Multiplication and Convolution of Distributions for Signal Processing Theory

Dai Akita

Graduate School of Life Science, Hokkaido University, N10W8, Sapporo, JAPAN

Abstract

In the theory of signal processing, signals are usually classified either by determining whether their time domain is discrete or continuous, or by determining whether they are periodic. However, no comprehensive definitions of multiplication and convolution exist that are consistent with the theories behind all classes, although some important theorems in signal processing involve multiplication and convolution. In order to unite the theories behind these classifications, we will consider tempered distributions. In this paper, we propose an approach to the multiplication and convolution of distributions that is appropriate to signal processing theory, and prove a well-known theorem regarding the impulse response of continuous linear time-invariant systems of tempered distributions in the context of this new approach.

Keywords: Theory of signal processing; multiplication of distributions; convolution of distributions; tempered distributions; LTI systems

1. Introduction

A *signal* is defined in standard textbooks as a complex-valued function, whose domain is called *time*. Signals are called *continuous-time signals* if the time is denoted by a real number, and *discrete-time signal* if the time is denoted by an integer. To emphasize that a signal f is a discrete-time signal, the expression $f[n]$ can be applied. Furthermore, signals can be classified based on whether they exhibit periodicity – that is, whether there exists a number $T > 0$ such that $f(t - T) = f(t)$ when f is continuous, or an integer $N > 0$ such that $f[n-N] = f[n]$ when f is discrete. Signals for which such T or N exist are called *periodic signals*; signals with no periodicity are called *non-periodic signals*. Because signals are classified by domain and periodicity, there are four methods of analyzing the frequency spectrum: *Fourier series* for continuous-time periodic signals, *Fourier transforms* for continuous-time non-periodic signals, *discrete Fourier transforms* for discrete-time periodic signals, and *discrete-time Fourier*

Email address: `akitad@mail.sci.hokudai.ac.jp` (Dai Akita)

transforms for discrete-time non-periodic signals. Corresponding to the theory of Fourier transforms, there are four different definitions of a convolution. However, while the relationships between Fourier transforms and convolutions are similar in each of the four cases, four separate theories of signal processing are presented in standard textbooks.

The theory of distributions that is described in detail in Section 2 integrates the four theories regarding the Fourier transform. This theory states that a discrete-time signal $f[n]$ can be expressed in terms of a delta function $\delta(x)$ and a sampling time T_s as

$$f(t) = \sum_{k=-\infty}^{\infty} f[k]\delta(t - kT_s). \tag{1}$$

Although $\delta(t)$ itself is defined in standard textbooks by the properties

$$\delta(t) = 0 \quad (t \neq 0) \tag{2}$$

and

$$\int_a^b \delta(t)dt = 1 \quad (a < 0 < b), \tag{3}$$

there is in fact no function that satisfies both conditions. Distribution theory provides a mathematically precise approach for dealing with $\delta(t)$, such that the distributional Fourier transform gives us the four Fourier transforms mentioned above [1].

Even if we consider discrete-time signals as distributions, multiplications between continuous-time signals and discrete-time signals are not preserved for existing definitions of multiplication for distributions, because the product $\delta \cdot \delta$ is not considered to be a distribution. Moreover, most conventional convolutions of distributions are only valid for distributions with a specific period, or only for distributions that have no period. In this paper, we propose a multiplication of distributions that preserves the multiplication of signals and admits a definition of a convolution of distributions that does not depend on whether the distributions have a period or not. To this end, the theory of distibutions and the results of previous studies on the multiplication and convolution of distributions are introduced in Section 2. Then, a multiplication and a convolution of distributions appropriate to signal processing theory are defined in Sections 3 and 4, respectively.

2. Tempered distributions

2.1. Conditions for signals to be tempered distributions

Schwartz [2] defined a distribution as a continuous linear functional, so that δ is defined as a functional $\delta : \varphi(t) \to \varphi(0)$ corresponding to the heuristic equation

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0). \tag{4}$$

However, we have to consider that the domain of the functional is a restricted function space.

A *rapidly decreasing function* is an infinitely differentiable complex-valued function $\varphi(t)$ satisfying

$$\sum_{n+k\leq N} \sup_{t\in\mathbb{R}} (1+|t|)^k \left|\varphi^{(n)}(t)\right| < \infty, \tag{5}$$

for any non-negative integer N. We express the space of rapidly decreasing functions as $\mathcal{S}$, and define convergence in $\mathcal{S}$ as follows. A sequence of a rapidly decreasing functions $\{\varphi_j\}$ is said to converge to φ if for any non-negative integer N,

$$\lim_{j\to\infty} \sum_{n+k\leq N} \sup_{t\in\mathbb{R}} (1+|t|)^k \left|\varphi_j^{(n)}(t) - \varphi^{(n)}(t)\right| = 0. \tag{6}$$

A *tempered distribution* F is a continuous linear functional of rapidly decreasing functions, where the value of F at φ is described as $\langle F, \varphi\rangle$. The continuity of a functional F in $\mathcal{S}$ is defined as follows. A functional F is continuous if for any sequence $\{\varphi_j\}$ which converges to 0 in $\mathcal{S}$,

$$\lim_{j\to\infty} \langle F, \varphi_j\rangle = 0. \tag{7}$$

If necessary, we employ the notation $\langle F(t), \varphi(t)\rangle$ to express the variable of φ explicitly. Moreover, we denote the space of a tempered distributions by $\mathcal{S}'$. Then, the *Fourier transform of a tempered distribution* $\mathcal{F}$ is defined as follows:

$$\langle \mathcal{F}[F], \varphi\rangle = \langle F, \mathcal{F}[\varphi]\rangle, \tag{8}$$

where $\varphi \in \mathcal{S}$, $F \in \mathcal{S}'$, and $\mathcal{F}[\varphi]$ on the right-hand side is the classical Fourier transform of the rapidly decreasing function

$$\mathcal{F}[\varphi](\omega) = \int_{-\infty}^{\infty} \varphi(t)\exp(-i\omega t)dt. \tag{9}$$

This Fourier transform is a one-to-one mapping of $\mathcal{S}'$ onto itself [3]. Here, and throughout this article, the Lebesgue integral is used.

Now, we describe how signals can be regarded as distributions. For a continuous-time signal f, consider a functional T_f such that for a given $\varphi \in \mathcal{S}$,

$$\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(t)\varphi(t)dt. \tag{10}$$

If T_f has a value for any $\varphi \in \mathcal{S}$ and continuity on $\mathcal{S}$, then T_f is a tempered distribution. Then, we also use the expression f to refer to T_f as a distribution.

In fact, there exist some sufficient conditions for a function f on $\mathbb{R}$ to be a tempered distribution T_f. The following theorem states one of these conditions [3].

Theorem 1. *Let f be a complex-valued, Lebesgue-measurable function on $\mathbb{R}$. If there exist $C > 0$ and a non-negative integer K such that*

$$|f(t)| \leq C(1+|t|)^K \tag{11}$$

for almost all t, then $T_f \in \mathcal{S}'$.

We let $\mathcal{Q}_M$ denote the space of functions satisfying the conditions of Theorem 1.

Before introducing the condition for a discrete-time signal to be a tempered distribution, we justify expression (1). As mentioned in the first part of this subsection, δ is defined as a tempered distribution, so that for any $\varphi \in \mathcal{S}$,

$$\langle \delta, \varphi \rangle = \varphi(0). \tag{12}$$

The *translation of distributions* τ_h is an operator parametrized with $h \in \mathbb{R}$, such that for any $F \in \mathcal{S}'$,

$$\langle \tau_h F, \varphi \rangle = \langle F, \varphi(t+h) \rangle . \tag{13}$$

We may employ the notation $\delta(t-h)$ for $\tau_h \delta$. In addition, we define the operator *reflection of distributions*, denoted by $\check{}$, as

$$\left\langle \check{F}, \varphi(t) \right\rangle = \langle F, \varphi(-t) \rangle . \tag{14}$$

Furthermore, we define the convergence of tempered distributions as follows. A sequence of tempered distributions $\{F_N\}$ is said to converge to $F \in \mathcal{S}'$ if

$$\forall \varphi \in \mathcal{S} \quad \lim_{N\to\infty} \langle F_N, \varphi \rangle = \langle F, \varphi \rangle . \tag{15}$$

In particular, for a sequence of tempered distributions $\{F_N\}$, $\sum_{k=-\infty}^{\infty} F_k$ denotes $\lim_{N\to\infty} \sum_{k=-N}^{N} F_k$. Using this definition, a discrete-time signal f is regarded as a tempered distribution with sampling time $T_s > 0$ if

$$\langle f, \varphi \rangle = \left\langle \sum_{k=-\infty}^{\infty} f[k]\delta(t-kT_s), \varphi \right\rangle = \lim_{N\to\infty} \sum_{k=-N}^{N} f[k]\varphi(kT_s) \tag{16}$$

has a value for any $\varphi \in \mathcal{S}$, and if $\langle f, \varphi_j \rangle \to 0$ whenever $\varphi_j \to 0$ in $\mathcal{S}$.

Just as arbitrary continuous-time signals cannot be regarded as tempered distributions, not all discrete-time signals are tempered distributions. However, in contrast to the case of continuous-time signals, there exists a necessary and sufficient condition for a discrete-time signal to be regarded as a tempered distribution. To show this, we introduce two theorems from [4].

Theorem 2. *Let $f[n]$ be a discrete-time signal. If there exist real numbers M and K such that*

$$|f[n]| \leq M|n|^K \tag{17}$$

for all non-zero $n \in \mathbb{Z}$, then for any $\omega_0 > 0$, $\lim_{N\to\infty} \sum_{k=-N}^{N} f[k] \exp(ik\omega_0 t)$ converges to a tempered distribution.

Theorem 3. *Let $F \in \mathcal{S}'$. If there exists $P > 0$ such that*

$$\tau_P F = F, \tag{18}$$

then there exists a discrete-time signal $f[n]$ such that

$$F = \sum_{k=-\infty}^{\infty} f[k] \exp\left(\frac{2\pi i k t}{P}\right). \tag{19}$$

Moreover, there exist real numbers M and K such that

$$|f[n]| \leq M|n|^K \tag{20}$$

for all non-zero $n \in \mathbb{Z}$.

Further, we will employ the following additional theorem from [3].

Theorem 4. *Let $\{F_N\}$ be a sequence of tempered distributions. Then, $\{F_N\}$ converges to the tempered distribution F if and only if $\{\mathcal{F}[F_N]\}$ converges to $\mathcal{F}[F]$.*

Applying these results, we can obtain a necessary and sufficient condition for a discrete-time signal to be regarded as a tempered distribution.

Corollary 1. *Let $f[n]$ be a discrete-time signal, and let T_s be a positive real number. Then, $\sum_{k=-\infty}^{\infty} f[k]\delta(t - kT_s)$ is a tempered distribution if and only if there exist real numbers M and K such that*

$$|f[n]| \leq M|n|^K \tag{21}$$

for all non-zero $n \in \mathbb{Z}$.

Proof. Let

$$T_N = \sum_{k=-N}^{N} f[k]\delta(t - kT_s) \tag{22}$$

and

$$S_N = \mathcal{F}[T_N] = \sum_{k=-N}^{N} f[k] \exp(ikT_s t). \tag{23}$$

Then, $\{T_N\}$ converges to a tempered distribution if and only if $\{S_N\}$ converges, from Theorem 4. Here, if there exist such M and K, then Theorem 2 implies that $\{S_N\}$ converges. Furthermore, if $\{S_N\}$ converges to a tempered distribution, then there also must exist such M and K according to Theorem 3, as

$$\begin{aligned}\tau_{2\pi/T_s} \sum_{k=-\infty}^{\infty} f[k] \exp(ikT_s t) &= \sum_{k=-\infty}^{\infty} f[k] \exp(ikT_s(t - 2\pi/T_s)) \\ &= \sum_{k=-\infty}^{\infty} f[k] \exp(ikT_s t). \end{aligned} \tag{24}$$

Thus, $\{T_N\}$ converges if and only if there exist such numbers M and K. $\square$

We let $\mathfrak{D}_M$ denote the space of discrete-time signals satisfying the conditions in Corollary 1.

