B;\, t)$ for $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$ does not depend on the choice of T and r. Indeed, if $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$, then the vector r is uniquely determined by B modulo $2\mathbb{Z}_p^{n-1}$, and the matrix $\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix}$ is uniquely determined by B up to $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$-equivalence. Thus by Proposition 3.1, $b_p^{(1)}(B;\, t)$ is uniquely determined by B. If $T \notin \mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$, then we have $b_p^{(1)}(B;\, t) = 0$. Furthermore, if $B = 4T' - {}^t r' r'$ is another expression, then T' does not belong to $\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$ either. This proves the well-definedness of $b_p^{(1)}(B;\, t)$.

Now we put

$$\widetilde{b}_p^{(1)}(B;\, t) := \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p) \backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z}_p)} \pi_p(D)\, b_p^{(1)}(B[D^{-1}];\, t)\, (p^{n-1}\, t^2)^{\mathrm{ord}_p(\det D)}.$$

Then by Proposition 3.1, we obtain the following rationality theorem for the polynomial $\mathbf{B}_p^{(1)}(B;\, t)$ defined in §1:

Proposition 3.2. *For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$\mathbf{B}_p^{(1)}(B;\, p^{n-1/2}\, t)\, \widetilde{b}_p^{(1)}(B;\, p^{1/2}\, t) = \prod_{i=1}^{n-1} (1 - p^{2i} t^2).$$

Next, we study the standard L-function attached to a Hecke eigenform and some power series related to it. For a Hecke eigenform $\phi \in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$, and $D \in \mathbf{D}_p^{(n-1)}(\mathbb{Z})$, let

$$\phi|_{k,1} \Gamma_{n-1}^J \mathbf{d}_{n-1}(D) \Gamma_{n-1}^J = \lambda_\phi(D) \phi$$

with $\lambda_\phi(D) \in \mathbf{C}$. Then we define a power series $Z_p(t,\, \phi)$ by

$$Z_p(t,\, \phi) := \sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \lambda_\phi(D)\, t^{\mathrm{ord}_p(\det D)},$$

where $\mathbf{ED}_p^{(n-1)}(\mathbb{Z})$ denotes the set of all elementary divisors of the form $p^{\alpha_1} \perp \cdots \perp p^{\alpha_{n-1}}$ with $0 \le \alpha_1 \le \cdots \le \alpha_{n-1}$. The following statement is shown by Murase and Sugano:

Proposition 3.3 (cf. Lemma 6.5 in [20], see also Theorem 5.5 in [2])**.** *Let $\phi \in J_{k,1}(\Gamma_{n-1}^J)$ be a Hecke eigenform with Satake p-parameters $(\chi_\phi^{(1)}(p), \cdots, \chi_\phi^{(n-1)}(p)) \in \mathbb{C}^{n-1}$. Then we have*

$$Z_p(t,\, \phi) = \prod_{i=1}^{n-1} \frac{(1 - p^{2i} t^2)}{(1 - \chi_\phi^{(i)}(p) p^{n-1/2}\, t)(1 - \chi_\phi^{(i)}(p)^{-1} p^{n-1/2}\, t)}.$$

Let

$$\mathscr{Z}_p^{(n-1)} := \left\{ \begin{pmatrix} V \\ W \end{pmatrix} \in \mathrm{M}_{2n-2,n-1}(\mathbb{Z}) \;\middle|\; V, W \in \mathbf{D}_p^{(n-1)}(\mathbb{Z}),\ \gcd(V,W)=1 \right\},$$

where $\gcd(V,W)$ denotes the greatest common divisor of all entries of V and W. For each $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right) \in \mathscr{Z}_p^{(n-1)}$, $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])$ and $(\lambda_1,\lambda_2) \in \mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}$, we put

$$M_{V,W,R} := \left(\begin{array}{c|c} {}^tW^{-1}\,{}^tV & {}^tW^{-1}RV^{-1} \\ \hline \mathbf{0}_{n-1} & WV^{-1} \end{array} \right) \in G_{n-1}(\mathbb{Z}[p^{-1}])$$

and

$$[\lambda_1,\lambda_2] := [(\lambda_1,\lambda_2), \lambda_1{}^t\lambda_2] = \left(\begin{array}{cc|cc} 1 & \lambda_1 & 0 & \lambda_2 \\ 0 & \mathbf{1}_{n-1} & {}^t\lambda_2 & \mathbf{0}_{n-1} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \mathbf{0}_{n-1} & -{}^t\lambda_1 & \mathbf{1}_{n-1} \end{array} \right) \in H_{n-1}(\mathbb{Z}).$$

