A LOOP GROUP METHOD FOR AFFINE HARMONIC MAPS INTO LIE GROUPS

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Abstract. We generalize the Uhlenbeck-Segal theory for harmonic maps into compact semi-simple Lie groups to general Lie groups equipped with torsion free bi-invariant connection.

Introduction

Harmonic maps of Riemann surfaces into compact semi-simple Lie groups, equipped with a bi-invariant Riemannian metric, have been paid much attention to by differential geometers as well as by mathematical physicists. In fact, harmonic maps of Riemann surfaces into compact Lie groups equipped with a bi-invariant Riemannian metric are called *principal chiral models* and intensively studied as toy models of gauge theory in mathematical physics [63].

Uhlenbeck established a fundamental theory of harmonic maps into the unitary group U_n, [62]. In particular she proved a factorization theorem for harmonic 2-spheres, the so called *uniton factorization*. Segal showed that harmonic maps into U_n are obtained by holomorphic curve into the based loop group of U_n [58]. Uhlenbeck-Segal theory actually works for any compact Lie groups [12, 10]. Pedit, Wu and the first named author of the present paper generalized the Uhlenbeck-Segal theory to harmonic maps into compact Riemannian symmetric spaces, now referred as to the *generalized Weierstrass type representation* [25].

It turned out that the compactness of the target space is not necessary, as long as one only considers surfaces away from singularities and considers groups with bi-invariant metric, Riemannian or pseudo-Riemannian. From a global point of view the construction principle will generally produce surfaces with singularities. A typical example for this are the spacelike CMC surfaces in Minkowski 3-space. In this case one considers harmonic maps into the Riemannian symmetric space $\mathbb{H}^2 = \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$. Harmonic maps into $\mathbb{H}^2$ are closely related to the so-called tt*-geometry [21]. It is a very important and difficult problem to find (and describe) globally smooth solutions for non-compact target spaces.

If one wants to generalize the loop group approach for harmonic maps into symmetric spaces to general homogeneous spaces, where the Lie group only has a left-invariant metric, one encounters a completely new situation. Clearly, the case of harmonic maps into Lie groups

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is a first interesting case. Since abelian groups always have bi-invariant metrics, the next interesting case is the one of 3-dimensional Lie groups (with left-invariant metric).

As a matter of fact, and largely independent of the issues discussed above, during the last ten years or so, minimal surfaces into 3-dimensional Lie groups have been studied extensively. In particular, minimal surfaces in 3-dimensional Lie groups, equipped with a left-invariant metric, have been investigated intensively. With the exception of the space $\mathbb{S}^2 \times \mathbb{R}$, the other seven model spaces of Thurston geometries obviously have or can be given the structure of a Lie group. These are the following spaces; the Euclidean 3-space $\mathbb{E}^3$, the unit 3-sphere $\mathbb{S}^3$, the hyperbolic 3-space $\mathbb{H}^3$, the model space Nil_3 of nilgeometry, the universal covering group $\widetilde{\mathrm{SL}_2\mathbb{R}}$, the space Sol_3 of solvgeometry and the product space $\mathbb{H}^2 \times \mathbb{R}$. The metrics on these groups are generally only left-invariant with respect to the Lie group structure. Only the Euclidean 3-space $\mathbb{E}^3$ and the 3-sphere $\mathbb{S}^3$ admit bi-invariant Riemannian metrics.

Since the generalized Weierstrass type representation for harmonic maps usually requires a *bi-invariant metric* on some related Lie group, we can not expect this scheme to work unchanged for harmonic maps into Lie groups, if these Lie groups only carry a left-invariant metric. In fact, harmonic maps into general Lie groups do not even admit a zero-curvature representation in general. For instance, all the explicit examples in [24] of minimal surfaces (except vertical planes) in Nil_3 do not satisfy the zero-curvature representation described in Proposition 3.20. On the other hand, it should be remarked that the harmonicity equation makes sense for maps from Riemann surfaces into Lie groups equipped with any *affine connection.*

Higaki [32] pointed out that if a map $\varphi : M \to G/H$ of a Riemann surface into a reductive homogeneous space G/H is harmonic with respect to the *canonical connection* of G/H and if its torsion vanishes along φ, then φ is *equiharmonic*, that is, harmonic with respect to any G-invariant metric. This result indicates that there may exist a large class of harmonic maps with particularly nice properties. It is also natural to expect that harmonic maps in such a nice class may even admit an explicit construction scheme. From our point of view, harmonic maps in such a class should admit a loop group method.

In this paper we develop a loop group theory for harmonic maps into Lie groups which are equipped with a bi-invariant affine connection, namely *affine harmonic maps.* Instead of considering the Levi-Civita connection on Lie groups, we will use a natural, torsion free, bi-invariant connection, called *neutral connection*, to study maps from Riemann surfaces to Lie groups. We shall show that smooth maps into a general Lie group G equipped with the neutral connection admit a loop group formulation. Based on this fundamental result, we generalize the generalized Weierstrass type representation to harmonic maps into Lie groups equipped with the neutral connection. We note that a Lie group G, equipped with the neutral connection, is looked at as the affine symmetric space $G \times G/G$. If G is semi-simple, any neutral connection coincides with the Levi-Civita connection of the bi-invariant metric induced by the Killing form. In this sense, the present paper is a generalization of the loop group method originally developed by Uhlenbeck and Segal.

We would like to emphasize that we establish a loop group method for affine harmonic maps into *any* Lie group, in particular also into Lie groups which do not have any bi-invariant metric. As a consequence, in a sense, the particularly new feature of this paper is a treatment of harmonic maps into solvable Lie groups (and, as a special case, into nilpotent Lie groups), since, generally, these groups do not admit any bi-invariant metric.

In this context we would like to mention that harmonic maps into Lie groups, especially nilpotent or solvable Lie groups, equipped with left-invariant affine connections have applications to probability theory, since it is known that harmonic maps have a probabilistic characterization: A smooth map between Riemannian manifolds is a harmonic map if and only if it sends Brownian motions to martingales [42, 51]. Martingales and harmonic maps into Lie groups equipped with left-invariant affine connections have been studied in [3, 59].

This paper is organized as follows: In Section 1, we will briefly give preliminary results on vector bundle valued differential forms. In Section 2, affine harmonic maps from a Riemannian manifold into an affine manifold will be discussed. In Section 3, affine harmonic maps into any Lie group G will be considered. It is known that all left-invariant affine connections on a Lie group G are given by bilinear maps μ on the Lie algebra of G, and are therefore denoted by ∇^{μ}. Then ∇^{μ}-harmonic maps from Riemann surfaces into G, where G carries a left-invariant affine connection ∇^{μ} defined by a skew-symmetric map μ, will be characterized by a loop of flat connections in Theorem 3.14. In particular, we consider Lie groups with left-invariant metrics and harmonic maps from Riemann surfaces into these Lie groups with the Levi-Civita connection. We will give some simple and fundamental examples of neutral harmonic maps in Section 4. In Section 5, the generalized Weierstrass type representation for neutral harmonic maps into Lie groups will be presented. In this section, the Lie group is a connected real analytic Lie group admitting a faithful finite dimensional representation and its linear complexification is simply-connected. In the final Section 6, as an example of the generalized Weierstrass type representation, we will discuss the case of 3-dimensional solvable Lie groups in detail. In Theorem 6.5, we give a representation formula for neutral harmonic maps into 3-dimensional solvable Lie groups. The class of solvable Lie groups considered in Section 6 includes the following model spaces of Thurston geometry: $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and Sol_3. In particular we introduce a new class of surfaces in $\mathbb{H}^3$ which admit holomorphic constructions.

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1. Preliminaries

1.1. **Basic facts.** Let M be a manifold and E a vector bundle over M and denote by $\varGamma(E)$ the space of all smooth sections of the vector bundle E. The space $\varGamma(\wedge^r T^*M \otimes E)$ is denoted by $\Omega^r(E)$. An element of $\Omega^r(E)$ is called an *E-valued r-form* on M.

In case $E = M \times V$ is a trivial vector bundle over M with standard fiber V, then $\Omega^r(M \times V)$ is denoted by $\Omega^r(M; V)$. An element of $\Omega^r(M; V)$ is called a *V-valued r-form* on M. By definition, for $\alpha \in \Omega^r(M; V)$ and $X_1, X_2, \cdots, X_r \in \varGamma(TM)$, $\alpha(X_1, X_2, \cdots, X_r) \in C^{\infty}(M, V)$.

Next let G be a Lie group with Lie algebra $\mathfrak{g}$. Take a bilinear map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then for α, $\beta \in \Omega^1(M; \mathfrak{g})$, we define a $\mathfrak{g}$-valued 2-form $\mu(\alpha \wedge \beta)$ by

$$\mu(\alpha \wedge \beta)(X, Y) := \mu(\alpha(X), \beta(Y)) - \mu(\alpha(Y), \beta(X))$$

for any sections $X, Y \in \Gamma(TM)$. Moreover the *symmetric part* $\operatorname{sym}\mu$ and the *skew-symmetric part* $\operatorname{skew}\mu$ of μ are defined by

$$(\operatorname{sym}\mu)(X,Y) := \tfrac{1}{2}\mu(X,Y) + \tfrac{1}{2}\mu(Y,X), \quad (\operatorname{skew}\mu)(X,Y) := \tfrac{1}{2}\mu(X,Y) - \tfrac{1}{2}\mu(Y,X).$$

for any sections $X, Y \in \Gamma(TM)$. It is easy to check that the following relations hold for any $\alpha,\ \beta \in \Omega^1(M; \mathfrak{g})$:

$$(\operatorname{sym}\mu)(\beta\wedge\alpha) = -(\operatorname{sym}\mu)(\alpha\wedge\beta), \quad (\operatorname{skew}\mu)(\beta\wedge\alpha) = (\operatorname{skew}\mu)(\alpha\wedge\beta). \tag{1.1}$$

Let us denote by θ the left-invariant Maurer-Cartan form on a Lie group G. By definition, θ is a $\mathfrak{g}$-valued 1-form on G. The $\mathfrak{g}$-valued 2-form $[\theta\wedge\theta]$ is computed as

$$[\theta\wedge\theta](X,Y) = 2[\theta(X), \theta(Y)] = 2(\theta\wedge\theta)(X,Y).$$

The left Maurer-Cartan form θ satisfies the Maurer-Cartan equation:

$$d\theta + \frac{1}{2}[\theta\wedge\theta] = d\theta + \theta\wedge\theta = 0.$$

2. Affine harmonic maps

2.1. **Affine harmonic maps.** Let (M, g) be a Riemannian manifold and (N, ∇) an *affine manifold*, that is, a manifold with an affine connection ∇. We denote by ∇^M and T the Levi-Civita connection of (M, g) and the torsion of the connection ∇ on N, respectively. Let $\varphi : M \to N$ be a smooth map. Then the connection ∇ induces a unique connection ∇^φ on the pull-back tangent bundle φ^*TN which satisfies the condition

$$\nabla^\varphi_X(V\circ\varphi) = (\nabla_{d\varphi(X)}V)\circ\varphi$$

for any sections $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TN)$, see [27, p. 4]. The differential $d\varphi$ is naturally interpreted as a φ^*TN-valued 1-form.

The *exterior covariant derivative* of $d\varphi$ is given by

$$d^{\nabla^\varphi} d\varphi(X,Y) = \varphi^*T(X,Y).$$

Hence $d\varphi$ is a closed φ^*TN-valued 1-form if and only if $\varphi^*T = 0$. The *second fundamental form* $\nabla d\varphi$ of φ (with respect to ∇) is a section of $T^*M \otimes T^*M \otimes \varphi^*TN$ defined by

$$\nabla d\varphi(Y; X) := (\nabla^\varphi_X d\varphi)Y = \nabla^\varphi_X d\varphi(Y) - d\varphi(\nabla^M_X Y), \quad X, Y \in \Gamma(TM). \tag{2.1}$$

One can see that $\nabla d\varphi$ is symmetric if and only if $\varphi^*T = 0$. The *tension field* $\tau(\varphi, \nabla)$ of φ is the section of φ^*TN defined by

$$\tau(\varphi,\nabla) := \operatorname{tr}_g(\nabla d\varphi).$$

Here tr_g denotes the trace with respect to g. We now arrive at the following definition.

Definition 2.1.

(1) A smooth map $\varphi : (M, g) \to (N, \nabla)$ from a Riemannian manifold (M, g) into an affine manifold (N, ∇) is said to be an *affine harmonic map* or to be ∇-*harmonic map* if

$$\tau(\varphi,\nabla) = 0.$$

(2) Let us define an affine connection ${}^{\dagger}\nabla$ on an affine manifold (N, ∇) as follows:

$$ {}^{\dagger}\nabla_X Y := \nabla_X Y - \frac{1}{2}T(X, Y), \quad X, Y \in \Gamma(TN). $$

Then ${}^{\dagger}\nabla$ is torsion free and is called the *associated torsion free connection* of ∇.

Since the torsion T is skew-symmetric, it is easy to see that the tension fields $\tau(\varphi, \nabla)$ and $\tau(\varphi, {}^{\dagger}\nabla)$ are equal. Hence we have the following.

Proposition 2.1. *Let $\varphi : (M, g) \to (N, \nabla)$ be a smooth map of a Riemannian manifold into an affine manifold. Then φ is ∇-harmonic if and only if it is ${}^{\dagger}\nabla$-harmonic.*

Remark 2.2.

(1) In case $\dim M = 2$, the harmonic map equation $\tau(\varphi, \nabla) = 0$ is invariant under conformal changes of the metric of M, see [27]. Thus the affine-harmonicity makes sense for maps from a Riemann surface. Thus if M is orientable, then we can give M a complex structure such that the metric is isothermal and it suffices to consider harmonic maps from this setting. If M is not orientable, we consider the double cover.

(2) If (M, g) and (N, h) are Riemannian manifolds and if ∇ denotes the Levi-Civita connection of h, then the ∇-harmonicity of $\varphi : M \to N$ coincides with the notion of a harmonic map in the classical sense, that is, it is a critical point of the *energy functional*, [27]:

$$ E(\varphi) = \int_M \frac{1}{2}|d\varphi|^2 \, dv_g. $$

2.2. **Torsion-free affine harmonic maps.** More generally, we can introduce the notion of an affine $(1, 1)$-harmonic map on a complex manifold in the following manner. Let (M, J) be a complex manifold and $\varphi : M \to (N, \nabla)$ a smooth map into an affine manifold. With respect to the complex structure J, we decompose the complexified tangent bundle $T^{\mathbb{C}}M$ into the Whitney sum $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. By restricting the differential $d\varphi$ to $T^{(1,0)}M$ and $T^{(0,1)}M$, we obtain vector bundle morphisms

$$ \partial\varphi : T^{(1,0)}M \to \varphi^*TN^{\mathbb{C}}, \quad \bar{\partial}\varphi : T^{(0,1)}M \to \varphi^*TN^{\mathbb{C}}. $$

Then the $(0, 1)$-covariant derivative $\nabla''\partial\varphi$ of $\partial\varphi$ (also called the *Levi-form* of φ) is defined by

$$ (\nabla''\partial\varphi)(Z; \overline{W}) := (\nabla''_{\overline{W}}\partial\varphi)Z = \nabla^{\varphi}_{\overline{W}}\partial\varphi(Z) - \partial\varphi(\bar{\partial}_{\overline{W}}Z), \quad Z, W \in \Gamma(T^{(1,0)}M), $$

where $\bar{\partial}$ denotes the $\bar{\partial}$-operator on $T^{(1,0)}M$.

Definition 2.2. A smooth map $\varphi : (M, J) \to (N, \nabla)$ is said to be an *affine* $(1, 1)$*-harmonic* if $\nabla''\partial\varphi = 0$.

In case M is a Riemann surface, then affine $(1, 1)$-harmonicity is equivalent to affine harmonicity under the *torsion free condition.* More precisely, the following relation holds (also see Remark 3.9).

Proposition 2.3 (Theorem 5.1.1 in [43])**.** *Let M be a Riemann surface and $\varphi : M \to (N, \nabla)$ a smooth map into an affine manifold. Then φ is affine $(1, 1)$-harmonic if and only if φ is ∇-harmonic and $\varphi^*T = 0$.*

Sketch of proof. Take $(1,0)$ vector fields Z and W. Then Z and W are represented as

$$Z = X - \sqrt{-1}JX, \quad W = Y - \sqrt{-1}JY, \quad X, Y \in \Gamma(TM).$$

The $(0,1)$-covariant derivative of $\partial\varphi$ is computed as

$$\begin{aligned}(\nabla''\partial\varphi)(Z;\overline{W}) =& \{(\nabla d\varphi)(X,Y) + (\nabla d\varphi)(JX,JY)\} \\ &+ \sqrt{-1}\{(\nabla d\varphi)(X,JY) - (\nabla d\varphi)(JX,Y)\}\end{aligned}$$

This equation implies $\varphi^*T = 0$ and $\tau(\varphi) = 0$. More precisely, take a Hermitian metric g in the conformal class. Then for a local orthonormal frame field $\{e_1, e_2 = Je_1\}$ we choose $X = Y = e_1$. Then the real part of the equation $(\nabla''\partial\varphi)(Z;\overline{W}) = 0$ implies $\tau(\varphi,\nabla) = \mathrm{tr}_g(\nabla d\varphi) = 0$. The imaginary part implies $\varphi^*T(e_1, e_2) = 0$. Hence $\varphi^*T = 0$. □

Based on this characterization we give the following definition:

Definition 2.3. Let $\varphi : M \to (N, \nabla)$ be a smooth map of a Riemann surface into an affine manifold. Then φ is said to be *torsion-free affine harmonic* or *torsion-free ∇-harmonic* if φ is ∇-harmonic and $\varphi^*T = 0$, that is, it is an affine $(1,1)$-harmonic map on a Riemann surface.

Remark 2.4.

(1) Torsion-free harmonic maps were first observed by Burstall and Pedit [11]. In [43], torsion-free ∇-harmonic maps were called "strongly harmonic maps".
(2) For a Riemann surface M and a Riemannian manifold (N, h) with Levi-Civita connection ∇ of h, the affine $(1,1)$-harmonicity of $\varphi : M \to N$ coincides with the harmonicity in the classical sense, see [27] and (2) in Remark 2.2.

