



Title	Generalized Sabban Curves in the Euclidean n-Sphere and Spherical Duality
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Citation	Results in mathematics, 72(1-2), 401-417 <a href="https://doi.org/10.1007/s00025-017-0685-5">https://doi.org/10.1007/s00025-017-0685-5</a>
Issue Date	2017-09
Doc URL	<a href="http://hdl.handle.net/2115/71419">http://hdl.handle.net/2115/71419</a>
Rights	The final publication is available at <a href="http://link.springer.com">link.springer.com</a>
Type	article (author version)
File Information	generalSabbanrev.pdf



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# Generalized Sabban curves in the Euclidean $n$ -sphere and spherical duality

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April 24, 2017

## Abstract

In this paper, we define generalized Sabban frames of curves in  $S^n$  and investigate the singularities of the spherical duals of the curves by using invariants with respect to such frames.

## 1 Introduction

In this paper we consider regular curves in the unit hypersphere in the Euclidean  $n + 1$ -space ( $S^n \subset \mathbb{R}^{n+1}$ ) which is called generalized Sabban curves. We denote that  $\mathbf{a} \times \mathbf{b}$  is the vector product and  $\mathbf{a} \cdot \mathbf{b}$  is the canonical scalar product of  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{n+1}$ .

Let  $\gamma : I \longrightarrow S^2 \subset \mathbb{R}^3$  be a unit speed regular curve. Then we have an orthonormal frame  $\{\gamma, \mathbf{t}, \mathbf{n}\}$  along  $\gamma$ , where  $\mathbf{t}(s)$  is the unit tangent vector of  $\gamma$  at  $s$  and  $\mathbf{n}(s) = \gamma(s) \times \mathbf{t}(s)$ . We have the Frenet-type formula:

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = -\gamma(s) + \kappa_g(s)\mathbf{n}(s), \\ \mathbf{n}'(s) = -\kappa_g(s)\mathbf{t}(s), \end{cases}$$

where  $\kappa_g(s) = \mathbf{t}'(s) \cdot \mathbf{n}(s)$  is the geodesic curvature of  $\gamma$  at  $s$ . This frame was introduced by Sabban (cf. [8]) and it is called a *Sabban frame* of  $\gamma$ . For a unit speed curve in  $S^2$ , we always have the Sabban frame. However, we need some assumptions to define the Sabban type frame for regular curves in  $S^n$ , where  $n \geq 3$ . We say that a unit speed curve  $\gamma : I \longrightarrow S^3 \subset \mathbb{R}^4$  is a *Sabban curve* (or, a *spherical Frenet curve*) if  $\|\mathbf{t}'(s) + \gamma(s)\| \neq 0$  at any point  $s \in I$ . Then we have an orthonormal frame  $\{\gamma, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$  of  $\mathbb{R}^4$  along  $\gamma$ , where  $\mathbf{n}_1(s) = (\mathbf{t}'(s) + \gamma(s)) / \|\mathbf{t}'(s) + \gamma(s)\|$  and  $\mathbf{n}_2(s) = \gamma(s) \times \mathbf{t}(s) \times \mathbf{n}_1(s)$ . Here,  $\mathbf{a}_1 \times \mathbf{a}_2 \times \mathbf{a}_3$  is the generalized vector product in  $\mathbb{R}^4$  (cf. §2). Then we have

$$\begin{cases} \gamma'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = -\gamma(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}_1'(s) = -\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \mathbf{n}_2'(s) = -\kappa_2(s)\mathbf{t}(s), \end{cases}$$

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2010 Mathematics Subject classification. Primary 53A35; Secondary 58K35

Key Words and Phrases. Generalized Sabban frame, Curves in  $n$ -sphere, spherical height functions, Unfoldings of function-germs

where  $\kappa_1(s) = \|\mathbf{t}'(s) + \boldsymbol{\gamma}(s)\| \neq 0$  and  $\kappa_2(s) = \mathbf{n}'_1(s) \cdot \mathbf{n}_2(s)$ . We call  $\{\boldsymbol{\gamma}, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$  a *Sabban frame* along a Sabban curve  $\boldsymbol{\gamma}$ . In §2, we construct an orthonormal frame  $\{\boldsymbol{\gamma}, \mathbf{t}, \mathbf{n}_1, \dots, \mathbf{n}_{n-1}\}$  along  $\boldsymbol{\gamma} : I \rightarrow S^n \subset \mathbb{R}^{n+1}$  under some conditions on  $\boldsymbol{\gamma}$ . We call the above orthonormal frame a *generalized Sabban frame*. In this case  $\boldsymbol{\gamma}$  is called a *generalized Sabban curve*, which is an analogous notion of Frenet curves in  $\mathbb{R}^n$  (cf. [6, 7])

In this paper we consider spherical dual hypersurfaces of generalized Sabban curves. For a generalized Sabban curve  $\boldsymbol{\gamma}$  we define a hypersurface  $(\boldsymbol{\gamma})^*$  in  $S^n$  by

$$(\boldsymbol{\gamma})^* = \{\xi_1 \mathbf{n}_1(s) + \dots + \xi_{n-1} \mathbf{n}_{n-1}(s) \mid s \in I, \xi_1^2 + \dots + \xi_{n-1}^2 = 1\}.$$

Then we call  $(\boldsymbol{\gamma})^*$  a *spherical dual hypersurface* of  $\boldsymbol{\gamma}$ . In §4 we give an interpretation why  $(\boldsymbol{\gamma})^*$  can be called a spherical dual of  $\boldsymbol{\gamma}$  as an application of the theory of Legendrian dualities. The main purpose in this paper is to give classifications and characterizations of the singularities of  $(\boldsymbol{\gamma})^*$  by using the geometric properties of the generalized Sabban frame (cf. Theorems 5.6 and 6.2). These results are generalizations of some of the results on curves in  $S^2$  or  $S^3$  [2, 9, 10].

## 2 Notations and Definitions

In this section we consider a regular curve in the unit hypersphere in the Euclidean space. Let  $S^n$  be the  $n$ -dimensional unit sphere in the Euclidean space  $\mathbb{R}^{n+1}$ . Given a vector  $\mathbf{n} \in \mathbb{R}^{n+1} \setminus \{0\}$  and a real number  $c$ , the hyperplane with a normal vector  $\mathbf{n}$  is defined to be  $HP(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{n} \cdot \mathbf{x} = c\}$ , where  $\mathbf{v} \cdot \mathbf{w}$  is the canonical scalar product of  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n+1}$ . A sphere in  $S^n$  is given by

$$S^{n-1}(\mathbf{n}, c) = S^n \cap H(\mathbf{n}, c) = \{\mathbf{x} \in S^n \mid \mathbf{n} \cdot \mathbf{x} = c\}.$$

We say that  $S^{n-1}(\mathbf{n}, c)$  is a *great hypersphere* if  $c = 0$  and a *small hypersphere* if  $c \neq 0$ , respectively. Here, we call  $\mathbf{n}$  a *polar vector* of  $S^{n-1}(\mathbf{n}, c)$ . For any  $\mathbf{a}_i = (a_i^1, a_i^2, \dots, a_i^{n+1}) \in \mathbb{R}^{n+1}$  ( $i = 1, \dots, n$ ), the vector product  $\mathbf{a}_1 \times \mathbf{a}_2 \times \dots \times \mathbf{a}_n$  is defined by

$$\mathbf{a}_1 \times \mathbf{a}_2 \times \dots \times \mathbf{a}_n = \det \begin{pmatrix} \mathbf{e}^1 & \mathbf{e}^2 & \dots & \mathbf{e}^{n+1} \\ a_1^1 & a_1^2 & \dots & a_1^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \dots & a_n^{n+1} \end{pmatrix},$$

where  $\{\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n+1}\}$  is the canonical basis of  $\mathbb{R}^{n+1}$ . We can easily show that  $\mathbf{a}_1 \times \mathbf{a}_2 \times \dots \times \mathbf{a}_n$  is orthogonal to any  $\mathbf{a}_i$  ( $i = 1, \dots, n$ ).

