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Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
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Sapporo, 2018

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Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 21 through August 23 in 2018 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 40 years ago. Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei (Hokkaido University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
H. Kubo (Hokkaido University)
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T. Sakajo (Kyoto University)
K. Tsutaya (Hirosaki University)

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The 43rd Sapporo Symposium on Partial Differential Equations

第 43 回偏微分方程式論札幌シンポジウム

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Venue	7-310, 7-219/220, Faculty of Science Bld. No. 7, Hokkaido University
Organizers	Hideo Kubo, Hirotoshi Kuroda
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Hirotoshi Kuroda, Tohru Ozawa, Takashi Sakajo, Kimitoshi Tsutaya
URL	http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program180821.html

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Decay of Surface Waves of Navier-Stokes Equations.

Thomas Beale *), Takaaki Nishida **) and Yoshiaki Teramoto ***)
Duke University, Kyoto University, Setsunan University

1 Formulation

We consider free surface problems of viscous fluids in the horizontal strip domain $\Omega(t) = \{(x, y)\ ;\ \ x \in \mathbb{R}^2\ , -b < y < \eta(x, t)\}$, where $b > 0$ is the mean depth and the unknown free surface is given by $y = \eta(x,t)$, $x \in \mathbb{R}^2$, $t > 0$.
We denote the velocity field by $\boldsymbol{u}(x, y, t)$ and the perturbation of pressure from the static pressure by $p(x, y, t)$. Then the equations of motion of fluid is written as follows:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}, \nabla)\boldsymbol{u} - \nu\, \Delta \boldsymbol{u} + \nabla p = 0\ ,$$
$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in}\ \ \Omega(t)\ ,\ t > 0\ .$$

For the free surface we impose the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = u_3 - u_1 \frac{\partial \eta}{\partial x_1} - u_2 \frac{\partial \eta}{\partial x_2} \qquad \text{on} \quad y = \eta(x, t)\ ,\ x \in \mathbb{R}^2\ ,\ t > 0\ .$$

The balance of stress tensor at the free surface becomes

$$(-g\,\eta(x, t) + p)\ n_j - \nu \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right) n_k + \sigma \nabla_F \left(\left(1 + |\nabla_F \eta|^2\right)^{-\frac{1}{2}}\ \nabla_F \eta\right) n_j = 0 \quad \text{on} \quad y = \eta(x, t)\ ,$$

where g is the acceleration of gravity, ν is the viscosity, $\boldsymbol{n}$ is the outward unit normal to the free surface and σ is the coefficient of surface tension, and $\nabla_F = \left(\frac{\partial}{\partial x_1}\ ,\ \frac{\partial}{\partial x_2}\right)$ denotes the horizontal gradient. The condition on the bottom is

$$\boldsymbol{u} = 0 \quad \text{on} \quad y = -b\ .$$

A solution is uniquely determined by specifying the initial data

$$\eta(x, 0) = \eta_0(x)\ ,\ \boldsymbol{u}(x, y, 0) = \boldsymbol{u}_0(x, y)\ \ ,\ \ x \in \mathbb{R}^2\ \ -b < y < \eta_0(x)$$

subject to certain compatibility conditions and appropriate not large initial data.
Solonnikov and Scadilov (1973), Solonnikov (1978, 1990), Beale (1980, 1984), Tani (1996), Tanaka and Tani (1995), Hataya (2009).

In order to consider the decay of solutions for the system above, we transform the system for $\boldsymbol{u}$ in $\Omega(t) = \{(x,\, y)\ ;\ \ x \in \mathbb{R}^2\ , -b < y < \eta(x,\, t)\}$ to the system for $\boldsymbol{v}$ in the fixed domain $\Omega = \{(x,\, y)\ ;\ \ x \in \mathbb{R}^2\ ,\ -b < y < 0\ \}$ by Bok-Beale (Beale 1980) transformation, which conserves the divergence free condition. The system becomes the following.

$$\frac{\partial \eta}{\partial t} - v_3 = 0 \qquad \text{on} \quad S_F\ ,\ t > 0\ , \tag{1.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} - \nu\, \Delta \boldsymbol{v} + \nabla q = \boldsymbol{F}(\eta,\, \boldsymbol{v},\, \nabla q)\ , \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \ \ \Omega\ ,\ t > 0\ . \tag{1.2}$$

The conditions on the upper boundary $S_F = \left\{(x,0) \in \overline{\Omega}\ ;\ x \in \mathbb{R}^2\right\}$ are written as follows

$$\frac{\partial v_j}{\partial x_3} + \frac{\partial v_3}{\partial x_j} = g_j(\eta,\ \boldsymbol{v})\ , \quad j = 1, 2\ , \tag{1.3}$$

$$q - 2\nu \frac{\partial v_3}{\partial x_3} - g\,\eta + \sigma \Delta_F \eta = g_3(\eta,\, \boldsymbol{v}) \qquad \text{on}\ S_F\ ,\ t > 0 \tag{1.4}$$

The boundary condition on the bottom $S_B = \left\{(x, -b) \in \overline{\Omega}\ ;\ x \in \mathbb{R}^2\right\}$ is

$$\boldsymbol{v} = 0 \qquad \text{on} \quad S_B\ ,\ t > 0\ . \tag{1.5}$$

Here the linear terms are written on the left hand side of the equations and the all nonlinear terms are written on the right hand side, which are lengthy and not explicitly given here. (See Nishida-Teramoto-Yoshihara 2002.)

2 Linearized Equation

Now we consider the decay of the linear solutions under the conditions
(i) $g > 0$ and $\sigma > 0$ and (ii) $g = 0\ ,\ \sigma > 0$.
Since we consider the decay and global in time small solutions and the decrease of nonlinear terms are faster than the linear terms, the decreasing and global in time solutions for the nonlinear system can be obtained for the small initial data.
(See Beale and Nishida, 1985)
Thus we focus on the decay of solutions for the linear system here.

Remark The case of $g > 0$ and $\sigma = 0$ has different decay properties and needs a different treatment. (Hataya 2009)

Resolvent equation for the case of $g > 0$ and $\sigma > 0$.

$$\lambda\eta - v_3 = h \qquad \text{on} \quad S_F , \tag{2.6}$$

$$\lambda \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{F} , \tag{2.7}$$

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in} \quad \Omega , \tag{2.8}$$

$$\frac{\partial v_j}{\partial x_3} + \frac{\partial v_3}{\partial x_j} = g_j , \quad j = 1,2 , \tag{2.9}$$

$$p - 2\nu \frac{\partial v_3}{\partial x_3} - g\eta + \sigma \Delta_F \eta = g_3 \quad \text{on} \quad S_F , \tag{2.10}$$

$$\boldsymbol{v} = 0 \qquad \text{on} \quad S_B . \tag{2.11}$$

Let P be the projection on the subspace of solenoidal vectors orthogonal to the subspace $\mathcal{G}^0 = \{ \nabla\phi : \phi \in H^1(\Omega) , \ \phi = 0 \ \ on \ \ S_F \}$ of $H^0(\Omega) = L^2(\Omega)$, i.e., $H^0 = P H^0 \oplus \mathcal{G}^0$. $P H^1 = \{ \boldsymbol{v} \in H^1(\Omega) : \operatorname{div} \boldsymbol{v} = 0 , \ v_3|_{S_B} = 0 \ \}$.
After absorbing the terms $g_j , j = 1, 2$ in the term $\boldsymbol{F}$ we apply the projection P to (2.7)(2.8) and we have the equation : $\lambda \boldsymbol{v} - \nu P \Delta \boldsymbol{v} + P \nabla p = P \boldsymbol{F}$, where $P \nabla p = \nabla \pi_1 + \nabla \pi_2 + \nabla \pi_3$ is given by the following.

$$\pi_1 = 2\nu \partial v_3/\partial x_3 , \quad \pi_2 = g\eta - \sigma \Delta_F \eta , \quad \pi_3 = g_3 \ \ on \ \ S_F ,$$
$$\Delta \pi_j = 0 \quad in \ \ \Omega , \quad \partial \pi_j/\partial y = 0 \ \ on \ \ S_B \ \ j = 1,2,3 .$$

We denote

$$A\boldsymbol{v} = -\nu P \Delta \boldsymbol{v} + \nabla \pi_1 , \quad R\boldsymbol{v} = v_3|_{S_F} , \quad R^* ((g - \sigma \Delta_F) \eta) = \nabla \pi_2 .$$

Using these notations the resolvent equation has the form :

$$\lambda\eta - R\boldsymbol{v} = h , \tag{2.12}$$

$$\lambda \boldsymbol{v} + A\boldsymbol{v} + R^* ((g - \sigma \Delta_F) \eta) = \boldsymbol{f} , \tag{2.13}$$

where the domain is $\eta \in H^{5/2}(S_F)$ and $\boldsymbol{v} \in W^2(\Omega)$,

$$W^2(\Omega) = \{ \boldsymbol{v} \in H^2(\Omega) : \nabla \cdot \boldsymbol{v} = 0 \ \ in \ \ \Omega , \ \boldsymbol{v} = 0 \ \ on \ \ S_B ,$$
$$\partial v_j/\partial x_3 + \partial v_3/\partial x_j = 0 , \ j = 1,2 \ \ on \ \ S_F , \ \ R\boldsymbol{v} \in H^{5/2}(S_F) \} .$$

If we eliminate η for the system in the case of $h = 0$, we have the equation

$$\lambda \boldsymbol{v} + A \boldsymbol{v} + \lambda^{-1} B \boldsymbol{v} = \boldsymbol{f} , \qquad where \ \ B = R^* ((g - \sigma \Delta_F) R \boldsymbol{v}) . \tag{2.14}$$

By Theorem 1 of Beale (1984), if $\boldsymbol{f} \in P H^0$ and λ away from zero satisfies $0 < R_e \lambda \le \tau_0$, then $M(\lambda) \equiv \lambda \boldsymbol{v} + A \boldsymbol{v} + \lambda^{-1} B \boldsymbol{v} : W^2 \to P H^0$ is invertible and satisfies

$$|| M(\lambda)^{-1} \boldsymbol{f} ||_2 + | \lambda | \, || M(\lambda)^{-1} \boldsymbol{f} ||_0 \le C || \boldsymbol{f} ||_0 ,$$
$$|| \lambda^{-1} R M(\lambda)^{-1} \boldsymbol{f} ||_{5/2, S_F} \le C || \boldsymbol{f} ||_0 .$$

This can be extended to the left half plane.

Theorem

For any $\omega_0 > 0$ there exists $c_0 > 0$ such that the resolvent (2.14) extends for λ with $\lambda = \tau + i\omega$, $|\omega| > \omega_0$, $-c_0|\omega| < \tau < \tau_0$, $\tau_0 > 0$, as follows.

$$||\boldsymbol{v}||_2 + |\lambda|\,||\boldsymbol{v}||_0 + ||\lambda^{-1} R\boldsymbol{v}||_{5/2,S_F} \leq C\,||\boldsymbol{f}||_0 . \tag{2.15}$$

Extension to Full Resolvent

Consider equations (2.12), (2.13) with domain conditions $\eta \in H^{5/2}(S_F)$, $\boldsymbol{v} \in W^2(\Omega)$.

$$(\lambda - G)\begin{pmatrix}\eta\\ \boldsymbol{v}\end{pmatrix} = \begin{pmatrix}h\\ \boldsymbol{f}\end{pmatrix}, \quad \text{where} \quad G\begin{pmatrix}\eta\\ \boldsymbol{v}\end{pmatrix} = \begin{pmatrix}0 & R\boldsymbol{v}\\ -R^*(g - \beta\Delta)\eta & A\boldsymbol{v}\end{pmatrix}.$$

Recall R^* is bounded $H^s(S_F) \mapsto PH^{s-1/2}(\Omega)$. If $\boldsymbol{v} \in W^2$, $\eta \in H^{5/2}$, then $\boldsymbol{f} \in PH^0$, $h \in H^{5/2}$.

Full Resolvent for λ away from zero.

Let $\lambda \in \{\lambda = \tau + i\omega\,;\, -c_0|\omega| < \tau < \tau_0\,,\, \tau_0 > 0\} \setminus \{|\lambda| \leq \omega_0,\, \omega_0 > 0\}$.
After we eliminate η in (2.13) by (2.12), we can use Theorem to have the following.

$$||\boldsymbol{v}||_2 + |\lambda|\,||\boldsymbol{v}||_0 + ||\lambda^{-1} R\boldsymbol{v}||_{5/2} \leq C(||\boldsymbol{f}||_0 + ||h||_{5/2}),$$
$$||\eta||_{5/2} \leq C(||\boldsymbol{f}||_0 + ||h||_{5/2}), \quad |\lambda|\,||\eta||_{3/2} \leq C(||\boldsymbol{f}||_0 + ||h||_{5/2}).$$

Full Resolvent near $\lambda = 0$, ξ away from 0.

Consider problem (2.12) and (2.13) for λ near 0 with the Fourier transform in x $\hat{h}(\xi)$, $\hat{\boldsymbol{f}}(\xi, y)$ supported in $|\xi| > \xi_0 > 0$. Let

$$V^2(\Omega) = \{\boldsymbol{v} \in H^2(\Omega) : \nabla\cdot\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} = 0 \text{ on } S_B,$$
$$\partial v_j/\partial x_3 + \partial v_3/\partial x_j = 0,\ j = 1, 2 \text{ on } S_F,\ R\boldsymbol{v} \in H^{3/2}(S_F)\}.$$

$(\eta, \boldsymbol{v}) \mapsto (h, \boldsymbol{f})$ is bounded operator with $\eta \in H^{5/2}$, $\boldsymbol{v} \in V^2$ to $h \in H^{3/2}$, $\boldsymbol{f} \in PH^0$. If the inverse exists for $\lambda = 0$, it also exists for λ sufficiently small. If we remind lemmas 3.2, 3.3 (Beale 1984), we have the following.

For any $\xi_0 > 0$ there exists $r_0 > 0$ such that if $\lambda \in \{|\lambda| < r_0\}$ and the supports of $\hat{h}(\xi)$, $\hat{\boldsymbol{f}}(\xi, y)$ belong to $\{|\xi| \geq \xi_0\}$, then the resolvent equation has the solution η, $\boldsymbol{v}$ satisfying

$$||\boldsymbol{v}||_2 \leq C(||\boldsymbol{f}||_0 + ||h||_{3/2}),$$
$$||\eta||_{5/2,S_F} \leq C(||\boldsymbol{f}||_0 + ||h||_{3/2}).$$

Spectrum for ξ small, λ in a bounded set.

Let $\hat{G}(\xi)$ be the Fourier transform of G with respect to x .

There exist ξ_1 *and* r_1, r_2 $(\nu(\pi/(2b))^2 > r_2 > r_1 > 0)$ *such that* $(\lambda - \hat{G}(\xi))^{-1}$ *exists for* $\lambda \in \Lambda = \{r_1 < |\lambda| < r_2\}$, $|\xi| < \xi_1$. *Let* Γ *be a circle about* 0 *in* Λ .

$$\tilde{P}_r(\xi) = \frac{1}{2\pi i} \int_\Gamma (\lambda - \tilde{G}(\xi))^{-1} \, d\lambda$$

is a projection to the one-dimensional eigenspace depending analytically on ξ .

For $h(x), \boldsymbol{f}(x, y)$ with $\hat{h}, \hat{\boldsymbol{f}}$ supported in $|\xi| \le \xi_1$, we have for $\lambda \in \Lambda$

$$\begin{aligned} &(\lambda - G)^{-1}(h, \boldsymbol{f}) = (\eta, \boldsymbol{v}) \qquad \text{with} \\ &(\hat{\eta}(\xi), \hat{\boldsymbol{v}}(\xi, \cdot)) = \{\lambda - \hat{G}(\xi)\}^{-1} (\hat{h}(\xi), \hat{\boldsymbol{f}}(\xi, \cdot)) . \end{aligned} \tag{2.16}$$

With ξ fixed and $\boldsymbol{v} = (v_1, v_2, w)$ ($\hat{}$ is omitted) it holds (2.16) if and only if for some p

$$\begin{aligned} \lambda \eta - w &= h , \\ \lambda v_j - \nu(-|\xi|^2 + D_y^2) v_j + i\xi_j p &= f_j , \quad (j = 1, 2) , \\ \lambda w - \nu(-|\xi|^2 + D_y^2) w + p_{,y} &= f_3 , \\ i\xi_1 v_1 + i\xi_2 v_2 + w_{,y} &= 0 , \\ v_{j,y} + i\xi_j w &= s_j , \quad (j = 1, 2) , \\ p - 2\nu w_{,y} - (g + \sigma|\xi|^2)\eta &= s_3 \quad \text{on } S_F , \\ v_j = w &= 0 \quad \text{on } S_B , \qquad \text{with } s_j = 0 . \end{aligned}$$

The eigenvector with $\xi = 0$ for $\lambda = 0$ is $\eta = 1, \boldsymbol{v} = \boldsymbol{0}, p = g$ and the eigenspace is one-dimensional. There are no additional eigenvectors as long as $|\lambda| < \nu(\pi/(2b))^2$. The eigenvalue and eigenvectors as functions of ξ_1, ξ_2 are given by the following.

$$\begin{aligned} \lambda &= -\frac{g b^3}{3\nu} \xi_1^2 - \frac{g b^3}{3\nu} \xi_2^2 + \mathcal{O}(|\xi|^4) , \\ \eta &= 1 + b^2 \xi_1^2 + b^2 \xi_2^2 + \mathcal{O}(|\xi|^4) , \\ p &= g + (\frac{g}{2} z^2 + \sigma) \xi_1^2 + (\frac{g}{2} z^2 + \sigma) \xi_2^2 + \mathcal{O}(|\xi|^4) , \\ v_1 &= i \frac{g}{2\nu} (z^2 - b^2) \xi_1 + \mathcal{O}(|\xi|^3) \\ v_2 &= i \frac{g}{2\nu} (z^2 - b^2) \xi_2 + \mathcal{O}(|\xi|^3) \\ w &= \left(\frac{g}{6\nu} z^3 - \frac{g}{2\nu} b^2 z - \frac{g}{3\nu} b^3 \right) \xi_1^2 + \left(\frac{g}{6\nu} z^3 - \frac{g}{2\nu} b^2 z - \frac{g}{3\nu} b^3 \right) \xi_2^2 \\ &\quad + \mathcal{O}(|\xi|^4) . \end{aligned}$$

3 Rates of Decay for Linear Problem

Theorem

Given $\boldsymbol{u}_0 = (\eta_0, \boldsymbol{v}_0)$ *with* $\eta_0 \in H^{5/2}(S_F) \cap L^1(S_F)$ *and* $\boldsymbol{v}_0 \in P\, H^0(\Omega)$ *, then*

$$\boldsymbol{u}(t) = (\eta(t), \boldsymbol{v}(t)) = e^{G\,t}\,\boldsymbol{u}_0$$

satisfies for $t \geq T_1 > 0$, $0 \leq \alpha \leq 5/2$,

$$\begin{aligned} \| \partial_x^\alpha \eta(t) \|_{L^2(S_F)} &\leq C\, t^{-(1+\alpha)/2} (\| \eta_0 \|_{H^{5/2} \cap L1} + \| \boldsymbol{v}_0 \|_{H^0}) \\ \| \boldsymbol{v}(t) \|_{H^2(\Omega)} &\leq C\, t^{-1} (\| \eta_0 \|_{H^{5/2} \cap L^1} + \| \boldsymbol{v}_0 \|_{H^0}) \end{aligned}$$

Since from the resolvent estimates we have

$$\| (\lambda - G)^{-1} \boldsymbol{u}_0 \|_{H^1(S_F) \oplus PH^0(\Omega)} \leq C\, |\lambda|^{-1} \| \boldsymbol{u}_0 \|$$

$$|| (\lambda - G)^{-1} \boldsymbol{u}_0 ||_{H^{5/2} \oplus H^2(\Omega)} \leq C \| \boldsymbol{u}_0 \| , \qquad \| \boldsymbol{u}_0 \| = \| \eta \|_{5/2} + \| \boldsymbol{v}_0 \|_0 ,$$

in particular, $(\lambda - G)^{-1} \boldsymbol{u}_0$ with $R_e\, \lambda = \tau > 0$ is in $L^2(\mathbf{R}; \mathcal{H})$, we have

$$\boldsymbol{u}(t) = \lim_{\omega_* \to \infty} \frac{1}{2\pi i} \int_{\tau - i\omega_*}^{\tau + i\omega_*} e^{\lambda t} (\lambda - G)^{-1} \boldsymbol{u}_0 \, d\lambda$$

with limit in $L^2(T_1, T_2; \mathcal{H})$, uniformly in t for $T_1 \leq t \leq T_2$.
Now we can move the contour to $\gamma_-, \gamma_1, \gamma_2, \gamma_3, \gamma_+$, where

$$\begin{aligned} \gamma_\pm &= \{\lambda = \tau \pm i\omega \; : \; \tau = -c_0 |\omega| \, , \; |\omega| \geq \omega_0 \quad \textit{some } \omega_0 , c_0 > 0 \} \, . \\ \gamma_1 &= \{\lambda = \tau + i\omega \; : \; c_0\omega \leq \tau \leq \tau_0 , \; \omega = -\omega_0 \, , \; \tau_0 > 0\} \, , \\ \gamma_2 &= \{\lambda = \tau + i\omega \; : \; \tau = \tau_0 \, , \; -\omega_0 \leq \omega \leq \omega_0 \} \, , \\ \gamma_3 &= \{\lambda = \tau + i\omega \; : \; \tau_0 \geq \tau \geq -c_0\omega_0 \, , \; \omega = \omega_0 \} \, , \\ \gamma_4 &= \{\lambda = \tau + i\omega \; : \; \tau = -c_0\omega_0 \, , \; -\omega_0 \leq \omega \leq \omega_0 \} \, . \end{aligned}$$

The integral on the γ_- , γ_4 , γ_+ gives the exponential decay estimate. The integral on the $\gamma_1, \gamma_2, \gamma_3, -\gamma_4$ gives the argebraic decay estimate.

The case of $g = 0$, $\sigma > 0$ can be treated in a similar way, but the decreasing rate is a little different. It is mentioned at the symposium.

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Singular limit problem for the Navier–Stokes equations in a curved thin domain

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1 Introduction

We consider the three-dimensional incompressible Navier–Stokes equations in a bounded curved thin domain with Navier's slip boundary conditions

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon - \nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ \operatorname{div} u^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ u^\varepsilon \cdot n_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ 2\nu[D(u^\varepsilon)n_\varepsilon]_{\tan} + \gamma_\varepsilon u^\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon. \end{cases} \tag{1.1}$$

Here Ω_ε is a curved thin domain in $\mathbb{R}^3$ with very small width of order $\varepsilon \in (0,1)$ around a given closed two-dimensional surface Γ and Γ_ε is the boundary of Ω_ε. Also, $\nu > 0$ is the viscosity coefficient, n_ε is the unit outward normal vector to Γ_ε, $D(u^\varepsilon) := \{\nabla u^\varepsilon + (\nabla u^\varepsilon)^T\}/2$ is the strain rate tensor, $[D(u^\varepsilon)n_\varepsilon]_{\tan}$ is the tangential component of the stress vector $D(u^\varepsilon)n_\varepsilon$ on Γ_ε, and $\gamma_\varepsilon \geq 0$ is the friction coefficient.

In the study of the Navier–Stokes equations in thin domains we expect to show the global existence of a strong solution for large data, since a three-dimensional thin domain with very small width can be considered almost two-dimensional. It is also important to study a singular limit problem as a thin domain degenerates into a lower dimensional set, in which we are concerned with derivation of limit equations on the degenerate set and comparison of the original and limit equations. Raugel and Sell [6] first studied the Navier–Stokes equations in a flat thin product domain $\Omega_\varepsilon = (0,1)^2 \times (0,\varepsilon)$ with purely periodic or mixed Dirichlet-periodic boundary conditions and established the global existence of a strong solution. Later, Temam and Ziane [7] generalized the results in [6] and studied a singular limit problem in the case of a flat thin domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$ with a domain ω in $\mathbb{R}^2$ and other boundary conditions, which are the combinations of the Dirichlet, periodic, and free boundary conditions. Iftimie, Raugel, and Sell [3], Hoang [1], and Hoang and Sell [2] considered the Navier–Stokes equations in a flat thin domain with nonflat boundaries

$$\Omega_\varepsilon = \{x = (x', x_3) \mid x' \in (0,1)^2,\ \varepsilon g_0(x') < x_3 < \varepsilon g_1(x')\}, \quad g_0, g_1 \colon (0,1)^2 \to \mathbb{R}$$

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with periodic boundary conditions on the lateral boundaries and the (perfect or partial) slip boundary conditions on the top and bottom boundaries. Besides the above cited papers, there is a large amount of literature on the Naiver–Stokes equations in flat thin domains around a two-dimensional domain. Also, Temam and Ziane [8] studied the Navier–Stokes equations in a thin spherical shell

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < a + a\varepsilon\}, \quad a > 0$$

with free boundary conditions and established the global existence of a strong solution as well as the convergence of the average in the thin direction of a solution to the bulk equations towards a solution to limit equations on a sphere. However, there is no literature on the Navier–Stokes equations in a curved thin domain degenerating into a general closed surface. (We refer to [5] for the study on a reaction-diffusion equation in a curved thin domain around a lower dimensional manifold.)

In this talk, we deal with a curved thin domain around a given closed surface whose boundaries are parametrized by functions on the surface. We show the global existence of a strong solution to (1.1) and derive limit equations on the degenerate surface by showing the convergence of the average of a solution to (1.1) and characterizing the limit as a solution to the limit equations.

