A varifold formulation of mean curvature flow with Dirichlet or dynamic boundary conditions

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Abstract

We consider the sharp interface limit of the Allen-Cahn equation with Dirichlet or dynamic boundary conditions and give a varifold characterization of its limit which is formally a mean curvature flow with Dirichlet or dynamic boundary conditions. In order to show the existence of the limit, we apply the phase field method under the assumption that the discrepancy measure vanishes on the boundary. For this purpose, we extend the usual Brakke flow under these boundary conditions by the first variations for varifolds on the boundary.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bonuded set with smooth boundaries $\partial\Omega$. For a parameter $\sigma \in (0, \infty)$, let $\{M_t^\sigma\}_{t\geq 0}$ be a family of hypersurfaces in Ω such that $\partial M_t^\sigma \subset \partial\Omega$ and ∂M_t^σ is oriented. We consider the generalized solutions to mean curvature flow with the following boundary condition:

$$\begin{cases} \mathbf{v}^\sigma = \mathbf{H}^\sigma & \text{on } M_t^\sigma,\, t > 0, \\ \mathbf{v}_b^\sigma = \dfrac{\sigma}{\tan\theta}\mathbf{N}_b^\sigma & \text{on } \partial M_t^\sigma,\, t > 0, \end{cases} \tag{1.1}$$

where $\mathbf{v}^\sigma$ is the velocity vector of M_t^σ, $\mathbf{v}_b^\sigma$ and $\mathbf{N}_b^\sigma$ are the velocity vector of ∂M_t^σ and the unit normal vector of ∂M_t^σ on $\partial\Omega$, respectively, and θ is the contact angle formed by M_t^σ and $\mathbf{N}_b^\sigma$. Here $\mathbf{H}^\sigma$ denotes the mean curvature vector. The motivation for considering the σ-parametrized boundary condition (1.1) to formulate is that we are able to study the three boundary conditions: Dirichlet, dynamic, and Neumann boundary conditions by considering the case $\sigma = 0$, the case that $\sigma > 0$ is finite and the case $\sigma = \infty$, respectively. The dynamic boundary condition of various types has been studied from another point of view such as the theory of viscosity solutions or the semilinear elliptic problems (see, for instance, [3], [8], [11], [23], or [24]). Moreover, the motion by mean curvature of the graph or level-set with Dirichlet boundary conditions has been also investigated in [21] and [33].

Our goal in this paper is to consider the singular limit of the Allen-Cahn equation by applying the phase field method under the assuption that the discrepancy measure vanishes on the boundary $\partial\Omega$ and to formulate a Brakke flow with Dirichlet or dynamic boundary conditions reflecting (1.1). By a formal argument, we may interpret that the boundary conditon in (1.1) corresponds to Dirichlet, dynamic and right-angle Neumann boundary conditions as $\sigma \to 0$, σ is finite and positive, or $\sigma = \infty$, respectively. Motivated by this formal argument, we try to consider the formulation of a Brakke flow with Dirichlet or dynamic boundary conditions in the following way: first, we formulate the generalized solutions to the mean curvature flow (1.1) in the sense of Brakke and, secondly, we take the limit of σ to 0 or finite positive to obtain the definition of Dirichlet or dynamic boundary conditions, respectively in some weak sense. Note that this argument is not rigorous but just a formal argument and we will state the rigorous analysis for such limiting procedure in Section 4.

Here we briefly recall the mean curvature flow of closed hypersurfaces. We say that a family of hypersurfaces $\{M_t\}_{t\geq 0}$ in $\mathbb{R}^n$ moves by its mean curvature if the following equation holds:

$$\mathbf{v}(\cdot, t) = \mathbf{H}(\cdot, t), \quad \text{on } M_t,\, t > 0, \tag{1.2}$$

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where $v(\cdot, t)$ is the velocity vector of a hypersurface M_t and $H(\cdot, t)$ is the mean curvature vector of M_t. In this case, the hypersurface M_t evolves to minimize its area. The notion of mean curvature flow was proposed by Mullins [26] to describe the motion of grain boundaries. Generally, it is known that singularities such as cusps or vanishing may occur in finite time and the motion of M_t after the appearance of singuralities cannot be analyized as a classical solution. However, from the point of clarifying natural phenomena, we need to understand the properties of the motion even though the singularities happen.

Several generalized flows by mean curvaure have been introduced so far. One is the level set flow proposed by Otha, Jasnow and Kawasaki [28] or Osher and Sethian [27]; the latter introduced the level-set equation to study the motion by mean curvature numerically. Later, Chen, Giga, and Goto [7] and, independently, Evans and Spruck [10] rigorously introduced the generalized solutions to mean curvature flow in the viscosity sense and proved the existence and uniqueness of the viscosity solutions.

Another approach is a Brakke flow which Brakke [5] proposed by using geometric measure theory, especially the theory of varifolds. This approach made it possible to deal with the motion of hypersurfaces with a variety of singularities such as triple junctions. He proved the global-in-time existence of a Brakke flow in $\mathbb{R}^n$ with an approximation scheme and compactness-type theorems on varifolds if a general integral varifold defined on $\mathbb{R}^n$ is given as an initial data. One problem on Brakke's results in [5] was that the construction of the approximation scheme and the proof of existence theorem he obtained does not preclude the possibility of a trivial flow, for instance, the one that has a sudden loss of its mass for all time except an initial time. This preblem remained open for a long time. Fortunately, Kim and Tonegawa [20] recently have succeeded proving, for the first time, a global-in-time existence theorem of the nontrivial mean curvature flow of grain boundaries by reformulated and modified approximation scheme.

As a different view of a Brakke flow, by applying the phase field method via Allen-Cahn equations, Ilmanen [15] considered the nontrivial global-in-time solutions to a Brakke flow without boundaries. The phase field method is the method that we make an approximation of a hypersurface by the transition layer with a small width of an order $\varepsilon > 0$ and characterize it by considering the singular limit ($\varepsilon \downarrow 0$) of the transition layer. For the problem with a boundary condition, Mizuno and Tonegawa [22], for the first time, formulated mean curvature flow with right-angle Neumann boundary conditions in the sense of Brakke. They consider the singular limit of the Allen-Cahn equation when the concerned domain is strictly convex and bounded. Later, Kagaya [17] extended thier results to a non-convex bounded domain. However, as far as we know, the weak formulation of a Brakke flow with another boundary conditions has not been considered. In connection with the phase field method for the boundary problem, the authors of [19] studied the singular limit of the Allen-Cahn equation with right-angle Neumann boundary conditions on the convex domain and rigorously proved that the separating front moves by its mean curvature in the sense of viscosity solutions, not Brakke. Motivated by these works, we aim to consider the singular limit of the Allen-Cahn equations and formulate a Brakke flow with Dirichlet or dynamic boundary conditions. To attain this goal, as our first attempt, we study the singular limit under the assumption that the discrepancy measure vanishes on the boundary and characerize its limit.

For the problem how to define a Brakke flow with Dirichlet or dynamic boundary conditions, we need to obtain at least the following two sufficient conditions: Brakke's inequality and the condition that corresponds to Dirichlet or dynamic boundary conditions. First of all, Brakke's inequality is a weak formulation of mean curvature flow motivated by the transport equation for a family of smooth hypersurfaces with boundaries as follows; let $\{M_t\}_{t\in[0,\,\infty)}$ be a family of smooth hypersurfaces on Ω with a smooth boundary $\partial M_t \subset \partial\Omega$. If M_t moves by its mean curvature vector H, then, calculating the quantity that corresponds to the time derivative of the surface area, we can have

$$\frac{d}{dt}\int_{M_t} \phi \, d\mathcal{H}^{n-1} = \int_{M_t} \left(-\phi|\mathbf{H}|^2 + \nabla\phi \cdot \mathbf{H} + \partial_t \phi\right) \, d\mathcal{H}^{n-1} + \int_{\partial M_t} \phi \, \mathbf{v}_b \cdot \gamma \, d\mathcal{H}^{n-2} \tag{1.3}$$

for any test function ϕ and $t > 0$, where $\mathbf{v}_b$ is the velocity vector of ∂M_t on $\partial\Omega$ and γ is the unit conormal vector of ∂M_t (see Figure 2). Note that, in (1.3), one do not need the boundary conditions which M_t satisfies. Brakke originally considered the case that $\partial M_t = \emptyset$ and hence the Brakke's inequaltiy is firstly motivated by (1.3) without the last term.

As an analogy of this identity, considering the singular limit of the Allen-Cahn equations (1.10), we will define the following inequality as one of the conditions for a Brakke flow with Dirichlet or dynamic boundary conditions (see Section 3 for exact Brakke's inequality); let $\{V_t\}_{t\geq 0}$, α, and $\tilde{\mathbf{v}}_b$ be a family of $(n-1)$-varifolds on $\overline{\Omega} \subset \mathbb{R}^n$, a non-zero Radon measure on $\partial\Omega \times [0,\,\infty)$, and a vector-valued function on $\partial\Omega \times [0,\,\infty)$, respectively. Then we will define the motion of a Brakke flow for the triplet $(V_t,\, \alpha,\, \tilde{\mathbf{v}}_b)$ by the following inequality:

$$\int_{\Omega} \phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2} \int_{\Omega} (-\phi |\widetilde{\mathbf{H}}_V|^2 + \nabla\phi \cdot \widetilde{\mathbf{H}}_V + \partial_t \phi) \, d\|V_t\| \, dt - \frac{1}{\sigma} \int_{\partial\Omega \times [t_1, \, t_2]} \phi |\tilde{\mathbf{v}}_b|^2 \, d\alpha, \tag{1.4}$$

for any test function ϕ with some conditions, where $\|V_t\|$ is the mass measure of V_t, and $\widetilde{\mathbf{H}}_V$ is the modified generalized mean curvature vector (see Definition 3.1 or 3.4 in Section 3 for the details). Note that the inequality (1.4) is for the case of dynamic boundary conditions (in the case when σ is finite and positive). For the case of Dirichlet boundary conditions ($\sigma \to 0$), if we consider the singular limit of Allen-Cahn equations under some assumptions, especially the one (4.8) (see Subsection 4.1.1 of Section 4), then we may have that the second term of the right-hand side in (1.4) vanishes as $\sigma \to 0$. Accordingly, the inequality (1.4) is a natural condition for a formulation of a Brakke flow in the case of both Dirichlet and dynamic boundary conditions. In this analogy, we see that the mass $\|V_t\|$ of V_t, the measure α, and the function $\tilde{\mathbf{v}}_b$ roughly correspond to the measure $\mathcal{H}^{n-1}\lfloor_{M_t}$, the measure $(\sin\theta)\mathcal{H}^{n-2}\lfloor_{\partial M_t} \otimes \mathcal{L}^1_t$ where $\otimes$ is the product of measures which is exactly defined in Definition 2.2 of Section 2, and $\mathcal{L}^1_t$ is the 1D Lebesgue measure on $\mathbb{R}$. Here we refer to the work of Kasai and Tonegawa [20]; they proved the local-in-time regularity results for the varifold solutions in $\mathbb{R}^n$ satisfying the inequality similar to (1.4). Hence it makes sense to consider (1.4) as a weak formulation of mean curvature flow.

Secondly, we should determine the boundary motion of varifolds representing Dirichlet or dynamic boundary conditions. To do this, we first define two linear functionals, that is, *boundary functional* $\mathcal{S}_{\alpha, \, \mathbf{v}_b}$ on $(C_c(\partial\Omega \times [0, \, \infty)))^n$ for a Radon measure α on $\partial\Omega \times [0, \infty)$ and a vector-valued function $\mathbf{v}_b \in (L^2(\alpha))^n$. Roughly speaking, the total variations of these functionals are regarded as the L^2-norm of $\mathbf{v}_b$ with respect to α (see Definition 2.13 in Section 2). Then, as the boundary condition for varifolds, we, roughly speaking, define the absolute continuities by using the total variations of $\mathcal{S}_{\alpha, \, \mathbf{v}_b}$, the mass measures and the total variation measures for varifolds as follows:

- (Dirichlet boundary condition)

$$\|\mathcal{S}_{\alpha, \, \mathbf{v}_b}\| \ll \|V_t\| \otimes \mathcal{L}^1_t \quad \text{on } \partial\Omega \times [0, \, \infty). \tag{1.5}$$

- (Dynamic boundary condition for $\sigma = 1$)

$$\left\| \int_0^\infty \delta V_t \lfloor^T_{\partial\Omega} \, dt + \mathcal{S}_{\alpha, \, \mathbf{v}_b} \right\| \ll \|V_t\| \otimes \mathcal{L}^1_t \quad \text{on } \partial\Omega \times [0, \, \infty). \tag{1.6}$$

Here $\mathbf{H}^{\Omega}_V$ is the generalized mean curvature vector in $\Omega \times [0, \, \infty)$ (see Subsection 2 for the details), $\ll$ means the absolute continuity for measures, and $\delta V_t \lfloor^T_{\partial\Omega}$ is the tangential first variation restricted to $\partial\Omega$ (see Definition 2.12 for the details). We should note that the conditions (1.5) and (1.6) are natural as a result of considering the limit of σ-parametrized boundary condition (1.1) by taking $\sigma \downarrow 0$ or $\sigma \to 1$. However, since the assumtion named "Uniform upper bound for the solution of Allen-Cahn equations" (see Section 4 for the details) seems to be strong, we can actually obtain a stronger result than (1.5) (see Theorem 4.10 in Section 4). So far, we are not able to get rid of or relax this assumption because of the technical obstacle in deriving a priori estimates and the construction of the curves described in Remark 4.1 of Section 4.

Here is a formal explanation of these boundary conditions (1.5) and (1.6) if hypersurfaces M_t satisfying our definitions are sufficiently smooth; when we consider the characterization of the limit of the Allen-Cahn equation, we formally obtain the following correspondences:

$$\|V_t\| \approx \mathcal{H}^{n-1}\lfloor_{M_t}, \quad \alpha \approx (\sin\theta)\mathcal{H}^{n-2}\lfloor_{\partial M_t} \otimes \mathcal{L}^1_t, \quad \mathbf{v}_b \approx (\text{the velocity of } \partial M_t \text{ on } \partial\Omega). \tag{1.7}$$

In the case of Dirichlet boundary conditions, if we assume that $\|V_t\|(\partial\Omega) = 0$ for all $t > 0$ which means that the geometric interior of M_t, that is, $M_t \setminus \partial M_t$ does not exist on $\partial\Omega$ for all the time, then we obtain, from (1.5), $\|\mathcal{S}_{\alpha, \, \mathbf{v}_b}\|(\partial\Omega \times [0, \, \infty)) = 0$. Moreover, if we assume that the contact angle θ is not identically equal to zero, then we have that $\sin\theta$ is not identically equal to zero on $\partial\Omega$ for $t > 0$. Since we may regard the total variation of the functional $\mathcal{S}_{\alpha, \, \mathbf{v}_b}$ as the L^2-norm of $\mathbf{v}_b$ with respect to the measure $(\sin\theta)\mathcal{H}^{n-2}\lfloor_{\partial M_t} \otimes \mathcal{L}^1_t$, we have that $\mathbf{v}_b = 0$ in L^2 on $\partial\Omega \times [0, \, \infty)$. Hence it is natural to consider (1.5) as Dirichlet boundary conditions. Similary, in the case of dynamic boundary conditions, if we again assume that $\|V_t\|(\partial\Omega) = 0$ for all $t > 0$, then we obtain, from (1.6), $\| \int_0^\infty \delta V_t \lfloor^T_{\partial\Omega} \, dt + \mathcal{S}_{\alpha, \, \mathbf{v}_b}\|(\partial\Omega \times [0, \, \infty)) = 0$ (see Remark 6 in Section 3 for the details). Here, from the analogy between the classical and the measure theoretic first variation which are described later, we may regard the total variation of $\delta V_t \lfloor^T_{\partial\Omega}$ as the L^2-norm of γ^T with respect to $\mathcal{H}^{n-2}\lfloor_{\partial M_t} \otimes \mathcal{L}^1_t$, where γ is the outer unit conormal of ∂M_t and γ^T is the tangential projection of γ onto $\partial\Omega$. Therefore, from (1.7) and $\| \int_0^\infty \delta V_t \lfloor^T_{\partial\Omega} \, dt + \mathcal{S}_{\alpha, \mathbf{v}_b} \| \equiv 0$, we obtain $(\sin\theta)\mathbf{v}_b + \gamma^T = 0$ in $(L^2(\alpha))^n$ on $\partial\Omega$ and thus we conclude that $\mathbf{v}_b \cdot \mathbf{N}_b$ is equal to $(\tan\theta)^{-1}$ on $\partial\Omega$, where $\mathbf{N}_b$ is the outer unit normal vector of ∂M_t on $\partial\Omega$. Hence, it is reasonable to consider the condition (1.6) as dynamic boundary conditions.

These ideas of the formulation of the Brakke flow are considered as the generalization of the results by Mizuno and Tonegawa [22]. They proved that the associated varifold with the limit measure of μ_t^ε, which is defined later, and its first variation satisfies a proper absolute continuity on $\overline{\Omega}$, as a result of the singular limit of the Allen-Cahn equations with right-angle Neumann boundary conditions. Moreover, it is shown in their paper that the absolute continuity represents right-angle Neumann boundary conditions in a weak sense. The key idea is the analogy between the first variation for a hypersurface M and a varifold V with a locally bounded first variation δV in the following; let $\{\Psi_t^g\}$ be the one-parameter group of diffeomorphism generated by the vector fields $\mathbf{g} \in (C_c^\infty(\overline{\Omega}))^n$. Suppose that V has the locally bounded first variation. Then, from the definitions of the first variations, we have

$$\frac{d}{dt}\mathcal{H}^{n-1}(M_t)\Big|_{t=0} = \int_M \mathbf{g} \cdot (-\mathbf{H}) \, d\mathcal{H}^{n-1} + \int_{\partial M} \mathbf{g} \cdot \gamma \, d\mathcal{H}^{n-2}, \tag{1.8}$$

$$\delta V(\mathbf{g}) = \int_{\overline{\Omega}} \mathbf{g} \cdot \eta \, \frac{d\|\delta V\|_{ac}}{d\|V\|} \, d\|V\| + \int_{\overline{\Omega}} \mathbf{g} \cdot \eta \, d\|\delta V\|_{sing}, \tag{1.9}$$

where $M_t := \Psi_t^{\mathbf{g}}(M)$, and γ is the unit conormal vector of ∂M. Here η is a $\|\delta V\|$-measurable vector-valued function such that $|\eta| = 1$ $\|\delta V\|$-a.e. in $\overline{\Omega}$, and $\|\delta V\|_{ac}$ and $\|\delta V\|_{sing}$ are the absolute continuous and singular part of the measure $\|\delta V\|$ with respect to $\|V\|$, respectively. The existence of these quantity is derived by Riesz representation theorem and Radon-Nikodym theorem. The readers should refer to Remark 2.9 for details.

To show the existence of the singular limit for our Brakke flow with Dirichlet or dynamic boundary conditions, we will apply the phase field method as we mentioned before. In the phase field method, the singular limit of the following Allen-Cahn equations corresponding to the equations (1.1) should be studied:

$$\begin{cases} \partial_t u^{\varepsilon,\sigma} = \Delta u^{\varepsilon,\sigma} - \varepsilon^{-2} W'(u^{\varepsilon,\sigma}) & \text{in } \Omega \times (0, \infty), \\ \partial_t u^{\varepsilon,\sigma} + \sigma \nabla u^{\varepsilon,\sigma} \cdot \nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u^{\varepsilon,\sigma}(\cdot, 0) = u_0^{\varepsilon,\sigma}(\cdot) & \text{in } \Omega, \end{cases} \tag{1.10}$$

where $\varepsilon \in (0, 1)$, $\sigma \in (0, \infty)$, ν is the outer unit normal vector and $W(s) := \frac{1}{2}(1 - s^2)^2$ is the double-well potential. The Allen-Cahn equation without boundary conditions in (1.10) was first proposed by Allen and Cahn [1] in order to study the phase separation in alloys. They introduced the free energy functional

$$E[u^\varepsilon(\cdot, t)] = \int_\Omega \left(\frac{\varepsilon |\nabla u^\varepsilon(x, t)|^2}{2} + \frac{W(u^\varepsilon(x, t))}{\varepsilon} \right) dx \tag{1.11}$$

for an order parameter u^ε. The Allen-Cahn equation is a L^2-gradient flow of the energy functional (1.11) and, by considering this equation, Allen and Cahn also formally established the mean curvature flow (1.2) as the correct limiting law of motion for antiphase boundaries. Later, their analysis was justified rigorously by, for instance, Bronsard and Kohn [6]. They proved that the solution of the Allen-Cahn equation converges to a piecewise constant function whose surfaces of discontinuities move along (1.2). With these formal and rigorous analyses and by setting the Radon measure μ_t^ε as

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon |\nabla u^\varepsilon(\cdot, t)|^2}{2} + \frac{W(u^\varepsilon(\cdot, t))}{\varepsilon} \right) dx, \tag{1.12}$$

one may expect that the measure μ_t^ε behaves like surface measures of moving phase boundaries under the finiteness assumption for $E[u^\varepsilon(\cdot, t)]$ for sufficiently small $\varepsilon > 0$. For our problems, we consider the limit measure of $\mu_t^{\varepsilon,\sigma}$ defined by the solution $u^{\varepsilon,\sigma}$ to the equation (1.10), which is a slight modification of μ_t^ε. Then, by using the limiting measure of $\mu_t^{\varepsilon,\sigma}$, we characterize the motion by mean curvature in (1.1) in the sense of Brakke.

One of the interesting observations on the Allen-Cahn equations (1.10) is that the boundary condition in (1.1) may be obtained by considering the asymptotic analysis of the boundary condition in (1.10) as

$\varepsilon \to 0$. From the asymptotic analysis, we may have that, if ε is sufficiently close to 0, the following approximations hold:

$$\frac{-\partial_t u^{\varepsilon,\sigma}}{|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|}\frac{\nabla_{\partial\Omega} u^{\varepsilon,\sigma}}{|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|} \approx \mathbf{v}_b^{\sigma}, \quad \frac{|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|}{|\nabla u^{\varepsilon,\sigma}|} \approx \sin\theta^{\sigma}, \quad \frac{\nabla u^{\varepsilon,\sigma}}{|\nabla u^{\varepsilon,\sigma}|}\cdot\nu \approx \cos\theta^{\sigma} \quad \text{on } \partial\Omega, \tag{1.13}$$

where $\mathbf{v}_b^{\sigma}$ is the velocity vector of ∂M_t^{σ} on $\partial\Omega$ and θ^{σ} is the contact angle formed by M_t^{σ} and $\partial\Omega$ (see Remark 2.18 and Figure 4 in Section 4). From these approximations, it may hold that the boundary condition of (1.1) is obtained by taking the limit ($\varepsilon \to 0$) in the boundary condition of (1.10). Another one is that, as we mentioned before, the boundary condition in (1.1) may be regarded formally as Dirichlet and dynamic boundary conditions when $\sigma \to 0$ and $\sigma > 0$ is finite, respectively and as in the boundary condition in (1.10). Thus, when we consider a Brakke flow with Dirichlet or dynamic boundary conditions, it is natural to consider the singular limit ($\varepsilon \to 0$) of (1.10) first and then take the limit of the parameter σ which goes to 0 or positive finite σ', respectively. However, for technical reasons, we need to take the limit of both ε and σ simultaneously to characterize the limit in the case of Dirichlet boundary conditions. Moreover, for the purpose of simplifying arguments, the parameter σ is fixed with 1 when we consider the formulation of a Brakke flow with dynamic boundary conditions.

In the above situation, we intend to characterize the limit of the Allen-Cahn equations and to obtain a Brakke flow with Dirichlet or dynamic boundary conditions which satisfies Brakke's inequality as in (1.4) and the condition as in (1.5) or (1.6). One of the features to study the characterization is that we, for the first time, introduce a proper Radon measure $\alpha^{\varepsilon,\sigma}$ and a proper vector-valued function $\mathbf{v}_b^{\varepsilon,\sigma}(x, t)$ on $\partial\Omega \times [0, \infty)$ for any solutions $u^{\varepsilon,\sigma}$ to the Allen-Cahn equations (1.10), any $\varepsilon > 0$ and $\sigma > 0$. Moreover, we newly define a proper linear functional $\mathcal{S}_{\alpha,\mathbf{v}_b}$ defined on $\partial\Omega \times [0, \infty)$ for a Radon measure α which is the proper limit of $\alpha^{\varepsilon,\sigma}$ and a vector-valued function $\mathbf{v}_b$ as we mentioned in the above. Roughly speaking, $\alpha^{\varepsilon,\sigma}$ and $\mathbf{v}_b^{\varepsilon,\sigma}$ approximates the product measure of the weighted area measure of ∂M_t on $\partial\Omega$ and the Lebesgue measure on $[0, \infty)$ and the velocity vector of ∂M_t on $\partial\Omega$, respectively. Those quantities make it possible to control the boundary terms of integrals which do not appear in the case of right-angle Neumann boundary conditions and then obtain the proper singular limits. Another feature is that we apply the convergence theorem for measure-function pairs which is introduced by Hutchinson [14] in order to consider the convergence of $\alpha^{\varepsilon,\sigma}$ and $\mathbf{v}_b^{\varepsilon,\sigma}$ and to show that the limits satisfy the definition of our Brakke flow with Dirichlet or dynamic boundary conditions. Note that it is necessary to show that the limit measure α is not identically equal to zero to obtain our Brakke flow as the singular limit. Fortunately, we may prove the local positivity of α in the case that the boundary of Ω is connected and we impose some assumption on $u^{\varepsilon,\sigma}$ (see Lemma 4.6 or Lemma 4.14).

In our problems, we have mainly two difficulties to obtain the proper singular limit of the Allen-Cahn equations. One is to show the vanishing of the discrepancy measure up to the boundary $\partial\Omega$ as $\varepsilon \to 0$. The discrepancy measure for the solutions $u^{\varepsilon,\sigma}$ to (1.10), denoted by $\xi_t^{\varepsilon,\sigma}$, is defined by

$$d\xi_t^{\varepsilon,\sigma} := \left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2(\cdot,t)}{2} - \frac{W(u^{\varepsilon,\sigma}(\cdot,t))}{\varepsilon}\right) dx. \tag{1.14}$$

The vanishing in the interior of $\overline{\Omega}$ was proved by several authors, that is, if the support of the limit measure of $\mu_t^{\varepsilon,\sigma}$ does not exist on $\partial\Omega$, then we can show that the limit measure of $\xi_t^{\varepsilon,\sigma}$ is equal to zero in Ω. For instance, Ilmanen [15] proved the vanishing of the discrepancy measure in the case that the domain Ω is $\mathbb{R}^n$ and, although they treated an elliptic problem, Röger and Schätzle [29] proved the vanishing of the discrepancy measure in the case that the domain Ω is an open subset of two or three dimensional Euclidian space. In the case of right-angle Neumann boundary conditions, Mizuno and Tonegawa [22] and Kagaya [17] proved the vanishing of the discrepancy measure by constructing the Huisken-type monotonicity formula via the reflection arguments. However, except the case of right-angle Neumann boundary conditions, the vanishing up to the boundary remains to be solved in the case of other boundary conditions. The other difficulty is, in the case of Dirichlet boundary conditions, to obtain the uniform upper bound of the Dirichlet energy of $u^{\varepsilon,\sigma}$ along ν on $\partial\Omega$ in ε and σ. One possible problem is that the Dirichlet energy of $u^{\varepsilon,\sigma}$ on the boundary $\partial\Omega$ can blow up as $\sigma \to 0$ due to the form of the boundary condition in (1.10). Thus, after we consider the limit ($\varepsilon \to 0$), the boundary condition in (1.1) may not approximate Dirichlet boundary condition as $\sigma \to 0$. To see this, we succeed in constructing a family of the curves moving along the motion (1.1) and this construction indicates that the boundary condition in (1.1) may not approximate Dirichlet boundary condition if σ converges to 0 (see Subsection 4.1.1 of Section 4). Because of this example, it is reasonable to put some assumption on the Dirichlet energy of $u^{\varepsilon,\sigma}$ along to ν and this assumption may prevent the occurrence of irregular motions.