We now define generalized Sabban frame of a spherical curve in  $S^n$ . Let  $\boldsymbol{\gamma} : I \rightarrow S^n$  be a regular curve, where  $I$  is an open interval in  $\mathbb{R}$ . We can reparametrize  $\boldsymbol{\gamma}$  by the arc-length. Hence, we may assume that  $\boldsymbol{\gamma}(s)$  is a unit speed curve, so that we have the tangent vector  $\mathbf{t}(s) = \boldsymbol{\gamma}'(s) = (d\boldsymbol{\gamma}/ds)(s)$  with  $\|\mathbf{t}(s)\| = 1$ . In the case when  $\|\mathbf{t}'(s) + \boldsymbol{\gamma}(s)\| \neq 0$ , we have a unit vector  $\mathbf{n}_1(s) = (\mathbf{t}'(s) + \boldsymbol{\gamma}(s))/\|\mathbf{t}'(s) + \boldsymbol{\gamma}(s)\|$ . We can easily show that  $\mathbf{n}_1(s)$  is orthogonal to  $\boldsymbol{\gamma}(s)$  and  $\mathbf{t}(s)$  by a straight forward calculation. We write  $\kappa_1(s) = \|\mathbf{t}'(s) + \boldsymbol{\gamma}(s)\|$ . Next we consider  $\kappa_2(s) = \|\mathbf{n}_1(s) + \kappa_1(s)\mathbf{t}(s)\|$ . In the case when  $\kappa_2(s) \neq 0$ , we have another unit vector  $\mathbf{n}_2 = (\mathbf{n}_1(s) + \kappa_1(s)\mathbf{t}(s))/\|\mathbf{n}_1(s) + \kappa_1(s)\mathbf{t}(s)\|$ . By repeating the method similar to the above,

we have the following functions and unit vectors;

$$\kappa_i(s) = \|\mathbf{n}'_{i-1}(s) + \kappa_{i-1}(s)\mathbf{n}_{i-2}(s)\|,$$

$$\mathbf{n}_i(s) = \frac{\mathbf{n}'_{i-1}(s) + \kappa_{i-1}(s)\mathbf{n}_{i-2}(s)}{\|\mathbf{n}'_{i-1}(s) + \kappa_{i-1}(s)\mathbf{n}_{i-2}(s)\|} = \frac{\mathbf{n}'_{i-1}(s) + \kappa_{i-1}(s)\mathbf{n}_{i-2}(s)}{\kappa_i(s)}$$

for  $i = 1, \dots, n-2$ , where we assume that  $\kappa_i(s) \neq 0$  for all  $i$ . Finally, we define

$$\mathbf{n}_{n-1}(s) = \frac{\boldsymbol{\gamma}(s) \times \mathbf{t}(s) \times \mathbf{n}_1(s) \times \dots \times \mathbf{n}_{n-2}(s)}{\|\boldsymbol{\gamma}(s) \times \mathbf{t}(s) \times \mathbf{n}_1(s) \times \dots \times \mathbf{n}_{n-2}(s)\|},$$

$$\kappa_{n-1}(s) = \mathbf{n}'_{n-2}(s) \cdot \mathbf{n}_{n-1}(s).$$

We call  $\kappa_i(s)$  a *ith-curvature* of  $\boldsymbol{\gamma}(s)$ .

**Lemma 2.1.** *With the above notation, vectors  $\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s), \mathbf{n}_{n-1}(s)$  are orthogonal to each other.*

*Proof.* By definition,  $\mathbf{n}_{n-1}(s)$  is orthogonal to  $\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s)$ . For other vectors we can prove by a straight forward calculation. For example,

$$(*) \quad \boldsymbol{\gamma}(s) \cdot \mathbf{n}_1(s) = \frac{1}{\kappa_1}(\boldsymbol{\gamma}(s) \cdot \mathbf{t}'(s) + \boldsymbol{\gamma}(s) \cdot \boldsymbol{\gamma}(s))$$

Because  $\boldsymbol{\gamma}$  is a spherical curve,  $\boldsymbol{\gamma}(s) \cdot \boldsymbol{\gamma}(s) = 1$ , so that we have  $\boldsymbol{\gamma}(s) \cdot \mathbf{t}(s) = 0$ . Then  $(\boldsymbol{\gamma}(s) \cdot \mathbf{t}(s))' = \boldsymbol{\gamma}'(s) \cdot \mathbf{t}(s) + \boldsymbol{\gamma}(s) \cdot \mathbf{t}'(s) = 0$ , so

$$\begin{aligned} \boldsymbol{\gamma}(s) \cdot \mathbf{t}'(s) &= -\boldsymbol{\gamma}'(s) \cdot \mathbf{t}(s) \\ &= -\|\mathbf{t}(s)\|^2 = -1. \end{aligned}$$

Then

$$(*) = \frac{1}{\kappa_1}(-1 + 1) = 0.$$

Therefore  $\boldsymbol{\gamma}(s)$  and  $\mathbf{t}(s)$  are orthogonal. We can prove that all vectors are orthogonal to each other by the method similar to the above calculation.  $\square$

By the above lemma, the set of vectors  $\{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-1}(s)\}$  is an orthonormal frame of  $\mathbb{R}^{n+1}$  along  $\boldsymbol{\gamma}$ . We call this frame a *generalized Sabban frame* along  $\boldsymbol{\gamma}$ . We also call a spherical curve  $\boldsymbol{\gamma}$  a *generalized Sabban curve* when  $\kappa_i(s) \neq 0$  for  $i = 1, 2, \dots, n-2$ . We remark that  $\kappa_{n-1}(s)$  might be equal to 0.

Here we expect the following *Frenet-Serret type formula* :

$$\left\{ \begin{array}{lcl} \boldsymbol{\gamma}'(s) & = & \mathbf{t}(s), \\ \mathbf{t}'(s) & = & -\boldsymbol{\gamma}(s) + \kappa_1(s)\mathbf{n}_1(s), \\ \mathbf{n}'_1(s) & = & -\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s), \\ \dots & = & \dots \\ \mathbf{n}'_i(s) & = & -\kappa_i(s)\mathbf{n}_{i-1}(s) + \kappa_{i+1}(s)\mathbf{n}_{i+1}(s), \\ \dots & = & \dots \\ \mathbf{n}'_{n-2}(s) & = & -\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s) \\ \mathbf{n}'_{n-1}(s) & = & -\kappa_{n-1}(s)\mathbf{n}_{n-2}(s). \end{array} \right.$$

We obtain equations for  $\gamma'(s), \dots, \mathbf{n}'_{n-3}(s)$  by definition. On the other hand we need a little calculation to get the equations for  $\mathbf{n}'_{n-2}(s), \mathbf{n}'_{n-1}(s)$  since the definitions of  $\kappa_{n-1}(s)$  and  $\mathbf{n}_{n-1}(s)$  are different from other  $\kappa_i(s)$  and  $\mathbf{n}_i(s)$ .

**Lemma 2.2.** *We have the following equations :*

$$\begin{aligned}\mathbf{n}'_{n-2}(s) &= -\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s) \\ \mathbf{n}'_{n-1}(s) &= -\kappa_{n-1}(s)\mathbf{n}_{n-2}(s).\end{aligned}$$

*Proof.* We denote that  $\mathbf{n}'_{n-2}(s) = \lambda\gamma(s) + \mu\mathbf{n}(s) + \xi_1\mathbf{n}_1(s) + \dots + \xi_{n-1}\mathbf{n}_{n-1}(s)$ . Since  $\gamma(s), \dots, \mathbf{n}_{n-1}(s)$  are orthogonal,  $\mathbf{n}_{n-2}(s) \cdot \gamma(s) = 0$ . Then, it follows that,

$$\begin{aligned}(\mathbf{n}_{n-2}(s) \cdot \gamma(s))' &= \mathbf{n}'_{n-2}(s) \cdot \gamma(s) + \mathbf{n}_{n-2}(s) \cdot \gamma'(s) \\ &= \lambda + \mathbf{n}_{n-2}(s) \cdot \mathbf{t}(s) \\ &= \lambda,\end{aligned}$$

so  $\lambda = 0$ . We can prove that  $\mu, \xi_1, \dots, \xi_{n-4}$  and  $\xi_{n-2}$  are equal to 0 by the same way as the above. By definition,  $\mathbf{n}'_{n-2}(s) \cdot \mathbf{n}_{n-1}(s) = \xi_{n-1} = \kappa_{n-1}(s)$ . Finally,

$$\begin{aligned}(\mathbf{n}_{n-2}(s) \cdot \mathbf{n}_{n-3}(s))' &= \mathbf{n}'_{n-2}(s) \cdot \mathbf{n}_{n-3}(s) + \mathbf{n}_{n-2}(s) \cdot \mathbf{n}'_{n-3}(s) \\ &= \xi_{n-3} + \mathbf{n}_{n-2}(s) \cdot (\kappa_{n-3}(s)\mathbf{n}_{n-4}(s) + \kappa_{n-2}(s)\mathbf{n}_{n-2}(s)) \\ &= \xi_{n-3} + \kappa_{n-2}(s).\end{aligned}$$

Since  $\mathbf{n}_{n-2}(s) \cdot \mathbf{n}_{n-3} = 0$ ,  $\xi_{n-3} = -\kappa_{n-2}(s)$ . We can prove the formula for  $\mathbf{n}'_{n-1}(s)$  by the same way as the above. This completes the proof.  $\square$

By Lemma 2.2, we have the Frenet-Serret type formulae for the generalized Sabban frame of a spherical curve. We can write them as follows:

$$\begin{pmatrix} \gamma'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'_1(s) \\ \vdots \\ \mathbf{n}'_{n-2}(s) \\ \mathbf{n}'_{n-1}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & \kappa_1(s) & 0 & 0 & \dots & 0 & 0 \\ 0 & -\kappa_1(s) & 0 & \kappa_2(s) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -\kappa_{n-2}(s) & 0 & \kappa_{n-1}(s) \\ 0 & 0 & \dots & 0 & 0 & 0 & -\kappa_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mathbf{t}(s) \\ \mathbf{n}_1(s) \\ \vdots \\ \mathbf{n}_{n-2}(s) \\ \mathbf{n}_{n-1}(s) \end{pmatrix}$$

We can interpret the geometric meaning of the  $(n-1)$ th-curvature  $\kappa_{n-1}(s)$  of  $\gamma(s)$ .

