RIGOROUS JUSTIFICATION OF THE HYDROSTATIC APPROXIMATION FOR THE PRIMITIVE EQUATIONS BY SCALED NAVIER-STOKES EQUATIONS

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Abstract. Consider the anisotropic Navier-Stokes equations as well as the primitive equations. It is shown that the horizontal velocity of the solution to the anisotropic Navier-Stokes equations in a cylindrical domain of height ε with initial data $u_0 = (v_0, w_0) \in B^{2-2/p}_{q,p}$, $1/q + 1/p \le 1$ if $q \ge 2$ and $4/3q+2/3p \le 1$ if $q \le 2$, converges as $\varepsilon \to 0$ with convergence rate $\mathcal{O}(\varepsilon)$ to the horizontal velocity of the solution to the primitive equations with initial data v_0 with respect to the maximal-L^p-L^q-regularity norm. Since the difference of the corresponding vertical velocities remains bounded with respect to that norm, the convergence result yields a rigorous justification of the hydrostatic approximation in the primitive equations in this setting. It generalizes in particular a result by Li and Titi for the L^2-L^2-setting. The approach presented here does not rely on second order energy estimates but on maximal L^p-L^q-estimates for the heat equation.

1. Introduction

The primitive equations for the ocean and atmosphere are considered to be a fundamental model for geophysical flows, see e.g. the survey article [13]. The mathematical analysis of these equations has been pioneered by Lions, Teman and Wang in their articles [14–16], where they proved the existence of global, weak solutions to the primitive equations. Their uniqueness remains an open problem until today. Global strong well-posedness of the primitive equations for initial data in H^1 was shown by Cao and Titi in [3] using energy methods. A different approach, based on the theory of evolution equations, was introduced by Hieber and Kashiwabara in [10] and subsequent works [6–8, 11].

It is the aim of this paper to show that the primitive equations can be obtained as the limit of anisotropically scaled Navier-Stokes equations. The scaling parameter $\varepsilon > 0$ represents the ratio of the depth to the horizontal width. Such an approximation is motivated by the fact that for large-scale oceanic dynamics, this aspect ratio ε is rather small and implies anisotropic viscosity coefficients (see e.g. [17]). For an aspect ratio ε, i.e., in the case where the spacial domain can be represented as $\Omega_\varepsilon = G \times (-\varepsilon, +\varepsilon)$ for some $G \subset \mathbb{R}^2$, and a horizontal and vertical eddy viscosity 1 and ε^2, respectively, the system can be rescaled into the form

$$
\left\{
\begin{array}{rl}
\partial_t v_\varepsilon + u_\varepsilon \cdot \nabla v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = & 0, \\
\varepsilon(\partial_t w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon - \Delta w_\varepsilon) + \frac{1}{\varepsilon}\partial_z p_\varepsilon = & 0, \\
\operatorname{div} u_\varepsilon = & 0.
\end{array}
\right. \tag{1.1}
$$

in the time-space domain $(0, T) \times \Omega_1$, which is *independent* of the aspect ratio. We refer to [12] for more details on this rescaling procedure. Here the horizontal and vertical velocities v_ε and w_ε describe the three-dimensional velocity u_ε, while p_ε denotes the pressure of the fluid. Here ∂_z denotes the vertical-derivative, ∇_H and div_H the horizontal gradient and divergence, whereas div, ∇ and Δ stand for the usual three-dimensional spatial divergence, gradient, and Laplacian.

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First convergence results for the above system in the steady state case go back to Besson and Laidy [2]. The convergence of the above system has been studied first by Azérad and Guillén in [1] in the setting of weak convergence, no uniform convergence rate was given.

Recently, Li and Titi [12] investigated the strong convergence of the above system within the L^2-L^2-setting for horizontal initial velocities belonging to H^1 and H^2. In addition, they showed a convergence rate of order $\mathcal{O}(\varepsilon)$.

It is the aim of this paper to show convergence results of the above system in the strong sense within the L^p-L^q-setting. Our method is very different from the one introduced by [12], whereas they rely on second order energy estimates, our approach is based on maximal L^p-L^q-regularity estimates for the heat equation. This allows us to give a very short proof of the convergence result in the more general L^p-L^q-setting, which even in the L^2-L^2-setting allows for a slightly larger class of initial data compared to the one introduced by Li and Titi in [12] by using energy estimates.

2. Preliminaries

Consider the cylindrical domain $\Omega := (0,1)^2 \times (-1,1)$. Let $u = (v,w)$ be the solution of the primitive equations

$$
\text{(PE)} \qquad \begin{cases}
\partial_t v + u\cdot\nabla v - \Delta v + \nabla_H p = & 0 \quad\quad \text{in } (0,T)\times\Omega,\\
\partial_z p = & 0 \quad\quad \text{in } (0,T)\times\Omega,\\
\operatorname{div} u = & 0 \quad\quad \text{in } (0,T)\times\Omega,\\
p \text{ periodic in } x,y & \\
v,w \text{ periodic in } x,y,z, & \text{even and odd} \quad \text{in } z,\\
u(0) = & u_0 \quad\quad \text{in } \Omega,
\end{cases}
$$

and $u_\varepsilon = (v_\varepsilon, w_\varepsilon)$ be the solution of the anisotropic Navier-Stokes equations

$$
(\text{NS}_\varepsilon) \qquad \begin{cases}
\partial_t v_\varepsilon + u_\varepsilon\cdot\nabla v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = & 0 \quad\quad \text{in } (0,T)\times\Omega,\\
\partial_t w_\varepsilon + u_\varepsilon\cdot\nabla w_\varepsilon - \Delta w_\varepsilon + \frac{1}{\varepsilon^2}\partial_z p_\varepsilon = & 0 \quad\quad \text{in } (0,T)\times\Omega,\\
\operatorname{div} u_\varepsilon = & 0 \quad\quad \text{in } (0,T)\times\Omega,\\
p_\varepsilon \text{ periodic in } x,y,z, & \text{even} \quad \text{in } z,\\
v_\varepsilon, w_\varepsilon \text{ periodic in } x,y,z, & \text{even and odd} \quad \text{in } z,\\
u_\varepsilon(0) = & u_0 \quad\quad \text{in } \Omega.
\end{cases}
$$