Assuming that signals that are classified into the four classes mentioned above can be expressed as tempered distributions in this way, the Fourier transform defined by (8) gives us Fourier series, the Fourier transform of a function, the discrete Fourier transform, and the discrete-time Fourier transform, depending on the classification [1]. Thus, the four classifications of signals and their corresponding Fourier transforms may be integrated into a single theory using tempered distributions.

In fact, tempered distributions include elements that are not covered by the four theories of signal processing. For example, the almost periodic function $f(t) = \sin t + \sin \sqrt{2}t$, despite being non-periodic, is not usually discussed in textbooks because its Fourier transform as function

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \tag{25}$$

cannot be defined. However, as tempered distribution $f(t)$ has a Fourier transform $(\delta(t-1) - \delta(t+1) + \delta(t-\sqrt{2}) - \delta(t+\sqrt{2}))/2i$. Recognizing signals as tempered distributions enables us to treat mathematically troublesome problems of this sort.

To conclude this subsection, we define some sets relating to distributions. A tempered distribution F is said to vanish on an open set Ω if

$$\forall \varphi \in \mathcal{S}, \quad \text{supp}\ \varphi \subset \Omega \Rightarrow \langle F, \varphi \rangle = 0, \tag{26}$$

and we denote the union of the open sets on which F vanishes by $Z(F)$. Then, the support of F is defined as

$$\text{supp}\ F = \mathbb{R} \setminus Z(F). \tag{27}$$

2.2. Preservation of multiplication and integration of convolution

Let $\mathcal{T}$ be a mapping from $\mathcal{Q}_M$ to $\mathcal{S}'$ such that

$$\forall f \in \mathcal{Q}_M, \varphi \in \mathcal{S} \quad \langle \mathcal{T}[f], \varphi \rangle = \int_{-\infty}^{\infty} f(t)\varphi(t) dt, \tag{28}$$

and for $T_s > 0$, let $\mathfrak{T}_{T_s}$ be a mapping from $\mathfrak{D}_M$ to $\mathcal{S}'$ such that

$$\forall f \in \mathfrak{D}_M, \varphi \in \mathcal{S} \quad \langle \mathfrak{T}_{T_s}[f], \varphi \rangle = \sum_{k=-\infty}^{\infty} f[k]\varphi(kT_s). \tag{29}$$

We let $\mathcal{T}[\mathcal{Q}_M] = \{\mathcal{T}[f] | f \in \mathcal{Q}_M\}$ and $\mathfrak{T}_{T_s}[\mathfrak{D}_M] = \{\mathfrak{T}_{T_s}[f] | f \in \mathfrak{D}_M\}$. Then, continuous-time signals in $\mathcal{Q}_M$ and discrete-time signals in $\mathfrak{D}_M$ can be regarded as tempered distributions given by $\mathcal{T}$ and $\mathfrak{T}_{T_s}$ respectively, according to the subsection 2.1. Moreover, the Fourier transforms of signals correspond to the Fourier transforms of their representations as tempered distributions.

However, while both $\mathcal{Q}_M$ and $\mathfrak{O}_M$ constitute rings, the usual multiplication of distributions holds only for very restricted distributions. For example, in many textbooks on distributions, we can find the following definition of multiplication for a tempered distribution and a slowly increasing function, where a *slowly increasing function* is an infinitely differentiable function $f \in \mathcal{Q}_M$. Let $\mathcal{O}_M$ denote the space of a slowly increasing function. Then, the product of $F \in \mathcal{S}'$ and $f \in \mathcal{O}_M$ is defined as

$$\langle fF, \varphi \rangle = \langle F, f\varphi \rangle \tag{30}$$

for any $\varphi \in \mathcal{S}$, because $f\varphi \in \mathcal{S}$. However, this definition of multiplication cannot be applied for $f, g \in \mathfrak{T}_{T_s}[\mathfrak{O}_M]$.

Consider defining a multiplication in a subset of $\mathcal{S}'$ containing $\mathcal{T}[\mathcal{Q}_M]$ and $\mathfrak{T}_{T_s}[\mathfrak{O}_M]$ for any $T_s > 0$, such that $\mathcal{T}$ and $\mathfrak{T}_{T_s}$ are ring homomorphisms. Then, we find that $\delta \cdot \delta = \delta$. To explain this, set

$$f[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases} \tag{31}$$

so that $\mathfrak{T}_{T_s}[f] = \delta$. Then, we see that

$$\delta = \mathfrak{T}_{T_s}[f] = \mathfrak{T}_{T_s}[f \cdot f] = \mathfrak{T}_{T_s}[f] \cdot \mathfrak{T}_{T_s}[f] = \delta \cdot \delta. \tag{32}$$

As will be described later, the existing definition of multiplication for distributions considers restricted distributions, and the majority of studies do not define $\delta \cdot \delta$ as a distribution.

Moreover, the convolution of distributions is only defined for a restricted subset of distributions [5, 6, 7, 8], and usually the definition of a convolution only holds for non-periodic distributions, as follows. For $F \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$,

$$F * \varphi(x) = \langle F(t), \varphi(x - t) \rangle . \tag{33}$$

For $F, G \in \mathcal{S}'$, if supp G is compact,

$$\langle F * G, \varphi \rangle = \left\langle F, \check{G} * \varphi \right\rangle . \tag{34}$$

The proof that these convolutions have values in $\mathcal{S}'$ can be found in certain textbooks on distributions (e.g., [9, 10]), and in fact, the convolution of $F \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ given by equation (33) belongs to $\mathcal{O}_M$. Smith [4] provides a rigorous and concise review of periodic tempered distributions, and defines a convolution for periodic tempered distributions. However, this cannot be applied to non-periodic tempered distributions. It is preferable to have a theory with one simple definition of convolution for an integrated treatment of signals, as with Fourier transforms, rather than separate definitions for the four cases mentioned above.

These difficulties concerning multiplication and convolution are frustrating, especially considering that there is a single important general formula relating the Fourier transform to multiplication and convolution in signal processing theory,

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g], \tag{35}$$

which appears in the theories of the respective signal classifications. If we define multiplication and convolution in a comprehensive way, this theorem can be described without the classification of signals.

In fact, the two problems are very closely related: finding integrated definition of convolution means determining a multiplication that meets the criterion of formula (35), in which the right-hand side denotes the multiplication of the spectrum. If the signal is not periodic, then with some exceptions the spectrum is expressed by a function on $\mathbb{R}$, whereas if the signal does have a period, then the spectrum is expressed by a function on $\mathbb{Z}$ [1]. Therefore, if we want to calculate the convolution of periodic signals such that equation (35) holds, we must justify the equation $\delta \cdot \delta = \delta$ [1]. That is, these problems are solved if there exists a multiplication such that $\delta \cdot \delta = \delta$, and we can define convolution as

$$f * g = \mathcal{F}^{-1} \left[\mathcal{F}[f]\mathcal{F}[g] \right], \tag{36}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

The difficulty of defining the multiplication of any two distributions was considered by Schwartz [11], who showed that there is no vector space that allows multiplication, derivation, and the existence of δ, so the expression $\delta \cdot \delta$ does not make sense. Therefore, although some studies have extended the domain of multiplication of distributions, none of these includes $\delta \cdot \delta$ (e.g., [6, 12, 13]). Thus, δ^2 has been excluded from the distribution space, but studies concerning δ^2 have proceeded by using the limit of a sequence of functions [14]. As a result of such studies, an extended space that includes the space of distributions has begun to be considered, termed *generalized functions* [2], and various multiplications of the generalized functions have been proposed [15, 16, 17]. The expression δ^2, in terms of the generalized functions, has been justified in various ways [15, 16, 18, 19]. In some of these, $\delta^2 = \delta$. For example, according to Vladimirov [19], $\delta^2 = C\delta$, where C is an arbitrary constant, so that $\delta^2 = \delta$ if we let $C = 1$. Bagarello [16] proposed a multiplication of generalized functions with some parameters, and we can choose these parameters such that $\delta^2 = \delta$. In this paper, however, these theories of generalized functions are not employed. A new multiplication is proposed, in which $\delta^2 = \delta$ in the distribution sense. The reason for not using generalized functions is that we can utilize the theory of the Fourier transform for tempered distributions, which has been studied for many years. Moreover we can avoid the complex theory of generalized functions. Although, in addition to the theory of generalized functions, non-standard analysis has been used to justify δ^2 [20, 21], our proposed definition of multiplication does not require either of these theories.

[1] In [4], where Smith reviewed periodic tempered distributions, he stated a convolution theorem for periodic tempered distributions. However, equation (35) in the theorem was not for the Fourier transform of tempered distributions, but for the finite Fourier transform of periodic tempered distributions.

[2] Although the space of generalized functions may or may not include distributions, depending on its definition, we refer to it as a set that contains the space of distributions.

In addition to the above literature, we should refer to Oberguggenberger's book [22], where many studies on multiplication of distributions are reviewed. He describes three main approaches to defining multiplication. The first is called "regular intrinsic operations", in which regular objects such as functions are treated and the operations are defined classically. The second is "irregular intrinsic operations", in which a product for particular pairs of singular distributions is defined. The last approach, "extrinsic products and algebras containing the distributions", uses objects outside of distributions. Furthermore, he describes a list of methods relating to the second approach: "(a) the duality method; (b) the method of Fourier transform and convolution; (c) molification and passage to the limit; and (d) analytic and harmonic regularization and passage to the limit or even extraction of the finite part." The approach in this paper is similar to (b), with some differences. In the method (b), a convolution of distributions is first defined, and then a multiplication is derived by

$$FG = \mathcal{F}\left[\mathcal{F}^{-1}[F] * \mathcal{F}^{-1}[G]\right]. \tag{37}$$

In contrast, we first define a multiplication of distributions and then derive a convolution by equation (36). As it has been noted by Oberguggenberger [22] that it is not possible to simultaneously have full generality, freedom in differential-algebraic manipulations, and coherence with classical definitions, the domain of the multiplication in this paper is not closed under differentiation, as mentioned in subsection 4.1.

3. δ-series distributions and multiplication

3.1. Definition

In this section, we will present our general definition of signal multiplication, in which the product of two discrete-time signals of δ-series form, equation (1), is the δ-series whose coefficients are products of the corresponding coefficients in the two series. In addition, in this general definition, the product of two continuous-time signals or the product of a continuous-time signal and discrete-time signal will involve a limiting process, also based on δ-series. Before presenting this general definition of multiplication, we first show that continuous-time signals may be expressed as a limit of δ-series.

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous-time signal, and let

$$f_N = \sum_{k=-\infty}^{\infty} \frac{1}{2^N} f\left(\frac{k}{2^N}\right) \delta\left(t - \frac{k}{2^N}\right). \tag{38}$$

From Corollary 1, we have $f_N \in \mathcal{S}'$ for all $N \in \mathbb{N}$. Considering the limit of f_N as $N \to \infty$, we obtain the following theorem.

Theorem 5. *Let f be a complex-valued Lebesgue-measurable right continuous function on $\mathbb{R}$. If there exist $C > 0$ and a non-negative integer K such that*

$$|f(t)| \le C(1 + |t|)^K \tag{39}$$

for almost all t, then the sequence $\{f_N\}$ defined by (38) converges to f.