2 Notations on a surface and a thin domain

In this section we fix several notations on a closed surface and a curved thin domain. Let Γ be a two-dimensional closed (i.e. compact and without boundary), connected, oriented, and smooth surface in $\mathbb{R}^3$. We denote by $n = (n_1, n_2, n_3)$ the unit outward normal vector field of Γ and set $P = (P_{ij})_{i,j} := I_3 - n \otimes n$, where $n \otimes n := (n_i n_j)_{ij}$ is the tensor product of n with itself. The matrix P is the orthogonal projection onto the tangent plane of Γ. We define the tangential gradient and tangential derivatives of $\eta \in C^1(\Gamma)$ by

$$\nabla_\Gamma \eta := P \nabla \tilde{\eta}, \quad \underline{D}_i \eta := \sum_{j=1}^{3} P_{ij} \partial_j \tilde{\eta} \quad \text{on} \quad \Gamma,\ i = 1, 2, 3$$

so that $\nabla_\Gamma \eta = (\underline{D}_1 \eta, \underline{D}_2 \eta, \underline{D}_3 \eta)$. Here $\tilde{\eta}$ is an extension of η to an open neighborhood of Γ in $\mathbb{R}^3$ such that $\tilde{\eta}|_\Gamma = \eta$. For a vector field $v = (v_1, v_2, v_3) \in C^1(\Gamma)^3$ we define the tangential gradient matrix and the surface divergence of v as

$$\nabla_\Gamma v := \begin{pmatrix} \underline{D}_1 v_1 & \underline{D}_1 v_2 & \underline{D}_1 v_3 \\ \underline{D}_2 v_1 & \underline{D}_2 v_2 & \underline{D}_2 v_3 \\ \underline{D}_3 v_1 & \underline{D}_3 v_2 & \underline{D}_3 v_3 \end{pmatrix}, \quad \mathrm{div}_\Gamma v := \sum_{i=1}^{3} \underline{D}_i v_i \quad \text{on} \quad \Gamma.$$

Also, for a matrix-valued function

$$A = (A_{ij})_{i,j} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in C^1(\Gamma)^{3\times 3}$$

we denote by $\mathrm{div}_\Gamma A$ a vector field on Γ whose j-th component is given by

$$[\mathrm{div}_\Gamma A]_j := \sum_{i=1}^{3} \underline{D}_i A_{ij} \quad \text{on} \quad \Gamma,\ j = 1, 2, 3.$$

For $\eta, \xi \in C^1(\Gamma)$ we have an integration by parts formula

$$\int_\Gamma (\eta \underline{D}_i \xi + \xi \underline{D}_i \eta)\, d\mathcal{H}^2 = -\int_\Gamma \eta \xi H n_i \, d\mathcal{H}^2, \quad i = 1, 2, 3,$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure and $H := -\mathrm{div}_\Gamma n$ is (twice) the mean curvature of Γ. Based on this formula we define the weak tangential derivatives of a function in $L^2(\Gamma)$ and the Sobolev spaces $H^m(\Gamma)$, $m \in \mathbb{N}$ as in the case of a flat domain. We also define the Sobolev spaces of tangential vector fields on Γ

$$H^m(\Gamma, T\Gamma) := \{v \in H^m(\Gamma)^3 \mid v \cdot n = 0 \text{ on } \Gamma\}, \quad m \geq 0 \quad (H^0 = L^2)$$

and denote by $H^{-1}(\Gamma, T\Gamma)$ for the dual of $H^1(\Gamma, T\Gamma)$ (via the $L^2(\Gamma)$-inner product). For $v \in H^1(\Gamma)^3$ we define the surface strain rate tensor

$$D_\Gamma(v) := P \left(\frac{\nabla_\Gamma v + (\nabla_\Gamma v)^T}{2} \right) P \quad \text{on} \quad \Gamma$$

and set $\mathcal{K}(\Gamma) := \{v \in H^1(\Gamma, T\Gamma) \mid D_\Gamma(v) = 0 \text{ on } \Gamma\}$. A vector field $v \in \mathcal{K}(\Gamma)$ satisfies

$$\overline{\nabla}_X v \cdot Y + X \cdot \overline{\nabla}_Y v = 0 \quad \text{on} \quad \Gamma$$

for all tangential vector fields X and Y on Γ, where $\overline{\nabla}_X v := P(X \cdot \nabla_\Gamma) v$ is the covariant derivative of v along X. Such a vector field generates a one-parameter group of isometries of Γ and is called a Killing vector field (see e.g. [4]).

Next we introduce notations on a curved thin domain. Let g_0 and g_1 be smooth functions on Γ such that $g := g_1 - g_0 \geq c$ on Γ with some constant $c > 0$. For sufficiently small $\varepsilon > 0$ we define a curved thin domain in $\mathbb{R}^3$ by

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma,\, \varepsilon g_0(y) < r < \varepsilon g_1(y)\}.$$

By Γ_ε we denote the boundary of Ω_ε with unit outward normal vector field n_ε. It is the union of the inner and outer boundaries Γ_ε^0 and Γ_ε^1 given by

$$\Gamma_\varepsilon^i := \{y + \varepsilon g_i(y) n(y) \mid y \in \Gamma\}, \quad i = 0, 1.$$

We assume that the friction coefficient γ_ε appearing in the boundary conditions of (1.1) takes different values on the inner and outer boundaries, i.e.

$$\gamma_\varepsilon := \gamma_\varepsilon^i \quad \text{on} \quad \Gamma_\varepsilon^i,\, i = 0, 1,$$

where γ_ε^0 and γ_ε^1 are nonnegative constants. Let

$$L^2_\sigma(\Omega_\varepsilon) = \{u \in L^2(\Omega_\varepsilon)^3 \mid \mathrm{div}\, u = 0 \text{ in } \Omega_\varepsilon,\, u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$$

be the solenoidal space on Ω_ε. We denote by $\mathbb{P}_\varepsilon$ be the Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$. Also, let A_ε be the Stokes operator on $L^2_\sigma(\Omega_\varepsilon)$ associated with slip boundary conditions and $D(A_\varepsilon)$ be its domain. For a function φ on Ω_ε we define the average of φ in the normal direction of Γ (i.e. the thin direction) by

$$M\varphi(y) := \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y))\, dr, \quad y \in \Gamma.$$

We also write $M_\tau u := PMu$ for the tangential average of a vector field u on Ω_ε.

3 Main results and the idea for the proof

To establish our main results, we need to show that the bilinear form corresponding to the Stokes problem on Ω_ε with slip boundary conditions is uniformly in ε bounded and coercive on $V_\varepsilon := L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$. For this purpose, we make the following assumptions on the friction coefficients γ_ε^0, γ_ε^1 and the degenerate surface Γ.

Assumption 1. There exists a constant $c > 0$ such that $\gamma_\varepsilon^i \leq c\varepsilon$ for all $\varepsilon > 0$ and $i = 0, 1$.

Assumption 2. Either of the following conditions is satisfied:

(A1) There exists a constant $c > 0$ such that $\max_{i=0,1} \gamma_\varepsilon^i \geq c\varepsilon$ for all $\varepsilon > 0$.

(A2) The function space $\mathcal{K}_g(\Gamma) := \{v \in \mathcal{K}(\Gamma) \mid v \cdot \nabla_\Gamma g = 0 \text{ on } \Gamma\}$ contains only a trivial vector field, i.e. $\mathcal{K}_g(\Gamma) = \{0\}$.

Note that the condition (A2) is necessary in the case of the perfect slip boundary conditions, i.e. $\gamma_\varepsilon^0 = \gamma_\varepsilon^1 = 0$. Also, $V_\varepsilon = D(A_\varepsilon^{1/2})$ and the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2}u$ is uniformly equivalent in ε to the canonical $H^1(\Omega_\varepsilon)$-norm of $u \in V_\varepsilon$.

Now let us state our main results. First we give the global existence of a strong solution for large data.

Theorem 3.1. *Under Assumptions 1 and 2 there exist constants $\varepsilon_0, c_0 \in (0,1)$ such that for $\varepsilon \in (0, \varepsilon_0)$ if $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ satisfy*

$$\|A_\varepsilon^{1/2}u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))} + \|M_\tau u_0^\varepsilon\|^2_{L^2(\Gamma)} + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))} \leq c_0\varepsilon^{-1},$$

then the Navier–Stokes equations (1.1) *admits a global-in-time strong solution*

$$u^\varepsilon \in C([0,\infty); V_\varepsilon) \cap L^2_{loc}([0,\infty); D(A_\varepsilon)) \cap H^1_{loc}([0,\infty); L^2_\sigma(\Omega_\varepsilon)).$$

We also derive several estimates for a strong solution in terms of ε.

Theorem 3.2. *Let $c_1, c_2, \alpha, \beta > 0$. Under Assumptions 1 and 2, there exists $\varepsilon_1 \in (0,1)$ such that for $\varepsilon \in (0, \varepsilon_1)$ if $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ satisfy*

$$\|A_\varepsilon^{1/2}u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))} \leq c_1\varepsilon^{-1+\alpha},$$
$$\|M_\tau u_0^\varepsilon\|^2_{L^2(\Gamma)} + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))} \leq c_2\varepsilon^{-1+\beta},$$

then there exists a global strong solution u^ε to (1.1) *satisfying*

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \leq c(\varepsilon^{1+\alpha} + \varepsilon^\beta), \quad \int_0^t \|u^\varepsilon(s)\|^2_{H^1(\Omega_\varepsilon)}\,ds \leq c(\varepsilon^{1+\alpha} + \varepsilon^\beta)(1+t),$$
$$\|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon)} \leq c(\varepsilon^{-1+\alpha} + \varepsilon^{-1+\beta}), \quad \int_0^t \|u^\varepsilon(s)\|^2_{H^2(\Omega_\varepsilon)}\,ds \leq c(\varepsilon^{-1+\alpha} + \varepsilon^{-1+\beta})(1+t)$$

for all $t \geq 0$, where $c > 0$ is a constant independent of ε, u^ε, and t.

Based on the estimates in Theorem 3.2, we prove the convergence of the average of a strong solution to (1.1) and characterize the limit as a unique weak solution to limit equations on the degenerate surface. We define weighted solenoidal spaces

$$
\begin{aligned}
L^2_{g\sigma}(\Gamma, T\Gamma) &:= \{v \in L^2(\Gamma, T\Gamma) \mid \mathrm{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}, \\
V_g &:= L^2_{g\sigma}(\Gamma, T\Gamma) \cap H^1(\Gamma, T\Gamma).
\end{aligned}
$$

They play a fundamental role in the study of limit equations.

Theorem 3.3. *For $\varepsilon \in (0,1)$ let $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$. Under Assumptions 1 and 2, suppose further that the following conditions are satisfied:*

(a) *There exist $c > 0$, $\alpha \in (0,1)$, and $\varepsilon_2 \in (0,1)$ such that*

$$
\|A_\varepsilon^{1/2} u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))} \leq c\varepsilon^{-1+\alpha}
$$

for all $\varepsilon \in (0, \varepsilon_2)$.

(b) *There exist $v_0 \in L^2(\Gamma, T\Gamma)$ and $f \in L^\infty(0,\infty; H^{-1}(\Gamma, T\Gamma))$ such that*

$$
\begin{aligned}
\lim_{\varepsilon\to 0} M_\tau u_0^\varepsilon &= v_0 \quad \text{weakly in} \quad L^2(\Gamma, T\Gamma), \\
\lim_{\varepsilon\to 0} M_\tau \mathbb{P}_\varepsilon f^\varepsilon &= f \quad \text{weakly-}\star \text{ in} \quad L^\infty(0,\infty; H^{-1}(\Gamma, T\Gamma)).
\end{aligned}
$$

(c) *For $i = 0, 1$ there exists $\gamma^i \geq 0$ such that $\lim_{\varepsilon\to 0} \varepsilon^{-1}\gamma^i_\varepsilon = \gamma^i$.*

Then there exists a constant $\varepsilon_3 \in (0,1)$ such that the Navier–Stokes equations (1.1) admits a global strong solution u^ε for $\varepsilon \in (0, \varepsilon_3)$ and

$$
\lim_{\varepsilon\to 0} Mu^\varepsilon \cdot n = 0 \quad \text{strongly in} \quad C([0,\infty); L^2(\Gamma)).
$$

Moreover, there exists a vector field

$$
v \in C([0,\infty); L^2_{g\sigma}(\Gamma, T\Gamma)) \cap L^2_{loc}([0,\infty); V_g) \cap H^1_{loc}([0,\infty); H^{-1}(\Gamma, T\Gamma))
$$

such that

$$
\lim_{\varepsilon\to 0} M_\tau u^\varepsilon = v \quad \text{weakly in} \quad L^2(0,T; H^1(\Gamma, T\Gamma))
$$

for all $T > 0$ and v is a unique weak solution to the equations

$$
\begin{aligned}
g\Big(\partial_t v + \overline{\nabla}_v v\Big) - 2\nu \left\{ P\mathrm{div}_\Gamma[gD_\Gamma(v)] - \frac{1}{g}(\nabla_\Gamma g \otimes \nabla_\Gamma g)v \right\} \\
+ (\gamma^0 + \gamma^1)v + g\nabla_\Gamma q = gf \quad \text{on} \quad \Gamma \times (0,\infty) \quad (3.1)
\end{aligned}
$$

with an associated pressure q and

$$
\mathrm{div}_\Gamma(gv) = 0 \quad \text{on} \quad \Gamma \times (0,\infty), \quad v|_{t=0} = v_0 \quad \text{on} \quad \Gamma. \quad (3.2)
$$

We also derive an estimate for the difference between $M_\tau u^\varepsilon$ and v to prove the strong convergence result. Moreover, we compare a strong solution u^ε to (1.1) with a weak solution v to (3.1)–(3.2) in Ω_ε (here we omit details).

Let us explain the idea for the proofs of our main results. In the proof of the global existence of a strong solution to (1.1), we follow the arguments in [1] and [2] to show that the $H^1(\Omega_\varepsilon)$-norm of a strong solution is bounded uniformly in time by a standard energy method. To this end, we derive a good estimate for the trilinear term $((u\cdot\nabla)u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)}$ for $u\in D(A_\varepsilon)$. A key idea for derivation of the estimate is to decompose u into the almost two-dimensional average part u^a and the residual part $u^r = u - u^a$ satisfying the impermeable boundary condition $u^r\cdot n_\varepsilon = 0$ on Γ_ε. Such decomposition enables us to use a good product estimate for u^a

$$\| |u^a|\,\varphi\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u^a\|_{H^1(\Omega_\varepsilon)}^{1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2}, \quad \varphi\in H^1(\Omega_\varepsilon)$$

and a good L^∞-estimate for u^r

$$\|u^r\|_{L^\infty(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/2}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right).$$

To prove the convergence of the tangential average of a strong solution u^ε to (1.1) and characterize the limit as a weak solution to (3.1)–(3.2), we use a change of variables formula to transform a weak formulation for (1.1) into that for the tangential average $M_\tau u^\varepsilon$ and then derive the energy estimate for $M_\tau u^\varepsilon$. In transformation of the weak formulation, we need to construct a test function of the weak formulation for (1.1) which is a vector field in V_ε from a test function of the weak formulation for (3.1)–(3.2) that is a surface vector field in V_g. To this end, we use an extension of a surface vector field to Ω_ε that satisfies the impermeable boundary condition on Γ_ε and a uniform estimate for the gradient part of the Helmholtz–Leray decomposition on Ω_ε. Also, in derivation of the energy estimate for $M_\tau u^\varepsilon$ we cannot substitute $M_\tau u^\varepsilon$ itself for its weak formulation since it is not in the space V_g of test functions (i.e. it is not weighted solenoidal) in general. To overcome this difficulty, we replace $M_\tau u^\varepsilon$ in its weak formulation with its weighted solenoidal part by using the weighted Helmholtz–Leray decomposition on Γ. Then we show the energy estimate for the weighted solenoidal part by substituting it for its weak formulation and derive the estimate for $M_\tau u^\varepsilon$ by using estimates for the gradient part of the weighted Helmholtz–Leray decomposition on Γ.

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Kobayashi–Warren–Carter type models of grain boundary motions with dynamic boundary conditions

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1 Introduction

In this talk, we take a nonnegative constant $\varepsilon \geq 0$ to consider the following coupled system of parabolic type PDEs, denoted by $(\mathrm{S})_\varepsilon$.

$(\mathrm{S})_\varepsilon$:

$$\partial_t \eta - \varDelta \eta + g(\eta) + \alpha'(\eta)|\nabla \theta| = 0 \quad \text{in } Q, \tag{1.1}$$

$$\alpha_0(\eta)\partial_t \theta - \mathrm{div}\left(\alpha(\eta)\frac{\nabla \theta}{|\nabla \theta|} + \nu^2 \nabla \theta\right) = 0 \quad \text{in } Q, \tag{1.2}$$

$$\nabla \eta_{|_\Gamma} \cdot n_\Gamma = 0 \quad \text{on } \Sigma, \tag{1.3}$$

$$\partial_t \theta_\Gamma - \varDelta_\Gamma(\varepsilon^2 \theta_\Gamma) + (\alpha(\eta)\tfrac{\nabla \theta}{|\nabla \theta|} + \nu^2 \nabla \theta)_{|_\Gamma} \cdot n_\Gamma = 0, \text{ and } \theta_{|_\Gamma} = \theta_\Gamma \quad \text{on } \Sigma, \tag{1.4}$$

$$\eta(0, \cdot) = \eta_0, \text{ and } \theta(0, \cdot) = \theta_0 \quad \text{in } \Omega, \tag{1.5}$$

$$\theta_\Gamma(0, \cdot) = \theta_{\Gamma,0} \quad \text{on } \Gamma. \tag{1.6}$$

Here, let $\nu > 0$ be a fixed constant. Let $\Omega \subset \mathbb{R}^N$ be a bounded spatial domain with a smooth boundary $\Gamma := \partial\Omega$ $(1 < N \in \mathbb{N})$, and let n_Γ be the unit outer normal on Γ. Also, we denote by $Q := (0, \infty) \times \Omega$ a product set of a time-interval $(0, \infty)$ and the spatial domain Ω, and we also put $\Sigma := (0, \infty) \times \Gamma$.

The above system is a modified version of the so-called *Kobayashi–Warren–Carter type model of grain boundary motion*, proposed by Kobayashi–Warren–Carter (cf. [9,10]), and the principal modifications are in the point that:

– the quasilinear diffusion with singularity in (1.2) includes the regularization term $\nu^2 \nabla \theta$ with a small constant $\nu > 0$;

– the boundary data θ_Γ is governed by the dynamic boundary conditions (1.4), which consist of parabolic type PDE on the boundary Σ and the transmission condition $\theta_{|_\Gamma} = \theta_\Gamma$ on Σ.

In the original model [9,10], the spatial domain Ω is settled as a two-dimensional domain $(N = 2)$, and the main focus is to reproduce the dynamics of the crystalline orientation by the time and spatial variation of a vector field

$$(t, x) \in Q \mapsto \varpi(t, x) := \eta(t, x)^{\mathrm{t}}\big(\cos \theta(t, x), \sin \theta(t, x)\big);$$

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consisting of two order parameter $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$. The variation of $\varpi = \varpi(t,x)$ is supposed to be governed by the gradient flow of the following energy functional, called *free-energy*, and in the case of $(\mathrm{S})_\varepsilon$, the corresponding free-energy is provided as follows.

$$\begin{aligned}[\eta,\theta,\theta_\Gamma] \mapsto \mathscr{F}_\varepsilon(\eta,\theta,\theta_\Gamma) := \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega G(\eta)\,dx \\ + \int_\Omega \alpha(\eta)|\nabla\theta|\,dx + \frac{\kappa^2}{2}\int_\Omega |\nabla\theta|^2\,dx + \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma,\end{aligned} \quad \text{for } \varepsilon \geq 0, \qquad (1.7)$$

with the effective domain

$$D(\mathscr{F}_\varepsilon) := \left\{ [\eta,\theta,\theta_\Gamma] \;\middle|\; \begin{array}{l} \eta \in H^1(\Omega),\ \theta \in H^1(\Omega),\ \theta_\Gamma \in H^{1/2}(\Gamma), \\ \varepsilon\theta_\Gamma \in H^1(\Gamma) \text{ and } \theta_{|_\Gamma} = \theta_\Gamma \text{ in } H^{1/2}(\Gamma) \end{array} \right\}. \qquad (1.8)$$

In the context, the unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ are supposed to be order parameters which correspond to the orientation order and the orientation angle in a polycrystal, respectively. In particular, η is supposed to satisfy the range constraint $0 \leq \eta \leq 1$ in Q, and the threshold values 1 and 0 indicate the completely oriented phase and the disoriented phase of orientation, respectively. $g = g(\eta)$ in (1.1) is given locally Lipschitz function on $\mathbb{R}$, having its nonnegative primitive $G = G(\eta)$. $0 < \alpha_0 = \alpha_0(\eta)$ in (1.2) is a given locally Lipschitz function while $0 < \alpha = \alpha(\eta)$ in (1.1)–(1.2) is given a C^2-convex function, and $\alpha' = \alpha'(\eta)$ is η-differential such that $\alpha'(0) = 0$. Besides, "$_{|_\Gamma}$" denotes the trace on Γ for a Sobolev function, $d\Gamma$ denotes the area element on Γ, ∇_Γ denotes the surface gradient on Γ, and $\mathit{\Delta}_\Gamma$ denotes the Laplacian on the surface, i.e. the so-called Laplace–Beltrami operator (cf. [13]). The equations in (1.5)–(1.6) are initial conditions with given initial data η_0, θ_0, and $\theta_{\Gamma,0}$ for the component η, θ, and θ_Γ, respectively.

The objective of this study is to develop the mathematical analysis for Kobayashi–Warren–Carter type model of grain boundary motion. The theoretical results of the systems of Kobayashi–Warren–Carter type models have been established by a lot of mathematicians (cf. [4–6,8,11,12,14–17]). However, the most of previous mathematical models were based on the original model which was subject to the standard boundary conditions for the parabolic PDEs (1.1) and (1.2). From a physical point of view, the dynamics of η and θ are supposed to be governed by the changes of environments around the crystalline materials. On this basis of such backgrounds, we consider the system of Kobayashi–Warren–Carter type models including dynamic boundary conditions to reproduce the more precise physical and dynamical situations.

On the other hand, in [3], the authors dealt with the quasilinear diffusion equations with singularities and dynamic boundary conditions as the transmission systems. The previous work [3] was concerned with the well-posedness for the transmission systems, such as the existence and uniqueness of the solution, as well as with some continuous dependence results, also with respect to $\varepsilon \geq 0$. Actually, the case when $\varepsilon > 0$ and $\varepsilon = 0$ could exhibit different features from various point of view (as for the large-time behavior), and also we note that the presence or absence of the term $\frac{\varepsilon^2}{2}\int_\Gamma |\nabla_\Gamma\theta_\Gamma|^2 d\Gamma$ brings here the gap of effective domains $D(\mathscr{F}_\varepsilon)$ between the case when $\varepsilon > 0$ and $\varepsilon = 0$.

Based on these, in this talk, we attempt to establish the unified mathematical theory of Kobayashi–Warren–Carter type models with dynamic boundary conditions, and our principal results will be stated in forms of the following three Main Theorems, which are concerned with qualitative properties of the system $(\mathrm{S})_\varepsilon$, for $\varepsilon \geq 0$.

Main Theorem 1: the existence of solutions to $(\mathrm{S})_\varepsilon$, for any $\varepsilon \geq 0$.

Main Theorem 2: the continuous dependence of solutions to $(\mathrm{S})_\varepsilon$ with respect to $\varepsilon \geq 0$.

Main Theorem 3: the large-time behavior of the solutions to $(\mathrm{S})_\varepsilon$.

2 Main Theorems

In this Section, we state the Main Theorems in this talk. Firstly, for any $\varepsilon \geq 0$, let us set product Hilbert spaces:

$$\begin{aligned} &H := L^2(\Omega) \times L^2(\Gamma), \ \mathscr{H} := L^2(\Omega)^2 \times L^2(\Gamma), \\ &V_\varepsilon := \left\{ [w, w_\Gamma] \in H \;\middle|\; \begin{array}{l} w \in H^1(\Omega),\ w_\Gamma \in H^{1/2}(\Gamma), \\ \varepsilon w_\Gamma \in H^1(\Gamma), \text{ and } w_{|_\Gamma} = w_\Gamma \text{ on } \Gamma \end{array} \right\}, \\ &\mathscr{V}_\varepsilon := \left\{ [z, w, w_\Gamma] \in \mathscr{H} \;\middle|\; \begin{array}{l} z \in H^1(\Omega),\ w \in H^1(\Omega),\ w_\Gamma \in H^{1/2}(\Gamma), \\ \varepsilon w_\Gamma \in H^1(\Gamma), \text{ and } w_{|_\Gamma} = w_\Gamma \text{ on } \Gamma \end{array} \right\}, \end{aligned} \tag{2.1}$$

endowed with the standard inner products, respectively.

Next, we list the assumptions for the spatial domain, the given functions $g = g(\eta)$, $\alpha_0 = \alpha_0(\eta)$, $\alpha = \alpha(\eta)$ and the initial data in our systems.

(A1) $\nu > 0$ is a fixed constant. Besides, $1 < N \in \mathbb{N}$ is a constant of dimension, and Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\Gamma := \partial\Omega$ and the unit outer normal n_Γ.

(A2) $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function, such that

$$g(0) \leq 0 \ \text{ and } \ g(1) \geq 0. \tag{2.2}$$

Besides, g is supposed to have a nonnegative potential $G : \mathbb{R} \to [0, \infty)$.