Then by combining Lemma 2.1 and some easy calculation (cf. [4]), we obtain the following:

Lemma 3.3. *We have*

$$\Gamma_{n-1}^J G_{n-1}(\mathbb{Z}[p^{-1}])\Gamma_{n-1}^J = \bigcup_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \Gamma_{n-1}^J \mathbf{d}_{n-1}(D) \Gamma_{n-1}^J$$

$$= \bigsqcup_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} \bigsqcup_{R} \bigsqcup_{(\lambda_1,\lambda_2)} \Gamma_{n-1}^J [M_{V,W,R}] \cdot [\lambda_1,\lambda_2],$$

where $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)$, R *and* (λ_1,λ_2) *run over all representatives of* $(\mathbf{1}_{n-1} \perp \mathrm{GL}_{n-1}(\mathbb{Z}))\backslash \mathscr{Z}_p^{(n-1)}/\mathrm{GL}_{n-1}(\mathbb{Z})$, $\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W$, *and* $(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})+(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}$, *respectively. Furthermore, if* $M_{V,W,R} \in \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J$ *with* $D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})$, *then we have* $\mathrm{ord}_p(\det D) = \mathrm{ord}_p(\det V \det W\, \mu_p(R))$.

Therefore, we get the following explicit formula for the actions of Hecke operators:

Corollary. *For each* $\phi \in J_{k,1}(\Gamma_{n-1}^J)$, *we have*

$$\begin{aligned} \sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \left(\phi\,|_{k,1}\Gamma_{n-1}^J\mathbf{d}_{n-1}(D)\Gamma_{n-1}^J\right)(\tau,z) \quad &= \quad \sum_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} \sum_{R} p^{(-2n+3)\delta_{V,W,R}} \det V^{k-1} \det W^{-k} \\ &\times \sum_{(\lambda_1,\lambda_2) \in (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1}{}^tV \oplus \mathbb{Z}^{n-1})} \mathbf{e}(\tau[{}^t\lambda_1] + 2\lambda_1{}^tz) \\ &\times \phi(\tau[VW^{-1}] + R[W^{-1}],\, (z+\lambda_1\tau+\lambda_2)VW^{-1}), \end{aligned}$$

where $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)$ *and* R *run over the sets stated above, and* $\delta_{V,W,R} = \mathrm{ord}_p(\det V \det W \mu_p(R))$.

Proof. For each $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right) \in \mathscr{Z}_p^{(n-1)}$ and $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])$, we have

$$\Gamma_{n-1}^J M_{V,W,R} \Gamma_{n-1}^J = \Gamma_{n-1}^J \mathbf{d}_{n-1}(D) \Gamma_{n-1}^J$$

for some $D = p^{\alpha_1}\perp\cdots\perp p^{\alpha_{n-1}} \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})$. Then we have

$$\begin{aligned}&(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}\\ \simeq\ &(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})\mathbf{d}_{n-1}(D)/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})\mathbf{d}_{n-1}(D)\\ \simeq\ &\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D.\end{aligned}$$

It follows from Lemma 3.3 that $\#(\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D) = p^{\delta_{V,W,R}}$ and $e_1,\dots,e_r \le \delta_{V,W,R}$. Thus we have a natural surjection π from $(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1\,t}V\oplus\mathbb{Z}^{n-1})$ to $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D$, and we have $\#\ker(\pi) = p^{2(n-3)\delta_{V,W,R}}\det V$. Thus the assertion holds. □

By the above corollary, we obtain the following conclusion:

Proposition 3.4. *Let $\phi \in J_{k,1}(\Gamma^J_{n-1})$ be a Hecke eigenform. If the associated form $\sigma(\phi) \in M^+_{k-1/2}(\Gamma_0^{(n-1)}(4))$ under the Eichler-Zagier-Ibukiyama correspondence possesses a Fourier expansion*

$$\sigma(\phi)(\tau) = \sum_{B\in \mathrm{Sym}^*_{n-1}(\mathbb{Z}_p)_{\ge 0}} C_{\sigma(\phi)}(B)\,\mathbf{e}(\mathrm{tr}(B\tau)),$$

then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, we have

$$\begin{aligned}&\prod_{i=1}^{n-1}\frac{1-p^{2i}t^2}{(1-\chi_\phi^{(i)}(p)p^{n-1/2}\,t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}\,t)}\,C_{\sigma(\phi)}(B)\\ &\quad= \sum_{\binom{V}{W}} b_p^{(1)}(B[{}^tV^{-1}];\,t)\,C_{\sigma(\phi)}(B[{}^tV^{-1}][W])\,p^{-(k-n-1)}p^{k\,\mathrm{ord}_p(\det V)}t^{\mathrm{ord}_p(\det V\det W)},\end{aligned}$$

where $\binom{V}{W}$ runs over the set stated in Lemma 3.3.