**Proposition 2.3.** *Let  $\gamma : I \rightarrow S^n$  be a generalized Sabban curve. Then there exists a great hypersphere  $S^{n-1}(\mathbf{n}, 0)$  such that  $\gamma(I) \subset S^{n-1}(\mathbf{n}, 0)$  if and only if  $\kappa_{n-1} \equiv 0$ .*

*Proof.* Suppose that  $\kappa_{n-1} \equiv 0$ . By the Frenet-Serret type formulae (\*\*),  $\mathbf{n}_{n-1}$  is a constant vector. We denote that  $\mathbf{n}_{n-1}(s) = \mathbf{n}$ . We consider a function  $f : I \rightarrow \mathbb{R}$  defined by  $f(s) = \gamma(s) \cdot \mathbf{n}$ . Then we have  $f(s) = \gamma(s) \cdot \mathbf{n} = \gamma(s) \cdot \mathbf{n}_{n-1}(s) = 0$  and  $f'(s) = \mathbf{t}(s) \cdot \mathbf{n} = \mathbf{t}(s) \cdot \mathbf{n}_{n-1}(s) = 0$ . Therefore  $f(s)$  is constantly equal to 0, so that  $\gamma(s) \in S^n \cap H(\mathbf{n}, 0) = S^{n-1}(\mathbf{n}, 0)$ . For the converse, suppose that there exists  $S^{n-1}(\mathbf{n}, 0)$  such that  $\gamma(I) \subset S^{n-1}(\mathbf{n}, 0)$ . Then the function  $f$  defined as the above is constantly equal to 0. It follows that  $f'(s) = \mathbf{t}(s) \cdot \mathbf{n} = 0$ . Thus,  $0 = f''(s) = \mathbf{t}'(s) \cdot \mathbf{n} = (-\gamma(s) + \kappa_1(s)\mathbf{n}_1(s)) \cdot \mathbf{n} = \kappa_1(s)\mathbf{n}_1(s) \cdot \mathbf{n}$ . Since  $\kappa_1(s) \neq 0$ , we have  $\mathbf{n}_1(s) \cdot \mathbf{n} = 0$ . It follows that

$$0 = \mathbf{n}'_1(s) \cdot \mathbf{n} = (-\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s)) \cdot \mathbf{n} = \kappa_2(s)\mathbf{n}_2(s) \cdot \mathbf{n}.$$

Since  $\kappa_2(s) \neq 0$ , we have  $\mathbf{n}_2(s) \cdot \mathbf{n} = 0$ . It also follows that

$$0 = \mathbf{n}'_2(s) \cdot \mathbf{n} = (-\kappa_2(s)\mathbf{n}_1(s) + \kappa_3(s)\mathbf{n}_3(s)) \cdot \mathbf{n} = \kappa_3(s)\mathbf{n}_3(s) \cdot \mathbf{n}.$$

Since  $\kappa_3(s) \neq 0$ , we have  $\mathbf{n}_3(s) \cdot \mathbf{n} = 0$ . We continue this procedure. Finally, we have  $\kappa_{n-1}(s)\mathbf{n}_{n-1}(s) \cdot \mathbf{n} = 0$ . If  $\mathbf{n}_{n-1}(s) \cdot \mathbf{n} = 0$ , then  $\mathbf{n}$  is orthogonal to all vectors of the generalized Sabban frame  $\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-1}(s)\}$  which contradicts to the fact that the generalized Sabban frame is a basis of  $\mathbb{R}^{n+1}$  and  $\mathbf{n} \neq \mathbf{0}$ . Thus  $\kappa_{n-1}(s) = 0$  for any  $s \in I$ .  $\square$

### 3 Spherical height functions

In this section we introduce a family of functions on a curve in the sphere that is useful for the study of invariants of generalized Sabban curve. For a generalized Sabban curve  $\gamma : I \rightarrow S^n$ , we define a function  $H : I \times S^n \rightarrow \mathbb{R}$  by  $H(s, \mathbf{v}) = \gamma(s) \cdot \mathbf{v}$ . We call  $H$  a *spherical height function* on  $\gamma$ . We write  $h_{\mathbf{v}_0}(s) = H_{\mathbf{v}_0}(s) = H(s, \mathbf{v}_0)$  for any fixed vector  $\mathbf{v}_0 \in S^n$ . Then we have the following proposition.

**Proposition 3.1.** *Let  $\gamma : I \rightarrow S^n$  be a generalized Sabban curve. Then we have the following:*

- (1)  $h_{\mathbf{v}_0}(s_0) = 0$  if and only if there exist  $\mu, \xi_1, \dots, \xi_{n-1} \in \mathbb{R}$  such that  $\mathbf{v}_0 = \mu\mathbf{t}(s_0) + \xi_1\mathbf{n}_1(s_0) + \dots + \xi_{n-1}\mathbf{n}_{n-1}(s_0)$  and  $\mu^2 + \xi_1^2 + \dots + \xi_{n-1}^2 = 1$ ,
- (2) for  $k < n$ ,  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \dots = h_{\mathbf{v}_0}^{(k)}(s_0) = 0$  if and only if there exist  $\xi_k, \dots, \xi_{n-1} \in \mathbb{R}$  such that  $\mathbf{v}_0 = \xi_k\mathbf{n}_k(s_0) + \dots + \xi_{n-1}\mathbf{n}_{n-1}(s_0)$  and  $\xi_k^2 + \dots + \xi_{n-1}^2 = 1$ ,
- (3)  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \dots = h_{\mathbf{v}_0}^{(n)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm\mathbf{n}_{n-1}(s_0)$  and  $\kappa_{n-1}(s_0) = 0$ ,
- (4) for  $k > n$ ,  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \dots = h_{\mathbf{v}_0}^{(k)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm\mathbf{n}_{n-1}(s_0)$  and  $\kappa_{n-1}(s_0) = \kappa'_{n-1}(s_0) = \dots = \kappa_{n-1}^{(k-n)}(s_0) = 0$ .

By the definition of spherical height function,  $h_{\mathbf{v}_0}^{(k)}(s_0) = (\gamma(s_0) \cdot \mathbf{v}_0)^{(k)} = \gamma^{(k)}(s_0) \cdot \mathbf{v}_0$ . We consider  $\gamma^{(k)}(s_0)$ . By the Frenet-Serret type formulae, we have the following lemma.

**Lemma 3.2.**  $\gamma^{(k)}(s)$  has the following form:

- (1)  $\gamma'(s) = \mathbf{t}(s)$ ,
- (2)  $\gamma''(s) = -\gamma(s) + \kappa_1(s)\mathbf{n}_1(s)$ ,
- (3) For  $3 \leq k \leq n$ , there exist functions  $\bar{\lambda}(s), \bar{\mu}(s), \bar{\xi}_1(s), \dots, \bar{\xi}_{k-2}(s)$  such that

$$(**) \quad \gamma^{(k)}(s) = \bar{\lambda}(s)\gamma(s) + \bar{\mu}(s)\mathbf{t}(s) + \bar{\xi}_1(s)\mathbf{n}_1(s) + \dots + \bar{\xi}_{k-2}(s)\mathbf{n}_{k-2}(s) + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-1}(s)\mathbf{n}_{k-1}(s).$$

*Proof.* By the definition of  $\mathbf{t}, \mathbf{n}_1$  and  $\kappa_1$ , (1) and (2) hold.

We prove assertion (3) by using induction. Let  $k = 3$ . By the formula (2),

$$\begin{aligned} \gamma^{(3)}(s) &= \{-\gamma(s) + \kappa_1(s)\mathbf{n}_1(s)\}' \\ &= -\mathbf{t}(s) + \kappa'_1(s)\mathbf{n}_1(s) + \kappa_1(s)\mathbf{n}'_1(s) \\ &= -\mathbf{t}(s) + \kappa'_1(s)\mathbf{n}_1(s) + \kappa_1(s)(-\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s)) \\ &= -(1 + \kappa_1^2(s))\mathbf{t}(s) + \kappa'_1(s)\mathbf{n}_1(s) + \kappa_1(s)\kappa_2(s)\mathbf{n}_2(s) \end{aligned}$$

The coefficient of  $\mathbf{n}_2$  is  $\kappa_1(s)\kappa_2(s)$ . Then (\*\*) holds for  $k = 3$ .

