GLOBAL PROPERTIES OF DIRICHLET FORMS IN TERMS OF GREEN'S FORMULA

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Abstract. We study global properties of Dirichlet forms such as uniqueness of the Dirichlet extension, stochastic completeness and recurrence. We characterize these properties by means of vanishing of a boundary term in Green's formula for functions from suitable function spaces and suitable operators arising from extensions of the underlying form. We first present results in the framework of general Dirichlet forms on σ-finite measure spaces. For regular Dirichlet forms our results can be strengthened as all operators from the previous considerations turn out to be restrictions of a single operator. Finally, the results are applied to graphs, weighted manifolds, and metric graphs, where the operators under investigation can be determined rather explicitly, and certain volume growth criteria can be (re)derived.

Contents

Date: April 11, 2017.

Introduction

We study global properties of Dirichlet forms. In this introduction we survey the structure and the results of this paper without going into detail. The precise definitions and statements can then be found in the upcoming sections.

The global properties under investigation of a Dirichlet form on a σ-finite measure space (X, m) are the following:

- Uniqueness of Dirichlet extensions.
- Stochastic completeness.
- Recurrence.

Such properties have attracted an enormous amount of attention. In particular, substantial efforts have been devoted to establishing geometric criteria for these properties in the case of manifolds, graphs and metric measure spaces. This includes [1, 7, 8, 9, 13, 15, 22, 30, 32, 34, 38].

Here, our point of view is completely different. We do not focus on geometry. We rather show that these properties can be characterized in terms of different domains of Laplace type operators. More specifically, we present a rather complete study of these properties via vanishing of Green's formula

$$\int_X \Delta u \, dm = 0. \tag{GF}$$

Here the function u is in a suitable class of functions and Δ is a suitable extension of the operator arising from a Dirichlet form. In this way each of the three global properties is governed by its own specific version of the Laplace operator.

Our change in perspective does not only provide rather smooth abstract characterizations but also ample applications to geometric settings such as manifolds, graphs and quantum graphs.

Our study is inspired by very recent work of Grigor'yan/Masamune [16] on weighted manifolds. However, while their study is based on methods from geometry our approach is rather different. Indeed, we used techniques from functional analysis and the full advantage of the modern abstract Dirichlet form theory. As a consequence, we will obtain some new characterizations even for weighted manifolds. Our main strategy in dealing with the lack of geometric structure in the general case is to consider extensions of the Dirichlet form under investigation to suitable spaces and to associate generators, which are carefully chosen to optimize the results according to the problems.

We study these questions on three levels of abstraction while the statements become the more explicit the more specific we get. The most abstract level are general Dirichlet forms on

σ-finite measure spaces. Secondly, we focus on regular Dirichlet forms and, finally, we present the results in the concrete contexts of graphs, weighted manifolds, and metric graphs.

Let us go into a bit more detail. We start with the most abstract level and discuss the three properties listed above.

In Section 2 we determine whether a Dirichlet extension $Q^{\#}$ of a Dirichlet form Q is different from Q by existence of a positive subharmonic, but non-harmonic, L^1 function u. In this case u is such that (GF) fails and Δ can be understood as the "Gaffney Laplacian" with respect to Q and $Q^{\#}$ and it will be denoted by L'. The main result (Theorem 2.8) is the characterization of the uniqueness of the Dirichlet extensions under the presence of a maximum principle (see Definition 2.6). The maximum principle will be explored further in Section 2.2 and a criterion for the maximum principle is given (Theorem 2.13). The criterion is quite general, e.g., it can be applied to various extensions (Silverstein extensions) of regular Dirichlet forms.

Secondly, we consider stochastic completeness in Section 3. A Dirichlet form is called stochastically complete if $(L+1)^{-1}1 = 1$, see Definition 1.1 and the discussion below for background. We obtain characterizations for stochastic completeness in terms of (GF) with Δ being the L^2-generator L or the L^1-generator $L^{(1)}$ and we obtain another characterization by the dual of $L^{(1)}$ on L^∞ (Theorem 3.1). The statement for the L^2-generator is an extension of the corresponding result in [16] to general Dirichlet forms while the other characterizations are new even in the case of manifolds.

Finally, in Section 4, we study recurrence. A Dirichlet form is called recurrent if $\int_0^\infty e^{-tL} f\,dt$ is equal to 0 or ∞ almost everywhere for all non-negative L^1 functions f, see Definition 1.2. We characterize recurrence by (GF) with Δ being an extension of L related to the extended Dirichlet space which will be denoted by L_e.

We remark that there is a relationship among these global properties. In general, recurrence implies stochastic completeness. Furthermore, stochastic completeness implies $Q = Q^{\#}$ when both forms satisfy a maximum principle. These implications are well-known and easily follow from our considerations.

After having established the theory in the general setting we zoom in to the case of regular Dirichlet forms in Section 5. Under this situation, we are able to extend all the above generators to one operator denoted by $\mathcal{L}$, and apply it to improve the results obtained in the previous sections. This application is preceded by a version of Fatou's lemma for the reflected Dirichlet form which allows for the definition of the operator $\mathcal{L}$. Then, in Subsection 5.2, we give two independent criteria (Theorems 5.13 and 5.14) for the uniqueness of Dirichlet extensions. The first one is stated in terms of the existence of positive subharmonic functions, while the second one is phrased via the validity of a Green formula. The last two subsections, Subsections 5.3 and 5.4, are devoted to the study of stochastic completeness and recurrence, respectively, using $\mathcal{L}$.

In Sections 6, 7, and 8, we apply the abstract results obtained of the previous sections to more concrete Dirichlet forms.

Specifically, in Section 6, we study graphs which have the prominent feature that all of operators are restrictions of a formal operator $\widetilde{L}$. This will allow us to understand these problems in a unified way. We emphasize that we do not need to assume local finiteness in any of our results. Part of the application to graphs is based on [31].

In Section 7, we study a general Dirichlet form on a weighted manifold. The main result here is the determination of the reflected Dirichlet space which generalizes a recent result of [4] to the manifold setting. In particular, this result implies that in this situation all of the introduced operators are restrictions of some weighted version of the Laplace Beltrami operator.

In Section 8, a general Dirichlet form on a metric graph is studied. Part of this application is based on [17].

The main thrust of our investigations is conceptual in nature. However, our results can also be used to (re)derive certain volume growth criteria. We will discuss details in Section 9.

List of some relevant notation. As we have explained above, we will use various operators associated to the Dirichlet form and its extensions. Below, we will list them to serve as an index (all of them except $\mathcal{L}$ are defined for a general Dirichlet form):

- $L : \mathcal{D}(L) \to L^2(X,m)$ – the L^2-generator of the "minimal" Dirichlet form Q (Section 2).
- $L^{\#} : \mathcal{D}(L^{\#}) \to L^2(X,m)$ – the L^2-generator of the "maximal" Dirichlet form $Q^{\#}$ (Section 2).
- $L' : \mathcal{D}(L') \to L^2(X,m)$ – an extension of both L and $L^{\#}$ which will be used to characterize the agreement of L and $L^{\#}$ (see Proposition 2.1).
- $L^{(1)}$ – the $L^1(X,m)$-generator which will be used to characterize stochastic completeness (Section 3).
- $L_e : D(Q)_e \to L^2(X,m)$ – an extension of L which will be used to characterize recurrence (Section 4).
- $\mathcal{L} : D(\mathcal{L}) \to L^1_{\mathrm{loc}}(X,m)$ – an extension of all of the above operators, which will be defined for a regular Dirichlet form (Section 5).

Acknowledgments. M.K. and D.L. gratefully acknowledge partial support from German Research Foundation (DFG). J.M. gratefully acknowledges partial support from the Japan-Korea Basic Scientific Cooperation Program "Non-commutative Stochastic Analysis: New Prospects of Quantum White Noise and Quantum Walk" (2015-2016) and Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number 26400062 (2014-2016), "Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers". M.S. has been financially supported by the Graduiertenkolleg 1523/2 : Quantum and gravitational fields and by the European Science Foundation (ESF) within the project Random Geometry of Large Interacting Systems and Statistical Physics.

The authors gratefully acknowledge the constructive comments on the paper by the anonymous referee, which improved the quality of the paper significantly.

1. The set up

As mentioned in the introduction this paper is concerned with global properties of symmetric Dirichlet forms. In this section we fix some notation and briefly recall the relevant notions and objects. For further discussions and proofs we refer the reader to the textbooks [11] and [4].

Throughout, we let m be a σ-finite measure on a measurable space X. For any number $1 \le p \le \infty$, we denote by $L^p(X,m)$ the corresponding real valued Lebesgue space with norm $\|\cdot\|_p$. The scalar product on $L^2(X,m)$ is denoted by $\langle \cdot, \cdot \rangle$. The space of all measurable m-a.e. defined real valued functions is denoted by $L^0(X,m)$. For $f, g \in L^0(X,m)$ we will write $f \wedge g$ to denote the minimum of f and g and $f \vee g$ to denote the maximum of f and g.

A function $C : \mathbb{R} \to \mathbb{R}$ is a *normal contraction* if $C(0) = 0$ and $|C(x) - C(y)| \leq |x - y|$ holds for all $x, y \in \mathbb{R}$. A densely defined, non-negative, closed symmetric bilinear form

$$Q : D(Q) \times D(Q) \to \mathbb{R}$$

on $L^2(X, m)$ is called a *Dirichlet form* if Q satisfies the *contraction property*, that is, for each normal contraction C and each $u \in D(Q)$ we have $C \circ u \in D(Q)$ and the inequality

$$Q(C \circ u, C \circ u) \leq Q(u, u)$$

holds. We will write $Q(u) := Q(u, u)$ for $u \in D(Q)$ and set $Q(u) := \infty$ if $u \notin D(Q)$. The *form norm* is given by $\|u\|_Q := (Q(u) + \|u\|_2^2)^{1/2}$ and the corresponding *form inner product* will be denoted by $\langle \cdot, \cdot \rangle_Q$.

A Dirichlet form gives rise to a positive self-adjoint operator L on $L^2(X, m)$. This operator is uniquely determined by the equality

$$Q(u, v) = \langle Lu, v \rangle,$$

for each $u \in D(L)$ and each $v \in D(Q)$. Here $D(L)$ is the domain of L. Each such operator coming from a Dirichlet form yields a strongly continuous resolvent $(G_\alpha)_{\alpha > 0}$ and a strongly continuous semigroup $(T_t)_{t > 0}$ viz

$$G_\alpha := (L + \alpha)^{-1} \quad \text{and} \quad T_t := e^{-tL}.$$

Both this resolvent and this semigroup are *markovian*, that is for each $e \in L^2(X, m)$ with $0 \leq e \leq 1$ the inequalities

$$0 \leq \alpha G_\alpha e \leq 1 \text{ and } 0 \leq T_t e \leq 1$$

hold. Thus, the resolvent and the semigroup can be extended to all $L^p(X, m)$ spaces via monotone approximations with L^2-functions. The resulting operators are contractions on $L^p(X, m)$. If not stated otherwise we will abuse notation and write T_t and G_α for the extended semigroup and resolvent on $L^p(X, m)$. The corresponding generators on $L^p(X, m)$ will be denoted by $L^{(p)}$ with domain $D(L^{(p)})$.

Definition 1.1 (Stochastic completeness)**.** A Dirichlet form Q is called *stochastically complete* if the associated L^∞ semigroup satisfies

$$T_t 1 = 1, \text{ for each } t > 0.$$

Stochastic completeness is equivalent to the validity of the equality

$$\alpha G_\alpha 1 = 1, \text{ for one/each } \alpha > 0.$$

For $f \in L^1_+(X, m)$, we introduce the *Green operator* as

$$Gf := \lim_{n \to \infty} \int_0^n T_s f ds,$$

where the integral is understood in the Bochner sense. Note that by the positivity of T_s this limit exists as an m-a.e. defined function which might be infinite on a set of positive measure.

Definition 1.2 (Recurrence/transience)**.** A Dirichlet form is called *recurrent* if for any $f \in L^1_+(X, m)$ we have $Gf = 0$ m-a.e. or $Gf = \infty$ m-a.e. It is called *transient* if for each $f \in L^1_+(X, m)$ the inequality $Gf < \infty$ m-a.e. holds.

Note that an arbitrary Dirichlet form might not be recurrent or transient. Thus, let us recall that a measurable set A is called *Q-invariant* if for each $f \in D(Q)$ we have $1_A f \in D(Q)$ and the equality

$$Q(f) = Q(1_A f) + Q(1_{X \setminus A} f)$$

holds. Here 1_A is the indicator function of the set A. A Dirichlet form Q is called *irreducible* if each Q-invariant set A satisfies $m(A) = 0$ or $m(X \setminus A) = 0$. For irreducible Dirichlet forms a dichotomy holds. They are either recurrent or transient but not both. For recent results on irreducible Dirichlet forms we refer the reader to [28].

2. Extensions of Dirichlet forms and a maximum principle

In concrete applications of Dirichlet forms one is often given two forms viz one form corresponding to Dirichlet boundary conditions and the other corresponding to Neumann boundary conditions. Then, the form with Neumann boundary conditions is an extension of the form with Dirichlet boundary conditions. Under suitable geometric conditions these two forms will actually agree. In this section we provide an abstract study of such a situation. More precisely, we study a pair of Dirichlet forms with one form extending the other. We seek for conditions ensuring that the two forms (or, equivalently, their domains of definition) are actually equal.

2.1. **Hilbert space theory.** Throughout, we assume the following situation (S):

(S) Let (X, m) be a σ-finite measure space. Let Q with domain $\mathcal{D}$ and $Q^{\#}$ with domain $\mathcal{D}^{\#}$ be Dirichlet forms on (X, m) such that $\mathcal{D} \subseteq \mathcal{D}^{\#}$ and Q and $Q^{\#}$ agree on $\mathcal{D}$. The generators of Q and $Q^{\#}$ are denoted by L and $L^{\#}$ respectively.

Under the assumption (S), the inclusion

$$j : \mathcal{D} \subseteq L^2(X, m) \longrightarrow \mathcal{D}^{\#}, u \mapsto u,$$

gives rise (by taking adjoints w.r.t. $Q^{\#}$) to the operator L' with domain

$$D(L') = \{u \in \mathcal{D}^{\#} \,|\, \text{there exists } w \in L^2(X, m) \text{ s.t. } Q^{\#}(u, v) = \langle w, v \rangle \text{ for all } v \in \mathcal{D}\}$$

via

$$L'u = w.$$

The following is an immediate consequence of the definitions.

Proposition 2.1. *Assume (S). Then, L' is an extension of both L and $L^{\#}$, i.e., both the domain of L and of $L^{\#}$ are contained in the domain of L' and L' agrees with L and $L^{\#}$ respectively on their domains.*

Remark 2.2. In general the operator L' will not be injective. For example, if X is a compact manifold and Q is the form associated to the Neumann-Laplacian, then 1 will be an eigenfunction to the eigenvalues 0 of L' (as L' is an extension of the Neumann operator by the preceding proposition).

We define the space of 1-harmonic functions by

$$\mathcal{H} := \{u \in \mathcal{D}^{\#} \,|\, L'u = -u\}.$$

Recall that – as $Q^{\#}$ is a Dirichlet form – the space $\mathcal{D}^{\#}$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{Q^{\#}} := Q^{\#}(u, v) + \langle u, v \rangle.$$

Here is the main result on the structure of $Q^\#$ in terms of Q (see Chapter 3, Section 3.3 of [11] for related results).

Theorem 2.3 (Decomposition theorem)**.** *Assume (S). Then, both $\mathcal{H}$ and $\mathcal{D}$ are closed subspaces of the Hilbert space $(\mathcal{D}^\#, \langle\cdot,\cdot\rangle_{Q^\#})$. Moreover, they are orthogonal to each other and*

$$\mathcal{D}^\# = \mathcal{H} \oplus \mathcal{D}$$

holds.

Proof. By construction the space $\mathcal{D}$ is a closed subspace of $\mathcal{D}^\#$. The remaining statements follow if we show that $\mathcal{H}$ is just the orthogonal complement of $\mathcal{D}$ in $\mathcal{D}^\#$. Thus, let $u \in \mathcal{H}$ and $v \in \mathcal{D}$ be given. Then, invoking the definition of L', we obtain

$$\langle u, v\rangle_{Q^\#} = Q^\#(u,v) + \langle u, v\rangle = \langle (L'+1)u, v\rangle = 0.$$

Thus, $\mathcal{H}$ is orthogonal to $\mathcal{D}$. Conversely, assume that $u \in \mathcal{D}^\#$ is orthogonal to $\mathcal{D}$. Then, $0 = \langle u, v\rangle_{Q^\#} = Q^\#(u,v) + \langle u, v\rangle$ holds for all $v \in \mathcal{D}$. By definition of L' this gives $L'u = -u$ and, hence, $u \in \mathcal{H}$ follows. □

We note the following immediate corollary of the theorem.

Corollary 2.4. *Assume (S). Then, the following assertions are equivalent:*

(i) $\mathcal{D} = \mathcal{D}^\#$.

(ii) *There does not exist a nontrivial $u \in D(L')$ with $L'u = -u$.*

We are now heading towards studying a different aspect of validity of $Q^\# = Q$.

Lemma 2.5. *Assume (S). If there exists an $u \in D(L') \cap L^1(X,m)$ with $u \geq 0$ and $L'u \leq 0$ and $L'u \neq 0$, then $\mathcal{D} \neq \mathcal{D}^\#$.*

Proof. To simplify the argument we use the notation $a < b$ for real valued measurable functions on a, b on X to mean that $a(x) \leq b(x)$ for m almost every x and that a and b differ on a set of positive m measure.