3. Affine harmonic maps into Lie groups

In this section we discuss affine harmonic maps from a Riemann surface into Lie groups. Let G be a connected real Lie group and denote by $\mathfrak{g}$ the Lie algebra of G, that is, the tangent space of G at the unit element $\mathrm{id} \in G$. Hereafter we restrict our attention to left-invariant affine connections on G.

3.1. Left-invariant connections.

Take a bilinear map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then we can define a left-invariant affine connection ∇^μ on G by its value at the unit element $\mathrm{id} \in G$ by

$$\nabla^\mu_X Y = \mu(X,Y), \quad X, Y \in \mathfrak{g}.$$

From [53] we know that all left-invariant affine connections are obtained in this way.

Proposition 3.1 ([53]). *Let $m_{\mathfrak{g}}$ be the vector space of all $\mathfrak{g}$-valued bilinear maps on $\mathfrak{g}$ and a_G the affine space of all left-invariant affine connections on G. Then the map*

$$m_{\mathfrak{g}} \ni \mu \longmapsto \nabla^\mu \in a_G$$

is a bijection between $m_{\mathfrak{g}}$ and a_G. The torsion T^μ of ∇^μ is given by

$$T^\mu(X,Y) = -[X,Y] + \mu(X,Y) - \mu(Y,X) \tag{3.1}$$

for all X, $Y \in \mathfrak{g}$.

Definition 3.1. We define a one parameter family $\{{}^{(t)}\nabla \mid t \in \mathbb{R}\}$ of bi-invariant connections ${}^{(t)}\nabla := \nabla^{\mu(t)}$ by $\mu(t) = {}^{(t)}\mu$, where

$$
{}^{(t)}\mu(X,Y) := \frac{1}{2}(1+t)[X,Y], \quad X,Y \in \mathfrak{g}. \tag{3.2}
$$

There are three particular connections in the family $\{{}^{(t)}\nabla \mid t \in \mathbb{R}\}$:

(1) The *canonical connection*: ${}^{(-1)}\nabla$ defined by setting $t = -1$.
(2) The *anti-canonical connection*: ${}^{(1)}\nabla$ defined by setting $t = 1$.
(3) The *neutral connection*: ${}^{(0)}\nabla$ defined by setting $t = 0$.

Remark 3.2.

(1) The canonical connection and the anti-canonical connection have been discussed in [45, 1, 43]. In [45], the connections ${}^{(-1)}\nabla$, ${}^{(1)}\nabla$ and ${}^{(0)}\nabla$ have been called *Cartan-Schouten's* $(-)$-*connection*, $(+)$-*connection* and (0)-*connection*, respectively.
(2) The set of all *bi-invariant* connections on G is parametrized by

$$
m_{\mathfrak{g}}^{bi} = \{\mu \in m_{\mathfrak{g}} \mid \mu(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = \mathrm{Ad}(g)\mu(X,Y), \quad \text{for any } X,Y \in \mathfrak{g},\ g \in G\}.
$$

Laquer [47] proved that for any compact simple Lie group G, the set $m_{\mathfrak{g}}^{bi}$ is 1-dimensional except for the case $G = \mathrm{SU}_n$ with $n \geq 3$. More precisely, for any compact Lie group G, except for $G = \mathrm{SU}_n$ $(n \geq 3)$, the set $m_{\mathfrak{g}}^{bi}$ is given by

$$
m_{\mathfrak{g}}^{bi} = \{{}^{(t)}\mu \mid t \in \mathbb{R}\},
$$

where ${}^{(t)}\mu$ is defined in (3.2). Thus the corresponding set of bi-invariant connections is given by $a_G^{bi} = \{{}^{(t)}\nabla \mid t \in \mathbb{R}\}$. In case $G = \mathrm{SU}_n$ with $n \geq 3$, $m_{\mathfrak{su}_n}^{bi}$ is 2-dimensional. The set $m_{\mathfrak{su}_n}^{bi}$ is parametrized by

$$
m_{\mathfrak{su}_n}^{bi} = \{{}^{(t,s)}\mu \mid t,s \in \mathbb{R}\}
$$

with ${}^{(t,s)}\mu(X,Y) = \frac{1}{2}(1+t)[X,Y] + \sqrt{-1}\, s\left((XY+YX) - \frac{2}{n}\mathrm{tr}\,(XY)\,\mathrm{id}\right)$.

We note the following elementary, but important Lemma.

Lemma 3.3.

(1) *The bi-invariant connection* ${}^{(t)}\nabla$ *is torsion-free if and only if it is the neutral connection* ${}^{(0)}\nabla$.
(2) *Assume that* μ *is skew-symmetric, then the torsion free connection* ${}^{\dagger}\nabla^{\mu}$ *associated to* ∇^{μ} *coincides with* ${}^{(0)}\nabla$.
(3) *Let* M *be a Riemann surface,* G *a Lie group and* ∇^{μ} *a connection on* G *with* μ *skew-symmetric. Then a map* $\varphi : M \to G$ *is* ∇^{μ}*-harmonic if and only if it is* ${}^{(0)}\nabla$*-harmonic.*

Proof. (1): It is straightforward to verify that using (3.1) the torsion of ${}^{(t)}\nabla$ is given by $T^{{}^{(t)}\mu}(X,Y) = t[X,Y]$. Thus the claim immediately follows.

(2): Assume that μ is skew-symmetric, then $T^\mu(X,Y) = -[X,Y] + 2\mu(X,Y)$ and the associated connection ${}^\dagger\nabla^\mu$ is computed as

$$ {}^\dagger\nabla^\mu_X Y = \nabla^\mu_X Y - \frac{1}{2}T^\mu(X,Y) = \mu(X,Y) - \frac{1}{2}\left(-[X,Y] + 2\mu(X,Y)\right) = \frac{1}{2}[X,Y] $$

for all X, $Y \in \mathfrak{g}$. Thus ${}^\dagger\nabla^\mu = {}^{(0)}\nabla$.

(3): Follows immediately from Proposition 2.1. □

3.2. **Affine harmonic maps.** Now let $\varphi : M \to G$ be a smooth map of a Riemann surface M into a Lie group G. Then the pull-back 1-form $\alpha := \varphi^*\theta$ of the left Maurer-Cartan form θ by φ satisfies the *Maurer-Cartan equation*:

$$ d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0. $$

With respect to the conformal structure of M, we decompose α as $\alpha = \alpha' + \alpha''$. Then the Maurer-Cartan equation is rephrased as

$$ \bar\partial\alpha' + \partial\alpha'' + [\alpha' \wedge \alpha''] = 0. \tag{3.3} $$

Remark 3.4. When G is a linear Lie group, then $\alpha = \varphi^*\theta$ has the form $\alpha = \varphi^{-1}d\varphi$. Even if G is not a linear Lie group (*e.g.*, the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of $\mathrm{SL}_2\mathbb{R}$), we can represent $\varphi^*\theta$ as $\varphi^{-1}d\varphi$ by using the tangent group structure of G, [44].

From now on we equip G with a left-invariant connection ∇^μ and take a Hermitian metric g in the conformal class of M. Then the second fundamental form $\nabla^\mu d\varphi$ with respect to ∇^μ is related to the second fundamental form ${}^{(-1)}\nabla d\varphi$ with respect to the canonical connection by

$$ \theta((\nabla^\mu_X d\varphi)Y) = \theta(({}^{(-1)}\nabla_X d\varphi)Y) + \mu(\alpha(X), \alpha(Y)), $$

where $\alpha = \varphi^*\theta$. Hence the tension field $\tau(\varphi, \nabla^\mu)$ with respect to ∇^μ is given by

$$ \theta(\tau(\varphi, \nabla^\mu)) = \theta(\tau(\varphi, {}^{(-1)}\nabla)) + \mathrm{tr}_g\, \mu(\alpha, \alpha). $$

Proposition 3.5. *Let $\varphi : M \to (G, \nabla^\mu)$ be a smooth map of a Riemann surface into a Lie group and $\alpha = \varphi^*\theta = \alpha' + \alpha''$ the Maurer-Cartan form. Then φ is ∇^μ-harmonic if and only if α satisfies*

$$ \bar\partial\alpha' - \partial\alpha'' + 2(\mathrm{sym}\,\mu)(\alpha'' \wedge \alpha') = 0. \tag{3.4} $$

Proof. Take a Hermitian metric $g = e^u dz d\bar z$ with a conformal coordinate z in the conformal class of M. With respect to this metric g, since $*dz = -\sqrt{-1}dz$, $*dz = \sqrt{-1}d\bar z$ and $*(dz \wedge d\bar z) = -2\sqrt{-1}e^{-u}$, we obtain

$$ \theta(\tau_g(\varphi, {}^{(-1)}\nabla)) = *d * \alpha = -\sqrt{-1} * (\bar\partial\alpha' - \partial\alpha''). $$

On the other hand,

$$ \mathrm{tr}_g\, \mu(\alpha, \alpha) = -2\sqrt{-1} * (\mathrm{sym}\,\mu)(\alpha'' \wedge \alpha'). $$

Hence we obtain

$$ \theta(\tau_g(\varphi, \nabla^\mu)) = -\sqrt{-1} * \left\{\bar\partial\alpha' - \partial\alpha'' + 2(\mathrm{sym}\,\mu)(\alpha'' \wedge \alpha')\right\}. $$

Thus the claim follows.

□

Corollary 3.6. *Let $\varphi : M \to (G, \nabla^\mu)$ be a smooth map of a Riemann surface into a Lie group and $\alpha = \varphi^*\theta$ the Maurer-Cartan form of φ. Then the following properties are mutually equivalent:*

(1) *The map φ is ∇^μ-harmonic.*
(2) *The Maurer-Cartan form α satisfies $d * \alpha + (\operatorname{sym}\mu)(*\alpha \wedge \alpha) = 0$, or equivalently, for $\alpha = \alpha' + \alpha''$,*

$$\bar{\partial}\alpha' - \partial\alpha'' + 2(\operatorname{sym}\mu)(\alpha'' \wedge \alpha') = 0. \tag{3.5}$$

(3) *The Maurer-Cartan form α satisfies $d\alpha + d * \alpha + \frac{1}{2}[\alpha \wedge \alpha] + (\operatorname{sym}\mu)(*\alpha \wedge \alpha) = 0$, or equivalently, for $\alpha = \alpha' + \alpha''$,*

$$2\bar{\partial}\alpha' + [\alpha' \wedge \alpha''] + 2(\operatorname{sym}\mu)(\alpha'' \wedge \alpha') = 0. \tag{3.6}$$

Proof. The equivalence of (1) and (2) has been already shown. To show the equivalence of (2) and (3), we add the Maurer-Cartan equation (3.3) and the harmonicity equation (3.5). Then the equation (3.6) follows. Conversely, we subtract the Maurer-Cartan equation (3.3) from the equation (3.6), then the harmonicity equation (3.5) follows. □

Then the following fundamental theorem is obtained.

Theorem 3.7. *Let ∇^μ be a left-invariant affine connection on a Lie group G determined by a bilinear map μ and $\varphi : M \to G$ a ∇^μ-harmonic map from a Riemann surface M into G. Then $\alpha = \varphi^*\theta = \alpha' + \alpha''$ satisfies* (3.6)*. Conversely, let $\mathbb{D} \subset \mathbb{C}$ be a simply-connected domain and $\alpha = \alpha' + \alpha''$ a $\mathfrak{g}$-valued 1-form satisfying* (3.6)*. Then there exist a ∇^μ-harmonic map $\varphi : \mathbb{D} \to G$ such that $\varphi^*\theta = \alpha$.*

Proof. We only need to prove the converse statement. Let $\alpha = \alpha' + \alpha''$ be a $\mathfrak{g}$-valued 1-form on $\mathbb{D}$ satisfying (3.6). Then subtraction and addition of the complex conjugate of (3.6) to itself gives the integrability condition (3.3) and the harmonicity condition (3.5), respectively. □

3.3. **Torsion-free affine harmonic maps.** In this subsection, we compute the *torsion-free* ∇^μ-harmonicity condition for a smooth map $\varphi : M \to (G, \nabla^\mu)$ in terms of the Maurer-Cartan form $\alpha = \varphi^*\theta$. Moreover, we relate ∇^μ-harmonicity to torsion-free ∇^μ-harmonicity.

By Proposition 3.1, the torsion T^μ along φ is given by

$$\varphi^* T^\mu(\alpha' \wedge \alpha'') = -[\alpha' \wedge \alpha''] + 2(\operatorname{skew}\mu)(\alpha' \wedge \alpha''). \tag{3.7}$$

Then we have the following theorem.

Theorem 3.8. *Let $\varphi : M \to (G, \nabla^\mu)$ be a smooth map. Then the following statements are equivalent*:

(1) *φ is a torsion free ∇^μ-harmonic map.*
(2) *The Maurer-Cartan form $\alpha = \varphi^*\theta$ satisfies*

$$(\operatorname{skew}\mu)(\alpha' \wedge \alpha'') - \frac{1}{2}[\alpha' \wedge \alpha''] = 0, \tag{3.8}$$

$$\bar{\partial}\alpha' + \mu(\alpha'' \wedge \alpha') = 0. \tag{3.9}$$

Proof. (1) $\Rightarrow$ (2): Since the torsion T^μ in (3.7) is assumed to vanish, we obtain (3.8). Moreover, since $(\operatorname{skew}\mu)(\alpha'\wedge\alpha'') = (\operatorname{skew}\mu)(\alpha''\wedge\alpha')$ and the ∇^μ-harmonicity is characterized by the equation (3.6), we conclude the equation (3.9), proving (2).

(2) $\Rightarrow$ (1): Starting from (3.9) we can rephrase it in the form

$$\bar\partial\alpha' - (\operatorname{sym}\mu)(\alpha'\wedge\alpha'') + (\operatorname{skew}\mu)(\alpha'\wedge\alpha'') = 0. \tag{3.10}$$

Comparing this equation to (3.8), we obtain the ∇^μ-harmonicity equation (3.6). Moreover the equation (3.8) is just the torsion-free condition, whence (1).

□

Remark 3.9. Theorem 3.8 can be obtained by computing the Levi-form of the map φ. From [12], the canonical connection and the left Maurer-Cartan form satisfy the following relation:

$$\theta({}^{(-1)}\nabla_X Y) = X\theta(Y) - [\theta(X), \theta(Y)].$$

Since $\nabla^\mu = {}^{(-1)}\nabla + \mu$, we have $\theta(\nabla^\mu_X Y) = X\theta(Y) - [\theta(X), \theta(Y)] + \mu(\theta(X), \theta(Y))$. Setting $\varphi^*\theta = \alpha$, we have

$$\theta\left(\nabla^{\mu''}\partial\varphi\right) = -\sqrt{-1} * \left\{\bar\partial\alpha' - [\alpha''\wedge\alpha'] + \mu(\alpha''\wedge\alpha')\right\}.$$

Thus $\nabla^{\mu''}\partial\varphi = 0$ if and only if

$$\bar\partial\alpha' - [\alpha''\wedge\alpha'] + \mu(\alpha''\wedge\alpha') = 0. \tag{3.11}$$

Adding and subtracting the complex conjugate of (3.11) to itself, we observe that (3.11) is equivalent to the two equations

$$2(\operatorname{skew}\mu)(\alpha'\wedge\alpha'') - [\alpha'\wedge\alpha''] = 0,$$
$$\bar\partial\alpha' - \partial\alpha'' + 2(\operatorname{sym}\mu)(\alpha''\wedge\alpha') = 0,$$

which are equivalent with the conditions (3.8) and (3.9). We note that we have used here $\mu = \operatorname{sym}\mu + \operatorname{skew}\mu$, $(\operatorname{sym}\mu)(\alpha''\wedge\alpha') = -(\operatorname{sym}\mu)(\alpha'\wedge\alpha'')$, $(\operatorname{skew}\mu)(\alpha''\wedge\alpha') = (\operatorname{skew}\mu)(\alpha'\wedge\alpha'')$ and $[\alpha''\wedge\alpha'] = [\alpha'\wedge\alpha'']$.

Among the three connections ${}^{(-1)}\nabla, {}^{(1)}\nabla$ and ${}^{(0)}\nabla$ in Definition 3.1, the ${}^{(0)}\nabla$ connection automatically satisfies the conditions of the theorem above. Thus we have the following.

Corollary 3.10. *A smooth map $\varphi : M \to (G, {}^{(0)}\nabla)$ is torsion-free ${}^{(0)}\nabla$-harmonic if and only if it is ${}^{(0)}\nabla$-harmonic.*

3.4. **Zero-curvature representations for admissible affine harmonic maps.** In this subsection, we introduce *admissible* ∇^μ-harmonic maps and show that admissible ∇^μ-harmonic maps admit a zero-curvature representation. As a corollary we obtain immediately the important fact that all ∇^μ-harmonic maps admit a zero-curvature representation if μ is skew-symmetric. This applies in particular to the connections ${}^{(t)}\nabla$. We first introduce the following definition.

Definition 3.2. A ∇^μ-harmonic map φ from a Riemann surface M into a Lie group G with the Maurer-Cartan form $\alpha = \varphi^*\theta = \alpha' + \alpha''$ satisfying the condition

$$(\operatorname{sym}\mu)(*\alpha\wedge\alpha) = 0, \quad \text{or equivalently,} \quad (\operatorname{sym}\mu)(\alpha'\wedge\alpha'') = 0, \tag{3.12}$$

is called an *admissible ∇^μ-harmonic map.*

Remark 3.11. If μ is skew-symmetric, then (3.12) is automatically satisfied. This applies, in particular, to the connections ${}^{(t)}\nabla$, $(t \in \mathbb{R})$. Thus a smooth map $\varphi : M \to (G, {}^{(t)}\nabla)$ is admissible ${}^{(t)}\nabla$-harmonic if and only if it is ${}^{(t)}\nabla$-harmonic.

We now characterize an admissible ∇^μ-harmonic map into a Lie group in terms of a family of flat connections.

Proposition 3.12. *Let $\varphi : M \to (G, \nabla^\mu)$ be an admissible ∇^μ-harmonic map into a Lie group G equipped with a left-invariant connection ∇^μ and $\alpha = \varphi^*\theta = \alpha' + \alpha''$ the Maurer-Cartan form of φ. Then the loop of connections $d + \alpha_\lambda$ defined by*

$$\alpha_\lambda := \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha'' \tag{3.13}$$

is flat for all $\lambda \in \mathbb{S}^1$.