We assume that (\*\*) holds for  $k - 1$ ; that is, there exist  $\bar{\lambda}(s), \bar{\mu}(s), \bar{\xi}_1(s), \dots, \bar{\xi}_{k-3}(s)$  such that

$$\gamma^{(k-1)}(s) = \bar{\lambda}(s)\gamma(s) + \bar{\mu}(s)\mathbf{t}(s) + \bar{\xi}_1(s)\mathbf{n}_1(s) + \dots + \bar{\xi}_{k-3}(s)\mathbf{n}_{k-3}(s) + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\mathbf{n}_{k-2}(s).$$

Here we calculate  $\gamma^{(k)}$  by using the Frenet-Serret type formulae:

$$\begin{aligned}
\gamma^{(k)}(s) &= \{\gamma^{(k-1)}\}'(s) \\
&= \bar{\lambda}'(s)\gamma(s) + \bar{\lambda}(s)\gamma'(s) + \bar{\mu}'(s)\mathbf{t}(s) + \bar{\mu}(s)\mathbf{t}'(s) + \cdots + \bar{\xi}_{k-3}'(s)\mathbf{n}_{k-3}(s) + \bar{\xi}_{k-3}(s)\mathbf{n}_{k-3}'(s) \\
&\quad + \{\kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\}'\mathbf{n}_{k-2}(s) + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\mathbf{n}_{k-2}'(s) \\
&= \{\bar{\lambda}'(s) - \bar{\mu}(s)\}\gamma(s) + \{\bar{\lambda}(s) + \bar{\mu}'(s) - \kappa_1(s)\}\mathbf{t}(s) + \cdots \\
&\quad + \{\bar{\xi}_{k-4}(s)\kappa_{k-3}(s) + \bar{\xi}_{k-3}'(s) + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\}\mathbf{n}_{k-3}(s) \\
&\quad + \{\bar{\xi}_{k-3}(s)\kappa_{k-2}(s) + (\kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s))'\}\mathbf{n}_{k-2}(s) \\
&\quad + \kappa_1(s)\kappa_2(s) \cdots \kappa_{k-2}(s)\kappa_{k-1}(s)\mathbf{n}_{k-1}(s)
\end{aligned}$$

The coefficient of  $\mathbf{n}_{n-1}(s)$  is  $\kappa_1(s)\kappa_2(s) \cdots \kappa_{k-1}(s)$ . Then  $(**)$  holds for all  $k = 3, 4, \dots, n$ .  $\square$

*Proof of Proposition 3.1.* By using the generalized Sabban frame, there exist  $\lambda, \mu, \xi_1, \dots, \xi_{n-1} \in \mathbb{R}$  such that:

$$\mathbf{v}_0 = \lambda\gamma(s_0) + \mu\mathbf{t}(s_0) + \xi_1\mathbf{n}_1(s_0) + \cdots + \xi_{n-1}\mathbf{n}_{n-1}(s_0).$$

Then we have

$$h_{\mathbf{v}_0}(s_0) = \gamma(s_0) \cdot \mathbf{v}_0 = \lambda.$$

Therefore assertion (1) holds.

We prove assertion (2) inductively. Let  $k = 1$ . By assertion (1) of Lemma 3.2, we have

$$h'_{\mathbf{v}_0}(s_0) = \gamma'(s_0) \cdot \mathbf{v}_0 = \mathbf{t}(s_0) \cdot \mathbf{v}_0 = \mu.$$

Thus  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = 0$  means  $\lambda = \mu = 0$ , then (2) holds for  $k = 1$ . We now assume that (2) holds for  $k - 1$ . This means that  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \cdots = h_{\mathbf{v}_0}^{k-1}(s_0) = 0$  if and only if there exist  $\xi_k, \dots, \xi_{n-1} \in \mathbb{R}$  such that  $\mathbf{v}_0 = \xi_{k-1}\mathbf{n}_{k-1}(s_0) + \cdots + \xi_{n-1}\mathbf{n}_{n-1}(s_0)$ . By assertion (3) of Lemma 3.2, we have

$$h_{\mathbf{v}_0}^k(s_0) = \gamma^{(k)}(s) \cdot \mathbf{v}_0 = (\bar{\lambda}(s)\gamma(s_0) + \cdots + \kappa_1(s_0)\kappa_2(s_0) \cdots \kappa_{k-1}(s_0)\mathbf{n}_{k-1}(s_0)) \cdot \mathbf{v}_0.$$

Therefore,  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \cdots = h_{\mathbf{v}_0}^{(k)}(s_0) = 0$  if and only if

$$0 = h_{\mathbf{v}_0}^{(k)}(s_0) = \gamma^{(k)}(s) \cdot (\xi_{k-1}\mathbf{n}_{k-1}(s_0) + \cdots + \xi_{n-1}\mathbf{n}_{n-1}(s_0)) = \kappa_1(s_0)\kappa_2(s_0) \cdots \kappa_{k-1}(s_0) \cdot \xi_{k-1}.$$

Since  $k - 1 < n - 1$ , we have  $\kappa_1(s_0)\kappa_2(s_0) \cdots \kappa_{k-1}(s_0) \neq 0$ , so that  $\xi_{k-1} = 0$ . We also have  $1 = \mathbf{v}_0 \cdot \mathbf{v}_0 = \xi_k^2 + \cdots + \xi_{n-1}^2$ . This completes the induction step. By assertion (2),  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \cdots = h_{\mathbf{v}_0}^{(n-1)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm\mathbf{n}_{n-1}(s_0)$ . It follows from (3) of Lemma 3.2 that  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \cdots = h_{\mathbf{v}_0}^{(n)}(s_0) = 0$  if and only if

$$0 = h_{\mathbf{v}_0}^{(n)}(s_0) = \gamma^{(n)}(s_0) \cdot (\pm\mathbf{n}_{n-1}(s_0)) = \pm\kappa_1(s_0)\kappa_2(s_0) \cdots \kappa_{n-2}(s_0)\kappa_{n-1}(s_0)$$

This means that  $\kappa_{n-1}(s_0) = 0$ . This completes the proof of (3).

We now prove assertion (4) by induction. By assertion (3),  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \dots = h_{\mathbf{v}_0}^{(n+1)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$ ,  $\kappa_{n-1}(s_0) = 0$  and  $h_{\mathbf{v}_0}^{(n+1)}(s_0) = 0$ . By (3) of Lemma 3.2, we have

$$\begin{aligned}\gamma^{(n)}(s) &= \bar{\lambda}(s)\gamma(s) + \bar{\mu}(s)\mathbf{t}(s) + \bar{\xi}_1(s)\mathbf{n}_1(s) + \dots \\ &\quad + \bar{\xi}_{n-2}(s)\mathbf{n}_{n-2}(s) + \kappa_1(s)\kappa_2(s) \cdots \kappa_{n-1}(s)\mathbf{n}_{n-1}(s).\end{aligned}$$

If we put  $K(s) = \kappa_1(s)\kappa_2(s) \cdots \kappa_{n-2}(s)$ , then we have

$$\begin{aligned}\gamma^{(n+1)}(s) &= \bar{\lambda}'(s)\gamma(s) + \bar{\lambda}(s)\mathbf{t}(s) + \bar{\mu}'(s)\mathbf{t}(s) + \bar{\mu}(s)(-\gamma(s) + \kappa_1(s)\mathbf{n}_1(s)) \\ &\quad + \bar{\xi}_1'(s)\mathbf{n}_1(s) + \bar{\xi}_1(s)(-\kappa_1(s)\mathbf{t}(s) + \kappa_2(s)\mathbf{n}_2(s)) + \dots \\ &\quad + \bar{\xi}_{n-2}'(s)\mathbf{n}_{n-2}(s) + \bar{\xi}_{n-2}(s)(-\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s)) \\ &\quad + (K'(s)\kappa_{n-1}(s) + K(s)\kappa'_{n-1}(s))\mathbf{n}_{n-1}(s) - K(s)\kappa_{n-1}^2(s)\mathbf{n}_{n-2}(s) \\ &= V(s) + W(s),\end{aligned}$$

for some  $V(s) \in \langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s) \rangle_{\mathbb{R}}$  and

$$W(s) = (\bar{\xi}_{n-2}(s)\kappa_{n-1}(s) + K'(s)\kappa_{n-1}(s) + K(s)\kappa'_{n-1}(s))\mathbf{n}_{n-1}(s).$$

Since  $V(s) \cdot \mathbf{n}_{n-1}(s) = 0$  and  $\kappa_1(s)\kappa_2(s) \cdots \kappa_{n-2}(s) \neq 0$ ,  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$ ,  $\kappa_{n-1}(s_0) = 0$  and  $h_{\mathbf{v}_0}^{(n+1)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$ ,  $\kappa_{n-1}(s_0) = 0$  and  $\kappa'_{n-1}(s_0) = 0$ . Thus assertion (4) for  $k = n + 1$  holds. We now assume that the following conditions hold for  $k = n + (r - 1)$ :  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \dots = h_{\mathbf{v}_0}^{(k)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$  and  $\kappa_{n-1}(s_0) = \kappa'_{n-1}(s_0) = \dots = \kappa_{n-1}^{(r-1)}(s_0) = 0$  and  $\gamma^{(k)}(s) = V(s) + W(s)$ , where  $V(s) \in \langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s) \rangle_{\mathbb{R}}$  and there exist functions  $\eta_0(s), \eta_1(s), \dots, \eta_{r-2}(s)$  such that

$$W(s) = \left( \sum_{i=0}^{r-2} \eta_i(s)\kappa_{n-1}^{(i)}(s) + K(s)\kappa_{n-1}^{(r-1)}(s) \right) \mathbf{n}_{n-1}(s).$$