Here v and v_ε denote the (two-dimensional) horizontal velocities, w and w_ε the vertical velocities, and p and p_ε denote the pressure term for the primitive equations as well as the Navier-Stokes equations, respectively. These are functions of three space variables $x,y\in(0,1)$, $z\in(-1,1)$. The vertical periodicity and parity conditions correspond to an equivalent set of equations with vertical Neumann boundary conditions for the horizontal velocity and vertical Dirichlet boundary conditions for the vertical velocity (cf. e.g. [4]). Since w is odd, the divergence free condition for the primitive equation translates into $\operatorname{div}_H \overline{v} = 0$, where $\overline{v}(x,y) = \frac{1}{2}\int_{-1}^{1} v(x,y,z)\,\mathrm{d}z$, and

$$
w(\cdot,\cdot,z) = -\int_{-1}^{z} \operatorname{div}_H v(\cdot,\cdot,\zeta)\,\mathrm{d}\zeta.
$$

For $p,q\in(1,\infty)$ and $s\in[0,\infty)$ we define the Bessel potential and Besov spaces

$$
H^{s,p}_{per}(\Omega) = \overline{C^\infty_{per}(\overline{\Omega})}^{\|\cdot\|_{H^{s,p}}} \quad \text{and} \quad B^s_{p,q,per}(\Omega) = \overline{C^\infty_{per}(\overline{\Omega})}^{\|\cdot\|_{B^s_{p,q}}},
$$

where $C^\infty_{per}(\overline{\Omega})$ denotes the space of smooth functions that are periodic of any order (cf. [10, Section 2]) in all three directions on $\partial\Omega$. The space $H^{s,p}(\Omega)$ denotes the Bessel potential space of order s, with norm $\|\cdot\|_{H^{s,p}}$ defined via the restriction of the corresponding space defined on the whole space to Ω (cf. [19, Definition 3.2.2.]). Moreover, $B^s_{p,q}(\Omega)$ denotes a Besov space on Ω, which is defined by restrictions of functions on the whole space to Ω, see e.g. [19, Definition 3.2.2.]. Note that $L^p(\Omega) = H^{0,p}_{per}(\Omega)$ and $B^s_{p,2,per}(\Omega) = H^s_{p,per}(\Omega)$. The anisotropic structure of the primitive equations motivates the definition of the Bessel potential spaces $H^{s,p}_{xy} := H^{s,p}((0,1)^2)$ and $H^{s,p}_z := H^{s,p}(-1,1)$ for the horizontal and vertical variables, respectively. Similarly as above we write $L^p_{xy} := H^{0,p}_{xy}$ and $L^p_z := H^{0,p}_z$ and set $H^{s,p}_{xy}H^{r,q}_z := H^{s,p}((0,1)^2; H^{r,q}_z)$.

The divergence free conditions in the above sets of equations can be encoded into the space of solenoidal functions

$$L^p_\sigma(\Omega) = \overline{\{u \in C^\infty_{per}(\overline{\Omega})^3 : \operatorname{div} u = 0\}}^{\|\cdot\|_{L^p}} \quad \text{and} \quad L^p_{\overline{\sigma}}(\Omega) = \overline{\{v \in C^\infty_{per}(\overline{\Omega})^2 : \operatorname{div}_H \overline{v} = 0\}}^{\|\cdot\|_{L^p}}.$$

For given $p, q \in (1,\infty)$ we set

$$\begin{aligned}
&X_0 := L^q(\Omega), \quad X_1 := H^{2,q}_{per}(\Omega),\\
&X^v_0 := \{v \in L^q_{\overline{\sigma}}(\Omega) : v \text{ even in } z\}, \quad X^v_1 := \{v \in H^{2,q}_{per}(\Omega)^2 \cap L^q_{\overline{\sigma}}(\Omega) : v \text{ even in } z\},\\
&X^u_0 := \{(v_1, v_2, w) \in L^q_\sigma(\Omega) : v_1, v_2 \text{ even } w \text{ odd in } z\},\\
&X^u_1 := \{(v_1, v_2, w) \in H^{2,q}_{per}(\Omega)^3 \cap L^q_\sigma(\Omega) : v_1, v_2 \text{ even } w \text{ odd in } z\},
\end{aligned}$$

and consider the trace space X_γ defined by

$$X_\gamma = (X^u_0, X^u_1)_{1-1/p,p}.$$

Here $(\cdot,\cdot)_{1-1/p,p}$ denotes the real interpolation functor.

Following the lines of [11, Section 4] and [5] the trace space X_γ can be characterized as follows.

Lemma 2.1 (Characterization of the trace space)**.** *Let $p, q \in (1,\infty)$. Then*

$$X_\gamma = \begin{cases} \{(v_1, v_2, w) \in B^{2-2/p}_{q,p,per}(\Omega)^3 \cap L^q_\sigma(\Omega) : & v = (v_1, v_2) \text{ even, } w \text{ odd in } z,\\ & (\partial_z v, w) = 0 \text{ at } z = -1, 0, 1\}, \quad 1 > \frac{2}{p} + \frac{1}{q},\\ \{(v_1, v_2, w) \in B^{2-2/p}_{q,p,per}(\Omega)^3 \cap L^q_\sigma(\Omega) : & v = (v_1, v_2) \text{ even, } w \text{ odd in } z,\\ & w = 0 \text{ at } z = -1, 0, 1\}, \quad 1 < \frac{2}{p} + \frac{1}{q}. \end{cases}$$

For $p, q \in (1,\infty)$ and $T \in (0,\infty]$ we also define the maximal regularity spaces

$$\mathbb{E}_0(T) := L^p(0,T; X_0), \quad \mathbb{E}_1(T) := L^p(0,T; X_1) \cap H^{1,p}(0,T; X_0),$$

and analogously $\mathbb{E}^v_0(T)$, $\mathbb{E}^v_1(T)$ and $\mathbb{E}^u_0(T)$, $\mathbb{E}^u_1(T)$ with respect to X^v_0, X^v_1 and X^u_0, X^u_1, respectively. In order to simplify our notation we sometimes omit the subscripts u and v and write only $\mathbb{E}_0(T)$ and $\mathbb{E}_1(T)$.