Unfortunately, so far we are unable to prove the vanishing of the discrepancy measure on the boundary and moreover to find the way to avoid to assume the uniform upper bound of the Dirichlet energy of $u^{\varepsilon,\sigma}$ on $\partial\Omega$. Hence, in this paper, we assume these two properties to obtain the main results. We emphasize that, in the case of dynamic boundary conditions, we may obtain the main results without assuming the uniform upper bound of the Dirichlet energy of $u^{\varepsilon,\sigma}$ on $\partial\Omega$. Thus this upper-bound assumption is necessary only in the case of Dirichlet boundary conditions.

The organaization of this paper is as follows; in Section 2, we give several notations and definitions of varifolds to make a formulation of our Brakke flow and the approximation results of it. Moreover, we introduce two important linear functionals; the one is defined on $(C_c(\partial\Omega \times [0, \infty)))^n$ and the other is defined on $(C_c(\Omega \times [0, \infty)))^n$. We also introduce two important Radon measures for the solutions of the Allen-Cahn equations (1.10); the one is defined on $\partial\Omega\times[0, \infty)$ and the other is defined on $\Omega\times[0, \infty)$. These quantities play an important role to consider our problems especially when we formulate the boundary conditions for varifolds. We also describe the intuitive geometric meaning of this measure. In Section 3, we state the formulation of a Brakke flow with Dirichlet or dynamic boundary conditions and then we give the motivation of these formulations. In Section 4, we state a sequence of the main lemmas and theorem, that is, the results of the singular limit of the Allen-Cahn equations and its characterization. Before we mention the main results, we give several assumptions and an important hypothesis in each case. Besides, in the case of Dirichlet boundary conditions, we also show the example which implies that the motion in (1.1) may not necessarily approximate the motion of mean curvature flow with Dirichlet boundary conditions as $\sigma \to 0$. In Section 5 and Section 6, we prove that the singular limit of the Allen-Cahn equations actually satisfies the definition of our Brakke flow with Dirichlet or dynamic boundary conditions. In Section 5, we derive a priori estimates and then, in Section 6, we calculate the first variation of the associated varifolds with $\mu^{\varepsilon,\sigma}$ and finally, consider the limit of the varifolds.

2 Preliminaries

We recall several definitions and notations related to varifolds and geometric measure theory to fix the notations. See for instance [2] and [31] for more details.

Let $X \subset \mathbb{R}^n$ be an open or compact subset. Let $\mathbb{G}(n\,, k)$ be the set of n-dimensional subspaces of $\mathbb{R}^n$ equipped with the metric $d(S, T) := \|S - T\|_*$ where we denote $\|\cdot\|_*$ by the operator norm on the space of linear endomorphism of $\mathbb{R}^n$. We set $G_k(X) := X \times \mathbb{G}(n\,, k)$ for $n,\, k \in \mathbb{N}$ with $n > k \geq 1$. For any $S \in \mathbb{G}(n\,, k)$, we can identify S with the corresponding orthogonal projection of $\mathbb{R}^n$ onto S and its matrix representation.

We define $\mathbf{A} : \mathbf{B}$ for $(n \times n)$-matrices $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$ by

$$\mathbf{A} : \mathbf{B} := \sum_{i,\,j=1}^{n} A_{ij}B_{ij}. \tag{2.1}$$

Now we define the support of a measure μ on X by

$$\operatorname{spt}\mu := \{x \in X \mid \mu(B_r(x)) > 0 \quad \text{for all } r > 0\}, \tag{2.2}$$

where $B_r(x)$ is an open ball of a center x with a radius r. One may easily show that the set defined by the right-hand-side of (2.2) is a closed subset of X.

In the following, we state several definitions of function spaces we use in te present paper. Let $m \in \mathbb{N}$ with $m \geq 1$ and $p \in [1,\infty]$ and let μ be a measure on X. First we say that f belongs to $(L^p(\mu, X))^m$ for $p \in [1, \infty)$ if f is defined μ-a.e. on X with the values on $\mathbb{R}^m$ and

$$\|f\|_{L^p(\mu,\,X)} := \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} < \infty \tag{2.3}$$

holds. Moreover, we say that f belongs to $(L^\infty(\mu, X))^m$ if

$$\|f\|_{L^\infty(\mu,\,X)} := \inf\{\lambda \in \mathbb{R} \mid |f| \leq \lambda \quad \mu\text{-a.e. on } X\} < \infty. \tag{2.4}$$

In particular, we write $L^p(\mu, X)$ when $m = 1$.

Secondly, we say that f belongs to $(C^k(X))^m$ for any $k \in \mathbb{N} \cup \{\infty\}$ and $m \in \mathbb{N}$ if f is a C^k-function defined on X taking the values on $\mathbb{R}^m$. Finally we say $f \in (C^k_c(X))^m$ if $f \in (C^k(X))^m$ with a compact support in X.

Definition 2.1 (Convergence of Radon measures)**.** Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a family of Radon measures on X. We say that $\{\mu_k\}_{k\in\mathbb{N}}$ converges to a Radon measure μ on X in Radon measures if and only if

$$\lim_{k\to\infty}\int_X \phi\, d\mu_k = \int_X \phi\, d\mu \tag{2.5}$$

holds for all $\phi \in C_c(X)$. We write $\mu_k \rightharpoonup \mu$ as $k \to \infty$ if $\{\mu_k\}_{k\in\mathbb{N}}$ converges to μ in Radon measure and also often write $\mu_k(\phi) \to \mu(\phi)$ as $k \to \infty$ for any $\phi \in C_c(X)$, where we set $\mu(\phi)$ by

$$\int_X \phi\, d\mu. \tag{2.6}$$

for any Radon measure μ on X and any function $\phi \in C_c(X)$.

Definition 2.2 (Product of measures)**.** Let $\{\mu_t\}_{t\in[0,\,\infty)}$, $f(x,\,t)$, and $\mathcal{L}^1_t$ be a family of Radon measures on X parametrized by $t \in [0,\,\infty)$, a function on $X\times[0,\,\infty)$ such that $f(\cdot,\,t)$ is μ_t-integrable on X for a.e. t, and the one-dimensional Lebesgue measure on $\mathbb{R}$, respectively. Then, throughout this paper, we define $\mu_t \otimes \mathcal{L}^1_t$ by

$$(f\,\mu_t \otimes \mathcal{L}^1_t)(A \times [a,\,b)) := \int_a^b\int_A f(x,\,t)\, d\mu_t(x)\, dt, \tag{2.7}$$

for any $A \subset X$ and any $0 \le a < b \le \infty$.

Definition 2.3 (k-rectifiable and k-integral Radon measure)**.** We say that a Radon measure μ on X is k-rectifiable if there exists $\mathcal{H}^k$-measurable countably k-rectifiable set $M \subset X$ and a locally $\mathcal{H}^k$-integrable positive function θ defined on M such that

$$\mu(A) = \int_{M\cap A}\theta\, d\mathcal{H}^k \tag{2.8}$$

for any Borel set $A \subset X$. Moreover μ is k-integral if θ takes the values on $\mathbb{Z}_{>0}$ a.e. in M.

Definition 2.4 (General k-varifold)**.** A general k-varifold V in X is a Radon measure on $G_k(X)$. We denote the set of all k-varifolds by $\mathbb{V}_k(X)$.

Definition 2.5 (Rectifiable k-varifold)**.** Let $V \in \mathbb{V}_k(X)$. We say that V is a rectifiable k-varifold if there exist a $\mathcal{H}^k$-measurable countably k-rectifiable set $M \subset X$ and a locally $\mathcal{H}^k$-integrable function θ defined on M such that

$$V(\phi) := \int_X \phi(x,\,T_xM)\theta(x)\, d\mathcal{H}^k\lfloor_M, \tag{2.9}$$

for any $\phi \in C_c(G_k(X))$, where T_xM is the approximate tangent plane of M at x which exists $\mathcal{H}^k$-a.e. on M. The existence of the approximate tangent plane is due to the rectifiability of M.

Definition 2.6 (Mass measure)**.** For any $V \in \mathbb{V}_k(X)$, we define the mass measure $\|V\|$ of V on X as the push-forward of V by the projection $\pi : G_k(X) \to X$. In particular, if V is a rectifiable k-varifold, its mass measure is expressed by $\|V\| = \theta\mathcal{H}^k\lfloor_M$.

Remark 2.7. The rectfiable k-varifold is uniquely determined by its mass measure through the identity (2.9). Due to this, we say that a k-rectifiable varifold V associated with a k-rectifiable Radon measure μ is a varifold such that the mass measure of V is equal to μ.

Definition 2.8 (First variation of a varifold)**.** For $V \in \mathbb{V}_k(X)$, we define the first variation δV of V by

$$\delta V(\mathbf{g}) := \int_{G_k(X)} \mathrm{div}_S\mathbf{g}(x)\, dV(x,\,S) \tag{2.10}$$

for any $\mathbf{g} \in (C^1_c(X))^n$, where $\mathrm{div}_S\mathbf{g}(x)$ is defined by

$$\mathrm{div}_S\mathbf{g}(x) := \sum_{j=1}^n(\mathbf{S}(\nabla g_j(x))\cdot e_j) = \sum_{i,j=1}^n S_{ij}\partial_{x_i}g_j(x) = \nabla g(x) : \mathbf{S}, \tag{2.11}$$

for any $\mathbf{g} \in (C^1_c(X))^n$ and any $S \in \mathbb{G}(n,\,k)$ and $\{e_j\}_{j=1}^n$ is the canonical basis of $\mathbb{R}^n$.

Remark 2.9. If a varifold V has a locally bounded first variation, then we may extend the linear functional δV into a locally bounded linear functional on $(C_c(X))^n$. Thus, from Riesz representation theorem, we have that the total variation $\|\delta V\|$ is a Radon measure on X and there exists a $\|\delta V\|$-measurable function $\eta : X \to \mathbb{R}^n$ such that $|\eta| = 1$ $\|\delta V\|$-a.e. in X and

$$\delta V(\mathbf{g}) = \int_X \mathbf{g} \cdot \eta \, d\|\delta V\| \tag{2.12}$$

for every $\mathbf{g} \in (C_c(X))^n$. Then, from Lebesgue decomposition theorem, we may decompose $\|\delta V\|$ into the absolutely continuous part $\|\delta V\|_{ac}$ and the singular part $\|\delta V\|_{sing}$ with respect to $\|V\|$. Therefore, from Radon-Nikodym theorem, we obtain

$$\delta V(\mathbf{g}) = \int_X \mathbf{g} \cdot \eta \, \frac{d\|\delta V\|_{ac}}{d\|V\|} \, d\|V\| + \int_X \mathbf{g} \cdot \eta \, d\|\delta V\|_{sing} \tag{2.13}$$

for any $\mathbf{g} \in (C_c(X))^n$, where $\frac{d\|\delta V\|_{ac}}{d\|V\|}$ is the Radon-Nikodym derivative. If we set $\mathbf{H}_V := -\frac{d\|\delta V\|_{ac}}{d\|V\|}\eta$, $\mathbf{H}_V$ is called *the generalized mean curvature vector* of V. This definition is the analogy of the classical version: if $M \subset \mathbb{R}^n$ is a k-dimensional smooth embedded manifold, then, from the divergence theorem and the first variation of M with a vector field $\mathbf{g}$, we have

$$\frac{d}{d\varepsilon} \mathcal{H}^k(\Psi_\varepsilon^{\mathbf{g}}(M))\Big|_{\varepsilon=0} = \int_M \mathrm{div}_M \mathbf{g} \, d\mathcal{H}^k = -\int_M \mathbf{g} \cdot \mathbf{H} \, d\mathcal{H}^k + \int_{\partial M} \mathbf{g} \cdot \gamma \, d\mathcal{H}^{k-1} \tag{2.14}$$

for all $\mathbf{g} \in (C_c^\infty(X))^n$, where $\mathbf{H}$ is the mean curvature vector of M, γ is the outer unit normal vector of ∂M, tangential to M. Here a map $\Psi_\varepsilon^{\mathbf{g}} : M \to \mathbb{R}^n$ is defined by $\Psi_\varepsilon^{\mathbf{g}}(x) := x + \varepsilon \mathbf{g}(x)$ for all $x \in M$ and $\varepsilon \in (-1, 1)$.

Definition 2.10 (First variation of a varifold with time integral)**.** Let $\{V_t\}_{t \in [0,\infty)} \subset \mathbb{V}_k(X)$ be a family of varifolds. We define $(\int_0^\infty \delta V_t \, dt)(\mathbf{g})$ for every $\mathbf{g} \in (C_c^1(X \times [0, \infty)))^n$ by

$$\left(\int_0^\infty \delta V_t \, dt \right)(\mathbf{g}) := \int_0^\infty \delta V_t(\mathbf{g}) \, dt. \tag{2.15}$$

Remark 2.11. Since the first variation of a varifold with time integral is a linear functional, we can also define *the generalized mean curvature vector* in the following way: first, we assume that there exists a constant $C > 0$ such that

$$\left| \int_0^\infty \delta V_t \, dt(\mathbf{g}) \right| \le C \sup_{X \times [0,\infty)} |\mathbf{g}|, \tag{2.16}$$

for any $\mathbf{g} \in (C_c^1(X \times [0, \infty)))^n$. Then, we may extend the domain of $\int_0^\infty \delta V_t \, dt$ into $(C_c(X \times [0, \infty)))^n$. Thus, from Riesz representation theorem, we have that the total variation of $\int_0^\infty \delta V_t \, dt$ is a Radon measure on $X \times [0, \infty)$ and there exists a $\| \int_0^\infty \delta V_t \, dt\|$-measurable function $\boldsymbol{\eta}$ with $|\eta| = 1$ $\| \int_0^\infty \delta V_t \, dt\|$-a.e. in $X \times [0, \infty)$ such that

$$\int_0^\infty \delta V_t \, dt(\mathbf{g}) = \int_{X \times [0,\infty)} \mathbf{g} \cdot \boldsymbol{\eta} \, d \left\| \int_0^\infty \delta V_t \, dt \right\|, \tag{2.17}$$

for any $\mathbf{g} \in (C_c(X \times [0, \infty)))^n$. By decomposing $\| \int_0^\infty \delta V_t \, dt\|$ into the absolute and singular part with respect to $\|V_t\| \otimes \mathcal{L}_t^1$ and applying the Radon-Nikodym theorem, we can also define *the generalized mean curvature vector in space-time* $\mathbf{H}_V$, as we did in Remark 2.9, by

$$\mathbf{H}_V := -\frac{d\| \int_0^\infty \delta V_t \, dt\|_{ac}}{d(\|V_t\| \otimes \mathcal{L}_t^1)} \boldsymbol{\eta}, \tag{2.18}$$

where $\frac{d\| \int_0^\infty \delta V_t \, dt\|_{ac}}{d(\|V_t\| \otimes \mathcal{L}_t^1)}$ is a Radon-Nikodym derivative. Note that, in this paper, we will use the notation $\mathbf{H}_V$ in the sense of *the generalized mean curvature vector in space-time*, which we define in this remark.

In the following, we assume that Ω is an open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Definition 2.12. Let $V \in \mathbb{V}_{n-1}(\overline{\Omega})$ be a varifold with a locally bounded first variation on $\overline{\Omega}$. We define the first variation of varifold tangential to $\partial\Omega$, denoting $\delta V \lfloor_{\partial\Omega}^T$, by

$$\delta V \lfloor_{\partial\Omega}^T(\mathbf{g}) := \delta V \lfloor_{\partial\Omega}(\mathbf{g} - (\mathbf{g} \cdot \nu)\nu) \tag{2.19}$$

for any $\mathbf{g} \in (C^1(\partial\Omega))^n$, where ν is the outer unit normal vector of $\partial\Omega$.

Now we define two linear functionals, which we name *boundary functional* and *interior functional* defined on $(C_c(\partial\Omega \times [0, \infty)))^n$ and $(C_c(\Omega \times [0, \infty)))^n$, respectively. The boundary functional is one of the keys to do the weak formulation of Dirichlet or dynamic boundary conditions of a Brakke flow and to prove the existence of the singular limits of the Allen-Cahn equations (1.10). On the other hand, the interior functional is one of the keys to do the weak formulation of only Dirichlet boundary conditions.

Definition 2.13 (Boundary functional)**.** Let ω be a Radon measure on $\partial\Omega\times[0,\infty)$ and $\mathbf{p}$ be in $(L^2(\omega, \partial\Omega\times[0,\infty)))^n$. Then we define the boundary linear functional $\mathcal{S}_{\omega,\mathbf{p}}$ by

$$\mathcal{S}_{\omega,\mathbf{p}}(\mathbf{g}) := \int_{\partial\Omega\times[0,\infty)} \mathbf{g}\cdot\mathbf{p}\, d\omega, \tag{2.20}$$

for any $\mathbf{g} \in (C_c(\partial\Omega \times [0,\infty)))^n$. We denote the total variation of $\mathcal{S}_{\omega,\mathbf{p}}$ by $\|\mathcal{S}_{\omega,\mathbf{p}}\|$.

Remark 2.14. From its definition, the total variation $\|\mathcal{S}_{\omega,\mathbf{p}}\|$ is in fact a Radon measure on $\partial\Omega \times [0, \infty)$. Indeed, let $K \subset \partial\Omega\times[0, \infty)$ be a compact set and we take any $\mathbf{g} \in (C_c(\partial\Omega\times[0, \infty)))^n$ such that $\operatorname{spt}\mathbf{g} \subset K$ holds. Then, by using Cauchy-Schwarz inequality, we have $|\mathcal{S}_{\omega,\mathbf{p}}(\mathbf{g})| \le \|\mathbf{p}\|_{L^2(\omega,\partial\Omega\times[0,\infty))}(\omega(K))^{\frac{1}{2}}\,\|\mathbf{g}\|_{L^\infty} < \infty$. This estimate allows us to apply Riesz representation theorem to $\mathcal{S}_{\omega,\mathbf{p}}$ and then we obtain the conclusion.

Definition 2.15 (Interior functional)**.** Let ω be a Radon measure in $\Omega \times [0, \infty)$ and $\mathbf{p}$ be in $(L^2(\omega, \Omega \times [0, \infty)))^n$. Then we define the interior linear functional $\mathcal{S}^{\Omega}_{\omega,\mathbf{p}}$ by

$$\mathcal{S}^{\Omega}_{\omega,\mathbf{p}}(\mathbf{g}) := -\int_{\Omega\times[0,\infty)} \mathbf{g}\cdot\mathbf{p}\, d\omega, \tag{2.21}$$

for any $\mathbf{g} \in (C_c(\Omega \times [0, \infty)))^n$. We denote the total variation of $\mathcal{S}^{\Omega}_{\omega,\mathbf{p}}$ by $\|\mathcal{S}^{\Omega}_{\omega,\mathbf{p}}\|$.

Remark 2.16. From the same argument we show in Remark 2.14, we can also show that the total variation $\|\mathcal{S}^{\Omega}_{\omega,\mathbf{p}}\|$ is actually a Radon measure in $\Omega \times [0, \infty)$.

Now we define *the weighted boundary area measure* defined on $\partial\Omega$ for a solution $u^{\varepsilon,\sigma}$ of Allen-Cahn equations. This measure plays an important role when we formulate a Brakke flow with Dirichlet or dynamic boundary condition and prove its existence theorem. The reason we name the measure "weighted boundary area" is stated in Remark 2.18 right after the definition.

Definition 2.17 (Weighted boundary area measures)**.** Let $u^{\varepsilon,\sigma}$ be a solution to the equation (1.10). Then, for all $\varepsilon > 0$, $\sigma > 0$ and all $t > 0$, we define a weighted boundary area measure $\alpha^{\varepsilon,\sigma}_t$ on $\partial\Omega$ by

$$\alpha^{\varepsilon,\sigma}_t := \varepsilon|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}(\cdot, t)|^2\, \mathcal{H}^{n-1}\lfloor_{\partial\Omega}, \tag{2.22}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $\nabla_{\partial\Omega}$ is a tangential gradient on $\partial\Omega$. Moreover, we set $\alpha^{\varepsilon,\sigma} := \alpha^{\varepsilon,\sigma}_t \otimes \mathcal{L}^1_t$.

Remark 2.18. We brielfly give the geometric interpretation of the measure $\alpha^{\varepsilon,\sigma}_t$. For simplicity, we do not write the index σ in the following. Let $\{M_t\}_{t\ge 0}$ be a family of smooth hypersurfaces on $\overline{\Omega} \subset \mathbb{R}^n$ with a boundary $\partial M_t \subset \partial\Omega$ and we assume that M_t moves by mean curvature under some boundary condition. Since the 1-D stationary solution to Allen-Cahn equation has the form of $q^\varepsilon(s) = \tanh(\varepsilon^{-1}s)$, we may obtain $2^{-1}\varepsilon|(u^\varepsilon)'(x)|^2 = \varepsilon^{-1}W(u^\varepsilon(x))$ and $\varepsilon|(u^\varepsilon)'(x)|^2\,\mathcal{L}^1(x) \approx \mathcal{H}^0(\{0\})$ if $\varepsilon > 0$ is sufficiently small and then we may also expect that the solution u^ε in the general dimensions has the asymptotic form $u^\varepsilon \approx q^\varepsilon(r^\varepsilon)$, where r^ε is the signed distance function to the front. Thus, we may also expect $2^{-1}\varepsilon|\nabla u^\varepsilon|^2 \approx \varepsilon^{-1}W(u^\varepsilon)$ and, moreover, $\varepsilon|\nabla u^\varepsilon|^2\,\mathcal{L}^n \approx \mathcal{H}^{n-1}\lfloor_{M_t}$ if $\varepsilon > 0$ is sufficiently small. Then the measure μ^ε_t may be regarded as $\mathcal{H}^{n-1}\lfloor_{M_t}$ up to constants. To see this, we refer to, for example, the formal analysis done by Rubinstein, Sternberg, and Keller [30] or the rigorous analysis done by Soner [32]. They also show that, as $\varepsilon \to 0$, Ω separates into two regions P_t and N_t where $u^\varepsilon \approx +1$ and $u^\varepsilon \approx -1$ respectively and the separating front corresponds to M_t. This means that, for sufficiently small $\varepsilon > 0$, we may consider the hypersurface M_t as the zero level set of $u^\varepsilon(\cdot, t)$. Therefore, in the phase field method, evolving surfaces $\{M_t\}_{t\ge 0}$ are approximated by the thin transition layers of an order ε. As an analogue of this, we also have that the transition layer on $\partial\Omega$ which approximates ∂M_t is supposed to have the width of an order $\varepsilon(\sin\theta)^{-1}$ (see Figure 1).

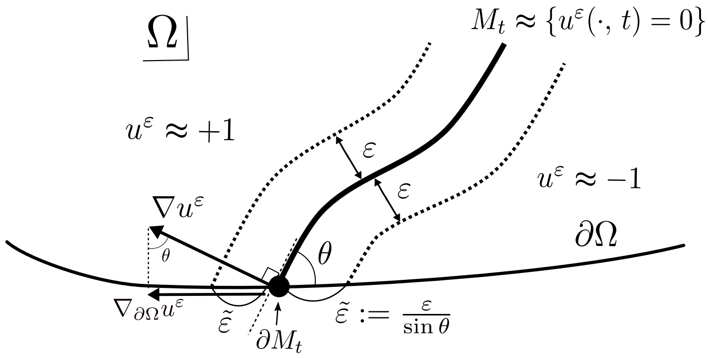

Figure 1: Approximated moving surfaces on $\partial\Omega$ with an order $\tilde{\varepsilon}$

As an analogy of the asymptotic analysis for the measure μ_t^ε, it is reasonable to expect that the measure $\tilde{\varepsilon}|\nabla_{\partial\Omega} u^\varepsilon|^2\, \mathcal{H}^{n-1}$ approximates the area measure of ∂M_t on $\partial\Omega$, that is, $\mathcal{H}^{n-2}\lfloor_{\partial M_t \cap \partial\Omega}$, where we set $\tilde{\varepsilon} := \varepsilon(\sin\theta)^{-1}$. Moreover, from Figure 1, we may compute as follows:

$$\sin\theta(x,\, t) = \frac{|\nabla_{\partial\Omega} u^\varepsilon(x,\, t)|}{|\nabla u^\varepsilon(x,\, t)|}. \tag{2.23}$$

Therefore, we obtain, as $\varepsilon \to 0$,

$$\alpha_t^\varepsilon = \varepsilon|\nabla_{\partial\Omega} u^\varepsilon|^2 \mathcal{H}^{n-1}\lfloor_{\partial\Omega} = (\sin\theta)\tilde{\varepsilon}|\nabla_{\partial\Omega} u^\varepsilon|^2\, \mathcal{H}^{n-1}\lfloor_{\partial\Omega} \approx (\sin\theta)\mathcal{H}^{n-2}\lfloor_{\partial M_t \cap \partial\Omega}. \tag{2.24}$$

3 Formulation of Brakke flow

In this section, we will give the definition of a Brakke flow with Dirichlet or dynamic boundary conditions in each subsection. Before stating the formulation of a Brakke flow, we will give the assumptions on the initial hypersurface $M_0 \subset \overline{\Omega}$ in the case of both Dirichlet and dynamic boundary conditions. Note that this initial condition allows you to have a variety of singularities on the initial hypersurface and to consider a wide rannge of mean curvature flow such as the flow of grain boundaries. The idea is based on the idea by Ilmanen (see [15]).

3.1 Initial hypersurface

We choose the initial hypersurface M_0 in the following manner; let E_0 be an open set in $\mathbb{R}^n$ with $E_0 \cap (\mathbb{R}^n \setminus \overline{\Omega}) \neq \emptyset$ and $E_0 \cap \Omega \neq \emptyset$. Defining $M_0 := \partial E_0 \cap \Omega (\neq \emptyset)$ with $\partial M_0 = \partial E_0 \cap \partial\Omega$, we assume that the density bound of M_0 and the pair $(E_0,\, M_0)$ can be approximated by smooth family of pairs $\{(E_0^l,\, M_0^l)\}_{l\in\mathbb{N}}$ (see [15]). More precisely, we assume that

1. There exists a constant $C > 0$ such that, for any $R > 0$ and $x \in \overline{\Omega}$,

$$\frac{\mathcal{H}^{n-1}(M_0 \cap B_R(x))}{\omega_{n-1} R^{n-1}} \leq C, \tag{3.1}$$

where ω_{n-1} is the area of $(n-1)$-dimensional sphere.