Proof. We put

$$\Lambda_p(t) = \sum_{D\in\mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \Gamma^J_{n-1}\mathbf{d}_{n-1}(D)\Gamma^J_{n-1}\,t^{\mathrm{ord}_p(\det D)}.$$

Then by Corollary of Lemma 3.3, we have

$$\begin{aligned}(\phi\,|_{k,1}\,\Lambda_p(t))(\tau,z) &= \sum_T\sum_r c_\phi(T,\,r)\\ &\times \sum_{\binom{V}{W}\in(\mathbf{1}_{n-1}\perp\mathrm{GL}_{n-1}(\mathbb{Z}))\backslash\mathscr{X}_p^{(n-1)}/\mathrm{GL}_{n-1}(\mathbb{Z})} p^{(k-1)\,\mathrm{ord}_p(\det V)-k\,\mathrm{ord}_p(\det W)}t^{\mathrm{ord}_p(\det V\det W)}\\ &\times\mathbf{e}(\mathrm{tr}(T[{}^tW^{-1\,t}V]\tau + {}^t(r\,{}^tW^{-1\,t}V)z))\\ &\times \sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W} \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\,t^{\mathrm{ord}_p(\mu_p(R))}\\ &\times \sum_{\lambda_1\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\cdot{}^tV} p^{-(2n-3)\delta_{V,W,R}}\mathbf{e}(\mathrm{tr}(2\,{}^t\lambda_1 z + {}^t(r\,{}^tW^{-1\,t}V+\lambda_1)\lambda_1\tau))\\ &\times \sum_{\lambda_2\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} \mathbf{e}(\mathrm{tr}({}^t(r\,{}^tW^{-1\,t}V+\lambda_1)\lambda_2)).\end{aligned}$$

Since

$$\sum_{\lambda_2\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}}\mathbf{e}(\mathrm{tr}({}^t(r\,{}^tW^{-1}\,{}^tV+\lambda_1)\lambda_2))=\begin{cases} p^{(n-1)\delta_{V,W,R}} & \text{if } r\,{}^tW^{-1}\in\mathbb{Z}^{n-1},\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W}\mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\,t^{\mathrm{ord}_p(\mu_p(R))}$$

$$=\begin{cases} (\det W)^n\displaystyle\sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})}\mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\,t^{\mathrm{ord}_p(\mu_p(R))} & \text{if } T[{}^tW^{-1}]\in\mathrm{Sym}^*_{n-1}(\mathbb{Z}),\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}(\phi\,|_{k,1}\,\Lambda_p(t))(\tau,z) &= \sum_T\sum_r\sum_{\binom{V}{W}}p^{k\,\mathrm{ord}_p(\det V)+(-k+n+1)\,\mathrm{ord}_p(\det W)}t^{\mathrm{ord}_p(\det V\det W)}\\ &\quad\times\sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})}\mathbf{e}(\mathrm{tr}(TR))\,(pt)^{\mathrm{ord}_p(\mu_p(R))}\\ &\quad\times\sum_{\lambda_1\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\cdot{}^tV}p^{-(n-1)\delta_{V,W,R}}c_\phi(T[{}^tW],\,r\,{}^tW)\\ &\quad\times\mathbf{e}(\mathrm{tr}({}^t(r\,{}^tV+2\lambda_1)z))\,\mathbf{e}(\mathrm{tr}((T[{}^tV]+{}^t(r\,{}^tV+\lambda_1)\lambda_1)\tau)).\end{aligned}$$

For a fixed $r_0\in\mathbb{Z}^{n-1}$, we put

$$\mathcal{S}_1(r_0)=\{\,\lambda_1\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\,{}^tV\mid 2\lambda_1\equiv r_0 \bmod \mathbb{Z}^{n-1}\,{}^tV\,\},$$

and

$$\mathcal{S}_2(r_0)=\{\,r\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\mid r\,{}^tV\equiv r_0 \bmod 2\mathbb{Z}^{n-1}\,\}.$$