Finally, we say that $u = (v, w)$ is a *strong solution to the primitive equations* (in the L^p-L^q-setting), if $v \in \mathbb{E}^v_1$ and (PE) holds almost everywhere. We say that u_ε is a *strong solution to the Navier-Stokes equations*, if $u \in \mathbb{E}^u_1$ and (NS$_\varepsilon$) holds almost everywhere.

3. Main Result

Roughly speaking, the idea of our approach consists of controling the maximal regularity norm of the differences $(v_\varepsilon - v, \varepsilon(w_\varepsilon - w))$ by the aspect ratio ε. To this end, we introduce the difference equations of (NS$_\varepsilon$) and (PE): setting $V_\varepsilon := v_\varepsilon - v$, $W_\varepsilon := w_\varepsilon - w$, $U_\varepsilon := (V_\varepsilon, W_\varepsilon)$ and $P_\varepsilon = p_\varepsilon - p$, we obtain

$$\tag{3.1} \begin{cases} \partial_t V_\varepsilon - \Delta V_\varepsilon = & F_H(V_\varepsilon, W_\varepsilon) - \nabla_H P_\varepsilon & \text{in } (0,T) \times \Omega,\\ \partial_t \varepsilon W_\varepsilon - \Delta \varepsilon W_\varepsilon = & \varepsilon F_z(V_\varepsilon, W_\varepsilon) - \frac{1}{\varepsilon}\partial_z P_\varepsilon & \text{in } (0,T) \times \Omega,\\ \operatorname{div} U_\varepsilon = & 0 & \text{in } (0,T) \times \Omega,\\ P_\varepsilon \text{ periodic in } x, y, z, & \text{even} & \text{in } z,\\ V_\varepsilon, W_\varepsilon \text{ periodic in } x, y, z, & \text{even and odd} & \text{in } z,\\ U_\varepsilon(0) = & 0 & \text{in } \Omega, \end{cases}$$

where the forcing terms F_H and F_z are given by

$$\begin{aligned}
F_H(V_\varepsilon, W_\varepsilon) :=& -U_\varepsilon \cdot \nabla v - u \cdot \nabla V_\varepsilon - U_\varepsilon \cdot \nabla V_\varepsilon,\\
F_z(V_\varepsilon, W_\varepsilon) :=& -U_\varepsilon \cdot \nabla w - u \cdot \nabla W_\varepsilon - U_\varepsilon \cdot \nabla W_\varepsilon - \partial_t w - u \cdot \nabla w + \Delta w.
\end{aligned}$$

Applying the maximal regularity estimate given in Proposition 4.7 to (3.1), we are able to estimate $\|U_\varepsilon\|_{\mathbb{E}_1}$ in terms of the right hand sides. The latter will be estimated in a series of lemmas in Section 4.

Like this we obtain a quadratic inequality for the norm of the differences (cf. Corollary 4.9) and we need to ensure that the constant term as well as the coefficient in front of the linear term are sufficiently small. This can be achieved provided the aspect ratio ε is small enough and provided the vertical and horizontal solution of the primitive equations exist globally in the maximal regularity class (cf. Proposition 4.8).

Assumption (A). Let $q \in \left(\frac{4}{3}, \infty\right)$ and $p \geq \max\left\{\frac{q}{q-1}; \frac{2q}{3q-4}\right\}$, i.e., $1 \geq \begin{cases} \frac{1}{p} + \frac{1}{q}, & \text{if } q \geq 2, \\ \frac{2}{3p} + \frac{4}{3q}, & \text{if } q \leq 2. \end{cases}$

We are now in the position to state our main result.

Theorem 3.1 (Main Theorem)**.** *Assume that p, q fulfill Assumption (A) and let $u_0 \in X_\gamma$, $T > 0$ and (v, w) and $(v_\varepsilon, w_\varepsilon)$ be solutions of (PE) and (NS$_\varepsilon$), respectively. Then there exists a constant $C > 0$, independent of ε, such that for ε sufficiently small it holds*

$$\|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1(T)} \leq C\varepsilon.$$

In particular

$$(v_\varepsilon, \varepsilon w_\varepsilon) \to (v, 0) \text{ in } L^p(0, T; H^{2,q}(\Omega)) \cap H^{1,p}(0, T; L^q(\Omega))$$

as $\varepsilon \to 0$ with convergence rate $\mathcal{O}(\varepsilon)$.

Remarks 3.2. a) If the solution $u = (v, w)$ of the primitive equations exists globally in time, the convergence rate is uniform for all $T \in (0, \infty]$, see Remark 4.10 b). For example, if $p = q = 2$ and the initial data are mean value free, one can show that the solution to the primitive equations exists globally in $\mathbb{E}_1(T)$ with $T = \infty$.

b) We note that the case $p = q = 2$, investigated before in [12], is covered by our result. More specifically, they assumed $v_0 \in H^2$ whereas for our purposes $v_0, \operatorname{div}_H v_0 \in H^1$ suffices.

c) The scaled Navier-Stokes equations are locally well-posed in the maximal regularity spaces on the torus and the parity conditions are preserved. Our main result, Theorem 3.1, yields that for each time T there exists an $\varepsilon > 0$ such that the solution exists on $(0, T)$.

d) The primitive equations are well-posed for all times $T > 0$ provided $u_0 \in X_\gamma$, see [5].

e) Our method can be adjusted to the case with perturbed initial data. That is, given initial data $(u_{0,\varepsilon})_{\varepsilon>0} \subset X_\gamma$ converging to u_0 in X_γ as $\varepsilon \to 0$ of order $\mathcal{O}(\delta_\varepsilon)$ for some null-sequence $(\delta_\varepsilon)_{\varepsilon>0}$, then Theorem 3.1 holds with $(v_\varepsilon, w_\varepsilon)$ replaced by the solution of (NS$_\varepsilon$) with initial data $v_{0,\varepsilon}$. In that case the maximal regularity norm of the differences is bounded by $C \max\{\varepsilon, \delta_\varepsilon\}$ and consequently the convergence rate is of order $\mathcal{O}(\max\{\varepsilon, \delta_\varepsilon\})$.