Conversely assume that $\mathbb{D} \subset \mathbb{C}$ is simply-connected and let $\alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha''$ be an $\mathbb{S}^1$-family of $\mathfrak{g}$-valued 1-forms satisfying the condition (3.12) and the zero-curvature condition:

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in \mathbb{S}^1$. Then there exists a 1-parameter family of maps $F_\lambda : \mathbb{D} \times \mathbb{S}^1 \to G$ such that $F_\lambda^*\theta = \alpha_\lambda$. The map F_λ is called the extended solution associated with α and it is admissible ∇^μ-harmonic for $\lambda = \pm 1$.

Proof. Let $\varphi : \mathbb{D} \to G$ be a smooth map and define the loop $\{d + \alpha_\lambda\}$ of connections by (3.13). Then a direct computation shows that $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda]$ is equal to

$$\bar\partial\alpha' + \partial\alpha'' + [\alpha' \wedge \alpha''] - \frac{\lambda^{-1}}{2}\left(2\bar\partial\alpha' + [\alpha' \wedge \alpha'']\right) - \frac{\lambda}{2}\left(2\partial\alpha'' + [\alpha'' \wedge \alpha']\right).$$

Assume that φ is a ∇^μ-harmonic map with condition (3.12), then we have the Maurer-Cartan equation (3.3) and the harmonicity equation (3.5). Hence $d + \alpha_\lambda$ is flat for all $\lambda \in \mathbb{S}^1$.

Conversely, let α_λ be an $\mathbb{S}^1$-family of $\mathfrak{g}$-valued 1-forms of the form (3.13) defined on a simply-connected $\mathbb{D}$. Assume that $d + \alpha_\lambda$ is flat for all $\lambda \in \mathbb{S}^1$. Then there exists a 1-parameter family of maps $F_\lambda : \mathbb{D} \times \mathbb{S}^1 \to G$ such that $F_\lambda^*\theta = \alpha_\lambda$. The flatness of all $d + \alpha_\lambda$ implies

$$\bar\partial\alpha' + \partial\alpha'' + [\alpha' \wedge \alpha''] = 0, \quad \bar\partial\alpha' + \frac{1}{2}[\alpha'' \wedge \alpha'] = 0, \quad \partial\alpha'' + \frac{1}{2}[\alpha' \wedge \alpha''] = 0.$$

The last two equations together with the condition (3.12) are nothing but the admissible ∇^μ-harmonic map equation for $F_{\lambda=\pm 1}$. □

Remark 3.13.

(1) It is easy to see, since $\alpha_{\lambda=1} = 0$, that the map $F_{\lambda=1}$ is a constant map.
(2) Take a smooth curve $\gamma : \mathbb{S}^1 \to G$ satisfying $\gamma(1) = \mathrm{id}$. Then there exists a unique map $F_\lambda : \mathbb{D} \times \mathbb{S}^1 \to G$ satisfying $F_\lambda^*\theta = \alpha_\lambda$ and the initial condition $F_\lambda(z_0) = \gamma(\lambda)$.
(3) Clearly, when assuming $F_{\lambda=1} = \mathrm{id}$ we obtain a map from $\mathbb{D}$ into the *based loop group*

$$\Omega G = \{\gamma : \mathbb{S}^1 \to G \mid \gamma(1) = \mathrm{id}\}.$$

Note that extended solutions are holomorphic curves into ΩG with respect to the canonical complex structure of ΩG.

Since ∇^μ-harmonic maps with skew-symmetric μ automatically satisfy the admissibility condition (3.12), by Remark 3.11 and by (3) of Lemma 3.3, ∇^μ-harmonicity is equivalent to ${}^{(0)}\nabla$-harmonicity. We thus have the following theorem as a corollary to Proposition 3.12.

Theorem 3.14. *Let M be a Riemann surface, G a Lie group, ∇^μ a left-invariant connection with skew-symmetric μ and ${}^{(0)}\nabla$ the bi-invariant connection stated Definition 3.1. Then a map $\varphi : M \to G$ is a ∇^μ-harmonic map if and only if it is a ${}^{(0)}\nabla$-harmonic map. Moreover let $\alpha = \varphi^*\theta = \alpha' + \alpha''$ be the Maurer-Cartan form of φ. Then the loop of connections $d + \alpha_\lambda$ defined by*

$$\alpha_\lambda := \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha'' \tag{3.14}$$

is flat for all $\lambda \in \mathbb{S}^1$.

Conversely, assume that $\mathbb{D}$ is simply-connected and let $\alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha''$ be an $\mathbb{S}^1$-family of $\mathfrak{g}$-valued 1-forms satisfying the zero-curvature condition:

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in \mathbb{S}^1$. Then there exists a 1-parameter family of maps $F_\lambda : \mathbb{D} \times \mathbb{S}^1 \to G$ such that $F_\lambda^*\theta = \alpha_\lambda$. The map F_λ is called an extended solution and it is ∇^μ-harmonic for any skew-symmetric μ, thus it is also ${}^{(0)}\nabla$-harmonic, for $\lambda = \pm 1$.

3.5. **Zero-curvature representations for torsion-free affine harmonic maps.** In this subsection, we show that the Maurer-Cartan form of an admissible ∇^μ-harmonic map satisfying $[\alpha' \wedge \alpha''] = 0$ admits a very strong condition which trivially implies the zero-curvature representation. As a corollary, we show that a torsion-free ${}^{(t)}\nabla$-harmonic ($t \neq 0$) map admits such a very special zero-curvature representation and give an explicit formula for such a map.

Proposition 3.15. *Let $\varphi : M \to (G, \nabla^\mu)$ be an admissible ∇^μ-harmonic map into a Lie group with a left-invariant connection ∇^μ and $\alpha = \varphi^*\theta = \alpha' + \alpha''$ the Maurer-Cartan form of φ. Further assume that*

$$[\alpha' \wedge \alpha''] = 0. \tag{3.15}$$

Then

$$\bar{\partial}\alpha' = \partial\alpha'' = [\alpha' \wedge \alpha''] = 0$$

holds. Moreover, the double loop of connections $d + \alpha^{(\lambda,\nu)}$ defined by

$$\alpha^{(\lambda,\nu)} := \lambda\alpha' + \nu\alpha'' \tag{3.16}$$

is flat for all $\lambda, \nu \in \mathbb{C}^*$. In this case we will say that the connection satisfies the double loop zero-curvature condition.

Conversely assume that $\mathbb{D}$ is simply-connected and let $\alpha^{(\lambda,\nu)} = \lambda\alpha' + \nu\alpha'', \lambda, \nu \in \mathbb{C}^*$, be a double loop family of $\mathfrak{g}$-valued 1-forms satisfying the admissibility condition (3.12) *and the double loop zero-curvature condition*:

$$d\alpha^{(\lambda,\nu)} + \frac{1}{2}[\alpha^{(\lambda,\nu)} \wedge \alpha^{(\lambda,\nu)}] = 0$$

for all $\lambda, \nu \in \mathbb{C}^*$. Then, for all $\lambda \in \mathbb{C}^*$, there exists a family of maps $\varphi_\lambda : \mathbb{D} \times \mathbb{C}^* \to G$ such that $\varphi_\lambda^*\theta = \alpha^{(\lambda,\bar{\lambda})}$. The maps φ_λ are admissible ∇^μ-harmonic maps satisfying the condition (3.15).

Proof. By Proposition 3.5, admissible ∇^μ-harmonicity satisfying condition (3.15) gives

$$\bar\partial\alpha' = \partial\alpha'' = [\alpha' \wedge \alpha''] = 0.$$

Moreover, it is easy now to check that the flatness of $d + \alpha^{(\lambda,\nu)}$ is exactly equivalent to the above three equations. Conversely, it is also easy to check that the flatness of $d+\alpha^{(\lambda,\nu)}$ under the conditions (3.12) gives admissible ∇^μ-harmonicity satisfying the condition (3.15). □

Moreover we can construct all admissible ∇^μ-harmonic maps satisfying (3.15) as follows.

Theorem 3.16. *Let G be a real Lie group with left-invariant connection ∇^μ. An admissible ∇^μ-harmonic map $\varphi : M \to G$ satisfying* (3.15) *is, up to left translation, a map of the form*

$$\varphi(z, \bar z) = \exp\left(\int_{z_*}^{z} \Phi(t)dt + \int_{z_*}^{\bar z} \bar\Phi(t)dt\right), \tag{3.17}$$

where z is a conformal coordinate on M, Φ is holomorphic and takes values in $\mathfrak{g}$. Moreover, $\bar\Phi$, the complex conjugate of Φ, satisfies

$$[\Phi, \bar\Phi] = 0 \ \text{ and } \ (\operatorname{sym}\mu)(\Phi, \bar\Phi) = 0. \tag{3.18}$$

Proof. We have seen in the proof of Proposition 3.15 that admissible ∇^μ-harmonic maps $\varphi : M \to G$ satisfying (3.15) are characterized by the equations

$$(\operatorname{sym}\mu)(\alpha' \wedge \alpha'') = [\alpha' \wedge \alpha''] = \bar\partial\alpha' = \partial\alpha'' = 0. \tag{3.19}$$

Set $\alpha' = \Phi dz$ and $\alpha'' = \bar\Phi d\bar z$, where Φ takes values in $\mathfrak{g}$ and $\bar\Phi$ denotes the complex conjugate. Then the conditions (3.19) are equivalent with that $(\operatorname{sym}\mu)(\Phi, \bar\Phi) = 0$, $[\Phi, \bar\Phi] = 0$ and Φ is holomorphic. Moreover, since $[\Phi, \bar\Phi] = 0$, we have $[\int_{z_*}^{z} \Phi(t)dt, \int_{z_*}^{\bar z} \bar\Phi(t)dt] = 0$, and the map φ is given by (3.17), up to left translation. □

Remark 3.17. This type of solutions has been investigated by [39] for complex projective spaces.

It should be recalled that by Proposition 3.1 the torsion of a map $\varphi : M \to (G, {}^{(t)}\nabla)$ can be computed as $\varphi^* T^{{}^{(t)}\mu}(\alpha' \wedge \alpha'') = t[\alpha' \wedge \alpha'']$. Thus a map $\varphi : M \to (G, {}^{(t)}\nabla), t \neq 0$, satisfies the torsion free condition if and only if

$$[\alpha' \wedge \alpha''] = 0.$$

Moreover the bilinear map ${}^{(t)}\mu$ in (3.1) is skew-symmetric, that is, the admissibility condition is automatically satisfied, and thus φ admits the zero-curvature representation. We summarize the above discussion as follows:

Corollary 3.18. *Let $\varphi : M \to (G, {}^{(t)}\nabla)$ be a torsion free ${}^{(t)}\nabla$-harmonic map $(t \neq 0)$ into a Lie group with a left-invariant connection ${}^{(t)}\nabla, t \neq 0$, and $\alpha = \varphi^*\theta = \alpha' + \alpha''$ the Maurer-Cartan form of φ. Then the double loop of connections $d + \alpha^{(\lambda,\nu)}$ defined by*

$$\alpha^{(\lambda,\nu)} := \lambda\alpha' + \nu\alpha'' \tag{3.20}$$

is flat for all $\lambda, \nu \in \mathbb{C}^*$.

Conversely assume that $\mathbb{D}$ is simply-connected and let

$$\alpha^{(\lambda,\nu)} := \lambda\alpha' + \nu\alpha'', \tag{3.21}$$

with $\lambda, \nu \in \mathbb{C}^*$, *be a double loop family satisfying the double loop zero-curvature condition*

$$d\alpha^{(\lambda,\nu)} + \frac{1}{2}[\alpha^{(\lambda,\nu)} \wedge \alpha^{(\lambda,\nu)}] = 0$$

for all $\lambda, \nu \in \mathbb{C}^*$. *Then there exists a family of maps* $\varphi_\lambda : \mathbb{D}\times\mathbb{C}^* \to G$ *such that* $\varphi_\lambda^*\theta = \alpha^{(\lambda,\bar{\lambda})}$.

The maps φ_λ *are torsion-free* ${}^{(t)}\nabla$*-harmonic maps for any* $t \neq 0$ *and* $\lambda \in \mathbb{C}^*$. *Moreover these torsion-free* ${}^{(t)}\nabla$*-harmonic maps* φ_λ *are, up to left translation, given by*

$$\varphi_\lambda(z, \bar{z}) = \exp\left(\lambda \int_{z_*}^{z} \varPhi(t)dt + \bar{\lambda} \int_{z_*}^{\bar{z}} \bar{\varPhi}(t)dt\right),$$

where $z \in \mathbb{D}$ *is a conformal coordinate and* $\varPhi$ *is holomorphic, and its conjugate* $\bar{\varPhi}$ *satisfies* $[\varPhi, \bar{\varPhi}] = 0$.

3.6. **Admissible harmonic maps into Lie groups with left-invariant metrics.** Clearly, a particularly interesting type of connections consists of the Levi-Civita connections of left-invariant metrics on semi-simple Lie groups. Therefore, and more generally, in this section we consider arbitrary Lie groups together with a left-invariant semi-Riemannian metric. We denote a non-degenerate symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of the real Lie group G by $\langle\cdot,\cdot\rangle$ and extend it to a left-invariant semi-Riemannian metric $ds^2 = \langle\cdot,\cdot\rangle$ on G. Next we introduce an *anti-commutator product* $\{\cdot,\cdot\}$ on $\mathfrak{g}$ by

$$\{X, Y\} := \nabla_X Y + \nabla_Y X, \quad X, Y \in \mathfrak{g}. \tag{3.22}$$

Clearly $\{X, Y\} = \{Y, X\}$. It is easy to see that ds^2 is right invariant if and only if $\{\cdot,\cdot\}$ vanishes. The Levi-Civita connection ∇ of (G, ds^2) is given by [14, Proposition 3.18],

$$\nabla_X Y = \frac{1}{2}[X, Y] + \frac{1}{2}\{X, Y\}, \quad X, Y \in \mathfrak{g}. \tag{3.23}$$

Hence ∇ is the left-invariant connection $\nabla = \nabla^\mu$ associated with the bilinear map μ given by

$$\mu(X, Y) = \frac{1}{2}[X, Y] + \frac{1}{2}\{X, Y\}.$$

This implies immediately the following Lemma.

Lemma 3.19. *The Levi-Civita connection* ∇ *of some left-invariant metric* ds^2 *on* G *is bi-invariant if and only if* $\nabla = {}^{(0)}\nabla$.

Thus the Levi-Civita connection ∇ is a left-invariant connection determined by the bilinear map μ such that

$$(\operatorname{skew}\mu)(X, Y) = \frac{1}{2}[X, Y], \quad (\operatorname{sym}\mu)(X, Y) = \frac{1}{2}\{X, Y\}. \tag{3.24}$$

Let us consider a smooth map $\varphi : M \to (G, ds^2)$ from a Riemann surface M into a Lie group G with a left-invariant metric ds^2, and let ∇ denote the Levi-Civita connection of ds^2.

From Corollary 3.6, the ∇-harmonicity is characterized by (3.6). Moreover since the symmetric part of μ is given in (3.24), we thus have

$$2\bar{\partial}\alpha' + \{\alpha'' \wedge \alpha'\} + [\alpha'' \wedge \alpha'] = 0. \tag{3.25}$$

In the study of harmonic maps of Riemann surfaces into compact semi-simple Lie groups equipped with a bi-invariant Riemannian metric, the zero curvature representation is the starting point of the loop group approach by Uhlenbeck [62] and Segal [58]. We recall that $\{\cdot,\cdot\}$ is the symmetric part of μ by (3.24), thus the admissibility condition (3.12) is satisfied if and only if

$$\{\alpha' \wedge \alpha''\} = 0. \tag{3.26}$$

Therefore, from Proposition 3.12 we obtain the following.

Proposition 3.20. *Let $\varphi : \mathbb{D} \to (G, ds^2)$ be an admissible harmonic map with respect to the Levi-Civita connection and $\alpha = \varphi^*\theta = \alpha' + \alpha''$ the Maurer-Cartan form. Then the loop of connections $d + \alpha_\lambda$ defined by*

$$\alpha_\lambda := \frac{1}{2}(1-\lambda^{-1})\alpha' + \frac{1}{2}(1-\lambda)\alpha'' \tag{3.27}$$

is flat for all $\lambda \in \mathbb{S}^1$.

Conversely assume that $\mathbb{D}$ is simply-connected. Let $\alpha_\lambda = \frac{1}{2}(1-\lambda^{-1})\alpha' + \frac{1}{2}(1-\lambda)\alpha''$ be an $\mathbb{S}^1$-family of $\mathfrak{g}$-valued 1-forms satisfying (3.26) *and the zero-curvature condition*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in \mathbb{S}^1$. Then there exists a 1-parameter family of maps $F_\lambda : \mathbb{D} \times \mathbb{S}^1 \to G$ such that $F_\lambda^*\theta = \alpha_\lambda$. The map F_λ is called the extended solution associated with α_λ and it is admissible harmonic with respect to the Levi-Civita connection for $\lambda = \pm 1$.

Remark 3.21. The extended solution F_λ is not harmonic for $\lambda \neq \pm 1$, but is a solution to the *harmonic map equation with Wess-Zumino term* [34, 43, 61]. In case, the metric is bi-invariant, then $\varphi_\lambda := F_{-\lambda}F_\lambda^{-1}$ is harmonic for all $\lambda \in \mathbb{S}^1$, see [20].

3.7. **The admissibility condition.** In the previous subsections we have characterized admissible affine harmonic maps by flat connections. Thus the admissibility condition (3.12) is important for the loop group method. So the following two questions naturally arise:

(1) Does there exist a *non* skew-symmetric μ such that *every* ∇^μ-harmonic map satisfies the admissibility condition (3.12)?
(2) Does there exist any non-skew-symmetric μ such that the connection ∇^μ admits an admissible ∇^μ-harmonic map?