(A3) $\alpha_0 : \mathbb{R} \to (0, \infty)$ is a locally Lipschitz function. Also, $\alpha : \mathbb{R} \to (0, \infty)$ is a C^2-function, such that $\alpha'(0) = 0$ and $\alpha'' \geq 0$ on $\mathbb{R}$, where α' and α'' are the first and the second differential of α, respectively. Hence, α finds out to be nonnegative and convex function on $\mathbb{R}$. Moreover, there exists a positive constant δ_α, such that

$$\alpha_0(\tau) \geq \delta_\alpha \ \text{ and } \ \alpha(\tau) \geq \delta_\alpha, \text{ for any } \tau \in \mathbb{R}.$$

(A4) For $\varepsilon \geq 0$, the initial data $[\eta_0, \theta_0, \theta_{\Gamma,0}]$ belongs to a class $D_\varepsilon \subset \mathscr{H}$, defined as:

$$D_\varepsilon := \left\{ [z, w, w_\Gamma] \in \mathscr{V}_\varepsilon \;\middle|\; 0 \leq \eta \leq 1 \text{ a.e. in } \Omega \text{ and } [w, w_\Gamma] \in L^\infty(\Omega) \times L^\infty(\Gamma) \right\}.$$

Remark 2.1 (Possible choice of given functions)**.** Referring to [9, 10] the setting

$$g(\tau) = \tau - 1 \text{ with } G(\tau) := \frac{1}{2}(\tau - 1)^2 \text{ and } \alpha_0(\tau) = \alpha(\tau) = \tau^2 + \delta_\alpha, \text{ for } \tau \in \mathbb{R},$$

provides a possible choice of given functions that fulfills the above assumptions. However, in the original models [9, 10], the presence of constant δ_α is not suppose. Also, we note that the condition (2.2) are lead to the range constraint $0 \leq \eta \leq 1$ in Ω, which enables us to suppose Lipschitz continuous for g, α_0, α without loss of generality.

Based on the above assumptions, the solution to the system $(\mathrm{S})_\varepsilon$, for $\varepsilon \geq 0$, is defined as follows.

Definition 2.1 (Definition of the solution)**.** For any $\varepsilon \geq 0$, a triplet of functions $[\eta, \theta, \theta_\Gamma] \in L^2_{\mathrm{loc}}([0,\infty); \mathscr{H})$ is called a *solution* to $(\mathrm{S})_\varepsilon$, if and only if the following items hold.

(S1) $[\eta, \theta, \theta_\Gamma] \in C([0,\infty); \mathscr{H}) \cap W^{1,2}_{\mathrm{loc}}((0,\infty); \mathscr{H}) \cap L^2_{\mathrm{loc}}([0,\infty); \mathscr{V}_\varepsilon) \cap L^\infty_{\mathrm{loc}}((0,\infty); \mathscr{V}_\varepsilon)$,

$$0 \leq \eta \leq 1, |\theta|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)}, \text{a.e. in } \Omega \text{ and } |\theta_\Gamma|_{L^\infty(\Sigma)} \leq |\theta_{\Gamma,0}|_{L^\infty(\Gamma)}, \text{a.e. on } \Gamma.$$

(S2) η solves the following variational equality:

$$\int_\Omega (\partial_t \eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)|)\varphi\,dx + \int_\Omega \nabla\eta(t)\cdot\nabla\varphi\,dx = 0,$$
$$\text{for any } \varphi \in H^1(\Omega), \text{ a.e. } t > 0,$$

subject to the initial condition $\eta(0) = \eta_0$ in $L^2(\Omega)$.

(S3) A pair of functions $[\theta, \theta_\Gamma]$ solves the following variational inequality:

$$\begin{aligned}
&\int_\Omega \alpha_0(\eta(t))\partial_t\theta(t)(\theta(t)-\psi)\,dx + \nu^2 \int_\Omega \nabla\theta(t)\cdot\nabla(\theta(t)-\psi)\,dx \\
&\quad + \int_\Gamma \partial_t\theta_\Gamma(t)(\theta_\Gamma(t)-\psi_\Gamma)\,d\Gamma + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma)\cdot\nabla_\Gamma(\varepsilon\psi_\Gamma)\,d\Gamma \\
&+ \int_\Omega \alpha(\eta(t))|\nabla\theta(t)|\,dx \leq \int_\Omega \alpha(\eta(t))|\nabla\psi|\,dx, \text{ for any } [\psi,\psi_\Gamma] \in V_\varepsilon, \text{ a.e. } t > 0,
\end{aligned}$$

subject to the initial conditions $[\theta(0), \theta_\Gamma(0)] = [\theta_0, \theta_{\Gamma,0}]$ in H.

Now, the main focus conclusion of this talk is stated as follows.
The first Main Theorem is concerned with the existence of the solutions.

Main Theorem 1 (Existence of solution to $(\mathrm{S})_\varepsilon$)**.** *Under the assumptions (A1)–(A4), the system $(S)_\varepsilon$ admits at least one solution $[\eta, \theta, \theta_\Gamma]$, for any $\varepsilon \geq 0$.*

The second Main Theorem is concerned with the continuous dependence of the solutions to $(\mathrm{S})_\varepsilon$ with respect to $\varepsilon \geq 0$.

Main Theorem 2 (ε-dependence of the solutions)**.** *Let us assume the assumptions (A1)–(A4), and let us take a nonnegative constant $\varepsilon_0 \geq 0$. Let $\{[\eta_0^\varepsilon, \theta_0^\varepsilon, \theta_{\Gamma,0}^\varepsilon] \in D_\varepsilon\}_{\varepsilon \geq 0} \subset \mathscr{H}$ be a sequence of initial triplet and for any $\varepsilon \geq 0$, let $[\eta^\varepsilon, \theta^\varepsilon, \theta_\Gamma^\varepsilon]$ be a solution to $(S)_\varepsilon$ corresponding to the initial triplet $[\eta_0^\varepsilon, \theta_0^\varepsilon, \theta_{\Gamma,0}^\varepsilon] \in D_{\varepsilon_0}$. If:*

$$[\eta_0^\varepsilon, \theta_0^\varepsilon, \theta_{\Gamma,0}^\varepsilon] \to [\eta_0, \theta_0, \theta_{\Gamma,0}] \text{ in } \mathscr{H}, \text{ as } \varepsilon \to \varepsilon_0.$$

Then, there exist a subsequence $\{\varepsilon_n\}_{n=1}^\infty \subset \{\varepsilon\}$, satisfying $\varepsilon_n \to \varepsilon_0$, as $n \to \infty$, and $[\eta, \theta, \theta_\Gamma] \in L^2_{\mathrm{loc}}([0,\infty); \mathscr{H})$ such that:

$$[\eta^{\varepsilon_n}, \theta^{\varepsilon_n}, \theta_\Gamma^{\varepsilon_n}] \to [\eta, \theta, \theta_\Gamma] \quad \begin{array}{l} \text{in } C_{\mathrm{loc}}([0,\infty); \mathscr{H}), \\ \text{in } L^2_{\mathrm{loc}}([0,\infty); \mathscr{V}_{\varepsilon_0}), \end{array} \quad \text{as } n \to \infty.$$

In particular, if $\varepsilon_0 > 0$, then:

$$\theta_\Gamma^{\varepsilon_n} \to \theta_\Gamma \text{ in } L^2_{\mathrm{loc}}([0,\infty); H^1(\Gamma)), \text{ as } n \to \infty.$$

The third Main Theorem is concerned with the large-time behavior of the solutions.

Main Theorem 3 (Large-time behavior of the solutions)**.** *Let us assume (A1)–(A4). Then, let $[\eta, \theta, \theta_\Gamma]$ be a solution to $(S)_\varepsilon$, for any $\varepsilon \geq 0$, and let $\omega(\eta, \theta, \theta_\Gamma)$ be the ω-limit set of $[\eta, \theta, \theta_\Gamma]$, by putting:*

$$\omega(\eta, \theta, \theta_\Gamma) := \left\{ [\eta^\infty, \theta^\infty, \theta_\Gamma^\infty] \in \mathscr{H} \;\middle|\; \begin{array}{l} [\eta(t_n), \theta(t_n), \theta_{\Gamma(t_n)}] \to [\eta^\infty, \theta^\infty, \theta_\Gamma^\infty] \text{ in } \mathscr{H}, \text{ as} \\ n \to \infty, \text{ for some } \{t_n\}_{n=1}^\infty \subset (0, \infty), \text{ satisfy-} \\ \text{ing } t_n \to \infty, \text{ as } n \to \infty \end{array} \right\}.$$

Then, the following two items hold.

(I) *$\omega(\eta, \theta, \theta_\Gamma) \neq \emptyset$ and $\omega(\eta, \theta, \theta_\Gamma)$ is compact in $\mathscr{H}$.*

(II) *For any $[\eta^\infty, \theta^\infty, \theta_\Gamma^\infty] \in \omega(\eta, \theta, \theta_\Gamma)$, fulfills that:*

$$\begin{cases} 0 \leq \eta^\infty \leq 1, \text{ a.e. in } \Omega \text{ and } -\Delta_N \eta^\infty + g(\eta^\infty) = 0 \text{ in } L^2(\Omega), \\ \theta^\infty \equiv C^\infty \text{ in } L^2(\Omega), \ \theta_\Gamma^\infty \equiv C^\infty \text{ in } L^2(\Gamma), \text{ for some constant } C^\infty \in \mathbb{R}, \end{cases}$$

where $\Delta_N : D(\Delta_N) := \{z \in H^2(\Omega) | \nabla z_{|_\Gamma} \cdot n_\Gamma = 0 \text{ in } L^2(\Gamma)\} \subset L^2(\Omega) \to L^2(\Omega)$ is the Laplacian with homogeneous Neumann boundary conditions.

3 Sketch of the Proofs of Main Theorems

In this Section, to prove our Main Theorems, we will propose the mathematical method that is different from the previous one. In any case of $\varepsilon \geq 0$, the keypoints of our mathematical analysis will be to reformulate the system $(S)_\varepsilon$ as the following Cauchy problem for an evolution equation:

$$\begin{cases} \mathcal{A}(U(t))U'(t) + \partial\Phi_\varepsilon(U(t)) + \mathcal{G}(U(t)) \ni 0 \text{ in } \mathscr{H}, \text{ a.e. } t > 0, \\ U(0) = U_0 \text{ in } \mathscr{H}, \end{cases} \tag{3.1}$$

which is governed by the subdifferential $\partial\Phi_\varepsilon$ of the following convex function Φ_ε on $\mathscr{H}$:

$$\begin{aligned} U &= [\eta, \theta, \theta_\Gamma] \in \mathscr{H} \mapsto \Phi_\varepsilon(U) = \Phi_\varepsilon(\eta, \theta, \theta_\Gamma) \\ &:= \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \eta|^2 \, dx + \dfrac{1}{2}\int_\Omega \left(\dfrac{1}{\nu}\alpha(\eta) + \nu|\nabla\theta| \right)^2 dx + \dfrac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2 \, d\Gamma, \\ \qquad\qquad \text{if } U = [\eta, \theta, \theta_\Gamma] \in \mathscr{V}_\varepsilon, \text{ for } \varepsilon \geq 0, \\ \infty, \qquad \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

In the context, the unknown $U \in C([0, \infty); \mathscr{H})$ corresponds to the solution triplet $[\eta, \theta, \theta_\Gamma]$ of the system $(S)_\varepsilon$, i.e. $U(t) = [\eta(t), \theta(t), \theta_\Gamma(t)]$ in $\mathscr{H}$, for any $t \geq 0$ with the initial triplet $U_0 = [\eta_0, \theta_0, \theta_{\Gamma,0}] \in \mathscr{H}$. Besides, $\mathcal{A}$ is a matrix-valued operator, by letting:

$$U = [\eta, \theta, \theta_\Gamma] \in \mathscr{H} \mapsto \mathcal{A}(U) = \mathcal{A}(\eta, \theta, \theta_\Gamma) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_0(\eta) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $\mathcal{G} : \mathscr{H} \to \mathscr{H}$ is a Lipschitz operator, defined by:

$$U = [\eta, \theta, \theta_\Gamma] \in \mathscr{H} \mapsto \mathcal{G}(U) = \mathcal{G}(\eta, \theta, \theta_\Gamma) := {}^{\mathrm{t}}[\Psi'(\eta), 0, 0] \in \mathscr{H},$$

where Ψ' denotes Fréchet differential of the functional Ψ:

$$\eta \in L^2(\Omega) \mapsto \Psi(\eta) := \int_\Omega G(\eta)\, dx - \frac{1}{2\nu^2} \int_\Omega (\alpha(\eta))^2\, dx. \tag{3.3}$$

For any $\varepsilon \geq 0$, we can see that the sum of the functionals Φ_ε and Ψ, given in (3.2) and (3.3), corresponds to the governing energies $\mathscr{F}_\varepsilon$, given in (1.7), and also $\mathscr{V}_\varepsilon$, given in (2.1) corresponds to the effective domain $D(\mathscr{F}_\varepsilon)$, given in (1.8). In addition, we can easily check that the functional Φ_ε is a proper, l.s.c. and convex function on $\mathscr{H}$, and the subdifferentials $\partial\Phi_\varepsilon$, for $\varepsilon \geq 0$, are maximal monotone graphs in $\mathscr{H}^2$. So, if $\alpha_0(\eta)$ is constant, then the well-posedness for the Cauchy problem (3.1) will be verified, immediately, by applying general theories for evolution equations, e.g. [1, 2, 7].

In view of this, the essential points in Main Theorems will be shown a certain association between our system $(\mathrm{S})_\varepsilon$ and the Cauchy problem (3.1), for any $\varepsilon \geq 0$. To this end, we will first show the solvability of the system $(\mathrm{S})_\varepsilon$, for any $\varepsilon \geq 0$, concerned with Main Theorem 1. Due to the unknown-dependent function $\alpha_0(\eta)$, we need to prepare another index $0 < \tau < 1$, related to a time-step, and for arbitrary $0 < \tau < 1$, we consider a time-discretized system, associated with $(\mathrm{S})_\varepsilon$. The solvability of the system $(\mathrm{S})_\varepsilon$ is obtained as a passage to the limit in the time-discretized system, as $\tau \to 0$. Then, we clarify the association between $(\mathrm{S})_\varepsilon$ and the Cauchy problem (3.1) by showing the following Key-Lemma, concerned with the representations of subdifferentials $\partial\Phi_\varepsilon$.

Key-Lemma 1 (Representation of the subdifferential $\partial\Phi_\varepsilon$)**.** *For any $\varepsilon \geq 0$, the following two items are equivalent.*

(O) $[\eta, \theta, \theta_\Gamma] \in D(\partial\Phi_\varepsilon)$ *and* $[\eta^*, \theta^*, \theta_\Gamma^*] \in \partial\Phi_\varepsilon(\eta, \theta, \theta_\Gamma)$ *in* $\mathscr{H}$.

(I) $[\eta, \theta, \theta_\Gamma] \in D(\Phi_\varepsilon)$ *and there exists* $z^* \in L^\infty(\Omega)^N$ *such that:*

$$\begin{cases} \bullet\ z^* \in \mathrm{Sgn}(\nabla\theta), \text{ a.e. in } \Omega, \\ \bullet\ -\Delta\eta + \alpha'(\eta)|\nabla\theta| + \frac{1}{\nu^2}\alpha(\eta)\alpha'(\eta) \in L^2(\Omega), \\ \bullet\ \alpha(\eta)z^* + \nu^2\nabla\theta \in L^2_{\mathrm{div}}(\Omega), \\ \bullet\ -\Delta_\Gamma(\varepsilon^2\theta_\Gamma) + \left[(\alpha(\eta)z^* + \nu^2\nabla\theta)_{|_\Gamma} \cdot n_\Gamma\right] \in L^2(\Gamma), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \bullet\ \eta^* = -\Delta\eta + \alpha'(\eta)|\nabla\theta| + \frac{1}{\nu^2}\alpha(\eta)\alpha'(\eta) \text{ in } L^2(\Omega), \\ \bullet\ \theta^* = -\mathrm{div}\big(\alpha(\eta)z^* + \nu^2\nabla\theta\big) \text{ in } L^2(\Omega), \\ \bullet\ \theta_\Gamma^* = -\Delta(\varepsilon^2\theta_\Gamma) + \left[(\alpha(\eta)z^* + \nu^2\nabla\theta) \cdot n_\Gamma\right] \text{ in } L^2(\Gamma). \end{cases} \tag{3.5}$$

Moreover, we need to check Mosco-convergences of energy Φ_ε, with respect to $\varepsilon \geq 0$, as follows.

Key-Lemma 2 (Mosco convergence for the convex energy)**.** *Let* $\varepsilon_0 \geq 0$ *be a fixed constant, and let us assume that arbitrary sequence* $\{\varepsilon_n\}_{n=1}^{\infty} \subset \{\varepsilon\}$ *satisfy* $\varepsilon_n \to \varepsilon_0$*. Then, the sequence of convex energy* $\{\Phi_{\varepsilon_n}\}_{n=1}^{\infty}$ *converges to* Φ_{ε_0} *on* $\mathscr{H}$*, in the sense of Mosco, as* $n \to \infty$*. More precisely:*

(Lower-bound): *If* $\check{U} = [\check{z}, \check{w}, \check{w}_\Gamma] \in \mathscr{H}$*,* $\{\check{U}_n = [\check{z}_n, \check{w}_n, \check{w}_{\Gamma,n}]\}_{n=1}^{\infty} \subset \mathscr{H}$ *and* $\check{U}_n \to \check{U}$ *weakly in* $\mathscr{H}$*, as* $n \to \infty$*, then* $\varliminf_{n\to\infty} \Phi_{\varepsilon_n}(\check{U}_n) \geq \Phi_{\varepsilon_0}(\check{U})$*.*

(Optimality): *For any* $\hat{U} = [\hat{z}, \hat{w}, \hat{w}_\Gamma] \in D(\Phi_{\varepsilon_0}) = \mathscr{V}_{\varepsilon_0}$*, there exists a sequence* $\{\hat{U}_n = [\hat{z}_n, \hat{w}_n, \hat{w}_{\Gamma,n}]\}_{n=1}^{\infty} \subset \mathscr{V}_{\varepsilon_n}$ *such that* $\hat{U}_n \to \hat{U}$ *and* $\Phi_{\varepsilon_n}(\hat{U}_n) \to \Phi_{\varepsilon_0}(\hat{U})$ *as* $n \to \infty$*.*

This Key-Lemma 2 can play an important role in the proof of Main Theorem 2, ε-dependence of the solutions, and in the observation of the approximating limit in Key-Lemma 1. Finally, to prove Main Theorem 3, we will show the energy-inequality by using time-discretized system.

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Endpoint Strichartz estimate for the damped wave equation and its application

Takahisa Inui

1 Introduction

We consider the damped wave equation.

$$\begin{cases} \partial_t^2\phi - \Delta\phi + \partial_t\phi = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ (\phi(0), \partial_t\phi(0)) = (\phi_0, \phi_1), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $d \in \mathbb{N}$, (ϕ_0, ϕ_1) are given complex valued functions. The space-time estimates for the solution ϕ of (1.1) in non-endpoint cases were obtained by the author [12]. In this talk, we will show the space-time estimates for the solution ϕ of (1.1) in an endpoint case.

Matsumura [21] applied the Fourier transform to (1.1) and obtained the formula

$$\phi(t,x) = \mathcal{D}(t)(\phi_0 + \phi_1) + \partial_t\mathcal{D}(t)\phi_0,$$

where $\mathcal{D}(t)$ is defined by

$$\mathcal{D}(t) := e^{-\frac{t}{2}}\mathcal{F}^{-1}L(t,\xi)\mathcal{F}$$

with

$$L(t,\xi) := \begin{cases} \dfrac{\sinh(t\sqrt{1/4-|\xi|^2})}{\sqrt{1/4-|\xi|^2}} & \text{if } |\xi| \le 1/2, \\[2ex] \dfrac{\sin(t\sqrt{|\xi|^2-1/4})}{\sqrt{|\xi|^2-1/4}} & \text{if } |\xi| > 1/2. \end{cases}$$

By this formula, Matsumura [21] proved the L^p-L^q type estimate:

$$\|\phi(t)\|_{L^p} \lesssim \langle t\rangle^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|(\phi_0,\phi_1)\|_{L^q\times L^q} + e^{-\frac{t}{4}}\left(\|\phi_0\|_{H^{\left[\frac{d}{2}\right]+1}} + \|\phi_1\|_{H^{\left[\frac{d}{2}\right]}}\right), \tag{1.2}$$

where $1 \le q \le 2 \le p \le \infty$ and $[d/2]$ denotes the integer part of $d/2$. Such L^p-L^q type estimates have been studied well. See [23, 8, 22] and references therein. The L^p-L^q type estimates for the heat equation and the wave equation are also well studied. We recall the L^p-L^q type estimate for the heat equation $\partial_t v - \Delta v = 0$:

$$\|\mathcal{G}(t)g\|_{L^p} \lesssim t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|g\|_{L^q},$$

This is a joint work with Yuta Wakasugi in Ehime university.

where $1 \leq q \leq p \leq \infty$ and $\mathcal{G}(t) := \mathcal{F}^{-1}e^{-t|\xi|^2}\mathcal{F}$. We also refer to the L^p-L^q type estimate for the wave equation $\partial_t^2 w - \Delta w = 0$:

$$\|\mathcal{W}(t)g\|_{L^p} \lesssim |t|^{-2d\left(\frac{1}{2}-\frac{1}{p}\right)} \|g\|_{\dot{W}^{\gamma-1,p'}},$$

for $2 \leq p < \infty$ and $(d+1)(1/2 - 1/p) \leq \gamma < d$, where $\mathcal{W}(t) := \mathcal{F}^{-1}\sin(t|\xi|)/|\xi|\mathcal{F}$. See [1]. Matsumura's estimate (1.2) shows that the solution of (1.1) behaves like the solution of the heat equation and the wave equation in some sense. More precisely, the low frequency part of the solution to the damped wave equation behaves like the solution of the heat equation and the high frequency part behaves like the solution of the wave equation but decays exponentially (see [10] for another L^p-L^q estimate).

For the heat equation and the wave equation, by using the L^p-L^q type estimates, we obtain the space-time estimates, what we call the Strichartz estimate. The Strichartz estimates for the heat equation is

$$\|v\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \|v_0\|_{L^2} + \|F\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

where v satisfies $\partial_t v - \Delta v = F$ with $v(0) = v_0$ and $2/q + d/r = 2/\tilde{q} + d/\tilde{r} = d/2$. See [27, 5]. We also have the Strichartz estimates for the wave equation as follows.

$$\|w\|_{L_t^q(I:L_x^p(\mathbb{R}^d))} \lesssim \|w_0\|_{\dot{H}^1} + \|w_1\|_{L^2} + \|F\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

where w satisfies $\partial_t^2 w - \Delta w = F$ with $(w(0), \partial_t w(0)) = (w_0, w_1)$ and $1/q + d/r = d/2 - 1 = 1/\tilde{q}' + d/\tilde{r}' - 2$. See [7]. Recently, Watanabe [26] obtained the Strichartz estimates for the damped wave equation when $d = 2, 3$ by an energy method. In the previous work [12], the author proved the following Strichartz estimates in the higher dimensional cases by the duality argument.

Proposition 1.1 (Homogeneous Strichartz estimates)**.** *Let $d \geq 2$, $2 \leq r < \infty$, and $2 \leq q \leq \infty$. Set $\gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d+1}{2}(1/2 - 1/r)\}$. Assume*

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \geq \frac{1}{q},$$

Then, we have

$$\|\mathcal{D}(t)f\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \left\|\langle\nabla\rangle^{\gamma-1} f\right\|_{L^2},$$
$$\|\partial_t\mathcal{D}(t)f\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \|\langle\nabla\rangle^{\gamma} f\|_{L^2},$$
$$\left\|\partial_t^2\mathcal{D}(t)f\right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \left\|\langle\nabla\rangle^{\gamma+1} f\right\|_{L^2}.$$

Proposition 1.2 (Inhomogeneous Strichartz estimates)**.** *Let $d \geq 2$, $2 \leq r, \tilde{r} < \infty$, and $2 \leq q, \tilde{q} \leq \infty$. We set $\gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d+1}{2}(1/2 - 1/r)\}$ and $\tilde{\gamma}$ in the same manner. Assume that (q, r) and $(\tilde{q}, \tilde{r})$ satisfies*

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) > \frac{1}{q} + \frac{1}{\tilde{q}}.$$

or

$$\frac{d}{2}\left(\frac{1}{2}-\frac{1}{r}\right)+\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)=\frac{1}{q}+\frac{1}{\tilde{q}} \text{ and } 1<\tilde{q}'<q<\infty.$$

Moreover, we exclude the wave endpoint case, that is, we assume $(q,r)\neq(2,2(d-1)/(d-3))$ *and* $(\tilde{q},\tilde{r})\neq(2,2(d-1)/(d-3))$ *when* $d\geq 4$. *Then, we have*

$$\left\|\int_0^t \mathcal{D}(t-s)F(s)ds\right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))}\lesssim\left\|\langle\nabla\rangle^{\gamma+\tilde{\gamma}+\delta-1}F\right\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$
$$\left\|\int_0^t \partial_t\mathcal{D}(t-s)F(s)ds\right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))}\lesssim\left\|\langle\nabla\rangle^{\gamma+\tilde{\gamma}+\delta}F\right\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

where $\delta=0$ *when* $\frac{1}{\tilde{q}}(1/2-1/r)=\frac{1}{q}(1/2-1/\tilde{r})$ *and in the other cases* $\delta\geq 0$ *is defined in the table 1 below.*

δ	$\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$	$\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)>\frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$
$\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\geq\frac{1}{q}$ $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\geq\frac{1}{\tilde{q}}$	0	0
$\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\geq\frac{1}{q}$ $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)<\frac{1}{\tilde{q}}$	×	$\frac{\tilde{q}}{q}\left\{\frac{1}{\tilde{q}}-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\right\}$
$\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}$ $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\geq\frac{1}{\tilde{q}}$	$\frac{q}{\tilde{q}}\left\{\frac{1}{q}-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\right\}$	×
$\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}$ $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)<\frac{1}{\tilde{q}}$	$\frac{1}{\tilde{q}}\frac{d-1}{2}\left\{\tilde{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)-q\left(\frac{1}{2}-\frac{1}{r}\right)\right\}$	$\frac{1}{q}\frac{d-1}{2}\left\{q\left(\frac{1}{2}-\frac{1}{r}\right)-\tilde{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\right\}$

Table 1: The value of δ. (× means that the case does not occur.)

In Proposition 1.1, we obtain the homogeneous Strichartz estimate when (q,r) is the wave endpoint pair, *i.e.* $(q,r)=(2,2(d-1)/(d-3))$. On the other hand, we have not obtained the inhomogeneous Strichartz estimate in the wave endpoint case.