4. Proof of the Main Theorem

The proof of Theorem 3.1 relies upon estimates on the terms F_H and F_z in equations (3.1) within the L^p-L^q-framework. These estimates imply eventually a quadratic inequality for the difference of the velocities, see Corollary 4.9. In order to establish these estimates we need to ensure that the solution of the primitive equations belongs to the maximal regularity class (see Proposition 4.8) and that the nonlinear terms can be estimated in $\mathbb{E}_0(T)$, see Lemma 4.3 and 4.5. We hence subdivide our proof in three steps. Throughout this section let $T < \infty$.

4.1. **Nonlinear estimates.** In this subsection, we estimate the bilinear terms and keep track of the T-dependence of the norms involved.

Lemma 4.1. *Let $s \geq s' \geq 0$ and $p \in [1, \infty)$. Then $H^{s,p}(0, T) \overset{s-s'}{\hookrightarrow} H^{s',p}(0, T)$ where $\overset{\eta}{\hookrightarrow}$ stands for an embedding with embedding constant CT^η, $C > 0$ independent of T.*

Proof. Set $m := \lfloor p(s - s') \rfloor$ and $1/r := m + 1/p + s' - s \in [1/p, 1)$. Sobolev embeddings and Hölder's inequality yield

$$H^{s,p} \hookrightarrow H^{m+s',r} \overset{1-\frac{1}{r}}{\hookrightarrow} H^{m+s',1} \hookrightarrow H^{m-1+s',\infty} \overset{1}{\hookrightarrow} H^{m-1+s',1} \hookrightarrow \ldots \hookrightarrow H^{s',\infty} \overset{\frac{1}{p}}{\hookrightarrow} H^{s',p}. \qquad \square$$

We will make use of the following classical Mixed Derivative Theorem, see e.g. [18, Corollary 4.5.10].

Proposition 4.2 (Mixed Derivative Theorem)**.** *If $\theta \in [0,1]$, then*
$$\mathbb{E}_1(T) \hookrightarrow H^{\theta,p}(0,T;H^{2-2\theta,q}(\Omega)).$$

Lemma 4.3. *Let $p,q \in (1,\infty)$ such that $2/3p + 1/q \leq 1$. Then for all $v_1, v_2 \in \mathbb{E}_1(T)$, $\partial \in \{\partial_x, \partial_y, \partial_z\}$ and $\eta \in \left[0, \frac{3}{2}\left(1 - \frac{2}{3p} - \frac{1}{q}\right)\right]$ there exists a constant $C > 0$ such that*
$$\|v_1 \partial v_2\|_{\mathbb{E}_0(T)} \leq C T^\eta \|v_1\|_{\mathbb{E}_1(T)} \|v_2\|_{\mathbb{E}_1(T)}, \quad C > 0.$$

Proof. Set $\theta_1 = \frac{2\eta}{3} + \frac{2}{3p}$ and $\theta_2 = \frac{1}{2}\theta_1$. The Mixed Derivative Theorem, Lemma 4.1 and Sobolev's embedding yield
$$\mathbb{E}_1(T) \hookrightarrow H^{\theta_1,p}(0,T;H^{2-2\theta_1,q}) \overset{2\eta/3}{\hookrightarrow} H^{2/3p,p}(0,T;H^{2-2\theta_1,q}) \hookrightarrow L^{3p}(0,T;L^{3q}(\Omega)),$$
$$\mathbb{E}_1(T) \hookrightarrow H^{\theta_2,p}(0,T;H^{2-2\theta_2,q}) \overset{\eta/3}{\hookrightarrow} H^{1/3p,p}(0,T;H^{2-2\theta_2,q}) \hookrightarrow L^{3p/2}(0,T;H^{1,3q/2}(\Omega)).$$
Hölder's inequality thus implies
$$\|v_1 \partial v_2\|_{L^p(L^q)} \leq \|\|v_1\|_{L^{3q}} \|\partial v_2\|_{L^{3q/2}}\|_{L^p} \leq \|v_1\|_{L^{3p}(L^{3q})} \|v_2\|_{L^{3p/2}(H^{1,3q/2})} \leq C T^\eta \|v_1\|_{\mathbb{E}_1} \|v_2\|_{\mathbb{E}_1}. \qquad \square$$

Lemma 4.4. *Let $q \in (1,\infty)$, $v_1, v_2 \in H^{1+1/q,q}(\Omega)$ and $w_1 := \int_{-1}^z \operatorname{div}_H v_1$. Then there exists a constant $C > 0$ such that*
$$\|w_1 \partial_z v_2\|_{L^q} \leq C \|v_1\|_{H^{1+1/q,q}} \|v_2\|_{H^{1+1/q,q}}.$$

Proof. Similarly as in [10, Lemma 5.1] we obtain by anisotropic Hölder's inequality and Sobolev inequalities
$$\begin{aligned}\|w_1 \partial_z v_2\|_{L^q} &\leq \|w_1\|_{L^{2q}_{xy} L^\infty_z} \|\partial_z v_2\|_{L^{2q}_{xy} L^q_z} \\ &\leq C \|\operatorname{div}_H v_1\|_{L^{2q}_{xy} L^1_z} \|\partial_z v_2\|_{L^{2q}_{xy} L^q_z} \\ &\leq C \|v_1\|_{H^{1+1/q}_{xy} L^1_z} \|v_2\|_{H^{1/q,q}_{xy} H^{1,q}_z}. \qquad \square\end{aligned}$$

Lemma 4.5. *Let $p,q \in (1,\infty)$ such that $1/p + 1/q \leq 1$ and $\eta \in \left[0, 1 - \frac{1}{q} - \frac{1}{p}\right]$. Then for all $v_1, v_2 \in \mathbb{E}_1(T)$ and w_1 given by $w_1 := \int_{-1}^z \operatorname{div}_H v_1$ there exists a constant $C > 0$ such that*
$$\|w_1 \partial_z v_2\|_{\mathbb{E}_0(T)} \leq C T^\eta \|v_1\|_{\mathbb{E}_1(T)} \|v_2\|_{\mathbb{E}_1(T)}.$$