Proof. Let $\varphi \in \mathcal{S}$ and $I_{N,k} = [(k-1)/2^N, k/2^N)$. We consider

$$g(t) = f(t)\varphi(t), \tag{40}$$
$$g_N(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2^N}\right)\varphi\left(\frac{k}{2^N}\right)\mathbf{1}_{I_{N,k}}(t), \tag{41}$$

where $\mathbf{1}_{I_{N,k}}(t)$ is an indicator function such that

$$\mathbf{1}_{I_{N,k}}(t) = \begin{cases} 1, & t \in I_{N,k} \\ 0. & t \notin I_{N,k} \end{cases} \tag{42}$$

Here,

$$\langle f_N, \varphi\rangle = \sum_{k=-\infty}^{\infty} \frac{1}{2^N} f\left(\frac{k}{2^N}\right)\varphi\left(\frac{k}{2^N}\right) = \int_{-\infty}^{\infty} g_N(t)dt. \tag{43}$$

First, we show that $g_N(t)$ converges pointwise to $g(t)$. For an arbitrary number t_0, we can take a sequence $\{k_1, k_2, ...\}$ such that

$$\forall N \quad t_0 \in I_{N,k_N}. \tag{44}$$

Then, we see that

$$g_N(t_0) = f\left(\frac{k_N}{2^N}\right)\varphi\left(\frac{k_N}{2^N}\right). \tag{45}$$

Noting that $\{k_N/2^N\}$ is a monotonically decreasing sequence converging to t_0 and f is right continuous, we have

$$\lim_{N\to\infty} g_N(t_0) = \lim_{N\to\infty} f\left(\frac{k_N}{2^N}\right)\varphi\left(\frac{k_N}{2^N}\right) = f(t_0)\varphi(t_0) = g(t_0). \tag{46}$$

Now, we will show that $\int g_N(t)dt \to \int g(t)dt$ $(N \to \infty)$ by using Lebesgue's dominated convergence theorem [3]. Given that

$$h(t) = \sum_{k=-\infty}^{\infty} \sup_{\frac{k-1}{2} \le x < \frac{k}{2}} |g(x)|\mathbf{1}_{I_{1,k}}(t), \tag{47}$$

we have

$$\forall t \in \mathbb{R}, N \in \mathbb{N} \quad h(t) \ge g_N(t). \tag{48}$$

[3] This theorem can be found in many textbooks on the Lebesgue integral (e.g., [23]). The claim is as follows: Given that $(X, \mathfrak{B}, \mu)$ is a measure space and $E \in \mathfrak{B}$, let f_n be a function that is measurable on E for all $n \in \mathbb{N}$ and $f_n(x) \to f(x)$ $(n \to \infty)$. If there exists a function $\varphi(x)$ that is integrable over E and for any $x \in E$, $|f_n(x)| \le \varphi(x)$ $(n \in \mathbb{N})$, then $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$.

On the other hand, there exist C, K, and L such that

$$|f(t)| \leq C(1+|t|)^K, \tag{49}$$

$$(1+|t|)^{K+2}|\varphi(t)| < L. \tag{50}$$

Then,

$$\begin{aligned}\int_{-\infty}^{\infty} h(t)dt &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \sup_{\frac{k-1}{2}\leq x<\frac{k}{2}} |f(x)\varphi(x)| \leq \frac{1}{2}\sum_{k=-\infty}^{\infty} \sup_{\frac{k-1}{2}\leq x<\frac{k}{2}} |f(x)||\varphi(x)| \\ &< \frac{1}{2}\sum_{k=-\infty}^{\infty} \sup_{\frac{k-1}{2}\leq x<\frac{k}{2}} \frac{CL}{(1+|x|)^2} = \sum_{k=0}^{\infty} \frac{CL}{(1+\frac{k}{2})^2}, \end{aligned}\tag{51}$$

which means that $h(t)$ is integrable over $\mathbb{R}$. Hence, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} g_N(t)dt = \int_{-\infty}^{\infty} g(t)dt. \tag{52}$$

That is,

$$\lim_{N\to\infty} \langle f_N, \varphi\rangle = \langle f, \varphi\rangle . \tag{53}$$

□

Remark. The idea of a multiplication involving coefficients of δ is not new. For example, as described in [4], we can define a mapping $\mathcal{F}_P$ called a *finite Fourier transform* from a set of tempered distributions of period P to a set of sequences, such that the sequences correspond to Fourier coefficients. Moreover, there exists a convolution $*_P$ for tempered distributions of period P such that

$$\mathcal{F}_P[F *_P G] = P\mathcal{F}_P[F]\mathcal{F}_P[G]. \tag{54}$$

Here, $f, g \in \mathfrak{O}_M$, $F = \mathcal{F}^{-1}[\mathfrak{T}_{2\pi/P}[f]]$, and $G = \mathcal{F}^{-1}[\mathfrak{T}_{2\pi/P}[g]]$ with a period P. Therefore, $\mathcal{F}[F *_P G]$ denotes the product of the coefficients of $\mathfrak{T}_{2\pi/P}[f]$ and $\mathfrak{T}_{2\pi/P}[g]$, although this is valid only for a specific period.

Thus, any function that satisfies the conditions of Theorem 5 is a tempered distribution, as the limit of a δ-series. Because, as series of δ, continuous-time signals attain the same form as discrete-time signals, it follows that we can define a new series of δ as a product of distributions by using the product of the coefficients of δ. We now define these concepts in detail. In the following definition, $a\mathbb{Z} + c$ and $\mathbb{Z}/b + d$ denote $\{ak + c|k \in \mathbb{Z}\}$ and $\{k/b + d|k \in \mathbb{Z}\}$, respectively, for $a, b, c, d > 0$.

Definition 1. *A tempered distribution F is a δ-*series distribution *if there exist $f : \mathbb{R} \to \mathbb{C}$ and a sequence of sets $\{B_N\}_{N\in\mathbb{N}}$ satisfying the following conditions:*

(I) $\{B_N\} = \{\mathbb{Z}/2^N\}$, *otherwise the sequence* $\{B_N\}$ *is independent of* N *and there exists a finite number of positive real numbers* $T_1, ..., T_n, p_1, ..., p_n$ *such that*

$$B_N \subset \bigcup_{j=1}^{n} T_j\mathbb{Z} + p_j. \tag{55}$$

(II) $f(t)$ *is Lebesgue-measurable and right continuous, and there exist* $C > 0$ *and a non-negative integer* K *such that*

$$|f(t)| \leq C(1 + |t|)^K, \tag{56}$$

for almost all t*.*

(III) *Defining*

$$J_N = \begin{cases} 1/2^N, & \{B_N\} = \{\mathbb{Z}/2^N\} \\ 1, & \text{else} \end{cases} \tag{57}$$

the sequence of tempered distributions

$$F_N = \sum_{h \in B_N} J_N f(h)\delta(t - h) \tag{58}$$

converges to F *as* $N \to \infty$*. Here, in the case that* B_N *is an empty set we define* F_N *as* 0*, and in the case that* B_N *is an infinite set the summation is defined as follows:*

$$F_N = \lim_{M \to \infty} \sum_{h \in B_N \cap [-M,M]} J_N f(h)\delta(t - h). \tag{59}$$

Let $\mathcal{A}'_\delta$ denote the set of δ-series distributions. If f and $\{B_N\}$ satisfy the above conditions for $F \in \mathcal{A}'_\delta$, then we write $F = (f, \{B_N\})_\delta$. Moreover, we use superscripts for $\{B_N\}$ such as $\{B_N^f\}$, $\{B_N^g\}$, and $\{B_N^{fg}\}$ if we need to distinguish between different $\{B_N\}$. The following theorem is easy to derive.

Theorem 6. *For any* f *and* $\{B_N\}$ *satisfying the conditions (I) and (II) of Definition 1, there exists a tempered distribution* F *such that* $F = (f, \{B_N\})_\delta \in \mathcal{A}'_\delta$*.*

Proof. We have already treated the case in which $\{B_N\} = \{\mathbb{Z}/2^N\}$, so we assume that $\{B_N\} \neq \{\mathbb{Z}/2^N\}$ and omit the "N" in B_N and F_N, because B_N does not depend on N according to condition (I) of Definition 1. Then, there exists a finite number of positive real numbers $T_1, ..., T_n, p_1, ..., p_n$, according to that condition. Therefore, given that $\varphi \in \mathcal{S}$, we have

$$\sum_{h \in B} |f(h)\varphi(h)| \leq \sum_{j=1}^{n} \sum_{k=-\infty}^{\infty} |f(kT_j + p_j)||\varphi(kT_j + p_j)|. \tag{60}$$

The last term is guaranteed to have a finite value because of Corollary 1, so that F is a linear functional on $\mathcal{S}$. The continuity of F in $\mathcal{S}$ can also be derived from equation (60), by considering a sequence of rapidly decresing functions $\{\varphi_j\}$ that converges to 0. Thus, it is shown that $F = (f, \{B_N\})_\delta \in \mathcal{A}'_\delta$. □

Example 1. The Heaviside function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t \end{cases} \tag{61}$$

satisfies the condition (II) of Definition 1. Therefore, u is a δ-series distribution expressed by $(u, \{\mathbb{Z}/2^N\})_\delta$.

Example 2. Let $F = \sum_{k=-\infty}^{\infty} k\delta(t-k)$. Then, $F \in \mathcal{A}'_\delta$, because $f(t) = t$ and $\{B_N^f\} = \{\mathbb{Z}\}$ meet the conditions of Definition 1 for F. Therefore, F is denoted by $(t, \{\mathbb{Z}\})_\delta$.

Here, suppose that $g(t) = t\cos^2(\pi t)$ and $\{B_N^g\} = \{\mathbb{Z}/2\}$. Then, g and $\{B_N^g\}$ also satisfy the conditions of Definition 1, so that $(t\cos^2(\pi t), \{\mathbb{Z}/2\})_\delta \in \mathcal{A}'_\delta$. Given that $\varphi \in \mathcal{S}$, we have that

$$\begin{aligned} \left\langle (t\cos^2(\pi t), \{\mathbb{Z}/2\})_\delta, \varphi \right\rangle &= \sum_{k=-\infty}^{\infty} \frac{k}{2} \cos^2\left(\frac{k\pi}{2}\right) \varphi\left(\frac{k}{2}\right) \\ &= \sum_{k=-\infty}^{\infty} k\varphi(k) = \langle (t, \{\mathbb{Z}\})_\delta, \varphi \rangle . \end{aligned} \tag{62}$$

Thus, $(t\cos^2(\pi t), \{\mathbb{Z}/2\})_\delta$ also denotes F, and so the expression of $F \in \mathcal{A}'_\delta$ is not unique.

Example 3. $\delta(t-2) + \sum_{k=-\infty}^{\infty} \delta(t - \sqrt{2}k - 1)$ is a δ-series distribution expressed by $(1, \{\{2\} \cup (\sqrt{2}\mathbb{Z}+1)\})_\delta$. We can confirm that $\{B_N\} = \{\{2\} \cup (\sqrt{2}\mathbb{Z}+1)\}$ satisfies the condition (I), because

$$B_N \subset \mathbb{Z} \cup (\sqrt{2}\mathbb{Z}+1). \tag{63}$$

Example 4. Let $f[n] \in \mathfrak{O}_M$ and

$$g(t) = f\left[\left\lfloor \frac{t}{T_s} \right\rfloor\right], \tag{64}$$

where $\lfloor t \rfloor$ is the floor function. Then, $\mathfrak{T}_{T_s}[f]$ is a δ-series distribution $(g, \{T_s\mathbb{Z}\})_\delta$

Example 5. Let $f \in \mathcal{Q}_M$ be right continuous. Then, $\mathcal{T}[f] = (f, \{\mathbb{Z}/2^N\})_\delta \in \mathcal{A}'_\delta$ according to Theorem 5.

According to the definition of a δ-series distribution, $\{B_N\}$ indicates whether F is continuous or discrete. That is, if $\{B_N\} = \{\mathbb{Z}/2^N\}$, then F is described by a function, otherwise F is described by a summation of $\delta(t-h)$. We denote the closure of $\bigcup_{N=1}^{\infty} B_N$ by B_∞. If $\{B_N\} = \{\mathbb{Z}/2^N\}$ then $B_\infty = \mathbb{R}$, otherwise $B_1 = B_2 = \cdots = B_\infty$ and the series contains no accumulation point of $\mathbb{R}$ according to condition (I) of Definition 1. We use this fact in the proofs of some lemmas in the subsection 3.2.

For these δ-series distributions, a new multiplication of distributions is defined.

Definition 2. *Given $F = (f, \{B_N^f\})_\delta$ and $G = (g, \{B_N^g\})_\delta$, their product is defined as $FG = (fg, \{B_N^{fg}\})_\delta$, where we choose $\{B_N^{fg}\}$ satisfying condition (I) of Definition 1 and*

$$B_\infty^{fg} = B_\infty^f \cap B_\infty^g. \tag{65}$$

In Definition 2, it is obvious that fg satisfies condition (II) of Definition 1, and we can choose $\{B_N^{fg}\}$ satisfying condition (I) of Definition 1 and equation (65). Therefore, to demonstrate well-definedness of the multiplication, we will illustrate later that FG is independent of the choice of f, g, $\{B_N^f\}$, and $\{B_N^g\}$ for $F = (f, \{B_N^f\})_\delta$ and $G = (g, \{B_N^g\})_\delta$. Deferring the proof of the well-definedness to the subsection 3.2, we present some examples.