Therefore, we have  $V(s) = \bar{V}(s) + \zeta(s)\mathbf{n}_{n-2}(s)$  for some  $\bar{V}(s) \in \langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-3}(s) \rangle_{\mathbb{R}}$ . It follows that

$$V'(s) = \bar{V}'(s) + \zeta'(s)\mathbf{n}_{n-2}(s) + \zeta(s)(-\kappa_{n-2}(s)\mathbf{n}_{n-3}(s) + \kappa_{n-1}(s)\mathbf{n}_{n-1}(s))$$

and

$$\begin{aligned}W'(s) &= \left( \sum_{i=0}^{r-2} (\eta_i'(s)\kappa_{n-1}^{(i)}(s) + \eta_i(s)\kappa_{n-1}^{(i+1)}(s)) + K'(s)\kappa_{n-1}^{(r-1)}(s) + K(s)\kappa_{n-1}^{(r)}(s) \right) \mathbf{n}_{n-1}(s) \\ &\quad - \left( \sum_{i=0}^{r-2} \eta_i(s)\kappa_{n-1}^{(i)}(s) + K(s)\kappa_{n-1}^{(r-1)}(s) \right) \kappa_{n-2}(s)\mathbf{n}_{n-2}(s).\end{aligned}$$

It follows that there exist functions  $\bar{\eta}_0(s), \dots, \bar{\eta}_{r-1}(s)$  and  $\tilde{V}(s) \in \langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-2}(s) \rangle_{\mathbb{R}}$  such that

$$\gamma^{(k+1)}(s) = \tilde{V}(s) + \left( \sum_{i=0}^{r-1} \bar{\eta}_i(s)\kappa_{n-1}^{(i)}(s) + K(s)\kappa_{n-1}^{(r)}(s) \right) \mathbf{n}_{n-1}(s).$$

Since  $K(s) \neq 0$  and  $\mathbf{n}_{n-1}(s) \cdot \tilde{V}(s) = 0$ ,  $h_{\mathbf{v}_0}(s_0) = h'_{\mathbf{v}_0}(s_0) = \dots = h_{\mathbf{v}_0}^{(k+1)}(s_0) = 0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$  and  $\kappa_{n-1}(s_0) = \kappa'_{n-1}(s_0) = \dots = \kappa_{n-1}^{(r)}(s_0) = 0$ . This completes the proof for assertion (4).  $\square$

## 4 Spherical Legendrian duality

According to the results of Proposition 3.1, we now define a mapping  $D_\gamma : I \times S^{n-2} \longrightarrow S^n$  by

$$D_\gamma(s, \xi) = \xi_1 \mathbf{n}_1(s) + \cdots + \xi_{n-1} \mathbf{n}_{n-1}(s),$$

where  $\xi = (\xi_1, \dots, \xi_{n-1}) \in S^{n-2}$ . Then  $(\gamma)^* = D_\gamma(I \times S^{n-2})$  is called a *spherical dual hypersurface* of  $\gamma$ . In this section we clarify the reason why  $(\gamma)^*$  is called the spherical dual of  $\gamma$ . For the purpose we now briefly review some properties of contact manifolds and Legendrian submanifolds. Let  $W$  be a  $2n + 1$ -dimensional smooth manifold and  $K$  be a tangent hyperplane field on  $W$ . Locally such a field is defined as the field of zeros of a 1-form  $\alpha$ . If tangent hyperplane field  $K$  is non-degenerate, we say that  $(W, K)$  is a *contact manifold*. Here  $K$  is said to be *non-degenerate* if  $\alpha \wedge (d\alpha)^n \neq 0$  at any point of  $W$ . In this case  $K$  is called a *contact structure* and  $\alpha$  is a *contact form*. Let  $\phi : W \longrightarrow W'$  be a diffeomorphism between contact manifolds  $(W, K)$  and  $(W', K')$ . We say that  $\phi$  is a *contact diffeomorphism* if  $d\phi(K) = K'$ . Two contact manifolds  $(W, K)$  and  $(W', K')$  are *contact diffeomorphic* if there exists a contact diffeomorphism  $\phi : W \rightarrow W'$ . A submanifold  $i : L \subset W$  of a contact manifold  $(W, K)$  is a *Legendrian submanifold* if  $\dim L = n$  and  $di_p(T_p L) \subset K_{i(p)}$  at any point  $p \in L$ . We consider a smooth fiber bundle  $\pi : N \rightarrow A$ . The fiber bundle  $\pi : N \rightarrow A$  is called a *Legendrian fibration* if its total space  $W$  is furnished with a contact structure and its fibers are Legendrian submanifolds. Let  $\pi : N \rightarrow A$  be a Legendrian fibration. For a Legendrian submanifold  $i : L \subset N$ , a map  $\pi \circ i : L \rightarrow A$  is called a *Legendrian map*. The image of the Legendrian map  $\pi \circ i$  is called a *wave front* of  $i$  which is denoted by  $W(i)$ . For any  $p \in W$ , it is known that there is a local coordinate system  $(x_1, \dots, x_n, p_1, \dots, p_n, z)$  around  $p$  such that  $\pi(x_1, \dots, x_n, p_1, \dots, p_n, z) = (x_1, \dots, x_n, z)$  and the contact structure is given by the 1-form  $\alpha = dz - \sum_{i=1}^n p_i dx_i$  (cf [1], Part III).

We now consider the following double fibrations of  $S^n$ :

$$\begin{aligned} \Delta &= \{(\mathbf{v}, \mathbf{w}) \in S^n \times S^n \mid \mathbf{v} \cdot \mathbf{w} = 0\}, \\ \pi_1 : \Delta \ni (\mathbf{v}, \mathbf{w}) &\longmapsto \mathbf{v} \in S^n, \quad \pi_2 : \Delta \ni (\mathbf{v}, \mathbf{w}) \longmapsto \mathbf{w} \in S^n, \\ \theta_1 &= d\mathbf{v} \cdot \mathbf{w}|_\Delta, \quad \theta_2 = \mathbf{v} \cdot d\mathbf{w}|_\Delta. \end{aligned}$$

Here,  $d\mathbf{v} \cdot \mathbf{w} = \sum_{i=0}^n w_i dv_i$  and  $\mathbf{v} \cdot d\mathbf{w} = \sum_{i=0}^n v_i dw_i$ . Since  $d(\mathbf{v} \cdot \mathbf{w}) = d\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot d\mathbf{w}$  and  $\mathbf{v} \cdot \mathbf{w} = 0$  on  $\Delta$ ,  $\theta_1^{-1}(0)$  and  $\theta_2^{-1}(0)$  define the same tangent hyperplane field over  $\Delta$  which is denoted by  $K$ . Since  $\Delta$  can be identified with the unit tangent bundle of  $S^n$ , we have the following result (cf. [3]).

**Theorem 4.1.** *Under the above notation,  $(\Delta, K)$  is a contact manifold and both of  $\pi_i$  are Legendrian fibrations.*

By definition,  $\Delta$  is a smooth submanifold in  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$  and each  $\pi_i$  ( $i = 1, 2$ ) is a smooth fibration. Moreover, by the definition of the contact forms  $\theta_1, \theta_2$ , all fibers of  $\pi_1$  and  $\pi_2$  are Legendrian submanifolds in  $(\Delta, K)$ . If we have a Legendrian submanifold  $L \subset \Delta$ , then we say that  $\pi_1(L) \subset S^n$  and  $\pi_2(L) \subset S^n$  are *spherical Legendrian dual* to each other. Since both of  $\mathbf{v} \times S(\mathbf{v}, 0) = \pi_1^{-1}(\mathbf{v})$  and  $S(\mathbf{v}, 0) \times \mathbf{v} = \pi_2^{-1}(\mathbf{v})$  are Legendrian submanifolds of  $\Delta$ , a point  $\mathbf{v} \in S^n$  and the great hypersphere  $S(\mathbf{v}, 0) \subset S^n$  are spherical Legendrian dual to each other. This means that this duality is analogous to the classical duality in the projective geometry. We have the following theorem.

**Theorem 4.2.** *Let  $\gamma : I \longrightarrow S^n$  be a generalized Sabban curve. Then  $\gamma(I)$  and  $(\gamma)^*$  are spherical Legendrian dual to each other.*

*Proof.* We define a mapping  $\mathcal{L} : I \times S^{n-2} \rightarrow \Delta$  by  $\mathcal{L}(s, \xi) = (\gamma(s), D_\gamma(s, \xi))$ . By definition,  $\mathcal{L}$  is a well defined mapping. Moreover, we have

$$\frac{\partial \mathcal{L}}{\partial s} = \left( \mathbf{t}, \frac{\partial D_\gamma}{\partial s} \right), \quad \frac{\partial \mathcal{L}}{\partial \xi_i} = (\mathbf{0}, \mathbf{n}_i(s)).$$

Therefore,  $\{\partial \mathcal{L}/\partial s, \partial \mathcal{L}/\partial \xi_1, \dots, \partial \mathcal{L}/\partial \xi_{n-1}\}$  are linearly independent. This means that  $\mathcal{L} : I \times \mathbb{R}^{n-1} \rightarrow \Delta$  is immersive. Then the restriction of the above mapping to  $I \times S^{n-2}$  is also immersive. Moreover, we have  $\mathcal{L}^* \theta_1 = \mathbf{t}(s) ds \cdot D_\gamma(s, \xi) = 0$ , so that  $\mathcal{L}(I \times S^{n-2})$  is a Legendrian submanifold of  $\Delta$ . This completes the proof.  $\square$

Then we have the following corollary:

**Corollary 4.3.** *For any generalized Sabban curve  $\gamma : I \rightarrow S^n$ ,  $(\gamma)^*$  is a wave front of  $\mathcal{L}(I \times S^{n-2})$  with respect to the Legendrian fibration  $\pi_2$ .*