Proof. Setting $\theta = \frac{\eta}{2} + \frac{1}{p}$, Lemma 4.1 and the Mixed Derivative Theorem, Proposition 4.2, yield
$$\mathbb{E}_1(T) \hookrightarrow H^{\theta,p}(0,T;H^{2-2\theta,q}) \overset{\frac{\eta}{2}}{\hookrightarrow} H^{1/2p,p}(0,T;H^{2-2\theta,q}) \hookrightarrow L^{2p}(0,T;H^{1+1/q,q}(\Omega)),$$
Putting $X := H^{1+1/q,q}$ and $L^p(L^q) := \mathbb{E}_0(T)$, Lemma 4.4 and the above embeddings imply
$$\|w_1 \partial_z v_2\|_{L^p(L^q)} \leq C \|\|v_1\|_X \|v_2\|_X\|_{L^p} \leq C \|v_1\|_{L^{2p}(X)} \|v_2\|_{L^{2p}(X)} \leq C T^\eta \|v_1\|_{\mathbb{E}_1(T)} \|v_2\|_{\mathbb{E}_1(T)}. \qquad \square$$

4.2. **Maximal regularity results.** In this subsection we prove that the vertical and horizontal solution of the primitive equations belong to the maximal regularity class $\mathbb{E}_1^u(T)$ as well as that the solution the linearized system associated with (3.1) fulfills a maximal regularity result estimate. We start by considering the linearization of (3.1). It corresponds to the difference equation of (NS$_\varepsilon$) and (PE).

Given $F \in \mathbb{E}_0(T)$ and initial data $U_0 \in X_\gamma$ we consider the linear problem

$$(4.1) \qquad \left\{\begin{array}{rll} \partial_t U - \Delta U = & F - \nabla_\varepsilon P & \text{in } (0,T) \times \Omega, \\ \nabla_\varepsilon \cdot U = & 0 & \text{in } (0,T) \times \Omega, \\ U, P & \text{periodic} & \text{in } x, y, z, \\ U(0) = & U_0 & \text{in } \Omega, \end{array}\right.$$

where $\nabla_\varepsilon := (\partial_x, \partial_y, \varepsilon^{-1}\partial_z)^T$. The functions U and P are the unknowns and represent the velocity and pressure differences.

We now aim to prove a maximal regularity estimate for U, where the constants are *independent* of the aspect ratio and the pressure gradient.

Lemma 4.6. *Let $\varepsilon > 0$, $q \in (1,\infty)$ and assume that $F = (f_H, f_z) \in L^q(\Omega)$ and $P \in H^{1,q}_{per}(\Omega)$ are satisfying the equation $-(\Delta_H + \varepsilon^{-2}\partial_z^2)P = \operatorname{div}(f_H, \varepsilon^{-1} f_z)$ for $\varepsilon > 0$. Then there exists a constant $C > 0$, independent of ε, such that*

$$\|(\nabla_H P, \varepsilon^{-1}\partial_z P)\|_q \le C\|F\|_q.$$

Proof. Denote by $\mathcal{F}$ the Fourier transform. For $k_\varepsilon = (k_1, k_2, \varepsilon^{-1}k_3)^T$ and $m(k) = -\frac{k\otimes k}{|k|^2} \in \mathbb{R}^{3\times 3}$ set $m_\varepsilon(k) = m(k_\varepsilon)$. Then

$$(\nabla_H, \varepsilon^{-1}\partial_z)P = \mathcal{F}^{-1} m_\varepsilon \mathcal{F} F$$

and $k\nabla m_\varepsilon(k) = k_\varepsilon \nabla m(k_\varepsilon)$. Hence,

$$\sup_{\gamma\in\{0,1\}^3} \sup_{k\neq 0} |k^\gamma D^\gamma m_\varepsilon(k)| = \sup_{\gamma\in\{0,1\}^3} \sup_{k_\varepsilon\neq 0} |k_\varepsilon^\gamma D^\gamma m(k_\varepsilon)| = 1,$$

and Mikhlin's theorem in the period setting, see e.g. [9, Proposition 4.5], implies that m_ε is an L^p-Fourier multiplier satisfying $\|\mathcal{F}^{-1} m_\varepsilon \mathcal{F}\|_{\mathcal{L}(L^q(\Omega))} \le C$ for some $C = C(q) > 0$. □

Proposition 4.7. *Let $p, q \in (1,\infty)$, $T > 0$, $F \in \mathbb{E}_0(T)$, $U_0 \in X_\gamma$ and $\varepsilon > 0$. Then there is a unique solution U, P to the equation (4.1) with $U \in \mathbb{E}_1(T)$ and $\nabla_\varepsilon P \in \mathbb{E}_0(T)$, where P is unique up to a constant. Moreover, there exist constants $C > 0$ and $C_T > 0$, independent of ε, such that*

$$\|U\|_{\mathbb{E}_1(T)} \le C\|F\|_{\mathbb{E}_0(T)} + C_T\|U_0\|_{X_\gamma}.$$

Proof. First, one defines the ε-dependent Helmholtz projection

$$\mathbb{P}_\varepsilon := \mathrm{Id} - \nabla_\varepsilon \Delta_\varepsilon^{-1} \operatorname{div}_\varepsilon, \quad \text{where } \Delta_\varepsilon = \nabla_\varepsilon \cdot \nabla_\varepsilon,$$

By Lemma 4.6 this is a bounded projection with uniform norm bound independent of ε.

First, we we apply this to (4.1). Taking into account that due to periodicity $\mathbb{P}_\varepsilon \Delta = \Delta \mathbb{P}_\varepsilon$ and $\mathbb{P}_\varepsilon \nabla_\varepsilon P = 0$ hold, the equation (4.1) reduces to the heat equation with right hand side $\mathbb{P}_\varepsilon F$.

Maximal L^p-regularity of the three-dimensional Laplacian in the periodic setting yields

$$\|U\|_{\mathbb{E}_1(T)} \le C\|\mathbb{P}_\varepsilon F\|_{\mathbb{E}_0(T)} + C_T\|U_0\|_{X_\gamma} \le C\|F\|_{\mathbb{E}_0(T)} + C_T\|U_0\|_{X_\gamma}.$$ □

Finally, we prove that the solution $u = (v, w)$ of the primitive equations belongs to the maximal regularity class $\mathbb{E}_1^u(T)$.