For each $\lambda_1\in\mathcal{S}_1(r_0)$, the map $\lambda_1\mapsto(2\lambda_1-r_0)\,{}^tV^{-1}$ induces a bijection between $\mathcal{S}_1(r_0)$ and $\mathcal{S}_2(r_0)$. Thus we have

$$\begin{aligned}(\phi|_{k,1}\,\Lambda_p(t))(\tau,\,z) &= \sum_T\sum_{r_0}\sum_{\binom{V}{W}}p^{k\,\mathrm{ord}_p(\det V)+(-k+n+1)\,\mathrm{ord}_p(\det W)}t^{\mathrm{ord}_p(\det V\det W)}\\ &\quad\times\sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})}\mathbf{e}(\mathrm{tr}(TR))\,(pt)^{\mathrm{ord}_p(\mu_p(R))}\,p^{-(n-1)\delta_{V,W,R}}\\ &\quad\times\sum_{r\in\mathcal{S}_2(r_0)}c_\phi(T[{}^tW],\,r\,{}^tW)\,\mathbf{e}(\mathrm{tr}({}^tr_0z))\,\mathbf{e}(\mathrm{tr}(T[{}^tV]+({}^tr_0r_0-{}^t(r\,{}^tV)(r\,{}^tV))/4)\tau))\\ &= \sum_{T_0}\sum_{r_0}\mathbf{e}(\mathrm{tr}(T_0\tau+{}^t\,r_0z))\sum_{\binom{V}{W}}\sum_{r\in\mathcal{S}_2(r_0)}p^{k\,\mathrm{ord}_p(\det V)+(-k+n+1)\,\mathrm{ord}_p(\det W)}p^{-(n-1)\delta_{V,W,R}}\\ &\quad\times c_\phi((T_0-{}^tr_0r_0/4)[{}^tV^{-1}][{}^tW]+({}^trr/4)[{}^tW],\,r\,{}^tW)\\ &\quad\times\sum_{R\in\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})}\mathbf{e}(\mathrm{tr}((T_0-{}^tr_0r_0/4)[{}^tV^{-1}]+{}^t\,rr/4)R))\,(pt)^{\mathrm{ord}_p(\mu_p(R))}.\end{aligned}$$

Then for a fixed $r\in\mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}$, the map

$$(r+2\mathbb{Z}^{n-1})+2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}/2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\ni r+2u\mapsto u\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}$$

is a bijection, and we have

$$c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t(r+2u)(r+2u)/4)[{}^tW],\ (r+2u){}^tW)$$
$$= c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t rr/4)[{}^tW],\ r{}^tW).$$

Thus we have

$$\begin{aligned}
&(\phi|_{k,1}\Lambda_p(t))(\tau,\, z)\\
&= \sum_{T_0}\sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^t r_0 z)) \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V\det W)}\\
&\quad\times \sum_{R\in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} (pt)^{\mathrm{ord}_p(\mu_p(R))}\\
&\quad\times \sum_{\substack{r\in\mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1},\\ r{}^tV\equiv r_0 \bmod 2\mathbb{Z}^{n-1}}} c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t rr/4)[{}^tW],\ r{}^tW)\\
&\quad\times \sum_{u\in \mathbb{Z}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} p^{-(n-1)\delta_{V,W,R}} \mathbf{e}(\mathrm{tr}((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}] + {}^t rr/4 + {}^t uu + {}^t ur/2 + {}^t ru/2)R)).
\end{aligned}$$

We easily see for an element $r\in\mathbb{Z}^{n-1}$ that the summation

$$\sum_{R\in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} (pt)^{\mathrm{ord}_p(\nu(R))} \sum_{u\in\mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} p^{-(n-1)\delta_{V,W,R}}$$
$$\times \mathbf{e}(\mathrm{tr}(((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}] + {}^t rr/4 + {}^t uu + {}^t ur/2 + {}^t ru/2)R))$$

equals $b_p^{(1)}((4T_0 - {}^t r_0 r_0)[{}^tV^{-1}];\, t)$ or 0 according as $(T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}] + {}^t rr/4 \in \mathrm{Sym}^*_{n-1}(\mathbb{Z}_p)$ or not, namely, according as $(4T_0 - {}^t r_0 r_0)[{}^tV^{-1}] \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ or not. In the former case, the vector r is uniquely determined by T_0, r_0, and V, which will be denoted by $r_1 = r_1(T_0, r_0, V)$. Furthermore we have