Let $u \in D(L') \cap L^1(X,m)$ with $u \geq 0$ and $L'u \leq 0$ and $L'u \neq 0$ be given and assume $Q = Q^\#$. Then, by construction, L' agrees with L and, in particular, $Lu = L'u < 0$ holds. As $(L+1)^{-1}$ is positivity preserving, we then find

$$(**) \quad 0 > (L+1)^{-1}Lu = L(L+1)^{-1}u = u - (L+1)^{-1}u.$$

As u belongs to $L^1(X,m)$, so does $(L+1)^{-1}u$. As $(L+1)^{-1}$ is a contraction on $L^1(X,m)$ and $u \geq 0$, we infer

$$\int_X (L+1)^{-1}u\, dm \leq \int_X u\, dm.$$

Using this we obtain by integrating $(**)$

$$0 > \int_X (u - (L+1)^{-1}u)\, dm = \int_X u dm - \int_X (L+1)^{-1}u\, dm \geq \int_X u\, dm - \int_X u\, dm = 0.$$

This is a contradiction. □

To provide a converse of this lemma, we will need one further concept.

Definition 2.6. Assume (S). Then, the *maximum principle* (MP) is said to hold if

$$(L+1)^{-1}f \leq (L^\# + 1)^{-1}f$$

holds for all $0 \leq f \in L^2(X,m)$.

Lemma 2.7. *Assume (S) and (MP). If $\mathcal{D} \neq \mathcal{D}^{\#}$, then there exists a nontrivial $u \in L^1(X,m) \cap L^\infty(X,m) \cap \mathcal{H}$ with $u \geq 0$ and $L'u \in L^1(X,m)$. In particular, such a function u satisfies the inequality*

$$\int_X L'u \, dm \neq 0.$$

Proof. By $\mathcal{D} \neq \mathcal{D}^{\#}$ the operators $(L+1)^{-1}$ and $(L^{\#}+1)^{-1}$ are different. As $L^1(X,m) \cap L^\infty(X,m)$ is dense in $L^2(X,m)$, there must exist an $f \in L^1(X,m) \cap L^\infty(X,m)$ with $f \geq 0$ and

$$0 \neq (L^{\#}+1)^{-1} f - (L+1)^{-1} f =: u.$$

Then, $u \geq 0$ holds by (MP). As Q and $Q^{\#}$ are Dirichlet forms and f belongs to $L^1(X,m) \cap L^\infty(X,m)$, we infer that u belongs to $L^1(X,m) \cap L^\infty(X,m)$. Moreover, by Proposition 2.1 we have

$$(L'+1)u = f - f = 0$$

and, hence,

$$L'u = -u \in L^1(X,m).$$

As $u \geq 0$ with $u \neq 0$ holds, this shows the first statement. As for the last statement, we note that obviously

$$\int_X L'u \; dm = -\int_X u \; dm < 0$$

holds. This finishes the proof. □

Combining the previous two lemmas we immediately infer the following theorem.

Theorem 2.8 (Characterization of $Q = Q^{\#}$)**.** *Assume (S) and (MP). Then, the following assertions are equivalent:*

(i) $\mathcal{D} \neq \mathcal{D}^{\#}$.

(ii) *There exists a nontrivial $u \in L^1(X,m) \cap D(L')$ with $u \geq 0$ and $L'u \leq 0$ and $L'u \neq 0$.*

If the assertions hold, then in (ii) *the function u can be chosen in $L^1(X,m) \cap L^\infty(X,m)$. Such a function u satisfies the inequality*

$$\int_X L'u \, dm \neq 0.$$

Remark 2.9. Let X be a locally compact, σ-compact topological space. Let $Q^{\#}$ with domain $\mathcal{D}^{\#}$ be a Dirichlet form on X such that $\mathcal{D}^{\#} \cap C_c(X)$ is dense in $C_c(X)$ (with respect to the supremum norm). Define Q to be the closure of the restriction of $Q^{\#}$ to $\mathcal{D}^{\#} \cap C_c(X)$. Then, Q is a regular Dirichlet form and Q and $Q^{\#}$ form a pair of Dirichlet forms satisfying (S). In such a situation (MP) is often known to hold (e.g., for manifolds, metric graphs and graphs) and our main result can be applied. For a detailed discussion see Section 2.2 and Proposition 5.10.

2.2. **An approximation criterion for the maximum principle.** In this section we will prove the maximum principle (MP) (see Definition 2.6) for the situation that the resolvents can be approximated via restrictions to an exhausting sequence (G_n) in X. For this some preparations are needed. Let $(Q, D(Q))$ be a Dirichlet form. For any measurable $G \subseteq X$, we consider the form Q_G given by

$$D(Q_G) = \{u \in D(Q) \,|\, u \equiv 0 \; m\text{-a.e. on } X \setminus G\}, \quad Q_G(u) = Q(u).$$

Then, $(Q_G, D(Q_G))$ is a Dirichlet form in the wide sense on $L^2(X,m)$, i.e., $D(Q_G)$ is not necessarily dense in $L^2(X,m)$, see [11, Theorem 1.3.2] and the discussion preceding it. The associated resolvent on $L^2(X,m)$, denoted by $(L_G+\alpha)^{-1}$, may not be strongly continuous and, thus, may not give rise to a densely defined self-adjoint operator. However, below we work with the resolvent and the form only. The following proposition is taken from [33]. We include a proof for the convenience of the reader.

Proposition 2.10. *Assume the situation described above. For all nonnegative $f \in L^2(X,m)$ the following estimate holds*

$$(L_G+\alpha)^{-1}f \le (L+\alpha)^{-1}f.$$

Proof. To simplify notation we only consider the case $\alpha = 1$. The case of general $\alpha > 0$ can be treated similarly. Let P be the orthogonal projection of $D(Q)$ onto $D(Q_G)$ in $(D(Q), \langle\cdot,\cdot\rangle_Q)$ and let $0 \le f \in L^2(X,m)$. Then,

$$(L_G+1)^{-1}f = P(L+1)^{-1}f$$

since for $u \in D(Q_G)$ the equation

$$\langle (L+1)^{-1}f, u\rangle_Q = \langle f, u\rangle = \langle (L_G+1)^{-1}f, u\rangle_{Q_G}$$

holds. Resolvents of Dirichlet forms are positivity preserving. Therefore, the above implies that showing $Pu \le u$ for all positive $u \in D(Q)$ settles the claim. Given $u \ge 0$, the function $u \wedge Pu$ belongs to $D(Q_G)$. Thus, we conclude

$$\begin{aligned}\|u - u\wedge Pu\|_Q^2 &= \|(u-Pu)_+\|_Q^2 = Q((u-Pu)_+) + \|(u-Pu)_+\|^2\\ &\le Q(u-Pu) + \|u-Pu\|^2 = \|u-Pu\|_Q^2.\end{aligned}$$

Since Pu is the unique distance minimizing element in $D(Q_G)$, we obtain $Pu = u \wedge Pu$. This finishes the proof. □

With the proposition at hand, we can prove an approximation result for general Dirichlet forms.

Proposition 2.11. *Let $(Q, D(Q))$ be a Dirichlet form on (X,m). Suppose (G_n) is an increasing sequence of subsets of X and set*

$$\mathcal{C} = \bigcup_{n=1}^{\infty} D(Q_{G_n}).$$

Let $Q_{\mathcal{C}}$ be the restriction of Q to the closure of $\mathcal{C}$ in $(D(Q), \|\cdot\|_Q)$ and let $(L_{\mathcal{C}}+\alpha)^{-1}$ be the associated resolvent. Then, for any $f \in L^2(X,m)$ and $\alpha > 0$,

$$(L_{G_n}+\alpha)^{-1}f \to (L_{\mathcal{C}}+\alpha)^{-1}f \text{ as } n \to \infty.$$

For nonnegative f this convergence is monotone. In particular, if $\mathcal{C}$ is dense in $D(Q)$, then the resolvent of Q can be approximated as above.

Proof. After decomposing f into positive and negative part, we can restrict our attention to $f \ge 0$. As the norm of $(L_{G_n}+\alpha)^{-1}$ is uniformly bounded by $\frac{1}{\alpha}$ and $L^2 \cap L^\infty$ is dense in L^2, we may assume that f is bounded. We set $u_n := (L_{G_n}+\alpha)^{-1}f$ and notice $u_n \in \mathcal{C}$, $n \ge 1$.

We first show that u_n converges to a function $u \in D(Q_{\mathcal{C}})$. Proposition 2.10 implies that the sequence u_n is m-a.e. monotone increasing. Furthermore, standard Dirichlet form theory implies $0 \le u_n \le \frac{1}{\alpha}\|f\|_\infty$. This shows that u_n is almost surely convergent to a bounded function u. The construction of resolvents from forms yields $\|u_n\| \leq \frac{1}{\alpha}\|f\|$. Therefore, the convergence $u_n \to u$ also holds in $L^2(X,m)$. Let us compute for $n \geq m$

$$\begin{aligned}
&Q(u_n - u_m) + \alpha\|u_n - u_m\|^2 \\
&= Q(u_n) + \alpha\|u_n\|^2 + Q(u_m) + \alpha\|u_m\|^2 - 2(Q(u_n,u_m) + \alpha\langle u_n, u_m\rangle) \\
&= Q(u_n) + \alpha\|u_n\|^2 + Q(u_m) + \alpha\|u_m\|^2 - 2\langle f, u_m\rangle \\
&= Q(u_n) + \alpha\|u_n\|^2 - \langle f, u_m\rangle \\
&= \langle f, u_n - u_m\rangle \\
&\leq \|f\|^2\|u_n - u_m\|^2 \to 0 \text{ as } n,m \to \infty.
\end{aligned}$$

This shows that (u_n) is a Cauchy sequence in $D(Q_{\mathcal{C}})$. Since $D(Q_{\mathcal{C}})$ is complete, we conclude $u \in D(Q_{\mathcal{C}})$ and $u_n \to u$ in $D(Q_{\mathcal{C}})$. Next, we prove that u is a minimizer of

$$q : D(Q_{\mathcal{C}}) \longrightarrow [0,\infty), \quad v \mapsto Q(v) + \alpha\|v - \frac{1}{\alpha}f\|^2.$$

This yields the statement since such a minimizer is known to be unique and to agree with the resolvent by a variational characterization of resolvents: Let $w \in \mathcal{C}$ be arbitrary. Then, there exists an n_0 such that $w \in D(Q_{G_n})$ for all $n \geq n_0$. Since the u_n are also resolvents and, thus, minimizers of q on $D(Q_{G_n})$ we obtain

$$Q(u) + \alpha\|u - \frac{1}{\alpha}f\|^2 = \lim_{n\to\infty}\left\{Q(u_n) + \alpha\|u_n - \frac{1}{\alpha}f\|^2\right\} \leq Q(w) + \alpha\|w - \frac{1}{\alpha}f\|^2.$$

Since $w \in \mathcal{C}$ was arbitrary and $\mathcal{C}$ is dense in $D(Q_{\mathcal{C}})$, we obtain the statement. □

Remark 2.12. With the above theorem we established Mosco convergence of the forms Q_{G_n} to $Q_{\mathcal{C}}$ by simple monotonicity arguments. This idea may already be found in the proof of Proposition 2.7 in [24]. Instead we could have also used a more abstract characterization of this convergence. See the Appendix of [5].

Theorem 2.13 (Sufficient condition for maximum principle)**.** *Assume the forms $(Q, D(Q))$ and $(Q^{\#}, D(Q^{\#}))$ satisfy (S). Let (G_n) be an increasing sequence of subsets of X and let*

$$\mathcal{C} = \bigcup_{n=1}^{\infty} D(Q^{\#}_{G_n}).$$

Assume $\mathcal{C} \subseteq D(Q)$ and that the closure of $\mathcal{C}$ coincides with $D(Q)$. Then, Q and $Q^{\#}$ satisfy the maximum principle (MP).

Proof. Our assumptions and Proposition 2.11 imply that $(L^{\#}_{G_n} + \alpha)^{-1}$ converge strongly to $(L + \alpha)^{-1}$. Furthermore, Proposition 2.10 shows that for nonnegative $f \in L^2(X,m)$ this convergence is monotone and

$$(L^{\#}_{G_n} + \alpha)^{-1} f \leq (L^{\#} + \alpha)^{-1} f.$$

This proves the claim. □

3. Stochastic completeness

In this section we present a characterization of stochastic completeness (see Definition 1.2) in Theorem 3.1. We give a proof by three lemmas which immediately imply the theorem.

Theorem 3.1 (Characterization of stochastic completeness)**.** *Let Q be a Dirichlet form with associated self-adjoint operator L. Let $L^{(1)}$ with domain $D(L^{(1)})$ be the generator of the L^1-semigroup associated to Q. Let $L^{(\infty)}$ be the adjoint of $L^{(1)}$. Then, the following assertions are equivalent:*

(i) *Q is stochastically complete.*

(ii) *For all $u \in D(L^{(1)})$ the following equality holds*

$$\int_X L^{(1)}u \, dm = 0.$$

(iii) *The constant function 1 belongs to the domain of $L^{(\infty)}$ and $L^{(\infty)}1 = 0$ holds.*

(iv) *For all $u \in D(L) \cap L^1(X,m)$ such that $Lu \in L^1(X,m)$ the following equality holds*

$$\int_X Lu \, dm = 0.$$

The proof follows from the subsequent three lemmas.

Remark 3.2.
- In the context of stochastic completeness, (iv) is already discussed in the literature for weighted manifolds [16].
- It can be shown that the adjoint of $L^{(1)}$ is indeed the generator of the L^∞ resolvent which is associated with Q.

The following lemma is certainly well known. For an L^1-version see [2, Prop.2.4.2]. It may be useful in other contexts as well. We include a proof for the convenience of the reader.

Lemma 3.3. *Let Q be a Dirichlet form and let L be the associated self-adjoint operator. For $p \in [1,\infty)$, let $L^{(p)}$ with domain $D(L^{(p)})$ be the generator of the L^p-semigroup associated to Q. Then,*

$$S := \{f \in L^p(X,m) \cap D(L) \,|\, Lu \in L^p(X,m)\}$$

is contained in $D(L) \cap D(L^{(p)})$ and L agrees with $L^{(p)}$ on S and for any $f \in D(L^{(p)})$ there exists a sequence (f_n) in S with $f_n \to f$ in $L^p(X,m)$ and $Lf_n \to L^{(p)}f$ in $L^p(X,m)$.

Proof. We first show $S \subset D(L^{(p)})$ and $Lf = L^{(p)}f$ for $f \in S$, (which implies the inclusion $S \subset D(L) \cap D(L^{(p)})$): For $f \in S$ the function $g := (L+1)f$ belongs to $L^p(X,m) \cap L^2(X,m)$. Thus, we can apply both $(L+1)^{-1}$ and $(L^{(p)}+1)^{-1}$ to g and by consistency of the resolvents we obtain

$$f = (L+1)^{-1}g = (L^{(p)}+1)^{-1}g.$$

This gives $f \in D(L^{(p)})$ as well as

$$Lf = (L+1-1)(L+1)^{-1}g = g - (L+1)^{-1}g = g - (L^{(p)}+1)^{-1}g = L^{(p)}f.$$

We now show denseness: Let $f \in D(L^{(p)})$ be given. Then, $g := (L^{(p)}+1)f$ exists and $f = (L^{(p)}+1)^{-1}g$ holds. By σ-finiteness we can choose a sequence $g_n \in L^p(X,m) \cap L^2(X,m)$ with $g_n \to g$ in $L^p(X,m)$. By continuity of resolvents we then have

$$f_n := (L^{(p)}+1)^{-1}g_n \to (L^{(p)}+1)^{-1}g = f$$

in $L^p(X,m)$. By consistency of resolvents we furthermore obtain

$$f_n = (L+1)^{-1}g_n = (L^{(p)}+1)^{-1}g_n \in D(L) \cap D(L^{(p)})$$

and

$$Lf_n = g_n - f_n = L^{(p)}f_n.$$

Putting these statements together we infer

$$Lf_n = g_n - f_n \to g - f = L^{(p)}f$$

in $L^p(X,m)$. This finishes the proof. □

Lemma 3.4. *Let Q be a Dirichlet form with associated self-adjoint operator L. Then, the following assertions are equivalent:*

(i) *Q is stochastically complete.*
(ii) *For all $u \in D(L) \cap L^1(X,m)$ with $Lu \in L^1(X,m)$ the following equality holds*

$$\int_X Lu\, dm = 0.$$

Proof. (i) $\Longrightarrow$ (ii): We choose a sequence $(g_n) \subset L^2(X,m)$ such that $0 \le g_n \le g_{n+1} \le 1$ and $g_n \to 1$ m-almost everywhere. Furthermore, let $e_n = (L+1)^{-1}g_n$. By stochastic completeness (e_n) converges to 1 m-almost everywhere. Thus, for any $u \in D(Q) \cap L^1(X,m)$, we obtain by Lebesgue's theorem

$$\lim_{n\to\infty} Q(e_n, u) = \lim_{n\to\infty} \langle g_n - e_n, u\rangle = 0.$$

For u that satisfies additionally $u \in D(L)$ and $Lu \in L^1(X,m)$, we obtain by Lebesgue theorem

$$0 = \lim_{n\to\infty} Q(e_n, u) = \lim_{n\to\infty} \langle e_n, Lu\rangle = \lim_{n\to\infty} \int_X e_n Lu dm = \int_X Lu\, dm.$$

This finishes the proof of this implication.