Proposition 4.8. *Let p, q fulfill Assumption (A) and let v be the strong solution of the primitive equations associated to v_0 satisfying $(v_0, w_0) \in X_\gamma$. Then*

$$u = (v, w) \in \mathbb{E}_1^u(T) \text{ for all } T > 0.$$

Proof. It was shown in [5, Theorem 3.3c] that the primitive equations admit a unique solution $v \in \mathbb{E}_1^v(T)$, which satisfies in addition $v \in C^\infty((0,T), C^\infty(\overline{\Omega})^2)$ and hence $w \in C^\infty((0,T), C^\infty(\overline{\Omega}))$ for any $T > 0$. It remains to show that w belongs to the maximal regularity class $\mathbb{E}_1(T^*)$ for some $T^* > 0$.

Applying $\int_{-1}^z \operatorname{div}_H(\cdot)$ to (PE) yields

$$\partial_t w - \Delta w = f(v, w) \text{ in } (0,\infty)\times\Omega,$$

where $f(v,w) = -\int_{-1}^z \operatorname{div}_H(\nabla_H p + u\cdot\nabla v)$. Using $\operatorname{div}_H \overline{v} = 0$, for $z = 1$ we obtain $2\Delta_H p = -\operatorname{div}_H \int_{-1}^1 u\cdot \nabla v$ and thus

$$f(v,w) = \frac{1}{2}\int_z \operatorname{div}_H u\cdot\nabla v = \frac{1}{2}\int_z \operatorname{div}_H \operatorname{div} u\otimes v,$$

where $\int_z := \int_z^1 - \int_{-1}^z + z\int_{-1}^1$. Observe that

$$\operatorname{div}_H \operatorname{div} u\otimes v = \partial_z(w \operatorname{div}_H v + v\cdot\nabla_H w) + (\operatorname{div}_H v)^2 + 2v\cdot\nabla_H \operatorname{div}_H v + \nabla_H v\cdot(\nabla_H v)^T.$$

Hence, $f(v,w) =: f_1(v,w) + f_2(v) + f_3(v,w)$ with

$$f_1 = (w \operatorname{div}_H v - v\cdot\nabla_H w)|_z^1, \quad f_2 = \frac{1}{2}\int_z \left(\nabla_H v\cdot(\nabla_H v)^T + (\operatorname{div}_H v)^2\right), \quad f_3 = \int_z \partial_z v\cdot\nabla_H w.$$

Here we used the fact that $\int_z v\cdot\nabla_H \operatorname{div}_H v = -2(v\cdot\nabla_H w)|_z^1 + \int_z \partial_z v\cdot\nabla_H w$, which follows by integration by parts. By Lemma 4.3 we obtain $\|f_1\|_{\mathbb{E}_0(T)} \le C\|v\|_{\mathbb{E}_1(T)}\|w\|_{\mathbb{E}_1(T)}$ and moreover

$$\begin{aligned}\|f_2\|_{L^p(L^q)} &\le C\|\partial_z f_2\|_{L^p(L^q_{xy}L^1_z)} \le C\|v\|^2_{L^{2p}(H^{1,2q}_{xy}L^2_z)},\\ \|f_3\|_{L^p(L^q)} &\le C\|\partial_z f_3\|_{L^p(L^q_{xy}L^1_z)} \le C\|v\|_{L^{2p}(L^{2q}_{xy}H^{1,2}_z)}\|w\|_{L^{2p}(H^{1,2q}_{xy}L^2_z)}.\end{aligned}$$

For $1 \ge 1/p + 1/q$ we find by the Mixed Derivative Theorem and Sobolev's embeddings

$$\mathbb{E}_1 \hookrightarrow H^{1/2p,p}(H^{2-1/p,q}) \hookrightarrow H^{1/2p,p}(H^{1+1/q,q}_{xy}L^q_z \cap H^{1/q,q}_{xy}H^{1,q}_z) \hookrightarrow L^{2p}(H^{1,2q}_{xy}L^q_z \cap L^{2q}_{xy}H^{1,q}_z).$$

If additionally $q \ge 2$, the above embedding implies $\mathbb{E}_1 \hookrightarrow L^{2p}(H^{1,2q}_{xy}L^2_z \cap L^{2q}_{xy}H^{1,2}_z)$. Similarly, for $q < 2$ and $1 \ge 4/3q + 2/3p$ we find

$$\mathbb{E}_1 \hookrightarrow H^{\frac{1}{2p},p}\left(H^{2-\frac{1}{p},q}\right) \hookrightarrow H^{\frac{1}{2p},p}\left(H^{1+\frac{1}{q},q}_{xy}H^{\frac{1}{q}-\frac{1}{2},q}_z \cap H^{\frac{1}{q},q}_{xy}H^{1+\frac{1}{q}-\frac{1}{2},q}_z\right) \hookrightarrow L^{2p}\left(H^{1,2q}_{xy}L^2_z \cap L^{2q}_{xy}H^{1,2}_z\right).$$

The above embeddings imply

$$\|f_2(v)\|_{\mathbb{E}_0(T)} \le C\|v\|^2_{\mathbb{E}_1(T)} \text{ and } \|f_1(v,w)\|_{\mathbb{E}_0(T)} + \|f_3(v,w)\|_{\mathbb{E}_0(T)} \le C\|v\|_{\mathbb{E}_1(T)}\|w\|_{\mathbb{E}_1(T)}.$$

In particular $f_2(v) \in \mathbb{E}_0(T)$. By maximal regularity there exists a solution operator $\mathcal{S} : X_\gamma \times \mathbb{E}_0(T) \to \mathbb{E}_1(T)$ such that $u := \mathcal{S}(w_0, f_2(v))$ satisfies

$$\partial_t u - \Delta u = f_2 \text{ in } (0,\infty)\times\Omega, \quad u(0) = w_0.$$

Setting now $B_v = -f_1(v,\cdot) - f_3(v,\cdot) \in \mathcal{L}(\mathbb{E}_1(T), \mathbb{E}_0(T))$, i.e. it is a bounded linear operator from $\mathbb{E}_1(T)$ to $\mathbb{E}_0(T)$, and adding $B_v u$ on both sides we see that

$$\partial_t u - \Delta u + B_v u = f_2 + B_v u = [\mathbb{I} + B_v\mathcal{S}(0,\cdot)]f_2 + B_v\mathcal{S}(w_0, 0) \text{ in } (0,\infty)\times\Omega.$$