## 5 Unfoldings of function-germs

We use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [4]. Let  $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$  be a function germ. We call  $F$  an  $r$ -parameter unfolding of  $f$ , where  $f(s) = F_{x_0}(s, x_0)$ . We say that  $f$  has an  $A_k$ -singularity at  $s_0$  if  $f^{(p)}(s_0) = 0$  for all  $1 \leq p \leq k$ , and  $f^{(k+1)}(s_0) \neq 0$ . Let  $F$  be an unfolding of  $f$  and  $f(s)$  has an  $A_k$ -singularity ( $k \geq 1$ ) at  $s_0$ . We write the  $(k-1)$ -jet of the partial derivative  $\frac{\partial F}{\partial x_i}$  at  $s_0$  by  $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j$  for  $i = 1, \dots, r$ . Then  $F$  is called an  $\mathcal{R}$ -versal unfolding if the  $k \times r$  matrix of coefficients  $(\alpha_{ji})_{j=0, \dots, k-1; i=1, \dots, r}$  has rank  $k$  ( $k \leq r$ ). We introduce an important set concerning the unfoldings relative to the above notions. A discriminant set of  $F$  is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r \mid \exists s \text{ such that } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

We also define

$$\mathcal{D}_F^i = \{x \in \mathbb{R}^r \mid \exists s \text{ such that } F = \frac{\partial F}{\partial s} = \dots = \frac{\partial^i F}{\partial s^i} = 0 \text{ at } (s, x)\}$$

which is called an  $i$ th-order discriminant set of  $F$ . By definition, the first-order discriminant set is the discriminant set. Then we are interested in classification of  $\mathcal{D}_F^i$  by diffeomorphisms. We say that two function germs  $f, g : (\mathbb{R}, 0) \rightarrow \mathbb{R}$  are  $\mathcal{R}$ -equivalent if there exists a diffeomorphism germ  $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  such that  $f \circ \phi(s) = g(s) + (f(0) - g(0))$  for any  $s \in (\mathbb{R}, 0)$ . If  $f : (\mathbb{R}, 0) \rightarrow \mathbb{R}$  has an  $A_k$ -singularity at 0, then  $f$  is  $\mathcal{R}$ -equivalent to  $g(s) = \pm s^{k+1}$  (cf. [4, Theorem 3.3]). We can also easily show the following proposition (cf. [4, 6.6]).

**Proposition 5.1.** *The unfolding  $G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$  given by*

$$G(s, x) = \pm s^{k+1} + x_1 + x_2 s + \dots + x_k s^{k-1}$$

*is an  $\mathcal{R}$ -versal unfolding of  $g(s) = \pm s^{k+1}$  at 0.*

Let  $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}, 0)$  be unfoldings of  $f, g : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ , respectively. We say that  $F, G$  are  $P$ - $\mathcal{R}$ -equivalent if there exists a diffeomorphism germ  $\Phi : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R} \times \mathbb{R}^r, (0, 0))$  of the form  $\Phi(s, x) = (\phi_1(s, x), \phi_2(x))$  such that  $F \circ \Phi = G$ . By definition, if  $F, G$  are  $P$ - $\mathcal{R}$ -equivalent, then  $f, g$  are  $\mathcal{R}$ -equivalent. By the uniqueness of  $\mathcal{R}$ -versal unfolding (cf. [5]), we have the following theorem.

**Theorem 5.2.** *Let  $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \longrightarrow (\mathbb{R}, 0)$  be unfoldings of  $f, g : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ , respectively. Suppose that  $F, G$  are  $\mathcal{R}$ -versal unfoldings of  $f, g$  respectively. If  $f, g$  are  $\mathcal{R}$ -equivalent, then  $F, G$  are  $P\text{-}\mathcal{R}$ -equivalent.*

By straightforward calculations and an induction, we can show the following proposition.

**Proposition 5.3.** *Let  $F, G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \longrightarrow (\mathbb{R}, 0)$  be unfoldings of  $f, g : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ , respectively. If  $F, G$  are  $P\text{-}\mathcal{R}$ -equivalent, then there exists a diffeomorphism germ  $\phi : (\mathbb{R}^r, 0) \longrightarrow (\mathbb{R}^r, 0)$  such that  $\phi(\mathcal{D}_G^i) = \mathcal{D}_F^i$  as set germs for any  $i$ .*

For the unfolding  $G : (\mathbb{R} \times \mathbb{R}^r, (0, 0)) \longrightarrow (\mathbb{R}, 0)$  given by

$$G(s, x) = \pm s^{k+1} + x_1 + x_2 s + \cdots + x_k s^{k-1},$$

we can show that  $x = (x_1, \dots, x_k) \in \mathcal{D}_G^i$  ( $1 \leq i \leq k-1$ ) if and only if

$$(***) \quad \begin{cases} x_1 = \mp s^{k+1} - s x_2 - \cdots - s^{k-1} x_k, \\ x_2 = \mp (k+1) s^k - 2 s x_3 - \cdots - (k-1) s^{k-2} x_k, \\ x_3 = \mp \frac{(k+1)k}{2} s^{k-1} - 3 s x_4 - \cdots - \frac{(k-1)(k-2)}{2} s^{k-3} x_k, \\ \vdots \\ x_{i+1} = \mp \frac{(k+1)k \cdots (k-i+3)}{i!} s^{k+1-i} - (i+1) s x_{i+2} - \cdots - \frac{(k-1)(k-2) \cdots (k-i+1)}{(i-1)!} s^{k-i} x_k. \end{cases}$$

We now consider a map-germ  $DA_k^\pm : (\mathbb{R}^{k-1}, 0) \longrightarrow (\mathbb{R}^k, 0)$  defined by

$$DA_k^\pm(u_1, \dots, u_{k-1}) = (\pm u_1^{k+1} + \sum_{i=2}^{k-1} (i-1) u_1^i u_i, \mp (k+1) u_1^k - \sum_{i=2}^{k-1} i u_1^{i-1} u_i, u_2, \dots, u_{k-1}).$$

We remark that  $(\text{Im } DA_2^\pm, 0)$  is diffeomorphic to the cusp  $C = \{(t^2, t^3) \mid t \in (\mathbb{R}, 0)\}$  and  $(\text{Im } DA_3^\pm, 0)$  is diffeomorphic to the swallowtail  $SW = \{(3u^4 + u^2 v, 4u^3 + 2uv, v) \mid (u, v) \in (\mathbb{R}^2, 0)\}$  as set-germs. By Theorem 5.2 and Proposition 5.3, we have the following classification.

**Theorem 5.4.** *Let  $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$  be an  $r$ -parameter unfolding of  $f(s)$  which has an  $A_k$  singularity ( $k \leq r$ ) at  $s_0$ . Suppose that  $F$  is an  $\mathcal{R}$ -versal unfolding. Then  $(\mathcal{D}_F, (s_0, x_0))$  is diffeomorphic to  $(\text{Im } DA_k^\pm \times \mathbb{R}^{r-k}, 0)$  as set-germs. Moreover,  $(\mathcal{D}_F^{k-1}, (s_0, x_0))$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, k, k+1] \times \mathbb{R}^{r-k}, 0)$  as set-germs, where  $\sigma[2, 3, \dots, k, k+1] : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^k, 0)$  is a curve defined by*

$$\sigma[2, 3, \dots, k, k+1](t) = (t^2, t^3, \dots, t^k, t^{k+1}).$$

*Proof.* By the system of equations  $(***)$  for  $i = 1$ , we have

$$\begin{cases} x_1 = \mp s^{k+1} - s x_2 - \cdots - s^{k-1} x_k, \\ x_2 = \mp (k+1) s^k - 2 s x_3 - \cdots - (k-1) s^{k-2} x_k. \end{cases}$$

Then we have

$$\begin{cases} x_1 = \pm s^{k+1} + s^2 x_3 + \cdots + (k-2) s^{k-1} x_k, \\ x_2 = \mp (k+1) s^k - 2 s x_3 - \cdots - (k-1) s^{k-2} x_k. \end{cases}$$

If we put  $s = u_1, x_3 = u_2, \dots, x_k = u_{k-1}$ , the above system of equations means that  $(\mathcal{D}_G, 0) = (\text{Im } DA_k^\pm \times \mathbb{R}^{r-k}, 0)$ . Since  $f(s)$  has an  $A_k$  singularity at  $s = s_0$ ,  $f$  is  $\mathcal{R}$ -equivalent to  $\pm t^{k+1}$ . By

Theorem 5.2,  $F$  and  $G$  are  $\mathcal{R}$ -equivalent, so that  $(\mathcal{D}_F, (s_0, x_0))$  and  $(\mathcal{D}_G, 0)$  are diffeomorphic as set-germs.