$$(4T_0 - {}^t r_0 r_0)[{}^tV^{-1}] + {}^t rr)[{}^tV] = (4T_0 - {}^t r_0 r_0) + {}^t(r{}^tV) r{}^tV \in 4\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p),$$

and we have $r{}^tV \equiv r_0 \bmod 2\mathbb{Z}^{n-1}$. Thus we have

$$(\phi|_{k,1}\Lambda_p(t))(\tau,\, z) = \sum_{T_0}\sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^t r_0 z)) \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V\det W)}$$
$$\times b_p^{(1)}((4T_0 - {}^t r_0 r_0)[{}^tV^{-1}];\, t)\, c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t r_1 r_1/4)[{}^tW],\ r_1{}^tW).$$

Now we take an element $B\in\mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ so that $B = 4T_0 - {}^t r_0 r_0$ with $T_0\in\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p)$ and $r_0\in\mathbb{Z}^{n-1}$. Then we have

$$c_\phi(T_0, r_0) = C_{\sigma(\phi)}(B),\ c_\phi((T_0 - {}^t r_0 r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t r_1 r_1/4)[{}^tW],\ r_1^t W) = C_{\sigma(\phi)}(B[{}^tV^{-1}][{}^tW]),$$

and

$$b_p^{(1)}((4T_0 - {}^t r_0 r_0)[{}^tV^{-1}];\, t) = b_p^{(1)}(B[{}^tV^{-1}];\, t).$$

Since $\phi|_{k,1}\Lambda_p(t) = Z_p(t,\,\phi)\,\phi$, the assertion follows immediately from Proposition 3.3. □

For each $B\in\mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, let $\widetilde{G}_{\phi,p}(B;\, t)$ be the polynomial in t defined in §1. Then by making use of the same argument as in [4] combined with Propositions 3.2 and 3.4, we obtain the following:

Theorem 3.1. *Let n and k be positive even integers such that $k > n+1$, and let $\phi \in J_{k,1}(\Gamma_{n-1}^J)$ be a Hecke eigenform with Satake p-parameters $(\chi_\phi^{(1)}(p), \cdots, \chi_\phi^{(n-1)}(p)) \in \mathbb{C}^{n-1}$. Then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)}$, we have*

$$\frac{\mathbf{B}_p^{(1)}(B;\, p^{n-1/2}\, t)\, \widetilde{G}_{\phi,\,p}(B;\, t)}{\prod_{i=1}^{n-1}(1-\chi_\phi^{(i)}(p)p^{n-1/2}\, t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}\, t)}$$
$$= \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W])\, p^{-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det W)}.$$

For each $D \in \mathrm{M}_{n-1}(\mathbb{Z}) \cap \mathrm{GL}_{n-1}(\mathbb{Q})$, we define the generalized global Möbius function $\pi(D)$ as $\prod_p \pi_p(D)$, where π_p is the local Möbius function defined in §1. We easily see that this is a finite product of $\pi_p(D)$. Then for each $B \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_{>0}^{(1)}$, we put

$$\widetilde{H}_\phi(B;\, s) = \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathrm{M}_{n-1}(\mathbb{Z})\cap \mathrm{GL}_{n-1}(\mathbb{Q})} \pi(D)\, C_{\sigma(\phi)}(B[D^{-1}])\, \det D^{-s+k} \quad (s \in \mathbb{C}),$$

which is a finite sum, and we have $\widetilde{H}_\phi(B;\, s) = \prod_p \widetilde{G}_{\phi,\,p}(B;\, p^{-s})$. In addition, we also put $\mathbf{B}^{(1)}(B;\, s) = \prod_p \mathbf{B}_p^{(1)}(B;\, p^{-s})$. Then Theorem 3.1 can be restated globally as follows:

Theorem 3.2. *Under the same situation as above, we have*

$$\mathbf{B}^{(1)}(B;\, s)\, L(s,\, \phi,\, \mathrm{St})\, \widetilde{H}_\phi(B;\, s+n-1/2)$$
$$= \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathrm{M}_{n-1}(\mathbb{Z})\cap \mathrm{GL}_{n-1}(\mathbb{Q})} C_{\sigma(\phi)}(B[W])\, (\det W)^{-s-k+3/2}.$$