Example 6. Let $F = (t, \{\mathbb{Z}/2^N\})_\delta$ and $G = (\sin t, \{\mathbb{Z}/2^N\})_\delta$. Here, $\{B_N^{fg}\} = \{\mathbb{Z}/2^N\}$, because B_∞^{fg} must be $B_\infty^f \cap B_\infty^g = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. Hence, $FG = (t \sin t, \{\mathbb{Z}/2^N\})_\delta$.

Example 7. Let $f \in \mathcal{Q}_M$ and $G = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = (1, \{T_s\mathbb{Z}\})_\delta$. Then, $fG = (f, \{T_s\mathbb{Z}\})_\delta = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s)$.

Example 8. Let $F = (t, \{\mathbb{Z}\})_\delta$ and $G = (\sin t, \{\mathbb{Z}/2^N\})_\delta$. In this case, B_∞^{fg} must be $B_\infty^f \cap B_\infty^g = \mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$. Note that $B_\infty^{fg} \neq \mathbb{R}$, $\{B_N^{fg}\} = \{\mathbb{Z}\}$. Thus, $FG = (t \sin t, \{\mathbb{Z}\})_\delta$.

Consider the additional expression $(t \cos^2(\pi t), \{\mathbb{Z}/2\})_\delta$ for F, which is shown in Example 2. Then, $FG = (t \sin(t) \cos^2(\pi t), \{\mathbb{Z}/2\})_\delta$. Given that $\varphi \in \mathcal{S}$, we see that

$$\begin{aligned}\left\langle (t \sin(t) \cos^2(\pi t), \{\mathbb{Z}/2\})_\delta, \varphi \right\rangle &= \sum_{k=-\infty}^{\infty} \frac{k}{2} \sin\left(\frac{k}{2}\right) \cos^2\left(\frac{k\pi}{2}\right) \varphi\left(\frac{k}{2}\right) \\ &= \sum_{k=-\infty}^{\infty} k \sin(k) \varphi(k) = \langle (t \sin t, \{\mathbb{Z}\})_\delta, \varphi \rangle .\end{aligned} \tag{66}$$

This is an example of the well-definedness.

Example 9. Let $F = (\exp(2\pi i t/3), \{2\mathbb{Z}\})_\delta$ and $G = (\exp(-\pi i t), \{3\mathbb{Z}\})_\delta$ and consider their product. Then, $\{B_N^{fg}\} = \{6\mathbb{Z}\}$. Therefore,

$$FG = (\exp(-\pi i t/3), \{6\mathbb{Z}\})_\delta = (1, \{6\mathbb{Z}\})_\delta. \tag{67}$$

The next two examples show that $\mathcal{T}$ and $\mathfrak{T}_{T_s}$ are ring homomorphisms.

Example 10. Let $\mathcal{Q}_m$ be a subset of $\mathcal{Q}_M$, the functions of which are right continuous, and let $\mathcal{T}[\mathcal{Q}_m] = \{\mathcal{T}[f] | f \in \mathcal{Q}_m\}$. For any $f, g \in \mathcal{Q}_m$, we can easily confirm that $fg \in \mathcal{Q}_m$ and

$$\mathcal{T}[f]\mathcal{T}[g] = (fg, \{\mathbb{Z}/2^N\})_\delta = \mathcal{T}[fg]. \tag{68}$$

Hence, the mapping $\mathcal{T} : \mathcal{Q}_m \to \mathcal{T}[\mathcal{Q}_m]$ is a ring homomorphism.

Example 11. For any $f, g \in \mathfrak{O}_M$ and $T_s > 0$,

$$\mathfrak{T}_{T_s}[f]\mathfrak{T}_{T_s}[g] = (fg, \{T_s\mathbb{Z}\})_\delta = \mathfrak{T}_{T_s}[fg], \tag{69}$$

which means that the mapping $\mathfrak{T}_{T_s} : \mathfrak{O}_M \to \mathfrak{T}_{T_s}[\mathfrak{O}_M]$ is a ring homomorphism.

This new multiplication does not contradict the multiplication defined by equation (30). The proof is given by calculating the multiplication of $f \in \mathcal{O}_M$ and $G = (g, \{B_N^g\})_\delta \in \mathcal{A}'_\delta$. Noting that $f = (f, \{\mathbb{Z}/2^N\})_\delta$, the product of the new multiplication is

$$fG = (fg, \{B_N^g\})_\delta. \tag{70}$$

On the other hand, the product according to equation (30) is

$$\begin{aligned} \langle fG, \varphi \rangle &= \langle G, f\varphi \rangle = \langle (g, \{B_N^g\})_\delta, f\varphi \rangle \\ &= \lim_{N\to\infty} \sum_{h \in B_N^g} J_N g(h) f(h) \varphi(h) = \langle (fg, \{B_N^g\})_\delta, \varphi \rangle . \end{aligned} \tag{71}$$

Hence, we can also use the multiplication for $\mathcal{O}_M$ and $\mathcal{S}'$, in addition to the new multiplication, if necessary.

Functions in $\mathcal{A}'_\delta$ are not allowed to have singular points. For example, the following tempered distributions are not in $\mathcal{A}'_\delta$ because of the condition (II) of Definition 1:

$$\left\langle \sqrt{t_+}, \varphi \right\rangle = \int_0^\infty \frac{\varphi(t)}{\sqrt{t}} dt, \tag{72}$$

$$\left\langle \sqrt{t_-}, \varphi \right\rangle = \int_{-\infty}^0 \frac{\varphi(t)}{\sqrt{-t}} dt, \tag{73}$$

$$\left\langle \mathrm{vp}\ \frac{1}{t}, \varphi \right\rangle = \lim_{\varepsilon \downarrow 0} \int_{|t|>\varepsilon} \frac{\varphi(t)}{t} dt. \tag{74}$$

These tempered distributions are known for being problematic, in that $\sqrt{t_+}\sqrt{t_-}$ results in different definitions when the product is defined using different molifiers, and the term vp $1/t$ causes non-associativity in the multiplication if $1 \cdot \delta$, $t \cdot \delta$, and $t \cdot \mathrm{vp}\ 1/t$ are defined [22]. Meanwhile, the multiplication on $\mathcal{A}'_\delta$ is associative, as shown in Corollary 2 of the subsection 3.2.

3.2. Well-definedness of multiplication

First, we will demonstrate the relationship between B_∞ and the support of the distribution.

Lemma 1. *For any* $F = (f, \{B_N\})_\delta \in \mathcal{A}'_\delta$,

$$\mathrm{supp}\ F \subset \mathrm{supp}\ f \cap B_\infty. \tag{75}$$

Proof. If $\{B_N\} = \{\mathbb{Z}/2^N\}$, then the lemma is obvious, because $B_\infty = \mathbb{R}$. Therefore, we assume that $\{B_N\} \neq \{\mathbb{Z}/2^N\}$, so that

$$F = \sum_{h \in B_\infty} f(h)\delta(t - h). \tag{76}$$

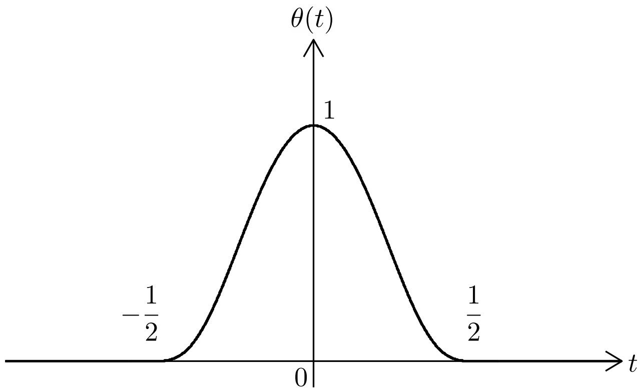

Figure 1: Graph of $\theta(t)$

Here,

$$\text{supp } f(h)\delta(t-h) \subset \text{supp } f \cap \{h\}. \tag{77}$$

Then, noting that supp $(F_1 + F_2) \subset \text{supp } F_1 \cup \text{supp } F_2$ for any $F_1, F_2 \in \mathcal{S}'$, we have that

$$\text{supp } F \subset \bigcup_{h \in B_\infty} \text{supp } f \cap \{h\} = \text{supp } f \cap B_\infty. \tag{78}$$

□

Lemma 2. *For any* $F = (f, \{B_N\})_\delta \in \mathcal{A}'_\delta$, $f(t) = 0$ *on* $B_\infty \setminus \text{supp } F$.

Proof. For this proof, we introduce an infinitely differentiable function $\theta(t)$ that has compact support. Given that

$$\nu(t) = \begin{cases} \exp(-1/t), & t > 0 \\ 0, & t \leq 0 \end{cases} \tag{79}$$

we define $\theta(t)$ as

$$\theta(t) = e^4 \nu\left(\frac{1}{2} + t\right) \nu\left(\frac{1}{2} - t\right). \tag{80}$$

It is easy to confirm that $\theta(t)$ has compact support, as the graph of $\theta(t)$ in Fig. 1 shows that

$$\text{supp } \theta = \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{81}$$

(i) $\{B_N\} = \{\mathbb{Z}/2^N\}$

Suppose that there exists $x \in B_\infty \setminus \text{supp } F$ such that $f(x) \neq 0$, and without loss of generality, let $f(x) > 0$. Because f is right continuous and $B_\infty \setminus \text{supp } F = \mathbb{R} \setminus \text{supp } F$ is an open set, there exists $\varepsilon > 0$ for which

$$\forall t \quad x < t < x + \varepsilon \Rightarrow f(t) > 0, \tag{82}$$

and

$$(x, x+\varepsilon) \subset \mathbb{R} \setminus \mathrm{supp}\ F. \tag{83}$$

Therefore, given that

$$\varphi(t) = \theta\left(\frac{t-x}{\varepsilon} - \frac{1}{2}\right), \tag{84}$$

we find that

$$\langle (f, \{B_N\})_\delta, \varphi \rangle = \int_x^{x+\varepsilon} f(t)\varphi(t)dt > 0. \tag{85}$$

On the other hand, because supp $\varphi \subset \mathbb{R} \setminus$ supp F, $\langle F, \varphi \rangle$ must be equal to 0, and so a contradiction is derived.

(ii) $\{B_N\} \neq \{\mathbb{Z}/2^N\}$

We assume that B_∞ is not an empty set, because the lemma states nothing about $f(t)$ if B_∞ is empty. Similar to (i), suppose that there exists $x \in B_\infty \setminus$ supp F such that $f(x) > 0$. As we remarked, there is no accumulation point of $\mathbb{R}$ in B_∞, and supp $F \subset B_\infty$, as in Lemma 1. Therefore, we can find $\varepsilon > 0$ for which

$$(x-\varepsilon, x+\varepsilon) \subset \mathbb{R} \setminus \mathrm{supp}\ F, \tag{86}$$

and

$$(x-\varepsilon, x+\varepsilon) \cap B_\infty = \{x\}. \tag{87}$$

Therefore, given that

$$\varphi(t) = \theta\left(\frac{t-x}{2\varepsilon}\right), \tag{88}$$

we find that

$$\langle (f, \{B_N\})_\delta, \varphi \rangle = f(x)\theta(0) > 0. \tag{89}$$

On the other hand, because supp $\varphi \subset \mathbb{R} \setminus$ supp F, $\langle F, \varphi \rangle$ must be 0, giving a contradiction. □

Lemma 3. *Let $F = (f, \{B_N^f\})_\delta$, $G = (g, \{B_N^g\})_\delta$, and $FG = (fg, \{B_N^{fg}\})_\delta$, as in Definition 2. Then,*

$$Z(FG) \supset Z(F) \cup Z(G). \tag{90}$$

That is,

$$\mathrm{supp}\ FG \subset \mathrm{supp}\ F \cap \mathrm{supp}\ G. \tag{91}$$

Proof. Suppose that $Z(F)$ is not empty and let $\varphi \in \mathcal{S}$ such that supp $\varphi \subset Z(F)$. According to Lemma 2, $f(t) = 0$ on $B_\infty^f \cap$ supp φ. Then,

$$\begin{aligned} |\langle FG, \varphi \rangle| &\leq \lim_{N\to\infty} \sum_{h \in B_N^{fg}} J_N |f(h)g(h)\varphi(h)| \\ &\leq \lim_{N\to\infty} \sum_{h \in B_N^{f}} J_N |f(h)g(h)\varphi(h)| \\ &= \lim_{N\to\infty} \sum_{h \in B_N^{f} \cap \mathrm{supp}\ \varphi} J_N |f(h)g(h)\varphi(h)| = 0, \end{aligned} \tag{92}$$

which means that $Z(FG) \supset Z(F)$. Even if $Z(F)$ is empty, $Z(FG) \supset Z(F)$. $Z(FG) \supset Z(G)$ is also shown in the same way, and it follows that $Z(FG) \supset Z(F) \cup Z(G)$. □

Using the derived lemmas, we consider the arbitrariness of our choice of B_N.