The answer to the first question is as follows:

Proposition 3.22. *Let G be a Lie group and ∇^μ a left-invariant connection determined by a bilinear map μ. Assume that every ∇^μ-harmonic map into (G, ∇^μ) satisfies the admissibility condition* (3.12)*. Then μ is skew-symmetric.*

Proof. Let $z = x + \sqrt{-1}y$ be a conformal coordinate on $\mathbb{D} \subset \mathbb{C}$, and set $\alpha' = A(x)dz$ and $\alpha'' = A(x)d\bar{z}$, where $A(x)$ is an arbitrary function of x which takes values in $\mathfrak{g}$. Then it is clear that the Maurer-Cartan equation (3.3) is automatically satisfied. Moreover, the ∇^μ-harmonicity equation (3.4) is equivalent with the ordinary differential equation for $A(x)$:

$$\frac{d}{dx}A(x) + 2\mu(A(x), A(x)) = 0. \tag{3.28}$$

Consider $A(x)$ with (arbitrarily chosen) initial condition $A_0 = A(0) \in \mathfrak{g}$. Therefore there exists a ∇^μ-harmonic map satisfying $\alpha = \varphi^*\theta = A(x)dz + A(x)d\bar{z}$.

On the other hand, by the admissibility condition (3.12) we have $\mu(A(x), A(x)) = 0$. In particular $\mu(A(0), A(0)) = 0$. Since the initial condition $A_0 = A(0)$ for (3.28) was chosen arbitrary, we conclude that

$$\mu(A_0, A_0) = 0$$

for any $A_0 \in \mathfrak{g}$. Thus μ is skew-symmetric. $\square$

To answer the second question, we introduce the notion of a *primitive map* into k-symmetric Lie groups ($k > 2$). Let G be a connected Lie group with a Lie group automorphism τ of order $k > 2$ such that the unit element is the only element of G fixed by τ. The pair (G, τ) is referred to as a *k-symmetric Lie group*. A k-symmetric Lie group $(G, {}^{(-1)}\nabla)$ equipped with the canonical connection is said to be *affine k-symmetric* if τ is an affine transformation with respect to the canonical connection. Note that affine k-symmetric Lie groups are affine k-symmetric spaces in the sense of Kowalski [46]. Note that affine k-symmetric Lie groups are solvable [46, Proposition V.9].

If we equip G with a left invariant semi-Riemannian metric ds^2 such that τ is an isometry with respect to this metric, then the resulting semi-Riemannian homogeneous space $(G/\{\mathrm{Id}\}, \tau, ds^2)$ is a semi-Riemannian k-symmetric space in the sense of [46].

The eigenvalues of $d\tau$ on $\mathfrak{g}$ are contained in the set $\{\omega^j \mid j \in \mathbb{Z}/k\mathbb{Z}\}$, where $\omega = e^{2\pi i/k}$ is the primitive k-th root of unity. We have an eigenspace decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}/\mathbb{Z}_k} \mathfrak{g}_j^{\mathbb{C}},$$

where $\mathfrak{g}_j^{\mathbb{C}}$ is the eigenspace of $d\tau$ corresponding to the eigenvalue ω^j. Clearly, $\mathfrak{g}_0^{\mathbb{C}} = \{\mathbf{0}\}$. Moreover we have

$$[\mathfrak{g}_j^{\mathbb{C}},\ \mathfrak{g}_\ell^{\mathbb{C}}] = \mathfrak{g}_{j+\ell}^{\mathbb{C}} \quad \mod k.$$

Note that $\mathfrak{g}_1^{\mathbb{C}} \cap \mathfrak{g}_{-1}^{\mathbb{C}} = \{\mathbf{0}\}$ since $k > 2$.

Definition 3.3. Let $\varphi : M \to (G, \tau)$ be a smooth map from a Riemann surface into a k-symmetric Lie group. Then φ is said to be a *primitive map* if

$$\alpha'(T^{(1,0)}M) \subset \mathfrak{g}_{-1}^{\mathbb{C}}.$$

Here α' is the $(1,0)$-part of $\alpha = \varphi^*\theta$.

Note that the notion of primitivity does not depend on any choice of left-invariant affine connection. If we express the Maurer-Cartan form $\alpha = \varphi^{-1}d\varphi$ as $\alpha = \alpha' + \alpha''$ with $\alpha' = \Phi dz$, then φ is primitive if and only if $\Phi \in \mathfrak{g}_{-1}^{\mathbb{C}}$.

We obtain the following.

Proposition 3.23. *Let G be a k-symmetric Lie group with automorphism τ of order $k > 2$. If a smooth map $\varphi : M \to G$ of a Riemann surface M is primitive with respect to τ then φ has the form*

$$\varphi(z, \bar{z}) = a \exp\left(\int_{z_*}^{z} \Phi(t)dt + \int_{z_*}^{\bar{z}} \bar{\Phi}(t)dt\right), \tag{3.29}$$

where $a \in G$, and Φ is holomorphic, takes values in $\mathfrak{g}^{\mathbb{C}}$, and satisfies $[\Phi, \bar{\Phi}] = 0$.

Proof. Let $\alpha = \Phi dz + \bar{\Phi} d\bar{z}$ be the Maurer-Cartan form of the primitive map φ. By definition, $\Phi \in \mathfrak{g}^{\mathbb{C}}_{-1}$, so we have $[\Phi, \bar{\Phi}] \in \mathfrak{g}^{\mathbb{C}}_0 = \{\mathbf{0}\}$. Next α satisfies the Maurer-Cartan equation

$$\bar{\partial}\alpha' + \partial\alpha'' + [\alpha' \wedge \alpha''] = 0.$$

Since φ is a primitive map, we obtain

$$\bar{\partial}\alpha' + \partial\alpha'' = 0.$$

Split this equation according to the eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to τ, we get $\bar{\partial}\alpha' = \partial\alpha'' = 0$. Hence by Theorem 3.16, φ has the form (3.29). □

We now answer the second question posed at the beginning of this subsection as follows:

Proposition 3.24. *Let (G, τ, ds^2) be Lie group equipped with the structure of semi-Riemannian k-symmetric space $(k > 2)$. Assume that the metric ds^2 is only left invariant. Then the Levi-Civita connection of ds^2 is of the form ∇^{μ} with non skew-symmetric μ. The admissibility condition $\{\alpha' \wedge \alpha''\} = 0$ is satisfied for every primitive map from a Riemann surface M into (G, τ, ds^2).*

For example, the space Sol_3, which will be defined in Section 4.3, has the structure of a Riemannian 4-symmetric space [46], see Section 4.4. Thus every primitive map of a Riemann surface into Sol_3 is harmonic with respect to its standard left invariant metric ds^2 and satisfies the admissibility condition.

4. Basic Examples

In the discussions of the previous sections we encountered several interesting basic types of classes of affine harmonic maps into Lie groups G.

(1) The class of ${}^{(0)}\nabla$-harmonic maps into G.
(2) The class of admissible harmonic maps with respect to a (fixed) left-invariant metric.
(3) The class of torsion-free ${}^{(t)}\nabla$-harmonic maps for any $t \in \mathbb{R} \setminus \{0\}$.

All harmonic maps of these classes admit zero-curvature representations.

In the first case, the Lie group G is regarded as an *affine symmetric space* $(G \times G/G, {}^{(0)}\nabla)$. This case contains the Uhlenbeck-Segal theory, when G is compact and semi-simple. In fact, if G is compact and semi-simple, ${}^{(0)}\nabla$ coincides with the Levi-Civita connection of the (actually bi-invariant) Killing metric.

If G is not semi-simple, G may not have any bi-invariant semi-Riemannian metric (like in the case of the Heisenberg group, see below). Even if we equip G with a left-invariant metric, ${}^{(0)}\nabla$-harmonic maps are not necessarily harmonic with respect to the Levi-Civita connection, see Section 4.2. Only in the bi-invariant case these coincide.

In the second case, unfortunately, we only know very simple examples, like the map given by (3.17) and the primitive harmonic maps discussed in Section 3.7.

In the third case, torsion-free ${}^{(t)}\nabla$-harmonic maps ($t \neq 0$), we need to demand the additional condition "torsion-free", that is, $[\alpha' \wedge \alpha''] = 0$, otherwise we do not obtain a zero curvature representation. Torsion-free ${}^{(t)}\nabla$-harmonic maps have the advantage that they are harmonic with respect to any left-invariant metric. However, unfortunately, the torsion-free condition is a very strong restriction on ${}^{(t)}\nabla$-harmonic maps. In fact such maps have been classified in Corollary 3.18.

Therefore the first case is the only candidate for a generalization of the Uhlenbeck-Segal theory. Among all Lie groups, two particularly interesting cases occur, the semi-simple Lie groups and the solvable Lie groups. Since the semi-simple case has been studied intensively already, in this section we exhibit some interesting examples of solvable Lie groups. We will start, however, with a completely general example.

4.1. **An equivariant map.** Let G be any Lie group and X, Y any pair of vectors in $\mathfrak{g}$. We define a map $\varphi : \mathbb{C} \to G$ by

$$\varphi(x, y) = \exp(xX)\exp(yY).$$

Then the Maurer-Cartan form α of φ is given by $\alpha = \mathrm{Ad}(\exp(-yY))X dx + Y dy = \alpha' + \alpha''$, where $\alpha' = \frac{1}{2}(\mathrm{Ad}(\exp(-yY))X - \sqrt{-1}Y)dz$ and $z = x + \sqrt{-1}y$ is a conformal coordinate. From this we get $\partial\alpha'' - \bar{\partial}\alpha' = 0$. Hence, by (3.4) we infer that φ is ${}^{(0)}\nabla$-harmonic. When G admits a bi-invariant semi-Riemannian metric $ds^2 = \langle\cdot, \cdot\rangle$, then φ is harmonic and the induced metric is computed as

$$\langle X, X\rangle dx^2 + 2\langle X, Y\rangle dxdy + \langle Y, Y\rangle dy^2.$$

If in addition we take X, Y so that $\langle X, X\rangle = \langle Y, Y\rangle > 0$ and $\langle X, Y\rangle = 0$, then φ is a conformally harmonic immersion. For $t \neq 0$, φ is torsion free ${}^{(t)}\nabla$-harmonic if and only if $[X, Y] = 0$, see Theorem 3.16.

Remark 4.1. When $[X, Y] = 0$, $\varphi(x, y) = \exp(xX)\exp(yY)$ is called a *vacuum solution*, [11].

4.2. **The Heisenberg group.** We consider the $(2n + 1)$-dimensional Heisenberg group Nil_{2n+1} in $\mathbb{R}^{2n+1}$ with multiplication

$$\begin{aligned}&(x^1, \cdots, x^{2n}, x^{2n+1}) \cdot (\tilde{x}^1, \cdots, \tilde{x}^{2n}, \tilde{x}^{2n+1}) \\ &\qquad = \left(x^1 + \tilde{x}^1, \cdots, x^{2n} + \tilde{x}^{2n}, x^{2n+1} + \tilde{x}^{2n+1} + \frac{1}{2}\sum_{i=1}^{n}(x^i\tilde{x}^{n+i} - \tilde{x}^i x^{n+i})\right).\end{aligned}$$

The natural left-invariant metric is

$$ds^2 = \sum_{i=1}^{2n}(dx^i)^2 + \left\{dx^{2n+1} + \sum_{i=1}^{n}(x^{n+i}dx^i - x^i dx^{n+i})\right\}^2.$$

It is known that Nil_{2n+1} has no bi-invariant metric [50]. For a smooth map $\varphi = (\varphi^1, \ldots, \varphi^{2n+1}) : \mathbb{C} \to \mathrm{Nil}_{2n+1}$, the ${}^{(0)}\nabla$-harmonicity equation is

$$\varphi^j_{z\bar{z}} = 0, \ \ (1 \leqq j \leqq 2n), \ \ \left\{\varphi^{2n+1} + \frac{1}{2}\left(\sum_{i=1}^{n}\varphi^i\varphi^{n+i}\right)\right\}_{z\bar{z}} - \frac{1}{2}\sum_{i=1}^{n}\left(\varphi^i_z\varphi^{n+i}_{\bar{z}} + \varphi^i_{\bar{z}}\varphi^{n+i}_z\right) = 0.$$

This system is equivalent to $\varphi^1_{z\bar{z}} = \varphi^2_{z\bar{z}} = \cdots = \varphi^{2n+1}_{z\bar{z}} = 0$. Thus every ${}^{(0)}\nabla$-harmonic map can be represented in the form

$$\varphi = (\varphi^1, \dots \varphi^{2n+1}) = (2\,\mathrm{Re}\, f^1, \dots, 2\,\mathrm{Re}\, f^{2n+1}),$$

where $f^1, \dots, f^{2n+1}$ are holomorphic functions.

Conversely, any such φ is a ${}^{(0)}\nabla$-harmonic map. For example, we consider $n = 1$ and take $f^1(z) = \frac{1}{2}z$, $f^2(z) = -\frac{\sqrt{-1}}{2}z$ and $f^3(z) = -\frac{\sqrt{-1}}{8}z^2$, where $z = x + \sqrt{-1}y$. Then the resulting map $\varphi = (x, y, \frac{1}{2}xy)$ is a ${}^{(0)}\nabla$-harmonic map and this map gives a hyperbolic paraboloid $x^3 = \frac{1}{2}x^1x^2$ which is a standard example of minimal surfaces in Nil_3 with respect to the canonical left-invariant metric, see [24]. Note that the hyperbolic paraboloid is represented as

$$\varphi(x, y) = \exp\left(x \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \exp\left(y \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

The canonical connection ${}^{(-1)}\nabla$ coincides with the *Tanaka-Webster connection* in CR-geometry. In [16] the following result was obtained.

Proposition 4.2. *Let (M, g) be a Riemannian 2-manifold and $\varphi : M \to (\mathrm{Nil}_3, ds^2)$ an isometric immersion. Then φ is ${}^{(-1)}\nabla$-harmonic if and only if φ is locally congruent to the vertical plane. In particular φ is harmonic with respect to the canonical left-invariant metric.*

For instance, the vertical plane $x_2 = 0$ in Nil_3 is represented as

$$\varphi(z, \bar{z}) = \exp(z\Phi + \bar{z}\bar{\Phi}), \quad \Phi = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\sqrt{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can easily check that $[\Phi, \bar{\Phi}] = 0$. Hence the vertical plane is a torsion-free ${}^{(-1)}\nabla$-harmonic surface.

4.3. **An interesting family of solvable Lie groups.** Let us consider the following 2-parameter family, $(\mu_1, \mu_2) \in \mathbb{R}^2 \setminus (0, 0)$, of 3-dimensional Lie groups:

$$G(\mu_1, \mu_2) = \left\{ (x^1, x^2, x^3) = \begin{pmatrix} e^{\mu_1 x^3} & 0 & x^1 \\ 0 & e^{\mu_2 x^3} & x^2 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_3\mathbb{R}. \tag{4.1}$$

The Lie group $G(\mu_1, \mu_2)$ is solvable and non-unimodular if $\mu_1 + \mu_2 \neq 0$. Note that $G(\mu_1, \mu_2)$ is originally defined as $\mathbb{R}^3(x^1, x^2, x^3)$ with multiplication law:

$$(x^1, x^2, x^3) \cdot (\hat{x}^1, \hat{x}^2, \hat{x}^3) = (x^1 + e^{\mu_1 x^3}\hat{x}^1, x^2 + e^{\mu_2 x^3}\hat{x}^2, x^3 + \hat{x}^3)$$

for any $(\mu_1, \mu_2) \in \mathbb{R}^2$. In case $(\mu_1, \mu_2) \neq (0, 0)$, $G(\mu_1, \mu_2)$ is realized as the matrix Lie group given by (4.1). We note that in case $\mu_1 = \mu_2 = 0$, $G(\mu_1, \mu_2)$ is just the abelian group $\mathbb{R}^3$ and one can not realize this group as a matrix group in (4.1). The ${}^{(0)}\nabla$-harmonic map equation for $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ is

$$\varphi^k_{z\bar{z}} - \frac{1}{2}\mu_k(\varphi^k_z\varphi^3_{\bar{z}} + \varphi^k_{\bar{z}}\varphi^3_z) = 0, \ (k = 1, 2), \quad \varphi^3_{z\bar{z}} = 0. \tag{4.2}$$

We equip $G(\mu_1,\mu_2)$ with the left-invariant metric

$$ds^2 = e^{-2\mu_1 x^3}(dx^1)^2 + e^{-2\mu_2 x^3}(dx^2)^2 + (dx^3)^2.$$

The resulting family of Riemannian homogeneous spaces includes Euclidean 3-space $\mathbb{E}^3 = G(0,0)$, hyperbolic 3-space $\mathbb{H}^3(-c^2) = G(c,c)$ of curvature $-c^2 < 0$, the model space $\mathrm{Sol}_3 = G(1,-1)$ of *solvgeometry* in the sense of Thurston and the Riemannian product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1 = G(0,c)$. We also note that for a group $G(\mu_1,\mu_2)$ the metric introduced above is bi-invariant if and only if $\mu_1 = \mu_2 = 0$. In general, the harmonic map equation with respect to this metric is

$$\varphi^k_{z\bar{z}} - \mu_k(\varphi^k_z\varphi^3_{\bar{z}} + \varphi^k_{\bar{z}}\varphi^3_z) = 0,\ (k=1,2), \quad \varphi^3_{z\bar{z}} + \sum_{k=1}^{2}\mu_k e^{-2\mu_k\varphi^3}\varphi^k_z\varphi^k_{\bar{z}} = 0. \tag{4.3}$$

4.4. **Primitive maps into Sol_3.** Let $G = \mathrm{Sol}_3$ and $\mathfrak{g} = \mathrm{Lie}(\mathrm{Sol}_3)$. Then (G,τ,ds^2) is a Riemannian 4-symmetric space with automorphism τ defined by $\tau(x^1,x^2,x^3) = (-x^2,x^1,-x^3)$. Since the primitive 4-th root of unity is $\omega = \sqrt{-1}$, the eigenvalues of $d\tau$ are ± 1 and $\pm\sqrt{-1}$. One can determine the eigenspaces of $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\omega^j, (j=0,1,2,-1)$, explicitly:

$$\mathfrak{g}^{\mathbb{C}}_0 = \{\mathbf{0}\},\ \mathfrak{g}^{\mathbb{C}}_1 = \mathbb{C}\begin{bmatrix} 1 \\ -\sqrt{-1} \\ 0 \end{bmatrix},\ \mathfrak{g}^{\mathbb{C}}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathfrak{g}^{\mathbb{C}}_{-1} = \mathbb{C}\begin{bmatrix} 1 \\ \sqrt{-1} \\ 0 \end{bmatrix}.$$

Direct computations show that $\varphi = (\varphi^1,\varphi^2,\varphi^3) : \mathbb{D} \to \mathrm{Sol}_3$ is primitive if and only if

$$\varphi^3_z = 0, \quad \sqrt{-1}e^{-\varphi^3}\varphi^1_z = e^{\varphi^3}\varphi^2_z.$$

The first equation implies that x^3 is constant. The second equation is rewritten as

$$\frac{\partial}{\partial z}\left(\varphi^1 - \sqrt{-1}e^{2\varphi^3}\varphi^2\right) = 0.$$

Thus we obtain the following.