2 Main result

In this talk, we give the inhomogeneous Strichartz estimate in the wave endpoint case. Namely, we show the following theorem.

Theorem 2.1 (Inhomogeneous Strichartz estimates in the wave endpoint case)**.** *Let $d \geq 2$, $2 \leq r, \tilde{r} < \infty$, and $2 \leq q, \tilde{q} \leq \infty$. We set $\gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d+1}{2}(1/2 - 1/r)\}$ and $\tilde{\gamma}$ in the same manner. Assume that (q, r) and $(\tilde{q}, \tilde{r})$ satisfies*

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) > \frac{1}{q} + \frac{1}{\tilde{q}}.$$

or

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } 1 < \tilde{q}' < q < \infty.$$

Moreover, we may take the wave endpoint pair, that is, we may assume $(q, r) = (2, 2(d-1)/(d-3)))$ or $(\tilde{q}, \tilde{r}) = (2, 2(d-1)/(d-3)))$ when $d \geq 4$. Then, we have

$$\left\| \int_0^t \mathcal{D}(t-s)F(s)ds \right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta - 1} F \right\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

$$\left\| \int_0^t \partial_t \mathcal{D}(t-s)F(s)ds \right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \left\| \langle \nabla \rangle^{\gamma + \tilde{\gamma} + \delta} F \right\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

where $\delta = 0$ when $\frac{1}{\tilde{q}}(1/2 - 1/r) = \frac{1}{q}(1/2 - 1/\tilde{r})$ and in the other cases $\delta \geq 0$ is defined in the table 1 above.

The proof is based on the idea of Keel–Tao [18] (see also [20]). Keel and Tao showed the endpoint Strichartz estimates for the general unitary operator such as Schrödinger and wave. We apply this method to the dissipative operator $\mathcal{D}(t)$.

Moreover, we apply this endpoint Strichartz estimate to prove the unconditional uniqueness in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ of the following energy critical nonlinear damped wave equation.

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^{\frac{4}{d-2}} u, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d, \end{cases} \tag{NLDW}$$

where $d \geq 3$, (u_0, u_1) is given, and u is an unknown complex valued function. The proof of this unconditional uniqueness is based on Bulut–Czubak–Li–Pavlović–Zhang [2].

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On the weak shocks and strong shocks for the supersonic flow past a sharp cone

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Abstract

The following problem was posed in the book “Supersonic Flow and Shock Waves” by Courant and Friedrichs (pages 313-314, 414): if there is a supersonic steady flow which comes from minus infinity, and the flow hits a sharp cone along its axis direction, then it follows from the Rankine-Hugoniot conditions and the physical entropy condition that there will appear a weak shock or a strong shock attached at the vertex of the cone, which corresponds to the supersonic shock or the transonic shock, respectively. A long-standing open problem is that only the weak shock could occur, and the strong shock is unstable. However, a convincing proof of this instability has apparently never been given. In this talk, we will give systematic studies on the stability of weak shocks and instability of strong shocks for the supersonic flow past a sharp cone. These works are joint with Prof.Li Jun, Prof.Xu Gang and so on.

Keywords: Supersonic flow, compressible Euler equations, potential equation, weak shock, strong shock, sharp cone

Mathematical Subject Classification 2000: 35L70, 35L65, 35L67, 76N15

§0 **Introduction**

In this talk, we are concerned with the multi-dimensional steady conic shock wave problem for a perturbed incoming supersonic flow past an infinitely long circular cone. This problem is fundamental in gas dynamics. It is also one of the basic models for the discussion of the theory of weak solutions to quasilinear conservation laws in multi-dimensions (see [1], [10], [11], [16], [18], [25]). Under suitable conditions on the incoming supersonic flow with a small spherically symmetric perturbation and a symmetric pointed body, there is an extensive literature studying supersonic flow past a pointed body (see [4], [5], [7], [8], [11], [15], [19]). In addition, for the supersonic flow past a sharp wedge, the corresponding results can be referred to [3], [9], [13], [14], [17], [23] and [24]. The first rigorous mathematical analysis was given in [6] by Courant and Friedrichs, who proved that, for a uniform supersonic flow $(0, 0, q_0)$ with constant density $\rho_0 > 0$ which approaches from minus infinity, when the flow hits the sharp circular cone $\sqrt{x_1^2 + x_2^2} = b_0 x_3$, $b_0 > 0$, in direction of the x_3-axis (see Figure 1 below), then there appears a supersonic (weak) conic shock $\sqrt{x_1^2+x_2^2}=s_0x_3$, $s_0>b_0$, attached to the tip of the cone provided that b_0 is less than some critical value $b^*>0$, which is determined by the parameters of the incoming flow. On the other hand, it was indicated in pages 313-314 and page 414 of [6] that if a steady supersonic flow comes from minus infinity and hits a sharp cone, then it follows from the Rankine-Hugoniot conditions and the physical entropy condition that there possibly appear a weak shock (see Figure 2 below) or a strong conic shock attached (see Figure 3 below) at the vertex of the sharp cone, which corresponds to a supersonic shock or a transonic shock, respectively. The question arises which of the two actually occurs. It has frequently been stated that the strong one is unstable and that, therefore, only the weak one could occur. However, a convincing proof of this instability has apparently never been given. In this talk, we will give systematic studies on the stability of weak shocks and instability of strong shocks for the supersonic flow past a sharp cone.

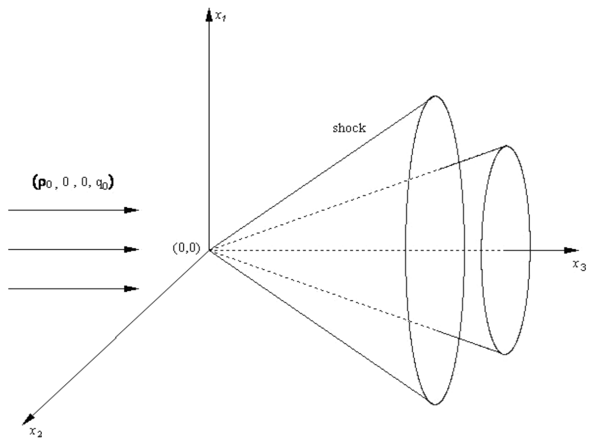

Figure 1: **Weak circular conic shock**

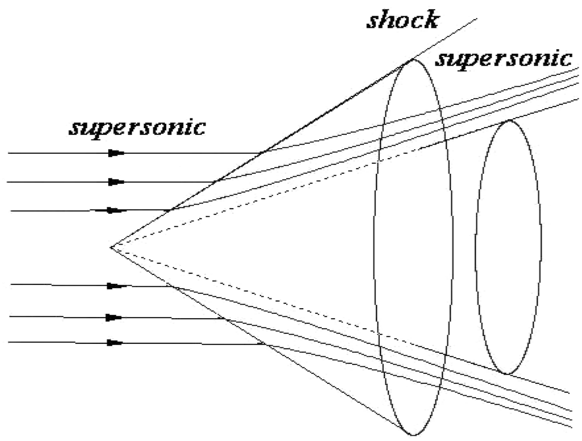

Figure 2: **A weak conic shock for the uniform supersonic flow past a sharp cone**

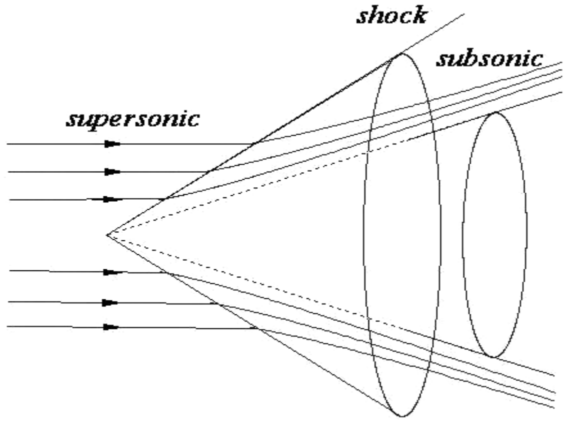

Figure 3: **A strong conic shock for the uniform supersonic flow past a sharp cone**

§1. The case of potential flow equations

The steady and isentropic compressible 3-D Euler system is described by

$$\begin{cases} \sum_{j=1}^{3} \partial_j(\rho u_j) = 0, \\ \sum_{j=1}^{3} \partial_j(\rho u_i u_j) + \partial_i P = 0, \qquad i = 1, 2, 3, \end{cases} \tag{1.1}$$

where $\rho > 0$ is the density, $u = (u_1, u_2, u_3)$ is the velocity, and $P = A\rho^\gamma$ $(1 < \gamma < 3)$ is the pressure with $A > 0$ a fixed constant. In addition, $c(\rho) = \sqrt{P'(\rho)}$ is called the local sound speed.

In this section, we will use the potential equation to describe the motion of the gases. Let $\varphi(x)$ be the potential of velocity $u = (u_1, u_2, u_3)$, i.e., $u_i = \partial_i \varphi$, then it follows from the Bernoulli's law that

$$\frac{1}{2}|\nabla_x \varphi|^2 + h(\rho) = C_0, \tag{1.2}$$

here $\nabla_x = (\partial_1, \partial_2, \partial_3)$, $h(\rho) = \dfrac{c^2(\rho)}{\gamma - 1}$ is the specific enthalpy, and $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ is the Bernoulli's constant which is determined by the uniform supersonic incoming flow from the minus infinity with the constant velocity $(0, 0, q_0)$ and the constant density $\rho_0 > 0$ (see Figure 1 above).

By (1.2) and the implicit function theorem, the density function $\rho(x)$ can be expressed as

$$\rho = h^{-1}(C_0 - \frac{1}{2}|\nabla_x \varphi|^2) \equiv H(\nabla_x \varphi), \tag{1.3}$$

where h^{-1} stands for the inverse function of $h(\rho)$.

Substituting (1.3) into the equation $\sum_{i=1}^{3}\partial_i(\rho u_i) = 0$, which expresses the conservation of mass, yields

$$\sum_{i=1}^{3}\partial_i\big(H(|\nabla\Phi|)\partial_i\Phi\big) = 0. \tag{1.4}$$

More intuitively, for any C^2-solution Φ, (1.4) can be rewritten as a second-order quasilinear equation,

$$\sum_{i=1}^{3}((\partial_i\Phi)^2 - c^2)\partial_i^2\Phi + 2\sum_{1\le i<j\le 3}\partial_i\Phi\partial_j\Phi\partial_{ij}^2\Phi = 0.$$

here $c = c(\rho) = c(H(|\nabla\Phi|))$. It is easy to verify that (1.4) is strictly hyperbolic with respect to the x_3-direction in case $u_3 > c(\rho)$ holds, and (1.4) is strictly elliptic in the case of $|u| < c(\rho)$.

Due to the geometry of the conic surface, it is convenient to work in the cylindrical coordinates (z, r, θ), where

$$(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z), \tag{1.5}$$

$r = \sqrt{x_1^2 + x_2^2}$, and $0 \le \theta \le 2\pi$. Under the change of coordinates (1.5), Eq. (1.4) becomes

$$(\partial_r, \partial_\theta, \partial_z)\cdot(r\rho\partial_r\Phi, \frac{\rho\partial_\theta\Phi}{r}, r\rho\partial_z\Phi) = 0 \tag{1.6}$$

with $\rho = H(|\nabla\Phi|)$.

Let $\Phi^-(z, r, \theta)$ and $\Phi^+(z, r, \theta)$ denote the velocity potential for the flow ahead and past the resulting shock front $r = \chi(z, \theta)$, respectively, where $\chi(0, \theta) = 0$. Then (1.6) splits into two equations. That is, $\Phi^\pm(z, r, \theta)$ satisfy the following equations in their corresponding domains,

$$(\partial_r, \partial_\theta, \partial_z)\cdot(r\rho_-\partial_r\Phi^-, \frac{\rho_-\partial_\theta\Phi^-}{r}, r\rho_-\partial_z\Phi^-) = 0 \quad \text{in} \quad \Omega_- \tag{1.7}$$

and

$$(\partial_r, \partial_\theta, \partial_z)\cdot(r\rho_+\partial_r\Phi^+, \frac{\rho_+\partial_\theta\Phi^+}{r}, r\rho_+\partial_z\Phi^+) = 0 \quad \text{in} \quad \Omega_+, \tag{1.8}$$

here $\rho_\pm = H(|\nabla\Phi^\pm|)$, $\Omega_- = \{(z, r, \theta)\colon r > \chi(z, \theta), 0 \le \theta \le 2\pi, z > 0\}$, and $\Omega_+ = \{(z, r, \theta)\colon b_0 z \le r < \chi(z, \theta), 0 \le \theta \le 2\pi, z > 0\}$.

On the conic surface $r = b_0 z$, Φ^+ satisfies the boundary condition

$$\partial_r\Phi^+ - b_0\partial_z\Phi^+ = 0 \quad \text{on} \quad r = b_0 z, \tag{1.9}$$

while on the conic shock $\Gamma = \{r = \chi(z, \theta)\}$, by Eq. (1.4) and the change of coordinates (1.5), the Rankine-Hugoniot condition becomes

$$[H(|\nabla\Phi|)\partial_r\Phi] - [H(|\nabla\Phi|)\partial_z\Phi]\partial_z\chi = \frac{1}{r^2}[H(|\nabla\Phi|)\partial_\theta\Phi]\partial_\theta\chi \quad \text{on} \quad \Gamma. \tag{1.10}$$

Moreover, the potential $\Phi(z, r, \theta)$ is continuous across the shock, i.e.,

$$\Phi^+\big(z, \chi(z, \theta), \theta\big) = \Phi^-\big(z, \chi(z, \theta), \theta\big) \quad \text{on} \quad \Gamma. \tag{1.11}$$

In addition, it follows from the physical entropy condition that

$$\rho^-(x_3, \chi(x_3, \theta), \theta) < \rho^+(x_3, \chi(x_3, \theta), \theta). \tag{1.12}$$

For the strong shock, the velocity field behind the shock admits a determined state:

$$\lim_{r=sx_3, x_3\to+\infty} |\nabla_x \Phi^+(x_3, sx_3, \theta)| \quad \text{exists for } b_0 \le s \le \frac{\chi(x_3,\theta)}{x_3} \tag{1.13}$$

Finally, we impose initial conditions on $\Phi^-(z, r, \theta)$,

$$(\Phi^-, \partial_z \Phi^-)(0, r, \theta) = (\varepsilon\Phi_0^-, q_0 + \varepsilon\Phi_1^-)(r, \theta), \tag{1.14}$$

where $\varepsilon > 0$ is a small parameter, $q_0 > c(\rho_0)$, and $\Phi_0^-, \Phi_1^- \in C_0^\infty\big((0,l)\times[0,2\pi]\big)$ for some fixed number $l > 0$.

For the weak conic shock, we have the following global stability result (see [5] and [12]):

Theorem 1.1. (Global stability of weak conic shock for potential flow equation) *For small $b_0 > 0$ and a sufficiently large speed q_0, there exists a small constant $\varepsilon_0 > 0$ depending on q_0, ρ_0, b_0, and γ such that problem (1.7)-(1.8) together with (1.9)-(1.12) and (1.14) possesses a global C^∞* **supersonic shock solution** $(\Phi^\pm(z, r, \theta), \chi(z, \theta))$ *for any $\varepsilon < \varepsilon_0$. Moreover, $(\nabla_x \Phi^+, \dfrac{\chi(z,\theta)}{z})$ approaches the corresponding quantities for the incoming uniform supersonic flow $(0, 0, q_0)$ with density ρ_0 past the sharp circular cone $r = b_0 z$ with rate $(1+z)^{-m_0}$ for any positive number $m_0 < \dfrac{1}{2}$.*

For the strong conic shock, we also have the following global stability result (see [20]):

Theorem 1.2.(Global stability of strong conic shock for potential flow equation) *If the perturbed supersonic incoming flow with the initial state (1.14) hits the circular cone $\sqrt{x_1^2 + x_2^2} = b_0 x_3$ along the x_3-direction, where q_0 is an appropriately large constant and $b_0 > 0$ is a fixed constant, then there exist a small constant ε_0 and a suitable constant δ_0 with $0 < \delta_0 < 1$ such that problem problem (1.7)-(1.8) together with (1.9)-(1.14) has a global C^{1,δ_0} multidimensional* **transonic shock solution** $(\Phi_\pm(x_3, r, \theta), \chi(x_3, \theta))$ *as $\varepsilon < \varepsilon_0$. Moreover, $(\nabla_x \Phi_\pm(x_3, r, \theta), \partial_3 \chi(x_3, \theta))$ tend to the corresponding ones for the uniform supersonic incoming flow $(0, 0, q_0; \rho_0)$ past the circular cone $r = b_0 x_3$ with the rate $(1 + x_3)^{-\delta_0}$ at infinity and $|\frac{\partial_\theta \chi(x_3,\theta)}{r}| \le \frac{C\varepsilon}{(1+x_3)^{\delta_0}}$ holds true.*

§2. **The case of full Euler system**

In Section 1, we have shown the global stability of weak or strong conic shock for potential flow equation. In this section, under some suitable assumptions, because of the essential influence of the rotation of Euler flow, we show that a global transonic conic shock is actually unstable when the surface of the conic body is perturbed.

The steady full compressible Euler system is described as

$$\begin{cases} \sum_{j=1}^{3} \partial_j(\rho u_j) = 0, \\ \sum_{j=1}^{3} (\rho u_i u_j) + \partial_i P = 0, \quad i = 1, 2, 3, \\ \sum_{j=1}^{3} \partial_j((\rho e + \frac{1}{2}\rho|u|^2 + P)u_j) = 0, \end{cases} \tag{2.1}$$

where ρ, $u = (u_1, u_2, u_3)$; and P, e and S stand for the density, velocity, pressure, internal energy, and specific entropy, respectively. Moreover, the pressure function $P = P(e, S)$ and the internal energy function $e = e(\rho, S)$ are smooth in their arguments, which satisfy $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$. In addition, $c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)}$ stands for the sound speed. For the ideal polytropic gases, the equations of state are given by

$$P = A\rho^\gamma exp(\frac{S}{c_v}) \quad \text{and} \quad e = \frac{P}{(\gamma - 1)\rho},$$

where A, c_v and γ are positive constants and $1 < \gamma < 3$ (especially $\gamma = 1.4$ with respect to the air).

It is assumed that there is a uniform supersonic incoming flow with the constant state $(\rho_0, 0, 0, q_0, P_0)$, and that the flow hits the perturbed conic body along the x_3-direction, whose surface equation is denoted by $r = b(x_3)$, where $r = \sqrt{x_1^2 + x_2^2}$, $b(x_3) = b_0 x_3 + \varepsilon\varphi(x_3)$, and $\varepsilon > 0$ is a small constant, $\varphi(x_3) \in C_0^\infty(0, l)$ with some fixed positive number $l > 0$. In particular, we point out that $b_0 > 0$ is less than some critical value b_* so that the resulted shock will attach at the vertex of the conic body (see Figure 3). Because of the symmetric property of the perturbed conic surface, it is convenient to introduce the following cylindrical coordinates x_3, r to study our problem:

$$x_3 = x_3, \quad r = \sqrt{x_1^2 + x_2^2}. \tag{2.2}$$

For the polytropic gases and axisymmetric solution to (2.1), which has the form $(\rho(x), u_1(x), u_2(x), u_3(x), P(x)) \equiv (\rho(x_3, r), U(x_3, r)\frac{x_1}{r}, U(x_3, r)\frac{x_2}{r}, u_3(x_3, r), P(x_3, r))$, (2.1) can be reduced into

$$\begin{cases} \partial_r(r\rho U) + \partial_3(r\rho u_3) = 0, \\ \partial_r(r\rho U^2) + \partial_3(r\rho U u_3) + r\partial_r P = 0, \\ \partial_r(r\rho U u_3) + \partial_3(r\rho u_3^2) + r\partial_r P = 0 \end{cases} \tag{2.3}$$

and

$$\frac{1}{2}(U^2 + u_3^2) + \frac{\gamma P}{(\gamma - 1)\rho} = \frac{1}{2}q_0^2 + \frac{\gamma P_0}{(\gamma - 1)\rho_0} \equiv C_0. \tag{2.4}$$

Suppose that the flow field behind the possible shock $r = \chi(x_3)$ is denoted by $(\rho^+(x_3, r), U^+(x_3, r), u_3^+(x_3, r), P^+(x_3, r))$, Then, in the domain $\Omega_+ \equiv \{(x_3, r) : x_3 > 0, b(x_3 < r < \chi(x_3))\}$, $(\rho^+, U^+,$

u_3^+, P^+) satisfies:

$$\begin{cases} \partial_r(r\rho^+U^+) + \partial_3(r\rho^+u_3^+) = 0, \\ \partial_r(r\rho^+(U^+)^2) + \partial_3(r\rho^+U^+u_3^+) + r\partial_r P^+ = 0, \\ \partial_r(r\rho^+U^+u_3^+) + \partial_3(r\rho^+(u_3^+)^2) + r\partial_r P^+ = 0, \\ \frac{1}{2}((U^+)^2 + (u_3^+)^2) + \frac{\gamma P^+}{(\gamma-1)\rho^+} = C_0. \end{cases} \tag{2.5}$$

On the shock $r = \chi(x_3)$, the Rankine-Hugoniot conditions imply

$$\begin{cases} [\rho U] - \chi'(x_3)[\rho u_3] = 0, \\ [P + \rho U^2] - \chi'(x_3)[\rho U u_3] = 0, \\ [\rho U u_3] - \chi'(x_3)[P + \rho u_3^2] = 0. \end{cases} \tag{2.6}$$

Meanwhile, the physical entropy condition holds true:

$$P_0 < P^+(x_3, \chi(x_3)). \tag{2.7}$$

Because of the fixed wall condition, we have that on the conic surface $r = b(x_3)$

$$U^+ = b'(x_3)u_3^+. \tag{2.8}$$

In addition, from the physical point of view, when a subsonic flow in an unbounded domain is called stable, it should admit a determined state at infinity. Thus, along each stream line starting from the shock curve, we naturally pose

$$\lim_{x_3 \to \infty \text{ along stream line}} (\rho^+, U^+, u_3^+, P^+) \quad \text{exists for } b(x_3) \le r \le \chi(x_3). \tag{2.9}$$

The main result in this section can be stated as follows.

Theorem 2.1 (Instability of a global transonic shock). *Under the assumptions above, there exists an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, problem (2.5) with (2.6)-(2.9) has no global solution $(\rho^+(x), U^+(x), u_3^+(x), P^+(x))$ which admits the following properties:*

(i) $\chi(x_3) \in C^2(0,\infty) \cap Lip[0,\infty)$, and

$$\|\chi'(x_3) - s_0\|_{L^\infty(0,\infty)} \le C(\varepsilon_0),$$

where below the generic positive function $C(\varepsilon_0)$ is a suitably small quantity, which is independent of ε.

(ii) $(\rho^+, U^+, u_3^+, P^+) \in C^1(\bar{\Omega}_+ \setminus (0,0)) \cap L^\infty(\Omega_+)$,

$$\|(\rho^+, U^+, u_3^+, P^+)(x_3, r) - (\hat{\rho}, \hat{U}, \hat{u}_3, \hat{P})(\frac{r}{x_3})\|_{L^\infty(\Omega_+)} \le C(\varepsilon_0).$$

In addition, if we denote by the domain $\Omega_{+,\delta} \equiv \{(x_3, r) : x_3 > \delta, b(x_3 < r < \chi(x_3))\}$, with $\delta > 0$ any fixed constant, then there exists a quantity $C_\delta(\varepsilon_0) > 0$ depending only on δ and ε_0 such that

$$\|(\rho^+, U^+, u_3^+, P^+)(x_3, r) - (\hat{\rho}, \hat{U}, \hat{u}_3, \hat{P})(\frac{r}{x_3})\|_{C^1(\Omega_{+,\delta})} \le C_\delta(\varepsilon_0).$$

Here $(\hat{\rho}, \hat{U}, \hat{u}_3, \hat{P})(\frac{r}{x_3})$ *stands for the extension of the self-similar downstream subsonic state* $(\tilde{\rho}, \tilde{U}, \tilde{u}_3, \tilde{P})(\frac{r}{x_3})$ *behind the transonic shock* $r = s_0 x_3$*, which is formed by the supersonic incoming flow* $(\rho_0, 0, 0, q_0, P_0)$ *past the cone* $\{x : r < b_0 x_3\}$.

(iii) Denoting by the stream line equation $r = r(x_3, y)$*, which starts from the point* $(y, \chi(y))$ *of the shock with* $y > 0$*, and setting* $\lim_{x_3\to\infty}(\rho^+, U^+, u_3^+, P^+)(x_3, r(x_3, y)) = (\rho_\infty(y), U_\infty(y), u_{3,\infty}(y), P_\infty(y))$*, then* $(\rho_\infty(y), U_\infty(y), u_{3,\infty}(y), P_\infty(y)) \in C^1(0,\infty)$ *and*

$$\lim_{x_3\to\infty} \nabla_{x_3,y}((\rho^+, U^+, u_3^+, P^+)(x_3, r(x_3, y)) - (\rho_\infty(y), U_\infty(y), u_{3,\infty}(y), P_\infty(y))) = 0.$$

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Penetration of bistable fronts through a perforated wall

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1. Introduction

We consider a bistable reaction-diffusion equation on $\mathbb{R}^N \setminus K$, where K represents an obstacle that can be regarded as a wall of infinite span with periodically arrayed holes. More precisely, K is a closed subset of $\mathbb{R}^N$ with smooth surface such that its projection onto the x_1-axis is bounded, while it is periodic in the rest of variables $y := (x_2, \ldots, x_N)$. We assume that $\mathbb{R}^N \setminus K$ is connected. Our goal is to study what happens when a planar traveling front coming from $x_1 = +\infty$ meets the wall K.

We first show that there is clear dichotomy between *propagation* and *blocking*. In other words, the traveling front either completely penetrates the wall or is totally blocked, and that there is no intermediate behavior (Theorem 1). This dichotomy result will be proved by what we call a De Giorgi type lemma for the elliptic equation $\Delta v + f(v) = 0$ on $\mathbb{R}^N$, which may be of interest in its own right (Lemma 2.2). Then we will discuss sufficient conditions for blocking, and those for propagation.