Lemma 4. *Let F be a δ-series distribution expressed by $(f, \{\mathbb{Z}/2^N\})_\delta$. If F can be also denoted by $(g, \{B_N^g\})_\delta$ where $B_N^g \neq \{\mathbb{Z}/2^N\}$, then $F = 0$, i.e., if $F \neq 0$, then $\{B_N^g\} = \{\mathbb{Z}/2^N\}$.*

Proof. We omit the case where B_∞^g is an empty set, because the lemma is obvious for that case. Then, as in the proof of Lemma 2, we can choose $\varepsilon > 0$ for any $x \in B_\infty^g$ such that

$$(x-\varepsilon, x+\varepsilon) \cap B_\infty^g = \{x\}, \tag{93}$$

and let

$$\varphi(t) = \theta\left(\frac{t-x}{2\varepsilon}\right). \tag{94}$$

Then, we see that

$$\langle F, \varphi\rangle = g(x)\varphi(x) = g(x), \tag{95}$$

while

$$\langle F, \varphi\rangle = \int_{-\infty}^{\infty} f(t)\varphi(t)dt = \int_{x-\varepsilon}^{x+\varepsilon} f(t)\theta\left(\frac{t-x}{2\varepsilon}\right)dt. \tag{96}$$

Considering $|g(x)|$, we have

$$\begin{aligned} |g(x)| &= \left|\int_{x-\varepsilon}^{x+\varepsilon} f(t)\theta\left(\frac{t-x}{2\varepsilon}\right)dt\right| \leq \int_{x-\varepsilon}^{x+\varepsilon} |f(t)|\left|\theta\left(\frac{t-x}{2\varepsilon}\right)\right|dt \\ &\leq 2\varepsilon \max_{t\in[x-\varepsilon,x+\varepsilon]} |f(t)| \to 0 \quad (\varepsilon \to 0). \end{aligned} \tag{97}$$

Hence, $g(x) = 0$ for all $x \in B_\infty^g$, which means that for any $\varphi \in \mathcal{S}$,

$$\langle F, \varphi\rangle = \langle (g, B_N^g)_\delta, \varphi\rangle = \sum_{h\in B_N^g} g(h)\varphi(h) = 0. \tag{98}$$

□

Lemma 5. *If $F = (f, \{B_N^f\})_\delta = (g, \{B_N^g\})_\delta \in \mathcal{A}'_\delta$, then $f = g$ on* supp F. *In particular, $f = g$ on $\mathbb{R}$ if $\{B_N^f\} = \{\mathbb{Z}/2^N\}$.*

Proof. If $F = 0$, then supp F is an empty set. Therefore, we assume that $F \neq 0$ and both B_∞^f and B_∞^g are not empty sets.

(i) $\{B_N^f\} = \{\mathbb{Z}/2^N\}$
From Lemma 4, we have $\{B_N^g\} = \{\mathbb{Z}/2^N\}$. Hence, for any $\varphi \in \mathcal{S}$,

$$\langle (f, \{B_N^f\})_\delta, \varphi\rangle - \langle (g, \{B_N^g\})_\delta, \varphi\rangle = \int_{-\infty}^{\infty} (f(t) - g(t))\,\varphi(t)dt = 0. \tag{99}$$

Let $r(t) = f(t) - g(t)$. Then, we can show that $r(t) = 0$ for all $t \in \mathbb{R}$.

We assume that there exists x such that $r(x) \neq 0$, and let $r(x) > 0$ without loss of generality. Because $r(x)$ is right continuous, there exists $\varepsilon > 0$ such that

$$\forall t \quad x < t < x + \varepsilon \Rightarrow r(t) > 0. \tag{100}$$

Then, given that

$$\varphi(t) = \theta\left(\frac{t-x}{\varepsilon} - \frac{1}{2}\right), \tag{101}$$

we have that

$$\int_{-\infty}^{\infty} r(t)\varphi(t)dt > 0. \tag{102}$$

This contradicts equation (99), so that $r(t) = 0$, i.e., $f = g$ on $\mathbb{R}$.

(ii) $\{B_N^f\} \neq \{\mathbb{Z}/2^N\}$

Lemma 4 implies that $\{B_N^g\} \neq \{\mathbb{Z}/2^N\}$, and any x in supp F belongs to both B_∞^f and B_∞^g, from Lemma 1. Therefore, we can choose $\varepsilon > 0$ for which

$$(x-\varepsilon, x+\varepsilon) \cap \left(B_\infty^f \cup B_\infty^g\right) = \{x\}. \tag{103}$$

Then, given that

$$\varphi(t) = \theta\left(\frac{t-x}{2\varepsilon}\right), \tag{104}$$

we find, by considering $\langle F, \varphi\rangle$, that

$$f(x) = g(x). \tag{105}$$

□

The well-definedness can be derived as follows.

Theorem 7. *The multiplication in Definition 2 is well-defined.*

Proof. Let $F = (f, \{B_N^f\})_\delta, G = (g, \{B_N^g\})_\delta$. First, we consider the case in which $F = 0$. Then, supp F is an empty set, so that supp FG is also empty according to Lemma 3. That is, $FG = 0$ if one of F and G is equal to 0.

Hence, we assume that F and G are both not equal to 0 and are, respectively, expressed as follows:

$$F = (f_1, \{B_N^{f_1}\})_\delta = (f_2, \{B_N^{f_2}\})_\delta, \tag{106}$$
$$G = (g_1, \{B_N^{g_1}\})_\delta = (g_2, \{B_N^{g_2}\})_\delta. \tag{107}$$

We will show that the products derived from these two expressions are the same, i.e., letting F_1G_1 denote $(f_1g_1, \{B_N^{f_1g_1}\})_\delta$ and F_2G_2 denote $(f_2g_2, \{B_N^{f_2g_2}\})_\delta$, we will prove that $F_1G_1 = F_2G_2$. Here, if both $\{B_N^{f_1}\}$ and $\{B_N^{g_1}\}$ are equal to $\{\mathbb{Z}/2^N\}$, then both F and G are expressed in a unique way, according to Lemmas

4 and 5. Therefore, we further assume that $\{B_N^{f_1}\} \neq \{\mathbb{Z}/2^N\}$. It follows that $\{B_N^{f_2}\} \neq \{\mathbb{Z}/2^N\}$ because $F \neq 0$.

Using Lemma 3, we see that

$$\text{supp } F_1G_1 \subset \text{supp } F \cap \text{supp } G, \tag{108}$$

and

$$\text{supp } F_2G_2 \subset \text{supp } F \cap \text{supp } G. \tag{109}$$

Moreover, according to Lemma 5, $f_1 = f_2$ on supp F and $g_1 = g_2$ on supp G, so that $f_1g_1 = f_2g_2$ on supp $F \cap$ supp G.

Next, we will show that supp $F_1G_1 =$ supp F_2G_2. For this purpose, suppose that supp $F_1G_1 \neq$ supp F_2G_2, and derive a contradiction. Noting that neither of $\{B_N^{f_1g_1}\}$ and $\{B_N^{f_2g_2}\}$ are equal to $\{\mathbb{Z}/2^N\}$, we can assume without loss of generality that there exists $x \in$ supp $F_2G_2 \setminus$ supp F_1G_1. Here, we choose $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \cap (\text{supp } F_1G_1 \cup \text{supp } F_2G_2) = \{x\}, \tag{110}$$

and set φ as

$$\varphi(t) = \theta\left(\frac{t-x}{2\varepsilon}\right). \tag{111}$$

Then, we find that

$$\langle F_1G_1, \varphi\rangle = 0, \tag{112}$$

$$\langle F_2G_2, \varphi\rangle = f_2(x)g_2(x), \tag{113}$$

which means that $f_2(x)$ and $g_2(x)$ are both not equal to 0. Meanwhile, $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$ are also not equal to 0, although $\langle F_1G_1, \varphi\rangle = 0$. Thus, we have $x \notin B_\infty^{f_1g_1}$, so that at least one of $B_\infty^{f_1}$ and $B_\infty^{g_1}$ does not contain x. Let $x \notin B_\infty^{f_1}$, without loss of generality. Because neither of $\{B_N^{f_1}\}$ and $\{B_N^{f_2}\}$ are equal to $\{\mathbb{Z}/2^N\}$, there exists η for which

$$(x-\eta, x+\eta) \cap \left(B_\infty^{f_1} \cup B_\infty^{f_2}\right) = \{x\}. \tag{114}$$

Then, given that

$$\psi(t) = \theta\left(\frac{t-x}{2\eta}\right), \tag{115}$$

we see that

$$\langle F, \psi\rangle = \langle (f_1, \{B_N^{f_1}\})_\delta, \psi\rangle = 0, \tag{116}$$

while

$$\langle F, \psi\rangle = \langle (f_2, \{B_N^{f_2}\})_\delta, \psi\rangle = f_2(x) \neq 0. \tag{117}$$

This contradiction shows that supp $F_1G_1 =$ supp F_2G_2.

Finally, for any $\phi \in \mathcal{S}$, we have

$$\begin{aligned}\langle F_1G_1, \phi\rangle = \sum_{h\in B_\infty^{f_1g_1}} f_1(h)g_1(h)\phi(h) &= \sum_{h\in \text{supp } F_1G_1} f_1(h)g_1(h)\phi(h) \\ = \sum_{h\in \text{supp } F_2G_2} f_2(h)g_2(h)\phi(h) &= \sum_{h\in B_\infty^{f_2g_2}} f_2(h)g_2(h)\phi(h) = \langle F_2G_2, \phi\rangle.\end{aligned} \tag{118}$$

That is, FG is the unique product of F and G. □

We conclude this subsection with the following corollary.

Corollary 2. *The multiplication of Definition 2 is associative.*

Proof. Because of the well-definedness, it is sufficient to demonstrate the associativity for a single instance. Let $F_1 = (f_1, \{B_N^{f_1}\})_\delta$, $F_2 = (f_2, \{B_N^{f_2}\})_\delta$, and $F_3 = (f_3, \{B_N^{f_3}\})_\delta$. Then, this corollary follows immediately from the facts that $(f_1 f_2) f_3 = f_1(f_2 f_3)$ and $(B_\infty^{f_1} \cap B_\infty^{f_2}) \cap B_\infty^{f_3} = B_\infty^{f_1} \cap (B_\infty^{f_2} \cap B_\infty^{f_3})$. □

4. exp-series distribution and convolution

4.1. Definition

In this subsection, we define the convolution corresponding to the multiplication defined in Section 3. First, we define a class of distributions for convolution.