On the other hand, if we continue the above calculation until  $i = k - 1$ , we can show that

$$x_1 = \lambda_1(k)s^{k+1}, x_2 = \lambda_2(k)s^k, x_3 = \lambda_3(k)s^{k-1}, \dots, x_k = \lambda_k(k)s^2,$$

for some  $\lambda_i(k) \in \mathbb{Q} \setminus \{0\}$ . By an affine coordinate change on  $\mathbb{R}^r$ , we have

$$x_1 = s^2, x_2 = s^3, \dots, x_{k-1} = s^k, x_k = s^{k+1}.$$

This means that  $(\mathcal{D}_G^{k-1}, 0)$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, k, k+1] \times \mathbb{R}^{r-k}, 0)$  as set-germs. By Theorem 5.2 and Proposition 5.3, we have the assertion. This completes the proof.  $\square$

We remark that we can calculate  $\mathcal{D}_G^i$  for  $1 < i < k - 1$ . It is, however, rather complicated, so that we omit the further arguments here.

We now consider spherical height functions on generalized Sabban curve. Let  $\gamma : I \rightarrow S^n$  be a generalized Sabban curve and  $H : I \times S^n \rightarrow \mathbb{R}$  the spherical height function on  $\gamma$ . For  $(s_0, \mathbf{v}_0) \in I \times S^n$ , we consider the function germ  $H : (I \times S^n, (s_0, \mathbf{v}_0)) \rightarrow \mathbb{R}$  and  $H$  can be considered as an  $n$ -parameter unfolding of  $h_{\mathbf{v}_0}$ .

**Proposition 5.5.** *For  $\mathbf{v}_0 \in S^n$ , suppose that  $h_{\mathbf{v}_0}$  has  $A_k$ -singularity at  $s_0$  for any  $k \leq n$ . Then  $H$  is an  $\mathcal{R}$ -versal unfolding of  $h_{\mathbf{v}_0}$ .*

*Proof.* We now consider an orthonormal basis  $\{\gamma(s_0), \mathbf{t}(s_0), \mathbf{n}_1(s_0), \dots, \mathbf{n}_{n-1}(s_0)\}$  of  $\mathbb{R}^{n+1}$ . We have the local representation of  $\gamma : I \rightarrow S^n$  around  $s_0$  by

$$\gamma(s) = x_1(s)\gamma(s_0) + x_2(s)\mathbf{t}(s_0) + x_3(s)\mathbf{n}_1(s_0) + \dots + x_{n+1}(s)\mathbf{n}_{n-1}(s_0).$$

We also write that  $\mathbf{v}_0 = \lambda\gamma(s_0) + \mu\mathbf{t}(s_0) + \sum_{i=1}^{n-1} \xi_i \mathbf{n}_i(s_0)$  with  $\lambda^2 + \mu^2 + \sum_{i=1}^{n-1} \xi_i^2 = 1$ . Then we have

$$H(s, \mathbf{v}_0) = \lambda x_1(s) + \mu x_2(s) + \xi_1 x_3(s) + \dots + \xi_{n-1} x_{n+1}(s).$$

Suppose that  $\xi_{n-1} > 0$  and  $\xi_{n-1} = \sqrt{1 - \lambda^2 - \mu^2 - \sum_{i=1}^{n-2} \xi_i^2}$ . Then we have

$$\begin{aligned} \frac{\partial H}{\partial \lambda}(s, \mathbf{v}_0) &= x_1(s) - \frac{\lambda}{\xi_{n-1}} x_{n+1}(s), \\ \frac{\partial H}{\partial \mu}(s, \mathbf{v}_0) &= x_2(s) - \frac{\mu}{\xi_{n-1}} x_{n+1}(s), \\ \frac{\partial H}{\partial \xi_1}(s, \mathbf{v}_0) &= x_3(s) - \frac{\xi_1}{\xi_{n-1}} x_{n+1}(s), \\ &\vdots \\ \frac{\partial H}{\partial \xi_{n-2}}(s, \mathbf{v}_0) &= x_n(s) - \frac{\xi_{n-2}}{\xi_{n-1}} x_{n+1}(s). \end{aligned}$$

We now consider the following matrix:

$$A = \begin{pmatrix} x_1(s_0) - \frac{\lambda}{\xi_{n-1}} x_{n+1}(s_0) & x_2(s_0) - \frac{\mu}{\xi_{n-1}} x_{n+1}(s_0) & \cdots & x_n(s_0) - \frac{\xi_{n-2}}{\xi_{n-1}} x_{n+1}(s_0) \\ x'_1(s_0) - \frac{\lambda}{\xi_{n-1}} x'_{n+1}(s_0) & x'_2(s_0) - \frac{\mu}{\xi_{n-1}} x'_{n+1}(s_0) & \cdots & x'_n(s_0) - \frac{\xi_{n-2}}{\xi_{n-1}} x'_{n+1}(s_0) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1}(s_0) - \frac{\lambda}{\xi_{n-1}} x_{n+1}^{n-1}(s_0) & x_2^{n-1}(s_0) - \frac{\mu}{\xi_{n-1}} x_{n+1}^{n-1}(s_0) & \cdots & x_n^{n-1}(s_0) - \frac{\xi_{n-2}}{\xi_{n-1}} x_{n+1}^{n-1}(s_0) \end{pmatrix}.$$

If we put  $\mathbf{a}_i = {}^t(x_i(s_0), x'_i(s_0), \dots, x_i^{n-1}(s_0))$ , then we have

$$A = \left( \mathbf{a}_1 - \frac{\lambda}{\xi_{n-1}} \mathbf{a}_{n+1}, \mathbf{a}_2 - \frac{\mu}{\xi_{n-1}} \mathbf{a}_{n+1}, \mathbf{a}_3 - \frac{\xi_1}{\xi_{n-1}} \mathbf{a}_{n+1}, \dots, \mathbf{a}_n - \frac{\xi_{n-2}}{\xi_{n-1}} \mathbf{a}_{n+1} \right).$$

It follows that

$$\begin{aligned} \det A &= \det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) + \frac{\lambda}{\xi_{n-1}} \det(\mathbf{a}_{n+1}, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n) \\ &+ \frac{\mu}{\xi_{n-1}} \det(\mathbf{a}_1, \mathbf{a}_{n+1}, \mathbf{a}_3, \dots, \mathbf{a}_n) + \dots + \frac{\xi_{n-2}}{\xi_{n-1}} \det(\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{a}_{n+1}) \\ &= \frac{(-1)^n}{\xi_{n-1}} \mathbf{v}_0 \cdot (\gamma(s_0) \times \gamma'(s_0) \times \dots \times \gamma^{(n-1)}(s_0)). \end{aligned}$$

By a straightforward calculation, we have

$$\gamma(s_0) \times \gamma'(s_0) \times \dots \times \gamma^{(n-1)}(s_0) = \kappa_1^{n-3}(s_0) \kappa_2^{n-2}(s_0) \dots \kappa_{n-2}(s_0) \mathbf{n}_{n-1}(s_0),$$

so that

$$\det A = (-1)^n \kappa_1^{n-3}(s_0) \kappa_2^{n-2}(s_0) \dots \kappa_{n-2}(s_0) \neq 0.$$

This means that  $\text{rank } A = n$ . Thus the assertion for  $k = n$  holds. It follows that the assertion for  $k < n$  also holds.  $\square$

By Proposition 3.1, (1) and (2), we have  $\mathcal{D}_H = (\gamma)^*$  and  $\mathcal{D}_H^{n-1} = \{\pm \mathbf{n}_{n-1}(s) \mid s \in I\}$ . Moreover, Proposition 3.1, (2) asserts that  $h_{\mathbf{v}_0}$  has an  $A_k$  singularity at  $s = s_0$  for  $k < n$  if and only if there exist  $(0, \dots, 0, \xi_k^0, \xi_{k+1}^0, \dots, \xi_{n-1}^0) \in S^{n-2}$  such that  $\xi_k^0 \neq 0$  and

$$\mathbf{v}_0 = \xi_k^0 \mathbf{n}_k(s_0) + \dots + \xi_{n-1}^0 \mathbf{n}_{n-1}(s_0).$$

We now define a set  $(\gamma)_k^* \subset S^n$  by

$$(\gamma)_k^* = \{\xi_k \mathbf{n}_k(s) + \dots + \xi_{n-1} \mathbf{n}_{n-1}(s) \mid \xi_k^2 + \dots + \xi_{n-1}^2 = 1, \text{ and } s \in I\},$$

so that  $(\gamma)_k^* = \mathcal{D}_H^k$ . By Theorem 5.4 and Proposition 5.5, we have the following theorem.