(ii) $\Longrightarrow$ (i): By σ-finiteness of (X,m) and Proposition A.1 there exists a sequence (e_n) in $D(Q)$ with $0 \le e_n \le 1$ and $e_n \to 1$ m-almost surely. Now, choose an arbitrary $f \in L^1(X,m) \cap L^2(X,m)$ with $f > 0$. (Such a choice is possible by σ-finiteness). Set $v := (L+1)^{-1}f$. By construction, v belongs to the domain of L. Moreover, as $(L+1)^{-1}$ is a contraction on $L^p(X,m)$ for any $p \ge 1$ and f belongs to $L^1(X,m) \cap L^2(X,m)$, we infer $v \in L^1(X,m) \cap L^2(X,m)$. Furthermore, we obviously have

$$Lv = (L+1-1)v = f - v \in L^1(X,m).$$

Thus, by (ii) we then obtain

$$\int_X Lv\, dm = 0.$$

This gives

$$0 = \int_X Lv dm = \lim_{n\to\infty} \int_X e_n Lv dm = \lim_{n\to\infty} \langle e_n, f - (L+1)^{-1}f\rangle = \lim_{n\to\infty} \langle e_n - (L+1)^{-1}e_n, f\rangle.$$

Since f was chosen strictly positive, this yields $(L+1)^{-1}1 = 1$ which is equivalent to stochastic completeness. □

Remark 3.5. The above proof shows that stochastic completeness is equivalent to the existence of a sequence (e_n) in $D(Q)$ satisfying $0 \leq e_n \leq 1$, $e_n \to 1$ m-almost everywhere and

$$\lim_{n\to\infty} Q(e_n, u) = 0,$$

for any $u \in D(Q) \cap L^1(X, m)$. This part of the proof is taken from Theorem 1.6.6 of [11].

Lemma 3.6. *Let Q be a Dirichlet form. Let $L^{(1)}$ with domain $D(L^{(1)})$ be the generator of the L^1-semigroup associated to Q. Let $L^{(\infty)}$ be the adjoint of $L^{(1)}$. Then, the following assertions are equivalent:*

(i) *For all $u \in D(L^{(1)})$ the equality $\int_X L^{(1)} u \, dm = 0$ holds.*

(ii) *The constant function 1 belongs to the domain of $L^{(\infty)}$ and $L^{(\infty)} 1 = 0$ holds.*

Proof. This is immediate from the definitions. □

Proof of Theorem 1. The equivalence of (i) and (iv) is shown in Lemma 3.4 and (ii) $\Longleftrightarrow$ (iv) follows from Lemma 3.3. Finally, the equivalence of (ii) and (iii) is given by Lemma 3.6. □

Remark 3.7. We proved the equivalence of (i) and (ii) in Theorem 3.1 via (iv) and a denseness argument. Using semigroup theory one could also proceed as follows: Let $(T_t^{(1)})$ denote the L^1-semigroup associated with Q and let $(T_t^{(\infty)})$ be its adjoint. For $u \in D(L^{(1)})$ the function $T_t^{(1)} u$ is a solution to the heat equation on L^1. Therefore,

$$\begin{aligned} \int_X L^{(1)} u \, dm &= -\frac{d}{dt} \int_X T_t^{(1)} u \, dm \bigg|_{t=0} \\ &= -\frac{d}{dt} \int_X u T_t^{(\infty)} 1 \, dm \bigg|_{t=0}. \end{aligned}$$

If Q is stochastically complete, then the right hand side of the above equation vanishes and (ii) follows. If (ii) holds the above shows that $\int_X u T_t^{(\infty)} 1 dm$ is constant for all $u \in D(L^{(1)})$. This then easily implies $T_t^{(\infty)} 1 = 1$.

4. Recurrence

In this section we characterize recurrence of Dirichlet forms (see Definition 1.2). The crucial new ingredient in our considerations will be the operator L_e defined below.

To each Dirichlet form with domain $D(Q)$ we can associate the extended Dirichlet space $D(Q)_e$ which consists of all m-a.e. finite measurable functions f for which a Q-Cauchy sequence $(f_n) \subseteq D(Q)$ exists such that $f_n \to f$ m-a.e. Such a sequence is called *approximating sequence* for f. We can then extend Q to a quadratic form on $D(Q)_e$ by setting

$$Q(f) = \lim_{n\to\infty} Q(f_n).$$

For properties of this space and further details, we refer the reader to [4, Chapter 1]. Note that we denote the extended form by Q as well.

The inclusion

$$j : D(Q) \subseteq L^2(X, m) \to D(Q)_e, \quad u \mapsto u,$$

gives (via taking the adjoint) rise to the operator

$$L_e : D(Q)_e \to L^2(X, m)$$

with domain

$$D(L_e) := \{v \in D(Q)_e \mid \text{there exists } w \in L^2(X,m) \text{ s.t. } \langle w, u\rangle = Q(v,u) \text{ for all } u \in D(Q)\}$$

acting by

$$L_e v = w.$$

Remark 4.1. It is not hard to see that L_e is an extension of the operator L associated to Q in the sense that $D(L) \subseteq D(L_e)$ and $L_e f = Lf$ for $f \in D(L)$.

The proof of the next result strongly relies on results of [4].

Theorem 4.2 (Characterization recurrence)**.** *Let Q be an irreducible Dirichlet form. Then the following assertions are equivalent:*

(i) *Q is recurrent.*

(ii) *For all $u \in D(L_e)$ with $L_e u \in L^1(X,m)$ the following equality holds*

$$0 = \int_X L_e u \, dm.$$

(iii) *The constant function 1 belongs to the domain of L_e and $L_e 1 = 0$ holds.*

Proof. (i) $\Longrightarrow$ (ii): By assumption (i) and [4, Part (ii) of Theorem 2.1.8], there exists a sequence (e_n) in $D(Q)$ with $0 \leq e_n \leq 1$ and $e_n \to 1$ m-almost everywhere and

$$0 = \lim_{n\to\infty} Q(e_n, u)$$

for all $u \in D(Q)_e$. For u satisfying additionally $u \in D(L_e)$ and $L_e u \in L^1(X,m)$ we obtain by Lebesgue's theorem

$$0 = \lim_{n\to\infty} Q(e_n, u) = \lim_{n\to\infty} \langle e_n, L_e u\rangle = \lim_{n\to\infty} \int_X e_n L_e u \, dm = \int_X L_e u \, dm.$$

This finishes the proof of this implication.

(ii) $\Longrightarrow$ (i): Because of the irreducibility of Q, it suffices to show that transience implies the existence of a $u \in D(L_e)$ with $L_e u \in L^1(X,m)$ and

$$0 \neq \int_X L_e u \, dm.$$

By transience of Q the space $D(Q)_e$ with the inner product Q is a Hilbert space and there exists a strictly positive $g \in L^1(X,m)$ with

$$\int_X |v| g dm \leq Q(v)^{1/2} \qquad (*)$$

for all $v \in D(Q)_e$ (see [4, Theorem 2.1.5]). Without loss of generality we can assume $g \in L^2(X,m)$ as well. The functional

$$F_g : D(Q)_e \to \mathbb{R}, \quad v \mapsto \int_X gv \, dm,$$

is continuous by $(*)$. Thus, by Riesz representation theorem there exists $u \in D(Q)_e$ with

$$\langle g, v\rangle = \int gv \, dm = F_g(v) = Q(u,v)$$

for all $v \in D(Q)$. By definition of L_e this implies

$$L_e u = g.$$

As g is strictly positive, we obtain $\int_X L_e u\, dm > 0$. This is the desired statement.

(i) $\Longleftrightarrow$ (iii): By [4, Theorem 2.1.8] recurrence is equivalent to the function 1 belonging to $D(Q)_e$. In this case, one has $Q(1,u) = 0$ for all $u \in D(Q)$ (see [4] as well). This gives the desired equivalence. $\square$

Remark 4.3.
- The irreducibility of Q is needed in the previous theorem to ensure the dichotomy of recurrence and transience in our context (see [11, Lemma 1.6.4]).
- To put condition (iii) in perspective we define

$$(L_e)_1 : \{u \in D(L_e) \,|\, L_e u \in L^1(X,m)\} \to L^1(X,m), u \mapsto L_e u.$$

Then, (ii) is equivalent to $1 \in D((L_e)_1^*)$ and $(L_e)_1^* 1 = 0$. In this sense, (iii) can be understood as some form of "symmetry" of $(L_e)_1$.

5. Application to regular Dirichlet forms

In this section we apply the theory developed so far to a regular Dirichlet form. Using the Beurling-Deny decomposition each such form can be extended to the so called reflected Dirichlet space. Using this space we can then provide a unified treatment of all the operators and spaces which were used above. In fact, we will show that all the operators above are just restrictions of a single operator to suitable domains.

Let X be a locally compact separable metric space and m a Radon measure of full support. We consider a regular Dirichlet form $(Q, D(Q))$ on $L^2(X,m)$, where regular means that $C_c(X) \cap D(Q)$ is dense in $D(Q)$ with respect to $\|\cdot\|_Q$ and in $C_c(X)$ with respect to $\|\cdot\|_\infty$.

A function $f : X \to \mathbb{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there is an open set $U \subseteq X$ with capacity less than ε, i.e.,

$$\operatorname{cap}(U) := \inf\{\|v\|_Q \mid v \in D(Q),\, 1_U \le v\} \le \varepsilon,$$

such that $f|_{X\setminus U}$ is continuous (where $\inf \emptyset = \infty$ and 1_U is the characteristic function of U). For a regular Dirichlet form Q every $u \in D(Q)$ admits a quasi continuous representative, see [11, Theorem 2.1.3]. Moreover, we say a function satisfies a property *quasi everywhere*, q.e., if the property holds outside of a set $N \subseteq X$ of capacity zero, i.e., $\operatorname{cap}(N) = 0$, where $\operatorname{cap}(A) = \inf\{\operatorname{cap}(U) \mid A \subseteq U \text{ open}\}$ for $A \subseteq X$.

We can express the regular Dirichlet form Q using the *Beurling-Deny formula* [11, Theorem 3.2.1]. That is, for any function u belonging to $D(Q)$ the equation

$$Q(u,u) = \int_X d\mu^{(c)}(u) + \iint_{X\times X\setminus diag} (\tilde u(x) - \tilde u(y))^2\, J(dx,dy) + \int_X \tilde u^2 dk$$

holds. Here $\mu^{(c)}(u)$ is the *strongly local measure*, $J(dx,dy)$ is the *jump measure*, *diag* is the diagonal set of $X\times X$, and k is the *killing measure* associated with Q (see, e.g., [11, Chapter 3] for construction and properties of these objects). Furthermore, $\tilde u$ denotes a quasi-continuous representative of u. We denote by

$$D(Q)_{\mathrm{loc}} = \{u \in L^2_{\mathrm{loc}} \,|\, \forall G \subseteq X \text{ open, relatively compact } \exists v \in D(Q) \text{ with } u = v \text{ on } G\}$$

the space of functions locally belonging to the domain $D(Q)$. Note that also each $u \in D(Q)_{\mathrm{loc}}$ admits a quasi continuous representative $\tilde u$ (see, e.g., [10, Proposition 3.1]).

The Beurling-Deny representation allows one to extend the diagonal of Q to larger classes of functions. Since any function in $D(Q)_{\mathrm{loc}}$ has a quasi continuous representative and J and k

charge no set of capacity zero, we can extend the second and third summand of the Beurling-Deny formula to $D(Q)_{\mathrm{loc}}$ in an obvious way. Moreover, for $u \in D(Q)_{\mathrm{loc}}$ we introduce the Radon measure $d\mu^{(c)}(u)$ via the identity

$$\int_X \varphi \, d\mu^{(c)}(u) = \int_X \varphi \, d\mu^{(c)}(u_\varphi),$$

where $\varphi \in C_c(X)$ and $u_\varphi \in D(Q)$ are such that $u = u_\varphi$ on a neighborhood $\operatorname{supp} \varphi$. The local property of $\mu^{(c)}$ assures that this is well defined. Thus, the diagonal of the form Q can be extended to $D(Q)_{\mathrm{loc}}$. We will denote this extension by the same symbol Q (note that it may take the value ∞).

Let us mention some properties of this extension.

Proposition 5.1. *Let $u \in D(Q)_{\mathrm{loc}}$ be given.*

(a) *For any normal contraction C the function $C \circ u$ belongs to $D(Q)_{\mathrm{loc}}$ and $Q(C \circ u) \leq Q(u)$ holds.*
(b) *The sequence $(Q((u \wedge n) \vee (-n)))_n$ is monotone increasing with $Q(u) = \lim_{n\to\infty} Q((u \wedge n) \vee (-n))$.*
(c) *If u belongs to $L^\infty(X,m)$ and has compact support, then $u \in D(Q)$.*
(d) *If $v \in D(Q)_{\mathrm{loc}}$ the following inequalities hold*

$$Q(u \wedge v)^{1/2} \leq Q(u)^{1/2} + Q(v)^{1/2} \quad \text{and} \quad Q(u \vee v)^{1/2} \leq Q(u)^{1/2} + Q(v)^{1/2}.$$

Proof. Assertions (a) and (b) follow immediately from the definition of a regular Dirichlet form and its Beurling-Deny decomposition. Let us turn to statement (c). By the discussion in [10, Section 3.1, page 4772], the space $L^\infty(X,m) \cap D(Q)_{\mathrm{loc}}$ is included in the space referred to as $\mathcal{D}^*_{loc}$ in [10]. By [10, Theorem 3.5] the compactly supported functions in $\mathcal{D}^*_{loc}$ belong to $D(Q)$. The proof of the first inequality of (d) uses $u \wedge v = \frac{1}{2}(u + v + |u - v|)$, the triangle inequality for Q and the contraction property (a). More precisely, we estimate

$$Q(u \wedge v)^{1/2} = \frac{1}{2} Q(u + v + |u - v|)^{1/2} \leq \frac{1}{2}(Q(u+v)^{1/2} + Q(|u - v|)^{1/2}) \leq Q(u)^{1/2} + Q(v)^{1/2}.$$

The other inequality can be treated similarly. This finishes the proof. □

5.1. **Functions of finite energy.** In this section we introduce another space of importance. When equipped with the extension of the underlying Dirichlet form this space is referred to as the *reflected Dirichlet space*.

Recall that $L^0(X,m)$ denotes the space of m-a.e. defined functions. For $n \geq 1$ and $u \in L^0(X,m)$, we write $u^{(n)} = (u \wedge n) \vee (-n)$. Similarly, for nonnegative $f \in L^0(X,m)$, we let $u^f = (u \wedge f) \vee (-f)$. We extend Q to

$$D(Q)^\infty_{\mathrm{loc}} := \{u \in L^0(X,m) \mid u^{(n)} \in D(Q)_{\mathrm{loc}} \text{ for all } n \geq 1\},$$

by setting

$$\widetilde{Q}(u) := \lim_{n\to\infty} Q(u^{(n)}).$$

Here, the preceding limit exists as $(Q(u^{(n)}))_n$ is monotone. Indeed, this monotonicity can be directly inferred from Proposition 5.1 (b) as $(u^{(n)})^{(k)} = u^{(k)}$ for all $k \leq n$. Whenever $u \in L^0(X,m) \setminus D(Q)^\infty_{\mathrm{loc}}$, we let $\widetilde{Q}(u) = \infty$.

Definition 5.2 (Functions of finite energy)**.** We say

$$\widetilde{D}(Q) := \{u \in D(Q)^{\infty}_{\mathrm{loc}} \,|\, \widetilde{Q}(u) < \infty\}$$

is the space of *functions of finite energy* associated with Q. The pair $(\widetilde{Q}, \widetilde{D}(Q))$ is called its *reflected Dirichlet space.*

Remark 5.3. It is immediate from the definitions that

$$\widetilde{D}(Q) \cap L^{\infty}(X, m) = \{u \in D(Q)_{\mathrm{loc}} \cap L^{\infty}(X, m) \mid Q(u) < \infty\}$$

and that $\widetilde{Q}$ and Q agree on this space. In fact, by Proposition 5.1 (b) the inclusion

$$\{u \in D(Q)_{\mathrm{loc}} \mid Q(u) < \infty\} \subseteq \widetilde{D}(Q)$$

holds and if $u \in D(Q)_{\mathrm{loc}}$ with $Q(u) < \infty$ the equality $\widetilde{Q}(u) = Q(u)$ is satisfied. Note however, that the inclusion $D(Q)_{\mathrm{loc}} \subseteq L^2_{\mathrm{loc}}(X, m)$ is satisfied by the definition of $D(Q)_{\mathrm{loc}}$ while the same need not be true for $\widetilde{D}(Q)$.

We will now prove two structural theorems about the space $(\widetilde{Q}, \widetilde{D}(Q))$. Namely, we show that $\widetilde{Q}$ has the Fatou property on $L^0(X, m)$ and that it is a quadratic form satisfying the Markov property.

Theorem 5.4 (Fatou's Lemma for $\widetilde{Q}$ on L^0)**.** *Let (u_n) be a sequence in $L^0(X, m)$ and $u \in L^0(X, m)$ such that $u_n \to u$ m-almost everywhere. Then,*

$$\widetilde{Q}(u) \leq \liminf_{n\to\infty} \widetilde{Q}(u_n).$$

In particular, $\liminf_{n\to\infty} \widetilde{Q}(u_n) < \infty$ *implies* $u \in \widetilde{D}(Q)$.

Proof. It suffices to consider the case $\liminf_{n\to\infty} \widetilde{Q}(u_n) < \infty$. So, assume $u_n \in \widetilde{D}(Q)$ for all n and $\liminf_{n\to\infty} \widetilde{Q}(u_n) = \lim_{n\to\infty} \widetilde{Q}(u_n)$.

We prove the statement in two steps. First we show the statement for bounded functions and conclude the general statement afterwards.