Definition 3. *Given that f and $\{B_N\}$ satisfy conditions (I) and (II) of Definition 1, if $F \in \mathcal{S}'$ is expressed as*

$$F = \lim_{N\to\infty} \sum_{h\in B_N} J_N f(h) \frac{1}{2\pi} \exp(iht) \tag{119}$$

in terms of J_N from Definition 1, then we call F an exp-series distribution *and denote F by $(f, \{B_N\})_e$.*

Let us call the set of exp-series distributions $\mathcal{A}'_e$. The elements of $\mathcal{A}'_e$ and those of $\mathcal{A}'_\delta$ are in a one-to-one correspondence by the Fourier transform, i.e., for any $(f, \{B_N\})_e \in \mathcal{A}'_e$,

$$\mathcal{F}[(f, \{B_N\})_e] = (f, \{B_N\})_\delta. \tag{120}$$

Therefore, $\mathcal{A}'_e$ can be written as

$$\mathcal{A}'_e = \{\mathcal{F}^{-1}[F] | F \in \mathcal{A}'_\delta\}. \tag{121}$$

Remark. $\mathcal{A}'_e$ is not equal to $\mathcal{A}'_\delta$. For example, the derivative of δ, for which

$$\langle \delta', \varphi \rangle = -\varphi'(0), \tag{122}$$

is not in $\mathcal{A}'_\delta$. This can be derived from the fact stated in [2] that for any tempered distribution $T \in \mathcal{S}'$ that has support $\{0\}$, there uniquely exist $c_0, c_1, ...$ such that

$$T = \sum_{j=0}^{\infty} c_j \delta^{(j)}, \tag{123}$$

where $\delta^{(j)}$ denotes the j th derivative of δ. Meanwhile, $(f, \{B_N\})_\delta \in \mathcal{A}'_\delta$ is a linear combination of $\delta(t-h)$ if $\{B_N\} \neq \{\mathbb{Z}/2^N\}$, and is a function if $\{B_N\} = \{\mathbb{Z}/2^N\}$, according to Theorem 5. Therefore, δ' must not be in $\mathcal{A}'_\delta$.

However, $\delta' = (it, \{\mathbb{Z}/2^N\})_e \in \mathcal{A}'_e$, because

$$\left(it, \{\mathbb{Z}/2^N\}\right)_e = \mathcal{F}^{-1}\left[\left(it, \{\mathbb{Z}/2^N\}\right)_\delta\right] = \mathcal{F}^{-1}[it] = \delta'. \tag{124}$$

Conversely, $f(t) = it$ is not in $\mathcal{A}'_e$, because $\mathcal{F}[it] = -2\pi\delta'$ is not in $\mathcal{A}'_\delta$. We can determine whether or not a tempered distribution F is in $\mathcal{A}'_e$ by considering whether $\mathcal{F}[F]$ is in $\mathcal{A}'_\delta$.

Example 12. $\delta \in \mathcal{A}'_\delta$ belongs to $\mathcal{A}'_e$, because

$$\mathcal{F}[\delta] = 1 = (1, \{\mathbb{Z}/2^N\})_\delta \in \mathcal{A}'_\delta. \tag{125}$$

That is to say, $\delta = (1, \{\mathbb{Z}/2^N\})_e$.

Example 13. Let $f \in L^1$. Then, the Fourier transform of f is a bounded continuous function [9]. Therefore, $f = (\mathcal{F}[f], \{\mathbb{Z}/2^N\})_e \in \mathcal{A}'_e$. That is, $L^1 \subset \mathcal{A}'_e$.

Convolution is now defined for exp-series distributions:

Definition 4. *Given $F = (f, \{B_N^f\})_e$ and $G = (g, \{B_N^g\})_e$, the convolution of F and G is defined as $F * G = (fg, \{B_N^{fg}\})_e$, where we choose $\{B_N^{fg}\}$ satisfying condition (I) of Definition 1 and*

$$B_\infty^{fg} = B_\infty^f \cap B_\infty^g. \tag{126}$$

This convolution, in fact, is derived from

$$F * G = \mathcal{F}^{-1}[\mathcal{F}[F]\mathcal{F}[G]]. \tag{127}$$

Therefore, the relationship in equation (35) holds, and we can see that the convolution is well-defined, because the multiplication is well-defined and there is a one-to-one correspondence between $\mathcal{A}'_\delta$ and $\mathcal{A}'_e$. It is interesting that the convolution derived from equation (127) is the product of the coefficients of complex exponential functions, whereas the multiplication is the product of the coefficients of δ. We can find a convolution corresponding to the one described in Smith's paper [4] for tempered distributions F and G that have the same period P, where F and G are expressed with the coefficients a_n and b_n as

$$F = \sum_{k=-\infty}^{\infty} a_n \exp(2\pi t/P), \tag{128}$$

$$G = \sum_{k=-\infty}^{\infty} b_n \exp(2\pi t/P). \tag{129}$$

Then, Smith's convolution $F * G$ is given by

$$F * G = P \sum_{k=-\infty}^{\infty} a_n b_n \exp(2\pi t/P). \tag{130}$$

Thus, Smith's convolution also calculates the coefficients of complex exponential functions, and agrees with our convolution. Furthermore, our definition does not depend on the period of the signal, nor on whether the signal is continuous or discrete.

Example 14. Suppose that $f, g \in L^1$. Then, the convolution of f and g is defined as

$$f * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx, \tag{131}$$

for which the following formula holds for the Fourier transform of functions [9]:

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]. \tag{132}$$

Now, let us calculate $f * g$ according to Definition 4:

$$\begin{aligned} f * g &= (\mathcal{F}[f], \{\mathbb{Z}/2^N\})_e * (\mathcal{F}[g], \{\mathbb{Z}/2^N\})_e \\ &= (\mathcal{F}[f]\mathcal{F}[g], \{\mathbb{Z}/2^N\})_e = (\mathcal{F}[f * g], \{\mathbb{Z}/2^N\})_e. \end{aligned} \tag{133}$$

Thus, the convolution of $\mathcal{A}'_e$ is consistent with that of L^1.

Example 15. δ is the identity element of this convolution. In fact, for any $F = (f, \{B_N\})_e \in \mathcal{A}'_e$,

$$F * \delta = (f, \{B_N\})_e * (1, \{\mathbb{Z}/2^N\})_e = (f, \{B_N\})_e = F. \tag{134}$$

Example 16. We can calculate the convolution of distributions even if their periods are different. For example, given $f(t) = \exp(it)$ and $g(t) = \exp(i\sqrt{2}t)$,

$$f * g(t) = (2\pi, \{1\})_e * (2\pi, \{\sqrt{2}\})_e = (4\pi^2, \emptyset)_e = 0, \tag{135}$$

where $\emptyset$ denotes an empty set.

Example 17. Let us confirm that the convolution of exp-series distributions agrees with the convolution in formula (33). An arbitrary $\varphi \in \mathcal{S}$ is an exp-series distribution, because $\mathcal{F}[\varphi] = (\mathcal{F}[\varphi], \{\mathbb{Z}/2^N\})_\delta$, so that $\varphi = (\mathcal{F}[\varphi], \{\mathbb{Z}/2^N\})_e$. Therefore, we can calculate the convolution of $F = (f, \{B_N\})_e \in \mathcal{A}'_e$ and $\varphi \in \mathcal{S}$.

First, suppose that $\{B_N\} = \{\mathbb{Z}/2^N\}$. Then, for any $\psi \in \mathcal{S}$,

$$\begin{aligned} \langle F * \varphi, \psi \rangle &= \langle (f\mathcal{F}[\varphi], \{\mathbb{Z}/2^N\})_e, \psi \rangle = \langle (f\mathcal{F}[\varphi], \{\mathbb{Z}/2^N\})_\delta, \mathcal{F}^{-1}[\psi] \rangle \\ &= \int_{-\infty}^{\infty} f(\omega)\mathcal{F}[\varphi](\omega)\mathcal{F}^{-1}[\psi](\omega)d\omega \\ &= \int_{-\infty}^{\infty} f(\omega)\mathcal{F}[\varphi](\omega)\frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t)\exp(i\omega t)dtd\omega \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\omega)\frac{1}{2\pi}\mathcal{F}[\varphi](\omega)\exp(i\omega t)d\omega\psi(t)dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\omega)\mathcal{F}^{-1}[\varphi(t-x)](\omega)d\omega\psi(t)dt \\ &= \int_{-\infty}^{\infty} \langle (f, \{\mathbb{Z}/2^N\})_\delta, \mathcal{F}^{-1}[\varphi(t-x)] \rangle \psi(t)dt \\ &= \int_{-\infty}^{\infty} \langle (f, \{\mathbb{Z}/2^N\})_e, \varphi(t-x) \rangle \psi(t)dt \\ &= \int_{-\infty}^{\infty} \langle F(x), \varphi(t-x) \rangle \psi(t)dt = \langle \langle F(x), \varphi(t-x) \rangle, \psi(t) \rangle. \end{aligned} \tag{136}$$

Next, let $\{B_N\} \neq \{\mathbb{Z}/2^N\}$ and $T_1, ..., T_n, p_1, ..., p_n$ be positive real numbers such that

$$B_\infty \subset \bigcup_{j=1}^{n} T_j\mathbb{Z} + p_j. \tag{137}$$

Then, the convolution is calculated as follows:

$$\begin{aligned}\langle F * \varphi, \psi \rangle &= \sum_{h\in B_\infty} f(h)\mathcal{F}[\varphi](h)\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(t)\exp(iht)dt \\ &= \sum_{h\in B_\infty}\int_{-\infty}^{\infty} f(h)\mathcal{F}[\varphi](h)\frac{1}{2\pi}\psi(t)\exp(iht)dt. \end{aligned} \tag{138}$$

We will show that the order of the summation and the integration is interchangeable.

Noting that B_∞ is a countable set, take a sequence $\{h_j\}$ such that

$$B_\infty = \{h_j | j \in \mathbb{N}\}, \tag{139}$$

$$\forall j, k \in \mathbb{N} \quad j < k \Rightarrow (h_j \neq h_k \wedge |h_j| \leq |h_k|)\,, \tag{140}$$

and we set

$$g_j(t) = f(h_j)\mathcal{F}[\varphi](h_j)\frac{1}{2\pi}\psi(t)\exp(ih_jt). \tag{141}$$

If $\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} g_j(t)dt$ converges absolutely, then,

$$\sum_{h\in B_\infty}\int_{-\infty}^{\infty} f(h)\mathcal{F}[\varphi](h)\frac{1}{2\pi}\psi(t)\exp(iht)dt = \sum_{j=1}^{\infty}\int_{-\infty}^{\infty} g_j(t). \tag{142}$$

In fact,

$$\begin{aligned}\sum_{j=1}^{\infty}\left|\int_{-\infty}^{\infty} g_j(t)dt\right| &\leq \sum_{j=1}^{\infty}\int_{-\infty}^{\infty}|g_j(t)|dt \leq \sum_{j=1}^{\infty}\int_{-\infty}^{\infty}|f(h_j)||\mathcal{F}[\varphi](h_j)|\frac{1}{2\pi}|\psi(t)|dt \\ &= \sum_{j=1}^{\infty}|f(h_j)||\mathcal{F}[\varphi](h_j)|\frac{1}{2\pi}\int_{-\infty}^{\infty}|\psi(t)|dt. \end{aligned} \tag{143}$$

Moreover, because $\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}|g_j(t)|dt$ has a finite value [4], it follows that

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} g_j(t)dt = \int_{-\infty}^{\infty}\sum_{j=1}^{\infty} g_j(t)dt. \tag{146}$$

Therefore, we have

$$\begin{aligned}\langle F*\varphi,\psi\rangle &= \sum_{h\in B_\infty}\int_{-\infty}^{\infty} f(h)\mathcal{F}[\varphi](h)\frac{1}{2\pi}\psi(t)\exp(iht)dt \\ &= \int_{-\infty}^{\infty}\psi(t)\sum_{h\in B_\infty} f(h)\mathcal{F}[\varphi](h)\frac{1}{2\pi}\exp(iht)dt \\ &= \int_{-\infty}^{\infty}\psi(t)\left\langle (f,\{B_N\})_\delta, \mathcal{F}[\varphi](\omega)\frac{1}{2\pi}\exp(i\omega t)\right\rangle dt \\ &= \int_{-\infty}^{\infty}\psi(t)\langle (f,\{B_N\})_\delta, \mathcal{F}^{-1}[\varphi(t-x)](\omega)\rangle dt \\ &= \int_{-\infty}^{\infty}\psi(t)\langle (f,\{B_N\})_e, \varphi(t-x)\rangle dt \\ &= \int_{-\infty}^{\infty}\psi(t)\langle F(x), \varphi(t-x)\rangle dt = \langle\langle F(x),\varphi(t-x)\rangle,\psi(t)\rangle. \end{aligned}\tag{147}$$

Thus, convolution of exp-series distributions is consistent with the convolution given by the formula (33).