Next, note that $\|\mathcal{S}(0,\cdot)\|_{\mathcal{L}(\mathbb{E}_0(T),\mathbb{E}_1(T))}$ can be bounded uniformly for $T \le 1$. Moreover $\|B_v\|_{\mathcal{L}(\mathbb{E}_1(T),\mathbb{E}_0(T))} \le C\|v\|_{\mathbb{E}_1(T)}$ by the previous estimates on f_1 and f_3. Choosing now T^* small enough such that $\|v\|_{\mathbb{E}_1(T^*)} < \left(C\|\mathcal{S}(0,\cdot)\|_{\mathcal{L}(\mathbb{E}_0(T^*),\mathbb{E}_1(T^*))}\right)^{-1}$, we see that $\|B_v\mathcal{S}(0,\cdot)\|_{\mathcal{L}(\mathbb{E}_0(T^*))} < 1$. A Neumann series argument yields $[\mathbb{I} + B_v\mathcal{S}(0,\cdot)]^{-1} \in \mathcal{L}(\mathbb{E}_0(T^*))$, and thus

$$\tilde{u} := \mathcal{S}\big(w_0, [\mathbb{I} + B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0))\big) \in \mathbb{E}_1(T^*)$$

solves

$$\begin{aligned}&\partial_t\tilde{u} - \Delta\tilde{u} + B_v\tilde{u} = \ [\mathbb{I} + B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0)) + B_v\tilde{u}\\ &= [\mathbb{I} + B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0)) + B_v\mathcal{S}\big(0, [\mathbb{I} + B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0))\big) + B_v\mathcal{S}(w_0,0)\\ &= [\mathbb{I} + B_v\mathcal{S}(0,\cdot)][\mathbb{I} + B_v\mathcal{S}(0,\cdot)]^{-1}(f_2 - B_v\mathcal{S}(w_0,0)) + B_v\mathcal{S}(w_0,0) = f_2\end{aligned}$$

in $(0,T^*)\times\Omega$ with $\tilde{u}(0) = w_0$. Since $f(v,\cdot) = f_2(v) - B_v$ and since the heat equation is uniquely solvable, we finally obtain $w = \tilde{u} \in \mathbb{E}_1(T^*)$. Summing up, $w \in \mathbb{E}_1(T)$ for any $T > 0$. □

Corollary 4.9. *Let $\mathcal{T} > 0$ and $p, q \in (1,\infty)$ such that $1/p + 1/q \le 1$. Let $(V_\varepsilon, W_\varepsilon) \in \mathbb{E}_1^u(\mathcal{T})$ denote the solution of equation (3.1) for some $u = (v,w) \in \mathbb{E}_1(\mathcal{T})$ and initial data $U_0 \in X_\gamma$. Then $X_\varepsilon(T) = \|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1(T)}$ and for any $\eta \in [0, 1 - 1/p - 1/q]$ there exists a constant $C > 0$, independent of ε, such that*

$$X_\varepsilon(T) \le CT^\eta\left[X_\varepsilon(T)\|u\|_{\mathbb{E}_1(T)} + X_\varepsilon^2(T)\right] + \varepsilon C\left[\|u\|_{\mathbb{E}_1(T)} + T^\eta\|u\|^2_{\mathbb{E}_1(T)}\right] + C\|U_0\|_{X_\gamma},$$

for all $T \in [0, \mathcal{T}]$.

Proof. Since

$$F_H = -V_\varepsilon \cdot \nabla_H v - W_\varepsilon \partial_z v - v \cdot \nabla_H V_\varepsilon - w\partial_z V_\varepsilon - V_\varepsilon \cdot \nabla_H V_\varepsilon - W_\varepsilon \partial_z V_\varepsilon,$$

we obtain with the help of Lemma 4.3 and 4.5

$$\|F_H\|_{\mathbb{E}_0(T)} \leq CT^\eta \|V_\varepsilon\|_{\mathbb{E}_1(T)} (\|V_\varepsilon\|_{\mathbb{E}_1(T)} + \|(v,w)\|_{\mathbb{E}_1(T)}). \tag{4.2}$$

Similarly, since

$$\varepsilon F_z = \varepsilon(-V_\varepsilon \cdot \nabla_H w - w \operatorname{div}_H V_\varepsilon) - \varepsilon W_\varepsilon \operatorname{div}_H(v + V_\varepsilon) - (v + V_\varepsilon) \cdot \nabla_H \varepsilon W_\varepsilon - \varepsilon(\partial_t w - u \cdot \nabla w + \Delta w),$$

Lemma 4.3 yields

$$\|\varepsilon F_z\|_{\mathbb{E}_0(T)} \leq CT^\eta [\|V_\varepsilon\|_{\mathbb{E}_1(T)} \|w\|_{\mathbb{E}_1(T)} + \|\varepsilon W_\varepsilon\|_{\mathbb{E}_1(T)} (\|V_\varepsilon\|_{\mathbb{E}_1(T)} + \|v\|_{\mathbb{E}_1(T)})] + C\varepsilon[\|w\|_{\mathbb{E}_1(T)} + T^\eta \|w\|^2_{\mathbb{E}_1(T)}].$$

Combining this estimate with (4.2), Proposition 4.7 yields the assertion. □

4.3. **Proof of the main result.**

Proof of Theorem 3.1. Fix $\mathcal{T} > 0$ and denote by u the solution of equation (PE). Proposition 4.8 yields $u \in \mathbb{E}_1(\mathcal{T})$. We now show that

$$X_\varepsilon(T) := \|V_\varepsilon, \varepsilon W_\varepsilon\|_{\mathbb{E}_1(T)} \leq \varepsilon C(\|(v,w)\|_{\mathbb{E}_1(\cdot)}, \mathcal{T}, p, q)$$

for all $T \in [0, \mathcal{T}]$ and $\varepsilon > 0$ small enough. By the uniform continuity of $T \mapsto \|u\|_{\mathbb{E}_1(T)}$ on $[0, \mathcal{T}]$ there is a $T^* \in [0, \mathcal{T}]$ such that $\|u\|^p_{\mathbb{E}_1(T+T^*)} - \|u\|^p_{\mathbb{E}_1(T)} \leq (2C\mathcal{T}^\eta)^{-p}$ for all $T \in [0, \mathcal{T} - T^*]$ and where C denotes the constant given in Corollary 4.9. The latter with $U_0 = 0$ implies

$$CX^2_\varepsilon(T) - \tfrac{1}{2} X_\varepsilon(T) + \varepsilon \geq 0, \quad T \in [0, T^*]. \tag{4.3}$$

Observe that $(X_\varepsilon)^p : t \mapsto \left(\int_0^t \ldots\right)$ is continuous in $[0,T]$ and $X_\varepsilon(0) = 0$. Thus, for $\varepsilon < (16C)^{-1}$, we may solve this quadratic inequality and obtain $X_\varepsilon \leq 2\varepsilon$ on $[0, T^*]$.