2. There exists a family of pairs $\{(E_0^l,\, M_0^l)\}_{l\in\mathbb{N}}$ such that E_0^l is open, $M_0^l := \partial E_0^l$ is a smooth hypersurface and the convergences

$$\chi_{E_0^l} \xrightarrow[l\to\infty]{} \chi_{E_0} \quad \text{in } BV(\Omega), \tag{3.2}$$

$$\mathcal{H}^{n-1}\lfloor_{M_0^l} \xrightarrow[l\to\infty]{} \mathcal{H}^{n-1}\lfloor_{M_0} \quad \text{in Radon measures} \tag{3.3}$$

hold.

In the following, we state the formulation of a Brakke flow with Dirichlet or dynamic boundary conditions starting from the initial hypersurface M_0 given in Subsection 3.1.

3.2 Dirichlet boundary condition

We now state the definition of a Brakke flow with Dirichlet boundary conditions and then show the motivation of it. Specifically, we state the reason why our definition is reasonable for a weak formulation of Dirichlet boundary conditions.

Recall that we try to consider a weak solution of the following mean curvature flow with Dirichlet boundary conditions in the sense of Brakke:

$$\begin{cases} \mathbf{v}(\cdot,\, t) = \mathbf{H}(\cdot,\, t) & \text{on } M_t,\, t > 0, \\ \mathbf{v}_b(\cdot,\, t) = 0 & \text{on } \partial M_t,\, t > 0, \end{cases} \tag{3.4}$$

where $\mathbf{v}_b$ is the velocity vector of the boundary of a hypersurface M_t on $\partial\Omega$.

Definition 3.1. Let $\{V_t\}_{t\geq 0}$ be a family of varifolds on $\overline{\Omega}$ with locally bounded first variations and be $(n-1)$-rectifiable for a.e. $t \geq 0$. Let $\alpha \not\equiv 0$ and $\mathbf{v}_b$ be a Radon measure on $\partial\Omega \times [0,\, \infty)$ and a function in $(L^2(\alpha))^n$, respectively. Then we say that the triplet $(\{V_t\}_{t\geq 0},\, \alpha,\, \mathbf{v}_b)$ moves along *Brake flow with Dirichlet boundary conditions* startnig from V_0 with $\|V_0\| = \sigma_0 \mathcal{H}^{n-1}\lfloor_{M_0}$, where M_0 is as in Subsection 3.1, if the following conditions hold:

1. (Absolute continuity with Dirichlet boundary condition)
   There exists the generalized mean curvature vector $\mathbf{H}_V^\Omega$ in $\Omega\times[0,\, \infty)$ such that $\delta V_t\lfloor_\Omega = -\mathbf{H}_V^\Omega(\cdot,\, t)\|V_t\|$ holds in Ω for a.e. $t \in [0,\, \infty)$. In addition, the following absolute continuity is valid:
   $$\left\| \mathcal{S}_{\alpha,\, \mathbf{v}_b} + \mathcal{S}^\Omega_{\tilde{\mu},\, \mathbf{H}_V^\Omega} \right\| \ll \|V_t\| \otimes \mathcal{L}^1_t \quad \text{on } \overline{\Omega} \times [0,\, \infty), \tag{3.5}$$
   where $\tilde{\mu} := \|V_t\| \otimes \mathcal{L}^1_t$, and $\mathcal{S}_{\alpha,\, \mathbf{v}_b}$ and $\mathcal{S}^\Omega_{\tilde{\mu},\, \mathbf{H}_V^\Omega}$ are as in Definition 2.13 and 2.15.

2. (Modified generalized mean curvature vector)
   There exists a vector $\widetilde{\mathbf{H}}_V$ such that
   $$\mathcal{S}_{\alpha,\, \mathbf{v}_b} + \mathcal{S}^\Omega_{\tilde{\mu},\, \mathbf{H}_V^\Omega} = -\widetilde{\mathbf{H}}_V \|V_t\| \otimes \mathcal{L}^1_t \quad \text{on } \overline{\Omega} \times [0,\, \infty), \quad \widetilde{\mathbf{H}}_V\lfloor_\Omega = \mathbf{H}_V^\Omega \quad \text{in } \Omega \times [0,\, \infty), \tag{3.6}$$
   and $\widetilde{\mathbf{H}}_V$ belongs to $(L^2(\|V_t\| \otimes \mathcal{L}^1_t,\, \overline{\Omega} \times [0,\, \infty)))^n$. We call $\widetilde{\mathbf{H}}_V$ as *the modified generalized mean curvature vector* throughout this paper.

3. (Brakke's inequality)
   The inequality
   $$\int_{\overline{\Omega}} \phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{\overline{\Omega}} (-\phi |\widetilde{\mathbf{H}}_V|^2 + (\nabla\phi \cdot \widetilde{\mathbf{H}}_V) + \partial_t \phi) \, d\|V_t\| \, dt \tag{3.7}$$
   holds for any $\phi \in C^1(\overline{\Omega} \times [0, \infty))$ such that $\phi \geq 0$ and $\phi(\cdot,\, t) \in C^1_c(\overline{\Omega})$, and for any $0 < t_1 \leq t_2 < \infty$.

Remark 3.2. First of all, we remark that the exixstence of the modified generalized mean curvature vector $\widetilde{\mathbf{H}}_V$ can be obtained from (3.5), however, the important thing is that $\widetilde{\mathbf{H}}_V$ needs to belong to $(L^2(\|V_t\| \otimes \mathcal{L}^1_t,\, \overline{\Omega} \times [0,\, \infty)))^n$ and it coincides with the generalized mean curvature vector $\mathbf{H}_V^\Omega$ in the interior of the domain. Secondly, the absolute continuity (3.5) is a stronger condition than (1.5). Thus, in order to see how the boundary conditions look like, it is sufficient to consider the condition (3.5) whose domain is restricted to $\partial\Omega \times [0,\, \infty)$.

Remark 3.3. Now we show that the absolutely continuity in (3.5) corresponds to a formulation of Dirichlet boundary conditions in measure theoretic sense if we focus on $\partial\Omega \times [0,\, \infty)$. Indeed, let M_t be a smooth hypersurface corresponding to a varifold V_t for all $t \geq 0$. Assume that M_t moves along the motion describing in (3.5) and (3.7). Moreover, we impose the following assumptions:

(A1) $\|V_t\| = \mathcal{H}^{n-1}\lfloor_{M_t}$ on $\overline{\Omega}$ and $\alpha = (\sin\theta)(\mathcal{H}^{n-2}\lfloor_{\partial M_t} \otimes \mathcal{L}^1_t)$ on $\partial\Omega \times [0,\, \infty)$.

(A2) $\partial M_t \subset \partial\Omega$ for all $t \geq 0$.

(A3) $\mathcal{H}^{n-1}(\partial\Omega \cap M_t) = 0$ for all $t \geq 0$, meaning that the geometric interior of M_t is not on $\partial\Omega$ for all $t \geq 0$.

Here $\theta \in [0, \pi]$ in (A1) is the contact angle formed by the unit normal vector N_b of ∂M_t on $\partial\Omega$ and $-\gamma$, where γ is the outer unit normal vector of ∂M_t on $\partial\Omega$ and is tangential to M_t and thus we have $\cos\theta = -\gamma \cdot N_b$ (see Figure 2). The assumption (A1) comes from Remark 2.18 saying that the measure $\alpha^{\varepsilon,\sigma}$ defined in Definition 2.17 should be considered as $(\sin\theta)(\mathcal{H}^{n-2}\lfloor_{\partial M_t} \otimes \mathcal{L}^1_t)$ as $\varepsilon \to 0$.

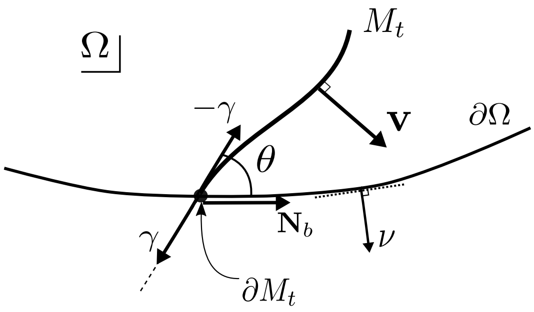

Figure 2: Definition of θ

Since (A1) and (A3) imply $(\|V_t\| \otimes \mathcal{L}^1_t)(\partial\Omega \times [0, \infty)) = 0$, we have from the absolute continuity (3.5)

$$\left\| \mathcal{S}_{\alpha,\,\mathbf{v}_b} + \mathcal{S}^{\Omega}_{\tilde{\mu},\,\tilde{\mathbf{H}}^{\Omega}_V} \right\| (\partial\Omega \times [0, \infty)) = 0. \tag{3.8}$$

Thus, for any $\mathbf{g} \in (C_c(\partial\Omega \times [0, \infty)))^n$ with $|\mathbf{g}| \le 1$, from the fact that spt $\|\mathcal{S}^{\Omega}_{\tilde{\mu},\,\tilde{\mathbf{H}}^{\Omega}_V}\| \subset \Omega \times [0, \infty)$, we have

$$0 = \mathcal{S}_{\alpha,\,\mathbf{v}_b}(\mathbf{g}) = \int_0^\infty \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}_b \, d\alpha = \int_0^\infty \int_{\partial M_t} \mathbf{g} \cdot \mathbf{v}_b \sin\theta \, d\mathcal{H}^{n-2} \, dt. \tag{3.9}$$

Moreover, if $\theta(x, t)$ is not identically equal to zero and π, then $\sin\theta(x, t)$ is not identically equal to zero on $\partial\Omega$ for $t \in [0, \infty)$. Hence we obtain $\mathbf{v}_b = 0$ on $\partial\Omega$ for a.e. $t \in [0, \infty)$. From the proof of Lemma 4.4 (see also the next section 4.1.2), we may regard $\mathbf{v}_b$ as the generalized velocity vector of ∂M_t on $\partial\Omega$ and thus we can say that this implies Dirichlet boundary conditions.

3.3 Dynamic boundary condition

We now state the definition of a Brakke flow with dynamic boundary conditions and then we show the reason why our boundary condition is reasonable for the weak formulation of Dirichlet boundary conditions.

Recall that we try to consider a weak solution of the following mean curvature flow with dynamic boundary conditions with $\sigma = 1$ in the sense of Brakke:

$$\begin{cases} \mathbf{v}(\cdot,\, t) = \mathbf{H}(\cdot,\, t) & \text{on } M_t,\ t > 0, \\ \mathbf{v}_b(\cdot,\, t) = \sigma(\tan\theta(\cdot,\, t))^{-1}\mathbf{N}_b(\cdot,\, t) & \text{on } \partial M_t,\ t > 0, \end{cases} \tag{3.10}$$

where $\mathbf{N}_b$ is the unit normal vector of ∂M_t on $\partial\Omega$, and θ is the contact angle formed by a hypersurface M_t and $\mathbf{N}_b$ on $\partial\Omega$. Note that, in this paper, we consider the formulation of a Brakke flow with dynamic boundary conditions mainly in the case $\sigma = 1$. However, in order to study the difference between dynamic and right-angle Neumann boundary conditions, which we will mention in Remark 3.7, we do not omit the index σ in this subsection. Actually, we may also prove the same results in the case $\sigma \in (0, 1) \cup (1, \infty)$ as in the case $\sigma = 1$.

Definition 3.4. Let $\{V_t\}_{t \ge 0}$ be a family of varifolds on $\overline{\Omega}$ with locally bounded first variations and be $(n-1)$-rectifiable for a.e. $t \ge 0$. Let $\alpha \not\equiv 0$ and $\mathbf{v}_b$ be a Radon measure on $\partial\Omega \times [0, \infty)$ and a function in $(L^2(\alpha))^n$, respectively. Then we say that the triplet $(\{V_t\}_{t \ge 0},\, \alpha,\, \mathbf{v}_b)$ moves along a *Brakke flow with dynamic boundary conditions* starting from V_0 with $\|V_0\| = \sigma_0 \mathcal{H}^{n-1}\lfloor_{M_0}$, where M_0 is as in Subsection 3.1, if the following conditions hold:

1. (Absolute continuity with dynamic boundary condition)
   The following absolute continuity holds.

$$\left\| \int_0^\infty \delta V_t \lfloor_{\Omega} \, dt + \int_0^\infty \delta V_t \lfloor^T_{\partial\Omega} \, dt + \sigma^{-1} \mathcal{S}_{\alpha,\,\mathbf{v}_b} \right\| \ll \|V_t\| \otimes \mathcal{L}^1_t \quad \text{on } \overline{\Omega} \times [0, \infty), \tag{3.11}$$

   where $\delta V_t \lfloor^T_{\partial\Omega}$ and $\mathcal{S}_{\alpha,\,\mathbf{v}_b}$ are as in Definition 2.12 and Definition 2.13, respectively.

2. (Modified generalized mean curvature vector)
   There exists a vector $\widetilde{\mathbf{H}}_V$ such that

$$\int_0^\infty \delta V_t\lfloor_\Omega\, dt + \int_0^\infty \delta V_t\lfloor_{\partial\Omega}^T\, dt + \sigma^{-1}\mathcal{S}_{\alpha,\,\mathbf{v}_b} = -\widetilde{\mathbf{H}}_V\|V_t\| \otimes \mathcal{L}_t^1 \quad \text{on } \overline{\Omega}\times[0,\,\infty) \tag{3.12}$$

   holds and $\widetilde{\mathbf{H}}_V$ belongs to $(L^2(\|V_t\|\otimes\mathcal{L}_t^1,\, \overline{\Omega}\times[0,\,\infty)))^n$. We call $\widetilde{\mathbf{H}}_V$ as *the modified generalized mean curvature vector* throughout this paper.

3. (Brakke's inequality)
   The inequality

$$\int_{\overline{\Omega}} \phi\, d\|V_t\|\Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2}\int_{\overline{\Omega}} \Big(-\phi|\widetilde{\mathbf{H}}_V|^2 + (\nabla\phi\cdot\widetilde{\mathbf{H}}_V) + \partial_t\phi\Big)\, d\|V_t\|\, dt - \frac{1}{\sigma}\int_{\partial\Omega\times[t_1,\,t_2]} \phi|\mathbf{v}_b|^2\, d\alpha \tag{3.13}$$

   holds for any $\phi \in C^1(\overline{\Omega}\times[0,\infty))$ such that $\phi\ge 0$ and $\nabla\phi(\cdot,\,t)\cdot\nu = 0$ on $\partial\Omega$ and $\phi(\cdot,\,t)\in C_c^1(\overline{\Omega})$, and for any $0<t_1\le t_2<\infty$.

Remark 3.5. First of all, we remark that the existence of the modified generalized mean curvature vector $\widetilde{\mathbf{H}}_V$ can be obtained from (3.11), however, the important thing is that $\widetilde{\mathbf{H}}_V$ needs to belong to $(L^2(\|V_t\|\otimes\mathcal{L}_t^1,\, \overline{\Omega}\times[0,\,\infty)))^n$. Secondly, we can also define, from (3.11), the generalized mean curvature vector $\mathbf{H}_V^\Omega$ defined in Remark 2.9 such that

$$\delta V_t\lfloor_\Omega = -\mathbf{H}_V^\Omega(\cdot,\,t)\,\|V_t\| \quad \text{in } \Omega \text{ for a.e. } t\in[0,\,\infty). \tag{3.14}$$

In addition, if we restrict the domain of the modified generalized mean curvature vector $\widetilde{\mathbf{H}}_V$ in (3.12) into $\Omega\times[0,\,\infty)$ (denoted by $\widetilde{\mathbf{H}}_V\lfloor_\Omega$), then we can show that $\widetilde{\mathbf{H}}_V\lfloor_\Omega$ coincides with $\mathbf{H}_V^\Omega$ in $\Omega\times[0,\,\infty)$. Thirdly, as we also mentioned in the case of Dirichlet boundary conditions, the absolutely continuity (3.11) is a stronger condition than (1.6). Thus, in order to see how the boundary conditions look like, it is sufficient to consider the condition (3.11) whose domain is restricted to $\partial\Omega\times[0,\,\infty)$.

Remark 3.6. Now we show that the absolute continuity (3.11) corresponds to a formulation of dynamic boundary conditions in measure theoretic sense if we focus on $\partial\Omega\times[0,\,\infty)$. Indeed, let M_t be a smooth hypersurface corresponding to a varifold V_t for all $t\ge 0$ and evolve by the motion described in (3.11) and (3.13). Note that we do not write the index σ for simplicity. Furthermore, we impose the same assumptions (A1), (A2) and (A3) in Remark 3.3 (see also Figure 2). Since (A1) and (A3) implies $\|V_t\|\otimes\mathcal{L}_t^1(\partial\Omega\times[0,\,\infty)) = 0$, from the absolute continuity (3.11), we have

$$\left\|\int_0^\infty \delta V_t\lfloor_{\partial\Omega}^T\, dt + \int_0^\infty \delta V_t\lfloor_\Omega\, dt + \sigma^{-1}\mathcal{S}_{\alpha,\,\mathbf{v}_b}\right\|(\partial\Omega\times[0,\,\infty)) = 0. \tag{3.15}$$

Then, for any $\mathbf{g}\in(C_c(\partial\Omega\times[0,\infty)))^n$ with $|\mathbf{g}|\le 1$, we have, from the divergence theorem, (A1), (A3) and (3.15),

$$\begin{aligned} 0 &= \int_0^\infty \delta V_t\lfloor_{\partial\Omega}^T(\mathbf{g})\, dt + \mathcal{S}_{\alpha,\,\mathbf{v}_b}(\mathbf{g}) \\ &= \int_0^\infty\int_{\partial M_t\cap\partial\Omega} (\mathbf{g}-(\mathbf{g}\cdot\nu)\nu)\cdot\gamma\, d\mathcal{H}^{n-2}\, dt + \sigma^{-1}\int_0^\infty\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}_b\, d\alpha, \end{aligned} \tag{3.16}$$

where ν is the outer unit normal vector of $\partial\Omega$. From the relation $\cos\theta = -\gamma\cdot N_b$, we may deduce $-\gamma^T\cdot N_b = -(\gamma-(\gamma\cdot\nu)\nu)\cdot N_b = -\gamma\cdot N_b = \cos\theta$, where γ^T is the ortogonal projection of γ onto $\partial\Omega$. Hence, from $(A1)$ and $(A2)$, we obtain

$$\begin{aligned} 0 &= \int_0^\infty\int_{\partial M_t} \mathbf{g}\cdot(\gamma-(\gamma\cdot\nu)\nu)\, d\mathcal{H}^{n-2}\, dt + \sigma^{-1}\int_0^\infty\int_{\partial M_t} \mathbf{g}\cdot\mathbf{v}_b(\sin\theta)\, d\mathcal{H}^{n-2}\, dt \\ &= \int_0^\infty\int_{\partial M_t} \mathbf{g}\cdot(\gamma^T+\sigma^{-1}\mathbf{v}_b(\sin\theta))\, d\mathcal{H}^{n-2}\, dt \end{aligned} \tag{3.17}$$

for all $\mathbf{g}\in(C_c^1(\overline{\Omega}\times[0,\infty)))^n$. This implies that $\gamma^T+\sigma^{-1}\mathbf{v}_b(\sin\theta) = 0$ on ∂M_t for all $t\ge 0$. Here it holds that $\sin\theta$ is not equal to 0, otherwise we have $\gamma^T = 0$ and this implies $\theta = \frac{\pi}{2}$ which contradicts $\sin\theta = 0$. Therefore we obtain

$$\mathbf{v}_b\cdot\mathbf{N}_b = \left(-\frac{\gamma^T}{\sigma^{-1}\sin\theta}\right)\cdot\mathbf{N}_b = \sigma\frac{\cos\theta}{\sin\theta} = \frac{\sigma}{\tan\theta} \tag{3.18}$$

on ∂M_t and we can say that this implies dynamic boundary condition.

Remark 3.7. Now we briefly refer to the formulation of right-angle Neumann boundary conditions, comparing to our formulation. We recall that the boundary condition of Allen-Cahn equations (1.10) corresponds to dynamic and right-angle Neumann boundary conditions in the case that σ is positive and finite and $\sigma = \infty$, respectively. In Subsection 4.2.2, we gave the conditions for a Brakke flow with dynamic boundary conditions as the absolute continuity (3.11) and Brakke's inequality (3.13) with a parameter σ. These results are also valid when σ is not only 1 but also in $(1, \infty)$ if we slightly modify the proofs of the case $\sigma = 1$. Hence, if we formally substitute $\sigma = \infty$ for (3.11) and (3.13), then we have

$$\Big\| \int_0^\infty \delta V_t \lfloor_{\partial\Omega}^T \, dt + \int_0^\infty \delta V_t \lfloor_\Omega \, dt \Big\| \ll \|V_t\| \otimes \mathcal{L}_t^1 \quad \text{on } \overline{\Omega} \times [0, \infty), \tag{3.19}$$

and

$$\int_{\overline{\Omega}} \phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{\overline{\Omega}} \Big(- \phi |\widetilde{\mathbf{H}}_V|^2 + (\nabla\phi \cdot \widetilde{\mathbf{H}}_V) + \partial_t \phi \Big) \, d\|V_t\| \, dt \tag{3.20}$$

for all $\phi \in C^1(\overline{\Omega} \times [0, \infty))$ with some conditions. Actually, these results essentially corresponds to those of Mizuno and Tonegawa [22]. The readers should refer to [22] for more details.

4 Existence results of sharp interface limit

Now we state the results of the sharp interface limits of our Brakke flow with Dirichlet or dynamic boundary conditions which we defined in the previous section. We emphasize that we applied the phase field method to show the approximation of our Brakke flow although one may obtain the similar results with ours by applying another method. In each subsection, we first state several assumptions and then we give a sequence of main lemmas and the main theorem.

4.1 Dirichlet boundary condition

Note that the assumptions described in "General assumptions" are mainly based on Ilmanen's work in [15]. Since Soner [32] later removed the technical assumptions made by Ilmanen [15], we may weaken these assumptions.

4.1.1 Assumptions, hypothesis and example

In this subsection, we will state three important assumtions in our study which consists of "General assumptions", "Vanishing hypothesis for the discrepancy measure", and "Uniform uppr bound for the solution of Allen-Cahn equations". Moreover, we will state one example which gives us the validity to impose the assumption "Uniform upper bound for the solution of Allen-Cahn equations".

General assumptions

Suppose that $n \geq 2$, Ω is a bounded domain with smooth boundaries and we define the potential function $W : \mathbb{R} \to \mathbb{R}$ by $W(s) = \frac{1}{2}(1 - s^2)^2$. Note that W is said to be a double-well potential and we may also apply the generalized W, which is defined in, for instance, [22] or [17].

Next we give the assumptions on the initial data of the solutions of the Allen-Cahn equations (1.10). We assume that there exists a subsequence $\{u_0^{\varepsilon_l, \sigma_l}\}_{l \in \mathbb{N}}$ such that

$$u_0^{\varepsilon_l, \sigma_l} \xrightarrow[l \to \infty]{} 2\chi_{E_0 \cap \Omega} - 1 \quad \text{in } BV(\Omega), \tag{4.1}$$

$$\mu_0^{\varepsilon_l, \sigma_l} \xrightarrow[l \to \infty]{} \sigma_0 \, \mathcal{H}^{n-1} \lfloor_{\partial E_0} \quad \text{in Radon measures,} \tag{4.2}$$

where E_0 is as in Subsection 3.1, $\sigma_0 := \int_{-1}^1 \sqrt{2W(u)} \, du$, $\mu_0^{\varepsilon, \sigma}$ is defined by

$$\mu_0^{\varepsilon, \sigma} := \Big(\frac{\varepsilon |\nabla u_0^{\varepsilon, \sigma}|^2}{2} + \frac{W(u_0^{\varepsilon, \sigma})}{\varepsilon} \Big) \mathcal{L}^n. \tag{4.3}$$

Now we suppose that the initial data $\{u_0^{\varepsilon, \sigma}\}_{\varepsilon, \sigma > 0}$ satisfy

$$\sup_\Omega |u_0^{\varepsilon, \sigma}| \leq 1, \tag{4.4}$$

and there exists $D > 0$ such that

$$\sup_{\varepsilon>0,\,\sigma>0} E^{\varepsilon,\,\sigma}[u_0^{\varepsilon,\,\sigma}] = \sup_{\varepsilon>0,\,\sigma>0} \mu_0^{\varepsilon,\,\sigma}(\Omega) \leq D, \tag{4.5}$$

and this indicates that the surface area of the initial hypersurface cannot blow up as $\varepsilon \downarrow 0$ or $\sigma \downarrow 0$. In addition, from (4.4), we can show that

$$\sup_{\Omega\times[0,\,\infty)} |u^{\varepsilon,\,\sigma}| \leq 1. \tag{4.6}$$

for any $\varepsilon,\, \sigma > 0$ (see Appendix B in Section 7 for the proof).

Next we should solve the equations (1.10) with $\sigma \in (0,\, 1)$. Since we have the boundedness of $u^{\varepsilon,\,\sigma}$ as shown in (4.6), we may obtain the desired regularity of the global-in-time solution to (1.10) by virtue of, for instance, Escher [9]. Hence, in this paper, we can assume that a solution $u^{\varepsilon,\,\sigma}$ to (1.10) exists, is not a constant function and it belongs to $L^2_{loc}([0,\,\infty);\, W^{2,\,2}(\Omega)) \cap L^2_{loc}([0,\,\infty);\, W^{1,\,2}(\partial\Omega,\, \mathcal{H}^{n-1}))$ and $\partial_t u^{\varepsilon,\,\sigma}$ belongs to $L^2_{loc}([0,\,\infty);\, L^2(\Omega)) \cap L^2_{loc}([0,\,\infty);\, L^2(\partial\Omega,\, \mathcal{H}^{n-1}))$. Note that, as far as we know, regarding the local-in-time classical solutions, Guidetti [12] proved the local-in-time existence and uniqueness of the classical solutions to the parabolic equations which is considered as generalized equations of (1.10) (see also [13]).

Vanishing hypothesis of the discrepancy measure

Let $\xi_t^{\varepsilon,\,\sigma}$ be as in (1.14) and recall that this measure is called discrepancy measure. Then, from (5.2) in Section 5, we can obatin the uniform boundedness of the measure $|\xi_t^{\varepsilon,\,\sigma}|$ in $\varepsilon,\, \sigma > 0$ and $t \in [0,\,\infty)$. Then, by applying the way we will use to prove Lemma 4.2 in the next section, we may choose a subsequence $\{\xi_t^{\varepsilon_j,\,\sigma_j}\}_{j\in\mathbb{N}}$ such that $|\xi_t^{\varepsilon_j,\,\sigma_j}| \rightharpoonup \xi_t$ as $j \to \infty$ for some Radon measure ξ_t.