Step 1: Assume $u \in L^{\infty}(X, m)$. Without loss of generality we assume $-1 \leq u \leq 1$. Using Proposition 5.1 (a) we may cut-off the u_n and assume $-1 \leq u_n \leq 1$ as well.

We show $u \in D(Q)_{\mathrm{loc}}$: Let G be open and relatively compact. By regularity of Q we choose a function $e \in D(Q) \cap C_c(X)$ such that $e \equiv 1$ on G. Set $u_n^e = (u_n \wedge e) \vee (-e)$. Since $u_n \in \widetilde{D}(Q) \cap L^{\infty}(X, m) \subseteq D(Q)_{\mathrm{loc}} \cap L^{\infty}(X, m)$ and e has compact support, we obtain $u_n^e \in D(Q)$ by Proposition 5.1 (c). Using Proposition 5.1 (d) we estimate

$$Q(u_n^e)^{1/2} \leq Q(u_n \wedge e)^{1/2} + Q(e)^{1/2} \leq Q(u_n)^{1/2} + 2Q(e)^{1/2}.$$

Therefore, $(u_n^e)_n$ is a bounded sequence in the Hilbert space $(D(Q), \|\cdot\|_Q)$. The Banach-Saks Theorem yields the existence of a subsequence $(u_{n_k}^e)$ and a $v \in D(Q)$ such that $v_N = \frac{1}{N}\sum_{k=1}^{N} u_{n_k}^e$ is $\|\cdot\|_Q$ convergent to v. From pointwise convergence of the u_n and, since e has compact support, we infer $v_N \to (u \wedge e) \vee (-e)$ in $L^2(X, m)$. Therefore, $(u \wedge e) \vee (-e) = v \in D(Q)$. Since $(u \wedge e) \vee (-e) = u$ on G, this shows $u \in D(Q)_{\mathrm{loc}}$.

Let us turn to proving the inequality: Since X is locally compact and separable, we find an increasing sequence of relatively compact open sets G_l such that $\overline{G}_l \subseteq G_{l+1}$. By regularity we can choose functions $e_l \in D(Q) \cap C_c(X)$, such that $e_l = 1$ on G_{l+1}. Using the above and a diagonal sequence argument, we may find a subsequence (u_{n_k}) such that for all l the sequence $v^l_N = \frac{1}{N}\sum_{k=1}^N u^{e_l}_{n_k}$ satisfies

$$\|v^l_N - u^{e_l}\|_Q \to 0, \text{ as } N \to \infty.$$

By [11, Theorem 2.1.4], we infer that each sequence $(\tilde v^l_N)_N$ has a q.e. convergent subsequence which converges to u^{e_l}. By a diagonal sequence argument, we may assume that $\tilde v_N = \frac{1}{N}\sum_{k=1}^N \tilde u_{n_k}$ is q.e. convergent towards u (otherwise take a subsequence). Since J and k charge no set of capacity zero, Fatou's Lemma yields

$$\begin{aligned}\iint_{X\times X\setminus diag} &(\tilde u(x)-\tilde u(y))^2\, J(dx,dy) + \int_X \tilde u^2 dk \\ &\le \liminf_{N\to\infty} \iint_{X\times X\setminus diag} (\tilde v_N(x)-\tilde v_N(y))^2\, J(dx,dy) + \int_X \tilde v_N^2 dk.\end{aligned}$$

For the strongly local part, we obtain

$$\begin{aligned}\int_X d\mu^{(c)}(u) &= \lim_{l\to\infty}\int_{G_l} d\mu^{(c)}(u)\\ &= \lim_{l\to\infty}\int_{G_l} d\mu^{(c)}(u^{e_l})\\ &= \lim_{l\to\infty}\lim_{N\to\infty}\int_{G_l} d\mu^{(c)}(v^l_N)\\ (\mu^{(c)} \text{ is local})\quad &= \lim_{l\to\infty}\lim_{N\to\infty}\int_{G_l} d\mu^{(c)}(v_N)\\ &\le \liminf_{N\to\infty}\int_X d\mu^{(c)}(v_N).\end{aligned}$$

The last inequality holds since for each N the convergence in l is monotone (see Lemma A.2). Altogether we obtain

$$Q(u)^{1/2} \le \liminf_{N\to\infty} Q^{1/2}(v_N) \le \liminf_{N\to\infty}\frac{1}{N}\sum_{k=1}^N Q(u_{n_k})^{1/2} = \lim_{n\to\infty} Q(u_n)^{1/2},$$

where the last step results from the assumption in the beginning of the proof.

Step 2: For arbitrary $u \in L^0(X,m)$ as in the statement of the theorem, the considerations of Step 1 applied to $u^{(k)}$ and the sequence $(u_n^{(k)})_n$ show $u^{(k)} \in D(Q)_{\mathrm{loc}}$ for any $k > 1$. Therefore, recalling the definition of $Q(u)$, we compute

$$\begin{aligned}\widetilde Q(u) &= \lim_{k\to\infty} Q(u^{(k)})\\ (\text{Step } 1) &\le \liminf_{k\to\infty}\liminf_{n\to\infty} Q(u_n^{(k)})\\ &\le \liminf_{n\to\infty}\liminf_{k\to\infty} Q(u_n^{(k)})\\ &\le \liminf_{n\to\infty} \widetilde Q(u_n).\end{aligned}$$

Here, we used in the third step that for each n the convergence in k is monotone (Proposition 5.1 (b)) and Lemma A.2. This finishes the proof. $\square$

Recall that a functional q on some real linear space F is called a *quadratic form*, if

$$q(f+g)+q(f-g)=2q(f)+2q(g) \text{ and } q(af)=a^2q(f)$$

for any $f,g\in F$ and $a\in\mathbb{R}$. Any quadratic form q induces a bilinear form via polarization which we also denote by q. The following theorem shows that we can apply this concept to $F=\widetilde{D}(Q)$ and $q=\widetilde{Q}$.

Theorem 5.5. *The map $\widetilde{Q}:\widetilde{D}(Q)\to[0,\infty)$ is a quadratic form. Furthermore, for any normal contraction $C:\mathbb{R}\to\mathbb{R}$ and any $u\in\widetilde{D}(Q)$ we have $C\circ u\in\widetilde{D}(Q)$ and*

$$\widetilde{Q}(C\circ u)\le\widetilde{Q}(u).$$

Proof. We first show the contraction property. Let $C:\mathbb{R}\to\mathbb{R}$ be a normal contraction. Now, Fatou's Lemma for $\widetilde{Q}$ and Proposition 5.1 (a) yields

$$\widetilde{Q}(C\circ u)\le\liminf_{n\to\infty}\widetilde{Q}(C\circ u^{(n)})=\liminf_{n\to\infty}Q(C\circ u^{(n)})\le\liminf_{n\to\infty}Q(u^{(n)})=\widetilde{Q}(u).$$

It remains to show that $\widetilde{Q}$ is a quadratic form. Let $a\in\mathbb{R}$ and $u\in\widetilde{D}(Q)$ be given. Fatou's Lemma for $\widetilde{Q}$ and the fact that Q is a quadratic form on $\{u\in D(Q)_{\mathrm{loc}}:Q(u)<\infty\}$ yields

$$\widetilde{Q}(u)\le\liminf_{n\to\infty}\widetilde{Q}(\frac{1}{a}(au)^{(n)})=\liminf_{n\to\infty}Q(\frac{1}{a}(au)^{(n)})=\frac{1}{a^2}\widetilde{Q}(au).$$

For the inequality $\widetilde{Q}(u)\ge\frac{1}{a^2}\widetilde{Q}(au)$ we note, that for each n the map $x\mapsto\frac{1}{a}(ax)^{(n)}$ is a normal contraction and compute

$$\frac{1}{a^2}\widetilde{Q}(au)=\lim_{n\to\infty}\frac{1}{a^2}Q((au)^{(n)})=\lim_{n\to\infty}Q(\frac{1}{a}(au)^{(n)})=\lim_{n\to\infty}\widetilde{Q}(\frac{1}{a}(au)^{(n)})\le\widetilde{Q}(u).$$

Now, let $u,v\in\widetilde{D}(Q)$ be given. Fatou's Lemma for $\widetilde{Q}$ and the fact that Q is a quadratic form on $\{u\in D(Q)_{\mathrm{loc}}:Q(u)<\infty\}$ yields

$$\begin{aligned}\widetilde{Q}(u+v)+\widetilde{Q}(u-v)&\le\liminf_{n\to\infty}(Q(u^{(n)}+v^{(n)})+(Q(u^{(n)}-v^{(n)}))\\&=\liminf_{n\to\infty}(2Q(u^{(n)})+2Q(v^{(n)}))\\&=2\widetilde{Q}(u)+2\widetilde{Q}(v).\end{aligned}$$

Since the above inequality is true for arbitrary functions, we can apply it to $u'=u+v$ and $v'=u-v$ to obtain

$$4\widetilde{Q}(u)+4\widetilde{Q}(v)=\widetilde{Q}(u'+v')+\widetilde{Q}(u'-v')\le2\widetilde{Q}(u')+2\widetilde{Q}(v')=2\widetilde{Q}(u+v)+2\widetilde{Q}(u-v).$$

The first equality is a consequence of $\widetilde{Q}(aw)=a^2\widetilde{Q}(w)$ which was proven above. This finishes the proof.

□

Remark 5.6.
- In general, $\widetilde{D}(Q)$ does not need to be included in $L^2(X,m)$ and, hence, $(\widetilde{Q},\widetilde{D}(Q))$ is not a Dirichlet form in the usual sense. See [27, 4] for the background of this definition and properties in the quasi-regular case.
- The importance of $\widetilde{D}(Q)$ stems from the fact that $\widetilde{Q}$ is a finite quadratic form on this space inducing a bilinear form by polarization (which will also be called $\widetilde{Q}$). It will follow from the previous theorems that this form does not only extend $(Q,D(Q))$ but also provides an extension of $(Q,D(Q)_e)$. Furthermore, it yields the well known fact that $(\widetilde{Q},\widetilde{D}(Q)\cap\ L^2(X,m))$ is a closed form (see, e.g., [27] and [3]).

- The above lower semi-continuity of Q on its reflected Dirichlet space with respect to pointwise convergence seems to be new. As $D(Q)_e \subseteq \widetilde{D}(Q)$ (see below), we obtain an extension of [4, Corollary 1.1.9] for regular Dirichlet forms.

We now come to a crucial definition for the subsequent considerations. In the previous sections we used various operators to characterize the investigated properties. In the regular setting we will see below that all these operators are restrictions of

$$\mathcal{L} : D(\mathcal{L}) \to L^1_{\mathrm{loc}}(X, m),$$

where

$$D(\mathcal{L}) = \{u \in \widetilde{D}(Q) \,|\, \exists f \in L^1_{\mathrm{loc}}(X, m)\ \forall v \in D(Q) \cap C_c(X)\ \widetilde{Q}(u, v) = \langle f, v\rangle\}$$

on which it acts by

$$\mathcal{L}u = f.$$

The following proposition is clear from the definitions.

Proposition 5.7. *The operator $\mathcal{L}$ is an extension of L. The domain of L satisfies*

$$D(L) = \{u \in D(Q) \cap D(\mathcal{L}) \ \mid\ \mathcal{L}u \in L^2(X, m)\}.$$

Remark 5.8. In a certain sense $\mathcal{L}$ is a distributional extension of L. In many situations its domain and action are known explicitly, see Sections 6, 7 and 8.

5.2. **Extensions of Dirichlet forms: The uniqueness of Silverstein's extension.** We will now apply the theory of Section 2 to the form Q on $\mathcal{D} = D(Q)$ and to $\widetilde{Q}$ on $\mathcal{D}^{\#} = D(Q^{\max}) := \widetilde{D}(Q) \cap L^2(X, m)$. We think of $D(Q)$ as encoding "Dirichlet boundary conditions" and of $D(Q^{\max})$ as encoding "Neumann type boundary conditions". We write $Q^{\max}$ whenever we refer to $\widetilde{Q}$ on $D(Q^{\max})$ and denote the associated positive operator by $L^{\max}$. The following propositions assure that the theory of Section 2 can be applied.

Proposition 5.9 ($Q^{\max}$ as Dirichlet form)**.** *The form $Q^{\max}$ is a Dirichlet form. The space $D(Q^{\max}) \cap L^\infty(X, m)$ is given by those $u \in D(Q)_{\mathrm{loc}} \cap L^2(X, m) \cap L^\infty(X, m)$ satisfying*

$$\int_X d\mu^{(c)}(u) + \iint_{X \times X \setminus diag} (\tilde{u}(x) - \tilde{u}(y))^2\, J(dx, dy) + \int_X \tilde{u}^2 dk < \infty.$$

Proof. The closedness of $Q^{\max}$ follows from Fatou's lemma, Theorem 5.4 while the fact that $Q^{\max}$ is a Markovian quadratic form is a consequence of Theorem 5.5. The fact about the action of $Q^{\max}$ on bounded function is a consequence of the fact that $\widetilde{Q}$ and the extension of Q to $D(Q)_{\mathrm{loc}}$ agree on $L^\infty(X, m)$. This finishes the proof. □

Proposition 5.10. *The forms $(Q, D(Q))$ and $(Q^{\max}, D(Q^{\max}))$ satisfy the maximum principle (MP), i.e., positive $f \in L^2(X, m)$ satisfy the inequality*

$$(L + \alpha)^{-1} f \leq (L^{\max} + \alpha)^{-1} f.$$

Proof. We use Theorem 2.13 with $\mathcal{D} = D(Q)$ and $\mathcal{D}^{\#} = D(Q^{\max})$. Choose (G_n) to be an increasing sequence of open, relatively compact sets such that $\overline{G_n} \subseteq G_{n+1}$ and $\cup G_n = X$. Then, the inclusion

$$C_c(X) \cap D(Q) \subseteq \mathcal{C} := \bigcup_{n \geq 1} D(Q^{\max}_{G_n})$$

holds, where $D(Q^{\max}_{G_n}) = \{u \in D(Q^{\max}) \mid u = 0 \ m\text{-a.e. on } X \setminus G_n\}$. Furthermore, we have $\mathcal{C} \subseteq D(Q)$. To see this, let $u \in D(Q^{\max}_{G_n})$ be given. Without loss of generality, we may assume $u \in L^\infty(X,m)$. Since $D(Q^{\max}) \cap L^\infty(X,m) \subseteq D(Q)_{\mathrm{loc}}$, there exists a function $v \in D(Q) \cap L^\infty(X,m)$ such that $v = u$ on G_{n+1}. By the regularity of Q there exists a function $\varphi \in D(Q) \cap L^\infty(X,m)$ such that $\varphi = 1$ on G_n and $\varphi = 0$ on G_{n+1}. Since $D(Q) \cap L^\infty(X,m)$ is an algbra, c.f. [11, Theorem 1.4.2], we obtain $u = u\varphi = v\varphi \in D(Q)$. Now, Theorem 2.13 shows the statement as $C_c(X) \cap D(Q)$ is dense in $D(Q)$. □

Remark 5.11.

- In the literature $(Q^{\max}, D(Q^{\max}))$ is called the active reflected Dirichlet space of $(Q, D(Q))$. This terminology stems from the following observation. If $(Q, D(Q))$ is the standard Dirichlet energy on an open subdomain of $\mathbb{R}^n$ considered with Dirichlet boundary conditions, then $(Q^{\max}, D(Q^{\max}))$ is the Dirichlet energy with Neumann boundary conditions. In terms of stochastic processes this means that $(Q, D(Q))$ corresponds to Brownian motion which is killed upon hitting the boundary while $(Q^{\max}, D(Q^{\max}))$ corresponds to Brownian motion which is reflected at the boundary.
- We use the notation $Q^{\max}$ as it is the maximal Silverstein extension of Q (see e.g. [27, 4]). Recall that an extension $\hat{Q}$ of Q is called a *Silverstein extension* if $u \cdot v \in D(Q)$ for all $u \in D(Q) \cap L^\infty(X)$, $v \in D(\hat{Q}) \cap L^\infty(X)$.

Recall the definition of the operator L' in Section 2.

Proposition 5.12. *If $\mathcal{D} = D(Q)$ and $\mathcal{D}^\# = D(Q^{\max})$, then the domain of L' satisfies*

$$D(L') = \{u \in D(\mathcal{L}) \cap L^2(X,m) \mid \mathcal{L}u \in L^2(X,m)\}.$$

Proof. This follows from the definitions and the regularity of Q. □

Therefore, our main statement of Section 2, Theorem 2.8, has now an immediate consequence.

Theorem 5.13. *Let Q be a regular Dirichlet form. Then the following assertions are equivalent.*

(i) $Q \neq Q^{\max}$.
(ii) *There exists a nontrivial $u \in D(\mathcal{L}) \cap L^1(X,m) \cap L^2(X,m)$ such that $\mathcal{L}u \in L^2(X,m)$ with $u \geq 0$, $\mathcal{L}u \leq 0$ and $\mathcal{L}u \neq 0$.*

If the assertions hold, then the function u in (ii) *can additionally be chosen to be bounded.*

We can now also give a characterization of $Q = Q^{\max}$ in terms of Green's formula.

Theorem 5.14 (Characterization of $Q = Q^{\max}$ for regular forms via Green's formula)**.** *Let Q be a regular Dirichlet form and L its associated operator. Then the following assertions are equivalent.*

(i) $Q = Q^{\max}$.
(ii) $D(L) = \{u \in D(\mathcal{L}) \cap L^2(X,m) \mid \mathcal{L}u \in L^2(X,m)\}$.
(iii) *For all $u \in D(\mathcal{L}) \cap L^2(X,m)$ such that $\mathcal{L}u \in L^2(X,m)$ and all $v \in \widetilde{D}(Q) \cap L^2(X,m)$ the following equality holds*

$$\widetilde{Q}(u,v) = \int_X \mathcal{L}uv \, dm.$$

Proof. (i) $\Longrightarrow$ (ii): The domain of L is given by

$$D(L) = \{u \in D(Q) \mid \text{there exists } w \in L^2(X,m) \text{ s.t. } Q(u,v) = \langle w, v\rangle \text{ for all } v \in D(Q)\}.$$

Since Q is regular, this implies $D(L) \subseteq \{u \in D(\mathcal{L}) \cap L^2(X,m) \mid \mathcal{L}u \in L^2(X,m)\}$.