Note that inequality (4.3) holds indeed on a time interval independent of ε. More specifically, if one replaces T^* by $T_\varepsilon < T^*$, where T_ε is the maximal existence time of (NS$_\varepsilon$) with initial data u_0, then it holds similarly as above that $X_\varepsilon \leq 2\varepsilon$ on $[0, T_\varepsilon]$ which implies a contradiction to the maximality of the existence time.

Assume there is some $m \in \mathbb{N}$ such that $mT^* < \mathcal{T}$ and $X_\varepsilon \leq \varepsilon 2K_m$ in $[0, mT^*]$, where $K_1 = 1$ and $K_m = 2^{1/p}\left[(2Cc_{mT^*} + 1)K_{m-1} + 1\right]$ and c_T denotes the embedding constant of $\mathbb{E}_1(T) \hookrightarrow L^\infty(0, T; X_\gamma)$. Let $(\tilde{V}_\varepsilon, \varepsilon\tilde{W}_\varepsilon)(T) = (V_\varepsilon, \varepsilon W_\varepsilon)(T + mT^*)$ be the unique solution of problem (3.1) with respect to $\tilde{u}(T) = u(T + mT^*)$ and initial data $U_0 = (V_\varepsilon, \varepsilon W_\varepsilon)(mT^*)$. Setting

$$\tilde{X}^p_\varepsilon(T) := \|(\tilde{V}_\varepsilon, \varepsilon\tilde{W}_\varepsilon)\|^p_{\mathbb{E}_1(T)} = X^p_\varepsilon(T + mT^*) - X^p_\varepsilon(mT^*),$$

Corollary 4.9 and the argument about the ε-independency of the time interval given above imply

$$C\tilde{X}^2_\varepsilon(T) - \tfrac{1}{2}\tilde{X}_\varepsilon(T) + \varepsilon + C\|U_0\|_{X_\gamma} \geq 0, \quad T \in [0, \min\{T^*; \mathcal{T} - mT^*\}].$$

By assumption $\|U_0\|_{X_\gamma} \leq c_{mT^*} X_\varepsilon(mT^*) \leq c_{mT^*}\varepsilon 2K_m$. Since $\tilde{X}_\varepsilon(0) = 0$ and $\tilde{X}_\varepsilon$ is continuous in $[0, \min\{T^*; \mathcal{T} - mT^*\}]$, we may solve the quadratic inequality for $\varepsilon < (16C(1 + 2Cc_{mT^*}K_m))^{-1}$ and obtain $\tilde{X}_\varepsilon \leq \varepsilon 2(1 + 2Cc_{mT^*}K_m)$ in $[0, \min\{T^*; \mathcal{T} - mT^*\}]$. Hence, by the assumption on m

$$X^p_\varepsilon(T) \leq (\varepsilon 2(1 + 2Cc_{mT^*}K_m))^p + X^p_\varepsilon(mT^*) \leq 2\left[2\varepsilon(1 + 2Cc_{mT^*}K_m) + \varepsilon 2K_m\right]^p = (\varepsilon 2K_{m+1})^p,$$

for all $T \in [mT^*, \min\{(m+1)T^*; \mathcal{T}\}]$. The assumption on m implies $X_\varepsilon \leq \varepsilon 2K_{m+1}$ in $[0, \min\{(m+1)T^*; \mathcal{T}\}]$. By induction we get $X_\varepsilon \leq \varepsilon 2K_M$ in $[0, \mathcal{T}]$ with $M = \left\lceil \frac{\mathcal{T}}{T^*} \right\rceil$. The proof of Theorem 3.1 is complete. □

Remarks 4.10. a) It is remarkable that in every induction step we do *not* rely upon the local well-posedness of equation (3.1). In fact, the boundedness of the difference and the long time well-posedness of the primitive equations as well as the local well-posedness of the Navier-Stokes equations are sufficient for our arguments. More specifically, let $X_\varepsilon \le \varepsilon 2K_m$ on $[0, mT^*]$. To construct a solution $(\tilde{V}_\varepsilon, \varepsilon\tilde{W}_\varepsilon)$ to (3.1) with inhomogeneous initial data $(V_\varepsilon, \varepsilon W_\varepsilon)(mT^*)$ we reconstruct the solution to the Navier-Stokes equations with initial data $(V_\varepsilon, \varepsilon W_\varepsilon)(mT^*) + u(mT^*)$, which exists locally, from the solution of the primitive equations with initial data $u(mT^*)$. This method ensures local existence of the difference solution $(\tilde{V}_\varepsilon, \varepsilon\tilde{W}_\varepsilon)$.

On the other hand every solution to (3.1) with initial data $(V_\varepsilon, \varepsilon W_\varepsilon)(mT^*)$ adds up with the solution of the primitive equations with initial data $u(mT^*)$ to a solution of the Navier-Stokes equation with initial data $(V_\varepsilon, \varepsilon W_\varepsilon)(mT^*) + u(mT^*)$.

b) Given $u \in \mathbb{E}_1(\infty)$, we may adjust the above proof in such a way that $X_\varepsilon(T) \le \varepsilon C$ *uniformly* for all $T \in (0, \infty]$. More precisely, there exist *finitely* many $0 = T_0 < T_1 < \ldots < T_m = \infty$ such that $\|u\|^p_{\mathbb{E}_1(T_i)} - \|u\|^p_{\mathbb{E}_1(T_{i-1})} \le (2C)^{-p}$ for $i = 1, \ldots m$. Proceeding as above, while using Corollary 4.9, $\eta = 0$ yields $X_\varepsilon(T) \le \varepsilon C$ for all $T \in [0, \infty)$, where C is *independent* of T. Taking the limit yields the assertion for $T = \infty$.

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