**Theorem 5.6.** *Let  $\gamma : I \rightarrow S^n$  be a generalized Sabban curve. Then we have the following:*

- (1) *for  $k < n - 1$ , the germ of the spherical dual  $(\gamma)^*$  of  $\gamma$  at  $(s_0, \xi^0) = (s_0, (\xi_1^0, \dots, \xi_{n-1}^0)) \in I \times S^{n-2}$  is diffeomorphic to  $(\text{Im } DA_k^\pm \times \mathbb{R}^{n-k}, 0)$  as set-germs if  $\xi_1^0 = \dots = \xi_{k-1}^0 = 0$  and  $\xi_k^0 \neq 0$ . In this case the germ of  $(\gamma)_k^*$  at  $(s_0, \xi^0) \in I \times S^{n-2}$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, k, k+1] \times \mathbb{R}^{n-k}, 0)$  as set-germs,*
- (2) *the germ of the spherical dual  $(\gamma)^*$  of  $\gamma$  at  $(s_0, \xi^0) \in I \times S^{n-1}$  is diffeomorphic to  $(\text{Im } DA_{n-1}^\pm \times \mathbb{R}, 0)$  as set-germs if  $\xi^0 = (0, \dots, 0, \pm 1)$  and  $\kappa_{n-1}(s_0) \neq 0$ . In this case the germ of the image of  $\mathbf{n}_{n-1}(s)$  at  $\mathbf{n}_{n-1}(s_0)$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, n-1, n] \times \mathbb{R}, 0)$  as set-germs.*
- (3) *the germ of the spherical dual  $(\gamma)^*$  of  $\gamma$  at  $(s_0, \xi^0) \in I \times S^{n-1}$  is diffeomorphic to  $(\text{Im } DA_n^\pm, 0)$  as set-germs if  $\xi^0 = (0, \dots, 0, \pm 1)$ ,  $\kappa_{n-1}(s_0) = 0$  and  $\kappa'_{n-1}(s_0) \neq 0$ . In this case the germ of the image of  $\mathbf{n}_{n-1}(s)$  at  $\mathbf{n}_{n-1}(s_0)$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, n, n+1], 0)$  as set-germs.*

*Proof.* By Proposition 5.5, if  $h_{\mathbf{v}_0}$  has an  $A_k$  singularity at  $s = s_0$  ( $k \leq n$ ), then  $H$  is an  $\mathcal{R}$ -versal unfolding of  $h_{\mathbf{v}_0}$ . Since  $(\gamma)_k^* = \mathcal{D}_H^k$ , assertion (1) follows from Theorem 5.4. By Proposition 3.1 ((2), (3)) and Theorem 5.4, assertion (2) holds. Similar to the other cases, assertion (3) holds applying Proposition 3.1 ((3), (4)) and Theorem 5.4. This completes the proof.  $\square$

## 6 Contact with great hyperspheres

By Proposition 2.3, there exists  $\mathbf{n} \in S^n$  such that  $\gamma(I) \subset S^{n-1}(\mathbf{v}, 0)$  if and only if  $\kappa_{n-1} \equiv 0$ . In this case  $\mathbf{n}_{n-1}(s)$  is a constant vector and  $\mathbf{n}_{n-1}(s) \equiv \mathbf{v}$ . By definition,  $\mathbf{v} \in \mathcal{D}_H$  if and only if there exists  $s_0 \in I$  such that  $h_{\mathbf{v}}(s_0) = h'_{\mathbf{v}}(s_0) = 0$ . If we consider a function  $\mathfrak{h}_{\mathbf{v}} : S^n \rightarrow \mathbb{R}$  defined by  $\mathfrak{h}_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$ , then  $\mathfrak{h}_{\mathbf{v}}^{-1}(0) = S^{n-1}(\mathbf{v}, 0)$  and  $\mathfrak{h}_{\mathbf{v}} \circ \gamma(s) = h_{\mathbf{v}}(s)$ , so that  $h_{\mathbf{v}}(s_0) = h'_{\mathbf{v}}(s_0) = 0$  if and only if  $S^{n-1}(\mathbf{v}, 0)$  is tangent to  $\gamma$  at  $s = s_0$ . We call such a hypersphere a *tangent great hypersphere* of  $\gamma$  at  $s = s_0$ . We remark that there are infinitely many tangent great hyperspheres of  $\gamma$  at  $s = s_0$ . We say that a tangent great hypersphere  $S^{n-1}(\mathbf{v}, 0)$  of  $\gamma$  at  $s = s_0$  has *at least  $k + 1$ -point contact with  $\gamma$*  if  $h_{\mathbf{v}}(s_0) = h'_{\mathbf{v}}(s_0) = \dots = h_{\mathbf{v}}^{(k)}(s_0) = 0$ . We also say that  $S^{n-1}(\mathbf{v}, 0)$  of  $\gamma$  at  $s = s_0$  has  *$k + 1$ -point contact with  $\gamma$*  if it has at least  $k + 1$ -point contact but does not have at least  $k + 2$ -point contact with  $\gamma$  at  $s = s_0$ . By Proposition 3.1, we have the following proposition.

**Proposition 6.1.** *Let  $\gamma : I \rightarrow S^n$  be a generalized Sabban curve and  $\mathbf{v}_0 \in S^n$ . Then we have the following:*

- (1)  *$S(\mathbf{v}_0, 0)$  is a tangent hypersphere of  $\gamma$  at  $s = s_0$  if and only if there exists  $\xi^0 = (\xi_1^0, \dots, \xi_{n-1}^0) \in S^{n-1}$  such that  $\mathbf{v}_0 = \xi_1^0 \mathbf{n}_1(s_0) + \dots + \xi_{n-1}^0 \mathbf{n}_{n-1}(s_0)$ ,*
- (2) *for  $k < n$ ,  $S(\mathbf{v}_0, 0)$  has at least  $k + 1$ -point contact with  $\gamma$  at  $s = s_0$  if and only if there exist  $(0, \dots, 0, \xi_k^0, \xi_{k+1}^0, \dots, \xi_{n-1}^0) \in S^{n-2}$  such that  $\mathbf{v}_0 = \xi_k^0 \mathbf{n}_k(s_0) + \dots + \xi_{n-1}^0 \mathbf{n}_{n-1}(s_0)$ ,*
- (3)  *$S(\mathbf{v}_0, 0)$  has at least  $n$ -point contact with  $\gamma$  at  $s = s_0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$ ,*
- (4) *for  $k > n$ ,  $S(\mathbf{v}_0, 0)$  has at least  $k + 1$ -point contact with  $\gamma$  at  $s = s_0$  if and only if  $\mathbf{v}_0 = \pm \mathbf{n}_{n-1}(s_0)$  and  $\kappa_{n-1}(s_0) = \kappa'_{n-1}(s_0) = \dots = \kappa_{n-1}^{(k-n)}(s_0) = 0$*

As a consequence, for  $k < n$ ,  $(\gamma)_k^*$  is the locus of the polar vectors  $\mathbf{v} \in S^n$  such that  $S(\mathbf{v}, 0)$  has at least  $k + 1$ -point contact with  $\gamma$  at  $s$ . In particular,  $(\gamma)_n^* = \{\pm \mathbf{n}_{n-1}(s) \mid s \in I\}$  is the locus of the polar vectors  $\mathbf{v} \in S^n$  such that  $S(\mathbf{v}, 0)$  has at least  $n + 1$ -point contact with  $\gamma$  at  $s$ . We call  $S(\mathbf{v}, 0)$  an *osculating great hypersphere* of  $\gamma$  at  $s = s_0$  if  $\mathbf{v} = \pm \mathbf{n}_{n-1}(s_0)$ . As a corollary of Theorem 5.6 and Proposition 6.1, we have the following theorem.

**Theorem 6.2.** *Let  $\gamma : I \rightarrow S^n$  be a generalized Sabban curve and  $\mathbf{v}_0 = \xi_1^0 \mathbf{n}_1(s_0) + \dots + \xi_{n-1}^0 \mathbf{n}_{n-1}(s_0) \in S^n$  (i.e.  $S(\mathbf{v}_0, 0)$  is a tangent great hypersphere of  $\gamma$  at  $s = s_0$ ). Then we have the following:*

- (1) *for  $k < n - 1$ , the germ of the spherical dual  $(\gamma)^*$  of  $\gamma$  at  $(s_0, \xi^0) = (s_0, (\xi_1^0, \dots, \xi_{n-1}^0)) \in I \times S^{n-2}$  is diffeomorphic to  $(\text{Im } DA_k^\pm \times \mathbb{R}^{n-k}, 0)$  as set-germs if  $S(\mathbf{v}_0, 0)$  has at least  $k + 1$ -point contact with  $\gamma$  at  $s = s_0$ , in this case the germ of  $(\gamma)_k^*$  at  $(s_0, \xi^0) \in I \times S^{n-2}$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, k, k + 1] \times \mathbb{R}^{n-k}, 0)$  as set-germs,*
- (2) *the germ of the spherical dual  $(\gamma)^*$  of  $\gamma$  at  $(s_0, \xi^0) \in I \times S^{n-1}$  is diffeomorphic to  $(\text{Im } DA_{n-1}^\pm \times \mathbb{R}, 0)$  as set-germs if  $S(\mathbf{v}_0, 0)$  has  $n + 1$ -point contact with  $\gamma$  at  $s = s_0$ . In this case the germ of the image of  $\mathbf{n}_{n-1}(s)$  at  $\mathbf{n}_{n-1}(s_0)$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, n - 1, n] \times \mathbb{R}, 0)$  as set-germs.*
- (3) *the germ of the spherical dual  $(\gamma)^*$  of  $\gamma$  at  $(s_0, \xi^0) \in I \times S^{n-1}$  is diffeomorphic to  $(\text{Im } DA_n^\pm, 0)$  as set-germs if  $S(\mathbf{v}_0, 0)$  has  $n + 2$ -point contact with  $\gamma$  at  $s = s_0$ . In this case the germ of the image of  $\mathbf{n}_{n-1}(s)$  at  $\mathbf{n}_{n-1}(s_0)$  is diffeomorphic to  $(\text{Im } \sigma[2, 3, \dots, n, n + 1], 0)$  as set-germs.*

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