Proposition 4.3. *Let $\varphi : \mathbb{D} \to G$ be a smooth map into Sol_3. Then φ is a primitive map if and only if $\varphi^3 =$ constant and φ^1 and $e^{2\varphi^3}\varphi^2$ are conjugate harmonic functions.*

4.5. **Generalization of Sol_3.** The following space is regarded as a higher dimensional generalization of Sol_3. Let us consider the solvable Lie group

$$G_n = \left\{ (w,u^1,\dots,u^n,v^1,\dots,v^n) = \begin{pmatrix} e^t & 0 & \cdots & 0 & w \\ 0 & e^{v^1} & \cdots & 0 & u^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{v^n} & u^n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{SL}_{n+2}\mathbb{R},$$

where $t = -(v^1 + v^2 + \cdots + v^n)$. For $n=1$, setting $(x_1,x_2,x_3) := (w,u^1,-v^1)$, we obtain $G_1 = \mathrm{Sol}_3$. The ${}^{(0)}\nabla$-harmonic map equation for $\varphi : \mathbb{C} \to G_n$ is

$$v^k_{z\bar{z}} = 0,\ u^k_{z\bar{z}} = \frac{1}{2}(u^k_z v^k_{\bar{z}} + u^k_{\bar{z}} v^k_z),\ (k=1,\cdots n),\quad w_{z\bar{z}} = -\frac{1}{2}\sum_{k=1}^{n}\left(w_{\bar{z}}v^k_z + w_z v^k_{\bar{z}}\right).$$

This space equipped with a left-invariant metric

$$ds^2 = \sum_{k=1}^{n} e^{-2v_k}(du^k)^2 + \sum_{i,j=1}^{n} dv^i dv^j + e^{-2t}(dw)^2$$

is a $(2n+1)$-dimensional Riemannian $(2n+2)$-symmetric space [9, 46].

4.6. **The Euclidean motion group.** The motion group of the Euclidean plane $\mathbb{E}^2$ is

$$\widetilde{\mathrm{SE}_2} = \left\{ (x^1, x^2, x^3) = \begin{pmatrix} \cos x^3 & -\sin x^3 & x^1 \\ \sin x^3 & \cos x^3 & x^2 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{SL}_3\mathbb{R}.$$

The ${}^{(0)}\nabla$-harmonic map equation for $\varphi = (\varphi^1, \varphi^2, \varphi^3) : \mathbb{C} \to \mathrm{SE}_2$ is the system:

$$\varphi^1_{z\bar{z}} + \frac{1}{2}(\varphi^2_z \varphi^3_{\bar{z}} + \varphi^2_{\bar{z}} \varphi^3_z) = 0, \ \varphi^2_{z\bar{z}} - \frac{1}{2}(\varphi^1_z \varphi^3_{\bar{z}} + \varphi^1_{\bar{z}} \varphi^3_z) = 0, \ \varphi^3_{z\bar{z}} = 0. \tag{4.4}$$

The standard left-invariant metric of SE_2 is the flat one $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. With respect to this flat metric, the harmonic map equation for φ is the system:

$$\varphi^1_{z\bar{z}} = \varphi^2_{z\bar{z}} = \varphi^3_{z\bar{z}} = 0.$$

It is known that SE_2 has no bi-invariant semi-Riemannian metric.

Remark 4.4. The 3-dimensional Lie groups with left-invariant Riemannian metric have been classified by Milnor [52]. Corresponding results for left-invariant Lorentzian metrics are given in [17]. Combining these two papers, we obtain the complete list of 3-dimensional Lie groups with bi-invariant semi-Riemannian metrics.

5. Generalized Weierstrass type representation

In this section we give the generalized Weierstrass type representation for neutral harmonic maps from a Riemann surface into any Lie group G. We assume that the Lie group G is a connected real analytic Lie group which admits a faithful finite dimensional representation. Moreover, where it is convenient, we will assume without loss of generality by [35, 36] that G is embedded into its linear complexification $G^{\mathbb{C}}$ and that $G^{\mathbb{C}}$ is simply-connected.

5.1. **Reductive decompositions of G.** Let G be a real Lie group as before. Then the product Lie group $G \times G$ acts transitively on G by

$$(a, b) \cdot g = agb^{-1}.$$

The isotropy subgroup of $G \times G$ at the unit element is the diagonal subgroup

$$\triangle = \{(a, a) \in G \times G \mid a \in G\}.$$

The homogeneous space $G \cong G \times G/\triangle$ is reductive. In fact there are three standard reductive decompositions of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ of $G \times G$:

$$\begin{aligned} \mathfrak{g} \oplus \mathfrak{g} &= \mathfrak{d} \oplus \mathfrak{p}_+, \quad \mathfrak{p}_+ = \{(0, X) \mid X \in \mathfrak{g}\}, \\ \mathfrak{g} \oplus \mathfrak{g} &= \mathfrak{d} \oplus \mathfrak{p}_-, \quad \mathfrak{p}_- = \{(X, 0) \mid X \in \mathfrak{g}\}, \\ \mathfrak{g} \oplus \mathfrak{g} &= \mathfrak{d} \oplus \mathfrak{p}_0, \quad \mathfrak{p}_0 = \{(X, -X) \mid X \in \mathfrak{g}\}, \end{aligned}$$

where $\mathfrak{d}$ is the Lie algebra of $\triangle$. The canonical connections of $G = G \times G/\triangle$ with respect to these reductive decompositions, taking $\mathfrak{p}_*, * \in \{+, -, 0\}$, are ${}^{(1)}\nabla$, ${}^{(-1)}\nabla$ and ${}^{(0)}\nabla$, respectively, see [45, pp.198–199]. Among these three reductive decompositions, only $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{d} \oplus \mathfrak{p}_0$ defines a symmetric pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{d})$. The corresponding involution σ of $G \times G$ is given by

$$\sigma(a, b) = (b, a). \tag{5.1}$$

Moreover, the projection $\pi : G \times G \to G$ is given by $\pi(g, h) = gh^{-1}$. Since symmetric spaces are particularly convenient for the loop group method discussed below, we will consider from now on exclusively the third reductive decomposition.

Remark 5.1. From another point of view, the neutral connection ${}^{(0)}\nabla$ is necessary for the generalized Weierstrass type representation: Our goal is to consider connections ∇^μ on G for which every ∇^μ-harmonic map is admissible. We have shown in Proposition 3.22 that each such connection comes from a skew-symmetric μ. But in (3) of Lemma 3.3 we have seen that a map is ∇^μ harmonic if and only if it is ${}^\dagger\nabla^\mu$-harmonic. Finally, from (2) of Lemma 3.3 we know that for skew-symmetric μ we have ${}^\dagger\nabla^\mu_X Y = {}^{(0)}\nabla_X Y := \frac{1}{2}[X, Y]$.

5.2. **Flat connections.** As we have seen in the preceding sections, a map $\varphi : \mathbb{D} \to G$ from a simply-connected domain $\mathbb{D} \subset \mathbb{C}$ is ${}^{(0)}\nabla$-harmonic if and only if $d + \alpha_\lambda$ is a family of flat connections for all $\lambda \in \mathbb{S}^1$, where

$$\alpha = \varphi^*\theta = \alpha' + \alpha'' \ \text{ and } \ \alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha''. \tag{5.2}$$

First we note that the map φ can be (globally) lifted to the frame

$$\mathcal{F} : \mathbb{D} \to \mathcal{G} := G \times G, \ \ p \mapsto (\mathrm{id}, \varphi(p)),$$

where id denotes the unit of G. As a consequence

$$\mathcal{A} = \mathcal{F}^*\theta_{\mathcal{G}} = (0, \varphi^*\theta) = (0, \alpha). \tag{5.3}$$

Here $\theta_{\mathcal{G}}$ is the left Maurer-Cartan form of $\mathcal{G} = G \times G$. Next we decompose $\mathcal{A}$ with respect to the eigenspaces of $d\sigma$. It is easy to verify that the fixed point algebra $\mathcal{K}$ of $d\sigma$ and the eigenspace $\mathcal{P}$ for the eigenvalue -1 are $\mathcal{K} = \{(X, X) \mid X \in \mathfrak{g}\}$ and $\mathcal{P} = \{(Y, -Y) \mid Y \in \mathfrak{g}\}$. As a consequence

$$\mathcal{A} = \mathcal{A}_{\mathcal{K}} + \mathcal{A}_{\mathcal{P}},$$

where $\mathcal{A}_{\mathcal{K}} = \frac{1}{2}(\alpha, \alpha)$ and $\mathcal{A}_{\mathcal{P}} = \frac{1}{2}(-\alpha, \alpha)$. After decomposing further we obtain

$$\mathcal{A}_{\mathcal{P}} = \mathcal{A}'_{\mathcal{P}} + \mathcal{A}''_{\mathcal{P}},$$

where $\mathcal{A}'_{\mathcal{P}}$ is a $(1, 0)$-form and $\mathcal{A}''_{\mathcal{P}}$ is a $(0, 1)$-form. We now introduce λ as usual:

$$\mathcal{A}_\lambda = \lambda^{-1}\mathcal{A}'_{\mathcal{P}} + \mathcal{A}_{\mathcal{K}} + \lambda\mathcal{A}''_{\mathcal{P}}, \ \ \lambda \in \mathbb{S}^1. \tag{5.4}$$

Then a straightforward computation shows

$$\mathcal{A}_\lambda = \left(\frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha'', \frac{1}{2}(1 + \lambda^{-1})\alpha' + \frac{1}{2}(1 + \lambda)\alpha''\right) = (\alpha_\lambda, \alpha_{-\lambda}). \tag{5.5}$$

Clearly, $\mathcal{A}_\lambda$ is integrable, that is, $d\mathcal{A}_\lambda + \frac{1}{2}[\mathcal{A}_\lambda \wedge \mathcal{A}_\lambda] = \mathbf{0}$, if and only if α_λ is integrable, that is, $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$. Hence there exits a map $\mathcal{F}_\lambda : \mathbb{D} \times \mathbb{S}^1 \to \mathcal{G}$ satisfying $\mathcal{F}_\lambda^*\theta_{\mathcal{G}} = \mathcal{A}_\lambda$. The map $\mathcal{F}_\lambda$ is called an *extended frame* of φ. The extended frame $\mathcal{F}_\lambda$ has the form

$$\mathcal{F}_\lambda = (F_\lambda, F_{-\lambda}),$$

where F_λ is an extended solution. The following theorem is an immediate consequence of Theorem 3.14.

Theorem 5.2. *Let M be a Riemann surface, G a Lie group and ${}^{(0)}\nabla$ its neutral connection. Moreover, let $\varphi : M \to (G, {}^{(0)}\nabla)$ be a smooth map and α_λ and $\mathcal{A}_\lambda$ the 1-forms defined in* (5.2) *and* (5.4). *Then the following statements are equivalent*:

(1) *φ is ${}^{(0)}\nabla$-harmonic.*
(2) *$d + \alpha_\lambda$ is a family of flat connections for all $\lambda \in \mathbb{S}^1$.*
(3) *$d + \mathcal{A}_\lambda$ is a family of flat connections for all $\lambda \in \mathbb{S}^1$.*

Remark 5.3. Every semi-Riemannian symmetric space G/K with semi-simple G is identified with the image of the Cartan immersion $\iota : G/K \hookrightarrow G$. For example, one obtains $\iota(\mathbb{S}^2) \subset \mathrm{SU}_2$ and $\iota(\mathbb{H}^2) \subset \mathrm{SU}_{1,1}$. Thus as a consequence of Theorem 5.2 we can establish the loop group formalism as for CMC surfaces in $\mathbb{E}^3$, spacelike CMC surfaces in Minkowski space and in many other cases, [18].

5.3. **Loop groups decompositions.** In the rest of this subsection we will follow the procedure presented in [25], even though our symmetric space $G = G \times G/\Delta$ is not necessarily compact and does, in general, not even carry any bi-invariant metric. But it has a "nice" bi-invariant connection, and this will allow us to produce all ${}^{(0)}\nabla$-harmonic maps into G by the loop group method.

Let G be a Lie group which admits a faithful finite dimensional representation and $G^{\mathbb{C}}$ its simply-connected linear complexification. Let $\Lambda G^{\mathbb{C}}$ be the (connected) loop group of $G^{\mathbb{C}}$:

$$\Lambda G^{\mathbb{C}} = \{\gamma : \mathbb{S}^1 \to G^{\mathbb{C}}\}.$$

We equip $\Lambda G^{\mathbb{C}}$ with a weighted Wiener topology [6] such that $\Lambda G^{\mathbb{C}}$ is a Banach Lie group and all the subgroups occurring in this paper will be Banach Lie subgroups. Let $\mathcal{D}$ be the unit disk in the complex plane and $\overline{\mathbb{C}}$ the extended complex plane. We now introduce the following subgroups of $\Lambda G^{\mathbb{C}}$:

$$\begin{aligned}
\Lambda G &= \{\gamma \in \Lambda G^{\mathbb{C}} \mid \gamma(\lambda) \in G\}, \\
\Lambda^+ G^{\mathbb{C}} &= \{\gamma \in \Lambda G^{\mathbb{C}} \mid \gamma \text{ and } \gamma^{-1}\text{extend holomorphically to } \mathcal{D}\}, \\
\Lambda^- G^{\mathbb{C}} &= \{\gamma \in \Lambda G^{\mathbb{C}} \mid \gamma \text{ and } \gamma^{-1}\text{extend holomorphically to } \overline{\mathbb{C}} \setminus \mathcal{D}\}, \\
\Lambda^-_* G^{\mathbb{C}} &= \{\gamma \in \Lambda^- G^{\mathbb{C}} \mid \gamma(\lambda = \infty) = \mathrm{id}\}.
\end{aligned}$$

Below we will recall two important decomposition theorems obtained in [6, 41, 57].

Remark 5.4. In our applications we consider frames with values in some real Lie group G and choose a convenient complexification (which may not necessarily be simply-connected). These frames are continuous images of connected surfaces M and attain the value id at some base point $z_0 \in M$. Hence all these frames take values in the connected component of $\mathrm{id} \in \Lambda G^{\mathbb{C}}$. Thus we are only interested in the decompositions of the identity component of our loop group. If $\tilde{G}^{\mathbb{C}}$ denotes the simply-connected cover of $G^{\mathbb{C}}$, then the loop group $\Lambda\tilde{G}^{\mathbb{C}}$ is connected and the canonical projection, induced from $\tilde{\pi} : \tilde{G}^{\mathbb{C}} \to G^{\mathbb{C}}$, has as image the connected component of $\Lambda G^{\mathbb{C}}$. Therefore, below we will write down the decomposition theorems for $\Lambda\tilde{G}^{\mathbb{C}}$, but will apply them later to the projection onto the connected component $(\Lambda G^{\mathbb{C}})^o$. It is not difficult to verify that the double cosets are parametrized by the same set of representatives and that a double coset in $(\Lambda G^{\mathbb{C}})^o$ is open if and only if the corresponding double coset in $\Lambda G^{\mathbb{C}}$ is open. It needs to be pointed out that the three references given above all use simply-connected $G^{\mathbb{C}}$, even though in [6] this was missed to state.

By the Levi theorem [35, Theorem 18.4.3], there exist a real analytic reductive subgroup H of G, and a simply-connected real analytic solvable subgroup B of G such that

$$G \cong H \ltimes B. \tag{5.6}$$

Note that B is a normal subgroup of G, and B can be represented in the form $B = A_1 \ltimes (A_2 \ltimes \cdots \ltimes N)$ with simply-connected 1-dimensional abelian Lie groups A_j and simply-connected unipotent Lie group N. Since $G = H \cdot B \cong H \ltimes B$, the complexified simply-connected group $G^{\mathbb{C}}$ satisfies $G^{\mathbb{C}} = H^{\mathbb{C}} \cdot B^{\mathbb{C}} \cong H^{\mathbb{C}} \ltimes B^{\mathbb{C}}$, we also have

$$\Lambda G^{\mathbb{C}} \cong \Lambda H^{\mathbb{C}} \cdot \Lambda B^{\mathbb{C}}, \quad \Lambda^- G^{\mathbb{C}} \cong \Lambda^- H^{\mathbb{C}} \cdot \Lambda^- B^{\mathbb{C}}, \quad \Lambda^+ G^{\mathbb{C}} \cong \Lambda^+ H^{\mathbb{C}} \cdot \Lambda^+ B^{\mathbb{C}}.$$

Below we will state two decomposition theorems. More precisely, we will describe (to some extent) the (double) cosets of the action of the product of some natural subgroups of the loop group $\Lambda G^{\mathbb{C}}$. Actually, there are several natural choices. Since in this paper we are mainly interested in the open cosets, these possibly different choices have little importance to us.

Let $\Lambda^d H^{\mathbb{C}}$ denote the set of representatives in the unique Birkhoff decomposition of [40, Corollary 5(c)] or [6, Theorem 4.4] of a simply-connected semi-simple Lie group $H^{\mathbb{C}}$:

$$\Lambda H^{\mathbb{C}} = \bigcup_{s \in \Lambda^d H^{\mathbb{C}}} (\Lambda_\sharp^- H^{\mathbb{C}})_s^- \cdot s \cdot C \Lambda_\sharp^+ H^{\mathbb{C}}, \tag{5.7}$$

where C is a Cartan subgroup of $G^{\mathbb{C}}$, $\Lambda_\sharp^\epsilon H^{\mathbb{C}}$ denotes the subgroup of $\Lambda^\epsilon H^{\mathbb{C}}$ defined such that the λ-independent term is in the maximal nilpotent subgroup of the (relative to C and a choice positive roots) opposite Borel subgroup of $H^{\mathbb{C}}$, if $\epsilon = -$ and in the maximal nilpotent subgroup of the Borel subgroup (relative to C and the same choice of positive roots) if $\epsilon = +$. Finally, $(\Lambda_\sharp^- H^{\mathbb{C}})_s^-$ is defined by

$$(\Lambda_\sharp^- H^{\mathbb{C}})_s^- = \left\{ h \in \Lambda_\sharp^- H^{\mathbb{C}} \mid s^{-1} h s \in \Lambda_\sharp^- H^{\mathbb{C}} \right\}.$$

Then note that $\Lambda^d H^{\mathbb{C}}$ can be identified with the Weyl group of $\Lambda G^{\mathbb{C}}$.