Moreover, by applying Theorem 3.1 to the Jacobi Eisenstein series $\mathfrak{E}_{k,\,1}^{(n-1)} = e_{k,\,1}^{(n-1)} \in J_{k,\,1}(\Gamma_{n-1}^J)$, we obtain the following conclusion:

Theorem 3.3. *Let n and k be as above. Then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$\frac{\mathbf{B}_p^{(1)}(B;\, p^{n-1/2}t)\, \widetilde{G}_p^{(1)}(B;\, p^{k-(n+1)/2},\, p^{(n+1)/2}\, t)}{\prod_{i=1}^{n-1}(1-p^{j-1}\, p^{k-(n+1)/2}\, p^{(n+1)/2}\, t)(1-p^{j-1}\, p^{-k+(n+1)/2}\, p^{(n+1)/2}\, t)}$$
$$= \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z}_p)} \widetilde{F}_p^{(1)}(B[W];\, p^{k-(n+1)/2})\, (p^{(n+1)/2}\, t)^{\mathrm{ord}_p(\det W)},$$

where $\widetilde{F}_p^{(1)}(B;\, X)$ and $\widetilde{G}_p^{(1)}(B;\, X,\, t)$ are polynomials defined in §1.

Proof. By Proposition 2.4, the B-th Fourier coefficient of $\sigma(e_{k,\,1}^{(n-1)}) \in M_{k-1/2}^+(\Gamma_0^{(n-1)}(4))$ is expressed as

$$\xi(n,\, k)L(1-k/2+n/2, \chi_{B^{(1)}})\, \mathfrak{f}(B^{(1)})^{k-(n+1)/2} \prod_{p \mid \mathfrak{f}(B^{(1)})} \widetilde{F}_p^{(1)}(B;\, p^{k-(n+1)/2}).$$

Thus the assertion follows from Theorem 3.1 and Corollary of Proposition 2.2. □

For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, let $R_p^{(1)}(B;\, X,\, t)$ be the formal power series in $X + X^{-1}$ and t, which is defined in §1. Then we obtain the rationality for $R_p^{(1)}(B;\, X,\, t)$ as follows:

Theorem 3.4. *Let n be a positive even integer. Then for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$R_p^{(1)}(B;\, X,\, t) = \frac{\mathbf{B}_p^{(1)}(B;\, p^{n/2-1}t)\, \widetilde{G}_p^{(1)}(B;\, X,\, t)}{\prod_{j=1}^{n-1}(1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)}.$$

Proof. We write the both-hand sides of the above equation as power series in t as

$$R_p^{(1)}(B;\, X,\, t) = \sum_{i=1}^{\infty} A_i(X)t^i,$$

and

$$\frac{\mathbf{B}_p^{(1)}(B;\, p^{n/2-1}t)\, \widetilde{G}_p^{(1)}(B;\, X,\, t)}{\prod_{j=1}^{n-1}(1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)} = \sum_{i=1}^{\infty} B_i(X)t^i,$$

where for each i, $A_i(X)$ and $B_i(X)$ are polynomials in $X + X^{-1}$. Then by Theorem 3.3, we have

$$A_i(p^{k-(n+1)/2}) = B_i(p^{k-(n+1)/2})$$

for infinitely many k. Thus we have

$$A_i(X) = B_i(X)$$

for each i. Therefore we complete the proof. □

Remark. For a given pair of positive even integers n and k as in Theorem 3.1, let $f \in S_{2k-n}(\Gamma_1)$ be a Hecke eigenform, which possesses a Fourier expansion

$$f(z) = \sum_{N=1}^{\infty} a_f(N)\, \mathbf{e}(Nz) \quad (z \in \mathfrak{H}_1)$$

normalized by $a_f(1) = 1$. For each rational prime p, we denote by α_p the Satake p-parameter of f, that is, an algebraic number determined by $\alpha_p + \alpha_p^{-1} = a_f(p)\, p^{-k+(n+1)/2}$ uniquely up to inversion. Then by substituting $X = \alpha_p$ in the main identity of Theorem 3.4, we can also derive a similar identity to Theorem 3.3 for a power series related to the first Fourier-Jacobi coefficient of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to f under Ikeda's lifting procedure (cf. [9]). We note that it will play an important role in a proof of Ikeda's conjecture on the period of such a F, which was proposed in [10] (cf. [13, 14]).

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