Now let $u \in D(\mathcal{L}) \cap L^2(X,m)$ with $\mathcal{L}u \in L^2(X,m)$ be given. Then, the definition of $\mathcal{L}$ and of $Q^{\max}$ together with the equality $Q = Q^{\max}$ imply

$$\langle \mathcal{L}u, v\rangle = Q^{\max}(u,v) = Q(u,v)$$

for all $v \in D(Q)$. This directly gives $u \in D(L)$ (and $Lu = \mathcal{L}u$).

(ii) $\Longrightarrow$ (iii): The definition of $L^{\max}$ shows $D(L^{\max}) \subseteq \{u \in D(\mathcal{L}) \cap L^2(X,m) : \mathcal{L}u \in L^2(X,m)\}$ and $\mathcal{L}u = L^{\max}u$ for $u \in D(L^{\max})$. Then (ii) gives that $L^{\max}$ is a restriction of L. As both L and $L^{\max}$ are selfadjoint, we infer $L = L^{\max}$ and this easily yields (iii).

(iii) $\Longrightarrow$ (i): Assume $Q \neq Q^{\max}$. Then, by Theorem 5.13, there exists $u \in D(\mathcal{L}) \cap L^2(X,m)$ such that $\mathcal{L}u \in L^2(X,m)$ with $u \geq 0$, $\mathcal{L}u \leq 0$ and $\mathcal{L}u \neq 0$. This u satisfies $Q(u,u) \geq 0$ and

$$\int_X \mathcal{L}u u \, dm < 0$$

which contradicts (iii). □

5.3. **Stochastic completeness.** In the regular setting we can give a more explicit characterization of stochastic completeness since we can compute the domain of L. For this the following proposition is needed.

Proposition 5.15. *Let a regular Dirichlet form Q be stochastically complete. Then $Q = Q^{\max}$.*

Proof. We show $(L+1)^{-1} = (L^{\max}+1)^{-1}$. Let (e_n) be a sequence in $D(Q) \cap C_c(X)$ such that $0 \leq e_n \leq 1$ and $e_n \uparrow 1$ m-almost everywhere. By the maximum principle (MP), Proposition 5.10, and stochastic completeness, we obtain

$$1 = (L+1)^{-1}1 = \lim_{n\to\infty} (L+1)^{-1}e_n \leq \lim_{n\to\infty} (L^{\max}+1)^{-1}e_n = (L^{\max}+1)^{-1}1 \leq 1.$$

This shows $(L^{\max}+1)^{-1}1 = 1$. Now, let $0 \leq f \in L^1(X,m) \cap L^2(X,m)$ be given. Since $(L^{\max}+1)^{-1}f - (L+1)^{-1}f \in L^1(X,m) \cap L^2(X,m)$, we obtain by Lebesgue's theorem

$$\begin{aligned} 0 &\leq \langle (L^{\max}+1)^{-1}f - (L+1)^{-1}f, 1\rangle \\ &\leq \lim_{n\to\infty} \langle (L^{\max}+1)^{-1}f - (L+1)^{-1}f, e_n\rangle \\ &= \lim_{n\to\infty} \langle f, (L^{\max}+1)^{-1}e_n - (L+1)^{-1}e_n\rangle \\ &= 0. \end{aligned}$$

This shows $(L+1)^{-1} = (L^{\max}+1)^{-1}$ and our claim follows. □

Remark 5.16. The previous proposition is known see, e.g., [27, Theorem 6.3]. Note however, that our proof only uses the estimate $(L+1)^{-1}f \leq (L^{\max}+1)^{-1}f$ and did not rely on the regularity of Q. Thus, it also holds for pairs of forms satisfying the maximum principle (MP) (c.f. Section 2).

With this at hand our main theorem on stochastic completeness reads as follows.

Theorem 5.17 (Characterization stochastic completeness for regular forms)**.** *Let Q be a regular Dirichlet form. Then the following assertions are equivalent.*

(i) *Q is stochastically complete.*
(ii) *For all $u \in D(\mathcal{L}) \cap L^1(X,m) \cap L^2(X,m)$ such that $\mathcal{L}u \in L^1(X,m) \cap L^2(X,m)$ the following equality holds*

$$\int_X \mathcal{L}u \, dm = 0.$$

Proof. (i) $\Longrightarrow$ (ii): Since Q is stochastically complete, $Q = Q^{\max}$ holds by the previous proposition. Then, Theorem 5.14 shows $D(L) = \{D(\mathcal{L}) \cap L^2(X,m) \mid \mathcal{L}u \in L^2(X,m)\}$. Therefore, Theorem 3.1 implies the statement.

(ii) $\Longrightarrow$ (i): This is an immediate consequence of Theorem 3.1 since $D(L) \subseteq \{D(\mathcal{L}) \cap L^2(X,m) \mid \mathcal{L}u \in L^2(X,m)\}$ is satisfied by Proposition 5.7. $\square$

5.4. **Recurrence.** We improve the results on recurrence in the regular setting. For this we need that the action of $\widetilde{Q}$ on the space of functions of finite energy is compatible with the action of Q on its extended Dirichlet space.

Lemma 5.18. *The inclusion $D(Q)_e \subseteq \widetilde{D}(Q)$ holds and the extension of Q to $D(Q)_e$ equals $\widetilde{Q}$ on $D(Q)_e$. Furthermore, if Q is recurrent, then $D(Q)_e = \widetilde{D}(Q)$.*

Proof. Recall the definition of the extension of Q to $D(Q)_e$ in the beginning of Section 4. Let $u \in D(Q)_e$ be given and let $(u_n) \subseteq D(Q)$ be an approximating sequence for u. Then, Theorem 5.4 shows $u \in \widetilde{D}(Q)$ and

$$|\widetilde{Q}(u)^{1/2} - \widetilde{Q}(u_n)^{1/2}| \le \widetilde{Q}(u-u_n)^{1/2} \le \liminf_{m\to\infty} \widetilde{Q}(u_m - u_n)^{1/2} = \liminf_{m\to\infty} Q(u_m-u_n)^{1/2}.$$

This shows $Q(u_n) \to \widetilde{Q}(u)$ which was the first claim.

Now, assume Q is recurrent. Then there exists a sequence $e_n \in D(Q) \cap C_c(X)$ such that $0 \le e_n \le 1$, $e_n \to 1$ m-a.e. and $Q(e_n) \to 0$ (see Appendix for details). Let $u \in \widetilde{D}(Q)$ be given. Without loss of generality we may assume that $0 \le u \le 1$ (else approximate, rescale, split in positive and negative part). The function $u \wedge e_n$ has compact support and it follows from the definition of $\widetilde{D}(Q)$ that it belongs to $D(Q)_{\mathrm{loc}} \cap L^\infty(X,m)$. Therefore, Proposition 5.1 (c) shows $u \wedge e_n \in D(Q)$. Since Proposition 5.1 (d) implies

$$Q(u \wedge e_n)^{1/2} \le Q(u)^{1/2} + Q(e_n)^{1/2},$$

the sequence $(u \wedge e_n)$ is bounded with respect to the inner product space $(Q, D(Q))$. Hence, by some version of the Banach-Saks Theorem, it has a subsequence $u \wedge e_{n_k}$ such that $v_N = \frac{1}{N}\sum_{k=1}^N u \wedge e_{n_k}$ is a Q-Cauchy sequence. Since $u \wedge e_n \to u$ m-a.e., we also obtain $v_N \to u$ m-a.e., and, therefore, $u \in D(Q)_e$. $\square$

With this at hand our main result on recurrence reads as follows.

Theorem 5.19 (Characterization recurrence for regular forms)**.** *Let Q be an irreducible regular Dirichlet form. Then the following assertions are equivalent.*

(i) *Q is recurrent.*
(ii) *For all $u \in D(\mathcal{L})$ such that $\mathcal{L}u \in L^1(X,m)$ the following equality holds*

$$\int_X \mathcal{L}u \, dm = 0.$$

(iii) *The killing measure k vanishes and for all $u \in D(\mathcal{L})$, such that $\mathcal{L}u \in L^1(X,m)$ and for all $v \in \widetilde{D}(Q) \cap L^\infty(X,m)$ the following equality holds*

$$\widetilde{Q}(u,v) = \int_X \mathcal{L}uv\, dm.$$

Proof. (ii) $\Longrightarrow$ (i): This immediately follows from Theorem 4.2 and the previous lemma.

(i) $\Longrightarrow$ (iii): The form Q is recurrent, therefore, the killing measure k vanishes. Let $v \in \widetilde{D}(Q) \cap L^\infty(X,m)$ be given. Since Q is recurrent, the previous Lemma shows $v \in D(Q)_e$, hence v possesses an approximating sequence (v_n) in $D(Q)$. By regularity we may choose (v_n) in $D(Q) \cap C_c(X)$. Furthermore, we may assume that the v_n are uniformly bounded by $\|v\|_\infty$. Using Lebesgue's theorem, we then obtain for $u \in D(\mathcal{L})$ with $\mathcal{L}u \in L^1$

$$\widetilde{Q}(u,v) = \lim_{n\to\infty} \widetilde{Q}(u,v_n) = \lim_{n\to\infty} \int_X \mathcal{L}uv_n\, dm = \int_X \mathcal{L}uv\, dm.$$

(iii) $\Longrightarrow$ (ii): Assume (ii) does not hold. Then, there exists $u \in D(\mathcal{L})$ with $\mathcal{L}u \in L^1$ such that

$$\int_X \mathcal{L}u\, dm \neq 0.$$

Since k vanishes, we obtain $1 \in \widetilde{D}(Q)$ and $\widetilde{Q}(1,u) = 0$. This contradicts (iii). □

5.5. **The relation between the concepts.** As an application of the above criteria we finish this section by discussing the relation of the various concepts.

Theorem 5.20 (Relation between the concepts)**.** *Suppose Q is a regular irreducible Dirichlet form. Consider the following statements.*

(i) *Q is recurrent.*
(ii) *Q is stochastically complete.*
(iii) *$Q = Q^{\max}$.*

Then, the implications (i) $\Longrightarrow$ (ii) *and* (ii) $\Longrightarrow$ (iii) *are always true. If the killing measure k vanishes and $m(X) < \infty$ the above assertions are equivalent.*

Proof. The implication (i) $\Longrightarrow$ (ii) immediately follows from Theorem 5.19 and Theorem 5.17. The implication (ii) $\Longrightarrow$ (iii) is the statement of Proposition 5.15. Suppose now $m(X) < \infty$ and $k = 0$. It remains to show the implication (iii) $\Longrightarrow$ (i). This follows from the fact that $m(X) < \infty$ implies $1 \in D(Q^{\max})$ and $k = 0$ implies $Q^{\max}(1) = 0$. □

Remark 5.21. The implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are well-known. However, the statement on the equivalence in case of finite measure seems to be interesting. See the proof of Theorem 6.12 for an application.

6. Application to graphs

In this section we apply the results obtained above to graphs. Here, we use the framework of regular Dirichlet forms on graphs discussed in various recent works. After a discussion of the background, we specify the space of the functions of finite energy. Then, we turn to extensions, stochastic completeness and to recurrence.

The salient feature of our discussion is that the operator $\mathcal{L}$ associated to a regular Dirichlet form in the previous section is known explicitly in the graph case. This makes it possible to present unified formulations of the results.

Unbounded Laplacians on graphs have become a focus of research in various recent works, see e.g., [6, 20, 24, 19, 36, 37], and references therein.

The subsequent discussion of the setting essentially follows [24] (see [18, 19] as well) to which we refer for further details and proofs. Let V be a countable set and $C(V)$ be the set of all real-valued functions on V. For a measure $m : V \to (0, \infty)$ let $\ell^2(V, m) = \{u : V \to \mathbb{R} \mid \sum_{x\in V} |u(x)|^2 m(x) < \infty\}$ and denote the corresponding scalar product by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\|\cdot\|$.

Let $b : V \times V \to [0, \infty)$ be symmetric with zero diagonal and assume $\sum_{y\in V} b(x, y) < \infty$ for all $x \in V$. Furthermore, let $c : V \to [0, \infty)$. We then call (b, c) a *weighted graph over* V and refer to V as the *vertex set*. Moreover, $x, y \in V$ are *connected by an edge with weight* $b(x, y)$ whenever $b(x, y) > 0$. In this case, we write $x \sim y$. Furthermore, c encodes one-way-edges from x whenever $c(x) > 0$.

We say a set $W \subseteq V$ is *connected* if for all $x, y \in W$ there exists a finite sequence of vertices $x = x_0, \ldots, x_n = y$ in W such that $x_j \sim x_{j+1}$, $j = 0, \ldots, n-1$. We call such a sequence of vertices a *path* from x to y.

Let $\widetilde{Q}_{b,c} : C(V) \to [0, \infty]$ be given by

$$\widetilde{Q}_{b,c}(u) = \frac{1}{2} \sum_{x,y\in V} b(x, y)(u(x) - u(y))^2 + \sum_{x\in V} c(x)u(x)^2.$$

We are interested in the space

$$\widetilde{D} = \{u \in C(V) \mid \widetilde{Q}_{b,c}(u) < \infty\}.$$

By the summability assumption on b the inclusion $C_c(V) \subseteq \widetilde{D}$ follows easily, where $C_c(V)$ is the space of finitely supported functions. By polarization $\widetilde{Q}_{b,c}$ extends to a symmetric bilinear form on $\widetilde{D} \times \widetilde{D}$. This bilinear form will again be denoted by $\widetilde{Q}_{b,c}$.

There is a regular Dirichlet form associated with (b, c) introduced next. Let Q be the restriction of $\widetilde{Q}_{b,c}$ to

$$D(Q) = \overline{C_c(V)}^{\|\cdot\|_Q},$$

where $\|\cdot\|_Q^2 = \|\cdot\|^2 + \widetilde{Q}_{b,c}(\cdot)$.

By Fatou's lemma $\widetilde{Q}_{b,c}$ is lower semi-continuous and, hence, every restriction is closable. Thus, the form Q is closed by definition of $D(Q)$. Moreover, $C_c(V) \subseteq D(Q)$ implies that Q is regular, i.e., $D(Q) \cap C_c(V)$ is dense in $C_c(V)$ with respect to the supremum norm $\|\cdot\|_\infty$ and $D(Q)$ with respect to $\|\cdot\|_Q$. One can check that $(Q, D(Q))$ is a Dirichlet form (see [11, Theorem 3.1.1]), which we call the *regular Dirichlet form associated to* (b, c).

6.1. **Functions of finite energy.** In view of the theory developed in Chapter 5, we aim at determining the associated space of functions of finite energy $\widetilde{D}(Q)$ and the maximal form $D(Q^{\max})$.

Proposition 6.1. *Let Q be the regular Dirichlet form on $\ell^2(V, m)$ associated to the graph (b, c). Then, $\widetilde{D}(Q) = \widetilde{D}$ and $\widetilde{Q}$ on $\widetilde{D}(Q)$ is given by $\widetilde{Q}_{b,c}$. Furthermore, $Q^{\max}$ is the restriction of $\widetilde{Q}_{b,c}$ to the domain $D(Q^{\max}) = \widetilde{D} \cap \ell^2(V, m)$.*

Proof. The equality $\widetilde{D}(Q) = \widetilde{D}$ follows from $D(Q)_{\mathrm{loc}} = C(V)$ and the monotone convergence theorem. The rest is clear from the definitions. $\square$

Let us now turn to the associated operators. We let

$$\widetilde{F} := \{w : V \to \mathbb{R} \mid \sum_{y \in V} b(x,y)|w(y)| < \infty \text{ for all } x \in V\ \}$$

and define $\widetilde{L} : \widetilde{F} \longrightarrow C(V)$ via

$$\widetilde{L}w(x) := \frac{1}{m(x)} \sum_{y \in V} b(x,y)(w(x) - w(y)) + \frac{c(x)}{m(x)} w(x).$$

Here, indeed the sum exists for each $x \in V$ due to $w \in \widetilde{F}$. Then L, the associated operator of Q, is a restriction of $\widetilde{L}$ with domain satisfying

$$D(L) \subset \{u \in \ell^2(V,m) \mid \widetilde{L}u \in \ell^2(V,m)\}.$$

As mentioned above, details can be found in [24]. Here, we just briefly discuss the crucial link between the operator $\widetilde{L}$ and the form Q. This link is given by the following Green-type-formula. Various variants can be found in [18, 24, 19].

Lemma 6.2 (Green type formula)**.** *(a) The set $\widetilde{D}$ is contained in $\widetilde{F}$.*

(b) For all $w \in \widetilde{F}$ and $v \in C_c(V)$, the following equality holds

$$\widetilde{Q}(w,v) = \sum_x (\widetilde{L}w)(x)v(x)m(x) = \sum_x w(x)(\widetilde{L}v)(x).$$

Proof. Statement (a) is part of [19, Proposition 2.8] and statement (b) is contained in [18, Lemma 4.7]. □

With this at hand we can identify the operator $\mathcal{L}$ for the regular Dirichlet form associated to the graph (b,c). Recall its definition on Page 20.