Further define the subgroups

$$\left(\Lambda_*^- B^{\mathbb{C}}\right)_s^\pm = \{ b \in \Lambda_*^- B^{\mathbb{C}} \mid s b s^{-1} \in \Lambda_*^\pm B^{\mathbb{C}} \},$$

and set $\Lambda_\sharp^\pm G^{\mathbb{C}} = \Lambda_\sharp^\pm H^{\mathbb{C}} \cdot \Lambda^\pm B^{\mathbb{C}}$ and $(\Lambda_\sharp^- G^{\mathbb{C}})_s^- = (\Lambda_\sharp^- H^{\mathbb{C}})_s^- \cdot (\Lambda^- B^{\mathbb{C}})_s^-$. Then from[6, Theorem 4.5], [41], we have:

Theorem 5.5 (Birkhoff decomposition)**.** *Assume $G^{\mathbb{C}}$ to be simply-connected, then the loop group $\Lambda G^{\mathbb{C}}$ can be decomposed into the disjoint union of double cosets*:

$$\Lambda G^{\mathbb{C}} = \bigcup_{s \in \Lambda^d H^{\mathbb{C}}} (\Lambda_\sharp^- G^{\mathbb{C}})_s^- \cdot s \left(\Lambda_*^- B^{\mathbb{C}}\right)_s^+ \cdot C \Lambda_\sharp^+ G^{\mathbb{C}}.$$

Moreover, the subset $Br_{G^{\mathbb{C}}} := \Lambda_*^- G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}}$ is called the (left) Birkhoff big cell *of $\Lambda G^{\mathbb{C}}$ and it is an open and dense subset of $\Lambda G^{\mathbb{C}}$. The multiplication map*

$$\Lambda_*^- G^{\mathbb{C}} \times \Lambda^+ G^{\mathbb{C}} \to \Lambda_*^- G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}} \subset \Lambda G^{\mathbb{C}}$$

provides a complex analytic diffeomorphism onto the Birkhoff big cell $Br_{G^{\mathbb{C}}}$.

Remark 5.6. As in [6] one can easily show that each element $g \in \Lambda G^{\mathbb{C}}$ can be represented uniquely in the form

$$g = (h_- s b_-^- s^{-1})(s b_-^+)(b_+ h_+), \tag{5.8}$$

where $s \in \Lambda^d H^{\mathbb{C}}$, $b_-^+ \in \left(\Lambda_*^- B^{\mathbb{C}}\right)_s^+$, $b_-^- \in \left(\Lambda_*^- B^{\mathbb{C}}\right)_s^-$, $b_+ \in \Lambda^+ B^{\mathbb{C}}$, $h_+ \in \Lambda_\sharp^+ H^{\mathbb{C}}$ and $h_- \in (\Lambda_\sharp^- H^{\mathbb{C}})_s^-$. From this it is not difficult to show that exactly one double coset is open, namely the one with $s = \mathrm{id}$.

In a similar fashion we obtain an Iwasawa decomposition of $\Lambda G^{\mathbb{C}}$. First we consider an Iwasawa decomposition (with disjoint cosets) of $H^{\mathbb{C}}$ as derived in [41, Chap. 4], [6, Theorem 6.1]:

$$\Lambda H^{\mathbb{C}} = \bigcup_{s \in \Lambda^m H^{\mathbb{C}}} \Lambda H \cdot s \cdot \Lambda^+ H^{\mathbb{C}}. \tag{5.9}$$

Here $\Lambda^m H^{\mathbb{C}}$ is a specific set of representatives for the double coset given in [54]. Then from [6, Theorem 6.5], we have:

Theorem 5.7 (Iwasawa decomposition)**.** *Let $G^{\mathbb{C}}$ be simply-connected, then the loop group $\Lambda G^{\mathbb{C}}$ can be decomposed into a disjoint union of double cosets*:

$$\Lambda G^{\mathbb{C}} = \bigcup_{s \in \Lambda^m H^{\mathbb{C}}} \Lambda G \cdot s\Lambda B \cdot \Lambda^+ G^{\mathbb{C}}.$$

Moreover, the subset $Iw_G^{\mathrm{id}} := \Lambda G \cdot \Lambda^+ G^{\mathbb{C}}$ is called the Iwasawa big cell (*containing the identity element* id)*. It is an open set in $\Lambda G^{\mathbb{C}}$.*

Remark 5.8.

(1) One can describe a unique decomposition of each loop [6], but we will not need these details for this paper.
(2) In general, there are several open Iwasawa cells Iw^{ω}. In this paper we will mostly use *the Iwasawa big cell.*
(3) Of course, one is interested to know when there is only one open Iwasawa cell. This happens if and only if Iw^{id} is open and dense in $\Lambda G^{\mathbb{C}}$.

About the denseness of Iw^{id} the following result is fundamental.

Theorem 5.9 (Theorem 7.2 and 7.3 in [6])**.** *Let $G^{\mathbb{C}}$ be simply-connected. Then, if the semi-simple part of a maximal compact subgroup of $H^{\mathbb{C}}$ is simply-connected, then the Iwasawa big cell Iw_H^{id} is dense in $\Lambda H^{\mathbb{C}}$ and we have*

$$Iw_G^{\mathrm{id}} = Iw_H^{\mathrm{id}} \cdot \Lambda B^{\mathbb{C}}.$$

In particular, $Iw_G^{\mathrm{id}} = \Lambda G^{\mathbb{C}}$ if and only if the semi-simple part S of H is compact.

5.4. **Generalized Weierstrass type representation.** We denote by $\mathcal{G}$ the direct product $G \times G$ and $(G \times G)^{\mathbb{C}} = G^{\mathbb{C}} \times G^{\mathbb{C}}$ by $\mathcal{G}^{\mathbb{C}}$. Consider the double loop group

$$\Lambda \mathcal{G}^{\mathbb{C}} := (\Lambda G \times \Lambda G)^{\mathbb{C}} = \Lambda G^{\mathbb{C}} \times \Lambda G^{\mathbb{C}}.$$

Then the *twisted loop group* $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$, twisted by σ (5.1), and its real form $\Lambda \mathcal{G}_{\sigma}$ are defined by

$$\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma} := \{(g(\lambda), g(-\lambda)) \mid g \in \Lambda G^{\mathbb{C}}\}, \quad \Lambda \mathcal{G}_{\sigma} := \{(g(\lambda), g(-\lambda)) \mid g \in \Lambda G\}.$$

According to the Levi decomposition $G \cong H \ltimes B$, we consider the twisted double loop groups of $B^{\mathbb{C}}$ and $H^{\mathbb{C}}$, and its real forms:

$$\Lambda \mathcal{B}^{\mathbb{C}}_{\sigma} = \left\{(b(\lambda), b(-\lambda)) \mid b \in \Lambda B^{\mathbb{C}}\right\}, \quad \Lambda \mathcal{H}^{\mathbb{C}}_{\sigma} = \{(h(\lambda), h(-\lambda)) \mid h \in \Lambda H^{\mathbb{C}}\},$$
$$\Lambda \mathcal{B}_{\sigma} = \{(b(\lambda), b(-\lambda)) \mid b \in \Lambda B\}, \quad \Lambda \mathcal{H}_{\sigma} = \{(h(\lambda), h(-\lambda)) \mid h \in \Lambda H\}.$$

For the "positive loop subgroup" and "negative loop subgroup" of $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$ we set:

$$\Lambda^{\pm} \mathcal{G}^{\mathbb{C}}_{\sigma} = \{(g(\lambda), g(-\lambda)) \mid g \in \Lambda^{\pm} G^{\mathbb{C}}\},$$

and we will need to define the following subgroup:

$$\Lambda^{-}_{*} \mathcal{G}^{\mathbb{C}}_{\sigma} = \{(g(\lambda), g(-\lambda)) \mid g \in \Lambda^{-} G^{\mathbb{C}} \text{ and } g(\lambda = \infty) = e\}.$$

According to the Birkhoff decomposition of $\Lambda G^{\mathbb{C}}$ in Theorem 5.5, we introduce the following subgroups of $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$:

$$\begin{aligned} \Lambda^{\pm}_{\sharp} \mathcal{G}^{\mathbb{C}}_{\sigma} &= \left\{(g(\lambda), g(-\lambda)) \mid g \in \Lambda^{\pm}_{\sharp} G^{\mathbb{C}}\right\}, \\ (\Lambda^{-}_{\sharp} \mathcal{G}^{\mathbb{C}}_{\sigma})^{-}_{s} &= \left\{(g(\lambda), g(-\lambda)) \mid g \in (\Lambda^{-}_{\sharp} G^{\mathbb{C}})^{-}_{s} \right\}, \\ (\Lambda^{-}_{*} \mathcal{B}^{\mathbb{C}}_{\sigma})^{+}_{s} &= \left\{(g(\lambda), g(-\lambda)) \mid g \in (\Lambda^{-}_{*} B^{\mathbb{C}}_{\sigma})^{+}_{s}\right\}. \end{aligned}$$

Now we consider the Birkhoff decomposition as well as the Iwasawa decomposition for the loop group $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma} (\cong \Lambda \mathcal{H}^{\mathbb{C}}_{\sigma} \cdot \Lambda \mathcal{B}^{\mathbb{C}}_{\sigma})$. Then the Birkhoff and Iwasawa decomposition Theorems 5.5 and 5.7 for $\Lambda G^{\mathbb{C}}$ induce the following decomposition theorems for $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$.

Theorem 5.10 (Birkhoff decomposition)**.** *The loop group $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$ can be decomposed into a disjoint union of double cosets:*

$$\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma} = \bigcup_{s \in \Lambda^{d} \mathcal{H}^{\mathbb{C}}_{\sigma}} (\Lambda^{-}_{\sharp} \mathcal{G}^{\mathbb{C}}_{\sigma})^{-}_{s} \cdot s (\Lambda^{-}_{*} \mathcal{B}^{\mathbb{C}}_{\sigma})^{+}_{s} \cdot \mathcal{C} \Lambda^{+}_{\sharp} \mathcal{G}^{\mathbb{C}}_{\sigma},$$

where $\mathcal{C} = (C, C)$ with a Cartan subgroup C and $\Lambda^{d} \mathcal{H}^{\mathbb{C}}_{\sigma} = \{(h(\lambda), h(-\lambda)) \mid h \in \Lambda^{d} H^{\mathbb{C}}\}$. Moreover, the subset $\mathcal{B}r_{\mathcal{G}^{\mathbb{C}}} = \Lambda^{-}_{*} \mathcal{G}^{\mathbb{C}}_{\sigma} \cdot \Lambda^{+} \mathcal{G}^{\mathbb{C}}_{\sigma}$ is called the (left) Birkhoff big cell *and it is an open and dense subset of $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$. The multiplication map*

$$\Lambda^{-}_{*} \mathcal{G}^{\mathbb{C}}_{\sigma} \times \Lambda^{+} \mathcal{G}^{\mathbb{C}}_{\sigma} \to \Lambda^{-}_{*} \mathcal{G}^{\mathbb{C}}_{\sigma} \cdot \Lambda^{+} \mathcal{G}^{\mathbb{C}}_{\sigma} \subset \Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$$

provides an analytic diffeomorphism onto the Birkhoff big cell $\mathcal{B}_{\mathcal{G}^{\mathbb{C}}}$.

Remark 5.11. As always, there is only one open (and dense) Birkhoff big cell.

Then we have the following Iwasawa decomposition theorem.

Theorem 5.12 (Iwasawa decomposition)**.** *The loop group $\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma}$ can be decomposed into a disjoint union of double cosets:*

$$\Lambda \mathcal{G}^{\mathbb{C}}_{\sigma} = \bigcup_{s \in \Lambda^{m} \mathcal{H}^{\mathbb{C}}} \Lambda \mathcal{G}_{\sigma} \cdot s \Lambda \mathcal{B} \cdot \Lambda^{+} \mathcal{G}^{\mathbb{C}}_{\sigma}.$$

Here $\Lambda^m\mathcal{H}^{\mathbb{C}} = \{(h(\lambda), h(-\lambda)) \mid h \in \Lambda^m H^{\mathbb{C}}\}$. *Moreover, the subset* $\mathcal{I}w^{\mathrm{id}}_{\mathcal{G}} := \Lambda\mathcal{G}_\sigma \cdot \Lambda^+\mathcal{G}^{\mathbb{C}}_\sigma$ *is called the* Iwasawa big cell *and it is an open set in* $\Lambda\mathcal{G}^{\mathbb{C}}_\sigma$.

5.5. From now on, we consider only the connected component of a twisted loop group and denote it by the same symbol. Now we can apply the usual loop group scheme. Starting from a ${}^{(0)}\nabla-$harmonic map φ, we consider the corresponding extended frame $\mathcal{F}_\lambda$. Where possible we perform a Birkhoff decomposition of the extended frame $\mathcal{F}_\lambda$ as described in Theorem 5.10:

$$\mathcal{F}_\lambda = \mathcal{F}_-\mathcal{V}_+, \tag{5.10}$$

where $\mathcal{F}_- \in \Lambda^-_*\mathcal{G}^{\mathbb{C}}_\sigma$ and $\mathcal{V}_+ \in \Lambda^+\mathcal{G}^{\mathbb{C}}_\sigma$. Then the usual argument shows the following, see [25].

Theorem 5.13. *Let* $\varphi : \mathbb{D} \to G$ *be a* ${}^{(0)}\nabla$*-harmonic map and* $\mathcal{F}_\lambda$ *its extended frame. Moreover let* $\mathcal{F}_\lambda = \mathcal{F}_-\mathcal{V}_+$ *be the Birkhoff decomposition given in* (5.10). *Then the following statements hold*:

(1) *There exists a discrete subset* $S \subset \mathbb{D}$ *such that* (5.10) *is defined for all* $z \in \mathbb{D} \setminus S$.
(2) *The map* $\mathcal{F}_-$ *only depends on* z. *It is a meromorphic* $\Lambda^-_*\mathcal{G}^{\mathbb{C}}_\sigma$*-valued matrix function.*
(3) *The Maurer-Cartan form* $\mathcal{N} = \mathcal{F}_-^{-1}d\mathcal{F}_-$ *of* $\mathcal{F}_-$ *has the form*

$$\mathcal{N}(z,\lambda) = \mathcal{F}_-(z,\lambda)^{-1}d\mathcal{F}_-(z,\lambda) = \left(\lambda^{-1}\xi(z), -\lambda^{-1}\xi(z)\right), \tag{5.11}$$

where ξ *is a meromorphic* 1*-form on* $\mathbb{D}$ *with values in* $\mathfrak{g}^{\mathbb{C}}$.

5.6. The converse procedure is as follows: Consider a meromorphic 1-form $\mathcal{N}$ on $\mathbb{D}$ of the form stated in (5.11).

Step 1. Solve the pair of ordinary differential equations:

$$d\mathcal{R}_- = \mathcal{R}_-\,\mathcal{N}$$

with any initial condition $\mathcal{R}_-(z_0) \in \mathcal{G}^{\mathbb{C}} (= G^{\mathbb{C}} \times G^{\mathbb{C}})$ at some base point $z_0 \in \mathbb{D}$ and assume that the solution $\mathcal{R}_-$ is meromorphic and takes values in $\Lambda\mathcal{G}^{\mathbb{C}}$.

Step 2. From the Iwasawa decomposition in Theorem 5.12 we obtain (for all $z \in \mathbb{D}$ for which $\mathcal{R}_- \in \tilde{\mathcal{P}}_{\mathcal{G}}$):

$$\mathcal{R}_- = \mathcal{F}\,\mathcal{W}_+, \ \ (\mathcal{F} \in \Lambda\mathcal{G}_\sigma,\ \mathcal{W}_+ \in \Lambda^+\mathcal{G}^{\mathbb{C}}_\sigma),$$

where $\mathcal{F}$ and $\mathcal{W}_+$ have the form

$$\mathcal{F}(z,\bar z,\lambda) = (F(z,\bar z,\lambda), F(z,\bar z,-\lambda))\,, \ \ \mathcal{W}_+(z,\bar z,\lambda) = (W_+(z,\bar z,\lambda), W_+(z,\bar z,-\lambda))\,.$$

There is freedom in this decomposition. Setting

$$\hat{\mathcal{F}}(z,\bar z,\lambda) = (F(z,\bar z,\lambda)F(z,\bar z,1)^{-1},\ F(z,\bar z,-\lambda)F(z,\bar z,1)^{-1}),$$

we obtain the following Iwasawa decomposition, where now

$$\mathcal{R}_- = \hat{\mathcal{F}}\,\hat{\mathcal{W}}_+ \ \text{ and } \ \hat{\mathcal{F}}(z,\bar z,1) = (e,\varphi).$$

In this case, $\hat{\mathcal{A}}_\lambda = \hat{\mathcal{F}}^{-1}d\hat{\mathcal{F}}$ has the form (5.4) and for $\lambda = 1$ we obtain $\hat{\mathcal{A}}_{\lambda=1} = (0,\alpha)$. Hence $\hat{\mathcal{A}}_\lambda$ is of the form (5.5). Setting $\hat{\mathcal{F}}(z,\bar z,\lambda) = (\hat F(z,\bar z,\lambda), \hat F(z,\bar z,-\lambda))$ we know $\hat F(z,\bar z,\lambda = 1) = \mathrm{id}$ and $\hat F(z,\bar z,\lambda=-1) = \varphi(z,\bar z)$.