1.1. **Formulation of the problem.** The equation we consider is given in the form

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where K is a closed subset of $\mathbb{R}^N$ with smooth boundary ∂K such that

$$K \subset \{x \in \mathbb{R}^N \mid -M \leq x_1 \leq 0\} \tag{1.2}$$

for some constant $M > 0$. Here each point $x \in \mathbb{R}^N$ will be written in the form

$$x = (x_1, x_2, \ldots, x_N) = (x_1, y), \; y \in \mathbb{R}^{N-1}.$$

We assume that K is periodic in y, namely,

$$K + L_i \boldsymbol{e}_i = K \quad \text{for some } L_i > 0 \; (i = 2, \ldots, N)$$

We assume that Ω is connected. The function f is a bistable nonlinearity satisfying, for some $0 < a < 1$,

$$f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0,$$
$$f(s) < 0 \; (s \in (0,a) \cup (1,\infty)), \quad f(s) > 0 \; (s \in (-\infty,0) \cup (a,1)).$$

We also assume that

$$\int_0^1 f(s)ds > 0.$$

This condition implies that the steady-state $u = 1$ has a lower potential than the steady-state $u = 0$; hence it ensures (see [5]) the existence of a traveling wave solution for the one-dimensional equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},$$

connecting 0 to 1, which is given in the form

$$u(x,t) = \phi(x+ct), \quad x \in \mathbb{R}, \quad t \in \mathbb{R},$$

where $c > 0$ is a constant and ϕ is a solution of the problem

$$\begin{cases} \phi''(z) - c\phi'(z) + f(\phi(z)) = 0 \quad (z \in \mathbb{R}), \\ 0 < \phi < 1, \quad \phi(-\infty) = 0, \; \phi(+\infty) = 1. \end{cases}$$

Since f is a bistable nonlinearity, it is known that such a traveling wave is unique up to translation. Under the normalization condition

$$\phi(0) = a,$$

the pair (ϕ, c) satisfying the above is uniquely determined.

Now we consider a **planar traveling front** on $\mathbb{R}^N$ that comes from the direction $x_1 = +\infty$ and propagates toward $x_1 = -\infty$. This is a solution of the equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N$$

that is given in the form

$$u(t,x) := u(t, x_1, y) = \phi(x_1 + ct).$$

The level surfaces of this solution, namely the set $\{x \in \mathbb{R}^N \mid u = \alpha\}$ for $0 < \alpha < 1$ is a hyperplane that is perpendicular to the x_1-axis. Our goal is to study the behavior of such a planar front in the presence of the wall K.

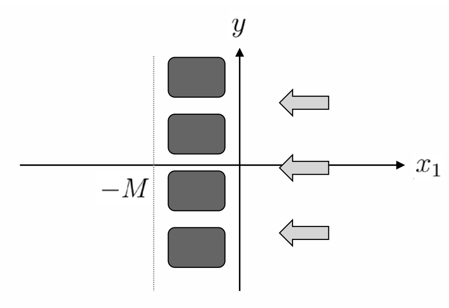

FIGURE 1. An image of a planar front approaching the wall K

In our previous paper [3], we studied the same question for the case where K is a compact obstacle. To study such a question, we first established the existence of a solution that represents a planar front approaching an obstacle K. This is given in the following lemma, which also holds in the present setting.

Lemma 1.1 ([3]). *There exists a unique entire solution $\bar{u}$ of* (1.1) *such that* $0 < \bar{u}(t,x) < 1$ $(x \in \overline{\Omega}, t \in \mathbb{R})$ *and*

$$\lim_{t \to -\infty} \sup_{x \in \Omega} |\bar{u}(t,x) - \phi(x_1 + ct)| = 0.$$

This solution satisfies $\bar{u}_t > 0$ for all $x \in \overline{\Omega}$, $t \in \mathbb{R}$.

The above lemma can be proved by using a pair of super- and subsolutions that are similar to those given in [6, 7], in which the existence of an entire solution exhibiting two colliding fronts was studied.

Since $\bar{u}$ is monotone increasing in t, the following limit exists:

$$v(x) := \lim_{t\to\infty} \bar{u}(t,x),$$

which we call the **limit profile**. The limit profile v satisfies

$$\begin{cases} \Delta v + f(v) = 0, & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The propagation and blocking are then characterized as follows:

$$\begin{cases} \text{invasion} & \Leftrightarrow \lim_{x_1\to-\infty} v(x_1,y) = 1, \\ \text{blocking} & \Leftrightarrow \lim_{x_1\to-\infty} v(x_1,y) = 0. \end{cases} \tag{1.4}$$

Note that there are many solutions of (1.3) that do not satisfy $v(x_1,y) \to 1$ nor $v(x_1,y) \to 0$ as $x_1 \to -\infty$. However, as far as the limit profile v is concerned, either of the above two always holds and there is no intermediate behavior (Theorem 1).

This lecture is organized as follows. In Section 2, we will prove Theorem 1 concerning the invasion / blocking dichotomy. This will be shown by using what we call a De Giorigi type lemma (Lemma 2.2). As a corollary to the above theorem, one can show that if blocking occurs for each wall K_j ($j = i, 2, 3, \ldots$), then the same is true of the limit wall $K = \lim_{j\to\infty} K_j$.

In Section 3, we will give a sufficient condition for the blocking. Roughly speaking, blocking occurs if the holes are sufficiently small (Theorem 2).

In Section 4, we present sufficient conditions for invasion (i.e., penetration). There are three types of such conditions, namely:

(a) Wall with large holes;
(b) Wall consisting of tiny debris;
(c) A skeleton wall.

The figure below shows a schematic image of those three types of walls. Note that the holes in (b), (c) are not necessarily small, but the separations between the holes look like either tiny particles or thin blades.

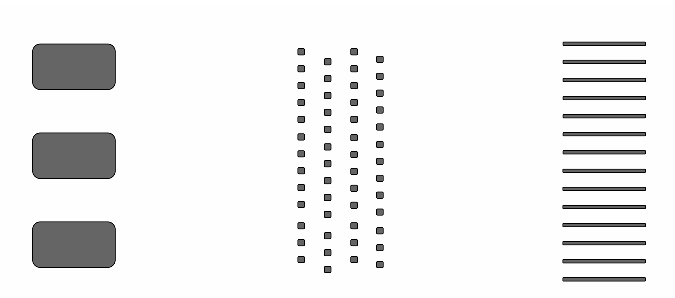

FIGURE 2. (a) wall with large holes (b) debris wall (c) skeleton wall

2. Invasion / blocking dichotomy

Here we state and prove the result that there is either blocking or propagation.The proof calls upon the result proved in the following section.

Theorem 1 (Dichotomy)**.** *The following alternatives holds for the limit profile:*

$$\lim_{x_1\to-\infty} v(x_1,y)=1 \quad \text{(invasion)}, \qquad \lim_{x_1\to-\infty} v(x_1,y)=0 \quad \text{(blocking)}$$

The above convergence is uniform with respect to $y\in\mathbb{R}^{N-1}$ *and also to* K *so long as* $K\subset\{x\in\mathbb{R}^N \mid -M\le x_1\le 0\}$ *for a fixed* $M>0$*.*

Corollary 2.1. *Let* $K_1,K_2,K_3,\ldots$ *be a sequence of walls satisfying*

$$K_j\subset\{x\in\mathbb{R}^N \mid -M\le x_1\le 0\} \quad (j=1,2,3,\ldots)$$

and converging to a wall K *in a certain appropriate sense. If blocking occurs for every* K_j $(j=1,2,3,\ldots)$ *then the same holds for* K*.*

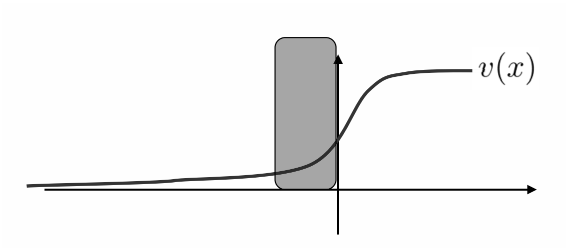

Figure 3. An image of v showing blocking of a front

2.1. **Proof of the dichotomy results.**

Proof of Theorem 1. It suffices to show that, for any δ with $0<\delta<\frac{1}{2}$, there exists $M_\delta>0$ such that the limit profile v satisfies

$$v(x)\in(0,\delta]\cup[1-\delta,1) \quad \text{for all} \quad x\in\Omega \quad \text{with} \quad x_1\le -M_\delta. \tag{2.1}$$

Suppose the contrary. Then there exists a sequence of walls

$$K_j\subset\{x\in\mathbb{R}^N \mid -M\le x_1\le 0\} \quad (j=1,2,3,\ldots)$$

and a sequence of real numbers $b_j\to\infty$ such that the limit profile v_j corresponding to the wall K_j satisfies

$$\delta<v_j(-b_j,y_j)<1-\delta \quad (j=1,2,3,\ldots)$$

for some $y_j\in\mathbb{R}^{N-1}$ within the unit cell of the periodicity of K (hence $\{y_j\}$ are bounded). Set $w_j(x_1,y)=v_j(x-b_j,y)$. Then w_j satisfies

$$\Delta w_j+f(w_j)=0 \quad (x_1\in(-\infty,-M+b_j]), \quad \delta<w_j(0,y_j)<1-\delta.$$

Since v_j are uniformly bounded, by the elliptic estimates, we can choose a convergent subsequence $w_j\to w_\infty$, $y_j\to y_\infty$, and the limit function w_∞ satisfies

$$\Delta w_\infty+f(w_\infty)=0 \quad (x\in\mathbb{R}^N), \quad \delta\le w_\infty(0,y_\infty)\le 1-\delta.$$

Furthermore, since each v_j is stable from below, it is stable (at least lineary neutrally stable) on every compact subset D of Ω. Since such stability is robust under spatial translation and limiting procedures, we see that w_∞ is also stable in the same sense. Consequently, by Lemma 2.2, w_∞ is a function of x_1 only, and it is monotone. Thus w_∞ coincides with 0 or a or 1, but since a is unstable, we have either $w_\infty = 0$ or $w_\infty = 1$. This, however, contradicts the inequality $\delta \le w_\infty(0, y_\infty) \le 1 - \delta$. This contradiction proves the Theorem. □

2.2. **De Giorgi type lemma.**

Lemma 2.2 (De Giorgi type lemma)**.** *Let $v(x) = v(x_1, y)$ be a non-negative bounded solution of $\Delta v + f(v) = 0$ in $\mathbb{R}^N$ that is stable and periodic with respect to y. Then v is a function of x_1 alone and is monotonic.*

Here, a solution v of $\Delta v + f(v) = 0$ is termed **(semi-)stable** if for any compact and Lipschitz domain $D \subset \mathbb{R}^N$, we have $\lambda_1[-\Delta - f'(v)], D] \ge 0$, where $\lambda_1[L, D]$ denotes the principal eigenvalue of the Dirichlet problem for the operator L in D. Note that the stability in such a sense is preserved by spatial translation and by the limiting procedure $v_j \to v$.

The proof of Lemma 2.2 is given by adopting an argument in [2]. We omit the proof, but roughly speaking the argument goes as follows. First, we note that there exists a generalized eigenfunction φ corresponding to $\lambda \ge 0$:

$$-\Delta\varphi - f'(v)\varphi = \lambda\varphi, \quad \varphi > 0 \text{ in } \mathbb{R}^N, \quad \varphi \text{ is periodic in } y.$$

We then show that, for each $i \in \{1, 2, \dots, N\}$ there is a constant c_i such that $\partial v/\partial x_i = c_j\varphi$. To prove this, we use some cut-off function and an estimate found in [2]. Since v is periodic in $y = (x_2, \dots, x_N)$, $\partial v/\partial x_i$ must change sign unless it is identically equal to 0. This shows that $\partial v/\partial x_i = 0$ $(j = 2, \dots, N)$, therefore v is a function of x_1 only. Once this is shown, the monotonicity of v follows easily from its stability.

3. Sufficient conditions for blocking

Theorem 2 (Blocking)**.** *Let the periodicity length $L_2, \dots, L_N$ be fixed. Let K_ε $(0 < \varepsilon < \varepsilon_0)$ be a family of walls sharing the same periodicity with periodically arrayed holes of size $\mathcal{O}(\varepsilon)$. Then blocking occurs for K_ε for all sufficiently small $\varepsilon > 0$.*

This theorem can be proved by using a variational argument similar to that found in [8], in which the existence of a nonconstant stationary solution in a dumbbel-shaped domain was proved. Similar arguments (though slightly different) are also found in the papers [3, 1].

4. Sufficient condition for invasion

In this section we discuss sufficient conditions for invasion. As we mentioned above, there are three types of walls that let the front pass through (see Figure 1):

(a) Wall with large holes;
(b) Wall consisting of tiny debris;
(c) A skeleton wall.

Here, the case (b) refers to a wall $K = \varepsilon$ consisting of small obstacles of size $\mathcal{O}(\varepsilon)$, whose positions are fixed. As $\varepsilon \to 0$, each small obstacle converges to a point. The case (c) refers to a wall K_ε that converges as $\varepsilon \to 0$ to a set consisting of 1-codimensional panels that are parallel to the x_1-axis. The thickness of the panels is $\mathcal{O}(\varepsilon)$.

Theorem 3 (Walls that allow invasion)**.** *As regards the wall of type (a), if one of its holes allows a ball of a certain critical size to pass through from one side of the wall to the other, then invasion occurs. As regards the walls of type (b) and type (c), invasion occurs for sufficiently small* $\varepsilon > 0$.

We remark that in the case (b) and (c), there may not exist a large hole. Nonetheless, invasion may still occur.

The proof of the above theorem for the case (a) is done by constructing a compactly supported subsolution.

The proof for cases (b) and (c) are completely different. We consider the limit $K_j \to K$, and we get a profile function $v(x)$, still in the blocking regime thanks to Corollary 2.1, outside of the limit set $K = \lim K_j$. Then we show that the singluar set K is removable, which leads to a contradiciton. The details are ommited in this abstract.

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THE UNIQUENESS OF WULFF SHAPE

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Abstract. An $\mathbb{S}^2$ immersed compact connected Constant Anisotropic Mean Curvature (CAMC) surface without boundary in $\mathbb{R}^3$ is unique up to translation and dilation for general energy density function γ that defines Minkowski norm on the tangent space of the surface in Finsler geometry. So it must be the boundary of Wulff shape and an affirmative reply to the uniqueness conjecture of Wulff shape is obtained. Moreover there are infinite many immersed CAMC surfaces that are not the boundary of Wulff shape, and for each $\mathbb{T}_k$ $(k \geq 1)$ immersed compact connected Constant Mean Curvature (CMC) surface there exists a unique $\mathbb{T}_k$ immersed compact connected CAMC surface in $\mathbb{R}^3$ up to translation and dilation.

1. from CMC to CAMC

An important isoperimetric problem is to classify the surfaces in Euclidean space $\mathbb{R}^3$ that have critical area energy subject to the requirement that they enclose a fixed volume. It is often called soap bubbles problem in physics. Although the soap bubbles, generally in physics, are anisotropic, only isotropic case is well understood. In the isotropic case, it is also called Hopf conjecture, and the solutions are constant mean curvature (CMC) surfaces. Delaunary (1841) [De] found the examples of CMC surfaces which were rotationally invariant, Hopf [Ho] proved that a CMC surface homeomorphic to $\mathbb{S}^2$ is a round sphere, and Aleksandrov (1956) [A] proved that only round sphere is the embedded compact constant mean curvature surface. Hsiang (1982) [Hs] constructed nonrounded CMC surfaces in higher dimensions, and Wente (1984) [We] gave the examples of CMC torus that are counterexamples to the Hopf conjecture. The CMC surfaces with higher genus were constructed by Kapouleas [K][K1][K2][K3].

To define an anisotropic version of CMC surface, an anisotropic structure on a surface M characterized by a function $\gamma|_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{R}^+$ on Gauss map n of the surface M is necessary. Here $\gamma|_{\mathbb{S}^2} \equiv 1$ corresponds to isotropic structure. In this

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paper we suppose $\gamma : \mathbb{R}^3 \to \mathbb{R}^+$ satisfying

$$(1.1)\quad \begin{aligned} &(1)\text{twice differentiable in } \quad \mathbb{R}^3 \setminus \{0\};\\ &(2)\text{homogeneity of degree one} : \forall \lambda \geq 0, \quad \gamma(\lambda p) = \lambda \gamma(p);\\ &(3)\text{strict convexity} : \forall p \in \mathbb{S}^2, \forall \eta \in \mathbb{R}^3 \setminus \{0\}, \text{if } \eta \cdot p = 0, \text{ then}\\ &\eta \cdot (\eta \nabla_p^2 \gamma(p)) > 0. \end{aligned}$$

Notice that mean curvature (MC) of a surface is the change ratio of surface area per change of volume enclosed by the surface. If a surface M has anisotropic structure, instead of area, it is natural to use the change ratio of surface energy

$$\int_M \gamma(n(x)) d\sigma(x)$$

per change of volume to define anisotropic mean curvature (AMC), where $d\sigma$ denotes the surface element, $n(x)$ denotes the Gauss map of the surface. A special constant AMC (CAMC) surface is the Wulff shape defined as (see [G])

$$\mathcal{W} = \cap_{\forall p \in \mathbb{S}^2} \{x \in \mathbb{R}^3 : \quad x \cdot p \leq \gamma(p)\}.$$

It is clear that if $\gamma|_{\mathbb{S}^2} \equiv 1$, the anisotropic mean curvature is the usual mean curvature, and a CAMC surface is just a CMC surface.

2. Special Finsler structure

Generally, in Finsler geometry [BCS], γ is called Finsler structure that determines a Minkowski norm provided that γ satisfies (1.1)(1)-(2) and

$$(3') \quad \text{strong convexity} : \ \{\frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)]\} > 0, \quad \forall p \in TM \setminus \{0\}.$$

The unit Finsler ball $\{p \in \mathbb{R}^3 : \gamma(p) \leq 1\}$ with respect to the Finsler structure γ is also called $Frank(\gamma)$ that is the convex conjugate of the Wulff shape $\mathcal{W}$[G1]. On the other hand, from the Wulff shape $\mathcal{W}$ we can define another Finsler structure F such that the Wulff shape $\mathcal{W}$ is just the unit Finsler ball $\mathcal{W} = \{p \in \mathbb{R}^3 : F(p) \leq 1\}$ with respect to F (see [BCS: Ex.1.2.7]).

In smooth case, the Wulff shape $\mathcal{W}$ with respect to $\gamma|_{\mathbb{S}^2}$ can also be derived from the derivatives of γ. Let $D\gamma|_{\mathbb{S}^2}$ and $D^2\gamma|_{\mathbb{S}^2}$ denote the gradient and Hessian with respect to the standard metric on $\mathbb{S}^2$. If

$$(D^2\gamma|_{\mathbb{S}^2} + \gamma|_{\mathbb{S}^2}\mathbb{I}) > 0 \quad \text{on} \quad \mathbb{S}^2$$

then $\{\gamma|_{\mathbb{S}^2} n + D^2\gamma|_{\mathbb{S}^2} : \ n \in \mathbb{S}^2\}$ is a convex surface in $\mathbb{R}^3$ that is just the boundary of Wulff shape $\mathcal{W}$.

From [BCS: Theorem 1.2.2 (1.2.5)(1.2.9)] (3′) implies (1.1)(3) and

$$(2.1) \qquad \inf_{|p|=1} \gamma(p) > 0,$$

because

$$\frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)] = \gamma(p)\frac{\partial^2}{\partial p_j \partial p_k}\gamma(p) + \frac{\partial}{\partial p_j}\gamma(p)\frac{\partial}{\partial p_k}\gamma(p)$$

and

$$\sum_{j,k} p_j p_k \frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)] = \gamma^2(p)$$

as well as

$$\begin{aligned}
&\eta \cdot (\eta \nabla_p^2 \gamma(p)) \\
&= \frac{1}{\gamma^3(p)}\{\gamma^2(p)\sum_{j,k}\eta_j\eta_k \frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)] - (\sum_{j,k} p_j \eta_k \frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)])^2\} \qquad (2.2)\\
&> 0, \quad \forall \eta \in \mathbb{R}^3 \setminus \{0\} : \ \eta \cdot p = 0
\end{aligned}$$

where the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left(\sum_{j,k} p_j \eta_k \frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)]\right)^2 \\
&< \left(\sum_{j,k} \eta_j \eta_k \frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)]\right)\left(\sum_{j,k} p_j p_k \frac{\partial^2}{\partial p_j \partial p_k}[\frac{1}{2}\gamma^2(p)]\right) \qquad (2.3)
\end{aligned}$$

is used in the last step.

On the other hand we have

Lemma 2.1. (1.1)(2) *implies*

$$p \cdot \nabla \gamma(p) = \gamma(p), \quad p \cdot \nabla^2 \gamma(p) = 0, \quad \forall p \in \mathbb{S}^2,$$

and (1.1)(3) implies there exists $c_0^\gamma > 0$, such that

$$\forall p \in \mathbb{S}^2, \forall \eta \in \mathbb{R}^3 \setminus \{0\}, \quad \textit{if } \eta \cdot p = 0, \textit{ then} \quad \eta \cdot (\eta \nabla^2 \gamma(p)) \geq c_0^\gamma |\eta|^2.$$

Moreover (1.1) implies

$$\begin{aligned}
&\sup_{|p|=1} |\nabla \gamma(p)| \leq \exists c_2^\gamma < \infty, \quad \sup_{|p|=1} |\nabla^2 \gamma(p)| \leq \exists c_3^\gamma < \infty, \qquad (2.4)\\
&\gamma(p) > 0 \quad \textit{for almost all } p \in \mathbb{S}^2 \textit{ except a finite set.}
\end{aligned}$$

Proof. The first part is induced from

$$\gamma(p) = \frac{d}{d\lambda}|_{\lambda=1}\gamma(\lambda p) = p \cdot \nabla \gamma(p),$$

and

$$0 = \frac{d^2}{d\lambda^2}\gamma(\lambda p) = p \cdot (p \cdot \nabla_q^2 \gamma(\lambda p)), \quad q := \lambda p.$$

To prove the second part, notice that $\gamma \in C^2(\mathbb{R}^3 \setminus \{0\})$, $\mathbb{S}^2$ is compact and for all $p \in \mathbb{S}^2$ if $\gamma(p) = 0$ then p is a minimizer and $D\gamma|_{\mathbb{S}^2}(p) = 0$. Moreover for all

$q \in \mathbb{S}^2$ in a neighborhood of p
(2.5)

$$\begin{aligned}\gamma(q) &= (q-p)\cdot\nabla\gamma(p) + (q-p)\cdot((q-p)\cdot\int_0^1 d\tau\int_0^\tau \nabla^2\gamma(p+t(q-p)))dt\\ &= ((q\cdot p)-1)p\cdot\nabla\gamma(p) + (q-(q\cdot p)p)\cdot\nabla\gamma(p)\\ &+ (q-p)\cdot((q-p)\cdot\int_0^1 d\tau\int_0^\tau \nabla^2\gamma(p+t(q-p)))dt\\ &= ((q\cdot p)-1)\gamma(p) + (q-(q\cdot p)p)\cdot\nabla\gamma(p)\\ &+ (q-p)\cdot((q-p)\cdot\int_0^1 d\tau\int_0^\tau \nabla^2\gamma(p+t(q-p)))dt\\ &= (q-(q\cdot p)p)\cdot\nabla\gamma(p)\\ &+\int_0^1 d\tau\int_0^\tau (q-(q\cdot p(t))p(t))^T\cdot\nabla^2\gamma(p(t))\cdot(q-(q\cdot p(t))p(t))dt > 0,\end{aligned}$$

provided $|q-p| \ll 1$. Here $p(t) := p+t(q-p)$, $q\cdot\nabla\gamma(q) = \gamma(q)$ and $q\cdot\nabla^2\gamma(q) = 0$ for all $q \in \mathbb{S}^2$(c.f. the first part of Lemma 1.1), as well as (1.1)(3) are used. So the zero points of γ are isolated. Since these neighborhoods of zero points cover $\mathbb{S}^2$ and $\mathbb{S}^2$ is compact, there exists a finite sub-covering. So we have the last claim. □

3. Wulff conjecture

G. Wulff (1901)[W] formulated the generalized isoperimetric problem as "**find a set minimizing the surface energy with fixed volume**" and conjectured that the answer is a dilation of the Wulff shape. A. Dinghas (1944)[D] gave a formal proof, and J. Taylor (1978)[T] gave a precise proof for very general surface energies defined by geometric measure theory. The minimizer of the generalized isoperimetric problem (Wulff's problem) is also unique up to translation (see [DP], [F], [FM], [KoP], [M][P] and [S] for detail). It is known that the anisotropic mean curvature of the boundary of the Wulff shape is constant(see [So][G]). But the converse problem seems to be open unless in the isotropic case[G][M]: If an embedded compact hypersurface has a constant anisotropic mean curvature(CAMC), is it the boundary of a Wulff shape up to translation and dilation? In [GZ] we proved that the CAMC immersion of the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ must be the boundary of Wulff shape provided that the function $\gamma|_{\mathbb{S}^2}$ is close to 1.

4. main results

Theorem 4.1. *Suppose (1.1) and (1.2). Then the $\mathbb{S}^2$ immersed compact connected weak CAMC surface without boundary in $\mathbb{R}^3$ w.r.t. γ is unique up to translation and dilation. That is, it must be a boundary of Wulff shape up to translation and dilation.*

Furthermore, there exist infinite many immersed CAMC surfaces in $\mathbb{R}^3$ that are not the boundary of Wulff shape.