In this study, we assume the vanishing of the descrepancy measure up to $\partial\Omega$, that is, we assume that

$$\xi_t \equiv 0 \quad \text{on } \overline{\Omega} \text{ for a.e. } t \in [0,\,\infty). \tag{4.7}$$

We emphasize that the vanishing of the discrepancy ξ_t in the interior of $\overline{\Omega}$ have been already proved by Ilmanen [15] ($\Omega = \mathbb{R}^n$), Röger and Schätzle [29] (in the elliptic problems) and so on. Namely, if the support of the limit measure of $\mu_t^{\varepsilon,\,\sigma}$ does not exist on $\partial\Omega$ for a.e. t, then we have that $\xi_t \equiv 0$ in $\overline{\Omega}$ for a.e. t. Moreover, in the case of right-angle Neumann boundary conditions, we also have the vanishing of the discrepancy up to the boundary by virtue of Mizuno and Tonegawa [22] (Ω is strictly convex) and Kagaya [17] (Ω is not necessarily convex).

Uniform upper bound for the solution of Allen-Cahn equations

We next assume the local-in-time uniform upper bound of the Dirichlet energy of the Allen-Cahn equations (1.10) on $\partial\Omega$ along ν. More precisely, we suppose that, for any $0 \leq t_1 < t_2 < \infty$, there exists a constant $C_0(t_1,\, t_2) > 0$ such that

$$\sup_{\varepsilon,\,\sigma>0} \int_{t_1}^{t_2} \int_{\partial\Omega} \frac{\varepsilon}{2}\Big(\frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu}\Big)^2 \, d\mathcal{H}^{n-1}\, dt \leq C_0(t_1,\, t_2) \tag{4.8}$$

holds. This assumption may not be removed due to our construction of the following example in Remark 4.1 in the sense of classical curvature flow.

Remark 4.1 (**Construction of the curves moving by the motion descibed in** (1.1)). As we mentioned, we now give a reason why it is reasonable to assume the uniform upper bound for $u^{\varepsilon,\,\sigma}$ to show the existence of the singular limit in the case of Dirichlet boundary conditions and study a Brakke flow which we defined in Section 3.

Generally speaking, it is necessary for us to make it clear which class of solutions to some equation is proper as the definition of the solutions. In our case, we have to clarify what kind of the solutions are suitable for the ones to a Brakke flow with Dirichlet boundary conditions. To see this, we consider the following example; first of all, we assume that Ω is a half space in $\mathbb{R}^2$. For each $\sigma > 0$ sufficiently small, we take an initial curve M_0^σ as a part of the $(1,\, -\sigma^{-1})$-centered circle with radius $R := \sqrt{1+\sigma^{-2}}$ and we set two points $x_0(0)$ and $x_1(0)$ on $\partial\Omega$ as $x_0(0) := 0$ and $x_1(0) := 2$ (see Figure 3).

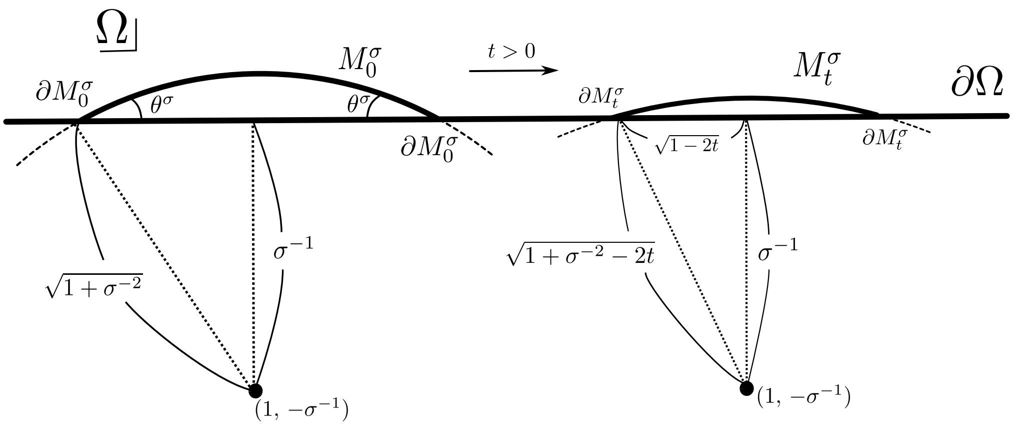

Figure 3: An initial curve and its motion by its curvature

In this setting, we may easily show that the curvature of a curve M_t^σ is $r^{-1}(t)$ where $r(t) := \sqrt{R^2-2t}$ is the radius of a circle, a part of which is M_t^σ. Then we see that a family of curves $\{M_t^\sigma\}_{t\geq 0}$ moves by their curvatures and moreover, we can easily show that the boundaries $\{\partial M_t\}_{t\geq 0}$ evolve under dynamical boundary conditions. Indeed, we can calculate the explicit forms of the velocities of $\partial M_t^\sigma := \{x_0^\sigma(t),\, x_1^\sigma(t)\}$ as follows: since M_t^σ is a part of the circle defined by $\{(x,\,y) \mid (x-1)^2+(y+\sigma^{-1})^2 = r(t)^2\}$ for each t, we can have the explicit forms of $x_0^\sigma(t)$ and $x_1^\sigma(t)$ by $-\sqrt{r(t)^2-\sigma^{-2}}+1$ and $\sqrt{r(t)^2-\sigma^{-2}}+1$, respectively. Hence we have that, for example, the velocity v_b^σ of ∂M_t at $x_0^\sigma(t)$ is

$$v_b^\sigma(x_0^\sigma,\,t) = (x_0^\sigma)'(t) = \frac{1}{\sqrt{r^2(t)-\sigma^{-2}}} = \frac{1}{\sqrt{1-2t}} \quad \text{on } \partial\Omega \text{ for } 0\leq t<\frac{1}{2} \tag{4.9}$$

$$v_b^\sigma(x_1^\sigma,\,t) = -(x_1^\sigma)'(t) = -\frac{-1}{\sqrt{r^2(t)-\sigma^{-2}}} = \frac{1}{\sqrt{1-2t}} \quad \text{on } \partial\Omega \text{ for } 0\leq t<\frac{1}{2}, \tag{4.10}$$

and

$$\frac{\sigma}{\tan\theta^\sigma} = \sigma\,\frac{\sigma^{-1}}{\sqrt{r^2(t)-\sigma^{-2}}} = \frac{1}{\sqrt{r^2(t)-\sigma^{-2}}} = \frac{1}{\sqrt{1-2t}} \quad \text{on } \partial\Omega\cap\partial M_t, \tag{4.11}$$

Thus, we obtain the equality that $v_b^\sigma = \sigma(\tan\theta)^{-1}$ on $\partial\Omega$ and this is exactly the same boundary condition as (1.1). From (4.9) and (4.10), we see that the velocities $(x_0^\sigma)'(t)$ and $(x_1^\sigma)'(t)$ on ∂M_t^σ are independent of σ. Hence, if the boundary condition of (1.1) corresponds to Dirichlet boundary conditions as $\sigma\to 0$, then the velocity of ∂M_t^σ should converge to 0 as $\sigma\to 0$, however, this contradicts the fact that the velocity of ∂M_t^σ is independent of σ. Therefore, this implies that, in the above example, the curvature flow with Dirichlet boundary conditions cannot be characterized by the one with boundary conditions (1.1) as $\sigma\to 0$. Indeed, the curve M_t^0 is actually the interval $[0,\,2]$ for all $t\geq 0$, which does not move for all the time.

Now we focus on how we can interpret this example on the level of the phase field method. To see this, we focus on the Dirichlet energy, one of the characteristic quantities in the phase field method. In the phase field method, a curve M_t^σ moving by its curvature can be expected to be the zero level set of $u^{\varepsilon,\sigma}$ satisfying the equation (1.10) and we may have that Ω is separated into the region that $u^{\varepsilon,\sigma}$ is almost $+1$ and the region that $u^{\varepsilon,\sigma}$ is almost -1 (see Figure 4).

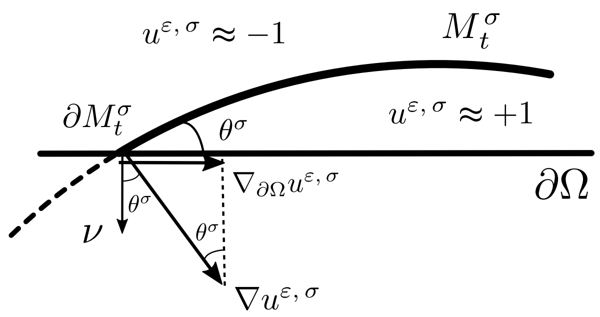

Figure 4: Interpretation in the phase field method

Thus we may calculate the Dirichlet energy on $\partial\Omega$ along ν as follows:

$$\begin{aligned}\int_{\partial\Omega} \varepsilon|\nabla u^{\varepsilon,\sigma}\cdot\nu|^2\, d\mathcal{H}^1 &= \int_{\partial\Omega} \varepsilon|\nabla u^{\varepsilon,\sigma}|^2 (\cos\theta^\sigma)^2\, d\mathcal{H}^1 \\ &= \int_{\partial\Omega} (\cos\theta^\sigma)^2 \frac{|\nabla u^{\varepsilon,\sigma}|^2}{|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|^2}\, \varepsilon|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|^2\, d\mathcal{H}^1 \\ &= \int_{\partial\Omega} \frac{(\cos\theta^\sigma)^2}{\sin\theta^\sigma} \frac{\varepsilon}{\sin\theta^\sigma} |\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|^2\, d\mathcal{H}^1, \end{aligned} \tag{4.12}$$

for each $0 \le t < \frac{1}{2}$. Recalling the phase field method and (2.24), we may have the following approximation;

$$\int_{\partial\Omega} \frac{\varepsilon}{\sin\theta^\sigma} |\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|^2\, d\mathcal{H}^1 \approx \int_{\partial\Omega} d\mathcal{H}^0 \lfloor_{\partial M_t^\sigma} = (\text{the number of elements of } \partial M_t^\sigma) = 2 \quad \text{as } \varepsilon \to 0. \tag{4.13}$$

Since we have $\sin\theta^\sigma = (\sqrt{1-2t+\sigma^{-2}})^{-1}\sqrt{1-2t}$ and $\cos\theta^\sigma = (\sqrt{1-2t+\sigma^{-2}})^{-1}\sigma^{-1}$, it holds that

$$\frac{(\cos\theta^\sigma)^2}{\sin\theta^\sigma} = \frac{\sigma^{-1}}{\sqrt{1-2t}} \frac{\sigma^{-1}}{\sqrt{1-2t+\sigma^{-2}}} \ge \frac{\sigma^{-1}}{\sqrt{2}}. \tag{4.14}$$

Therefore, from (4.13) and (4.14), we may obtain the following estimate:

$$\begin{aligned}\int_{\partial\Omega} \varepsilon|\nabla u^{\varepsilon,\sigma}\cdot\nu|^2\, d\mathcal{H}^1 &\ge \frac{1}{\sqrt{2}\sigma} \int_{\partial\Omega} \frac{\varepsilon}{\sin\theta^\sigma} |\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|^2\, d\mathcal{H}^1 \\ &\approx \frac{1}{\sqrt{2}\sigma} \mathcal{H}^0(\partial M_t^\sigma) = \frac{\sqrt{2}}{\sigma}, \end{aligned} \tag{4.15}$$

for each $t \in [0, \frac{1}{2})$. This implies that the Dirichlet energy on $\partial\Omega$ along ν goes to infinity as σ goes to zero for each t. Therefore, we may conclude that it is reasonable to assume (4.8) or this kind of the energy estimate for the Dirichlet energy in order to consider our definition of a Brakke flow.

4.1.2 Main theorem and important lemmas

Now we state a sequence of one definition, several lemmas and the main theorem.

Lemma 4.2. *Suppose that "General assumptions" and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold and $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset (0, 1)$ and $\{\sigma_j\}_{j\in\mathbb{N}} \subset (0, 1)$ are two families of parameters with $\varepsilon_i \to 0$ and $\sigma_j \to 0$. Let $\{u^{\varepsilon_i,\sigma_j}\}_{i,j\in\mathbb{N}}$ be the solutions of (1.10) and $\mu_t^{\varepsilon_i,\sigma_j}$ be as in (1.12). Then there exist a subsequence $\{\mu_t^{\varepsilon_j',\sigma_j'}\}_{j\in\mathbb{N}}$ and a family of Radon measures $\{\mu_t\}_{t\ge 0}$ on $\overline{\Omega}$ such that for all $t \ge 0$, $\mu_t^{\varepsilon_j',\sigma_j'} \rightharpoonup \mu_t$ as $j \to \infty$ on $\overline{\Omega}$. Moreover, for a.e. $t \ge 0$ and for all $j \in \mathbb{N}$, μ_t are $(n-1)$-rectifiable on $\overline{\Omega}$.*

Remark 4.3. As we mentioned in the case of Dirichlet boundary condition, the integrality of the Radon measure only in the interior Ω follows from the interior argument of Tonegawa [35] or Takasao and Tonegawa [34] by using the local monotonicity formula. Thus we have that $\sigma_0^{-1}\mu_t$ is a $(n-1)$-integral Radon measure in Ω for a.e. $t \ge 0$, where $\sigma_0 := \int_{-1}^1 \sqrt{2W(u)}\, du$.

Lemma 4.4. *Suppose that "General assumptions" and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold and* $\{\varepsilon'_i\}_{i\in\mathbb{N}}$ *and* $\{\sigma'_j\}_{j\in\mathbb{N}}$ *are as in Lemma 4.2. Let* $\{\alpha^{\varepsilon'_i,\,\sigma'_j}\}_{i,\,j\in\mathbb{N}}$ *be as in Definition 2.17. Then there exist a subsequence* $\{\alpha^{\varepsilon'_j,\,\sigma'_j}\}_{j\in\mathbb{N}}$ *(denoted the same index) and a Radon measure* α *on* $\partial\Omega\times[0,\infty)$ *such that* $\alpha^{\varepsilon'_j,\,\sigma'_j}\rightharpoonup\alpha$ *as* $j\to\infty$ *on* $\partial\Omega\times[0,\infty)$*. In addition, setting a vector-valued function* $\boldsymbol{v}_b^{\varepsilon'_j,\,\sigma'_j}:\partial\Omega\to\mathbb{R}^n$ *by*

$$\boldsymbol{v}_b^{\varepsilon'_j,\,\sigma'_j}:=\begin{cases}-\dfrac{\partial_t u^{\varepsilon'_j,\,\sigma'_j}}{|\nabla_{\partial\Omega}u^{\varepsilon'_j,\,\sigma'_j}|}\dfrac{\nabla_{\partial\Omega}u^{\varepsilon'_j,\,\sigma'_j}}{|\nabla_{\partial\Omega}u^{\varepsilon'_j,\,\sigma'_j}|} & \text{if } |\nabla_{\partial\Omega}u^{\varepsilon'_j,\,\sigma'_j}|\neq 0,\\ 0 & \text{otherwise,}\end{cases}\tag{4.16}$$

then $\boldsymbol{v}^{\varepsilon'_j,\,\sigma'_j}\xrightarrow[j\to\infty]{}0$ *in the sense that*

$$\lim_{j\to\infty}\int_{\partial\Omega\times[0,\,\infty)}\boldsymbol{g}\cdot\boldsymbol{v}_b^{\varepsilon'_j,\,\sigma'_j}\,d\alpha^{\varepsilon'_j,\,\sigma'_j}=0\tag{4.17}$$

for all $\boldsymbol{g}\in(C_c(\partial\Omega\times[0,\,\infty)))^n$.

Remark 4.5. In Lemma 4.4, the convergence of $\mathbf{v}^{\varepsilon'_j,\,\sigma'_j}$ means that there exists $\mathbf{v}_b\in(L^2(\alpha,\,\partial\Omega\times[0,\,\infty)))^n$ such that

$$\lim_{j\to\infty}\int_{\partial\Omega\times[0,\,\infty)}\mathbf{g}\cdot\mathbf{v}_b^{\varepsilon'_j,\,\sigma'_j}\,d\alpha^{\varepsilon'_j,\,\sigma'_j}=-\int_{\partial\Omega\times[0,\infty)}\mathbf{g}\cdot\mathbf{v}_b\,d\alpha\tag{4.18}$$

for any $\mathbf{g}\in(C_c(\partial\Omega\times[0,\,\infty)))^n$, and $\mathbf{v}_b=0$ in $(L^2(\alpha))^n$.

Lemma 4.6. *Suppose that "General assumptions" and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold and the space-dimension* n *is larger than 2. Let a subsequence* $\{\alpha^{\varepsilon'_j,\,\sigma'_j}\}_{j\in\mathbb{N}}$ *and* α *be as in Lemma 4.4. Moreover, setting*

$$w^j(x,\,t):=\Phi\circ u^{\varepsilon'_j,\,\sigma'_j}:=\int_0^{u^{\varepsilon'_j,\,\sigma'_j}(x,\,t)}\sqrt{2W(s)}\,ds=u^{\varepsilon'_j,\,\sigma'_j}(x,\,t)-\frac{1}{3}(u^{\varepsilon'_j,\,\sigma'_j}(x,\,t))^3\tag{4.19}$$

for any $j\in\mathbb{N}$ *and* $(x,\,t)\in\overline{\Omega}\times[0,\,\infty)$*, we assume that there exist* $0\le t_1<t_2<\infty$ *and a non-empty connected component* Γ_1 *of* $\partial\Omega$ *such that*

$$\limsup_{j\to\infty}\int_{t_1}^{t_2}\left|\int_{\Gamma_1}w^j\,d\mathcal{H}^{n-1}\right|\,dt<\frac{2}{3}\mathcal{H}^{n-1}(\Gamma_1)\,(t_2-t_1)<\infty.\tag{4.20}$$

holds. Then there exists a positive constant $0<\tilde{C}(t_1,\,t_2)<\infty$ *such that* $\alpha(\Gamma_1\times[t_1,\,t_2])\ge\tilde{C}(t_1,\,t_2)$*, which means that* α *is not identically zero.*

Remark 4.7. In Lemma 4.6, the assumption (4.20) means that there always exists a phase boundary on $\partial\Omega$, that is, there exists the boundary ∂M_t of the hypersurface M_t on $\partial\Omega$; otherwise we have that u^j is equal to either $+1$ or -1 almost everywhere on $\partial\Omega$ as $j\to\infty$ and thus this implies, from the definition of w^j,

$$\lim_{i\to\infty}\int_{t_1}^{t_2}\left|\int_{\Gamma}w^j\,d\mathcal{H}^{n-1}\right|\,dt=\frac{2}{3}\mathcal{H}^{n-1}(\Gamma)\,(t_2-t_1)\tag{4.21}$$

for any time $0<t_1<t_2<\infty$ and any connected component $\Gamma\subset\partial\Omega$, which contradicts (4.20).

Due to Lemma 4.2, we may define the unique rectifiable varifolds as follows:

Definition 4.8. We define a rectifiable varifold V_t associated with μ_t as follows: for $t\ge 0$ where μ_t is $(n-1)$-rectifiable on $\overline{\Omega}$,

$$V_t(\phi):=\int_{\overline{\Omega}}\phi(x,\,T_x\mu_t)\,d\mu_t(x)\tag{4.22}$$

for every $\phi\in C_c(G_{n-1}(\overline{\Omega}))$. For $t\ge 0$ where μ_t is not $(n-1)$-rectifiable, we define V_t by $V_t(\phi):=\int_{\overline{\Omega}}\phi(x,\,\mathbb{R}^{n-1}\times\{0\})\,d\mu_t(x)$ for every $\phi\in C_c(G_{n-1}(\overline{\Omega}))$. Here $T_x\mu_t$ is an approximate tangent space at x, which exists for μ_t-a.e. $x\in\overline{\Omega}$ because of the rectifiability of μ_t.

Lemma 4.9. *Suppose that "General assumptions", "Vanishing hypothesis for the discrepancy measure", and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold and let $\{V_t\}_{t\geq 0}$ be as in Definition 4.8. Then the following properties hold:*

$$\|\delta V_t\|(\overline{\Omega}) < \infty, \qquad \int_0^T \|\delta V_t\|(\overline{\Omega})\, dt < \infty \tag{4.23}$$

for a.e. $t \geq 0$ and all $T > 0$.

Now we state the main theorem of this subsection, that is, the approximation result of our Brakke flow with Dirichlet boundary conditions, which we defined in the previous section.

Theorem 4.10. *Suppose that "General assumptions", "Vanishing hypothesis for the discrepancy measure", and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold. Let V_t, α, and $\boldsymbol{v}_b$ be the quantities in Lemma 4.2 and 4.9, and Remark 4.5 (see also Definition 4.8), which are obtained from the singular limits of the Allen-Cahn equations* (1.10) *by taking ε, $\sigma \to 0$. Then the triplet $(\{V_t\}_{t\geq 0}, \alpha, \boldsymbol{v}_b)$ is a Brakke flow with Dirichlet boundary conditions in Definition 3.1 with $\|V_0\| = \sigma_0\, \mathcal{H}^{n-1}\lfloor_{M_0}$ where M_0 is as in Subsection 4.1.1. In addition, we have the estimate that*

$$\int_0^\infty \int_{\overline{\Omega}} |\widetilde{\boldsymbol{H}}_V|^2\, d\|V_t\|\, dt \leq D, \tag{4.24}$$

where $\widetilde{\boldsymbol{H}}_V$ is the modified generalized mean curvature vector.

Moreover, the assumption "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 actually leads us to obtain the stronger result that the total variation measure $\|\mathcal{S}_{\alpha,\, \boldsymbol{v}_b}\|$ is identically equal to zero on $\partial\Omega \times [0, \infty)$ (this is equivalent to the claim that $\boldsymbol{v}_b = 0$ in $(L^2(\alpha))^n$ on $\partial\Omega \times [0, \infty)$ as in Lemma 4.4).

4.2 Dynamic boundary condition

In this subsection, we will first give an assumption named "General assumptions" and a working hypothesis named "Vanishing hypothesis for the discrepancy measure" and then will state a sequence of the lemmas and the main theorem of the sharp interface limits of Allen-Cahn equations (1.10). Note that, in the following, we mainly consider the case $\sigma = 1$ for simplicity, which corresponds to the special case of dynamic boundary conditions in (3.10), however we do not omit the index σ in the following. One feature of the results in the case of dynamic boundary conditions is that we do not need to assume the uniform upper bound which we state in the assumption in the case of Dirichlet boundary conditions and it is described in Subsection 4.1.1.

4.2.1 Assumptions and hypothesis

General assumptions

We impose the general assumptions essentially same as those in Subsection 4.1.1. Note that, in the case of dynamic boundary conditions, the parameter $\sigma \in (0, \infty)$ is given and fixed.

Vanishing hypothesis for the discrepancy measure

As we mentioned in the case of Dirichlet boundary condition, we can show the existence of the convergent subsequence, independent of t, of the discrepancy measure $\{|\xi_t^{\varepsilon,\sigma}|\}_{\varepsilon>0}$ by virtue of Lemma 4.2 and the a priori estimate (5.2), which we will show later in Subsetion 4.2.2 and Section 5.

Then, in the case of dynamic boundary conditions, we also assume the vanishing of the discrepancy measure, that is,

$$\xi_t^\sigma \equiv 0 \quad \text{on } \overline{\Omega} \text{ for a.e. } t \in [0, \infty), \tag{4.25}$$

where ξ_t^σ is the limit measure of $|\xi_t^{\varepsilon,\sigma}|$ as $\varepsilon \to 0$. As we mentioned in the case of Dirichlet boundary condition, we may show the vanishing of ξ_t^σ in the interior of $\overline{\Omega}$ by virtue of Ilmanen and et al. and thus, if $\operatorname{spt} \mu_t^\sigma \cap \partial\Omega = \emptyset$, we have that $\xi_t^\sigma \equiv 0$ in $\overline{\Omega}$.

4.2.2 Main theorem and important lemmas

Now we state a sequence of several lemmas and the main theorem. As we mentioned before, in the following statements, we fix the index σ as 1 to show the results. However, we also try to consider how different the formulations between dynamic and right-angle Neumann boundary conditions are, which corresponds to the case that σ is positive and finite and the case $\sigma = \infty$, respectively. Thus, we do not omit the index σ for this purpose. In fact, if $\sigma \in (0, 1) \cup (1, \infty)$ and is fixed, we can prove the same lemmas and main theorem as those in the case $\sigma = 1$ by slightly modifying some of the proofs.

First of all, we give two important lemmas, which describe the convergence of the measures $\mu_t^{\varepsilon,\sigma}$ and $\alpha^{\varepsilon,\sigma}$. These lemmas are essentially the same results as Lemma 4.2 and 4.4 in Subsection 4.1.2.

Lemma 4.11. *Suppose that "General assumptions" in Subsection 4.2.1 holds and $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0, 1)$ is a family of parameters with $\varepsilon_j \to 0$ as $j \to \infty$. Let $\{u^{\varepsilon_j,\sigma}\}_{j\in\mathbb{N}}$ be a family of the solutions of (1.10) and $\mu_t^{\varepsilon_j,\sigma}$ be as in (1.12). Then there exist a subsequence $\{\mu_t^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$ and a family of Radon measures $\{\mu_t^\sigma\}_{t\geq 0}$ on $\overline{\Omega}$ such that for all $t \geq 0$, $\mu_t^{\varepsilon_j',\sigma} \rightharpoonup \mu_t^\sigma$ as $j \to \infty$ on $\overline{\Omega}$. Moreover, for a.e. $t \geq 0$, μ_t^σ is $(n-1)$-rectifiable on $\overline{\Omega}$.*

Remark 4.12. The integrality of the Radon measure μ_t^σ only in the interior Ω follows from the interior argument of Tonegawa [35] or Takasao and Tonegawa [34] by using the local monotonicity formula. Thus, we have that $\sigma_0^{-1}\mu_t^\sigma$ is a $(n-1)$-integral Radon measure in Ω for a.e. $t \geq 0$, where $\sigma_0 := \int_{-1}^1 \sqrt{2W(u)}\, du$.

Lemma 4.13. *Suppose that "General assumptions" in Subsection 4.2.1 holds and $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0, 1)$ is as in Lemma 4.11. Let $\{\alpha^{\varepsilon_j,\sigma}\}_{\varepsilon>0}$ be as in (2.22). Then there exist a subsequence $\{\alpha^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$ and a Radon measure α^σ on $\partial\Omega \times [0,\infty)$ such that $\alpha^{\varepsilon_j',\sigma} \rightharpoonup \alpha^\sigma$ as $j \to \infty$ on $\partial\Omega \times [0,\infty)$. In addition, there exists a function $\boldsymbol{v}_b^\sigma \in (L^2(\alpha^\sigma, \partial\Omega \times [0,\infty)))^n$ such that*

$$\lim_{j\to\infty} \int_0^\infty \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v}_b^{\varepsilon_j',\sigma}\, d\alpha^{\varepsilon_j',\sigma} = -\int\int_{\partial\Omega\times[0,\infty)} \boldsymbol{g} \cdot \boldsymbol{v}_b^\sigma\, d\alpha^\sigma \tag{4.26}$$

for all $g \in (C_c(\partial\Omega \times [0,\infty)))^n$ where $\boldsymbol{v}_b^{\varepsilon_j',\sigma}$ is as in (4.16), and such that

$$\int\int_{\partial\Omega\times[0,\,\infty)} |\boldsymbol{v}_b^\sigma|^2\, d\alpha^\sigma \leq \liminf_{j\to\infty} \int_0^\infty \int_{\partial\Omega} \varepsilon_j'(\partial_t u^{\varepsilon_j',\sigma})^2\, d\mathcal{H}^{n-1}\, dt. \tag{4.27}$$

Lemma 4.14. *Suppose that "General assumptions" in Subsection 4.1.1 holds and the space-dimension n is larger than 2. Let $\sigma = 1$ and let a subsequence $\{\alpha^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$ and α be as in Lemma 4.13. We set*

$$w^{j,\sigma}(x,\, t) := \Phi \circ u^{\varepsilon_j',\sigma} := \int_0^{u^{\varepsilon_j',\sigma}(x,\,t)} \sqrt{2W(s)}\, ds = u^{\varepsilon_j',\sigma}(x,\, t) - \frac{1}{3}(u^{\varepsilon_j',\sigma}(x,\, t))^3 \tag{4.28}$$

and $w_0^{j,\sigma}(x) := w^{j,\sigma}(x, 0)$ for any $j \in \mathbb{N}$ and $(x,\, t) \in \overline{\Omega} \times [0,\, \infty)$. Moreover, we assume the following two assumptions on the initial data;

1. *There exists a non-empty connected component Γ_2 of $\partial\Omega$ such that*

$$\limsup_{j\in\mathbb{N}} \left| \int_{\Gamma_2} w_0^{j,\sigma}\, d\mathcal{H}^{n-1} \right| < \frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2) \tag{4.29}$$

   holds.