Theorem 6.3. *Let Q be the regular Dirichlet form associated with the graph (b,c). Then, $\mathcal{L}$ is the restriction of $\widetilde{L}$ to the domain $D(\mathcal{L}) = \widetilde{D}$.*

Proof. This is a consequence of Proposition 6.1, the previous lemma and the definition of $\mathcal{L}$. □

6.2. **Extensions of Dirichlet forms.** The next result is a direct application of Theorem 5.13.

Theorem 6.4. *Let Q be the regular Dirichlet form associated to (b,c). Then the following assertions are equivalent.*

(i) $Q \neq Q^{\max}$.

(ii) *There exists a nontrivial $u \in D(Q^{\max}) \cap \ell^1(V,m) \cap \ell^\infty(V)$ such that $\widetilde{L}u \in \ell^2(V,m)$ with $u \geq 0$, $\widetilde{L}u \leq 0$ and $\widetilde{L}u \neq 0$.*

Remark 6.5. The previous theorem is a slight extension of [19, Corollary 4.3]. Specifically, it suffices to look for subsolutions in ℓ^1 for the direction (ii) $\Longrightarrow$ (i) is not found in [19].

Next, we come to the application of Theorem 5.14.

Theorem 6.6. *Let Q be the regular Dirichlet form associated to (b,c). Then, the following assertions are equivalent.*

(i) $Q = Q^{\max}$.

(ii) $D(L) = \{u \in D(Q^{\max}) \mid \widetilde{L}u \in \ell^2(V,m)\}$.
(iii) *For all* $u \in D(Q^{\max})$ *such that* $\widetilde{L}u \in \ell^2(V,m)$ *and all* $v \in D(Q^{\max})$,

$$\widetilde{Q}(u,v) = \sum_{x \in V} \widetilde{L}u(x)v(x)m(x).$$

6.3. **Stochastic Completeness.** We now come to the application of Theorem 5.17 concerning stochastic completeness. It is known (see [24]) that the generator $L^{(1)}$ of the ℓ^1 semigroup is a restriction of $\widetilde{L}$ to the domain $D(L^{(1)})$ which satisfies

$$D(L^{(1)}) \subseteq \{u \in \ell^1(V,m) \mid \widetilde{L}u \in \ell^1(V,m)\}.$$

Here, equality holds if furthermore the following assumption (A) is satisfied (compare [24, Theorem 5]):

(A) Every infinite path (x_n) of vertices has infinite measure, i.e., $\sum_n m(x_n) = \infty$.

With this the following theorem is then an immediate consequence of Theorem 5.17 and Theorem 6.3.

Theorem 6.7. *Let* Q *be the regular Dirichlet form associated to* (b,c)*. Consider the statements:*

(i) Q *is stochastically complete.*
(ii) *For all* $u \in \widetilde{D} \cap \ell^2(V,m) \cap \ell^1(V,m)$ *satisfying* $\widetilde{L}u \in \ell^1(V,m) \cap \ell^2(V,m)$ *we have*

$$\sum_{x \in V} \widetilde{L}u(x)m(x) = 0.$$

(iii) *For all* $u \in \ell^1(V,m)$ *satisfying* $\widetilde{L}u \in \ell^1(V,m)$ *we have*

$$\sum_{x \in V} \widetilde{L}u(x)m(x) = 0.$$

Then, (i) $\Longleftrightarrow$ (ii) *and* (iii) $\Longrightarrow$ (i)*. If (A) is satisfied, then* (i) $\Longleftrightarrow$ (iii)*.*

Remark 6.8. The equivalence between (i) and (iii) holds whenever $L^{(1)}$ is the maximal restriction of $\widetilde{L}$ on $\ell^1(V,m)$. Condition (A) comes into play as it ensures that $L^{(1)}$ is this maximal restriction.

6.4. **Recurrence.** As discussed in the general setting we will restrict our investigation to irreducible Dirichlet forms. We can characterize irreducibility in terms of connectedness of the underlying graph (see [28, 25] as well).

Lemma 6.9. *Let* Q *be the regular Dirichlet form associated to* (b,c)*. The following assertions are equivalent:*

(i) Q *is irreducible*
(ii) (b,c) *is connected*

Proof. (i) $\Longrightarrow$ (ii): Let W be a connected component of V with respect to (b,c). We show $1_W \cdot u \in D(Q)$ for any $u \in D(Q)$ and

$$Q(u,u) = Q(1_W u, 1_W u) + Q(1_{W^c} u, 1_{W^c} u) \tag{$\heartsuit$}$$

holds, where 1_W denotes the characteristic function of the set W. Since Q is a restriction of $\widetilde{Q}$ and W is a connected component, the formula ($\heartsuit$) follows as soon as we show $1_W \cdot u \in D(Q)$. This is immediate for $u \in C_c(V)$. Let $u \in D(Q)$ be arbitrary. By regularity there exists a sequence (u_n) in $C_c(V)$ such that $u_n \to u$ with respect to $\|\cdot\|_Q$. Since ($\heartsuit$) holds for compactly supported functions, we obtain $\|1_W \cdot u_n - 1_W \cdot u_m\|_Q \leq \|u_n - u_m\|_Q$ and, because Q is closed, we infer $1_W \cdot u \in D(Q)$. From irreducibility we conclude $W = \emptyset$ or $V \setminus W = \emptyset$ showing the connectedness.

(ii) $\Longrightarrow$ (i): It is not hard to see that ($\heartsuit$) can only be satisfied if W is connected. By the connectedness of the graph we infer $W = V$ or $W = \emptyset$ showing the irreducibility of Q. $\square$

With these preparations Theorem 5.19 reads in the graph situation as follows.

Theorem 6.10. *Let Q be the regular Dirichlet form associated to a connected graph (b, c). Then the following assertions are equivalent:*

(i) *Q is recurrent.*

(ii) *For all $u \in \widetilde{D}$ with $\tilde{L}u \in \ell^1(V, m)$ the following equality holds*
$$\sum_{x \in V} \widetilde{L}u(x)m(x) = 0.$$

(iii) *For all $u \in \widetilde{D}$ with $\tilde{L}u \in \ell^1(V, m)$ and all $v \in \widetilde{D} \cap \ell^\infty$ the following equality holds*
$$\widetilde{Q}(u, v) = \sum_{x \in V} (\widetilde{L}u)(x)v(x)m(x).$$

Remark 6.11. A result related to (i) $\Leftrightarrow$ (iii) of the previous theorem can be found in [23]. There, a boundary term of the form $Q(u, v) - \langle \widetilde{L}u, v \rangle$ is defined via a limiting procedure on a set of functions which is somewhat different from ours. It is then shown that recurrence is equivalent to this boundary term vanishing.

A result that is related but somewhat independent of the theory developed in Section 5 is the following. It shows that functions of finite energy can be replaced by bounded functions. The proof is given below.

Theorem 6.12. *Let Q be the regular Dirichlet form associated to a connected graph $(b, 0)$. Then the following assertions are equivalent:*

(i) *Q is recurrent.*

(ii) *For all $u \in \ell^\infty(V)$ with $\widetilde{L}u \in \ell^1(V, m)$ the following equality holds*
$$\sum_{x \in V} \widetilde{L}u(x)m(x) = 0.$$

Remark 6.13. The above characterization is a direct analogue to [16, Theorem 1.1].

In order to prove the theorem, we establish some notation and prepare a lemma. For any point $o \in V$, consider the inner product
$$\langle u, v \rangle_o = \widetilde{Q}(u, v) + u(o)v(o)$$
on V. If the graph $(b, 0)$ is connected the pair $(\widetilde{D}, \langle \cdot, \cdot \rangle_o)$ is a Hilbert space and pointwise evaluation of functions is continuous with respect to the corresponding norm, c.f. [12, Lemma 3.6]. Let $\{\Omega_n\}_{n \geq 1}$ be an exhaustion of V with finite sets such that each of Ω_n includes $o \in V$. Furthermore, for $x \in V$ and finite $G \subseteq V$, we use the notation
$$\operatorname{cap}(x) = \inf\{Q(u) \mid u \in D(Q), u(x) \geq 1\}$$
and
$$\operatorname{cap}(x, G) = \inf\{Q(u) \mid u \in D(Q), u(x) \geq 1, u|_{V \setminus G} \equiv 0\}.$$

Lemma 6.14. *Let Q be the regular Dirichlet form associated to a connected graph $(b, 0)$. Then, the following assertions hold.*

(a) *There exists a unique $e \in \overline{D(Q)}^{\|\cdot\|_o}$ such that $\widetilde{Q}(e) = \mathrm{cap}(o)$ and unique $e_n \in D(Q)$ such that $Q(e_n) = \mathrm{cap}(o, \Omega_n)$ for every $n \geq 1$. Furthermore, these functions satisfy $0 \leq e_n, e \leq 1$ and $e_n(o) = e(o) = 1$.*
(b) *The inequality $\widetilde{L}e_n(x) \geq 0$ holds for every $x \in \Omega_n$, $n \geq 1$.*
(c) $\mathrm{cap}(o, \Omega_n) \to \mathrm{cap}(o)$ *and $e_n \to e$ pointwise, as $n \to \infty$.*
(d) *If Q is recurrent, then* $\mathrm{cap}(x) = 0$ *for any $x \in V$.*

Proof. (a) Let $\| \cdot \|_o$ denote the norm induced by $\langle \cdot, \cdot \rangle_o$. Then, e and e_n are minimizers of the functional $u \mapsto \|u\|_o^2 - 1$ on the sets $\{u \in \overline{D(Q)}^{\|\cdot\|_o} \mid u(o) \geq 1\}$ and $\{u \in D(Q) \mid u(o) \geq 1, u|_{V \setminus G}\}$ respectively. The existence and uniqueness of e and e_n follow from the closedness and the convexity of these two sets (note that pointwise evaluation of functions is continuous w.r.t. $\|\cdot\|_o$) and standard Hilbert space theory. The furthermore statement follows from the fact that $\|(u \wedge 1) \vee 0\|_o \leq \|u\|_o$ for each $u \in \widetilde{D}$.

(b) Let $x \in \Omega_n$ be given and δ_x be the function which is $\frac{1}{m(x)}$ at x and 0 elsewhere. For each $\varepsilon > 0$, we obtain

$$Q(e_n) \leq Q(e_n + \varepsilon\delta_x) = Q(e_n) + \varepsilon^2 Q(\delta_x) + 2\varepsilon Q(e_n, \delta_x),$$

which together with Lemma 6.2 implies

$$\widetilde{L}e_n(x) = \langle \widetilde{L}e_n, \delta_x \rangle = Q(e_n, \delta_x) \geq 0.$$

(c) Obviously, we have $\mathrm{cap}(o, \Omega_n) \geq \mathrm{cap}(o)$. We next show the inequality $\limsup_{n\to\infty} \mathrm{cap}(o, \Omega_n) \leq \mathrm{cap}(o)$. Choose a sequence of finitely supported functions (φ_k) with

$$\|\varphi_k - e\|_o \to 0, \text{ as } k \to \infty.$$

This convergence implies pointwise convergence. Thus, we infer $\varphi_k(o) \to 1$. Since (Ω_n) is an exhausting sequence and each φ_k has finite support, we obtain

$$\limsup_{n\to\infty} \mathrm{cap}(o, \Omega_n) \leq Q\left(\frac{1}{\varphi_k(o)}\varphi_k\right) = \frac{1}{\varphi_k(o)^2} Q(\varphi_k) \to \widetilde{Q}(e) = \mathrm{cap}(o), \qquad \text{as } k \to \infty.$$

This gives the desired inequality. It remains to show the statement on pointwise convergence. Using the parallelogram identity and the convexity of $\{u \in \overline{D(Q)}^{\|\cdot\|_o} \mid u(o) \geq 1\}$, we obtain

$$\left\|\frac{e_n - e}{2}\right\|_o^2 \leq \frac{1}{2}\|e_n\|_o^2 + \frac{1}{2}\|e\|_o^2 - \mathrm{cap}(o) - 1 \to 0, \qquad \text{as } n \to \infty.$$

(d) This is a consequence of [35, Theorem 2.12] and the well known fact that the notion of recurrence given in [35] coincides with the one for Dirichlet forms used above, c.f. [31] for details. This finishes the proof. □

Now, we are in a position to prove the theorem.

Proof of Theorem 6.12. (i) $\Longrightarrow$ (ii): Let e_n and e be as in Lemma 6.14. Without loss of generality let $0 \leq u(x) \leq 1$ for every $x \in V$. Set $|\nabla u|^2(x) := \sum_{y \in V} b(x, y)\,(u(x) - u(y))^2$. Since e_n has finite support, Lemma 6.2 implies

$$\sum_{x \in V} e_n^2(x)\widetilde{L}u(x)m(x) = \sum_{x \in V} \widetilde{L}e_n^2(x)u(x)m(x).$$

Furthermore, the equation

$$(e_n^2(x) - e_n^2(y))u(x) = -u(x)(e_n(x) - e_n(y))^2 + 2e_n(x)u(x)(e_n(x) - e_n(y))$$

and the definition of $\widetilde{L}$ yields

$$\sum_{x\in V} \widetilde{L}e_n^2(x)u(x)m(x) = -\sum_{x\in V} |\nabla e_n(x)|^2 u(x) + 2\sum_{x\in\Omega_n} e_n(x)\widetilde{L}e_n(x)u(x)m(x).$$

By the assumption the form Q is recurrent. Thus, by Lemma 6.14 (d) $\operatorname{cap}(o) = 0$ and, therefore, $e \equiv 1$ holds. From the pointwise convergence of the e_n to e we then obtain $e_n(x) \to 1$ for all $x \in V$. Since the e_n are uniformly bounded, we infer from Lebesgue's Theorem that

$$\sum_{x\in V} e_n^2(x)\widetilde{L}u(x)m(x) \to \sum_{x\in V} \widetilde{L}u(x)m(x), \text{ as } n \to \infty.$$

Lemma 6.14 (c) and $0 \le u \le 1$ implies

$$\sum_{x\in V} |\nabla e_n(x)|^2 u(x) \le \sum_{x\in V} |\nabla e_n(x)|^2 = 2Q(e_n) \to 0, \text{ as } n \to \infty.$$

Altogether, these considerations yield

$$\sum_{x\in V} \widetilde{L}u(x)m(x) = \lim_{n\to\infty} 2\sum_{x\in\Omega_n} e_n(x)\widetilde{L}e_n(x)u(x)m(x).$$

Since e_n is superharmonic on Ω_n and u is positive, we obtain

$$\sum_{x\in V} \widetilde{L}u(x)m(x) \ge 0.$$

The same argumentation may be repeated with $1 - u$ in place of u, and we arrive at the conclusion.

(ii) $\Longrightarrow$ (i): It follows from Theorem 6.10 that recurrence is independent of the underlying measure m. Hence, we may assume $m(V) < \infty$. Now, assume Q is transient. From Theorem 5.20 we infer $Q^{\max} \neq Q$. Theorem 5.13 and the characterization of $\mathcal{L}$ in the graph case imply the existence of a nontrivial function $u \in \widetilde{D} \cap \ell^1(V,m) \cap \ell^\infty(V)$ such that $\widetilde{L}u \in \ell^2(V,m)$ with $\widetilde{L}u \le 0$ and $\widetilde{L}u \neq 0$. As m is a finite measure $\ell^2(V,m) \subseteq \ell^1(V,m)$ holds and, therefore, $\widetilde{L}u \in \ell^1(V,m)$. From the choice of u, we then infer

$$\sum_{x\in V} \widetilde{L}u(x)m(x) < 0,$$

which shows the claim. □

7. Application to weighted manifolds

In this section we apply the theory of Section 5 to the standard Dirichlet energy on a weighted manifold. This is exactly the situation that was studied in [16]. As this setting is standard we only give a brief introduction and refer the reader to [14] for a detailed discussion of the relevant analysis on manifolds.

Let (M, g) be a connected smooth Riemannian manifold and let Φ be a strictly positive smooth function on M. The triplet (M, g, m), where $m = \Phi d\mathrm{vol}_g$ and $d\mathrm{vol}_g$ is the Riemannian measure, is called a *weighted manifold*. Such a manifold is a metric measure space which has the same distance and shape as the underlying Riemannian manifold but its measure is arbitrary. A weighted manifold carries a second-order elliptic operator, called the *weighted Laplacian*, defined as

$$\Delta_\Phi u = \frac{1}{\Phi}\mathrm{div}(\Phi \nabla u),$$

where the involved operators ∇ and div are understood in the distributional sense. The energy form formally corresponding to this operator is given by

$$Q(u,v) = \int_M g(\nabla u, \nabla v) dm.$$

It is well known that $(Q, C_c^\infty(M))$ is closable on $L^2(M,m)$ and its closure, which will be denoted by $(Q, W_0^1(M,m))$, is a regular Dirichlet form.

Furthermore, we will need the space

$$W^1(M,m) = \{u \in L^2(M,m) \,|\, |\nabla u| \in L^2(M,m)\},$$

where $|X| = g(X,X)^{\frac{1}{2}}$.

7.1. **Functions of finite energy.** We determine the space of functions of finite energy $\widetilde{D}(Q)$ and the formal operator $\mathcal{L}$ associated with $(Q, W_0^1(M,m))$. We will need the definitions given in Section 5.1.

Proposition 7.1 (Reflected Dirichlet space and Beppo-Levi functions)**.** *The reflected Dirichlet space of $(Q, W_0^1(M,m))$ is given by the space of Beppo-Levi functions*

$$\mathrm{BL}(M,m) = \{u \in L^2_{\mathrm{loc}}(M) \,|\, |\nabla u| \in L^2(M,m)\}.$$

On this space the extended form $\widetilde{Q}$ acts as

$$\widetilde{Q}(u,v) = \int_M g(\nabla u, \nabla v)\, dm.$$

In particular, the Dirichlet form $Q^{\max}$ is the restriction of $\widetilde{Q}$ to the space $W^1(M,m)$. The domain of the operator $\mathcal{L}$ is given by

$$D(\mathcal{L}) = \{u \in \mathrm{BL}(M,m) \,|\, \Delta_\Phi u \in L^1_{\mathrm{loc}}(M)\}$$

on which it acts as $\mathcal{L}u = -\Delta_\Phi u$.