Theorem 4.2. *Suppose (1.1) and (1.2). Then for each* $k \geq 1$*, for each* $\mathbb{T}_k$ *immersed compact connected CMC surface without boundary, there exists a unique* $\mathbb{T}_k$ *immersed compact connected CAMC surface in* $\mathbb{R}^3$ *w.r.t.* γ *up to translation and dilation.*

Remark (1) It is well known that the only embedding compact CMC surface or constant mean curvature immersion of sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is round sphere, and Wente torus is an immersed CMC surface in $\mathbb{R}^3$. There are infinitely many CMC surfaces with higher genus [K3].

(2) The main results of this paper were first published in [42]. After that, M. Koiso and B. Parmer [28], and Y. J. He and H. Li [18] gave different steps to prove anisotoropic Hopf's theorem.

(3) Related to the results of this paper, Palmer [P] has obtained that a stable equilibrium smoothly immersed hypersurface in $\mathbb{R}^n$ is unique and is the boundary of Wulff shape up to homothety and translation. Morgan [M] has proved that in $\mathbb{R}^2$, an immersed closed rectifiable curve in equilibrium for fixed area must be the boundary of Wulff shape, without smoothness assumptions on the anisotropic structure. A local version of constant anisotropic curvature curve with boundary in plane was considered by Mucha and Rybka in [MR]. They proved that the local CAMC curves can be glued up and a uniquely defined closed curve can be obtained and up to a translation it is the boundary of Wulff shape. Koiso and Palmer [KoP] have showed that for a large class of rotationally symmetric energy functionals, the only stable equilibria supported on parallel planes are either cylinders or a part of the Wulff shape.

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Entire solutions to reaction-diffusion equations in a domain of star graph

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1 Introduction

There are large number of mathematical models describing front wave propagations. In particular, the Fisher-KPP equation and the Allen-Cahn equation are typical model equations which possess traveling waves exhibiting the propagation of front transition between two states. In this report we deal with reaction-diffusion equations with bistable nonlinearity,

$$u_t = \Delta u + f(u), \tag{1.1}$$

where $f(u)$ is class of C^2 on an open interval containing $[0, 1]$ and satisfies the bistable condition

$$\begin{aligned} &f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \\ &f(u) \neq 0 \ \text{ for } \ u \in (0, a) \cup (a, 1). \end{aligned} \tag{1.2}$$

The equation has a monotone planar traveling wave with nonzero speed connecting $u = 0$ and $u = 1$ if $\int_0^1 f(u)du \neq 0$ (see [1, 2, 10]). Henceforth we assume $\int_0^1 f(u)du > 0$. Then there is a monotone decreasing traveling front solution $\phi(x - ct)$ ($x \in \mathbb{R}$) with speed $c > 0$, where $\phi(\xi)$ ($\xi = x - ct$) is called the profile of the traveling wave and satisfies $\phi(-\infty) = 1$ and $\phi(\infty) = 0$. We remark that the reflected $\phi(-x - ct)$ is a monotone increasing traveling wave solution propagating in the opposite direction of $\phi(x - ct)$, in addition, arbitrarily shifted $\phi(x - ct - \theta)$ (or $\phi(-x - ct - \theta)$) is also a traveling wave solution.

On the infinite interval $\mathbb{R}$, the equation allows not only the propagation of monotone traveling waves but also several types of propagation patterns. For instance, we can perform a numerical simulation exhibiting that two facing fronts propagate from the left and right axes and eventually the solution converges to the uniform state after collision of the front transitions. In general, it is difficult to describe such transient dynamics in long time mathematically, If, however, we look at the asymptotic behavior time backward, we can find that the solution behaves as the front $\phi(x - ct)$ near $x = -\infty$ while $\phi(-x - ct)$ near $x = \infty$, where the two fronts sufficiently far from each other have almost no interaction. This observation naturally leads us a question if there is a solution $u(x,t)$ satisfying

$$\sup_{x\le 0}|u(x,t)-\phi(x-ct)|+\sup_{x\ge 0}|u(x,t)-\phi(-x-ct)|\to 0 \quad \text{as} \quad t\to -\infty, \tag{1.3}$$

namely a solution defined for all negative time in the whole space and it has the profile of front transitions in the far left and far right from the origin. We call an entire solution if it is a classical solution defined for all $(x,t)\in\mathbb{R}\times\mathbb{R}$. Besides the special entire solutions of constant stationary solutions and traveling wave solutions we are interested in the entire solution which satisfies (1.3). It is fortune that this kind of entire solutions are first found in [14] for the Fisher-KPP equation while in [21] for the bistable equation. Later, the works of [11], [12] and [4] developed the study.

Interesting thing is that the study for entire solutions has been much developed to the higher-dimensional case in [15] and [13], merging of front transitions in [17] and [6], Lotka-Volterra competition diffusion system [20] and so on (see some of other works in the survey article [18]). In addition, for the bistable equation one can also refer to [3] for entire solutions propagating in the presence of obstacles and [19] for a new type entire solution.

In this report, motivated by a recent work [9], we look for entire solution in a domain of multiple half-lines joined at a single junction, namely, multiple half-lines with the same endpoints. Precisely, we are concerned with the domain

$$\Omega=\left(\bigcup_{i=0}^{m}\Omega_i\right)\cup\{\mathcal{O}\},\qquad m\ge 2,$$

where Ω_i $(i=0,1,\dots m)$ are semi-infinite intervals and they are joined at the origin $\mathcal{O}$.

The purpose of our study is to establish entire solutions having front transitions of the reaction-diffusion equation in Ω.

2 Main results

Let ℓ be a nonnegative integer less than m. We coordinate Ω as

$$\Omega_i=\{-\infty<x_i<0\}\quad (0\le i\le \ell),\quad \Omega_i=\{0<x_i<\infty\}\quad (\ell+1\le i\le m). \tag{2.1}$$

By putting $u_i=u_{|\Omega_i}$, the equation reads the following reaction-diffusion equations joined at the junction of $m+1$ semi-infinite intervals:

$$\partial_t u_i=\partial_{x_i}^2 u_i+f(u_i)\qquad (x_i\in\Omega_i),\qquad i=0,1,\dots,m, \tag{2.2}$$

with

$$u_0(0,t) = \cdots = u_m(0,t), \quad \sum_{j=0}^{\ell} \partial_{x_i} u_i(-0,t) = \sum_{i=\ell+1}^{m} \partial_{x_i} u_i(+0,t). \tag{2.3}$$

As for the profile of the planar traveling front $\phi(\xi)$ of (1.1), it is given by a solution to

$$\begin{cases} \phi'' + c\phi' + f(\phi) = 0, \quad \phi(\xi) > 0 \quad (\xi \in \mathbb{R}), \\ \lim_{\xi \to -\infty} \phi(\xi) = 1, \quad \lim_{\xi \to \infty} \phi(\xi) = 0, \end{cases} \tag{2.4}$$

where $' = d/d\xi$, $'' = d^2/d\xi^2$. This solution is unique up to phase shift.

We discuss the existence and the asymptotic behavior of an ancient solution, which is define for all negative time; more precisely, the existence of the solution having the front transition profile as $t \to -\infty$ in several half-lines. Once we establish such a solution, the solution can be extended up to the whole time $-\infty < t < \infty$ by the theorem of parabolic type equations. We certainly have entire solutions of (2.2) with (2.3) as follows:

Theorem 2.1 *Assume that $0 \le \ell \le (m-1)/2$. Then arbitrarily given number θ there exists an entire solution $U^*(x,t)$ satisfying*

$$\max_{0 \le i \le \ell} \sup_{x_i \in \Omega_i} |U^*(x_i,t) - \phi(x_i - ct - \theta)| + \max_{\ell+1 \le i \le m} \sup_{x_i \in \Omega_i} |U^*(x_i,t)| \le M_1 e^{\mu ct} \quad (t \le 0).$$

where μ is a positive constant independent of θ while $M_1 > 0$ depends on θ.

Theorem 2.1 implies the existence of the entire solution with $\ell + 1$ transition fronts, whose number $\ell + 1$ is less than equal to $(m+1)/2$. In this case there is a chance that the propagating fronts are blocked in the presence of the junction. The lemma below tells the existence of stationary solutions under a certain condition.

Lemma 2.2 *Let $F(u) := \int_0^u f(s)ds$. Assume $1 \le \ell \le (m-1)/2$ and define $\tilde{m} := (m-\ell)/(\ell+1)$. Then there are stationary solutions $\tilde{w}_1(x) < \tilde{w}_2(x)$ to the equation* (2.2) *with* (2.3) *if $F(1) + (\tilde{m}^2 - 1)F(a) < 0$ while $\tilde{w}_1$ and $\tilde{w}_2$ coincides if $F(1) + (\tilde{m}^2 - 1)F(a) = 0$.*

Since $\tilde{m} > 1$ holds under the condition $1 \le \ell \le (m-1)/2$, we have the next result:

Theorem 2.3 *Let $U^*(x,t)$ be the entire solution obtained in Theorem 2.1. Assume the condition in Lemma 2.2. Then*

$$\lim_{t \to \infty} \sup_{x \in \Omega} |U^*(x,t) - \tilde{w}_1(x)| = 0. \tag{2.5}$$

where $\tilde{w}_1$ is the stationary solution stated in Lemma 2.2.

We next consider the case $\ell > (m-1)/2$. Then we have the following theorem:

Theorem 2.4 *Assume $\ell > (m-1)/2$ and ℓ is odd. Then arbitrarily given θ there is an entire solution $U_*(x,t)$ satisfying*

$$\lim_{t\to-\infty}\left[\max_{0\le i\le\ell}\sup_{x_i\in\Omega_i}|U_*(x_i,t)-\phi(x_i-ct-\theta)|+\max_{\ell+1\le i\le m}\sup_{x_i\in\Omega_i}|U_*(x_i,t)|\right]=0,$$

and

$$\lim_{t\to\infty}[\max_{0\le i\le\ell}\sup_{x_i\in\Omega_i}|U_*(x,t)-1|+\max_{\ell+1\le i\le m}\sup_{0\le x_i\le c_it}|U_*(x_i,t)-1|]=0$$

where c_i is any positive number less than c.

We remark that the assumption of the oddness on ℓ is a technical condition to prove the assertion. We expect that it could be removed.

3 Remarks

The proofs for Theorems 2.1 and 2.3 are performed by constructing appropriate pair of sub-super solution pair, which are defined for all negative time and those converge to the profile of the traveling front solution in each Ω_i $(i=0,\dots,\ell)$. See the details in [8].

If the condition $F(1)+(\tilde{m}^2-1)F(a)<0$ stated in Lemma 2.2 is met, then the linearized stability (resp. instability) of the stationary solution $\tilde{w}_1$ (resp. $\tilde{w}_2$) are can be shown. Applying the theorem [13] shows that the linearized stability/instability yield the nonlinear stability/instability of the solutions. On the other hand, when $F(1)+(\tilde{m}^2-1)F(a)<0$, the instability for $\tilde{w}_1(=\tilde{w}_2)$ is shown by a comparison method. Those results are given in [8].

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Asymptotic behavior of eigenfrequencies of a thin elastic rod with non-uniform cross-section

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1. Introduction

In this abstract we present the asymptotic behavior of small eigenvalues and eigenfunctions of the linearized elasticity eigenvalue problem of a thin rod with non-uniform cross-section (see Figure 1).

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. We want to study the oscillations of an elastic body with the shape of Ω.

We denote by $u = (u_1, u_2, u_3) : \Omega \longrightarrow \mathbb{R}^3$ the displacement vector field associated with the oscillations. Let $\lambda_1, \lambda_2 > 0$ be positive real constants corresponding to the mechanical properties of the elastic body. More precisely, λ_1 is associated with the repelling force when stretched (expanded or contracted) while λ_2 is the characteristic quantity that appears when the object is twisted. We define the tensors

$$
\begin{aligned}
e(u) &= (e_{ij}(u))_{1\le i,j\le 3} = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)_{1\le i,j\le 3}, \\
\sigma(u) &= \lambda_1 \operatorname{tr}(e(u)) \operatorname{Id}_3 + 2\lambda_2 e(u),
\end{aligned}
$$

where tr is the trace of a matrix and Id_3 is the 3×3 identity matrix. $e(u)$ is called the *linearized strain tensor* and $\sigma(u)$ is the *stress tensor* derived from Hooke's law in the case of a homogeneous isotropic elastic body (cf. Ciarlet [1]).

With this notation, the operator of the elastic equation is defined as the 2nd order linear elliptic operator

$$
L[u] = \operatorname{div}\sigma(u), \qquad \text{i.e.} \qquad (L[u])_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(u) \quad (1 \le i \le 3),
$$

and the oscillations of an elastic body can be described by the following wave equation

$$
\varrho \frac{\partial^2 u}{\partial t^2} = L[u] \tag{1}
$$

where $\varrho > 0$ is the density.

Now, we take $\varrho = 1$ and we assume that the oscillations are periodic of period $\frac{2\pi}{\omega}$ $(\omega > 0)$. In this case, we can write the displacement field as $u(x,t) = \mathrm{e}^{i\omega t} v(x)$. Thus, $\frac{\partial^2 u}{\partial t^2} = -\omega^2 u(x,t)$. Putting $\mu = \omega^2$, the wave equation (1) becomes the eigenvalue problem

$$
L[v] + \mu v = \mathbf{0}. \tag{2}
$$

We now prepare the mathematical setting of our problem. We start presenting the domain $\Omega_\varepsilon = \Omega$, where $\varepsilon > 0$ is a small parameter corresponding to the thickness of the elastic body. Let $l > 0$ and let $B \subseteq \mathbb{R}^2$ be a connected bounded domain with $m \in \mathbb{N}$ connected components and $\mathcal{C}^3$ boundary. We consider the sets

$$
\begin{aligned}
S &= B \times (0,l), \qquad & s_1^{(-)} &= \overline{B} \times \{0\}, \\
s_1^{(+)} &= \overline{B} \times \{l\}, \qquad & s_2 &= \partial B \times (0,l).
\end{aligned}
$$

Note that $\partial S = s_1^{(-)} \cup s_1^{(+)} \cup s_2$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^3-diffeomorphism which satisfies the following properties.

i) $F(z) = (F_1(z), F_2(z), z_3) \quad (z = (z_1, z_2, z_3) \in S)$.

ii) $F_i(0, 0, z_3) = 0 \quad (i = 1, 2, \quad 0 \le z_3 \le l)$.

iii) The determinant of the Jacobian matrix of F is positive for all $z \in S$.

Let $\varepsilon > 0$ be a small positive parameter and define $F^\varepsilon(z) = (\varepsilon F_1(z), \varepsilon F_2(z), z_3)$. With this notation, we consider the following sets in $\mathbb{R}^3$.

$$\Omega_\varepsilon = F^\varepsilon(S), \quad \Gamma_{1,\varepsilon}^{(-)} = F^\varepsilon(s_1^{(-)}), \quad \Gamma_{1,\varepsilon}^{(+)} = F^\varepsilon(s_1^{(+)}), \quad \Gamma_{2,\varepsilon} = F^\varepsilon(s_2).$$

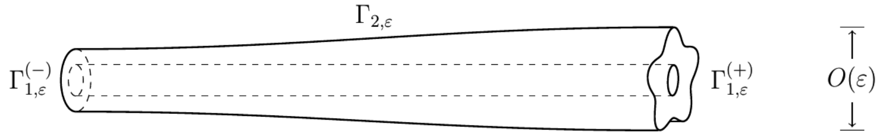

Figure 1: Example of Ω_ε

It is easy to see $\partial\Omega_\varepsilon = \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)} \cup \Gamma_{2,\varepsilon}$. Moreover, we obtain Ω_1, $\Gamma_{1,1}^{(-)}$, $\Gamma_{1,1}^{(+)}$, $\Gamma_{2,1}$ just by putting $\varepsilon = 1$ in the previous definition. Note that $\Omega_1 = F(S)$.

We want to study the small eigenvalues (low-frequency oscillations related to flexural vibrations) associated with the thin elastic body Ω_ε. We denote by $u = (u_1, u_2, u_3) : \Omega_\varepsilon \longrightarrow \mathbb{R}^3$ the displacement vector field associated with the oscillations.

With this notation, the main subject of the present paper is to study the eigenvalues and eigenfunctions when the parameter ε goes to zero of the following eigenvalue problems.

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega_\varepsilon \\ u = \mathbf{0} & \text{on } \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)} \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{2,\varepsilon} \end{cases} \tag{DD}$$

$$\begin{cases} L[u] + \mu u = \mathbf{0} & \text{in } \Omega_\varepsilon \\ u = \mathbf{0} & \text{on } \Gamma_{1,\varepsilon}^{(-)} \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{2,\varepsilon} \cup \Gamma_{1,\varepsilon}^{(+)} \end{cases} \tag{DN}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega_\varepsilon$. The case (DD) corresponds to a thin rod with both ends clamped while the case (DN), to a thin rod with only one clamped end.

2. Some notations

In order to state the main results we first introduce several notations.

Denote $\mathrm{d}y' = \mathrm{d}y_1\mathrm{d}y_2$ and define the set $\widehat{\Omega}(y_3)$ to be the cross-section of $\Omega_1 = F(S)$ at $y_3 \in [0, l]$. Furthermore, for $1 \le i, j \le 2$, we define the functions

$$H(y_3) = \int_{\widehat{\Omega}(y_3)} 1\mathrm{d}y', \quad K_i(y_3) = \int_{\widehat{\Omega}(y_3)} y_i \mathrm{d}y', \quad A_{ij}(y_3) = \int_{\widehat{\Omega}(y_3)} y_i y_j \mathrm{d}y' \quad (y_3 \in [0, l])$$

and write $Y = \frac{\lambda_2(3\lambda_1 + 2\lambda_2)}{\lambda_1 + \lambda_2}$, known as the *Young modulus.*

Remark 2.1. Note that the matrix $(A_{ij}(z_3))_{1\le i,j,\le 2}$ is positive definite.

If we denote by $\{\mu_k^{DD}(\varepsilon)\}_{k=1}^{+\infty}$ and $\{\mu_k^{DN}(\varepsilon)\}_{k=1}^{+\infty}$ the eigenvalues of problem (DD) and (DN) respectively, it is known that for any $\varepsilon > 0$ there are infinite discrete sequences of positive eigenvalues

$$0 < \mu_1^{DD}(\varepsilon) \le \mu_2^{DD}(\varepsilon) \le \cdots \le \mu_k^{DD}(\varepsilon) \le \mu_{k+1}^{DD}(\varepsilon) \le \cdots \text{ with } \lim_{k\to+\infty} \mu_k^{DD}(\varepsilon) = +\infty$$
$$0 < \mu_1^{DN}(\varepsilon) \le \mu_2^{DN}(\varepsilon) \le \cdots \le \mu_k^{DN}(\varepsilon) \le \mu_{k+1}^{DN}(\varepsilon) \le \cdots \text{ with } \lim_{k\to+\infty} \mu_k^{DN}(\varepsilon) = +\infty$$

which are arranged in increasing order, counting multiplicities (cf. Courant-Hilbert [3], Edmunds-Evans [5], Egorov-Kondratiev [6]).

3. Main results

Now we present the main results of the paper.

Theorem 3.1 (Both ends clamped)**.** *Let $\mu_k^{DD}(\varepsilon)$ be the k-th eigenvalue of problem* (DD)*. Then the following statements hold for each $k \in \mathbb{N}$.*

a) $\mu_k^{DD}(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$.

b) Moreover, we have the limit

$$\lim_{\varepsilon\to 0} \frac{\mu_k^{DD}(\varepsilon)}{\varepsilon^2} = \Lambda_k^{DD},$$

where Λ_k^{DD} denotes the k-th eigenvalue of the 4th order ordinary differential operator

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau)\end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} & (0<\tau<l), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) & (0<\tau<l), \\ \eta_3(0) = \eta_i(0) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(0) = 0 & (i=1,2), \\ \eta_3(l) = \eta_i(l) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(l) = 0 & (i=1,2). \end{cases}$$

Theorem 3.2 (Only one end clamped)**.** *Let $\mu_k^{DN}(\varepsilon)$ be the k-th eigenvalue of problem* (DN)*. Then the following statements hold for each $k \in \mathbb{N}$.*

a) $\mu_k^{DN}(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$.

b) Moreover, we have the limit

$$\lim_{\varepsilon\to 0} \frac{\mu_k^{DN}(\varepsilon)}{\varepsilon^2} = \Lambda_k^{DN},$$

where Λ_k^{DN} denotes the k-th eigenvalue of the 4th order ordinary differential operator

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau)\end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} & (0<\tau<l), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) & (0<\tau<l), \\ \eta_3(0) = \eta_i(0) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}z_3}(0) = 0 & (i=1,2), \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3}(l) = \dfrac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}(l) = \dfrac{\mathrm{d}^3\eta_i}{\mathrm{d}z_3^3}(l) = 0 & (i=1,2). \end{cases}$$

Remark 3.1. Let $\eta^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)}, \eta_3^{(k)})$ be the associated eigenfunction to Λ_k^{DD} and let $u_\varepsilon^{(k)}$ be the associated eigenfunction to $\mu_k^{DD}(\varepsilon)$. If we denote $U_\varepsilon^{(k)} = (U_{\varepsilon,1}^{(k)}, U_{\varepsilon,2}^{(k)}, U_{\varepsilon,3}^{(k)})$ and define $U_\varepsilon^{(k)}(z) := u_\varepsilon^{(k)}(\varepsilon F_1(z), \varepsilon F_2(z), z_3)$ with $\|U_\varepsilon^{(k)}\|_{L^2(S)} = 1$. Then

$$\begin{aligned} U_{\varepsilon,1}^{(k)} &\longrightarrow \eta_{\varepsilon,1}^{(k)} \\ U_{\varepsilon,2}^{(k)} &\longrightarrow \eta_{\varepsilon,2}^{(k)} \\ \frac{1}{\varepsilon}U_{\varepsilon,3}^{(k)} &\longrightarrow \eta_{\varepsilon,3}^{(k)} - F_1\frac{\mathrm{d}\eta_{\varepsilon,1}^{(k)}}{\mathrm{d}z_3} - F_2\frac{\mathrm{d}\eta_{\varepsilon,2}^{(k)}}{\mathrm{d}z_3} \end{aligned}$$

strongly in $H^1(S)$ as $\varepsilon \to 0$. We obtain a similar result with the eigenfunctions associated to $\mu_k^{DN}(\varepsilon)$.

Remark 3.2. Note that if the functions $K_i \equiv 0$ for $i = 1, 2$, then the ordinary differential equations in Theorem 3.1 and Theorem 3.2 get simpler. Using the corresponding boundary conditions, the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\frac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\frac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\frac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) \qquad (0<\tau<l)$$

yields $\eta_3 \equiv 0$, and hence the ODE in Theorem 3.1 and Theorem 3.2 simplifies to

$$Y\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{12}(\tau) & A_{22}(\tau)\end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}$$

with the respective boundary conditions.

4. Idea of the proof

We only state the ideas of the eigenvalue $\mu_k^{DD}(\varepsilon) = \mu_k(\varepsilon)$. The case for μ_k^{DN} is very similar.

Step 1

The domain Ω_ε depends on the parameter ε. The first thing we do is to change the variables with $(x_1, x_2, x_3) = (\varepsilon y_1, \varepsilon y_2, y_3)$ so that we work on $\Omega_1 = F(S)$, independent of ε.

Step 2

Using the above change of variables and the scaling $U_1 = u_1$, $U_2 = u_2$, $U_3 = \varepsilon u_3$ we can characterize the eigenvalues of the main problems with

$$\mu_k^{DD}(\varepsilon) = \sup_{\substack{Z \subset L^2(F(S),\mathbb{R}^3) \\ \dim Z = k-1}} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}.$$

where

$$\begin{aligned}
\widetilde{B}_\varepsilon[\Phi, \Phi] = \int_{F(S)} \Bigg\{ & \lambda_1 \left(E_{11}(\Phi) + E_{22}(\Phi) + \varepsilon^2 E_{33}(\Phi)\right)^2 \\
& + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\Phi)^2 + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(\Phi)^2 + \varepsilon^4 E_{33}(\Phi)^2 \right) \Bigg\} \varepsilon^2 \mathrm{d}y, \\
\widetilde{\mathcal{R}}_\varepsilon(\Phi) = {} & \frac{\widetilde{B}_\varepsilon[\Phi, \Phi]}{\displaystyle\int_{F(S)} \left(\varepsilon^2 \Phi_1^2 + \varepsilon^2 \Phi_2^2 + \varepsilon^4 \Phi_3^2\right) \varepsilon^2 \mathrm{d}y}, \\
E_{ij}(U) = \varepsilon e_{ij}(u), \quad & E_{i3}(U) = e_{i3}(u) \quad (1 \le i, j \le 2), \quad E_{33}(U) = \frac{1}{\varepsilon} e_{33}(U),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}_1 &= \{\Phi \in H^1(F(S), \mathbb{R}^3) \mid \Phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)} \cup \Gamma_{1,1}^{(+)}\}, \\
Z^{\perp_\varepsilon} &= \{\Phi \in \mathcal{W}_1 \mid \textstyle\int_{F(S)} (\Phi_1 \Psi_1 + \Phi_2 \Psi_2 + \varepsilon^2 \Phi_3 \Psi_3) \mathrm{d}y = 0 \text{ for all } \Psi \in Z\}.
\end{aligned}$$

We consider the function

$$\Upsilon(y) = \left(\Upsilon_1(y_3), \Upsilon_2(y_3), \Upsilon_3(y_3) - y_1 \Upsilon_1(y_3) - y_2 \Upsilon_2(y_3)\right).$$

With the help of this test function and the Max-Min method (the previous characterization of the eigenvalues) we conclude that $\mu_k(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$. Put $\lim_{\varepsilon \to 0} \frac{\mu_k(\varepsilon)}{\varepsilon^2} = \widetilde{\Lambda}_k$.

Step 3

We introduce Korn's inequality.

Theorem 4.1 (Korn's inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$. Let Γ_0 be a measurable subset of the boundary $\partial\Omega$ such that the 2-dimensional area of Γ_0 is positive. If we write $e_{ij}(v) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ for $1 \le i, j \le 3$, then there exists a constant $C > 0$ such that*

$$\|v\|_{H^1(\Omega)} \le C \left(\sum_{i,j=1}^{3} \|e_{ij}(v)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

for any $v = (v_1, v_2, v_3) \in H^1(\Omega)$ with $v|_{\Gamma_0} = \mathbf{0}$.