2. $\lim_{\gamma_0\downarrow 0} \sup_{j\in\mathbb{N}} \mu_0^j(\Omega \cap \{x \mid \mathit{dist}(x,\, \partial\Omega) < \gamma_0\}) = 0$.

Then, there exists a time $s \in (0,\, \infty)$ such that, for any $0 < t_1 < t_2 < s$, there exists a positive constant $0 < \tilde{C}(t_1,\, t_2) < \infty$ such that $\alpha^\sigma(\Gamma_2 \times [t_1,\, t_2]) \geq \tilde{C}(t_1,\, t_2)$, which means that α is not identically zero.

Remark 4.15. In Lemma 4.14, the first assumption of the initial data means that there always exists a phase boundary on $\partial\Omega$, that is, there exists the boundary ∂M_0 of the hypersurface M_0 on $\partial\Omega$; otherwise we have that u_0^j is equal to either $+1$ or -1 almost everywhere on $\partial\Omega$ as $j \to \infty$ and thus this implies, from the definition of w_0^j,

$$\lim_{i\to\infty} \left| \int_\Gamma w_0^j\, d\mathcal{H}^{n-1} \right| = \frac{2}{3}\mathcal{H}^{n-1}(\Gamma) \tag{4.30}$$

for any connected component $\Gamma \subset \partial\Omega$, which contradicts (4.29).

Regarding to the second assumption of the initial data, it can be interpreted that the geometric interior of M_0 does not exist on the boundary $\partial\Omega$. If this is not true, we have that there exists a constant $\delta > 0$ such that, for any $\gamma > 0$, $\mathcal{H}^{n-1}\lfloor_{M_0}(\partial\Omega \cap \{x \mid \mathrm{dist}(x, \partial\Omega) < \gamma\}) \geq \delta$. From the convergence of μ_0^j to $\mathcal{H}^{n-1}\lfloor_{M_0}$, up to constants, as $j \to \infty$, we obtain the approximation

$$\begin{aligned}\mu_0^j(\Omega \cap \{x \mid \mathrm{dist}(x, \partial\Omega) < \gamma\}) &\approx \mu_0(\overline{\Omega} \cap \{x \mid \mathrm{dist}(x, \partial\Omega) < \gamma\}) \\ &\geq \mathcal{H}^{n-1}\lfloor_{M_0}(\partial\Omega \cap \{x \mid \mathrm{dist}(x, \partial\Omega) < \gamma\}) \geq \delta \end{aligned} \tag{4.31}$$

for $j \in \mathbb{N}$ large enough.

Due to Lemma 4.11, we may define the unique rectifiable varifolds as follows:

Lemma 4.16. *Suppose that "General assumptions" and "Vanishing hypothesis for the discrepancy measure" in Subsection 4.2.1 hold and let $\{V_t^\sigma\}_{t\geq 0}$ be the associated varifold with μ_t^σ (see the definition stated in Definition 4.8). Then the following properties hold:*

$$\|\delta V_t^\sigma\|(\overline{\Omega}) < \infty, \qquad \int_0^T \|\delta V_t^\sigma\|(\overline{\Omega})\, dt < \infty \tag{4.32}$$

for a.e. $t \geq 0$ and all $T \geq 0$ respectively.

Now we state the main theorem of this subsection, that is, the approximation result of our Brakke flow with dynamic boundary conditions which we defined in the previous section.

Theorem 4.17. *Suppose that "General assumptions" and "Vanishing hypothesis for the discrepancy measure" in Subsection 4.2.1 hold. Let V_t^σ, α^σ, and $\boldsymbol{v}_b^\sigma$ be the quantities in Lemma 4.11, 4.13, and 4.16, which are obtained from the singular limits of the Allen-Cahn equations* (1.10) *by taking $\varepsilon \to 0$. Then the triplet $(\{V_t^\sigma\}_{t\geq 0}, \alpha^\sigma, \boldsymbol{v}_b^\sigma)$ is a Brakke flow with dynamic boundary conditions in Definition 3.4 with $\|V_0^\sigma\| = \sigma_0\, \mathcal{H}^{n-1}\lfloor_{M_0}$ where M_0 is as in Subsection 4.2.1. Moreover, we have the estimate*

$$\int_0^\infty \int_{\overline{\Omega}} |\widetilde{\boldsymbol{H}}_V^\sigma|^2\, d\|V_t^\sigma\|\, dt \leq D, \tag{4.33}$$

where $\widetilde{\boldsymbol{H}}_V^\sigma$ is the modified generalized mean curvature vector.

5 A priori estimates for Allen-Cahn equations

In this section, we derive a priori estimate of Allen-Cahn equations (1.10) in the case that σ is in positive and finite. This estimate is important to consider the characterization of the singular limit in the case of both Dirichlet and dynamic boundary conditions.

Proposition 5.1. *It holds that*

$$\sup_{\varepsilon>0,\, \sigma>0} \left(E^{\varepsilon,\sigma}[u^{\varepsilon,\sigma}(\cdot\,, T)] + \int_0^T \left(\int_\Omega \varepsilon(\partial_t u^{\varepsilon,\sigma})^2\, dx + \int_{\partial\Omega} \frac{\varepsilon}{\sigma}(\partial_t u^{\varepsilon,\sigma})^2\, d\mathcal{H}^{n-1} \right) dt \right) \leq D, \tag{5.1}$$

for all $T > 0$. Here D is as in (4.5). *Moreover we have*

$$\sup_{\varepsilon>0,\, \sigma>0} \mu_t^{\varepsilon,\sigma}(\overline{\Omega}) \leq D, \tag{5.2}$$

for all $t \geq 0$.

Proof. By integration by parts, we can calculate in the following manner.

$$\begin{aligned}\frac{d}{dt}E^{\varepsilon,\sigma}[u^{\varepsilon,\sigma}] &= \int_\Omega \left(-\varepsilon\Delta u^{\varepsilon,\sigma} + \frac{W'(u^{\varepsilon,\sigma})}{\varepsilon} \right) \partial_t u^{\varepsilon,\sigma}\, dx + \int_{\partial\Omega} \varepsilon \frac{\partial u^{\varepsilon,\sigma}}{\partial\nu} \partial_t u^{\varepsilon,\sigma}\, d\mathcal{H}^{n-1} \\ &= -\int_\Omega \varepsilon(\partial_t u^{\varepsilon,\sigma})^2\, dx - \int_{\partial\Omega} \frac{\varepsilon}{\sigma}(\partial_t u^{\varepsilon,\sigma})^2\, d\mathcal{H}^{n-1}. \end{aligned} \tag{5.3}$$

Thus, for any $T > 0$, $\varepsilon > 0$ and $\sigma > 0$, we have

$$\begin{aligned}&E^{\varepsilon,\sigma}[u^{\varepsilon,\sigma}(\cdot,\, T)] + \int_0^T \left(\int_\Omega \varepsilon(\partial_t u^{\varepsilon,\sigma}))^2\, dx + \int_{\partial\Omega} \frac{\varepsilon}{\sigma}(\partial_t u^{\varepsilon,\sigma})^2\, d\mathcal{H}^{n-1} \right) dt \\ &= E^{\varepsilon,\sigma}[u_0^{\varepsilon,\sigma}] \leq D. \end{aligned} \tag{5.4}$$

Therefore (5.1) follows by taking supremum with respect to $\varepsilon > 0$ and $\sigma > 0$. From (4.5), (5.2) also easily follows. ☐

5.1 The case $\sigma \in (0, 1)$

In this subsection, we show the energy estimates of Allen-Cahn equations (1.10) on the boundary $\partial\Omega$ in the case $\sigma \in (0, 1)$. This estimate plays an important role in considering the singular limit and formulate a Brakke flow especially with Dirichlet boundary conditions. Note that we only have the energy estimate in an integration form with respect to time $t > 0$ so far. Note that we assume "General assumptions" and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1 in this case.

Proposition 5.2. *There exists $C_1 = C_1(n, \partial\Omega, D) > 0$ such that*

$$\sup_{\varepsilon>0,\, \sigma\in(0,\,1)} \int_{t_1}^{t_2} \int_{\partial\Omega} \Big(\frac{\varepsilon|\nabla_{\partial\Omega} u^{\varepsilon,\,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\,\sigma})}{\varepsilon} \Big)\, d\mathcal{H}^{n-1} dt \leq C_1(t_2 - t_1 + 1) + C_0, \tag{5.5}$$

for any $0 \leq t_1 \leq t_2 < \infty$, where $C_0 = C_0(t_1, t_2) > 0$ is as in (4.8)*.*

Proof. For any $\phi \in C^2(\overline{\Omega})$ and by using integration by part and denoting $f^{\varepsilon,\,\sigma} := -\varepsilon\Delta u^{\varepsilon,\,\sigma} + \frac{W'(u^{\varepsilon,\,\sigma})}{\varepsilon}$, we may obtain

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \phi\, d\mu_t^{\varepsilon,\,\sigma}\right) &= \int_\Omega \phi \left(-\varepsilon\Delta u^{\varepsilon,\,\sigma} + \frac{W'(u^{\varepsilon,\,\sigma})}{\varepsilon}\right) \partial_t u^{\varepsilon,\,\sigma}\, dx - \int_\Omega \varepsilon(\nabla\phi \cdot \nabla u^{\varepsilon,\,\sigma})\partial_t u^{\varepsilon,\,\sigma}\, dx \\
&\quad + \int_{\partial\Omega} \varepsilon\phi \frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu} \partial_t u^{\varepsilon,\,\sigma}\, d\mathcal{H}^{n-1} \\
&= -\frac{1}{\varepsilon}\int_\Omega \phi(f^{\varepsilon,\,\sigma})^2\, dx + \int_\Omega f^{\varepsilon,\,\sigma}(\nabla\phi \cdot \nabla u^{\varepsilon,\,\sigma})\, dx \\
&\quad - \int_{\partial\Omega} \frac{\varepsilon}{\sigma}\phi(\partial_t u^{\varepsilon,\,\sigma})^2\, d\mathcal{H}^{n-1}.
\end{aligned} \tag{5.6}$$

By integration by parts again,

$$\begin{aligned}
\int_\Omega f^{\varepsilon,\,\sigma}(\nabla\phi \cdot \nabla u^{\varepsilon,\,\sigma})\, dx &= -\int_\Omega \Delta\phi\, d\mu_t^{\varepsilon,\,\sigma} + \int_\Omega \varepsilon(\nabla u^{\varepsilon,\,\sigma} \otimes \nabla u^{\varepsilon,\,\sigma} : \nabla^2\phi)\, dx \\
&\quad - \int_{\partial\Omega} \varepsilon(\nabla\phi \cdot \nabla u^{\varepsilon,\,\sigma}) \frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu}\, d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu}\Big(\frac{\varepsilon|\nabla u^{\varepsilon,\,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\,\sigma})}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}.
\end{aligned} \tag{5.7}$$

Therefore we can compute as follows.

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \phi\, d\mu_t^{\varepsilon,\,\sigma}\right) &= -\frac{1}{\varepsilon}\int_\Omega \phi(f^{\varepsilon,\,\sigma})^2\, dx - \int_\Omega \Delta\phi\, d\mu_t^{\varepsilon,\,\sigma} + \int_\Omega \varepsilon(\nabla u^{\varepsilon,\,\sigma} \otimes \nabla u^{\varepsilon,\,\sigma} : \nabla^2\phi)\, dx \\
&\quad - \int_{\partial\Omega} \varepsilon(\nabla\phi \cdot \nabla u^{\varepsilon,\,\sigma}) \frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu}\, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \varepsilon\,\sigma\phi \left(\frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} \left(\frac{\varepsilon|\nabla u^{\varepsilon,\,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1}.
\end{aligned} \tag{5.8}$$

Specifically, we can choose $\phi = d + 1$, where d is a signed distance function from the boundary $\partial\Omega$ which is positive in the domain Ω. However, since Ω is general open domain with smooth boundary, d is smooth only in some open neighborhood of the boundary $\partial\Omega$. Thus we have to extend d smoothly into $\mathbb{R}^n$ such that $|d|$ and $|\nabla^2 d|$ are uniformly bounded in Ω. This extension can be done by a simple argument.
Thus, by using $\nabla\phi = -\nu$ and $\phi = 1$ on $\partial\Omega$ and the fact $\sigma \in (0, 1)$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \phi\, d\mu_t^{\varepsilon,\,\sigma}\right) &\leq -\int_\Omega \Delta\phi\, d\mu_t^{\varepsilon,\,\sigma} + \int_{\Omega\cap\{|\nabla u^{\varepsilon,\,\sigma}|\neq 0\}} (a^{\varepsilon,\,\sigma} \otimes a^{\varepsilon,\,\sigma} : \nabla^2\phi)\varepsilon|\nabla u^{\varepsilon,\,\sigma}|^2\, dx \\
&\quad + \int_{\partial\Omega} \frac{\varepsilon}{2}\left(\frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1} - \int_{\partial\Omega}\left(\frac{\varepsilon|\nabla_{\partial\Omega} u^{\varepsilon,\,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1},
\end{aligned} \tag{5.9}$$

where $a^{\varepsilon,\,\sigma}$ is defined by $\frac{\nabla u^{\varepsilon,\,\sigma}}{|\nabla u^{\varepsilon,\,\sigma}|}$. Note that, in (5.9), we used the fact that

$$|\nabla u^{\varepsilon,\,\sigma}|^2 = |\nabla_{\partial\Omega} u^{\varepsilon,\,\sigma}|^2 + \Big(\frac{\partial u^{\varepsilon,\,\sigma}}{\partial\nu}\Big)^2, \quad \text{on } \overline{\Omega} \times [0, \infty). \tag{5.10}$$

Recalling the estimate (5.1) and the assumption (4.8), and integrating both members of the inequality (5.9) from time t_1 to t_2 , we obtain

$$\begin{aligned}\int_\Omega \phi\, d\mu_{t_2}^{\varepsilon,\sigma} - \int_\Omega \phi\, d\mu_{t_1}^{\varepsilon,\sigma} &\le \sup_\Omega |\nabla^2\phi| \int_{t_1}^{t_2} \mu_t^{\varepsilon,\sigma}(\Omega)\, dt + 2\sup_\Omega |\nabla^2\phi| \int_{t_1}^{t_2} \mu_t^{\varepsilon,\sigma}(\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|\neq 0\})\, dt \\ &\quad + \int_{t_1}^{t_2}\int_{\partial\Omega} \frac{\varepsilon}{2}\left(\frac{\partial u^{\varepsilon,\sigma}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1}\, dt - \int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1} \\ &\le 3D \sup_\Omega |\nabla^2\phi|\,(t_2-t_1) + C_0(t_1,\, t_2) - \int_{t_1}^{t_2}\int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1}. \end{aligned} \tag{5.11}$$

Since ϕ is bounded in C^2-norm and (5.2), we have

$$\int_{t_1}^{t_2}\int_{\partial\Omega}\Big(\frac{\varepsilon|\nabla_{\partial\Omega} u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\Big) d\mathcal{H}^{n-1}\, dt \le 3D\, \|\phi\|_{C^2(\overline{\Omega})}(t_2-t_1+1) + C_0(t_1,\, t_2) < +\infty. \tag{5.12}$$

Therefore (5.5) follows by taking the supremum with respect to $\varepsilon > 0$ and $\sigma > 0$ in (5.12). □

5.2 The case $\sigma \in [1,\, \infty)$

In this subsection, we show the energy estimate of Allen-Cahn equations on the boundary $\partial\Omega$ in the case $\sigma \in [1,\, \infty)$. This estimate plays an important role in considering the singular limit and formulate a Brakke flow with dynamic boundary conditions in the case $\sigma \in [1,\, \infty)$. Note that, as same as the case $\sigma \in (0,\, 1)$, we only have the energy estimate in an integration form with respect to time $t > 0$. Note that we only assume "General assumptions" in Subsection 4.2 in this case.

Proposition 5.3. *There exists $C_1 = C_1(n, \partial\Omega, D) > 0$ such that*

$$\sup_{\varepsilon>0,\, \sigma\in[1,\,\infty)} \int_{t_1}^{t_2}\int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1} dt \le C_1(t_2-t_1+1), \tag{5.13}$$

for any $0 \le t_1 \le t_2 < \infty$.

Proof. For any $\phi \in C^2(\overline{\Omega})$ and by applying the same argument in the proof of Proposition 5.2, we may obtain the identity (5.8). Here, as we stated in Proposition 5.2, we choose $\phi = d+1$, where d is a signed distance function from the boundary $\partial\Omega$ which is positive in the domain Ω. Note that this d is also smoothly extended into the function whose domain is $\mathbb{R}^n$. Thus, by using the properties of the signed distance function, we have

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega \phi\, d\mu_t^{\varepsilon,\sigma}\right) &\le -\int_\Omega \Delta\phi\, d\mu_t^{\varepsilon,\sigma} + \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|\neq 0\}} (a^{\varepsilon,\sigma}\otimes a^{\varepsilon,\sigma} : \nabla^2\phi)\varepsilon|\nabla u^{\varepsilon,\sigma}|^2\, dx \\ &\quad + \int_{\partial\Omega} \frac{\varepsilon}{\sigma^2}(\partial_t u^{\varepsilon,\sigma})^2\, d\mathcal{H}^{n-1} - \int_{\partial\Omega}\frac{\varepsilon}{\sigma}(\partial_t u^{\varepsilon,\sigma})^2\, d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1}, \end{aligned} \tag{5.14}$$

where $a^{\varepsilon,\sigma}$ is defined by $\frac{\nabla u^{\varepsilon,\sigma}}{|\nabla u^{\varepsilon,\sigma}|}$. Hence, by integrating both members of the inequality (5.14) from time t_1 to t_2 and using the estimate (5.1) and the fact that σ is in $[1,\, \infty)$, we have

$$\begin{aligned}\int_\Omega \phi\, d\mu_{t_2}^{\varepsilon,\sigma} - \int_\Omega \phi\, d\mu_{t_1}^{\varepsilon,\sigma} &\le \sup_\Omega |\nabla^2\phi| \int_{t_1}^{t_2} \mu_t^{\varepsilon,\sigma}(\Omega)\, dt + 2\sup_\Omega |\nabla^2\phi| \int_{t_1}^{t_2} \mu_t^{\varepsilon,\sigma}(\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|\neq 0\})\, dt \\ &\quad - \int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1} \\ &\le 3D \sup_\Omega |\nabla^2\phi|\,(t_2-t_1) - \int_{t_1}^{t_2}\int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1}. \end{aligned} \tag{5.15}$$

Thus, recalling the choice of ϕ, we may obtain

$$\int_{t_1}^{t_2}\int_{\partial\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1}\, dt \le 3D\, \|\phi\|_{C^2(\overline{\Omega})}(t_2-t_1+1) < +\infty. \tag{5.16}$$

Therefore (5.13) follows by taking the supremum with respect to $\varepsilon > 0$ in (5.16). □

6 Characterization of the limits

In this section, we will show the proofs of a sequence of the main results in each case of Dirichlet or dynamic boundary conditions.

6.1 Dirichlet boundary condition

In this section, we will prove a sequence of the main results which we stated in Section 4.1.2. We note that the positive constants C_1 and D are as in Proposition 5.1 and Proposition 5.2.

6.1.1 Convergence of the measures $\{\mu_t^{\varepsilon,\sigma}\}_{\varepsilon,\sigma>0,t\geq 0}$ (Dirichlet boundary conditions)

First of all, in order to prove the convergence of $\{\mu_t^{\varepsilon,\sigma}\}_{\varepsilon>0,\sigma>0}$ for all $t\geq 0$, we derive an estimate on the change of the diffuse surface area measures in time. The main idea in the following proof comes from Mugnai and Röger [25]. In the following, we set

$$\mu_t^{\varepsilon,\sigma}(\phi) := \int_\Omega \phi\, d\mu_t^{\varepsilon,\sigma} \tag{6.1}$$

for all $\phi\in C_c(\Omega)$.

Note that we assume that "General assumptions" in Subsection 4.1.1 holds through this subsection.

Lemma 6.1. *Let $T>0$ be arbitrary. Then we have, for all $\phi\in C_c^1(\overline{\Omega})$,*

$$\sup_{\varepsilon,\sigma>0}\int_0^T \left|\frac{d}{dt}\mu_t^{\varepsilon,\sigma}(\phi)\right| dt \leq D\Big(T+\frac{3}{2}\Big)\|\phi\|_{C^1(\overline{\Omega})}<\infty. \tag{6.2}$$

Proof. Let ϕ be in $C_c^1(\overline{\Omega})$ and $T>0$ be any time. From the calculation in the proof of Proposition 5.2, we have

$$\begin{aligned}\left|\frac{d}{dt}\mu_t^{\varepsilon,\sigma}(\phi)\right| &= \left|-\int_\Omega \varepsilon\phi(\partial_t u^{\varepsilon,\sigma})^2\,dx - \int_{\partial\Omega}\frac{\varepsilon}{\sigma}\phi(\partial_t u^{\varepsilon,\sigma})^2\,d\mathcal{H}^{n-1} - \int_\Omega \varepsilon\partial_t u^{\varepsilon,\sigma}\nabla u^{\varepsilon,\sigma}\cdot\nabla\phi\,dx\right|\\ &\leq \|\phi\|_{C^1(\overline{\Omega})}\left(\int_\Omega \varepsilon(\partial_t u^{\varepsilon,\sigma})^2\,dx + \int_{\partial\Omega}\frac{\varepsilon}{\sigma}\phi(\partial_t u^{\varepsilon,\sigma})^2\,d\mathcal{H}^{n-1}\right)\\ &\quad + \frac{1}{2}\|\phi\|_{C^1(\overline{\Omega})}\int_\Omega \left(\varepsilon(\partial_t u^{\varepsilon,\sigma})^2+\varepsilon|\nabla u^{\varepsilon,\sigma}|^2\right)\,dx.\end{aligned} \tag{6.3}$$

Thus by integrating 0 to T and using the estimates (5.1), we obtain the conclusion. $\square$

Proof of a part of Lemma 4.2. Let $T>0$ be fixed. Choose the countable family $\{\phi_k\}_{k\in\mathbb{N}}$ of $C_c^1(\overline{\Omega})$ which is dense in $C_c(\overline{\Omega})$. Since we have $\sup_{i\in\mathbb{N}}\|\mu_t^{\varepsilon_i,\sigma_i}(\phi_k)\|_{BV(0,T)}<\infty$ from (5.2) and Lemma 6.1, we may apply the compactness for BV functions and thus, by the diagonal argument, there exist a subsequence independent of $k\in\mathbb{N}$ (denoted the same index) and a family of BV functions $\{f_k\}_{k\in\mathbb{N}}$ such that

$$\mu_t^{\varepsilon_i,\sigma_i}(\phi_k)\xrightarrow[i\to\infty]{} f_k(t)\quad\text{for a.e. } t\in(0,T), \tag{6.4}$$

$$\left|\frac{d}{dt}\mu_t^{\varepsilon_i,\sigma_i}(\phi_k)\right|\mathcal{L}^1 \rightharpoonup |Df_k|\quad\text{in Radon measures on } (0,T) \tag{6.5}$$

as $i\to\infty$. We note that Df_k is a signed measure on $(0,T)$ derived by Riesz representation theorem and its total variation $|Df_k|$ is characterized by

$$|Df_k|(U)=\sup\left\{\int_0^T f_k\phi'\,dt\,\middle|\,\phi\in C_c^1(0,T),\ \operatorname{spt}\phi\subset U,\ \|\phi\|_\infty\leq 1\right\} \tag{6.6}$$

for all open set $U\subset(0,T)$. Generally, the set of discontinuous points for functions of bounded variations is at most countable and thus we can choose a countable set S such that, for all $k\in\mathbb{N}$, f_k is continuous on $(0,T)\setminus S$.

Next we claim that (6.4) holds on $(0,T)\setminus S$. To see this, we take an arbitrary $t\in(0,T)\setminus S$ and choose a sequence $\{t_l\}_{l\in\mathbb{N}}$ such that $t_l\to t$ and (6.4) holds for all t_l. We then have, by (6.5)

$$\lim_{l\to\infty}\limsup_{i\to\infty}\left(\left|\frac{d}{dt}\mu_t^{\varepsilon_i,\sigma_i}(\phi_k)\right|\mathcal{L}^1\right)([t_l,t])\leq\lim_{l\to\infty}|Df_k|([t_l,t])=0, \tag{6.7}$$

for all $k \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}|f_k(t) - \mu_t^{\varepsilon_i,\sigma_i}(\phi_k)| &\le |f_k(t) - f_k(t_l)| + |f_k(t_l) - \mu_{t_l}^{\varepsilon_i,\sigma_i}(\phi_k)| + |\mu_{t_l}^{\varepsilon_i,\sigma_i}(\phi_k) - \mu_t^{\varepsilon_i,\sigma_i}(\phi_k)| \\ &\le |f_k(t) - f_k(t_l)| + |f_k(t_l) - \mu_{t_l}^{\varepsilon_i,\sigma_i}(\phi_k)| + \left(\left| \frac{d}{dt}\mu_t^{\varepsilon_i,\sigma_i}(\phi_k) \right| \mathcal{L}^1 \right)([t_l,\, t]). \end{aligned} \tag{6.8}$$

Then we first take $i \to \infty$ and, after that, take $l \to \infty$ to conclude that (6.4) holds for all $t \in (0,\, T) \setminus S$.