Proof. $\mathrm{BL}(M,m) \subseteq \widetilde{D}(Q)$: Let $u \in \mathrm{BL}(M,m)$ be bounded and let a smooth function with compact support φ be given. It is well known that $u \cdot \varphi \in W_0^1(M,m)$ (see e.g. [14, Corollary 5.6]), which gives that u agrees locally with a function from $W_0^1(M,m)$. Now, the remaining statements are a rather direct consequence of the definitions and the Markov property of the weighted energy on $\mathrm{BL}(M,m)$ (compare proof of Theorem 5.4 for a similar reasoning).

$\widetilde{D}(Q) \subseteq \mathrm{BL}(M,m)$: Let $u \in \widetilde{D}(Q)$ and a relatively compact, open subset $G \subseteq M$ be given. As the positive and negative part of u belong to $\widetilde{D}(Q)$, we may assume $u \geq 0$. Recall that we set $u^{(n)} = (u \wedge n) \vee (-n)$. By definition $u^{(n)} \in \mathrm{BL}(M,m)$ for each $n \in \mathbb{N}$ and $\sup_n Q(u^{(n)}) < \infty$. Furthermore, as G is relatively compact, a Poincaré-type inequality holds on G, that is, there is a constant $C > 0$ (depending on G) such that

$$\int_G |v - \overline{v}_G|^2\, dm \leq C \int_M g(\nabla v, \nabla v)\, dm$$

for each $v \in \mathrm{BL}(M,m)$. Here, $\overline{v}_G = m(G)^{-1} \int_G v\, dm$. This rough Poincaré inequality holds since on compact sets the Ricci curvature is bounded from below and Φ^{-1} is locally bounded.

We define $f := |u-\overline{u}_G|^2$ and $f_n = |u^{(n)} - \overline{u^{(n)}}_G|^2$. The function u is positive and almost surely finite. Hence, f is well defined. Furthermore, f is almost surely finite if and only if $\overline{u}_G < \infty$. We deduce from monotone convergence and the positivity of u that $\overline{u}_G = \lim_n \overline{u^{(n)}}_G$. Using this observation and applying the Poincaré inequality, we obtain with the help of Fatou's lemma

$$\int_G f\,dm \leq \liminf_{n\to\infty} \int_G f_n\,dm \leq C \liminf_{n\to\infty} Q(u^{(n)}) < \infty.$$

This implies $\overline{u}_G < \infty$ and $u \in L^2(G)$. As G was arbitrary we obtain $u \in L^2_{\text{loc}}(M)$. Furthermore, $\nabla u^{(n)}$ is a bounded sequence in the Hilbert space $\vec{L}^2(M,m)$, i.e., the space of L^2 vector fields. Thus, it possesses a weakly convergent subsequence with weak limit $V \in \vec{L}^2(M,m)$. Choose a smooth vector field X with compact support. We obtain

$$\begin{aligned}\int_M g(V,X)\,dm &= \lim_{k\to\infty}\int_M g(\nabla u^{(n_k)}, X)\,dm \\ &= -\lim_{k\to\infty}\int_M u^{(n_k)}\frac{1}{\Phi}\text{div}(\Phi X)\,dm \\ &= -\int_M u\frac{1}{\Phi}\text{div}(\Phi X)\,dm.\end{aligned}$$

As X was arbitrary, we obtain $\nabla u = V \in \vec{L}^2(M,m)$ which show the claim.

The statement about the action of Q follows from the definition of the extension of Q to $\widetilde{D}(Q)$. Furthermore, the statement on $\mathcal{L}$ follows from the definition of Δ_Φ via distributions. □

Remark 7.2. For Brownian motion on an open subset of $\mathbb{R}^n$ the statement about the reflected Dirichlet space is one of the examples of Section 6.5 in [4]. For general Riemannian manifolds the statement seems to be new.

7.2. **Extensions of Dirichlet forms.** Theorem 5.13 now reads in the manifold setting.

Theorem 7.3. *The following assertions are equivalent.*

(i) $W^1(M,m) \neq W^1_0(M,m)$.
(ii) *There exists a nontrivial* $u \in W^1(M,m) \cap L^1(M,m) \cap L^\infty(M,m)$ *such that* $\Delta_\Phi u \in L^2(M,m)$ *with* $u \geq 0$, $\Delta_\phi u \geq 0$ *and* $\Delta_\Phi u \neq 0$.

Next, we come to the application of Theorem 5.14.

Theorem 7.4. *Let L be the self-adjoint operator associated with $(Q, W^1_0(M,m))$. Then, the following assertions are equivalent.*

(i) $W^1(M,m) = W^1_0(M,m)$.
(ii) $D(L) = \{u \in W^1(M,m) \mid \Delta_\Phi u \in L^2(M,m)\}$.
(iii) *For all* $u \in W^1(M,m)$ *such that* $\Delta_\Phi \in L^2(M,m)$ *and all* $v \in W^1(M,m)$ *we have*

$$\widetilde{Q}(u,v) = -\int_M \Delta_\Phi uv\,dm.$$

7.3. **Stochastic completeness.** The following is the application of Theorem 5.17 to the manifold setting.

Theorem 7.5. *The following assertions are equivalent:*

(i) $(Q, W^1_0(M,m))$ *is stochastically complete.*

(ii) *For all* $u \in L^1(M,m) \cap L^2(M,m)$ *such that* $|\nabla u| \in L^2(M,m)$ *and* $\Delta_\Phi \in L^1(M,m) \cap L^2(M,m)$ *the following equality holds*

$$\int_M \Delta_\Phi u \, dm = 0.$$

Remark 7.6. The previous theorem is basically the same as of [16, Theorem 1.2] after one realizes that stochastic completeness implies $W^1(M,m) = W_0^1(M,m)$ (which follows e.g. from the implication (ii)$\Longrightarrow$ (iii) in Theorem 5.20).

7.4. **Recurrence.** In the manifold setting Theorem 5.19 becomes:

Theorem 7.7. *The following assertions are equivalent.*

(i) $(Q, W_0^1(M,m))$ *is recurrent.*

(ii) *For each* $u \in \mathrm{BL}(M,m)$ *such that* $\Delta_\Phi u \in L^1(M,m)$ *the following equality holds*

$$\int_M \Delta_\Phi u \, dm = 0.$$

(iii) *For each* $u \in \mathrm{BL}(M,m)$ *such that* $\Delta_\Phi u \in L^1(M,m)$ *and each* $v \in \mathrm{BL}(M,m) \cap L^\infty(M,m)$ *the following equality holds*

$$\widetilde{Q}(u,v) = -\int_M \Delta_\Phi u v \, dm.$$

Remark 7.8. The previous theorem is an analogue to [16, Theorem 1.1]. There the equivalence of (i) and (ii) is also proven but, in contrast to our result, with the Beppo-Levi functions replaced by L^∞ functions. However, the advantage of using Beppo-Levi functions is that the very general Green type formula (iii) holds for them in the recurrent case.

8. Application to metric graphs

In this section we consider metric graphs and discuss applications of the abstract results of the previous sections. Metric graphs are in some sense a hybrid model between manifolds and discrete graphs and fit into the framework of regular Dirichlet forms. Most of the material presented here is based on the thesis [17] of one of the authors.

The basic idea of a metric graph is to view edges as intervals which are glued together according a graph structure.

Let l be a locally finite graph over a discrete countable vertex set V, i.e., $l : V \times V \to [0,\infty)$ is symmetric, has zero diagonal and $l(x,\cdot)$ vanishes for all but finitely many vertices. We define the set of combinatorial edges E to be the equivalence classes of $\{(x,y) \in V \times V \mid l(x,y) > 0\}$ under the equivalence relation that relates (x,y) and (y,x) for all $x,y \in V$. By symmetry of l the map l is well defined on E.

For $e \in E$, we define the *continuum edge* $\mathcal{X}_e = (0, l(e)) \times \{e\}$ and the *metric graph* to be the set

$$\mathcal{X}_\Gamma = V \cup \bigcup_{e \in E} \mathcal{X}_e.$$

Next, we equip $\mathcal{X}_\Gamma$ with a topology defined in terms of a certain subspace of continuous functions. We define an orientation on the combinatorial edges which is a map $E \to \{(x,y) \in V \times V \mid l(x,y) > 0\}$, $e \mapsto (\partial^+(e), \partial^-(e))$. We call $\partial^+(e)$ the *initial vertex* and the $\partial^-(e)$ *terminal vertex* of e. If x is the initial or terminal vertex of an edge e we say x and e are *adjacent* and we write $x \sim e$.

For a vertex x adjacent to an edge e and variables $t \in (0, l(e))$, we interpret $t \to x$ as $t \to 0$ if $x = \partial^+(e)$ and as $t \to l(e)$ if $x = \partial^-(e)$.

For a function $f : \mathcal{X}_\Gamma \to \mathbb{R}$, we denote by f_e the restriction of f to $\mathcal{X}_e$ which is essentially a function on $(0, l(e))$. The continuous functions $C(\mathcal{X}_\Gamma)$ are the functions $f : \mathcal{X}_\Gamma \to \mathbb{R}$ such that f_e is continuous for all $e \in E$ and for all $x \in V$ we have

$$f(x) = \lim_{t \to x} f_e(t)$$

for all $e \in E$ adjacent to x. The space $C(\mathcal{X}_\Gamma)$ gives rise to a topology on $\mathcal{X}_\Gamma$ such that $\mathcal{X}_\Gamma$ becomes a locally compact Hausdorff space. We denote by $C_c(\mathcal{X}_\Gamma)$ the subspace of continuous functions of compact support. For a function f such that f_e are weakly differentiable for all $e \in E$, we write $f' = (f'_e)_{e \in E}$ and $f'' = (f''_e)_{e \in E}$ similarly.

We introduce Lebesgue spaces $L^p(\mathcal{X}_\Gamma) = \bigoplus_{e \in E} L^p(0, l(e))$, $p \in \{1, 2\}$, where we neglect the vertices since points have Lebesgue measure zero. We denote the space of functions locally in $L^p(\mathcal{X}_\Gamma)$, $p \in \{1, 2\}$, by $L^p_{\mathrm{loc}}(\mathcal{X}_\Gamma)$. This space is not a direct sum since locally here means with respect to the topology of $\mathcal{X}_\Gamma$. For a more detailed description of the set up, we refer to [17, Chapter 1].

Now, let b be another locally finite graph over V such as in Section 6. Note that the combinatorial structures of l and b are completely independent. We introduce the quadratic form

$$\widetilde{Q}_{l,b}(u) = \sum_{e \in E} \int_0^{l(e)} |u'_e(t)|^2 dt + \frac{1}{2} \sum_{x,y \in V} b(x, y)(u(x) - u(y))^2$$

on the space which will turn out to be the space of functions of finite energy

$$\widetilde{D} = \{u \in C(\mathcal{X}_\Gamma) \mid u' \in L^2(\mathcal{X}_\Gamma),\, \widetilde{Q}_{l,b}(u) < \infty\}.$$

On this space $\widetilde{Q}_{l,b}$ has the Markov property, i.e., for each $u \in \widetilde{D}$ and each normal contraction $C : \mathbb{R} \to \mathbb{R}$ we have that $C \circ u \in \widetilde{D}$ and $\widetilde{Q}_{l,b}(C \circ u) \le \widetilde{Q}_{l,b}(u)$. We define Q to be the restriction of $\widetilde{Q}_{l,b}$ to

$$D(Q) = \overline{\widetilde{D} \cap C_c(\mathcal{X}_\Gamma)}^{\|\cdot\|_{\widetilde{Q}_{l,b}}},$$

where $\|\cdot\|^2_{\widetilde{Q}_{l,b}} = \|\cdot\|^2 + \widetilde{Q}_{l,b}(\cdot)$. It can be checked that Q is a regular Dirichlet form, for details see [17, Chapter 1, Section 3].

8.1. **Functions of finite energy.** We will deal with the functions of finite energy $\widetilde{D}(Q)$ arising from a regular Dirichlet form Q as introduced in Section 5.1.

Theorem 8.1. *Let Q be the regular Dirichlet form defined on a metric graph as above. Then, $\widetilde{D} = \widetilde{D}(Q)$ and $\widetilde{Q}$ on $\widetilde{D}(Q)$ is given by $\widetilde{Q}_{l,b}$.*

In order to prove the preceding theorem we need to characterize convergence in $\widetilde{D}$ which turns out to be a Hilbert space with respect to the norm $\|\cdot\|_o := (|u(o)|^2 + \widetilde{Q}_{l,b}(u))^{1/2}$ for arbitrary $o \in \mathcal{X}_\Gamma$ whenever the combinatorial graph $l + b$ over V is connected.

Lemma 8.2. *Let the graph $l + b$ be connected. The space $\widetilde{D}$ equipped with $\|\cdot\|_o$ for $o \in \mathcal{X}_\Gamma$ is a Hilbert space. A sequence (u_n) in $\widetilde{D}$ converges to $u \in \widetilde{D}$ with respect to $\|\cdot\|_o$ if and only if it converges pointwise to u and* $\limsup_n \widetilde{Q}_{l,b}(u_n) \le \widetilde{Q}_{l,b}(u)$.

Proof. Note that convergence with respect to the $\|\cdot\|_o$ implies pointwise convergence. This is due to a one-dimensional Sobolev embedding which can be deduced as follows. For an arbitrary $x \in \mathcal{X}_\Gamma$, let γ be a path connecting x and o (which consists of a mix of continuous paths along edges with respect to l and combinatorial paths along edges of b). Then, a combination of the fundamental theorem of calculus along the continuous parts of γ and a summation along the combinatorial parts of γ and Cauchy-Schwarz inequality lead to

$$|u(x)| \le C(\gamma)\widetilde{Q}_{l,b}(u)^{1/2},$$

for arbitrary $u \in \widetilde{D}$, with a constant $C(\gamma)$ independent of u. For more details we refer the reader to [17, Chapter 1, Section 2].

Now, the pointwise limit has finite energy by standard Fatou type arguments. Hence, $\widetilde{D}$ is a Hilbert space with norm $\|\cdot\|_o$.
Next, consider a sequence (u_n) converging to u with respect to $\|\cdot\|_o$. As mentioned already, this implies pointwise convergence of u_n to u and, clearly, $\widetilde{Q}_{l,b}(u_n) \to \widetilde{Q}_{l,b}(u)$ holds as well. For the other direction let $u_n \in \widetilde{D}$, $n \ge 0$, be a sequence as stated. In Hilbert spaces bounded sets are weakly compact. Since (u_n) is a bounded sequence, there is a weakly convergent subsequence. This weak limit u has to agree with the pointwise limit and we have $u_n \to u$ weakly. Finally, we arrive at

$$0 \le \widetilde{Q}_{l,b}(u-u_n) + (u(o)-u_n(o))^2 \le \widetilde{Q}_{l,b}(u) + u(o)^2 + \widetilde{Q}_{l,b}(u_n) + u_n(o)^2 - 2\langle u, u_n\rangle_o$$

which yields $u_n \to u$ in $\widetilde{D}$. □

Proof of Theorem 8.1. We will only prove the statement in the case when the graph $l+b$ is connected. The general case follows by considering connected components.

Recall that $\{u \in D(Q)_{\mathrm{loc}} \cap L^\infty(X,m) \mid Q(u) < \infty\} = \widetilde{D} \cap L^\infty(X,m)$ and that the extension of Q to $D(Q)_{\mathrm{loc}}$ and $\widetilde{Q}_{l,b}$ agree on this set. Let $u \in \widetilde{D}$ be given. As discussed, we obtain $u^{(n)} = (u \wedge n) \vee (-n) \in D(Q)_{\mathrm{loc}}$ and by the Markov property of $\widetilde{Q}_{l,b}$ we have $Q(u^{(n)}) = \widetilde{Q}_{l,b}(u^{(n)}) \le \widetilde{Q}_{l,b}(u)$ for each n. Therefore, $u \in \widetilde{D}(Q)$ holds.

On the other hand, for $u \in \widetilde{D}(Q)$, we conclude $u^{(n)} \in \widetilde{D}$ and $\widetilde{Q}_{l,b}(u^{(n)}) = \widetilde{Q}(u^{(n)}) \le \widetilde{Q}(u)$. Therefore, $(u^{(n)})$ is a bounded sequence in the Hilbert space $(\widetilde{Q}_{l,b}, \widetilde{D})$. Thus, it has a weakly convergent subsequence. By the pointwise convergence of $u^{(n)}$ towards u this limit must coincide with u, showing $u \in \widetilde{D}$. Since $\widetilde{Q}_{l,b}$ has the Markov property, we obtain $\widetilde{Q}_{l,b}(u^{(n)}) \le \widetilde{Q}_{l,b}(u)$. Now, the previous lemma implies

$$\widetilde{Q}_{l,b}(u) = \lim_{n\to\infty} \widetilde{Q}_{l,b}((u \wedge n) \vee (-n)) = \lim_{n\to\infty} Q((u \wedge n) \vee (-n)) = \widetilde{Q}(u),$$

where the last equality follows from the definition of $\widetilde{Q}$. This finishes the proof. □

We now turn to the associated operators. We denote by $\widetilde{F}$ the space from Section 6 for the graph b over V. Similarly, we let $\widetilde{L}$ be the generalized Laplacian from Section 6 for the graph b and the counting measure $m \equiv 1$.