With the help of this theorem we can see that the eigenfunctions of the main problem are bounded in $H^1(S)$.

Step 4

The variational form of the main problem is

$$\int_{F(S)} \Bigg\{ \lambda_1 \left(E_{11}(U) + E_{22}(U) + \varepsilon^2 E_{33}(U)\right) \left(E_{11}(V) + E_{22}(V) + \varepsilon^2 E_{33}(V)\right)$$
$$+ 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(U)E_{ij}(V) + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(U)E_{i3}(V) + \varepsilon^4 E_{33}(U)E_{33}(V) \right) \Bigg\} \mathrm{d}y$$
$$= \mu \int_{F(S)} \left(\varepsilon^2 U_1 V_1 + \varepsilon^2 U_2 V_2 + \varepsilon^4 U_3 V_3\right) \mathrm{d}y.$$

where μ is the eigenvalue, U the eigenfunction and V is a test function. Inserting the test function $V = (V_1(y_3), V_2(y_3), V_3(y_3) - y_1 \frac{\mathrm{d}V_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}V_2}{\mathrm{d}y_3})$ and taking the limit $\varepsilon \to 0$, we can see that the value $\widetilde{\Lambda}_k$ is an eigenvalue of the ordinary differential equation in Theorem 3.1. If we write Λ_k the set of eigenvalues of the previous ODE, we have $\widetilde{\Lambda}_k \geq \Lambda_k$.

Step 5

We refine the upper estimate of the eigenvalue we found in Step 2 to find that $\widetilde{\Lambda}_k \leq \Lambda_k$. More precisely, we consider again the Rayleigh quotient $\widetilde{\mathcal{R}}_\varepsilon(\Theta)$ introduced in Step 2. We now use test functions of the form

$$\begin{aligned} \Theta_i &= \eta_i + \varepsilon^2 \phi_i \qquad (i = 1, 2), \\ \Theta_3 &= \eta_3 - y_1 \frac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon \phi_3, \end{aligned} ,$$

where η is an eigenfunction of the limit eigenvalue problem. Using these new test functions we want to minimize

$$\mathcal{M}(\phi) = \int_{F(S)} \left(\lambda_1 \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} + N \right)^2 + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\phi)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\left(\frac{\partial \phi_3}{\partial y_i} \right)^2 + N^2 \right) \right) \right) \mathrm{d}y.$$

This function $\mathcal{M}$ is obtained as the limit of the numerator $\widetilde{B}_\varepsilon[\Theta, \Theta]$. We try some test functions

$$\begin{aligned} \phi_i(y) &= \sum_{p,q=1}^{2} \alpha_{pq}^{(i)} y_p y_q + \sum_{p=1}^{2} \beta_p^{(i)} y_p \qquad (i = 1, 2), \\ \phi_3(y) &= 0, \end{aligned}$$

where $\alpha_{pq}^{(i)}$, $\beta_p^{(i)}$ only depend on y_3 for $1 \leq p, q, i \leq 2$. We solve the minimization problem and after very long but simple calculations we find the coefficients

$$\alpha_{11}^{(1)} = \frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}^2 \eta_1}{\mathrm{d}y_3^2}, \quad \alpha_{12}^{(1)} = \frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}^2 \eta_2}{\mathrm{d}y_3^2}, \quad \alpha_{22}^{(1)} = -\frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}^2 \eta_1}{\mathrm{d}y_3^2},$$
$$\alpha_{11}^{(2)} = -\frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}^2 \eta_2}{\mathrm{d}y_3^2}, \quad \alpha_{12}^{(2)} = \frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}^2 \eta_1}{\mathrm{d}y_3^2}, \quad \alpha_{22}^{(2)} = \frac{1}{4} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}^2 \eta_2}{\mathrm{d}y_3^2},$$
$$\beta_1^{(1)} = -\frac{1}{2} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}, \quad \beta_2^{(2)} = -\frac{1}{2} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}.$$

Then

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Theta) = \Lambda_k.$$

With this plus the Max-min characterization of the eigenvalues, we prove $\widetilde{\Lambda}_k \leq \Lambda_k$. Combining this last inequality together with Step 4, we conclude that

$$\widetilde{\Lambda}_k = \Lambda_k.$$

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Hyperbolic solutions to Bernoulli's free boundary problem

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1 The Bernoulli problem

Let Ω be a bounded domain in $\mathbb{R}^2$ and $Q > 0$ a constant. The Bernoulli problem is the problem of finding an open subset $A \subset \Omega$ for which the following overdetermined problem is solvable:

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \overline{A}, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u = 1 & \text{on} \quad \partial A, \\ \dfrac{\partial u}{\partial \nu} = Q & \text{on} \quad \partial A, \end{cases} \tag{1.1}$$

where ν is the unit outer normal vector with respect to the annular domain $\Omega \setminus \overline{A}$. The first three equations comprise the classical Dirichlet problem, and it has a unique solution u. Thus (1.1) has an extra boundary condition, which makes a restriction on A for the solvability of (1.1).

Equations (1.1) arises in a shape optimization problem in which one wants to design the shape of the insulation layer of an electronic cable such that the current leakage is minimized subject to a given amount of insulation material. Then, u stands for the electrostatic potential and Ω is the cross-section of the cable with the insulation layer $\Omega \setminus \overline{A}$. Another physical interpretation of u is the stream potential of the stationary irrotational velocity field in the plane of an incompressible inviscid fluid which circulates around a bubble A of air in a given container Ω.

The existence of a solution A for prescribed Ω and Q is shown by various methods including the super and subsolution method of Beurling [3], a variational method by Alt and Caffarelli [2], and the inverse function theorem of Nash and Moser by Hamilton [6]. However, most of the results are concerned with a class of "stable", or "well-ordered", solutions called elliptic solutions (see Definition 2), where a solution A to the Bernoulli problem is called elliptic if, roughly speaking, the infinitesimal increase of the value of $Q > 0$ makes the corresponding solution A to expand. Indeed, the super and subsolution method only allows one to construct elliptic solutions, since the method constructs a solution A as the union of all subsolutions, where $A_{\mathrm{sub}} \subset \Omega$ is called a subsolution to (1.1) if there exists a solution u to (1.1) in $\Omega \setminus A_{\mathrm{sub}}$ with the last boundary condition replaced by $\partial_\nu u \leq Q$; and hence for $\tilde{Q} > Q$ the corresponding solution $\tilde{A}$ must be larger than A. Variational solutions constructed as minimizers are also elliptic. This can be seen by looking at the form of the variational functional (see [5, Section 5.3]). On the other hand, the inverse function theorem is, in principle, able to handle "unstable" solutions called hyperbolic solutions, for which the increase of $Q > 0$ makes A to shrink; but (1.1) has a regularity issue called "loss of derivatives", and this requires several estimates which are only (at least up to now) available for elliptic solutions.

The structure of solutions A to the Bernoulli problem is illustrated by the simplest situation where Ω is the unit ball $\mathbb{B} = \mathbb{B}_1$, Here, we denote by $\mathbb{B}_r$ the ball of radius $r > 0$ with center at the origin.

Example. For $\Omega = \mathbb{B}$, it is known that solutions A must be concentric balls. The function u_r satisfying the first three equations in (1.1) for $A = \mathbb{B}_r$ $(0 < r < 1)$ and its normal derivative at $|x| = r$ are

$$u_r(x) = \frac{\log|x|}{\log r}, \quad \frac{\partial u_r}{\partial \nu}(r) = -\frac{1}{r \log r}.$$

The function $Q(r) := \partial_\nu u_r(r)$ $(0 < r < 1)$ is convex and takes its minimum at $r = 1/e$. Therefore, for $Q = Q_0 := Q(1/e)$, the Bernoulli problem has a unique solution $A = \mathbb{B}_{1/e}$; while there are an elliptic solution $\mathbb{B}_{r_e(Q)}$ and a hyperbolic solution $\mathbb{B}_{r_h(Q)}$ with $r_e(Q) > 1/e > r_h(Q)$ for $Q > Q_0$; and no solution for $Q < Q_0$. Moreover, these solutions satisfy

$$\lim_{Q \to \infty} r_e(Q) = 1, \quad \lim_{Q \to \infty} r_h(Q) = 0.$$

Thus, the elliptic solutions $\mathbb{B}_{r_e(Q)}$ asymptotically approach to the prescribed domain $\Omega = \mathbb{B}$, and the hyperbolic solutions $\mathbb{B}_{r_h(Q)}$ shrink to the single point $\{0\}$.

One of the interesting questions is whether these asymptotic behavior of solutions can be observed for general convex domain Ω. As a matter of fact, Acker [1] proved that this is true for elliptic solutions. Our research was initiated towards the affirmative answer to the following conjecture.

Conjecture (Flucher and Rumpf [5])**.** *There exist hyperbolic solutions shrinking to the conformal center $z_0 \in \Omega$ for any convex domain Ω.*

The conformal centers of a simply-connected domain Ω are the points $z_0 \in \Omega$ at which the conformal radius

$$R(z) := |f_z'(0)| \quad (z \in \Omega)$$

takes its maximum, where $f_z : \mathbb{B} \to \Omega$ is the biholomorphic map satisfying $f(0) = z$ and $f'(0) > 0$. It is known that the conformal radius $R(z)$ is strictly concave if Ω is convex; and thus the conformal center is unique (see Cardaliaguet and Tahraoui [4]).

This conjecture has been open for decades. This is due to the fact that many arguments based on the maximum principles or the variational method fail for hyperbolic solutions. The inverse function theorem can possibly apply to capture hyperbolic solutions; however, the problem of "loss of derivatives" forced one to work with not only a Banach space, but also a graded family of spaces (see [6]). Thus, these existing methods are not suitable to pursue the behavior of hyperbolic solutions A when the value $Q > 0$ changes.

The main contribution of this work is to introduce a new "parabolic" approach, namely that we derive and analyze a flow equation describing the behavior of solutions $A = A(t)$ for varying data $Q = Q(t)$. This has a common feature with the inverse function theorem in [6], since the derivation of the flow equation is essentially based on the linearization of (1.1). But the parabolic approach has the advantage that "loss of derivatives" can be handled with the established theory of evolution equations; and hence a graded family of spaces is no longer needed and we can work with a fixed Banach space. Moreover, our method can apply in any space dimensions $n \geq 2$, and thus hereafter we consider (1.1) in $\mathbb{R}^n$.

2 Deformation flow

To explain how our approach can handle the regularity problem, let us consider the abstract functional equation

$$F(x, s) = 0 \quad (x \in X, \ s \in \mathbb{R}), \tag{2.1}$$

where X is a Banach space and F is a C^1-mapping from $X \times \mathbb{R}$ to another Banach space Y with $X \subset Y$. If $F(0, 0) = 0$ and the Fréchet derivative $\partial_x F(0, 0) \in \mathcal{L}(X, Y)$ (the space of bounded operators from X to Y) is invertible, then for each given small data s we can find a unique solution $x(s) \in X$ in a neighborhood of 0 by the implicit function theorem. In fact, the sequence of X-valued curves

$$x_1(s) := 0, \quad x_{j+1}(s) := x_j(s) - \partial_x F(0, 0)^{-1} F(x_j(s), s) \quad (-\varepsilon \leq s \leq \varepsilon) \tag{2.2}$$

converges to a C^1-curve $x(s)$ satisfying $x(0) = 0$ and $F(x(s), s) = 0$. However, the method would fail if we only have the regularity gain $\partial_x F(0, 0)^{-1} \in \mathcal{L}(Y, Y)$[1]; and thus $x_{j+1}(s)$ is merely Y-valued even if $x_j(s)$ is X-valued. This "loss of derivatives" actually happens in the Bernoulli problem (1.1), and one is required

[1]Of course, this happens only when $\partial_x F(0, 0)$ is not invertible. We will give the precise meaning of the "inverse" $\partial_x F(0, 0)^{-1}$ later.

to use the technique of Nash and Moser to overcome this regularity issue. We, instead, consider the evolution equation

$$x'(s) = -\partial_x F(x(s), s)^{-1} \partial_s F(x(s), s), \quad x(0) = 0. \tag{2.3}$$

This equation is formally derived by differentiating (2.1) in s with $x = x(s)$, and it is easy to see that $F(x(s), s) = 0$ if and only if $x = x(s)$ is a solution to (2.3). A natural regularity condition for this parabolic formulation is

$$x(s) \in C([0, \varepsilon), X) \cap C^1([0, \varepsilon), Y),$$

so that the "loss of derivatives" is no longer an issue and it is well-treated within the standard theory of evolution equations.

Now we set up the Bernoulli problem (1.1) as a functional equation of the form $F(\rho, Q) = 0$. Hereafter, Ω denotes a fixed bounded domain in $\mathbb{R}^n$ with $h^{2+\alpha}$-boundary, and Q is allowed to be a positive function in $h^{2+\alpha}(\mathbb{R}^n)$, where $h^{k+\alpha}(\Gamma)$ is the so-called little Hölder space on a domain (or hypersurface) Γ defined as the closure of $C^\infty(\Gamma)$ in the topology of the Hölder space $C^{k+\alpha}(\Gamma)$. Let us choose a reference domain $A_0 \subset \overline{A_0} \subset \Omega$ with smooth boundary ∂A_0, say of class $h^{4+\alpha}$, and identify $\rho \in \mathcal{U}_\gamma \subset h^{3+\alpha}(\partial A_0)$ with a perturbed domain A_ρ having $h^{3+\alpha}$-boundary

$$\partial A_\rho = \{\zeta + \rho(\zeta)\nu(\zeta) \mid \zeta \in \partial A_0\}, \tag{2.4}$$

where $\nu = \nu(\zeta)$ is the unit outer normal vector with respect to $\Omega \setminus \overline{A_0}$ and

$$\mathcal{U}_\gamma := \{\rho \in h^{3+\alpha}(\partial A_0) \mid \|\rho\|_{h^{3+\alpha}(\partial A_0)} < \gamma\}, \quad \gamma \leq a/4,$$

with $0 < a < \operatorname{dist}(\partial A_0, \partial\Omega)$ taken to be small such that $\theta(\zeta, r) := \zeta + r\nu(\zeta)$ defines a diffeomorphism from $\partial A_0 \times (-a, a)$ to its image. Denoting by ζ and r the components of the inverse map θ^{-1}, i.e., $\theta^{-1}(x) = (\zeta(x), r(x))$, we see that

$$\theta_\rho(x) := \begin{cases} \theta(\zeta(x), r(x) + \eta(r(x))\rho(\zeta(x))) & \text{if } x \in \theta(\partial A_0 \times (-a, a)), \\ x & \text{otherwise,} \end{cases}$$

defines an $h^{2+\alpha}$-diffeomorphism from $\Omega \setminus A_0$ to the annular domain $\Omega \setminus A_\rho$, where η is a smooth cut-off function satisfying

$$\eta(r) = \begin{cases} 1 & (|r| \leq a/4), \\ 0 & (|r| \geq 3a/4) \end{cases} \quad \text{and} \quad \left|\frac{d\eta}{dr}(r)\right| < \frac{4}{a}. \tag{2.5}$$

The diffeomorphism θ_ρ induces the pull-back and push-forward operators

$$\theta_\rho^* u := u \circ \theta_\rho, \quad \theta_*^\rho v := v \circ \theta_\rho^{-1}$$

for $u \in h^{k+\alpha}(\overline{\Omega} \setminus A_\rho)$ and $v \in h^{k+\alpha}(\overline{\Omega} \setminus A_0)$ $(0 \leq k \leq 2)$. For a given $\rho \in \mathcal{U}_\gamma$, the first three equations in (1.1) with $A = A_\rho$ comprise the Dirichlet problem and thus always have a unique solution $u_\rho \in h^{2+\alpha}(\overline{\Omega} \setminus A_\rho)$. Hence if we define

$$F(\rho, Q) := \theta_\rho^* \left(\frac{\partial u_\rho}{\partial \nu} - Q \right),$$

then A_ρ is a solution to (1.1) if and only if $F(\rho, Q) = 0$.

Proposition 1. *Suppose that* $Q \in h^{2+\alpha}(\mathbb{R}^n)$ *and* $\rho \in \mathcal{U}_\gamma$. *Then,*

(i) $F \in C^1(\mathcal{U}_\gamma \times \mathbb{R}, h^{1+\alpha}(\partial A_0))$.

(ii) *The Fréchet derivative of* F *with respect to* ρ *at* $\rho = 0$ *is given by*

$$\partial_\rho F(0,Q)[\tilde{\rho}] = \frac{\partial p}{\partial \nu} - HQ\tilde{\rho} - \frac{\partial Q}{\partial \nu}\tilde{\rho} \quad \left(\tilde{\rho} \in h^{3+\alpha}(\partial A_0)\right),$$

where $H = H_{\partial A_0} \in h^{1+\alpha}(\partial A_0)$ *is the mean curvature of* ∂A_0 *normalized in such a way that* $H = -(n-1)$ *if* $A_0 = \mathbb{B}$, *and* p *is the solution to*

$$\begin{cases} -\Delta p = 0 & \textit{in} \quad \Omega \setminus \overline{A_0}, \\ \quad p = 0 & \textit{on} \quad \partial\Omega, \\ \quad p = -Q\tilde{\rho} & \textit{on} \quad \partial A_0. \end{cases} \tag{2.6}$$

(iii) *The linear operator given above is extended to*

$$\partial_\rho F(0,Q) \in \mathcal{L}\left(h^{2+\alpha}(\partial A_0), h^{1+\alpha}(\partial A_0)\right).$$

The extension $\partial_\rho F(0,Q)$ in Proposition 1 (iii) with $Q(x) > 0$ has the bounded inverse $\partial_\rho F(0,Q)^{-1} \in \mathcal{L}(h^{1+\alpha}(\partial A_0), h^{2+\alpha}(\partial A_0))$ if the elliptic equation

$$\begin{cases} -\Delta p = 0 & \text{in} \quad \Omega \setminus \overline{A}, \\ p = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial p}{\partial \nu} + \left(H + \dfrac{\partial_\nu Q}{Q}\right) p = q & \text{on} \quad \partial A \end{cases} \tag{2.7}$$

with $A = A_0$ is uniquely solvable for any $q \in h^{1+\alpha}(\partial A_0)$. Moreover,

$$\partial_\rho F(0,Q)^{-1}[q] = -\frac{p}{Q} \in h^{2+\alpha}(\partial A_0) \quad \left(q \in h^{1+\alpha}(\partial A_0)\right),$$

where p is the unique solution to (2.7). Let us now recall some notions for solutions A to (1.1) in terms of the linearized problem (2.7).

Definition 1 (Non-degeneracy)**.** *We say that a domain* A *is non-degenerate if the linearized problem* (2.7) *with* $q = 0$ *has only the trivial solution* $p = 0$.

Remark 1. The non-degeneracy of A, in fact, guarantees the unique solvability of (2.7) for any inhomogeneous data q by the Fredholm theory.

Furthermore, a classification of solutions A in terms of the behavior of solutions p to (2.7) was introduced by Flucher and Rumpf [5] as an extension of Beurling's original definition [3].

Definition 2 (Elliptic, hyperbolic and parabolic solutions). *A solution A to the Bernoulli problem* (1.1) *is called elliptic (hyperbolic) if* (2.7) *has a solution for $q = 1$ and all the solutions p satisfy*

$$\int_{\partial A} p \, d\sigma > 0 \quad (< 0).$$

Otherwise, A is called parabolic. Moreover, an elliptic (hyperbolic) solution A is said to be monotone if $p > 0$ (< 0) holds everywhere on ∂A.

Remark 2. Elliptic (hyperbolic) solutions are interpreted as volume-increasing (decreasing) solutions $A(\varepsilon)$ for varying $Q(\varepsilon) = Q + \varepsilon$, since

$$\frac{d}{d\varepsilon}\left[\int_{A(\varepsilon)} dx\right] = \frac{1}{Q(\varepsilon)} \int_{\partial A(\varepsilon)} p \, d\sigma > 0 \quad (< 0).$$

The monotonicity implies that $A(\varepsilon)$ increases (decreases) in the sense of set inclusion, which actually corresponds to Beurling's original definiton.

If $F(\rho_0, Q_0) = 0$ and A_{ρ_0} is non-degenerate, i.e., $\partial_\rho F(\rho_0, Q_0)$ is invertible, one would proceed to the successive approximation procedure as (2.2) in order to construct a solution ρ to $F(\rho, Q) = 0$ for $Q \neq Q_0$; but it fails because of the loss of derivatives $\partial_\rho F(\rho_0, Q_0)^{-1} F(\rho, Q) \in h^{2+\alpha}(\partial A_0)$ for $\rho \in h^{3+\alpha}(\partial A_0)$ as presented at the beginning of this section. Instead, we take the alternative parabolic approach, namely, setting $Q(x,t) = Q_0(x) + tq(x) > 0$ and $\tilde{F}(\rho, t) = F(\rho, Q(t))$, we consider the evolution equation

$$\rho'(t) = -\partial_\rho \tilde{F}(\rho, t)^{-1} \left[\partial_t \tilde{F}(\rho, t)\right] \tag{2.8}$$

with $\rho(0) = \rho_0$ under the regularity condition

$$\rho \in C([0,T), h^{3+\alpha}(\partial A_0)) \cap C^1([0,T), h^{2+\alpha}(\partial A_0)). \tag{2.9}$$

In fact, this regularity assumption is suitable not only for treating loss of derivatives in (2.8), but also for applying the standard theory of evolution equations. Proposition 1 shows that, for $A(t) = A_{\rho(t)}$, (2.8) is represented by flow equation

$$\begin{aligned} & V = -\frac{p}{Q} \quad \text{on} \quad \partial A(t), \\ & \text{with} \quad \begin{cases} -\Delta p = 0 & \text{in} \quad \Omega \setminus \overline{A(t)}, \\ p = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial p}{\partial \nu} + \left(H + \dfrac{1}{Q}\dfrac{\partial Q}{\partial \nu}\right) p = q & \text{on} \quad \partial A(t), \end{cases} \end{aligned} \tag{2.10}$$

where V is the speed of moving surface $\partial A(t)$ in the outer normal direction with respect to $\Omega \setminus \overline{A(t)}$ to (1.1) for varying $Q(x,t)$. Summarizing the above argument, we obtain the following characterization of a family of solutions $A(t)$ to (1.1).

Theorem 1. *Let $Q(x,t) = Q_0(x) + tq(x) > 0$ and $Q_0, q \in h^{2+\alpha}(\mathbb{R}^n)$, and suppose that $A(0) = A_{\rho_0} \subset \Omega$ with $\rho_0 \in h^{3+\alpha}(\partial A_0)$ is a solution to* (1.1) *for Q_0. If* (2.9) *holds and $A(t) = A_{\rho(t)}$ are all non-degenerate, then the following are equivalent:*

(i) Each $A(t)$ is a solution to the Bernoulli problem (1.1) *for $Q(x,t)$;*

(ii) $\{A(t)\}_{0 \le t < T}$ is a solution to the flow equation (2.10).

Theorem 1 reduces the construction of solutions $A(t)$ of (1.1) to the solvability of the flow equation (2.10). The following theorem shows that (2.10) is, in fact, solvable locally in time.

Theorem 2. *Let Ω be a bounded domain with $h^{2+\alpha}$-boundary and $Q(x,t) = Q_0(x) + tq(x) > 0$, $Q \in h^{3+\alpha}(\mathbb{R}^n)$, and $q \in h^{2+\alpha}(\mathbb{R}^n)$, and suppose that $A_{\rho_0} \subset \Omega$ with $\rho_0 \in h^{3+\alpha}(\partial A_0)$ is a non-degenerate solution to* (1.1) *for $Q_0 > 0$.*

(i) If A_{ρ_0} is elliptic, monotone and $q < 0$, then there exists $T > 0$ such that, for all $0 \le t < T$, (1.1) *possesses a non-degenerate, elliptic and monotone solution $A(t) = A_{\rho(t)}$ for $Q(x,t)$ satisfying $\rho(0) = \rho_0$ and* (2.9).

(ii) If A_{ρ_0} is hyperbolic, monotone and $q > 0$, then there exists $T > 0$ such that, for all $0 \le t < T$, (1.1) *possesses a non-degenerate, hyperbolic and monotone solution $A(t) = A_{\rho(t)}$ for $Q(x,t)$ satisfying $\rho(0) = \rho_0$ and* (2.9).

Remark 3. We require the higher regularity $Q \in h^{3+\alpha}(\mathbb{R}^n)$, $q \in h^{2+\alpha}(\mathbb{R}^n)$ as compared to Theorem 1. This is due to the fact that we differentiate (2.8) (or (2.10)) with respect to ρ for the application of the semigroup theory.

Remark 4. Depending on the ellipticity/hyperbolicity of A_{ρ_0}, the linearized operator has the opposite sign, which reflects in the assumption $q \lesseqgtr 0$. Thus, in both cases (i), (ii), the moving domain $A(t)$ shrinks under the flow (2.10).

The proof of Theorem 2 is mainly based on harmonic analysis. For the functional analytic treatment of (2.10), we first derive the corresponding evolution equation in terms of ρ defined on a fixed surface ∂A_0. We then analyze the spectral properties of its linearized operator by representing its principal part as a Fourier multiplier operator, and prove that it generates a strongly continuous analytic semigroup.