Now let t be in $(0,\, T) \setminus S$. Since we have the estimate (5.1), there exist a subsequence of $\{\varepsilon_i\}_{i\in\mathbb{N}}$ (denoted the same index) and a Radon measure μ_t such that $\mu_t^{\varepsilon_i,\sigma_i} \rightharpoonup \mu_t$ as $i \to \infty$ on $\overline{\Omega}$. Hence we deduce that $\mu_t(\phi_k) = f_k(t)$ for all $k \in \mathbb{N}$. Since a family $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c^1(\overline{\Omega})$ is dense in $C_c(\overline{\Omega})$, it holds that, for all $\phi \in C_c(\overline{\Omega})$ and all $t \in (0,\, T) \setminus S$,

$$\mu_t^{\varepsilon_i,\sigma_i}(\phi) \to \mu_t(\phi) \quad \text{on } \overline{\Omega}. \tag{6.9}$$

After taking another subsequence (denoted the same index), we can also ensure that

$$\mu_0^{\varepsilon_i,\sigma_i} \rightharpoonup \mu_0 \tag{6.10}$$

in Radon measures on $\overline{\Omega}$. Therefore we can deduce that there exist a subsequence $\{\mu_t^{\varepsilon_i,\sigma_i}\}_{i\in\mathbb{N}}$ and a Radon measure μ_t on $\overline{\Omega}$ such that, for all $t \in [0,\, T) \setminus S$, we have $\mu_t^{\varepsilon_i,\sigma_i} \rightharpoonup \mu_t$ as $i \to \infty$ on $\overline{\Omega}$. Since the set S is countable, we may apply the further diagonal argument and then we can choose a further subsequence(denoted by the same index) such that $\mu_t^{\varepsilon_i,\sigma_i}$ converges to some Radon measure μ_t as $i \to \infty$ on $\overline{\Omega}$ for all $t \in [0,\, T)$.

Finally, we may conclude that there exist a subsequence and a Radon measure μ_t such that $\mu_t^{\varepsilon_i,\sigma_i} \rightharpoonup \mu_t$ on $\overline{\Omega}$ for all $t \in [0, \infty)$ from the fact $[0, \infty) = \bigcup_{n=1}^{\infty}[n-1,\, n)$ and by using the diagonal argument. □

6.1.2 Convergence of the measures $\{\alpha^{\varepsilon,\sigma}\}_{\varepsilon>0,\,\sigma>0}$ and proof of Lemma 4.4 and Lemma 4.6 (Dirichlet boundary conditions)

In this subsection, we show the existence of the convergent subsequence of a family of $\{\alpha^{\varepsilon,\sigma}\}_{\varepsilon,\,\sigma>0}$ and we also prove Lemma 4.4 and 4.6. Note that we assume that "General assumptions" and "Uniform upper bound for the solution of Allen-Cahn equations" hold in this subsection.

Proof of Lemma 4.4. Let $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ and $\{\sigma_j'\}_{j\in\mathbb{N}}$ be subsequences such that Lemma 4.2 holds. First of all, we show that there exist a subsequence (denoted the same index) $\{\alpha^{\varepsilon_j',\,\sigma_j'}\}_{j\in\mathbb{N}}$ and a Radon measure α on $\partial\Omega \times [0, \infty)$ such that $\alpha^{\varepsilon_j',\,\sigma_j'} \rightharpoonup \alpha$ as $j \to \infty$ on $\partial\Omega \times [0, \infty)$.

Let $T > 0$ be any time. Then, from Proposition 5.2, we have for all $j \in \mathbb{N}$

$$\begin{aligned}\alpha^{\varepsilon_j',\,\sigma_j'}(\partial\Omega \times [0,\, T]) &= \int_0^T \int_{\partial\Omega} \varepsilon_j' |\nabla_{\partial\Omega} u^{\varepsilon_j',\,\sigma_j'}|^2 \, d\mathcal{H}^{n-1}\, dt \\ &\le 2\int_0^T \int_{\partial\Omega} \left(\frac{\varepsilon_j' |\nabla_{\partial\Omega} u^{\varepsilon_j',\,\sigma_j'}|^2}{2} + \frac{W(u^{\varepsilon_j',\,\sigma_j'})}{\varepsilon_j'} \right) d\mathcal{H}^{n-1}\, dt \\ &\le 2C_1\, T + C_0(T). \end{aligned} \tag{6.11}$$

Thus we have the locally uniform boundedness, that is, $\sup_{j\in\mathbb{N}} \alpha^{\varepsilon_j',\,\sigma_j'}(K) < +\infty$ for any compact subset $K \subset \partial\Omega \times [0, \infty)$. Since this shows that we can apply the compactness theorem for Radon measures, we may conclude that there exist a subsequence (denoted the same index) $\{\alpha^{\varepsilon_j',\,\sigma_j'}\}_{j\in\mathbb{N}}$ and a Radon measure α on $\partial\Omega \times [0,\, \infty)$ such that $\alpha^{\varepsilon_j',\,\sigma_j'} \rightharpoonup \alpha$ as $j \to \infty$. This completes the proof of the first claim.

Secondly, we will prove the claims (4.17) and that $\mathbf{v}_b = 0$ in $(L^2(\alpha))^n$ as in Lemma 4.4. We take any $j \in \mathbb{N}$. From (4.16), (5.1), and the fact that $\sigma_j' \in (0,\, 1)$, we deduce that

$$\int_0^\infty \int_{\partial\Omega} |\mathbf{v}_b^{\varepsilon_j',\,\sigma_j'}|^2 \, d\alpha^{\varepsilon_j',\,\sigma_j'} \le \int_0^\infty \int_{\partial\Omega} \frac{\varepsilon_j'}{\sigma_j'} (\partial_t u^{\varepsilon_j',\,\sigma_j'})^2 \, d\mathcal{H}^{n-1}\, dt \le D. \tag{6.12}$$

Therefore $(\alpha^{\varepsilon_j',\,\sigma_j'},\ \mathbf{v}_b^{\varepsilon_j',\,\sigma_j'})$ is a measure-function pair which satisfies the L^2-uniform boundedness with respect to $j \in \mathbb{N}$. Since we have the convergence such that $\alpha^{\varepsilon_j',\,\sigma_j'} \rightharpoonup \alpha$ as $j \to \infty$ on $\partial\Omega \times [0, \infty)$ and we can apply the theorem [14, Theorem 4.4.2.], we may conclude that there exist a subsequence (denoted the same index) and a function $\mathbf{v}_b \in (L^2(\alpha,\, \partial\Omega \times [0,\infty)))^n$ such that

$$\begin{aligned}\lim_{j\to\infty} \int_0^\infty \int_{\partial\Omega} \mathbf{g}\cdot (\varepsilon_j' \partial_t u^{\varepsilon_j',\,\sigma_j'} \nabla_{\partial\Omega} u^{\varepsilon_j',\,\sigma_j'})\, d\mathcal{H}^{n-1}\, dt &= -\lim_{j\to\infty} \int_{\partial\Omega\times[0,\infty)} \mathbf{g}\cdot \mathbf{v}_b^{\varepsilon_j',\,\sigma_j'}\, d\alpha^{\varepsilon_j',\,\sigma_j'} \\ &= -\int_{\partial\Omega\times[0,\infty)} \mathbf{g}\cdot\mathbf{v}_b\, d\alpha \end{aligned} \tag{6.13}$$

for all $\mathbf{g} \in (C_c(\partial\Omega\times[0,\infty)))^n$ and moreover,

$$\int_{\partial\Omega\times[0,\,\infty)} |\mathbf{v}_b|^2\, d\alpha \le \liminf_{j\to\infty} \int_0^\infty \int_{\partial\Omega} \varepsilon_j' (\partial_t u^{\varepsilon_j',\,\sigma_j'})^2\, d\mathcal{H}^{n-1}\, dt \tag{6.14}$$

holds. Here, from (5.1), we have

$$\begin{aligned}\limsup_{j\to\infty} \int_0^\infty \int_{\partial\Omega} \varepsilon_j' (\partial_t u^{\varepsilon_j',\,\sigma_j'})^2\, d\mathcal{H}^{n-1}\, dt &= \limsup_{j\to\infty} \left(\sigma_j' \int_0^\infty \int_{\partial\Omega} \frac{\varepsilon_j'}{\sigma_j'} \left(\partial_t u^{\varepsilon_j',\,\sigma_j'}\right)^2 d\mathcal{H}^{n-1}\, dt \right) \\ &\le (\lim_{j\to\infty} \sigma_j')\, D < \infty \\ &= 0. \end{aligned} \tag{6.15}$$

Therefore, from (6.14) and (6.15), we obtain that $\mathbf{v}_b = 0$ in $(L^2(\alpha))^n$ on $\partial\Omega\times[0,\,\infty)$ and this completes the proof of Lemma 4.4 and also the proof of the claim in Remark 4.5. □

Next we prove Lemma 4.6 in the following.

Proof of Lemma 4.6. Since $W^{1,\,2}$ is dense in C^∞ with respect to $W^{1,\,2}$-topology, it is sufficient to show the claim when $u^{\varepsilon_j',\,\sigma_j'}$ is smooth on $\partial\Omega\times(0,\,T)$ for some $T > 0$. From the assumption in Theorem 4.6, we can choose $t_1,\, t_2 \in [0,\,T]$ and Γ_1 such that $0 \le t_1 < t_2 \le T$ and Γ_1 is a non-empty connected component of $\partial\Omega$ and then we fix these quantities. First of all, from the estimate (4.6), (5.1), (5.5) and the definition of w^j, we have the following two inequalities;

$$\sup_{j\in\mathbb{N}} \int_{t_1}^{t_2}\int_{\Gamma_1} |w^j|\, d\mathcal{H}^{n-1}\, dt \le \mathcal{H}^{n-1}(\Gamma_1)\,(t_2 - t_1) < \infty, \tag{6.16}$$

$$\sup_{j\in\mathbb{N}} \int_{t_1}^{t_2}\int_{\Gamma_1} \left(|\nabla_{\partial\Omega} w^j| + |\partial_t w^j|\right)\, d\mathcal{H}^{n-1}\, dt \le \tilde{C}_1(t_1,\, t_2,\, D) < \infty, \tag{6.17}$$

where $\tilde{C}_1(t_1,\, t_2,\, D)$ is some positive constant depending only on Ω, t_1, t_2, and D. Here $D > 0$ is as in Propositon 5.1. Indeed, we can show (6.17) in the following way; for any $j \in \mathbb{N}$ and $0 < t_1 < t_2 < T$, we have, from Schwarz inequality,

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Gamma_1} \left(|\nabla_{\partial\Omega} w^j| + |\partial_t w^j|\right)\, d\mathcal{H}^{n-1}\, dt &= \int_{t_1}^{t_2}\int_{\Gamma_1} \sqrt{2W(u^{\varepsilon_j',\,\sigma_j'})}\left(|\nabla_{\partial\Omega} u^{\varepsilon_j',\,\sigma_j'}| + \partial_t u^{\varepsilon_j',\,\sigma_j'}\right)\, d\mathcal{H}^{n-1}\, dt \\ &\le 2\int_{t_1}^{t_2}\int_{\Gamma_1} \left(\frac{\varepsilon_j |\nabla_{\partial\Omega} u^{\varepsilon_j',\,\sigma_j'}|^2}{2} + \frac{W(u^{\varepsilon_j',\,\sigma_j'})}{\varepsilon_j} \right) d\mathcal{H}^{n-1}\, dt \\ &\quad + \frac{1}{2}\int_{t_1}^{t_2}\int_{\Gamma_1} \varepsilon_j (\partial_t u^{\varepsilon_j',\,\sigma_j'})^2\, d\mathcal{H}^{n-1}\, dt \\ &\le 2C_1(t_2 - t_1) + 2C_1 + \frac{1}{2}D, \end{aligned} \tag{6.18}$$

where C_1 is as in Proposition 5.3. Hence, $\{w^j\}_{j\in\mathbb{N}}$ is a bounded sequence in $BV(\Gamma_1 \cap [t_1,\, t_2])$.

Since $\partial\Omega$ is a smooth $(n-1)$-dimensional embedded manifold in $\mathbb{R}^n$, we can choose one atlas $(V_a,\, \varphi_a)$ of $\partial\Omega$ at $p \in \Gamma_1$ such that $\varphi_a : V_a \to \varphi_a(V_a) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism. Replacing Γ_1 with one of the connected components of $\Gamma_1 \cap V_a$, if necessary, we may assume that $\Gamma_1 \subset V_a$ holds. Then, we can show that $\{w^j(\varphi_a^{-1})\}_{j\in\mathbb{N}}$ is also a bounded sequence in $BV(\varphi_a(\Gamma_1)\times[t_1,\, t_2])$. Thus, we may apply the compactness theorem for BV functions to $\{w^j(\varphi_a^{-1})\}_{j\in\mathbb{N}}$ and then we have that there exist a subsequence $\{w^{j_i}\}_{i\in\mathbb{N}}$ and $\tilde{w}^\infty \in BV(\varphi_a(\Gamma_1)\times[t_1,\, t_2])$ such that $w^{j_i}(\varphi_a^{-1}) \to \tilde{w}^\infty$ in L^1-topology as $i\to\infty$. Defining w^∞ by $\tilde{w}^\infty(\varphi_a)$ and taking another subsequence (denoted the same index), we have that

$w^{j_i}(\varphi_a^{-1}) \to w^\infty(\varphi^{-1})$ $(\mathcal{L}^{n-1}\otimes\mathcal{L}^1)$-a.e. in $\varphi_a(\Gamma_1)\times[t_1,\, t_2]$ as $i\to\infty$. Then, setting $u^{j_i} := u^{\varepsilon'_{j_i},\,\sigma'_{j_i}}$ and $u^\infty := \Phi^{-1}\circ w^\infty$, we have that $u^{j_i}(\varphi_a^{-1}) \to u^\infty(\varphi_a^{-1})$ $(\mathcal{L}^{n-1}\otimes\mathcal{L}^1)$-a.e. in $\varphi_a(\Gamma_1)\times[t_1,\, t_2]$. These imply that $w^{j_i}\to w^\infty$ and $u^{j_i}\to u^\infty$ $(\mathcal{H}^{n-1}\otimes\mathcal{L}^1_t)$-a.e. on $\Gamma_1\times[t_1,\, t_2]$.

Moreover, from a priori estimate (5.5) and the dominated convergence theorem, we have

$$0 \le \int_{t_1}^{t_2}\!\!\int_{\Gamma_1} W(u^\infty)\, d\mathcal{H}^{n-1}\, dt \xleftarrow[i\to\infty]{} \int_{t_1}^{t_2}\!\!\int_{\Gamma_1} W(u^{j_i})\, d\mathcal{H}^{n-1}\, dt \le \varepsilon_{j_i}\tilde{C}(t_1,\, t_2) \xrightarrow[i\to\infty]{} 0, \tag{6.19}$$

and thus we conclude that $u^\infty = \pm 1$ $(\mathcal{H}^{n-1}\otimes\mathcal{L}^1_t)$-a.e. on $\Gamma_1\times[t_1,\, t_2]$ and this yields that $w^\infty = \pm\frac{2}{3}$ $(\mathcal{H}^{n-1}\otimes\mathcal{L}^1_t)$-a.e. on $\Gamma_1\times[t_1,\, t_2]$.

Now, since $\Gamma_1\neq\emptyset$ is bounded and connected, from Poincaré-Wirtinger inequality on manifolds (see Lemma 7.2 in Appendix B of Section 7.2), it may follow that there exists a constant $C>0$ depending only on n such that

$$\|w^{j_i}(t) - w^{j_i}_{\Gamma_1}(t)\|_{L^1(\Gamma_1)} \le C\|\nabla_{\partial\Omega} w^{j_i}(t)\|_{L^1(\Gamma_1)} \tag{6.20}$$

for any $i\in\mathbb{N}$ and any $t\in[t_1,\, t_2]$, where we set

$$w^{j_i}_{\Gamma_1}(t) := \frac{1}{\mathcal{H}^{n-1}(\Gamma_1)}\int_{\Gamma_1} w^{j_i}(t)\, d\mathcal{H}^{n-1}. \tag{6.21}$$

From Cauchy-Schwarz inequality and (6.20), we may obtain the following calculation;

$$\begin{aligned}\int_{t_1}^{t_2}\!\!\int_{\Gamma_1} |w^{j_i} - w^{j_i}_{\Gamma_1}|\, d\mathcal{H}^{n-1}\, dt &\le C\int_{t_1}^{t_2}\!\!\int_{\Gamma_1} |\nabla_{\partial\Omega} w^{j_i}|\, d\mathcal{H}^{n-1}\, dt \\ &\le \delta\, C\int_{t_1}^{t_2}\!\!\int_{\Gamma_1} \frac{W(u^{\varepsilon'_j,\,\sigma'_j})}{\varepsilon_j}\, d\mathcal{H}^{n-1}\, dt + \frac{C}{2\delta}\alpha^{\varepsilon'_{j_i},\,\sigma'_{j_i}}(\Gamma_1\times[t_1,\, t_2]) \\ &\le \delta\, C\, C_1(t_2-t_1+1) + \frac{C}{2\delta}\alpha^{\varepsilon'_{j_i},\,\sigma'_{j_i}}(\Gamma_1\times[t_1,\, t_2]).\end{aligned} \tag{6.22}$$

where $\delta>0$ independent of $i\in\mathbb{N}$ will be chosen later. Hence, from (6.22) and the triangle inequality, we obtain

$$\int_{t_1}^{t_2}\!\!\int_{\Gamma_1} |w^{j_i}|\, d\mathcal{H}^{n-1}\, dt - \int_{t_1}^{t_2}\left|\int_{\Gamma_1} w^{j_i}\, d\mathcal{H}^{n-1}\right| dt \le \delta\, C' + \frac{C}{2\delta}\alpha^{\varepsilon'_{j_i},\,\sigma'_{j_i}}(\Gamma_1\times[t_1,\, t_2]), \tag{6.23}$$

where we put $C' := C\, C_1(t_2-t_1+1)$. Here, from the assumption (4.20), we can choose the constant $c_0>0$ such that

$$\frac{2}{3}\mathcal{H}^{n-1}(\Gamma_1)\,(t_2-t_1) - \limsup_{i\to\infty}\int_{t_1}^{t_2}\left|\int_{\Gamma_1} w^{j_i}\, d\mathcal{H}^{n-1}\right| dt > c_0 > 0. \tag{6.24}$$

Therefore, from Lemma 4.4 and (6.24), taking the limit $(i\to\infty)$ in (6.23), we have

$$\begin{aligned}\delta\, C' + \frac{C}{2\delta}\alpha(\Gamma_1\times[t_1,\, t_2]) &\ge \delta\, C' + \frac{C}{2\delta}\limsup_{i\to\infty}\alpha^{\varepsilon'_{j_i},\,\sigma'_{j_i}}(\Gamma_1\times[t_1,\, t_2]) \\ &\ge \frac{2}{3}\mathcal{H}^{n-1}(\Gamma_1)\,(t_2-t_1) - \limsup_{i\to\infty}\int_{t_1}^{t_2}\left|\int_{\Gamma_1} w^{j_i}\, d\mathcal{H}^{n-1}\right| dt \\ &> c_0 > 0.\end{aligned} \tag{6.25}$$

Taking δ such that $0<\delta<\frac{c_0}{2C'}$, we may conclude that the limit measure α is positive on $\Gamma_1\times[t_1,\, t_2]$. $\square$

6.1.3 First variations of associated varifolds and proof of Lemma 4.9 (Dirichlet boundary conditions)

In the previous subsection, we have already proved that there exists a convergent subsequence $\{\mu_t^{\varepsilon'_i,\,\sigma'_j}\}_{i,\,j\in\mathbb{N}}$ for all $t\ge 0$. Then, in this subsection, we mainly discuss the first variation of the associated varifold with $\mu_t^{\varepsilon'_i,\,\sigma'_j}$ and the proof of Lemma 4.9. Note that the first variation of a varifold plays a very important role to prove Lemma 4.9. Note that, through this subsection, we assume that "General assumptions", "Vanishing hypothesis for the discrepancy measure", and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold.

First of all, we associate a varifold with each $\mu_t^{\varepsilon'}$ as follows.

Definition 6.2. Let $\{u^{\varepsilon,\sigma}\}_{\varepsilon,\sigma>0}$ be the solutions to the equations (1.10) and $\mu_t^{\varepsilon,\sigma}$ be as in (1.12). Then for $\psi \in C_c(G_{n-1}(\overline{\Omega}))$ and any $t \geq 0$, define

$$V_t^{\varepsilon,\sigma}(\psi) := \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}(\cdot,t)|\neq 0\}} \psi(x, \mathbf{I} - \mathbf{a}^{\varepsilon,\sigma} \otimes \mathbf{a}^{\varepsilon,\sigma})\, d\mu_t^{\varepsilon,\sigma}(x), \tag{6.26}$$

where $\mathbf{a}^{\varepsilon,\sigma} := \frac{\nabla u^{\varepsilon,\sigma}}{|\nabla u^{\varepsilon,\sigma}|}$.

Note that from the definition, we can obtain $\|V_t^{\varepsilon,\sigma}\| = \mu_t^{\varepsilon,\sigma}\lfloor_{\{|\nabla u^{\varepsilon,\sigma}(\cdot,t)|\neq 0\}}$, hence, by the definition of the first variation, we may derive a formula for the first variation of $V_t^{\varepsilon,\sigma}$ up to the boundary.

Lemma 6.3. *Let $\{u^{\varepsilon,\sigma}\}_{\varepsilon,\sigma>0}$ and $\mu_t^{\varepsilon,\sigma}$ be as in Definition 6.2. Then, for any $\varepsilon > 0$, $\sigma > 0$, $t \geq 0$ and all $\boldsymbol{g} \in (C_c^1(\overline{\Omega}))^n$, we have*

$$\begin{aligned}\delta V_t^{\varepsilon,\sigma}(\boldsymbol{g}) = &\int_\Omega (\boldsymbol{g}\cdot\nabla u^{\varepsilon,\sigma})\left(\varepsilon\Delta u^{\varepsilon,\sigma} - \frac{W'(u^{\varepsilon,\sigma})}{\varepsilon}\right) dx + \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|\neq 0\}} \nabla\boldsymbol{g} : (\boldsymbol{a}^{\varepsilon,\sigma}\otimes\boldsymbol{a}^{\varepsilon,\sigma})\, d\xi_t^{\varepsilon,\sigma} \\ &+ \int_{\partial\Omega} (\boldsymbol{g}\cdot\nu)\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} + \frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right) d\mathcal{H}^{n-1} - \int_{\partial\Omega} \varepsilon(\boldsymbol{g}\cdot\nabla u^{\varepsilon,\sigma})\frac{\partial u^{\varepsilon,\sigma}}{\partial\nu}\, d\mathcal{H}^{n-1} \\ &- \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|=0\}} \nabla\boldsymbol{g} : \boldsymbol{I}\frac{W(u^{\varepsilon,,\sigma})}{\varepsilon}\, dx.\end{aligned} \tag{6.27}$$

Proof. From the definition of the first variation, we have

$$\delta V_t^{\varepsilon,\sigma}(\mathbf{g}) = \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}(\cdot,t)|\neq 0\}} \nabla\mathbf{g}(x) : (\mathbf{I} - \mathbf{a}^{\varepsilon,\sigma}\otimes\mathbf{a}^{\varepsilon,\sigma})\, d\mu_t^{\varepsilon,\sigma}. \tag{6.28}$$

Using integration by part, we have

$$\begin{aligned}\int_\Omega (\nabla\mathbf{g} : \mathbf{I})\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2}\, dx = &\int_{\partial\Omega}\left(\mathbf{g}\cdot\nu\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2} - \varepsilon(\mathbf{g}\cdot\nabla u^{\varepsilon,\sigma})\frac{\partial u^{\varepsilon,\sigma}}{\partial\nu}\right) d\mathcal{H}^{n-1} \\ &+ \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}(\cdot,t)|\neq 0\}} \nabla\mathbf{g} : (\mathbf{a}^{\varepsilon,\sigma}\otimes\mathbf{a}^{\varepsilon,\sigma})\varepsilon|\nabla u^{\varepsilon,\sigma}|^2\, dx \\ &+ \int_\Omega (\mathbf{g}\cdot\nabla u^{\varepsilon,\sigma})\varepsilon\Delta u^{\varepsilon,\sigma}\, dx.\end{aligned} \tag{6.29}$$

Similary by using integration by part, we get

$$\begin{aligned}\int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|\neq 0\}} \nabla\mathbf{g} : \mathbf{I}\frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\, dx = &\int_{\partial\Omega} (\mathbf{g}\cdot\nu)\frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\, d\mathcal{H}^{n-1} - \int_\Omega (\mathbf{g}\cdot\nabla u^{\varepsilon,\sigma})\frac{W'(u^{\varepsilon,\sigma})}{\varepsilon}\, dx \\ &- \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|=0\}} \nabla\mathbf{g}\cdot I\frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\, dx.\end{aligned} \tag{6.30}$$

By substituting (6.29) and (6.30) into (6.28) and recalling the definition of $\xi_t^{\varepsilon,\sigma}$ which is equal to $\varepsilon|\nabla u^{\varepsilon,\sigma}|^2\,\mathcal{L}^n - \mu_t^{\varepsilon,\sigma}$ where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure, we obtain (6.27). ☐

Remark 6.4. By recalling again the definition of $\xi_t^{\varepsilon,\sigma}$, we have

$$\frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\,\mathcal{L}^n = \frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2}\,\mathcal{L}^n - \xi_t^{\varepsilon,\sigma}. \tag{6.31}$$

Thus, we can rewrite the last term in the left-hand side in (6.27) as follows;

$$-\int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|=0\}} \nabla\mathbf{g}\cdot I\frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\, dx = \int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|=0\}} \mathrm{div}\mathbf{g}\, d\xi_t^{\varepsilon,\sigma}. \tag{6.32}$$

Proposition 6.5. *Let $\{\varepsilon'_j\}_{j\in\mathbb{N}}$ and $\{\sigma'_j\}_{j\in\mathbb{N}}$ be such that Lemma 4.2 and 4.4 hold and let $\{u^{\varepsilon'_j,\sigma'_j}\}_{j\in\mathbb{N}}$ be the solutions to the equations* (1.10)*. Then, defining $c(t)$ by*

$$\begin{aligned}c(t) := \liminf_{j\to\infty}\Bigg(&\int_\Omega |\nabla u^{\varepsilon'_j,\sigma'_j}|\left|\varepsilon'_j\Delta u^{\varepsilon'_j,\sigma'_j} - \frac{W'(u^{\varepsilon'_j,\sigma'_j})}{\varepsilon'_j}\right| dx + \int_{\partial\Omega} \frac{\varepsilon'_j|\nabla u^{\varepsilon'_j,\sigma'_j}|^2}{2} + \frac{W(u^{\varepsilon'_j,\sigma'_j})}{\varepsilon'}\, d\mathcal{H}^{n-1} \\ &+ \int_{\partial\Omega} \varepsilon'_j|\nabla u^{\varepsilon'_j,\sigma'_j}|\left|\frac{\partial u^{\varepsilon'_j,\sigma'_j}}{\partial\nu}\right| d\mathcal{H}^{n-1}\Bigg),\end{aligned} \tag{6.33}$$

we have $c \in L^1_{loc}([0,\infty))$ and $c(t) < \infty$ for a.e. $t \in [0,\infty)$.