Theorem 5.14. *The map* $\varphi(z,\bar z) := \hat F(z,\bar z,-1)$ *is* ${}^{(0)}\nabla$*-harmonic. Moreover,*

$$\hat\varphi(z,\bar z,\lambda) = \hat F(z,\bar z,-\lambda)\hat F(z,\bar z,\lambda)^{-1}$$

is a harmonic map into $(G, {}^{(0)}\nabla)$ *for all* $\lambda \in \mathbb{S}^1$*. The map* $\hat F(z,\bar z,\lambda)$ *is an extended solution of* φ*.*

For many purposes it is very convenient to start the construction scheme with a holomorphic potential (as opposed to a meromorphic potential). In this case one will need to admit more (usually infinitely many) powers of λ in a Fourier expansion of the potential. The construction scheme just outlined above works verbatim in the same way and produces a ${}^{(0)}\nabla$-harmonic map into G. To make sure that starting from some holomorphic potential we do not miss any harmonic maps we state that the proof given in the appendix of [25] can be carried out mutatis mutandis and we obtain:

Theorem 5.15. *Let* $\mathbb{D}$ *be a non-compact simply-connected Riemann surface* $\mathbb{D}$*, then for every harmonic map* φ *from* $\mathbb{D}$ *to a Lie group* G*, equipped with the neutral connection* ${}^{(0)}\nabla$*, there exists a holomorphic potential defined on* $\mathbb{D}$ *which generates this harmonic map* (*actually an* $\mathbb{S}^1$*-family of* ${}^{(0)}\nabla$*-harmonic maps* φ_λ *with* $\varphi = \varphi_{\lambda=1}$) *by the construction scheme outlined above.*

6. Examples

In this section, as an application of Section 5, we give the generalized Weierstrass type representation for ${}^{(0)}\nabla$-harmonic maps into the 3-dimensional solvable Lie groups. We first remark that the case of the 3-dimensional Heisenberg group Nil_3 was discussed in [5] in detail. Thus we omit this case.

6.1. 3-dimensional solvable Lie groups.

Let us consider the 2-parameter family of 3-dimensional solvable Lie groups $G(\mu_1,\mu_2)$ given in (4.1). For the loop group $\Lambda G(\mu_1,\mu_2)^{\mathbb{C}}$, the respective Birkhoff and Iwasawa decompositions in Theorem 5.5 and 5.7 are given explicitly as follows.

Theorem 6.1. *The Birkhoff decomposition of* $\Lambda G(\mu_1,\mu_2)^{\mathbb{C}}$ *is given by*

$$\Lambda G(\mu_1,\mu_2)^{\mathbb{C}} = \Lambda^-_* G(\mu_1,\mu_2)^{\mathbb{C}} \cdot \Lambda^+ G(\mu_1,\mu_2)^{\mathbb{C}}.$$

Every element

$$g(\lambda) = \begin{pmatrix} e^{\mu_1 x^3(\lambda)} & 0 & x^1(\lambda) \\ 0 & e^{\mu_2 x^3(\lambda)} & x^2(\lambda) \\ 0 & 0 & 1 \end{pmatrix}$$

of $\Lambda G(\mu_1,\mu_2)^{\mathbb{C}}$ *is globally decomposed as*

$$g(\lambda) = g_-(\lambda) g_+(\lambda),$$

where

$$g_\pm(\lambda) = \begin{pmatrix} e^{\mu_1 x^3_\pm(\lambda)} & 0 & x^1_\pm(\lambda) \\ 0 & e^{\mu_2 x^3_\pm(\lambda)} & x^2_\pm(\lambda) \\ 0 & 0 & 1 \end{pmatrix}.$$

Here using the expansion $x^3(\lambda)=\sum_{j=-\infty}^{\infty} x^3_j\lambda^j$, *the functions* $x^3_\pm(\lambda)$ *are given by*

$$x^3_+(\lambda)=\sum_{j\geq 0} x^3_j\lambda^j,\quad x^3_-(\lambda)=\sum_{j<0} x^3_j\lambda^j,$$

and using the expansion

$$\exp\left(-\mu_k x^3_-(\lambda)\right) x^k(\lambda)=\sum_{j=-\infty}^{\infty}\hat{x}^k_j\lambda^j\ \ (k=1,2),$$

$x^k_\pm(\lambda)$, $(k=1,2)$ *are given by*

$$\begin{cases} x^k_+(\lambda)=\sum_{j\geq 0}\hat{x}^k_j\lambda^j,\ \ (k=1,2),\\ x^k_-(\lambda)=(\sum_{j<0}\hat{x}^k_j\lambda^j)\exp(\mu_k x^3_-(\lambda)),\ \ (k=1,2).\end{cases}$$

Theorem 6.2. *The Iwasawa decomposition of* $\Lambda G(\mu_1,\mu_2)^{\mathbb{C}}$ *is given by*

$$\Lambda G(\mu_1,\mu_2)^{\mathbb{C}}=\Lambda G(\mu_1,\mu_2)\cdot\Lambda^+ G(\mu_1,\mu_2)^{\mathbb{C}}.$$

Every element

$$g(\lambda)=\begin{pmatrix} e^{\mu_1 x^3(\lambda)} & 0 & x^1(\lambda)\\ 0 & e^{\mu_2 x^3(\lambda)} & x^2(\lambda)\\ 0 & 0 & 1\end{pmatrix}$$

of $\Lambda G(\mu_1,\mu_2)^{\mathbb{C}}$ *is globally decomposed as*

$$g(\lambda)=\tilde{g}(\lambda)g_+(\lambda),$$

where

$$\tilde{g}(\lambda)=\begin{pmatrix} e^{\mu_1\tilde{x}^3(\lambda)} & 0 & \tilde{x}^1(\lambda)\\ 0 & e^{\mu_2\tilde{x}^3(\lambda)} & \tilde{x}^2(\lambda)\\ 0 & 0 & 1\end{pmatrix},\ \ g_+(\lambda)=\begin{pmatrix} e^{\mu_1 x^3_+(\lambda)} & 0 & x^1_+(\lambda)\\ 0 & e^{\mu_2 x^3_+(\lambda)} & x^2_+(\lambda)\\ 0 & 0 & 1\end{pmatrix}.$$

Here using the expansion $x^3(\lambda)=\sum_{j=-\infty}^{\infty} x^3_j\lambda^j$, *the functions* $\tilde{x}^3(\lambda)$ *and* $x^3_+(\lambda)$ *are given by,*

$$x^3_+(\lambda)=\sum_{j\geq 0}x^3_j\lambda^j-\sum_{j<0}\overline{x^3_j}\,\lambda^{-j},\quad \tilde{x}^3(\lambda)=\sum_{j<0}\left(x^3_j\lambda^j+\overline{x^3_j}\lambda^{-j}\right),$$

and using the expansions of

$$\exp\left(-\mu_k\tilde{x}_3(\lambda)\right)x^k(\lambda)=\sum_{j=-\infty}^{\infty}\hat{x}^k_j\lambda^j,\quad (k=1,2),$$

$\tilde{x}^k(\lambda)$ *and* $x^k_+(\lambda)$, $(k=1,2)$ *are given by*

$$x^k_+(\lambda)=\sum_{j\geq 0}\hat{x}^k_j\lambda^j-\sum_{j<0}\overline{\hat{x}^k_j}\lambda^{-j},\quad \tilde{x}^k(\lambda)=\sum_{j<0}\left(\hat{x}^k_j\lambda^j+\overline{\hat{x}^k_j}\lambda^{-j}\right)e^{\mu_k\tilde{x}^3(\lambda)},\quad (k=1,2).$$

Remark 6.3. It is shown in [6, Lemma 4.3 and Lemma 6.4], the Birkhoff and Iwasawa decompositions for the loop group $\Lambda B^{\mathbb{C}}$ of a simply-connected solvable Lie group $B^{\mathbb{C}}$ is *global*, that is, the Birkhoff and the Iwasawa decompositions are given respectively as follows:

$$\Lambda B^{\mathbb{C}}=\Lambda^-_* B^{\mathbb{C}}\cdot\Lambda^+ B^{\mathbb{C}},\quad \Lambda B^{\mathbb{C}}=\Lambda B\cdot\Lambda^+ B^{\mathbb{C}}.$$

Let φ be a ${}^{(0)}\nabla$-harmonic map parametrized as

$$\varphi(z,\bar z) = \begin{pmatrix} e^{\mu_1\varphi^3(z,\bar z)} & 0 & \varphi^1(z,\bar z) \\ 0 & e^{\mu_2\varphi^3(z,\bar z)} & \varphi^2(z,\bar z) \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{D} \to G(\mu_1,\mu_2) \subset \mathrm{GL}_3\mathbb{R}.$$

Here we assume that $\mathbb{D}$ is a simply-connected domain in $\mathbb{C}$ containing 0 and $z \in \mathbb{D}$ is a conformal coordinate. Then it is easy to see that

$$\varphi^{-1}d\varphi = \alpha = \alpha' + \alpha'' = \begin{pmatrix} \mu_1\varphi^3_z & 0 & e^{-\mu_1\varphi^3}\varphi^1_z \\ 0 & \mu_2\varphi^3_z & e^{-\mu_2\varphi^3}\varphi^2_z \\ 0 & 0 & 0 \end{pmatrix} dz + \begin{pmatrix} \mu_1\varphi^3_{\bar z} & 0 & e^{-\mu_1\varphi^3}\varphi^1_{\bar z} \\ 0 & \mu_2\varphi^3_{\bar z} & e^{-\mu_2\varphi^3}\varphi^2_{\bar z} \\ 0 & 0 & 0 \end{pmatrix} d\bar z. \tag{6.1}$$

From Corollary 3.6, $\varphi(z,\bar z)$ is ${}^{(0)}\nabla$-harmonic if and only if $\bar\partial\alpha' - \partial\alpha'' = 0$ which is equivalent to

$$2\varphi^j_{z\bar z} - \mu_j(\varphi^j_z\varphi^3_{\bar z} + \varphi^j_{\bar z}\varphi^3_z) = 0 \ \ (j=1,2), \ \ \varphi^3_{z\bar z} = 0.$$

We now set $\mathcal{A} = (0,\alpha)$ and $\mathcal{A}_{\mathcal{K}} = \frac{1}{2}(\alpha,\alpha)$, which takes values in the fixed point set of the derivative of the involution $\sigma(a,b) = (b,a)$ for $(a,b) \in G(\mu_1,\mu_2)\times G(\mu_1,\mu_2)$, and the complement $\mathcal{A}_{\mathcal{P}} = \frac{1}{2}(-\alpha,\alpha)$, that is, $\mathcal{A} = \mathcal{A}_{\mathcal{K}} + \mathcal{A}_{\mathcal{P}}$. Moreover we decompose $\mathcal{A}_{\mathcal{P}}$ into its $(1,0)$ and $(0,1)$-parts as $\mathcal{A}_{\mathcal{P}} = \mathcal{A}'_{\mathcal{P}} + \mathcal{A}''_{\mathcal{P}} = \frac{1}{2}(-\alpha',\alpha') + \frac{1}{2}(-\alpha'',\alpha'')$ and define $\mathcal{A}_\lambda$ as

$$\mathcal{A}_\lambda = \lambda^{-1}\mathcal{A}'_{\mathcal{P}} + \mathcal{A}_{\mathcal{K}} + \lambda\mathcal{A}''_{\mathcal{P}}.$$

Since φ is ${}^{(0)}\nabla$-harmonic, by Theorem 5.2, there exists a $\mathcal{F}_\lambda$, which is a solution to the equation $\mathcal{F}_\lambda^{-1}d\mathcal{F}_\lambda = \mathcal{A}_\lambda$. From now on, for convenience, we choose the base point to be $(0,0)$ and there we assume

$$\mathcal{F}_\lambda|_{(z,\bar z)=(0,0)} = (\mathrm{id},\ \mathrm{id}). \tag{6.2}$$

Then, decomposing $\mathcal{F}_\lambda$ by the Birkhoff decomposition Theorem 6.1, we obtain

$$\mathcal{F}_\lambda = \mathcal{F}_-\mathcal{V}_+. \tag{6.3}$$

From Theorem 5.13, we know that $\mathcal{F}_-$ depends only on z and moreover it is meromorphic on $\mathbb{D}$. Then a direct computation shows the following theorem.

Theorem 6.4 (The normalized potentials)**.** *Let $\mathcal{F}_-$ be the loop defined in* (6.3)*. Then $\mathcal{F}_-$ is meromorphic on $\mathbb{D}$ and the pair of normalized potentials*

$$\mathcal{N}_-(z,\lambda) = \mathcal{F}_-(z,\lambda)^{-1}d\mathcal{F}_-(z,\lambda) = \big(\lambda^{-1}\xi(z),\ \ -\lambda^{-1}\xi(z)\big)$$

is determined by

$$\xi(z) = \begin{pmatrix} \mu_1\xi^3(z) & 0 & \xi^1(z) \\ 0 & \mu_2\xi^3(z) & \xi^2(z) \\ 0 & 0 & 0 \end{pmatrix} dz, \tag{6.4}$$

where ξ^1, ξ^2 *and* ξ^3 *are meromorphic functions on* $\mathbb{D}$ *given*

$$\xi^1(z) = -\frac{1}{2} e^{-\frac{1}{2}\mu_1 \varphi^3(z,0)} \varphi^1_z(z,0) + \frac{\mu_1 \varphi^3_z(z,0)}{8\pi i} \int_{\mathbb{D}} \frac{e^{-\frac{1}{2}\mu_1 \varphi^3(\xi,\bar\xi)} \partial_{\bar\xi} \varphi^1(\xi,\bar\xi)}{\xi - z} d\xi \wedge d\bar\xi \Bigg|_{\bar z = 0},$$

$$\xi^2(z) = -\frac{1}{2} e^{-\frac{1}{2}\mu_2 \varphi^3(z,0)} \varphi^2_z(z,0) + \frac{\mu_2 \varphi^3_z(z,0)}{8\pi i} \int_{\mathbb{D}} \frac{e^{-\frac{1}{2}\mu_2 \varphi^3(\xi,\bar\xi)} \partial_{\bar\xi} \varphi^2(\xi,\bar\xi)}{\xi - z} d\xi \wedge d\bar\xi \Bigg|_{\bar z = 0},$$

$$\xi^3(z) = -\frac{1}{2} \varphi^3_z(z,0).$$

Proof. Let F be the extended solution given by $\mathcal{F}_\lambda = (F(\lambda), F(-\lambda))$ and V_+ the plus element given by $\mathcal{V}_+ = (V_+(\lambda), V_+(-\lambda))$. Then the Maurer-Cartan form for $\mathcal{F}_-$ can be computed as

$$\begin{aligned} \mathcal{F}_-^{-1} d\mathcal{F}_- &= \mathcal{V}_+ \mathcal{F}_\lambda^{-1} d\mathcal{F}_\lambda \mathcal{V}_+ - d\mathcal{V}_+ \mathcal{V}_+^{-1} \\ &= \left(\mathrm{Ad}(V_+(\lambda))(\alpha_\lambda) - dV_+(\lambda) V_+(\lambda)^{-1},\ s(\lambda) \right), \end{aligned} \tag{6.5}$$

where the second component $s(\lambda)$ is just the first component with the replacement λ to $-\lambda$. Thus we only consider the first component. Since the left hand side is a meromorphic 1-form on $\mathbb{D}$, the same is true for the right hand side. Using $\alpha_\lambda = \frac{1}{2}(1-\lambda^{-1})\alpha' + \frac{1}{2}(1-\lambda)\alpha''$, we can rephrase the first component of the right hand side of (6.5) as

$$-\frac{\lambda^{-1}}{2} \mathrm{Ad}(V_+)\alpha' + \frac{1}{2} \mathrm{Ad}(V_+)\alpha' - \partial_z V_+ V_+^{-1}\, dz, \quad \text{(the } (1,0)\text{-part)}, \tag{6.6}$$

and

$$-\frac{\lambda}{2} \mathrm{Ad}(V_+)\alpha'' + \frac{1}{2} \mathrm{Ad}(V_+)\alpha'' - \partial_{\bar z} V_+ V_+^{-1}\, d\bar z, \quad \text{(the } (0,1)\text{-part)}. \tag{6.7}$$

Since the normalized potential $\mathcal{F}_- d\mathcal{F}_-$ is the λ^{-1}-term of the $(1,0)$-part, we can compute it by using $V_+ = V_{+0} + V_{+1}\lambda + \cdots$;

$$\mathcal{F}_-^{-1} d\mathcal{F}_- = \left(-\frac{\lambda^{-1}}{2} \mathrm{Ad}(V_{+0})\alpha',\ \frac{\lambda^{-1}}{2} \mathrm{Ad}(V_{+0})\alpha' \right).$$

Set V_+ as

$$V_+(z,\bar z,\lambda) = \begin{pmatrix} e^{\mu_1 v^3_+(z,\bar z,\lambda)} & 0 & v^1_+(z,\bar z,\lambda) \\ 0 & e^{\mu_2 v^3_+(z,\bar z,\lambda)} & v^2_+(z,\bar z,\lambda) \\ 0 & 0 & 1 \end{pmatrix}.$$

Then using the matrix form of α' given in (6.1), we can rephrase (6.6) without the term $-\frac{1}{2}\lambda^{-1} \mathrm{Ad}(V_+)\alpha'$ as

$$\begin{cases} \mu_k(\frac{1}{2}\varphi^3_z - (v^3_+)_z) dz \ \text{ for the } (k,k)\text{-entry } (k=1,2), \\ \left(\frac{1}{2} e^{\mu_k(v^3_+ - \varphi^3)} \varphi^k_z - \mu_k(\frac{1}{2}\varphi^3_z - (v^3_+)_z) v^k_+ - (v^k_+)_z \right) dz \ \text{ for the } (k,3)\text{-entry } (k=1,2), \end{cases}$$

and zero for the other (k,ℓ)-entries. Since $\mathcal{F}_-$ takes values in $\Lambda^-_* \mathcal{G}^{\mathbb{C}}$, thus $\frac{1}{2}\varphi^3_z - (v^3_{+0})_z$ needs to vanish, where $v^3_+ = v^3_{+0} + v^3_{+1}\lambda + v^3_{+2}\lambda^2 + \cdots$. Moreover since $\mathcal{F}_-$ is also a meromorphic function on $\mathbb{D}$, that is, it is independent of $\bar z$. Thus setting $\bar z = 0$ on $\frac{1}{2}\varphi^3_z - (v^3_{+0})_z$, we have $\frac{1}{2}\varphi^3_z(z,0) - (v^3_{+0})_z(z,0) = 0$. Finally, using the initial condition $\varphi^3(0,0) = v^3_{+0}(0,0) = 0$, we obtain

$$v^3_{+0}(z,0) = \frac{1}{2}\varphi^3(z,0). \tag{6.8}$$

We now compute the $(0,1)$-part of (6.5). Then a similar computation shows that using the matrix form of α'' given in (6.1), we can rephrase (6.7) without the term $-\frac{1}{2}\lambda\,\mathrm{Ad}(V_+)\alpha'$ as

$$
(6.9)\quad \begin{cases} \mu_k(\frac{1}{2}\varphi^3_{\bar z} - (v^3_+)_{\bar z})d\bar z \ \text{ for the } (k,k)\text{-entry } (k=1,2), \\ \left(\frac{1}{2}e^{\mu_k(v^3_+-\varphi^3)}\varphi^k_{\bar z} - \mu_k(\frac{1}{2}\varphi^3_{\bar z} - (v^3_+)_{\bar z})v^k_+ - (v^k_+)_{\bar z}\right) d\bar z \ \text{ for the } (k,3)\text{-entry } (k=1,2), \end{cases}
$$

and zero for the other (k,ℓ)-entries. The constant term of the $(1,1)$-entry in (6.9) needs to vanish, that is, $(v^3_{+0})_{\bar z}(z,\bar z) = \frac{1}{2}\varphi^3_{\bar z}(z,\bar z)$. Thus using (6.8), we have

$$
v^3_{+0}(z,\bar z) = \frac{1}{2}\varphi^3(z,\bar z).
$$

Then the constant term of $(1,3)$- and $(2,3)$-entries in (6.9) can be rephrased as

$$
\frac{1}{2}e^{-\frac{1}{2}\mu_k\varphi^3}\varphi^k_{\bar z} - (v^k_{+0})_{\bar z} = 0, \quad (k=1,2).
$$

This equation is the inhomogeneous Cauchy-Riemann equation and the solution can be explicitly given by [37, Theorem 1.2.2]:

$$
v^k_{+0}(z,\bar z) = \frac{1}{4\pi i}\int_{\mathbb{D}} \frac{e^{-\frac{1}{2}\mu_k\varphi^3(\xi,\bar\xi)}\partial_{\bar\xi}\varphi^k(\xi,\bar\xi)}{\xi - z}\, d\xi\wedge d\bar\xi, \quad (k=1,2).
$$

Finally a straightforward computation shows that the normalized potential ξ is given by (6.4). $\square$

We now give the converse procedure.