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Dynamics of localized unimodal patterns in reaction-diffusion systems for cell polarization by extracellular signaling

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1 Introduction

Cell polarity is a general phenomenon observed in many stages of development processes where it is used for regulating cell migration, cell aggregation, or cell functions by inducing a different differentiation between two daughter cells. To understand the general mechanism of cell polarization, [3, 6] proposed a conceptual model of a reaction-diffusion system

$$\begin{cases} \dot{u} &= d_1 u_{xx} - \gamma_1 f(u,v) \\ \dot{v} &= d_2 v_{xx} + \gamma_2 f(u,v), \end{cases} \tag{1.1}$$

where u and v denote the concentrations of two internal chemicals in a cell; u and v correspond to chemicals in the membrane and cytosol, respectively. Since the diffusion in cytosol is faster than that in the membrane, the diffusion coefficients d_1 and d_2 are positive constants satisfying $d_2 > d_1$. Moreover, $f(u,v)$ is a smooth function, and the reaction rates γ_1 and γ_2 are positive constants. We consider (1.1) on an interval $I = (-K/2, K/2)$ for $K > 0$ under the periodic boundary condition, which implies that

$$\int_I (\gamma_2 u(x,t) + \gamma_1 v(x,t))\, dx \equiv \int_I (\gamma_2 u(x,0) + \gamma_1 v(x,0))\, dx \tag{1.2}$$

*Kobe University, kuwamura@main.h.kobe-u.ac.jp. This is a joint work with S. Seirin-Lee (Hiroshima University) and S.-I. Ei (Hokkaido University) holds for any (smooth) solutions of (1.1), i.e., the (weighted) total mass of u and v is conserved in a cell. Therefore, (1.1) is called a reaction-diffusion system with mass conservation, and its mathematical aspects are studied by [4, 5].

In this report, we consider a perturbed system of (1.1)

$$\begin{cases} \dot{u} &= d_1 u_{xx} - \gamma_1\{f(u,v) + \varepsilon g(x,u,v)\} \\ \dot{v} &= d_2 v_{xx} + \gamma_2\{f(u,v) + \varepsilon g(x,u,v)\} \end{cases} \tag{1.3}$$

for sufficiently small $\varepsilon > 0$, where g is a smooth function expressed as

$$g(x,u,v) = -g_1(u,v)g_2(x).$$

Notice that (1.3) satisfies the mass conservation property as (1.2).

The purpose of this report is to investigate the dynamics of a localized unimodal pattern in (1.3) (see Figure 1), which is regarded as an equilibrium of the unperturbed system (1.1). We rigorously derive the equation of motion of the localized pattern, which shows that the localized pattern moves to the maximum point of $g_2(x)$ under natural assumptions. From the viewpoint of biology, our result suggests conditions under which the location of a polarity peak is determined in the site where the maximal extracellular signal is present.

2 Basic assumptions and key lemmas

First, we suppose the existence and stability of a localized unimodal pattern in (1.1). Moreover, we propose assumptions concerning the perturbation term g in (1.3), which describes a spatial inhomogeneity.

Assumption 2.1 (1.1) *has a stable equilibrium* $S(x) = (p(x), q(x))$ *satisfying the following conditions:*
(i) p *and* q *are even periodic functions with period* K.
(ii) p *and* q *are strictly decreasing and increasing, respectively, in* x *for* $0 \leq x \leq K/2$.

Assumption 2.2 g_1 *and* g_2 *satisfy the following properties:*
(i) $g_1 \geq 0$ ($g_1 \not\equiv 0$).
(ii) g_2 *is an even periodic function with the period* K.
(iii) g_2 *is strictly decreasing in* x *for* $0 \leq x \leq K/2$.

We rewrite (1.1) and (1.3) as evolution equations on a Hilbert space $L^2(I) \times L^2(I)$, respectively as follows:

$$\mathbf{u}_t = \mathcal{L}(\mathbf{u}), \quad \mathbf{u} = (u, v) \tag{2.1}$$

and

$$\mathbf{u}_t = \mathcal{L}(\mathbf{u}) + \varepsilon G(x, \mathbf{u}), \quad \mathbf{u} = (u, v), \tag{2.2}$$

where

$$\mathcal{L}(\mathbf{u}) = \begin{pmatrix} d_1 u_{xx} - \gamma_1 f(u, v) \\ d_2 v_{xx} + \gamma_2 f(u, v) \end{pmatrix} \quad \text{and} \quad G(x, \mathbf{u}) = \begin{pmatrix} -\gamma_1 g(x, u, v) \\ \gamma_2 g(x, u, v) \end{pmatrix}.$$

Noting the mass conservation property, we see that the equations (2.1) and (2.2) define a semi-flow on a hyperplane X_ξ in $L^2(I) \times L^2(I)$, where $X_\xi = \{\, \mathbf{u} \in L^2(I) \times L^2(I) \,|\, \langle \mathbf{u}, \mathbf{a} \rangle = \xi \,\}$ for some $\xi \in \mathbf{R}$, and

$$\mathbf{a} = \frac{1}{\sqrt{K(\gamma_1^2 + \gamma_2^2)}} \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix}, \quad \langle \mathbf{a}, \mathbf{a} \rangle = 1$$

Let us denote by $\tilde{L}$ the linearized operator of the right hand side of (2.1) at $S = S(x) \in X_\xi$. Notice that $\tilde{L}$ is defined on a subspace X in $L^2(I) \times L^2(I)$, where $X = \{\, \mathbf{u} \in L^2(I) \times L^2(I) \,|\, \langle \mathbf{u}, \mathbf{a} \rangle = 0 \,\}$. That is, $\tilde{L} = L|_X : X \to X$ is a restriction of $L : L^2(I) \times L^2(I) \to L^2(I) \times L^2(I)$, which is defined by

$$L = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \partial_x^2 + \begin{pmatrix} -\gamma_1 \tilde{f}_u & -\gamma_1 \tilde{f}_v \\ \gamma_2 \tilde{f}_u & \gamma_2 \tilde{f}_v \end{pmatrix},$$

where $\tilde{f}_u = f_u(p, q)$, $\tilde{f}_v = f_v(p, q)$ and $S = (p, q) \in X_\xi$. Since $S_x \in X$ and $LS_x = \tilde{L}S_x = 0$, we see that $\tilde{L} : X \to X$ has an eigenvalue 0 and its associated eigenfunction S_x.

Assumption 2.3 (i) *0 is a simple eigenvalue of* $\tilde{L}$.
(ii) $\mathrm{Re}(\sigma(\tilde{L}) \setminus \{0\}) < -\delta$ *for some* $\delta > 0$.

The following lemmas play a key role in our analysis.

Lemma 2.1 *Let $\tilde{L}^* : X \to X$ be the adjoint operator of $\tilde{L} : X \to X$. Then, $\tilde{L}^*$ has a simple eigenvalue* 0 *and its associated eigenfunction is given by*

$$\Phi^* = \begin{pmatrix} \dfrac{-w(x) + \gamma_2 \varphi_2^*(x)}{\gamma_1} \\ \varphi_2^*(x) \end{pmatrix} \in X,$$

where

$$\varphi_2^*(x) = h_1(x) + Ax$$

is an odd function, and

$$h_1(x) = \int_0^x \int_0^y h_2(s) ds dy, \quad A = \int_0^{K/2} \left(\frac{2y}{K} - 1 \right) h_2(y) dy,$$

and

$$h_2(x) = -\frac{1}{d_2} \tilde{f}_v w(x), \quad \tilde{f}_v = f_v(p, q), \quad S = (p, q) \in X_\xi.$$

Lemma 2.2 *$\varphi_2^*(x) > 0$ for $0 < x < K/2$ if* (i) *$h_2(x) < 0$ for $0 < x < K/2$ or* (ii) *there exists $\alpha \in (0, K/2)$ such that $h_2(x) < 0$ for $0 < x < \alpha$ and $h_2(x) > 0$ for $\alpha < x < K/2$, and*

$$\int_0^{K/2} x h_2(x) dx < 0.$$

3 Dynamics of localized unimodal pattern

In this section, we investigate the dynamics of (1.3) around a manifold $M = \{ S(x - \ell) \,|\, \ell \in \mathbf{R} \} \subset X_\xi$. Notice that ℓ represents the peak position of a localized unimodal pattern.

Let $\mathbf{u} = \mathbf{u}(x, t)$ be a solution of (2.2), which is an evolution equation on X_ξ defined by (1.3). Noting that M is parameterized by ℓ, let

$$\mathbf{u}(x, t) = S(x - \ell(t)) + V(x - \ell(t), t) = S(z) + V(z, t), \tag{3.1}$$

where $z = x - \ell(t)$ and $V \in X$. Since solutions of (1.3) around M can be expressed by (3.1), we consider that the equation of $\ell(t)$ determines the dynamics of (1.3) around M under Assumption 2.3.

Differentiating (3.1) with respect to t, we have

$$\mathbf{u}_t = -\dot{\ell} S_z + V_t - \dot{\ell} V_z, \quad \dot{\ell} = d\ell/dt.$$

On the other hand, substituting (3.1) into the right hand side of (2.2), we have

$$\begin{aligned}\mathcal{L}(\mathbf{u}) + \varepsilon G(x, \mathbf{u}) &= \mathcal{L}(S+V) + \varepsilon G(z+\ell, S+V) \\ &= \mathcal{L}(S) + \tilde{L}V + O(||V||^2) + \varepsilon G(z+\ell, S) + O(\varepsilon ||V||) \\ &= \tilde{L}V + \varepsilon G(z+\ell, S) + O(\varepsilon^2 + ||V||^2).\end{aligned}$$

Therefore, we have

$$V_t - \dot{\ell} V_z = \tilde{L}V + \dot{\ell} S_z + \varepsilon G(z+\ell, S) + O(\varepsilon^2 + ||V||^2).$$

By neglecting higher-order error terms, we obtain

$$V_t = \tilde{L}V + \dot{\ell} S_z + \varepsilon G(z+\ell, S) \tag{3.2}$$

if $\max(||V||, ||V_z||) = O(\varepsilon)$ and $|\dot{\ell}| = O(\varepsilon)$. According to [2], the condition that solutions of (3.2) are uniformly bounded in X under Assumption 2.3 is

$$\langle \dot{\ell} S_z + \varepsilon G(z+\ell, S), \Phi^* \rangle = 0.$$

Hence, we obtain

$$\frac{d\ell}{dt} = \varepsilon H(\ell) + O(\varepsilon^2), \tag{3.3}$$

where

$$H(\ell) = -\frac{J(\ell)}{\langle S_z, \Phi^* \rangle} \quad \text{and} \quad J(\ell) = \langle G(z+\ell, S), \Phi^* \rangle.$$

This equation determines the dynamics of (1.3) around M. Noting that $|\dot{\ell}| = O(\varepsilon)$ by (3.3), we have $\max(||V||, ||V_z||) = O(\varepsilon)$ by (3.2) and $V \in H^1(I) \times H^1(I)$ under Assumption 2.3. This implies that the validity of (3.2) and (3.3) can be rigorously guaranteed by applying the same lines of argument in [1].

The following result gives a characterization of the dynamics defined by (3.3).

Theorem 3.1 *$d\ell/dt = \varepsilon H(\ell)$ has only two equilibria $\ell = 0$ and $\ell = K/2(= -K/2)$. Moreover, $\ell = 0$ is stable and $\ell = K/2$ is unstable under any of the following conditions:* (i) *$f_v(p(x), q(x)) < 0$ for $0 < x < K/2$,* (ii) *there exists $\alpha \in (0, K/2)$ such that $f_v(p(x), q(x)) < 0$ for $0 < x < \alpha$ and $f_v(p(x), q(x)) > 0$ for $\alpha < x < K/2$, and*

$$\int_0^{K/2} x f_v(p(x), q(x)) p'(x) dx > 0 \quad \text{or} \quad \int_0^{K/2} x f_v(p(x), q(x)) q'(x) dx < 0.$$

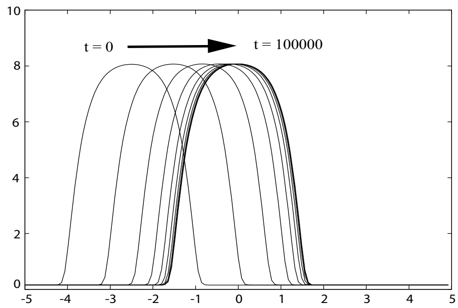

Figure 1: The motion of a localized unimodal pattern of (3.4) for $\varepsilon = 0.01$, where $a_0 = 0.3$, $a_1 = 0.25$, $a_2 = 0.1$, $a_3 = 1.0$, $a_4 = 6.225$, $\gamma = 0.2$, $b_0 = 1.0$, $b_1 = 1.0$, $d_1 = 0.0048$, $d_2 = 0.288$ and $K = 10.0$. The spatial profiles of $u(x, t)$ are given by the graphs of $u(x, t)$ on $-5 \le x \le 5$ for $t = 10000n$ ($n \in \mathbf{Z}$, $0 \le n \le 10$). The peak position of a localized unimodal pattern moves from $x = -2.5$ to $x = 0$. In this case, the value of C in (3.5) can be calculated as $C \approx 1.0 \times 10^{-2}$.

Example 3.1

$$\begin{cases} \dot{u} &= d_1 u_{xx} - \gamma\{F(u) - v - \varepsilon\chi(x)\} \\ \dot{v} &= d_2 v_{xx} + \gamma\{F(u) - v - \varepsilon\chi(x)\}, \end{cases} \tag{3.4}$$

where

$$F(u) = \left(a_0 + \frac{a_4}{a_1 + a_2 u + a_3 u^2}\right) u \quad \text{and} \quad \chi(x) = b_0 \cos\left(\frac{2\pi x}{K}\right) + b_1$$

with nonnegative constants b_k ($k = 1, 2$) and a_j ($j = 0, 1, 2, 3, 4$) satisfying $a_1^2 + a_2^2 + a_3^2 \neq 0$.

Figure 1 shows a numerical result under appropriate parameter values, which is supported by Theorem 3.1. In fact, condition (i) in Theorem 3.1 holds by

$$\frac{\partial}{\partial v} f(u,v) = \frac{\partial}{\partial v}(F(u) - v) \equiv -1 < 0.$$

Moreover, we see that the concrete expression of (3.3) is given by

$$\frac{d\ell}{dt} = -\varepsilon C \sin\left(\frac{2\pi\ell}{K}\right) + O(\varepsilon^2), \tag{3.5}$$

which shows that the peak position of a localized unimodal pattern approaches the maximal point of $\chi(x)$.

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Concentration points in stationary solutions of spatially heterogeneous reaction-diffusion equation

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1 Introduction

As a model of pattern formation in hydra, Gierer and Meinhardt proposed an activator-inhibitor system ([2]). Numerical simulations show that this system produces spiky patterns such that the distribution of solutions concentrates in a very narrow region around finitely many points. In this paper, we consider the stationary problem of the simplified Gierer-Meinhardt system. In particular, we are interested in such concentration phenomena for a spatially heterogeneous reaction-diffusion equation. In order to study the influences of spatial heterogeneity on concentration points, we consider the singularly perturbed Neumann problem with variable coefficients:

$$\text{(P)} \qquad \begin{cases} \varepsilon^2 \mathcal{A}(x)u - b(x)u + c(x)u^p = 0, \quad u > 0 & \text{in } \Omega, \\ \mathcal{B}(x)u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$ and ν is the unit outward normal vector to the boundary. Also, ε is a positive constant, p is a number satisfying $1 < p < (N+2)/(N-2)$ if $N \geqslant 3$ and $1 < p < \infty$ if $N = 1, 2$, $\mathcal{A}(x) = \sum_{i,j=1}^{N}(\partial/\partial x_i)a_{ij}(x)(\partial/\partial x_j)$ is a strictly and uniformly elliptic operator with $a_{ij} = a_{ji}$ and $\mathcal{B}(x) = \sum_{i,j=1}^{N} \nu_i(x)a_{ij}(x)(\partial/\partial x_j)$ is the co-normal differential operator, where $\nu = (\nu_1, \ldots, \nu_N)$. The coefficients a_{ij}, b and c are of class C^2 on $\overline{\Omega}$ and $b(x)$ and $c(x)$ are positive on $\overline{\Omega}$. In this paper, we consider the following problem:

(i) *Is there a solution concentrating at finitely many points as ε tends to zero?*;

(ii) *Where is the concentration points on the domain?*

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1.1 Known results

There are many works on concentration phenomena, and we mainly introduce concentration phenomena of semilinear elliptic equations.

First, let us mention several known results in spatially homogeneous case, given by $\varepsilon^2\Delta u - u + u^p = 0$ in Ω. Ni and Takagi [5] proved that the least-energy solution has only one local maximum point and the point converges to a maximum point of the mean cuvature function of the boundary as ε tends to zero. This is the first result that the location of the concentration point is obtained. By the later studies, Wei [11] proved that a concentration phenomenon occurs not only the least-energy solution but also a solution with a bounded energy. Moreover Gui and Wei [4] proved the existence of solutions with many finitely concentration points.

Second, let us introduce concentration phenomena for a heterogeneous equation, given by $\varepsilon^2\Delta u - u + c(x)u^p = 0$ in Ω. The spatial heterogeneity plays an important role in pattern formation because biological pattern formation takes place usually in spatially heterogeneous environments. Ren [7] proved that the concentration point becomes a maximum point of $c(x)$ in the interior of Ω or on the boundary of Ω by the size of the maxima of $c(x)$ over $\overline{\Omega}$ and on $\partial\Omega$.

On the other hand, in the case $\Omega = \mathbb{R}^N$ there are many works on concentration phenomena in bound states of nonlinear Schrödinger equations initiated by Floer and Weinstein [1]. In addition, see, e. g., Wang [9] and Wang and Zeng [10].

2 Preliminaries

Let us define the notation and the function in order to study solutions of (P) which mean that as $\varepsilon \downarrow 0$, the distribution of a solution concentrates around finitely many points on $\overline{\Omega}$.

First, we introduce an energy functional $J_\varepsilon(u)$ corresponding to (P):

$$J_\varepsilon(u) := \frac{1}{2}\int_\Omega \left(\varepsilon^2 \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + b(x)u^2 \right) dx - \frac{1}{p+1}\int_\Omega c(x) u_+^{p+1}\, dx$$

for $u \in W^{1,2}(\Omega)$ where $u_+(x) = \max\{u(x), 0\}$.

We can apply the Mountain Pass Lemma [6, Theorem 2.2] to J_ε and conclude as follows:

Lemma 2.1 (The least-energy solution)**.** *For each $\varepsilon > 0$, zero is a local minimum of J_ε in $W^{1,2}(\Omega)$. In addition, there exists an $e \in W^{1,2}(\Omega)$ such that $J_\varepsilon(e) < 0$. Let $\Gamma = \{h \in C^0([0,1]; W^{1,2}(\Omega)) \mid h(0) = 0, h(1) = e\}$. Then,*

$$c_\varepsilon = \inf_{h\in\Gamma} \max_{t\in[0,1]} J_\varepsilon(h(t))$$

is a positive critical point of J_ε. Moreover, c_ε is the least positive critical value of J_ε.

We remark that a critical point $u_c \in W^{1,2}(\Omega)$ of J_ε is a weak solution of (P). Then by the elliptic regularity theory we conclude that u_c is a classical solution of (P). In particular, $u_c \in C^{2,\alpha}(\overline{\Omega})$ (see [3, Theorem 6.31 and the remark immediately after its proof in p.130]). Clearly, a classical solution of (P) gives rise to a critical point of J_ε. Hence, finding a solution of (P) is equivalent to finding a critical point of J_ε. We call u_ε *the least-energy solution* of (P).

To be precise, we state the definition of "point concentration" for (P).

Definition 1. A family $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of solutions of (P) is said to exhibit a *point concentration phenomenon* if there exist M distinct points $\{P_{1,0}, \ldots, P_{M,0}\} \subset \overline{\Omega}$, a strictly decreasing sequence $\varepsilon_j \to 0$ ($j \to \infty$) and Msequences $\{P_{k,\varepsilon_j}\}_{j=1}^{\infty} \subset \overline{\Omega}$ with $P_{k,\varepsilon_j} \to P_{k,0}$, $k = 1, \ldots, M$, such that (i) u_{ε_j} achieves *strict* local maxima at $x = P_{k,\varepsilon_j}$ and (ii) $u_{\varepsilon_j}(x) - W_k((x - P_{k,\varepsilon_j})/\varepsilon_j) \to 0$ as $j \to \infty$ in $B_\rho(P_{k,0}) \cap \overline{\Omega}$, where ρ is a positive number and $W_k \in C^2(\mathbb{R}^N)$ is a positive function satisfying $W_k(0) = \max_{y \in \mathbb{R}^N} W_k(y) > 0$ and $W_k(y) \to 0$ as $|y| \to \infty$. We say that $P_{k,0} \in \overline{\Omega}$ is a *concentration point* of $\{u_\varepsilon\}$ if there is a sequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$ such that $\varepsilon_j \to 0$ and $P_{k,\varepsilon_j} \to P_{k,0}$ as $j \to \infty$ for some $1 \leqslant k \leqslant M$.

The purpose of this paper is (i) to show that the least-energy solutions $\{u_\varepsilon\}_{\varepsilon>0}$ exhibit a point concentration phenomenon, and they concentrate at exactly one point $P_0 \in \overline{\Omega}$; and (ii) to give a method to locate P_0 by introducing a *locator function*. We remark here that this type of function was introduced first by Wang and Zeng [10] when they considered a point concentration phenomenon for $h^2\Delta u - V(x)u + K(x)|u|^{p-1}u + Q(x)|u|^{q-1}u = 0$ in $\mathbb{R}^N$.

Definition 2. For any $x \in \overline{\Omega}$, let

$$\Phi(x) := b(x)^{1-N/2+2/(p-1)} c(x)^{-2/(p-1)} (\det A_x)^{1/2},$$

where $A_x := (a_{ij}(x))_{1 \leqslant i,j \leqslant N}$. We call $\Phi(x)$ *the locator function*.

3 Main Results

In this paper, the results to introduce from here include collaborative research [8] with Professor Izumi Takagi.

Theorem 1 ([8]). *Suppose that $P_0 \in \overline{\Omega}$ is a concentration point of a family $\{u_\varepsilon\}_{\varepsilon>0}$ of the least-energy solutions. Then, the following holds:*

(I) If $\min_{x\in\partial\Omega} \Phi(x) < 2\min_{x\in\overline{\Omega}} \Phi(x)$, then $P_0 \in \partial\Omega$. Moreover P_0 is a minimum point of the locator function Φ over $\partial\Omega$.

(II) If $\min_{x\in\partial\Omega} \Phi(x) > 2\min_{x\in\overline{\Omega}} \Phi(x)$, then $P_0 \in \Omega$. Moreover P_0 is a global minimum point of the locator function Φ over $\overline{\Omega}$.

Moreover, as $\varepsilon \downarrow 0$, the least-energy solution u_ε satisfies the following:

$$u_\varepsilon(x) - v_{P_\varepsilon}\left(\frac{x - P_\varepsilon}{\varepsilon}\right) \to 0 \qquad \textit{in } C^2(B_\rho(P_0) \cap \overline{\Omega}) \tag{1}$$

where $\rho > 0$ is arbitrary and P_ε is only one local maximum point of u_ε. Also, for each $Q \in \overline{\Omega}$, v_Q is a unique positive solution of the following boundary value problem:

$$\begin{cases} \mathcal{A}(Q)v - b(Q)v + c(Q)v^p = 0 & \textit{in } \mathbb{R}^N, \\ \lim_{|y|\to\infty} v(y) = 0, \quad v(0) = \max_{y \in \mathbb{R}^N} v(y), \end{cases}$$

where $\mathcal{A}(Q) = \sum_{i,j=1}^N a_{ij}(Q)(\partial^2/\partial y_i \partial y_j)$.

Consequently, we know the location of a concentration point P_0 by investigating sets of the minimum points of Φ over $\overline{\Omega}$ and $\partial\Omega$. We also remark that the least energy $c_\varepsilon/\varepsilon^N$ converges to $\Phi(P_0)I_0$ if $P_0 \in \Omega$ and $\Phi(P_0)(I_0/2)$ if $P_0 \in \partial\Omega$ as $\varepsilon \downarrow 0$, where I_0 is a positive constant depending on only the dimension N and p.

So far, we have stated a concentration phenomenon observed in a family of the least-energy solutions whose existence is guaranteed by the Mountain Pass Lemma. However, it is possible that solutions with higher energy $J_\varepsilon(u) > c_\varepsilon$ exist and exhibit a point concentration phenomenon as in the case of spatially homogeneous equations. The following results reveal the role of the locator function Φ in locating the concentration point.

Theorem 2 (Necessary condition for concentration points [12])**.** *Assume that*

1. *A solution u_ε of* (P) *is concentrating at $P_0 \in \Omega$,*
2. *A local maximum point P_ε of u_ε converges to P_0 as $\varepsilon \downarrow 0$.*

Then, $\nabla\Phi(P_0) = 0$ holds and u_ε satisfies the convergence (1) *for $\rho > 0$ sufficiently small.*

Theorem 3 (Sufficient condition for concentration points [12])**.** *Suppose that $a_{ij} = \delta_{ij}$, b and c are of class C^3 on $\overline{\Omega}$. Assume that $P_0 \in \Omega$ is a nondegenerate critical point of Φ. Then, for small $\varepsilon > 0$ there exists a solution u_ε concentrating at P_0. Moreover, a local maximum point P_ε of u_ε satisfies*

$$P_\varepsilon - P_0 = \varepsilon^\kappa\, [D^2\Phi(P_0)]^{-1}\, {}^T(C_1, \cdots, C_N) + o(\varepsilon^\kappa) \qquad \textit{as } \varepsilon \downarrow 0, \tag{2}$$

where $\kappa = \min(2, p)$, $D^2\Phi$ is the Hesse matrix of Φ and C_j is a complicated constant depending on b, c and these derivatives of order up to 3.

By Theorem 2, we see that the candidates for concentration points become critical points of Φ. By Theorem 3, if $P_0 \in \Omega$ is a nondegenerate critical point of Φ, then we can construct a solution of (P) concentrating at P_0 certainly. Furthermore, from (2), we have the asymptotic behavior of a local maximum point P_ε of u_ε as $\varepsilon \downarrow 0$. We note that it is possible to apply the theorems to the case that solutions have finitely many concentration points. Moreover, we see that the locations of concentration points depend on only the locator function Φ.

Consequently, the influences of spatial heterogeneity seem to be stronger than the geometry of the domain. Hence, choosing the coefficients a_{ij}, b and c appropriately, we obtain the spiky pattern concentrating at any point in the domain.

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