Proof. It is sufficient to show that (6.33) holds for a.e. $t \in [0, T]$ for every $T > 0$ because we have $\bigcup_{l=1}^{\infty}[0, l] = [0, \infty)$. Let $T > 0$ be arbitrary. For simplicity, we denote the parameters $(\varepsilon'_j, \sigma'_j)$ by (ε, σ) in this proof. We set

$$I_1^{\varepsilon, \sigma} := \int_{\Omega} |\nabla u^{\varepsilon, \sigma}| \left| \varepsilon \Delta u^{\varepsilon, \sigma} - \frac{W'(u^{\varepsilon, \sigma})}{\varepsilon} \right| dx \tag{6.34}$$

$$I_2^{\varepsilon, \sigma} := \int_{\partial\Omega} \left(\frac{\varepsilon |\nabla u^{\varepsilon, \sigma}|^2}{2} + \frac{W(u^{\varepsilon, \sigma})}{\varepsilon} \right) d\mathcal{H}^{n-1} \tag{6.35}$$

$$I_3^{\varepsilon, \sigma} := \int_{\partial\Omega} \varepsilon |\nabla u^{\varepsilon, \sigma}| \left| \frac{\partial u^{\varepsilon, \sigma}}{\partial \nu} \right| d\mathcal{H}^{n-1}. \tag{6.36}$$

From (5.1) and (5.2) and by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left(\int_0^T I_1^{\varepsilon, \sigma} \, dt \right)^2 &\leq \left(\int_0^T \int_{\Omega} \varepsilon |\nabla u^{\varepsilon, \sigma}|^2 \, dx \, dt \right) \left(\int_0^T \int_{\Omega} \varepsilon \left(\Delta u^{\varepsilon, \sigma} - \frac{W'(u^{\varepsilon, \sigma})}{\varepsilon} \right)^2 dx \, dt \right) \\ &\leq 2D \int_0^T \mu_t^{\varepsilon, \sigma}(\Omega) \, dt \leq 2D^2 \, T, \end{aligned} \tag{6.37}$$

and, from the assumption of uniform upper bound (4.8) and (5.5),

$$\int_0^T I_2^{\varepsilon, \sigma} \, dt \leq C_1 T + C_0(T), \tag{6.38}$$

and finally, from (4.8) and (5.5),

$$\left(\int_0^T I_3^{\varepsilon, \sigma} \, dt \right)^2 \leq \int_0^T \int_{\partial\Omega} \varepsilon |\nabla u^{\varepsilon, \sigma}|^2 \, d\mathcal{H}^{n-1} \, dt \leq 2C_1 T + 2C_0(T) \tag{6.39}$$

for any $\varepsilon, \sigma > 0$. Then by using Fatou's lemma we have

$$\int_0^T c(t) \, dt \leq \liminf_{\varepsilon, \sigma \to 0} \int_0^T (I_1^{\varepsilon, \sigma} + I_2^{\varepsilon, \sigma} + I_3^{\varepsilon, \sigma}) \, dt \leq C_2(T, D) < \infty, \tag{6.40}$$

where $C_2(T, D) := \sqrt{2} D \sqrt{T} + C_1 T + C_0(T) + \sqrt{2}\sqrt{C_1 T + C_0(T)}$. This shows that $c \in L^1_{loc}([0, \infty))$ and thus $c(t) < \infty$ holds for a.e. $t \in [0, T]$. This completes the proof. □

Next we will show that μ_t is actually $(n-1)$-rectifiable measure on $\overline{\Omega}$ for a.e. $t \geq 0$ and a proper subsequence of the associated varifolds $\{V_t^{\varepsilon, \sigma}\}_{\varepsilon, \sigma > 0}$ converges uniquely to the varifold V_t associated with μ_t.

Proposition 6.6. *For a.e. $t \geq 0$, μ_t is $(n-1)$-rectifiable on $\overline{\Omega}$ and any convergent subsequence $\{V^{\varepsilon''_j, \sigma'_j}\}_{j \in \mathbb{N}}$ of $\{V_t^{\varepsilon'_j, \sigma'_j}\}_{j \in \mathbb{N}}$, where $\{\varepsilon'_j\}_{j \in \mathbb{N}}$ and $\{\sigma'_j\}_{j \in \mathbb{N}}$ are such that Lemma 4.2 and 4.4 hold, converges to the unique $(n-1)$-rectifiable varifold V_t associated with μ_t. Moreover, we have*

$$\|\delta V_t\|(\overline{\Omega}) < \infty, \qquad \int_0^T \|\delta V_t\|(\overline{\Omega}) \, dt < \infty \tag{6.41}$$

for a.e. $t \geq 0$ and any $T > 0$ respectively.

Proof. Recalling the assumption in Subsection 4.1.1, that is, the vanishing of the discrepancy measure $\xi_t^{\varepsilon'_j, \sigma'_j}$ up to the boundary $\partial\Omega$, we now have that, for all $\phi \in C_c(\overline{\Omega})$,

$$\lim_{j \to \infty} \int_{\Omega} \phi \, d|\xi_t^{\varepsilon'_j, \sigma'_j}| = 0 \tag{6.42}$$

holds for a.e. $t \in [0, \infty)$.

First, as we mentioned in Subsection 4.1.2, we know the integrality of the measure μ_t in the interior of $\overline{\Omega}$ for a.e. $t \geq 0$ and thus there exist a $(n-1)$-rectifiable set $M_t \subset \Omega$ and a function $\theta_t \in L^1_{loc}(\mathcal{H}^{n-1}; M_t)$ such that θ_t takes the values on $\mathbb{N}$ and, for a.e. $t \geq 0$,

$$\mu_t(U) = \int_{M_t \cap U} \theta_t(x) \, d\mathcal{H}^{n-1}(x) \tag{6.43}$$

for any measurable set $U \subset \Omega$. From (6.43) we can prove that for a.e. $t \geq 0$, $\operatorname{spt}\mu_t \cap \Omega = \operatorname{spt}\mathcal{H}^{n-1}\lfloor_{M_t}$.

From the boundedness of Ω, we can show that $\mathcal{H}^{n-1}(\operatorname{spt}\mu_t) < \infty$ as follows:

$$\begin{aligned}\mathcal{H}^{n-1}(\operatorname{spt}\mu_t) &= \mathcal{H}^{n-1}(\operatorname{spt}\mu_t \cap \Omega) + \mathcal{H}^{n-1}(\operatorname{spt}\mu_t \cap \partial\Omega)\\ &\leq \mathcal{H}^{n-1}(M_t) + \mathcal{H}^{n-1}(\operatorname{spt}\mu_t \cap \partial\Omega)\\ &\leq \mu_t(\Omega) + \mathcal{H}^{n-1}(\partial\Omega)\\ &\leq \liminf_{j\to\infty} \mu_t^{\varepsilon_j', \sigma_j'}(\Omega) + \mathcal{H}^{n-1}(\partial\Omega) \leq D + \mathcal{H}^{n-1}(\partial\Omega) < \infty \end{aligned} \tag{6.44}$$

for a.e. $t \geq 0$. Note that, in the above calculation, we use the property of $\theta_t \geq 1$ for $\mathcal{H}^{n-1}$-a.e. in M_t. In addition, from a standard measure theory (see, for instance, [31, Chapter 3]), we have

$$\mu_t\left(\left\{x \in \operatorname{spt}\mu_t \mid \limsup_{r\downarrow 0} \frac{\mu_t(B_r(x))}{\omega_{n-1}r^{n-1}} \leq s\right\}\right) \leq 2^{n-1}s\mathcal{H}^{n-1}(\operatorname{spt}\mu_t) < \infty, \tag{6.45}$$

for any $0 < s < \infty$ and a.e. $t \geq 0$. Letting s go to zero, we have

$$\mu_t\left(\left\{x \in \operatorname{spt}\mu_t \mid \limsup_{r\downarrow 0} \frac{\mu_t(B_r(x))}{\omega_{n-1}r^{n-1}} = 0\right\}\right) = 0, \tag{6.46}$$

and thus we obatin

$$\mu_t \equiv \mu_t\lfloor_{\left\{x \mid \limsup_{r\downarrow 0} \frac{\mu_t(B_r(x))}{\omega_{n-1}r^{n-1}} > 0\right\}} \quad \text{on } \overline{\Omega} \text{ for a.e. } t. \tag{6.47}$$

Now we will prove that, for a.e. $t \geq 0$, the total variation $\|\delta V_t\|$ of the first variation of V_t is actually a Radon measure on $\overline{\Omega}$. Note that we only have to show this for any $T > 0$ and a.e. $0 \leq t \leq T$.

Let $T > 0$ be arbitrary and we fix $0 \leq t \leq T$ such that (6.33), (6.42), (6.44) and (6.47) hold. Let $\{V_t^{\varepsilon_j'', \sigma_j'}\}_{j\in\mathbb{N}}$ be any convergent subsequence of $\{V_t^{\varepsilon_j', \sigma_j'}\}_{j\in\mathbb{N}}$ and we denote its limit by $\tilde{V}_t$. From (6.27), (6.42) and the varifold convergence of $\{V_t^{\varepsilon_j'', \sigma_j'}\}_{j\in\mathbb{N}}$, we have for any $\mathbf{g} \in (C_c^1(\overline{\Omega}))^n$

$$\begin{aligned}\delta\tilde{V}_t(\mathbf{g}) = \lim_{j\to\infty} \delta V_t^{\varepsilon_j'', \sigma_j'}(\mathbf{g}) &= \lim_{j\to\infty} \int_\Omega \nabla\mathbf{g}(x) : (I - a^{\varepsilon_j'', \sigma_j'} \otimes a^{\varepsilon_j'', \sigma_j'})\, d\mu_t^{\varepsilon_j'', \sigma_j'}(x)\\ &= \lim_{j\to\infty}\left(\int_\Omega (\mathbf{g}\cdot\nabla u^{\varepsilon_j''\,\sigma_j'})\left(\varepsilon_j''\Delta u^{\varepsilon_j'', \sigma_j'} - \frac{W'(u^{\varepsilon_j'', \sigma_j'})}{\varepsilon_j''}\right)dx\right.\\ &\quad + \int_{\partial\Omega} (\mathbf{g}\cdot\nu)\left(\frac{\varepsilon_j''|\nabla u^{\varepsilon_j'', \sigma_j'}|^2}{2} + \frac{W(u^{\varepsilon_j'', \sigma_j'})}{\varepsilon_j''}\right)d\mathcal{H}^{n-1}\\ &\quad \left. - \int_{\partial\Omega} \varepsilon_j''(\mathbf{g}\cdot\nabla u^{\varepsilon_j'', \sigma_j'})\frac{\partial u^{\varepsilon_j'', \sigma_j'}}{\partial\nu}\, d\mathcal{H}^{n-1}\right). \end{aligned} \tag{6.48}$$

Then we have

$$|\delta\tilde{V}_t(\mathbf{g})| \leq c(t)\sup_{\overline{\Omega}} |\mathbf{g}| < \infty \tag{6.49}$$

for a.e. $t \geq 0$, where $c(t)$ is as in (6.33) in Proposition 6.5. This shows that we can extend the domain of the functional $\delta\tilde{V}_t$ into $(C_c(\overline{\Omega}))^n$ and hence from Riesz representation theorem, we have that the total variation $\|\delta\tilde{V}_t\|$ is a Radon measure on $\overline{\Omega}$. Since we have $\|V_t^{\varepsilon_j'', \sigma_j'}\| = \mu_t^{\varepsilon_j'', \sigma_j'}$ and the subsequence $\{\mu_t^{\varepsilon_j'', \sigma_j'}\}_{j\in\mathbb{N}}$ converges to μ_t, we have $\|\tilde{V}_t\| = \mu_t$, which is uniquely determined.

From (6.47) and (6.49), we can apply Allard's rectifiability theorem to $\tilde{V}_t$ and conclude that $\tilde{V}_t$ is $(n-1)$-rectifiable varifold on $\overline{\Omega}$. In particular, $\tilde{V}_t$ is determined uniquely by $\|\tilde{V}_t\| = \mu_t$, and therefore, this yields that the uniqueness of the limit $(n-1)$-rectifiable varifold V_t associated with μ_t (as in Definition 4.8) follows. This also shows that μ_t is $(n-1)$-rectifiable on $\overline{\Omega}$ for a.e. $0 \leq t \leq T$.

Furthermore, the calculation in the above shows that the boundedness of the total variation of the first variation of varifolds shown in (6.41) holds for a.e. $t \geq 0$ and $T \geq 0$. Thus we have completed the proof. □

Considering the all arguments in Lemma 4.2, and Proposition 6.5 and 6.6, we may conclude that Lemma 4.9 is valid.

6.1.4 Proof of Theorem 4.10 (Dirichlet boundary conditions)

In this section, we will prove Theorem 4.10, that is, the existence of the singular limits of the Allen-Cahn equations described in (1.10). Before proving Theorem 4.10, as a preparation, we will show three propositions. First of all, we show that the first variation in integral form $\int_0^\infty \delta V_t^{\varepsilon_j', \sigma_j'}\, dt$ converges to $\int_0^\infty \delta V_t\, dt$ locally in time as $j \to \infty$, where the subsequence $\{V_t^{\varepsilon_j', \sigma_j'}\}_{j\in\mathbb{N}}$ has the limit varifold V_t.

Note that, through this subsection, we assume that "General assumptions", "Vanishing hypothesis for the discrepancy measure", and "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1 hold. Moreover, we only consider the subsequence $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ and $\{\sigma_j'\}_{j\in\mathbb{N}}$ such that Lemma 4.2 and 4.4 hold.

First of all, we will show the folloing proposition;

Proposition 6.7. *Suppose that "General assumptions" and "Vanishing hypothesis for the discrepancy measure" in Subsection 4.1.1 hold, and $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ and $\{\sigma_j'\}_{j\in\mathbb{N}}$ are such that Lemma 4.2 and 4.4 hold. Define a Radon measure $\tilde{\mu}^{\varepsilon_j', \sigma_j'}$ on $\Omega\times[0,\infty)$ by*

$$\tilde{\mu}^{\varepsilon_j', \sigma_j'}(U) := \iint_U \varepsilon_j' |\nabla u^{\varepsilon_j', \sigma_j'}|^2\, dx\, dt \left(= \iint_U d\mu_t^{\varepsilon_j', \sigma_j'}\, dt + \iint_U d\xi^{\varepsilon_j', \sigma_j'}\, dt \right), \tag{6.50}$$

for any open set $U \subset \Omega\times[0,\infty)$. Then there exist a subsequence $\{\tilde{\mu}^{\varepsilon_j', \sigma_j'}\}$ (denoted the same index) and a function $\boldsymbol{v} \in (L^2(\mu_t\otimes\mathcal{L}_t^1, \Omega\times[0,\infty)))^n$ such that $\tilde{\mu}^{\varepsilon_j', \sigma_j'} \rightharpoonup \mu_t \otimes \mathcal{L}_t^1$ in $\Omega\times[0,\infty)$, where μ_t is as in Lemma 4.2, and

$$\lim_{j\to\infty} \int_0^\infty \int_\Omega \boldsymbol{g}\cdot(\varepsilon_j'\, \partial_t u^{\varepsilon_j', \sigma_j'}\, \nabla u^{\varepsilon_j', \sigma_j'})\, dx\, dt = -\iint_{\Omega\times[0,\infty)} \boldsymbol{g}\cdot\boldsymbol{v}\, d\mu_t\, dt, \tag{6.51}$$

for any $\boldsymbol{g} \in (C_c(\Omega\times[0,\infty)))^n$.

Proof. First of all, we show the convergence of the family of Radon measures $\{\tilde{\mu}^{\varepsilon_j', \sigma_j'}\}_{j\in\mathbb{N}}$ and also show that the limit of $\tilde{\mu}^{\varepsilon_j', \sigma_j'}$ is equal to $\mu_t\otimes\mathcal{L}_t^1$ in $\Omega\times[0,\infty)$. The convergence of the measures is readily obtained from its definition and Lemma 4.2. We denote its limit by $\tilde{\mu}$, that is, we obtain that there exists a subsequence (denote the same index) such that $\tilde{\mu}^{\varepsilon_j', \sigma_j'} \rightharpoonup \tilde{\mu}$ in $\Omega\times[0,\infty)$. Regarding to its limit, we should recall the vanishing of the discrepancy measure $\xi_t^{\varepsilon_j', \sigma_j'}$ as $j\to\infty$ on $\overline{\Omega}$ for a.e. $t\geq 0$. As we mentioned in Subsection 4.1.1, it is known that the vanishing in the interior of $\overline{\Omega}$ is valid by applying the monotonicity formula (see, for instance, [15]). Thus, we have that $\xi_t^{\varepsilon_j', \sigma_j'}\otimes\mathcal{L}_t^1 \rightharpoonup 0$ in $\Omega\times[0,\infty)$. Moreover, from the definitions, we may easily see that

$$\tilde{\mu}^{\varepsilon_j', \sigma_j'} = \mu_t^{\varepsilon_j', \sigma_j'}\otimes\mathcal{L}_t^1 + \xi_t^{\varepsilon_j', \sigma_j'}\otimes\mathcal{L}_t^1 \tag{6.52}$$

holds for any $j\in\mathbb{N}$. Therefore, letting $j\to\infty$, we may conclude that $\tilde{\mu} = \lim_{j\to\infty}\tilde{\mu}^{\varepsilon_j', \sigma_j'} = \mu_t\otimes\mathcal{L}_t^1$ in $\Omega\times[0,\infty)$ in the sense of Radon measure.

Next, we prove that there exists a function $\mathbf{v} \in (L^2(\tilde{\mu}, \Omega\times[0,\infty)))^n$ such that

$$\lim_{j\to\infty} \int_0^\infty \int_\Omega \mathbf{g}\cdot(\varepsilon_j'\, \partial_t u^{\varepsilon_j', \sigma_j'}\, \nabla u^{\varepsilon_j', \sigma_j'})\, dx\, dt = -\int_{\Omega\times[0,\infty)} \mathbf{g}\cdot\mathbf{v}\, d\tilde{\mu}\, (= \mathcal{S}^\Omega_{\tilde{\mu},\mathbf{v}}(\mathbf{g})), \tag{6.53}$$

for any $\mathbf{g} \in (C_c(\Omega\times[0,\infty)))^n$. To do this, we first define a function $\mathbf{v}^{\varepsilon_j', \sigma_j'}$, which can be regarded as the approximation with the normal velocity vector of a seperating front, by

$$\mathbf{v}^{\varepsilon_j', \sigma_j'} = \begin{cases} -\dfrac{\partial_t u^{\varepsilon_j', \sigma_j'}}{|\nabla u^{\varepsilon_j', \sigma_j'}|}\dfrac{\nabla u^{\varepsilon_j', \sigma_j'}}{|\nabla u^{\varepsilon_j', \sigma_j'}|} & \text{if } |\nabla u^{\varepsilon_j', \sigma_j'}| \neq 0. \\ 0 & \text{otherwise.} \end{cases} \tag{6.54}$$

Then, from the definitions, we may deduce that

$$\int_0^\infty \int_\Omega |\mathbf{v}^{\varepsilon_j', \sigma_j'}|^2\, d\tilde{\mu}^{\varepsilon_j', \sigma_j'} = \int_0^\infty \int_\Omega \varepsilon_j'\, (\partial_t u^{\varepsilon_j', \sigma_j'})^2\, dx\, dt \leq D < \infty. \tag{6.55}$$

This implies that the pair $(\tilde{\mu}^{\varepsilon_j', \sigma_j'}, \mathbf{v}^{\varepsilon_j', \sigma_j'})$ is the one which satisfies the L^2-uniform boundedness with respect to $i, j\in\mathbb{N}$. Since we have already had the convergence of $\tilde{\mu}^{\varepsilon_j', \sigma_j'}$, we can again apply the theorem [14, Theorem 4.4.2.] and thus we may conclude that there exist a subsequence (denoted the same index) and a function $\mathbf{v} \in (L^2(\tilde{\mu}, \Omega\times[0,\infty)))^n$ such that (6.53) holds for any $\mathbf{g} \in (C_c(\Omega\times[0,\infty)))^n$ and thus, this completes the proof. ∎

Secondly, we will show that interchanging limit processes and integral signs of the first variation of varifolds is valid.

Proposition 6.8. *Let $\{V_t^{\varepsilon'_j,\,\sigma'_j}\}_{j\in\mathbb{N}}$ be a family of associated varifolds with $\mu_t^{\varepsilon'_j,\,\sigma'_j}$ satisfying Proposition 6.5 and 6.6 for a.e. $t\geq 0$. Then we have*

$$\lim_{j\to\infty}\int_0^T \delta V_t^{\varepsilon'_j,\,\sigma'_j}(\boldsymbol{g})\,dt = \int_0^T \delta V_t(\boldsymbol{g})\,dt \tag{6.56}$$

for all $T>0$ and all $\boldsymbol{g}\in (C_c^1(\overline{\Omega}\times[0,\,\infty)))^n$.

Proof. Let $T>0$ be arbitrary. From the convergence of $\{\delta V_t^{\varepsilon'_j,\,\sigma'_j}\}_{j\in\mathbb{N}}$, Proposition 6.5 and (6.49) in Proposition 6.6, we have

$$\left|\limsup_{j\in\mathbb{N}}\delta V_t^{\varepsilon'_j,\,\sigma'_j}(\mathbf{g}(\cdot,\,t))\right| = |\delta V_t(\mathbf{g}(\cdot,\,t))| \leq \sup_{\overline{\Omega}\times[0,\,\infty)}|\mathbf{g}|\,c(t) \tag{6.57}$$

for any $\mathbf{g}\in (C_c^1(\overline{\Omega}\times[0,\,\infty)))^n$ and, from Proposition 6.5, we have $(\sup_{\overline{\Omega}\times[0,\,\infty)}|\mathbf{g}|)\,c\in L^1([0,\,T])$. Therefore, from dominated convergence theorem, we have the equality (6.56). ☐

Next we show the absolute continuities for total variation measures and L^2-estimate for the modified generalized mean curvature vector.

Proposition 6.9. *There exists the generalized mean curvature vector $\boldsymbol{H}_V^{\Omega}(\cdot,\,t)$ in Ω such that*

$$\boldsymbol{H}_V^{\Omega}\equiv \boldsymbol{v}\quad \text{in } (L^2(\|V_t\|\otimes\mathcal{L}_t^1,\,\Omega\times[0,\,\infty)))^n,\quad \delta V_t\lfloor_{\Omega} = -\boldsymbol{H}_V^{\Omega}(\cdot,\,t)\,\|V_t\|\quad \text{in } \Omega \text{ for a.e. } t\geq 0 \tag{6.58}$$

holds, where $\boldsymbol{v}$ is as in Proposition 6.7. In addition, assume that α and $\boldsymbol{v}_b$ are followed by Lemma 4.4. Let $\mathcal{S}_{\alpha,\,\boldsymbol{v}_b}$ and $\mathcal{S}^{\Omega}_{\tilde{\mu},\,\boldsymbol{H}_V^{\Omega}}$ be as in Definition 2.13 and 2.15 where $\tilde{\mu}:=\mu_t\otimes\mathcal{L}_t^1=\|V_t\|\otimes\mathcal{L}_t^1$. Then we have

$$\|\mathcal{S}_{\alpha,\,\boldsymbol{v}_b}+\mathcal{S}^{\Omega}_{\tilde{\mu},\,\boldsymbol{H}_V^{\Omega}}\| \ll \|V_t\|\otimes\mathcal{L}_t^1\quad \text{on } \overline{\Omega}\times[0,\,\infty), \tag{6.59}$$

and there exists the modified generalized mean curvature vector $\widetilde{\boldsymbol{H}}_V$ (see Definition 3.1) such that $\widetilde{\boldsymbol{H}}_V\lfloor_{\Omega}\equiv \boldsymbol{H}_V^{\Omega}$ in $(L^2(\|V_t\|\otimes\mathcal{L}_t^1,\,\Omega\times[0,\,\infty)))^n$, $\widetilde{\boldsymbol{H}}_V$ belongs to $(L^2(\|V_t\|\otimes\mathcal{L}_t^1,\,\overline{\Omega}\times[0,\infty)))^n$, and we have $\|\widetilde{\boldsymbol{H}}_V\|_{L^2}\leq D^{\frac{1}{2}}$. Besides, because of the assumption "Uniform upper bound for the solution of Allen-Cahn equations" in Subsection 4.1.1, we in fact obatin $\|\mathcal{S}_{\alpha,\,\boldsymbol{v}_b}\|\equiv 0$ on $\partial\Omega\times[0,\,\infty)$.

Moreover, for any $0<t_1\leq t_2<\infty$, we have

$$\int_{t_1}^{t_2}\int_{\Omega}\phi|\widetilde{\boldsymbol{H}}_V|^2\,d\|V_t\|\,dt \leq \liminf_{j\to\infty}\int_{t_1}^{t_2}\int_{\Omega}\varepsilon'_j\phi(\partial_t u^{\varepsilon'_j,\,\sigma'_j})^2\,dx\,dt \tag{6.60}$$

for all $\phi\in C_c(\overline{\Omega}\times[0,\,\infty))$ with $\phi\geq 0$.