Step 1: Let us choose a pair of normalized potentials $\mathcal{N}_-(z,\lambda) = (\lambda^{-1}\xi(z), -\lambda^{-1}\xi(z))$ with ξ in (6.4) and consider the following linear ordinary differential equation:

$$
dC = C(\lambda^{-1}\xi), \ \text{ with } \ C(z_*) = \mathrm{id},
$$

where z_* is some base point in some simply-connected domain $\mathbb{D}$ in $\mathbb{C}$. It is easy to see that the solution can be computed explicitly as

$$
C(z,\lambda) = \begin{pmatrix} e^{\lambda^{-1}\mu_1\Xi(z)} & 0 & \lambda^{-1}\int_{z*}^z \xi^1(t)e^{\lambda^{-1}\mu_1\Xi(t)}\,dt \\ 0 & e^{\lambda^{-1}\mu_2\Xi(z)} & \lambda^{-1}\int_{z*}^z \xi^2(t)e^{\lambda^{-1}\mu_2\Xi(t)}\,dt \\ 0 & 0 & 1 \end{pmatrix},
$$

where $\Xi(z) = \int_{z_*}^z \xi^3(t)\,dt$. Let $\mathcal{R}_-$ denote the pair

$$
\mathcal{R}_-(z,\lambda) = (C(z,\lambda), \ \ C(z,-\lambda)) \in \Lambda^-\mathcal{G}^{\mathbb{C}}.
$$

Step 2: The Iwasawa decomposition in Theorem 5.7 for $\mathcal{R}_-$ gives

$$
\mathcal{R}_- = \mathcal{F}\mathcal{W}_+, \ \ (\mathcal{F}\in\Lambda\mathcal{G}_\sigma, \ \mathcal{W}_+\in\Lambda^+\mathcal{G}^{\mathbb{C}}_\sigma),
$$

where $\mathcal{F}(z,\bar z,\lambda) = (F(z,\bar z,\lambda), F(z,\bar z,-\lambda))$ and $\mathcal{W}_+(z,\bar z,\lambda) = (W_+(z,\bar z,\lambda), W_+(z,\bar z,-\lambda))$. Here we define the *real part* of a loop $f(z,\bar z,\lambda)$ by

$$
\mathrm{Re}(f(z,\bar z,\lambda)) = \frac{1}{2}\left(f(z,\bar z,\lambda) + \overline{f(z,\bar z,1/\bar\lambda)}\right),
$$

which is of course real for $\lambda \in \mathbb{S}^1$. Then using Theorem 6.2 the first component $F(z, \bar{z}, \lambda)$ of $\mathcal{F}(z, \bar{z}, \lambda)$ is given explicitly as follows:

$$F(z, \bar{z}, \lambda) = \begin{pmatrix} e^{2\mu_1 \operatorname{Re}(\lambda^{-1}\Xi(z))} & 0 & e^{2\mu_1 \operatorname{Re}(\lambda^{-1}\Xi(z))}\tilde{f}^1(z, \bar{z}, \lambda) \\ 0 & e^{2\mu_2 \operatorname{Re}(\lambda^{-1}\Xi(z))} & e^{2\mu_2 \operatorname{Re}(\lambda^{-1}\Xi(z))}\tilde{f}^2(z, \bar{z}, \lambda) \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tilde{f}^k(z, \bar{z}, \lambda)$, $(k = 1, 2)$, are real parts of the Iwasawa decomposition of the scalar loop, that is,

$$g^k = \tilde{f}^k + g^k_+ \quad \text{for} \quad g^k(z, \bar{z}, \lambda) = \lambda^{-1} e^{-2\mu_k \operatorname{Re}(\lambda^{-1}\Xi(z))} \int_{z_*}^{z} \xi^k(t) e^{\lambda^{-1}\mu_k \Xi(t)}\, dt,$$

where g^k_+ denotes the positive element. Using the expansion $g^k(z, \bar{z}, \lambda) = \sum_{j=-\infty}^{\infty} g^k_j(z, \bar{z})\lambda^j$, $\tilde{f}^k$ and g^k_+ are given by

$$\begin{cases} \tilde{f}^k(z, \bar{z}, \lambda) = \sum_{j<0} \left(g^k_j(z, \bar{z})\lambda^j + \overline{g^k_j(z, \bar{z})}\, \lambda^{-j} \right), \\ g^k_+(z, \bar{z}, \lambda) = \sum_{j \geq 0} g^k_j(z, \bar{z})\lambda^j - \sum_{j<0} \overline{g^k_j(z, \bar{z})}\, \lambda^{-j}. \end{cases}$$

Step 3: Then the extended solution $\hat{F}(z, \bar{z}, \lambda) = F(z, \bar{z}, \lambda) F(z, \bar{z}, 1)^{-1} = (\hat{f}^1, \hat{f}^2, \hat{f}^3)$ is given by

$$\begin{cases} \hat{f}^k(z, \bar{z}, \lambda) & = e^{2\mu_k \operatorname{Re}(\lambda^{-1}\Xi(z))}(\tilde{f}^k(z, \bar{z}, \lambda) - \tilde{f}^k(z, \bar{z}, 1)), \ (k = 1, 2), \\ \hat{f}^3(z, \bar{z}, \lambda) & = 2 \operatorname{Re}((\lambda^{-1} - 1)\Xi(z)). \end{cases}$$

We summarize the discussion above in the following theorem.

Theorem 6.5. *Any ${}^{(0)}\nabla$-harmonic map $\varphi = (\varphi^1, \varphi^2, \varphi^3) : \mathbb{D} \to G(\mu_1, \mu_2)$ can be represented by some meromorphic functions $\xi^k(z)$, $(k = 1, 2, 3)$, as follows:*

$$\varphi^1(z, \bar{z}) = \exp\left(-2\mu_1 \operatorname{Re} \int_{z_*}^{z} \xi^3(t)\, dt \right) \left(\tilde{f}^1(z, \bar{z}, -1) - \tilde{f}^1(z, \bar{z}, 1) \right),$$
$$\varphi^2(z, \bar{z}) = \exp\left(-2\mu_2 \operatorname{Re} \int_{z_*}^{z} \xi^3(t)\, dt \right) \left(\tilde{f}^2(z, \bar{z}, -1) - \tilde{f}^2(z, \bar{z}, 1) \right),$$
$$\varphi^3(z, \bar{z}) = -4 \operatorname{Re} \int_{z_*}^{z} \xi^3(t)\, dt.$$

Here $\tilde{f}^k(z, \bar{z}, \lambda)$, $(k = 1, 2)$, is the real part of the Iwasawa decomposition of the scalar loop $g^k = \tilde{f}^k + g^k_+$ with

$$g^k(z, \bar{z}, \lambda) = \lambda^{-1} \exp\left(-2\mu_k \operatorname{Re}(\lambda^{-1} \int_{z_*}^{z} \xi^3(t)\, dt) \right) \int_{z_*}^{z} \xi^k(t) \exp\left(\lambda^{-1}\mu_k \int_{z_*}^{t} \xi^3(s)\, ds \right) dt.$$

Remark 6.6.

(1) An integral representation formula for conformal maps into $G(\mu_1, \mu_2)$ which are harmonic with respect to the metric was obtained in [38].
(2) In case $\mu_1 = \mu_2 = 0$, this formula reduces to the classical formula of harmonic functions:

$$\varphi^k(z, \bar{z}) = -4 \operatorname{Re} \int_{z_*}^{z} \xi^k(t)\, dt, \quad (k = 1, 2, 3).$$

Let us define 3×3 matrices E_{pq} with (i,j)-entry one if $(i,j)=(p,q)$ and zero otherwise. Let us consider a map $\varphi(x,y)=\exp(xE_{11})\cdot\exp(yE_{22})$. As we have seen before, φ is ${}^{(0)}\nabla$-harmonic. With respect to the left-invariant Riemannian metric $ds^2=e^{-2\mu_1 x^3}(dx^1)^2+e^{-2\mu_2 x3}(dx^2)^2+(dx^3)^2$, φ is a flat surface with constant mean curvature $(\mu_1+\mu_2)/2$. Moreover, one can check that this flat surface is the plane $x^3=$ constant.

(1) If $(\mu_1,\mu_2)=(0,0)$, then φ is a totally geodesic plane in the Euclidean 3-space $\mathbb{E}^3$.
(2) If $\mu_1=0,\mu_2=c\neq 0$, then φ is a cylinder over a horizontal horocycle in $\mathbb{H}^2(-c^2)$.
(3) If $\mu_1=\mu_2=c\neq 0$, then φ is a horosphere in the hyperbolic 3-space $\mathbb{H}^3(-c^2)$.
(4) If $\mu_1=-\mu_2\neq 0$, then φ is a non-totally geodesic minimal surface.

Thus horospheres in $\mathbb{H}^3(-c^2)$ are non-harmonic with respect to the metric, but harmonic with respect to the neutral connection. We can reconstruct these surfaces from normalized potentials with $\xi^3(z)=0$. It is easy to see that $C(z,\lambda)$ and the unitary part of its Iwasawa decomposition F given by $C=FV_+$ with $F\in\Lambda G(\mu_1,\mu_2)$ and $V_+\in\Lambda^+G(\mu_1,\mu_2)^{\mathbb{C}}$ are as follows:

$$C(z,\lambda)=\begin{pmatrix}1&0&\lambda^{-1}\int_{z*}^{z}\xi^1(t)\,dt\\0&1&\lambda^{-1}\int_{z*}^{z}\xi^2(t)\,dt\\0&0&1\end{pmatrix},\ F(z,\bar z,\lambda)=\begin{pmatrix}1&0&2\mathrm{Re}(\lambda^{-1}\int_{z*}^{z}\xi^1(t)\,dt)\\0&1&2\mathrm{Re}(\lambda^{-1}\int_{z*}^{z}\xi^2(t)\,dt)\\0&0&1\end{pmatrix}.$$

Hence we obtain the extended solution

$$\hat F(z,\bar z,\lambda)=F(z,\bar z,\lambda)F(z,\bar z,1)^{-1}=\begin{pmatrix}1&0&2\,\mathrm{Re}\left((\lambda^{-1}-1)\int_{z*}^{z}\xi^1(t)\,dt\right)\\0&1&2\,\mathrm{Re}\left((\lambda^{-1}-1)\int_{z*}^{z}\xi^2(t)\,dt\right)\\0&0&1\end{pmatrix}.$$

For instance, choosing $z_*=0,\xi^1(z)=-1/4$ and $\xi^2(z)=i/4$ we obtain the plane $\varphi(z,\bar z,\lambda=-1)=(x,y,0)$.

Remark 6.7. The generalized Weierstrass type representation of ${}^{(0)}\nabla$-harmonic maps into the solvable Euclidean motion group SE_2 can be given along the same line. Conjugated SE_2 by the element

$$\frac{1}{\sqrt2}\begin{pmatrix}1&\sqrt{-1}&0\\\sqrt{-1}&1&0\\0&0&\sqrt2\end{pmatrix},$$

we obtain the isomorphic solvable group

$$\mathrm{SE}_2=\left\{(x^1,x^2,x^3)=\begin{pmatrix}e^{\sqrt{-1}x^3}&0&\frac{1}{\sqrt2}(x^1+\sqrt{-1}x^2)\\0&e^{-\sqrt{-1}x^3}&\frac{\sqrt{-1}}{\sqrt2}(x^1-\sqrt{-1}x^2)\\0&0&1\end{pmatrix}\right\}=\mathrm{U}_1\ltimes\mathbb{C}.$$

This means that SE_2 is a real form of the complexified solvable Lie group $G(1,-1)^{\mathbb{C}}$. Moreover, the ${}^{(0)}\nabla$-harmonicity of a map $\varphi=(\varphi^1,\varphi^2,\varphi^3)$ can be rephrased as

$$\tilde\varphi^1_{z\bar z}-\frac12(\tilde\varphi^1_z\tilde\varphi^3_{\bar z}+\tilde\varphi^1_{\bar z}\tilde\varphi^3_z)=0,\ \ \tilde\varphi^2_{z\bar z}+\frac12(\tilde\varphi^2_z\tilde\varphi^3_{\bar z}+\tilde\varphi^2_{\bar z}\tilde\varphi^3_z)=0, \tag{6.10}$$

where $\tilde\varphi^1=\frac{1}{\sqrt2}(\varphi^1+\sqrt{-1}\varphi^2)$, $\tilde\varphi^2=\frac{1}{\sqrt2}(\varphi^1-\sqrt{-1}\varphi^2)$ and $\tilde\varphi^3=\sqrt{-1}\varphi^3$. This is the ${}^{(0)}\nabla$-harmonicity equation of the solvable Lie group $G(\mu_1,\mu_2)$ with $\mu_1=-\mu_2=1$ in (4.2). So the generalized Weierstrass type representation of ${}^{(0)}\nabla$-harmonic maps into SE_2 is quite similar to the case of the solvable Lie group $G(1,-1)$.

Remark 6.8. Solvable Lie groups equipped with a left invariant Riemannian metric play an important and fundamental role in many branches of mathematics. A Riemannian manifold is said to be a *solvmanifold* if it has a transitive solvable Lie group of isometries. It has been shown by Heintze [31] that connected homogeneous Riemannian manifolds of strictly negative curvature are solvable Lie groups with left invariant metrics. Moreover such spaces are diffeomorphic to $N \times \mathbb{R}^+$, where N is a simply connected nilpotent Lie group. Azencott and Wilson [4] showed that every connected, simply-connected, homogeneous Riemannian manifold of non-positive curvature admits a solvable Lie group acting simply-transitively by isometries. Any simply connected solvmanifold of non-positive curvature is *standard* in the sense of [4].

Solvable Lie groups with left invariant Riemannian metric provide many examples of homogeneous Einstein manifolds of non-positive curvature, for example, Damek-Ricci Einstein spaces and Boggino-Damek-Ricci Einstein spaces [8, 30]. Lauret proved that any Einstein solvmanifold is standard [48]. Alekseevskii [2] conjectured that for every non flat non-compact homogeneous Einstein manifold G/K, the isotropy subgroup K must be a maximal compact subgroup of G. If this conjecture is true, then the classification of non-compact homogeneous Einstein manifolds is reduced to the investigation of solvable Lie groups with left invariant Einstein metrics. This conjecture is still an open problem. Moreover solvable Lie groups and nilpotent Lie groups play fundamental roles to construct Riemannian manifolds with exceptional holonomy [15, 28] and to solve the Calabi-Yau equations on 4-manifolds [13, 29].

The class of Carnot spaces contains all rank one Riemannian symmetric spaces of non-compact type. According to Pansu [55], a simply-connected solvable Lie group G equipped with a left invariant Riemannian metric $ds^2 = \langle \cdot, \cdot \rangle$ is said to be a *k-step Carnot space* if G is the semidirect product of a nilpotent Lie group N and the abelian group $\mathbb{R}^+$. Then the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{n} = \mathrm{Lie}(N)$ satisfy: $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}\, H$,

$$\mathfrak{n} = \sum_{i=1}^{k} \mathfrak{n}_i, \quad \mathfrak{n}_i = \{X \in \mathfrak{n} \mid \mathrm{ad}(H)X = iX\}, \quad (i = 1, \ldots, k), \quad \langle H, H \rangle = 1, \quad H \perp \mathfrak{n}$$

and (G, ds^2) is of negative curvature [55]. For example, real hyperbolic space is a 1-step Carnot space and the other rank 1 symmetric spaces of non-compact type are 2-step Carnot spaces. Since harmonic maps into Riemannian symmetric spaces of non-compact type can be investigated by the generalized Weierstrass type representation [25], it seems to be interesting to study the generalized Weierstrass type representation for *non-symmetric* Carnot spaces.

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