Example 18. Let $G\in\mathcal{S}'$ have compact support. It is known that

$$\mathcal{F}[G]\in\mathcal{O}_M, \tag{148}$$

$$G*\varphi(t) = \langle G(x),\varphi(t-x)\rangle\in\mathcal{S}, \tag{149}$$

$$\mathcal{F}[G*\varphi] = \mathcal{F}[G]\mathcal{F}[\varphi], \tag{150}$$

for any $\varphi\in\mathcal{S}$ (see, e.g., [1]), where the convolution in (150) is in the context of the equation (149). Hence, $G = (\mathcal{F}[G],\{\mathbb{Z}/2^N\})_e\in\mathcal{A}'_e$, so that we can calculate

[4]Here, we used the following theorem: Given that $(X,\mathfrak{B},\mu)$ is a measure space and $E\in\mathfrak{B}$, if f_n is a measurable function on E for all $n\in\mathbb{N}$ and

$$\sum_{n=1}^{\infty}\int_E |f_n|d\mu < \infty, \tag{144}$$

then $\sum_{n=1}^{\infty} f_n$ is finite for almost all $x\in E$ and

$$\sum_{n=1}^{\infty}\int_E f_n d\mu = \int_E\sum_{n=1}^{\infty} f_n d\mu. \tag{145}$$

The proof can be found in textbooks on the Lebesgue integral (e.g., [23]).

the convolution of G and $F = (f, \{B_N\})_e \in \mathcal{A}'_e$. The result is as follows.

$$\begin{aligned}\langle F * G, \varphi\rangle &= \langle (f\mathcal{F}[G], \{B_N\})_e, \varphi\rangle = \left\langle (f\mathcal{F}[G], \{B_N\})_\delta, \mathcal{F}^{-1}[\varphi]\right\rangle \\ &= \left\langle (f, \{B_N\})_\delta, \mathcal{F}[G](t)\frac{1}{2\pi}\mathcal{F}[\varphi](-t)\right\rangle \\ &= \left\langle F, \frac{1}{2\pi}\mathcal{F}[\mathcal{F}[G](t)\mathcal{F}[\varphi](-t)]\right\rangle \\ &= \left\langle F, \mathcal{F}^{-1}[\mathcal{F}[G](-t)\mathcal{F}[\varphi](t)]\right\rangle \\ &= \left\langle F, \check{G} * \varphi\right\rangle, \end{aligned} \tag{151}$$

which means the convolution is consistent with the formula in (34).

4.2. Impulse response of continuous LTI systems

This definition of convolution enables us to derive an important theorem regarding continuous linear time-invariant (LTI) systems. First, we define a continuous LTI system:

Definition 5. *A mapping $A : \mathcal{S}' \to \mathcal{S}'$ is called a* continuous LTI system *if A meets the following conditions:*

(I) *Continuity:*
For any converging sequence of tempered distributions $\{F_N\} \to F$ $(N \to \infty)$,

$$\lim_{N\to\infty} A(F_N) = A(F). \tag{152}$$

(II) *Linearity:*
For any $\alpha, \beta \in \mathbb{C}$ and $F, G \in \mathcal{S}'$,

$$A(\alpha F + \beta G) = \alpha A(F) + \beta A(G). \tag{153}$$

(III) *Time-invariance:*
For any $h \in \mathbb{R}$ and $F \in \mathcal{S}'$,

$$A(\tau_h F) = \tau_h A(F). \tag{154}$$

To prove the theorem to follow, we introduce a lemma regarding continuous LTI systems.

Lemma 6. *Any continuous LTI system A admits $\exp(iat)$ as an eigenfunction for any $a \in \mathbb{R}$, and the eigenvalue $\lambda(a)$ is a continuous function. Moreover, there exist $C > 0$ and a non-negative integer K such that*

$$|\lambda(a)| \le C(1 + |a|)^K \tag{155}$$

for any a.

Proof. We denote $\exp(-iat)A(\exp(iat))$ by $H(t;a)$, where the product of $\exp(-iat)$ and $A(\exp(iat))$ is in the context of the multiplication of $\mathcal{O}_M$ and $\mathcal{S}'$. Then,

$$A(\exp(iat)) = \exp(iat)H(t;a). \tag{156}$$

We will show that $H(t;a)$ is in fact independent of t.

Considering $A(\tau_h \exp(iat))$ for a real number h, we see that

$$\begin{aligned} A(\tau_h \exp(iat)) &= \tau_h \left(\exp(iat)H(t;a)\right), & (157)\\ \exp(-iah)A(\exp(iat)) &= \exp(ia(t-h))H(t-h;a), & (158)\\ A(\exp(iat)) &= \exp(iat)H(t-h;a). & (159) \end{aligned}$$

Therefore, we have

$$\forall h \in \mathbb{R} \quad H(t;a) = H(t-h;a), \tag{160}$$

which means that $H(t;a)$ is independent of t, and in fact takes a constant value that depends on a [5]. Therefore, for any $a \in \mathbb{R}$, $\exp(iat)$ is an eigenfunction of A. Moreover, because A is continuous, it holds for any $a_0 \in \mathbb{R}$ that

$$\lim_{a \to a_0} A(\exp(iat)) = \lambda(a_0)\exp(ia_0 t). \tag{161}$$

This means that $\lambda(a)$ is a continuous function.

Consider the condition that there exist $C > 0$ and a non-negative integer K such that

$$\forall a \in \mathbb{R} \quad |\lambda(a)| \le C(1+|a|)^K. \tag{162}$$

Let

$$\lambda_x = 1 + \max_{a \in [-1,1]} |\lambda(a)|, \tag{163}$$

and let K' be a non-negative integer larger than K for which

$$2^{K'-K}\lambda_x > C. \tag{164}$$

Then, we have

$$\forall a \in \mathbb{R} \quad |\lambda(a)| \le \lambda_x(1+|a|)^{K'}. \tag{165}$$

That is, the existence of C and K for which equation (162) holds is equivalent to the existence of K' in equation (165). Hence, we suppose that there exists no non-negative integer K' satisfying equation (165), and derive a contradiction to prove that $\lambda(a)$ is slowly increasing.

According to this assumption, for any non-negative integer K there exists a set $J \subset \mathbb{R}$ such that

$$\forall a \in J \quad |\lambda(a)| > \lambda_x(1+|a|)^K. \tag{166}$$

[5] More correctly, because the derivative of $H(t;a)$ is 0, $H(t;a)$ is constant. See also Schwartz [2].

Here, J is not bounded, because if there exists a positive number r for which $J \subset [-r, r]$, then given

$$\lambda_r = 1 + \max_{a \in [-r,r]} |\lambda(a)| \tag{167}$$

we have

$$\forall a \in \mathbb{R} \quad |\lambda(a)| \leq \lambda_r \lambda_x (1 + |a|)^K, \tag{168}$$

which contradicts the assumption. Thus we can take a sequence $\{a_j\}$ ($j \in \mathbb{N}$) such that

$$\forall j, k \in \mathbb{N} \quad j < k \Rightarrow |a_j| + 1 < |a_k| \tag{169}$$

and

$$\forall j \in \mathbb{N} \quad |\lambda(a_j)| > \lambda_x (1 + |a_j|)^j. \tag{170}$$

Then, we consider the following function:

$$\psi(t) = \sum_{j=1}^{\infty} \frac{1}{|\lambda(a_j)|} \theta(t - a_j). \tag{171}$$

We will show that $\psi \in \mathcal{S}$. For any non-negative integers n and k,

$$\sup_{t \in \mathbb{R}} (1 + |t|)^k |\psi^{(n)}| \leq \left(\sup_{t \in \mathbb{R}} \theta^{(n)}(t) \right) \left(\sup_{j \in \mathbb{N}} \frac{(1 + |a_j| + 1/2)^k}{|\lambda(a_j)|} \right). \tag{172}$$

Here, $\sup_{t \in \mathbb{R}} \theta^{(n)}(t)$ is finite, because $\theta(t) \in \mathcal{S}$. Thus, we consider the latter term. Because

$$|\lambda(a_j)| > \lambda_x (1 + |a_j|)^k \tag{173}$$

for $j \geq k$, we see that

$$\sup_{j \geq k} \frac{(1 + |a_j| + 1/2)^k}{|\lambda(a_j)|} < \sup_{j \geq k} \frac{1}{\lambda_x} \frac{(1 + |a_j| + 1/2)^k}{(1 + |a_j|)^k} = \frac{1}{\lambda_x} \left(1 + \frac{1}{2(1 + |a_k|)} \right)^k. \tag{174}$$

Therefore,

$$\begin{aligned} \sup_{j \in \mathbb{N}} \frac{(1 + |a_j| + 1/2)^k}{|\lambda(a_j)|} &< \frac{1}{\lambda_x} \left(1 + \frac{1}{2(1 + |a_k|)} \right)^k + \max_{1 \leq j \leq k-1} \frac{(1 + |a_j| + 1/2)^k}{|\lambda(a_j)|} \\ &< \infty. \end{aligned} \tag{175}$$

It follows that $\psi \in \mathcal{S}$.

Furthermore, we set

$$F = \sum_{j=1}^{\infty} \delta(t - a_j), \tag{176}$$

and will show that $F \in \mathcal{S}'$. Take an arbitrary $\varphi \in \mathcal{S}$. Then, there exists L such that

$$|\varphi(t)| < \frac{L}{(1 + |t|)^2}, \tag{177}$$

for all t. It follows that

$$\sum_{j=1}^{\infty} |\varphi(a_j)| \leq \sum_{j=1}^{\infty} \frac{L}{(1+|a_j|)^2} \leq \sum_{j=1}^{\infty} \frac{L}{(1+\lfloor |a_j| \rfloor)^2} \leq \sum_{j=1}^{\infty} \frac{L}{(1+j)^2}, \tag{178}$$

so $\langle F, \varphi \rangle = \sum_{j=1}^{\infty} \varphi(a_j)$ converges. For a sequence of rapidly decreasing functions $\{\varphi_l\}$ that converges to 0, we can take a decreasing sequence $\{L_l\}$ such that

$$|\varphi_l(t)| < \frac{L_l}{(1+|t|)^2}, \tag{179}$$

where $L_l \to 0$ as $l \to \infty$. Then,

$$|\langle F, \varphi_l \rangle| \leq L_l \sum_{j=1}^{\infty} \frac{1}{(1+j)^2} \to 0 \quad (l \to \infty), \tag{180}$$

so that $F \in \mathcal{S}'$. Although $\mathcal{F}[A(\mathcal{F}^{-1}[F])]$ must be in $\mathcal{S}'$ too, we see that

$$\begin{aligned} &\langle \mathcal{F}[A(\mathcal{F}^{-1}[F])], \psi \rangle = \left\langle \mathcal{F}\left[A\left(\sum_{j=1}^{\infty} \frac{1}{2\pi} \exp(i a_j t) \right) \right], \psi \right\rangle \\ &= \left\langle \mathcal{F}\left[\sum_{j=1}^{\infty} \lambda(a_j) \frac{1}{2\pi} \exp(i a_j t) \right], \psi \right\rangle = \left\langle \sum_{j=1}^{\infty} \lambda(a_j) \delta(t-a_j), \psi \right\rangle = \sum_{j=1}^{\infty} \frac{\lambda(a_j)}{|\lambda(a_j)|}, \end{aligned} \tag{181}$$

which does not converge. Therefore, $\mathcal{F}[A(\mathcal{F}^{-1}[F])]$ does not have any value at ψ, giving a contradiction. □

In the following, we present a theorem regarding the impulse response $A(\delta)$, which is a well-known theorem in signal processing.