For u such that $u_e'' \in L^2(0, l(e))$, $e \in E$, the derivatives $u_e'(\partial^+(e))$ and $u_e'(\partial^-(e))$ exist for all $e \in E$ and we define normal derivative in a vertex $x \in V$ by

$$\partial_n u(x) = \sum_{e \sim x} \sum_{\partial^+(e)=x} u_e'(0) - \sum_{\partial^-(e)=x} u_e'(l(e)).$$

We say $u \in W^{1,2}_{\text{loc}}(\mathcal{X}_\Gamma)$ satisfies the Kirchoff conditions if $u \in C(\mathcal{X}_\Gamma)$, $u|_V \in \widetilde{F}$ and

$$\text{(KC)} \qquad \partial_n u(x) = \widetilde{L}u(x), \qquad x \in V.$$

We will need the operator $\mathcal{L}$ defined at the end of Section 5.1.

Theorem 8.3. *The operator $\mathcal{L}$ acts as*

$$(\mathcal{L}u)_e = -u_e'', \qquad e \in E,$$

on the domain

$$D(\mathcal{L}) = \{u \in C(\mathcal{X}_\Gamma) \mid u' \in L^2(\mathcal{X}_\Gamma), u'' \in L^1_{\text{loc}}(\mathcal{X}_\Gamma), u \textit{ satisfies (KC)}\}.$$

Proof. The domain $D(\mathcal{L})$ is given as

$$\{u \in \widetilde{D}(Q) \mid \text{ there exists } f \in L^1_{\text{loc}}(\mathcal{X}_\Gamma) \text{ s.t. } \widetilde{Q}(u,v) = \langle f, v\rangle \text{for all } v \in D(Q) \cap C_c(\mathcal{X}_\Gamma)\}.$$

For u in this domain, we get from Theorem 8.1 $u \in \widetilde{D}(Q) = \widetilde{D}$ which implies $u' \in L^2(\mathcal{X}_\Gamma)$. Furthermore, by testing with functions v supported within the edges, we get $f = u'' \in L^1_{\text{loc}}(\mathcal{X}_\Gamma)$. Finally, we see using integration by parts and Lemma 6.2,

$$\langle -u'', v\rangle = \widetilde{Q}(u,v) = \langle -\widetilde{\partial}_n u, v\rangle - \langle u'', v\rangle + \langle \widetilde{L}u, v\rangle.$$

Hence, u satisfies (KC). The other inclusion follows by similar considerations. □

Remark 8.4. Let us note that the values of b do not play a role in the action of $\mathcal{L}$. They only enter when it comes to domains.

8.2. **Extensions of Dirichlet forms.** By Theorem 8.1 the form $Q^{\max}$ introduced in Section 5.2 is a restriction of $\widetilde{Q}$ to

$$D(Q^{\max}) = \widetilde{D} \cap L^2(\mathcal{X}_\Gamma).$$

The following is an application of Theorem 5.13.

Theorem 8.5. *The following assertions are equivalent.*

(i) $Q \neq Q^{\max}$.

(ii) *There exists a nontrivial $u \in D(\mathcal{L}) \cap L^1(\mathcal{X}_\Gamma) \cap L^2(\mathcal{X}_\Gamma)$ such that $u'' \in L^2(\mathcal{X}_\Gamma)$ with $u \geq 0$, $u'' \leq 0$ and $u'' \neq 0$.*

The following is an application of Theorem 5.14.

Theorem 8.6. *The following assertions are equivalent.*

(i) $Q = Q^{\max}$.

(ii) $D(L) = \{u \in D(\mathcal{L}) \cap L^2(\mathcal{X}_\Gamma) \mid u'' \in L^2(\mathcal{X}_\Gamma)\}$.

(iii) *For all $u \in D(\mathcal{L}) \cap L^2(\mathcal{X}_\Gamma)$ such that $u'' \in L^2(\mathcal{X}_\Gamma)$ and all $v \in \widetilde{D}(Q) \cap L^2(X,m)$*

$$\widetilde{Q}(u,v) = -\sum_{e \in E} \int_0^{l(e)} u_e''(t) v(t)\, dt.$$

8.3. **Stochastic completeness.** Next, we come to the application of Theorem 5.17.

Theorem 8.7. *The following assertions are equivalent.*

(i) *Q is stochastically complete.*

(ii) *For all $u \in D(\mathcal{L}) \cap L^1(\mathcal{X}_\Gamma) \cap L^2(\mathcal{X}_\Gamma)$ and $u'' \in L^1(\mathcal{X}_\Gamma) \cap L^2(\mathcal{X}_\Gamma)$ we have*

$$\sum_{e \in E} \int_0^{l(e)} u_e''(t)\, dt = 0.$$

8.4. **Recurrence.** In Section 6 we discussed what it means that a graph is connected. The form Q has two underlying graphs. The graph l giving rise to $\mathcal{X}_\Gamma$ and the graph b giving rise to the jumping part of Q.

The Dirichlet form Q is irreducible if the graph $l+b$ is connected. This follows by the same argument as in Lemma 6.9. With this at hand we can apply Theorem 5.19.

Theorem 8.8. *Assume the graph $l+b$ is connected. Then the following assertions are equivalent.*

(i) *Q is recurrent.*

(ii) *For all $u \in D(\mathcal{L})$ and $u'' \in L^1(\mathcal{X}_\Gamma)$ we have*

$$\sum_{e\in E} \int_0^{l(e)} u_e''(t)\, dt = 0.$$

(iii) *For all $u \in D(\mathcal{L})$ and $u'' \in L^1(\mathcal{X}_\Gamma)$ and for all $v \in \widetilde{D} \cap L^\infty(\mathcal{X}_\Gamma)$ we have*

$$\widetilde{Q}(u,v) = -\sum_{e\in E} \int_0^{l(e)} u_e''(t) v_e(t)\, dt.$$

9. Connection to volume growth criteria

The main thrust of our investigations is conceptual in nature. However, our results can also be used to (re)derive certain volume growth criteria - at least - in a radially symmetric setting. Details are discussed in this section. We treat the cases of manifolds, discrete graphs and quantum graphs.

Model manifolds A warp-product $M = [0,\infty) \times \mathbb{S}^n$ with the canonical coordinate (r,θ) is called a *model manifold* if the Riemannian metric has the form:

$$g = dr^2 + \sigma^2(r) d\theta^2,$$

where $\sigma = \sigma(r)$ is a smooth function of $r \geq 0$ such that $\sigma(r) > 0$ if $r > 0$, $\sigma(0) = 0$ and $\sigma'(0) = 1$, and $d\theta^2$ is the standard metric on the n-sphere $\mathbb{S}^n$. We set

$$r(x) = d(x, x_0),$$

where $x_0 = r^{-1}(0)$,

$$B(R) = \{x \in M \mid r(x) < R\}$$

for $R > 0$, and

$$V(r) = \int_0^r S(\tau)\, d\tau$$

with $S(r) = c_n \sigma^n(r)$, where c_n is the voulume of the n-sphere $\mathbb{S}^n$. We note that $V(R)$ equals the Riemannian volume of the ball $B(R)$.

With this notation, the celebrated volume growth criteria for model manifolds state the following (see [14] and references within):

(SI) $\int_1^\infty \frac{V}{S} dr < \infty$ implies that M is stochastically incomplete.

(R) $\int_1^\infty \frac{1}{S} dr < \infty$ implies that M is non recurrent.

Here, we say that M is stochastically incomplete if the associated regular Dirichlet form (cf. Section 7) is stochastically incomplete.

In order to derive these criteria from our main results we consider an arbitrary function $u \in C^\infty(M)$ which coincides on $M \setminus B(1)$ with

$$x \mapsto \int_{r(x)}^{\infty} \frac{dr}{S(r)}.$$

Since

$$\Delta u = \frac{1}{S}\frac{\partial}{\partial r}\left(S\frac{\partial u}{\partial r}\right) = 0, \quad \text{in } M \setminus B(1),$$

$\Delta u \in L^1 \cap L^2$ and

$$\int_M \Delta u \, d\mathrm{vol}_g = \int_{B(1)} \Delta u \, d\mathrm{vol}_g = -\int_{\partial B(1)} \frac{1}{S(1)} S(1) d\theta < 0.$$

After this short preliminary consideration we can quickly derive (SI) and (R):

As for (R) we now note that due to the volume growth assumption of M,

$$\int_M |\nabla u|^2 \, d\mathrm{vol}_g = \int_{B(1)} |\nabla u|^2 \, d\mathrm{vol}_g + \int_{\mathbb{S}^n} \int_1^\infty \left(\frac{1}{S(r)}\right)^2 S(r) \, dr d\theta < \infty,$$

which implies that M is non recurrent by our main result, Theorem 7.7.

As for (SI), we now note that due to the volume growth assumption of M and by the Fubini theorem, $u \in L^1(M)$ and $|\nabla u| \in L^2(M)$. As $u(x) \to 0$ as $r(x) \to \infty$, we have $u \in L^1(M) \cap L^2(M)$. By our main result, Theorem 7.5, M is then stochastically incomplete.

Remark 9.1. It is known that the opposite implications in Proposition hold true [14].

Graphs. Very similar considerations apply for graphs. We refrain from giving all the details but just present the framework and write down the corresponding quantities.

Let $(b, 0)$ be a connected weighted graph over V and let $m : V \to (0, \infty)$. By d we denote the shortest path metric of $(b, 0)$, i.e., for $x, y \in V$ we let $d(x, y)$ be the minimal number of edges in a path between x and y. We fix $x_0 \in V$ and consider the spheres and balls

$$S_r := S_r(x_0) := \{x \in V \mid d(x, x_0) = r\} \text{ and } B_r := B_r(x_0) := \bigcup_{i=0}^{r} S_r(x_0)$$

around x_0 of radius $r \in \mathbb{N}$. The outer and inner curvatures $\kappa_\pm : V \to [0, \infty)$ (with respect to x_0) are given by

$$\kappa_\pm(x) := \frac{1}{m(x)} \sum_{y \in S_{r\pm 1}} b(x, y) \text{ for } x \in S_r.$$

Here, we let $S_{-1} = \emptyset$. Following [26], we call a graph $(b, 0)$ with measure m *weakly spherically symmetric* (with respect to x_0) if the functions $\kappa_\pm$ are spherically symmetric, i.e., the value $\kappa_\pm(x)$ only depends on the distance of x to x_0. For a weakly spherically symmetric graph $(b, 0)$ with measure m, similar considerations as in the manifold case show the following for the associated regular Dirichlet form Q, cf. Section 6.

(SI) $\sum_{r=0}^\infty \frac{m(B_r)}{\kappa_+(r) m(S_r)} < \infty$ implies that Q is stochastically incomplete.

(R) $\sum_{r=0}^\infty \frac{1}{\kappa_+(r) m(S_r)} < \infty$ implies that Q is not recurrent.

Here, we wrote $\kappa_+(r)$ for $\kappa_+(x)$ with $x \in S_r$, which is independent of the concrete x. It is known that for the previous criterion also the converse implication holds true, see [26, Theorem 5] and [21, Proposition 6.1].

Quantum graphs Completely analogous considerations can be carried our for corresponding quantum graphs. We leave the details to the reader.

Remark 9.2. In these examples a key role is played by variants of the function

$$x \mapsto \int_{r(x)}^{\infty} \frac{dr}{S(r)}.$$

This is not a coincidence. Indeed, this function is the Green function up to a constant with pole at x_0.

Appendix A. Construction of the sequence (e_n) and a lemma on sequences

The subsequent two results are certainly known in one way or other. We include a proof in this appendix in order to keep the paper self-contained and as they may be useful for further references as well.

Proposition A.1. *Let Q be a Dirichlet form on a σ-finite space (X, m). Then, the following holds:*

(a) *There exists a sequence (e_n) in $D(Q)$ with $0 \le e_n \le 1$ and $e_n \to 1$ m-a.e.*
(b) *If Q is regular, the sequence (e_n) from (a) can be chosen in $D(Q) \cap C_c(X)$.*
(c) *If Q is regular and recurrent, then (e_n) from (b) can be chosen to satisfy $e_n \to 1$ in the sense of Q_e, i.e., $Q(e_n) \to 0$.*

Proof. (a) By our assumptions there exists an increasing sequence of sets of finite measure (B_k) such that $X = \bigcup_k B_k$. Because $D(Q)$ is dense in $L^2(X, m)$, we can choose $f_n \in D(Q)$ satisfying

$$\|f_n - \chi_{B_n}\|_2 \to 0, \text{ as } n \to \infty.$$

Let $e_n = (0 \vee f_n) \wedge 1$. Since Q is a Dirichlet form, we infer $e_n \in D(Q)$. Furthermore by construction we see $0 \le e_n \le 1$ and

$$\|e_n - \chi_{B_n}\|_2 \le \|f_n - \chi_{B_n}\|_2.$$

We want to show that (e_n) possesses a subsequence converging to 1 $m-$almost surely. For $k, n \in \mathbb{N}$ and $\delta > 0$ let

$$A_{k,n,\delta} = \{x \in B_k : |e_n(x) - 1| \ge \delta\}.$$

By the Markov-inequality we observe for $n \ge k$

$$m(A_{k,n,\delta}) \le \delta^{-2} \|(1 - e_n)\chi_{B_k}\|_2^2 \le \delta^{-2} \|\chi_{B_n} - e_n\|_2^2.$$

This allows us to choose a subsequence e_{n_l}, such that for any k

$$\sum_{l=1}^{\infty} m(A_{k,n_l,l^{-1}}) < \infty.$$

Let $N = \bigcup_k \bigcap_{j \geq 1} \bigcup_{l \geq j} A_{k,n_l,l^{-1}}$ and $x \in X \setminus N$. It is easily verified that $e_{n_l}(x) \to 1$ as $l \to \infty$. To prove (a), it remains to show $m(N) = 0$, which can be checked directly by computing

$$m(N) \leq \sum_k \lim_{j \to \infty} m\Big(\bigcup_{l \geq j} A_{k,n_l,l^{-1}}\Big) \leq \sum_k \limsup_{j \to \infty} \sum_{n \geq j} m(A_{k,n_l,l^{-1}}) = 0$$

(b) Because of the regularity of Q, we know that $C_c(X) \cap D(Q)$ is dense in $D(Q)$ (hence, in $L^2(X,m)$) with respect to $L^2(X,m)$ convergence. Thus, in the proof of (a) we can replace $f_n \in D(Q)$ by $g_n \in C_c(X) \cap D(Q)$ to obtain (b).

(c) By recurrence there exists a sequence $h_n \in D(Q)$ such that $0 \leq h_n \leq 1$, $h_n \to 1$ pointwise m-almost surely and

$$\lim_{n \to \infty} Q(h_n) = 0.$$

By regularity of Q we can choose $\tilde{e}_n \in C_c(X) \cap D(Q)$ satisfying

$$\|\tilde{e}_n - h_n\|_Q \to 0 \text{ as } n \to \infty.$$

Let $e_n = (0 \vee h_n) \wedge 1$ such that $0 \leq e_n \leq 1$. We will show, that e_n has a subsequence converging to 1 m-almost everywhere and

$$\lim_{n \to \infty} Q(e_n) = 0.$$

Let $A_{k,n,\delta}$ be sets defined as in the proof of (a). The first assertion follows as above, using

$$\begin{aligned} m(A_{k,n,\delta}) &\leq \delta^{-2} \|(e_n - 1)\chi_{B_k}\|_2^2 \\ &\leq \delta^{-2} \|(\tilde{e}_n - 1)\chi_{B_k}\|_2^2 \\ &\leq \delta^{-2} \left[\|(\tilde{e}_n - h_n)\chi_{B_k}\|_2 + \|(1 - h_n)\chi_{B_k}\|_2\right]^2 \\ &\leq \delta^{-2} \left[\|(\tilde{e}_n - h_n)\|_2 + \|(1 - h_n)\chi_{B_k}\|_2\right]^2. \end{aligned}$$

The second statement can be deduced by

$$Q(e_n)^{1/2} \leq Q(\tilde{e}_n)^{1/2} \leq Q(\tilde{e}_n - h_n)^{1/2} + Q(h_n)^{1/2} \leq \|\tilde{e}_n - h_n\|_Q + Q(h_n)^{1/2}.$$

This finishes the proof. $\square$

Lemma A.2. *Let $(a_{n,m})_{n,m \in \mathbb{N}}$ be a sequence of real numbers satisfying $a_{n+1,m} \geq a_{n,m}$ for each $n, m \in \mathbb{N}$. Then,*

$$\liminf_{n \to \infty} \liminf_{m \to \infty} a_{n,m} \leq \liminf_{m \to \infty} \liminf_{n \to \infty} a_{n,m}.$$

Proof. Suppose $\liminf_{m \to \infty} \liminf_{n \to \infty} a_{n,m} < \infty$. Let $\varepsilon > 0$ be arbitrary. Choose an increasing sequence of indices (m_l) such that

$$\liminf_{n \to \infty} a_{n,m_l} \leq \liminf_{m \to \infty} \liminf_{n \to \infty} a_{n,m} + \varepsilon$$

for each $l \geq 1$. This and the monotonicity in n imply

$$a_{n,m_l} \leq \liminf_{m \to \infty} \liminf_{n \to \infty} a_{n,m} + \varepsilon.$$

As $(a_{n,m_l})_{l \geq 1}$ is a particular subsequence of $(a_{n,m})_{m \geq 1}$, we infer

$$\liminf_{m \to \infty} a_{n,m} \leq \lim_{l \to \infty} \liminf_{n \to \infty} a_{n,m_l} + \varepsilon$$

which proves the claim. $\square$

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