Theorem 8. *Let A be a continuous LTI system. Then, given $F \in \mathcal{A}'_e$,*

$$A(F) = F * A(\delta). \tag{182}$$

Proof. Noting that $\delta = (1, \{\mathbb{Z}/2^N\})_e$, we have

$$A(\delta) = \lim_{N\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2^N} \frac{1}{2\pi} A\left(\exp\left(\frac{ikt}{2^N} \right) \right). \tag{183}$$

Here, it follows from Lemma 6 that there exists a function $\lambda(a)$ such that $A(\exp(iat)) = \lambda(a)\exp(iat)$. Therefore, we find

$$\begin{aligned} A\delta &= \lim_{N\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2^N} \frac{1}{2\pi} A\left(\exp\left(\frac{ikt}{2^N} \right) \right) \\ &= \lim_{N\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2^N} \lambda\left(\frac{k}{2^N} \right) \frac{1}{2\pi} \exp\left(\frac{ikt}{2^N} \right) = \left(\lambda, \{\mathbb{Z}/2^N\} \right)_e. \end{aligned} \tag{184}$$

Then, letting $F = (f, \{B_N\})_e$, we find that

$$
\begin{aligned}
A(F) &= \lim_{N\to\infty} \sum_{h\in B_N} J_N f(h) A\left(\exp(iht)\right) \\
&= \lim_{N\to\infty} \sum_{h\in B_N} J_N f(h)\lambda(h)\exp(iht) \\
&= (f\lambda, \{B_N\})_e = (f, \{B_N\})_e * (\lambda, \{\mathbb{Z}/2^N\})_e = F * A(\delta). \qquad (185)
\end{aligned}
$$

□

In his book [10], Krabs described the following similar theorem, where the conditions are different.

Theorem 9. *Let A be a continuous LTI system. Then, if $A(\delta)$ has a compact support,*

$$A(F) = F * A(\delta) \qquad (186)$$

for any $F \in \mathcal{S}'$.

Furthermore, he gave an example of a continuous LTI system A such that $A(\delta)$ does not have compact support, and such that equation (186) holds when applied to any rapidly decreasing function belonging to the set

$$\chi_0 = \left\{\varphi \in \mathcal{S} \,\middle|\, \int_{-\infty}^{\infty} \varphi(t)dt = 0\right\}. \qquad (187)$$

The following example, in contrast, presents a case in which supp $A(\delta)$ is not compact and the codomain of A is $\mathcal{S}'$.

Example 19. Choose a rapidly decreasing function ψ and let the mapping $A : \mathcal{S}' \to \mathcal{S}'$ be defined by

$$\langle A(F), \varphi\rangle = \lim_{K\to\infty} \sum_{k=-K}^{K} \langle \psi(k)\tau_{-k}F, \varphi\rangle \qquad (188)$$

for all $\varphi \in \mathcal{S}$. First, we will show that A is a continuous LTI system. Given that $F \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, we have

$$
\begin{aligned}
\langle A(F), \varphi\rangle &= \lim_{K\to\infty} \sum_{k=-K}^{K} \psi(k)\left\langle F, \varphi(t-k)\right\rangle = \lim_{K\to\infty} \sum_{k=-K}^{K} \psi(k) F * \check{\varphi}(k) \\
&= \left\langle \sum_{k=-\infty}^{\infty} F * \check{\varphi}(k)\delta(t-k), \psi \right\rangle, \qquad (189)
\end{aligned}
$$

where $\check{\varphi}(t) = \varphi(-t)$ and the convolutions are defined as in formula (33). Using the fact that $F * \check{\varphi}(t) \in \mathcal{O}_M$, we see that $\langle A(F), \varphi\rangle$ has a finite value, according to Corollary 1.

It is straightforward to show that $\lim_{K\to\infty}\sum_{k=-K}^{K}\psi(k)\varphi(t-k)$ converges to a function $\phi(t)$, and that $\lim_{K\to\infty}\sum_{k=-K}^{K}\psi(k)\varphi^{(n)}(t-k)$ converges to $\phi^{(n)}(t)$, for all positive integers n. In addition, $\phi(t)=\sum_{k=-\infty}^{\infty}\psi(k)\varphi(t-k)$ is a rapidly decreasing function, because for any non-negative integers n and K, there exist C_1 and C_2 such that

$$\forall t\in\mathbb{R}\quad (1+|t|)^{K+2}|\psi(t)|<C_1, \tag{190}$$

$$\forall t\in\mathbb{R}\quad (1+|t|)^{K}|\varphi^{(n)}(t)|<C_2. \tag{191}$$

That is,

$$\begin{aligned}&(1+|t|)^K|\phi^{(n)}(t)|=(1+|t|)^K\left|\sum_{k=-\infty}^{\infty}\psi(k)\varphi^{(n)}(t-k)\right|\\&\le\sum_{k=-\infty}^{\infty}(1+|k|)^K|\psi(k)|(1+|t-k|)^K|\varphi^{(n)}(t-k)|\le\sum_{k=-\infty}^{\infty}\frac{C_1C_2}{(1+|k|)^2},\end{aligned} \tag{192}$$

for all t. Moreover, let C_3 be a number for which

$$\forall t\in\mathbb{R}\quad (1+|t|)^{K+2}|\varphi^{(n)}(t)|<C_3. \tag{193}$$

Then, given that N is a natural number, it holds that

$$\begin{aligned}&(1+|t|)^K\left|\phi^{(n)}-\sum_{k=-N}^{N}\psi(k)\varphi^{(n)}(t-k)\right|=(1+|t|)^K\left|\sum_{|k|>N}\psi(k)\varphi^{(n)}(t-k)\right|\\&\le\sum_{|k|>N}(1+|k|)^K|\psi(k)|(1+|t-k|)^K|\varphi^{(n)}(t-k)|\\&\le\left(\sum_{|k|>N}|\psi(k)|(1+|k|)^K\right)\left(\sum_{k=-\infty}^{\infty}\frac{C_3}{(1+|k|+1/2)^2}\right),\end{aligned} \tag{194}$$

for all t. Noting that $\sum_{k=-\infty}^{\infty}|\psi(k)|(1+|k|)^K$ converges, we have

$$\lim_{N\to\infty}\sup_{t\in\mathbb{R}}(1+|t|)^K\left|\phi^{(n)}-\sum_{k=-N}^{N}\psi(k)\varphi^{(n)}(t-k)\right|=0, \tag{195}$$

which means that $\sum_{k=-N}^{N}\psi(k)\varphi(t-k)$ converges to $\phi(t)$ in $\mathcal{S}$.

Therefore, given a sequence of tempered distributions $\{G_N\}$ that converges to G, it follows that

$$\begin{aligned}&\lim_{N\to\infty}\langle A(G_N),\varphi\rangle=\lim_{N\to\infty}\left\langle G_N,\sum_{k=-\infty}^{\infty}\psi(k)\varphi(t-k)\right\rangle=\lim_{N\to\infty}\langle G_N,\phi\rangle\\&=\langle G,\phi\rangle=\left\langle G,\sum_{k=-\infty}^{\infty}\psi(k)\varphi(t-k)\right\rangle=\langle A(G),\varphi\rangle,\end{aligned} \tag{196}$$

which means A is continuous. The linearity and time-invariance are obvious, giving that A is a continuous LTI system.

According to the proof of Theorem 8, it holds that

$$\lambda = \mathcal{F}[A(\delta)] = \sum_{k=-\infty}^{\infty} \psi(k) \exp(ikt). \tag{197}$$

For example, let an input of A be $\sin t = (-\pi i t, \{-1, 1\})_e$. Then, we see that

$$\begin{aligned}
&A(\delta) * (\sin t) = (\lambda, \{\mathbb{Z}/2^N\})_e * (-\pi i t, \{-1, 1\})_e = (-\pi i t \lambda, \{-1, 1\})_e \\
&= \left(\pi i \sum_{k=-\infty}^{\infty} \psi(k) \exp(-ik)\right) \frac{1}{2\pi} \exp(-it) + \left(-\pi i \sum_{k=-\infty}^{\infty} \psi(k) \exp(ik)\right) \frac{1}{2\pi} \exp(it) \\
&= \sum_{k=-\infty}^{\infty} \psi(k) \frac{\exp(i(t+k)) - \exp(-i(t+k))}{2i} = \sum_{k=-\infty}^{\infty} \psi(k) \sin(t+k) = A(\sin t).
\end{aligned} \tag{198}$$

5. Conclusion

We have defined a new multiplication of $\mathcal{A}'_\delta \subset \mathcal{S}'$ and convolution of $\mathcal{A}'_e \subset \mathcal{S}'$ that do not contradict the theories of continuous-time periodic signals, continuous-time non-periodic signals, discrete-time periodic signals, and discrete-time non-periodic signals. Under the multiplication, both continuous-time signals and discrete-time signals are transformed to subsets of tempered distributions by ring homomorphisms. Moreover, this proposed convolution is consistent with conventional convolutions of periodic distributions and non-periodic distributions.

As a result of these new definitions of operations, we have obtained a proof of a well-known theorem regarding the impulse response of continuous LTI systems of tempered distributions, under the assumption that the input of the system satisfies the conditions for the convolution. However, because $\mathcal{A}'_\delta \neq \mathcal{A}'_e$, not all elements of $\mathcal{A}'_\delta$ are suitable for calculating the convolution. For applications, it is more useful to have a convolution that is defined for $\mathcal{A}'_\delta$, because some noise filters are defined using a convolution. This extension of the convolution will be the subject of further research.

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[1] D. Kammler, *A First Course in Fourier Analysis*, Cambridge University Press, 2nd ed. (2008).

[2] L. Schwartz, *Théorie des distributions*, Hermann (1966).

[3] D. Smith, An introduction to distribution theory for signals analysis, *Digital Signal Processing*, **13**, 2, pp. 201–232 (2003).

[4] D. Smith, An introduction to distribution theory for signals analysis. Part III. Periodic tempered distributions, *Digital Signal Processing*, **41**, pp. 1–21 (2015).

[5] H. Youn and I. Richards, On the general definition of convolution for distributions, *Journal of the Korean Mathematical Society*, **17**, pp. 13–37 (1980).

[6] H. Youn and I. Richards, *The Theory of Distributions*, Cambridge University Press (1995).

[7] R. Shiraishi, On the definition of convolutions for distributions, *Journal of science of the Hiroshima University. Series A*, **23**, pp 19–32 (1959).

[8] D. Smith, An introduction to distribution theory for signals analysis. Part II. The convolution, *Digital Signal Processing*, **16**, 4, pp. 419–444 (2006).

[9] H. Arai, *Fourier kaisekigaku (Fourier analysis)*, Asakura Shoten (2003).

[10] W. Krabs, *Mathematical foundations of signal theory*, Heldermann Verlag (1995).

[11] L. Schwartz, Sur l'impossibilité de la multiplication des distributions, *Comptes Rendus de l'Académie des Sciences*, **239**, pp. 847–848 (1954).

[12] R. Shiraishi, M. Itano, On the multiplicative products of distributions, *Journal of Science of the Hiroshima University, Series A-I*, **28**, 2, pp. 223–235 (1964).

[13] I. Richards, H. Youn, Localization and multiplication of distributions, *Journal of the Korean Mathematical Society*, **37**, 3, pp. 371–389 (2000).

[14] J. Mikusiński, On the square of the dirac delta-distribution, *Bulletin de l'Académie polonaise des sciences. Série des sciences mathématiques, astronomiques et physiques*, **14**, 9, pp. 511–513 (1966).

[15] J. Colombeau, A multiplication of distributions, *Journal of Mathematical Analysis and Applications*, **94**, 1, pp. 96–115 (1983).

[16] F. Bagarello, Multiplication of distributions in any dimension: Applications to δ-function and its derivatives, *Journal of Mathematical Analysis and Applications*, **337**, 2, pp. 1337–1344 (2008).

[17] J. Colombeau, *New Generalized Functions and Multiplication of Distributions* , Elsevier Science Ltd (1984).

[18] Y. Egorov, A contribution to the theory of generalized functions, *Russian Mathematical Surveys*, **45**, 5, pp. 1–49 (1990).

[19] V. Vladimirov, *Obobshchennye funktsii i ikh primeneniya (Generalized functions and their applications)*, Znanie (1990).

[20] Li Bang-He, Nonstandard analysis and multiplication of distributions, *Scientia Sinica*, **21** pp. 561–585 (1978).

[21] T. Nitta, T. Okada, Infinitesimal fourier transformation for the space of functionals, *Proceedings of the Conference on Differential & Difference Equations and Applications*, pp. 871–883 (2006).

[22] M. Oberguggenberger, *Multiplication of Distributions and Application to Partial Differential Equations*, Longman Sc & Tech (1992).

[23] S. Ito, *Lebesgue sekibun nyuumon (Introduction to Lebesgue integral)*, Shokabo (1963).

Dai Akita received his Bachelor of Engineering and his Master of Engineering degrees from Osaka University in 2012 and 2014. He is now a doctoral student at Hokkaido University studying information processing of organisms through experiments on slime mold.