Proof. Take any time $t\geq 0$ such that (6.33) and (6.42) hold. Let $\{V_t^{\varepsilon'_j,\,\sigma'_j}\}_{j\in\mathbb{N}}$ be a subsequence converging to V_t. From (5.1), (6.27) and (6.42), we have, for any $\mathbf{g}\in (C_c^1(\Omega))^n$ and any $t\geq 0$ such that Proposition 6.5 holds,

$$\begin{aligned}
|\delta V_t\lfloor_{\Omega}(\mathbf{g})| &= \lim_{j\to\infty}\left|\delta V_t^{\varepsilon'_j,\,\sigma'_j}(\mathbf{g})\right| \\
&\leq \liminf_{\varepsilon'\to 0}\left(\int_{\Omega}\varepsilon'_j(\partial_t u^{\varepsilon'_j,\,\sigma'_j})^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega}|\mathbf{g}|^2\,d\|V_t\|\right)^{\frac{1}{2}} \\
&\leq D^{\frac{1}{2}}\left(\int_{\Omega}|\mathbf{g}|^2\,d\|V_t\|\right)^{\frac{1}{2}} < \infty.
\end{aligned} \tag{6.61}$$

This shows that $\|\delta V_t\lfloor_{\Omega}\|\ll\|V_t\|$ in Ω for a.e. $t\geq 0$. Moreover, from Riesz representation theorem and Radon-Nikodym theorem, it holds that, for a.e. $t\geq 0$ such that Proposition 6.5 holds, there exists a $(\|V_t\|\otimes\mathcal{L}_t^1)$-integrable vector-valued function $\mathbf{H}_V^{\Omega}(\cdot,\,t)$ such that

$$(\delta V_t\lfloor_{\Omega})\,(\mathbf{g}) = -\int_{\Omega}\mathbf{H}_V^{\Omega}(\cdot,\,t)\cdot\mathbf{g}\,d\|V_t\| \tag{6.62}$$

for all $\mathbf{g} \in (C_c(\Omega))^n$. Note that, from (6.61) and (6.62), it also holds that $\mathbf{H}_V^\Omega \in (L^2(\|V_t\| \otimes \mathcal{L}_t^1,\, \Omega \times [0,\, \infty)))^n$. Moreover, from Proposition 6.7, Proposition 6.8, (6.27), and (6.62), we may obtain the following calculation:

$$\begin{aligned}\int_0^\infty \int_\Omega \mathbf{g} \cdot \mathbf{H}_V^\Omega \, d\|V_t\| \, dt = -\left(\int_0^\infty \delta V_t \lfloor_\Omega \, dt\right)(\mathbf{g}) &= -\lim_{j\to\infty} \int_0^\infty \int_\Omega (\mathbf{g} \cdot \nabla u^{\varepsilon_j',\, \sigma_j'})\, \varepsilon_j' \,(\partial_t u^{\varepsilon_j',\, \sigma_j'})\, dx\, dt \\ &= \lim_{j\to\infty} \int_0^\infty \int_\Omega \mathbf{g} \cdot \left(-\frac{\partial_t u^{\varepsilon_j',\, \sigma_j'}}{|\nabla u^{\varepsilon_j',\, \sigma_j'}|} \frac{\nabla u^{\varepsilon_j',\, \sigma_j'}}{|\nabla u^{\varepsilon_j',\, \sigma_j'}|}\right) \varepsilon_j' |\nabla u^{\varepsilon_j',\, \sigma_j'}|^2 \, dx\, dt \\ &= \lim_{j\to\infty} \int_0^\infty \int_\Omega \mathbf{g} \cdot \mathbf{v}^{\varepsilon_j',\, \sigma_j'} \, d\tilde{\mu}^{\varepsilon_j',\, \sigma_j'} = \int_{\Omega\times[0,\,\infty)} \mathbf{g} \cdot \mathbf{v} \, d\tilde{\mu}.\end{aligned} \tag{6.63}$$

for any $\mathbf{g} \in (C_c^1(\Omega \times [0,\, \infty)))^n$. Here we note that the following identity holds; for any $f \in (L^2(\|V_t\| \otimes \mathcal{L}_t^1))^n$,

$$\begin{aligned}&\|f\|_{L^2(\|V_t\|\otimes\mathcal{L}_t^1,\, X\times[0,\,\infty))} \\ &= \sup\left\{\int_0^\infty \int_X f \cdot \mathbf{g} \, d\|V_t\| \, dt \;\middle|\; \mathbf{g} \in (C_c^1(X \times [0,\, \infty)))^n,\; \|\mathbf{g}\|_{L^2(\|V_t\|\otimes\mathcal{L}_t^1)} \le 1\right\}\end{aligned} \tag{6.64}$$

holds where X is either Ω or $\overline{\Omega}$. The readers should refer to Ilmanen [16] for more details on this identity. Recalling that $\tilde{\mu} = \|V_t\| \otimes \mathcal{L}_t^1$, from (6.63), and (6.64), we have that $\mathbf{H}_V^\Omega \equiv \mathbf{v}$ in $(L^2(\tilde{\mu},\, \Omega \times [0,\, \infty)))^n$. Moreover, from (6.63), we actually obtain $\|\int_0^\infty \delta V_t \lfloor_\Omega \, dt\| \equiv \|\mathcal{S}_{\tilde{\mu},\, \mathbf{H}_V^\Omega}^\Omega\|$ in $\Omega \times [0,\, \infty)$.

Now let $U \subset\subset \overline{\Omega}$ be a relatively open set, $T > 0$ be arbitrary number, and $\mathbf{g} \in (C_c^1(U \times [0,\, T)))^n$ be any test function such that $|\mathbf{g}| \le 1$. In order to prove (6.59), we need to compute the following: from (6.13), (6.27), and (6.53), we have that

$$\begin{aligned}\left(\mathcal{S}_{\alpha,\, \mathbf{v}_b} + \mathcal{S}_{\tilde{\mu},\, \mathbf{H}_V^\Omega}^\Omega\right)(\mathbf{g}) &= \int_0^\infty \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}_b \, d\alpha - \int_0^\infty \int_\Omega \mathbf{g} \cdot \mathbf{H}_V^\Omega \, d\|V_t\| \, dt \\ &= -\lim_{j\to\infty} \int_0^\infty \int_{\partial\Omega} (\mathbf{g} \cdot \nabla_{\partial\Omega} u^{\varepsilon_j',\, \sigma_j'}) \varepsilon_j' (\partial_t u^{\varepsilon_j',\, \sigma_j'}) \, d\mathcal{H}^{n-1} \, dt \\ &\quad + \lim_{j\to\infty} \int_0^\infty \int_\Omega (\mathbf{g} \cdot \nabla u^{\varepsilon_j',\, \sigma_j'})\, \varepsilon_j' (\partial_t u^{\varepsilon_j',\, \sigma_j'})\, dx\, dt.\end{aligned} \tag{6.65}$$

Recalling the boundary condition $\partial_t u^{\varepsilon_j',\, \sigma_j'} + \sigma_j' \nabla u^{\varepsilon_j',\, \sigma_j'} \cdot \nu = 0$, from Cauchy-Schwartz inequality and Lemma 4.4, we can show that

$$\begin{aligned}-\lim_{j\to\infty} \int_0^\infty \int_{\partial\Omega} (\mathbf{g} \cdot \nabla_{\partial\Omega} u^{\varepsilon_j',\, \sigma_j'}) \varepsilon_j' (\partial_t u^{\varepsilon_j',\, \sigma_j'}) \, d\mathcal{H}^{n-1} \, dt &= \lim_{j\to\infty} \int_0^\infty \int_{\partial\Omega} (\mathbf{g} \cdot \nabla_{\partial\Omega} u^{\varepsilon_j',\, \sigma_j'}) \varepsilon_j' \, \sigma_j' \frac{\partial u^{\varepsilon_j',\, \sigma_j'}}{\partial\nu} \, d\mathcal{H}^{n-1} \, dt \\ &\le (\lim_{j\to\infty} \sigma_j') \lim_{j\to\infty} \left(\int_0^\infty \int_{\partial\Omega} |\mathbf{g}|^2 \varepsilon_j' |\nabla_{\partial\Omega} u^{\varepsilon_j',\, \sigma_j'}|^2 \, d\mathcal{H}^{n-1} \, dt\right)^{\frac{1}{2}} \\ &\qquad \times \sup_{j\in\mathbb{N}} \left(\int_0^T \int_{\partial\Omega} \varepsilon_j' \left(\frac{\partial u^{\varepsilon_j',\, \sigma_j'}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1} \, dt\right)^{\frac{1}{2}} \\ &= (\lim_{j\to\infty} \sigma_j') \left(\int_0^\infty \int_{\partial\Omega} |\mathbf{g}|^2 \, d\alpha\right)^{\frac{1}{2}} C_0(t_1,\, t_2) \\ &\le (\lim_{j\to\infty} \sigma_j') \, (\alpha((\partial\Omega \cap U) \times [0,\, T]))^{\frac{1}{2}} \, C_0(t_1,\, t_2) < \infty \\ &= 0\end{aligned} \tag{6.66}$$

Thus, we obtain, from (6.65), (6.66), and Cauchy-Schwartz inequality,

$$\begin{aligned}\left(\mathcal{S}_{\alpha,\, \mathbf{v}_b} + \mathcal{S}_{\tilde{\mu},\, \mathbf{H}_V^\Omega}^\Omega\right)(\mathbf{g}) &\le 0 + \liminf_{j\to\infty} \left(\int_0^\infty \int_\Omega \varepsilon_j' (\partial_t u^{\varepsilon_j',\, \sigma_j'})^2 \, dx\, dt\right)^{\frac{1}{2}} \left(\int_0^\infty \int_{\overline{\Omega}} |\mathbf{g}^\delta|^2 \, d\|V_t\| \, dt\right)^{\frac{1}{2}} \\ &\le D^{\frac{1}{2}} \left(\|V_t\| \otimes \mathcal{L}_t^1(U \times [0,\, T))\right)^{\frac{1}{2}}.\end{aligned} \tag{6.67}$$

Of course, (6.66) also yeilds that $\mathcal{S}_{\alpha,\, \mathbf{v}_b}(\mathbf{g}) = 0$ for any $\mathbf{g} \in (C_c(\partial\Omega \times [0,\, \infty)))^n$, and thus $\|\mathcal{S}_{\alpha,\, \mathbf{v}_b}\| \equiv 0$ on $\partial\Omega \times [0,\, \infty)$.

Therefore, taking the supremum of the both side of (6.67) with respect to $\mathbf{g}$, we obtain the absolute continuity (6.59) from the arbitrariness of $U \subset\subset \overline{\Omega}$ and $T > 0$. Moreover, from (6.59) and Radon-Nikodym theorem, we can show that there exists a $(\|V_t\| \otimes \mathcal{L}_t^1)$-integrable vector-valued function $\widetilde{\mathbf{H}}_V$ such that

$$\left(\mathcal{S}_{\alpha,\,\mathbf{v}_b} + \mathcal{S}^{\Omega}_{\tilde{\mu},\,\mathbf{H}^{\Omega}_V}\right)(\mathbf{g}) = -\int_0^\infty \int_{\overline{\Omega}} \mathbf{g} \cdot \widetilde{\mathbf{H}}_V \, d\|V_t\| \, dt, \tag{6.68}$$

for any $\mathbf{g} \in (C_c(\overline{\Omega} \times [0,\, \infty)))^n$.

Next we need to show the three claims, that is, the claims that $\widetilde{\mathbf{H}}_V \lfloor_\Omega \equiv \mathbf{H}^\Omega_V$ in $(L^2(\|V_t\| \otimes \mathcal{L}_t^1,\, \Omega \times [0,\, \infty)))^n$, $\widetilde{\mathbf{H}}_V$ belongs to $(L^2(\|V_t\| \otimes \mathcal{L}_t^1,\, \overline{\Omega} \times [0, \infty)))^n$, and we have the estimate (6.60). First of all, we show that $\widetilde{\mathbf{H}}_V \lfloor_\Omega \equiv \mathbf{H}^\Omega_V$ in $(L^2(\|V_t\| \otimes \mathcal{L}_t^1,\, \Omega \times [0,\, \infty)))^n$. From the absolute continuity (6.59), the fact that $\mathbf{H}^\Omega_V$ is equal to $\mathbf{v}$ in $\Omega \times [0,\, \infty)$ in $(L^2(\|V_t\| \otimes \mathcal{L}_t^1))^n$, (6.61), (6.62), and (6.68), we have that, for any $\mathbf{g} \in (C_c(\Omega \times [0,\, \infty)))^n$,

$$\begin{aligned}
\int_0^\infty \int_\Omega \widetilde{\mathbf{H}}_V \lfloor_\Omega \cdot \mathbf{g} \, d\|V_t\| \, dt &= -\mathcal{S}^{\Omega}_{\tilde{\mu},\,\mathbf{H}^{\Omega}_V}(\mathbf{g}) + \mathcal{S}_{\alpha,\,\mathbf{v}_b}(\mathbf{g}) \\
&= -\mathcal{S}^{\Omega}_{\tilde{\mu},\,\mathbf{v}}(\mathbf{g}) + 0 \\
&= \lim_{j \to \infty} \int_0^\infty \int_\Omega (\mathbf{g} \cdot \nabla u^{\varepsilon'_j,\,\sigma'_j})\, \varepsilon'_j\, (\partial_t u^{\varepsilon'_j,\,\sigma'_j}) \, dx \, dt \\
&= -\int_0^\infty \delta V_t \lfloor_\Omega \, dt(\mathbf{g}) \\
&= \int_0^\infty \int_\Omega \mathbf{H}^\Omega_V \cdot \mathbf{g} \, d\|V_t\| \, dt.
\end{aligned} \tag{6.69}$$

Note that since the support of $\mathbf{g}$ is in Ω, we have that $\mathcal{S}_{\alpha,\,\mathbf{v}_b}(\mathbf{g}) = 0$. Thus, from (6.64) and (6.69) and the arbitrariness of $\mathbf{g}$, we have that $\widetilde{\mathbf{H}}_V \lfloor_\Omega \equiv \mathbf{H}^\Omega_V$ in $(L^2(\|V_t\| \otimes \mathcal{L}_t^1,\, \Omega \times [0,\, \infty)))^n$.

To prove the second and third claims, we will apply the identity (6.64). Now take any $\mathbf{g} \in (C_c(\overline{\Omega} \times [0,\, \infty)))^n$ with $\|\mathbf{g}\|_{L^2(\|V_t\| \otimes \mathcal{L}_t^1,\, \overline{\Omega} \times [0,\, \infty))} \leq 1$. From (6.67) and (6.68), we have the following estimate:

$$\begin{aligned}
\int_0^\infty \int_\Omega \widetilde{\mathbf{H}}_V \cdot \mathbf{g} \, d\|V_t\| \, dt &= -\left(\mathcal{S}_{\alpha,\,\mathbf{v}_b} + \mathcal{S}^{\Omega}_{\tilde{\mu},\,\mathbf{H}^{\Omega}_V}\right)(\mathbf{g}) \\
&\leq \liminf_{j \to \infty} \left(\int_0^\infty \int_\Omega \varepsilon'_j (\partial_t u^{\varepsilon'_j,\,\sigma'_j})^2 \, dx \, dt\right)^{\frac{1}{2}} \left(\int_0^\infty \int_{\overline{\Omega}} |\mathbf{g}|^2 \, d\|V_t\| \, dt\right)^{\frac{1}{2}} \\
&\leq D^{\frac{1}{2}}
\end{aligned} \tag{6.70}$$

Therefore, carrying out the approximation (6.64), we have

$$\int_0^\infty \int_{\overline{\Omega}} |\widetilde{\mathbf{H}}_V|^2 \, d\|V_t\| \, dt \leq D. \tag{6.71}$$

Finally, we need to prove (6.60). To do this, we may carry out the approximation argument which is conducted by Ilmanen in [15] and we are able to apply this method to our problem since the associated varifold V_t is rectifiable for a.e. $t \geq 0$. Therefore, considering all the above, we may conclude that Proposition 6.9 follows. □

Finally, we will prove the Brakke's inequaltiy (3.7) and this completes the proof of Theorem 4.10. First of all, we show that the $(\varepsilon,\, \sigma)$-approximated velocity vector $\mathbf{v}^{\varepsilon,\,\sigma}$ converges to the modified generalized mean curvature vector $\widetilde{\mathbf{H}}_V$ on $\overline{\Omega}$.

Proposition 6.10. *Let $\{V_t^{\varepsilon'_j,\,\sigma'_j}\}_{j \in \mathbb{N}}$ be as in Proposition 6.6, 6.8 and 6.9. Let $\widetilde{\boldsymbol{H}}_V$ be as in Proposition 6.9. Then we have*

$$-\lim_{j \to \infty} \int_0^\infty \int_\Omega (\boldsymbol{g} \cdot \nabla u^{\varepsilon'_j,\,\sigma'_j}) \varepsilon'_j (\partial_t u^{\varepsilon'_j,\,\sigma'_j}) \, dx \, dt = \int_0^\infty \int_{\overline{\Omega}} \boldsymbol{g} \cdot \widetilde{\boldsymbol{H}}_V \, d\|V_t\| \, dt \tag{6.72}$$

for all $\boldsymbol{g} \in (C_c^1(\overline{\Omega} \times [0,\, \infty)))^n$.

Proof. Let $\mathbf{g}$ be in $(C_c^1(\overline{\Omega}\times[0,\infty)))^n$. From Proposition 6.6, we have already known that $V_t^{\varepsilon_j',\sigma_j'}$ converges to $(n-1)$-rectifiable varifold V_t associated with μ_t for a.e. $t \geq 0$ in the sense of varifolds and thus we have $\lim_{j\to\infty}\delta V_t^{\varepsilon_j',\sigma_j'} = \delta V_t$. From (6.59) in Proposition 6.9, there exists the modified generalized mean curvature vector $\widetilde{\mathbf{H}}_V$ such that

$$\int_0^\infty \int_{\overline{\Omega}} \mathbf{g}\cdot\widetilde{\mathbf{H}}_V \, d\|V_t\|\, dt = -\left(\mathcal{S}_{\alpha,\mathbf{v}_b} + \mathcal{S}^{\Omega}_{\tilde{\mu},\mathbf{H}_V^{\Omega}}\right)(\mathbf{g}). \tag{6.73}$$

Thus, from (6.65) and (6.66), we have

$$\int_0^\infty \int_{\overline{\Omega}} \mathbf{g}\cdot\widetilde{\mathbf{H}}_V \, d\|V_t\|\, dt = -\lim_{j\to\infty}\int_0^\infty\int_\Omega (\mathbf{g}\cdot\nabla u^{\varepsilon_j',\sigma_j'})\varepsilon_j'(\partial_t u^{\varepsilon_j',\sigma_j'})\,dx\,dt. \tag{6.74}$$

This implies (6.72) and completes the proof of Proposition 6.10. □

Finally, considering Proposition 6.8, 6.9, and 6.10, we are prepared to prove Theorem 4.10.

Proof of Theorem 4.10. Since we have already shown Lemma 4.2, 4.4, 4.6, 4.9 and 6.7, it is sufficient to prove Brakke's inequality (3.7). From Lemma 4.2, 4.4, 4.6, and 6.7, we can take the same subsequence $\varepsilon_j'\to 0$ and $\sigma_j'\to 0$ as $j\to\infty$ such that all the claims in Lemma 4.2, 4.4, 4.6, and 6.7 hold. Thus, in the following, it is sufficient to consider such a subsequence. Let ϕ be in $C_c^1(\overline{\Omega}\times[0,\infty))$ such that $\phi\geq 0$. For any $0\leq t_1<t_2<\infty$, recalling (5.6) in Proposition 5.2 and the notation $f^{\varepsilon_j',\sigma_j'} := -\varepsilon_j'\Delta u^{\varepsilon_j',\sigma_j'} + \frac{W'(u^{\varepsilon_j',\sigma_j'})}{\varepsilon_j'}$, we have

$$\begin{aligned}
\mu_t^{\varepsilon_j',\sigma_j'}(\phi)\Big|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2}\int_\Omega \Big(-\frac{1}{\varepsilon_j'}(f^{\varepsilon_j',\sigma_j'})^2\phi + f^{\varepsilon_j',\sigma_j'}\nabla\phi\cdot\nabla u^{\varepsilon_j',\sigma_j'}\Big)\,dx\,dt \\
&\qquad + \int_{t_1}^{t_2}\int_\Omega \partial_t\phi\, d\mu_t^{\varepsilon_j',\sigma_j'}\,dt - \int_{t_1}^{t_2}\int_{\partial\Omega}\varepsilon_j'\,\sigma_j'\,\phi\left(\frac{\partial u^{\varepsilon_j',\sigma_j'}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1}\,dt \\
&\leq \int_{t_1}^{t_2}\int_\Omega \Big(-\frac{1}{\varepsilon_j'}(f^{\varepsilon_j',\sigma_j'})^2\phi + f^{\varepsilon_j',\sigma_j'}\nabla\phi\cdot\nabla u^{\varepsilon_j',\sigma_j'}\Big)\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega \partial_t\phi\, d\mu_t^{\varepsilon_j',\sigma_j'}dt.
\end{aligned} \tag{6.75}$$

Since we have already proved $\mu_t^{\varepsilon_j',\sigma_j'} \rightharpoonup \mu_t = \|V_t\|$ for all $t\geq 0$ on $\overline{\Omega}$, the left-hand side of (6.75) converges to that of (3.7) and so does the third term of the right hand side of (6.75). Hence, combinig (6.60) and (6.72) with (6.75) and taking $j\to\infty$, we may obtain

$$\begin{aligned}
\int_{\overline{\Omega}}\phi\, d\|V_t\|\Big|_{t=t_1}^{t_2} &\leq -\liminf_{j\to\infty}\left(\int_{t_1}^{t_2}\int_\Omega \varepsilon_j'\phi(\partial_t u^{\varepsilon_j',\sigma_j'})^2\,dx\,dt\right) \\
&\qquad - \liminf_{j\to\infty}\left(\int_{t_1}^{t_2}\int_\Omega \nabla\phi\cdot\nabla u^{\varepsilon_j',\sigma_j'}\varepsilon_j'(\partial_t u^{\varepsilon_j',\sigma_j'})\,dx\,dt\right) \\
&\qquad\qquad + \int_{t_1}^{t_2}\int_{\overline{\Omega}}\partial_t\phi\, d\|V_t\|\,dt \\
&\leq \int_{t_1}^{t_2}\int_{\overline{\Omega}}\Big(-\phi|\widetilde{\mathbf{H}}_V|^2 + \nabla\phi\cdot\widetilde{\mathbf{H}}_V\Big)\,d\|V_t\|\,dt + \int_{t_1}^{t_2}\int_{\overline{\Omega}}\partial_t\phi\,d\|V_t\|\,dt
\end{aligned} \tag{6.76}$$

This completes the proof of Brakke's inequality and thus we obtain Theorem 4.10. □

6.2 Dynamic boundary condition

In this section, we will prove a sequence of the main theorems in the case of dynamic boundary conditions, which we stated in Section 4.2.2. We note that the positive constants C_1 and D are as in Proposition 5.1 and Proposition 5.3 and also note that, for simplicity, we only consider the case that the parameter σ is equal to 1, which is fixed in the following.

6.2.1 Convergence of the measures $\{\mu^{\varepsilon,\sigma}\}_{\varepsilon>0}$ (dynamic boundary conditions)

The convergence of the measure $\{\mu_t^{\varepsilon,\sigma}\}_{\varepsilon>0}$ for all $t\geq 0$ can be proved in the similar manner shown in Subsection 6.1.1, that is, we can show almost the same lemma as Lemma 6.1 by using the same arguments. Therefore, we do not repeat the proof of the convergence of $\{\mu_t^{\varepsilon,\sigma}\}_{\varepsilon>0}$ again here.

6.2.2 Convergece of the measures $\{\alpha^{\varepsilon,\sigma}\}_{\varepsilon>0}$ and proof of Lemma 4.13 and Lemma 4.14 (dynamic boundary conditions)

From Propostion 5.1 and 5.3, we can also show, in the same manner described in Subsection 6.1.2, the convergence of the measures $\{\alpha^{\varepsilon,\sigma}\}_{\varepsilon>0}$ and Lemma 4.13. Moreover, we can also prove Lemma 4.14 in the case $\sigma \geq 1$ in the similar manner to the proof of Lemma 4.6. However, as we impose the different asumptions from Lemma 4.6 on Lemma 4.14, we will state the precise proof for clarification and thus show the proof of Lemma 4.14 in the following. Note that we assume that "General assumptions" is valid in this subsection.

Proof of Lemma 4.14. Let σ be equal to 1. First of all, we should say that we can apply the proof of Lemma 4.6 if we have the following claim; there exist $0 < s' < \infty$ and Γ_2 such that Γ_2 is a non-empty connected component of $\partial\Omega$, and the inequality

$$\limsup_{j\to\infty} \int_{t_1}^{t_2} \left| \int_{\Gamma_2} w^j \, d\mathcal{H}^{n-1} \right| dt < \frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2)\,(t_2 - t_1) \tag{6.77}$$

holds for any $0 < t_1 < t_2 < s'$. Thus, it is sufficient to show this claim to prove Lemma 4.14.

Now, from the first assumption of the initial data in Lemma 4.14, we can choose the constants $c_0 > 0$ and $\delta_0 > 0$ such that

$$\limsup_{j\to\infty} \left| \int_{\Gamma_2} w_0^j \, d\mathcal{H}^{n-1} \right| < c_0 < c_0 + 5\delta_0 < \frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2). \tag{6.78}$$

Then, from the second assumption, we can also choose the constant $\gamma_0 > 0$ such that

$$\sup_{j\in\mathbb{N}} \mu_0^j(\Omega \cap \{\mathrm{dist}(x,\,\partial\Omega) < \gamma_0\}) < \delta_0. \tag{6.79}$$

Then, we may take $s_0 > 0$ such that

$$0 < s_0 < \frac{\frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2) - c_0 - 5\delta_0}{2C_3(\gamma_0)}, \tag{6.80}$$

where $C_3(\gamma_0)$ is a positive constant depending only on γ_0, Γ_2, and D in (5.2), which we will choose later.

Next, we will derive the local energy estimate on the boundary which is important to show (6.77). First of all, we choose a bounded open set U_2 such that $\Gamma_2 \subset U_2$ and $U_2 \cap (\partial\Omega \setminus \Gamma_2) = \emptyset$ since $\mathbb{R}^n$ is normal space and Γ_2 is a connected component of $\partial\Omega$. Moreover, we may choose $\rho > 0$ such that $\{x \in \mathbb{R}^n \mid \mathrm{dist}(x,\,\partial\Omega) < \rho\} \subsetneq U_2$. Then, recalling the calculation of the a priori estimate in Proposition 5.2 or 5.3, and taking the proper test function $\phi_{\gamma_0} \in C_c^2(\overline{\Omega})$ such that $0 < \phi_{\gamma_0} \leq 2$ in Ω, $\phi_{\gamma_0} = 1$ on $\partial\Omega$ and $\operatorname{spt}\phi_{\gamma_0} \subset U_2$, we may obtain, from (5.8) and (5.15),

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \phi_{\gamma_0}\, d\mu_t^j\right) \leq& -\int_\Omega \Delta\phi_{\gamma_0}\, d\mu_t^j + \int_\Omega \varepsilon_j(\nabla u^{j,\sigma} \otimes \nabla u^{j,\sigma} : \nabla^2\phi_{\gamma_0})\, dx \\
&- \int_{\partial\Omega} \varepsilon_j(\nabla\phi_{\gamma_0}\cdot\nabla u^{j,\sigma})\frac{\partial u^{j,\sigma}}{\partial\nu}\, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \varepsilon\,\sigma\phi_{\gamma_0}\left(\frac{\partial u^{j,\sigma}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1} \\
&+ \int_{\partial\Omega} \frac{\partial\phi_{\gamma_0}}{\partial\nu}\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1} \\
\leq& \, 3D\sup_\Omega|\nabla^2\phi_{\gamma_0}| - \int_{\partial\Omega} \varepsilon_j(\nabla\phi_{\gamma_0}\cdot\nabla u^{j,\sigma})\frac{\partial u^{j,\sigma}}{\partial\nu}\, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \varepsilon_j\,\sigma\phi_{\gamma_0}\left(\frac{\partial u^{j,\sigma}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1} \\
&+ \int_{\partial\Omega} \frac{\partial\phi_{\gamma_0}}{\partial\nu}\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1}.
\end{aligned} \tag{6.81}$$

The way we construct the proper test function ϕ_{γ_0} is as follows; let $d_{\partial\Omega}$ be the signed distance function from $\partial\Omega$ which is positive in Ω. Then, because of the smoothness of $\partial\Omega$, we can choose $\rho' > 0$ such that $d_{\partial\Omega}$ is smooth in $\{|d_{\partial\Omega}| < \rho'\}$ and, moreover, setting $\tilde\rho := \min\{2,\,\rho,\,2^{-1}\rho'\}$, we can extend $d_{\partial\Omega}+1$ into the function $\tilde d_{\partial\Omega}$ such that $\tilde d_{\partial\Omega}$ is smooth on $\overline{\Omega}$, $\tilde d_{\partial\Omega}$ is equal to $d_{\partial\Omega}+1$ in $\{x \in \overline{\Omega} \mid |d_{\partial\Omega}| < \tilde\rho\}$ and $|\tilde d_{\partial\Omega}| \leq 2$ on $\overline{\Omega}$. Then, setting $\tilde\gamma_0 := \min\{\gamma_0,\,\tilde\rho\}$, and $\phi_{\gamma_0} := \eta_{\tilde\gamma_0}\tilde d_{\partial\Omega}$ where $\eta = \eta_{\tilde\gamma_0}$ is the cut-off function such that $\operatorname{spt}\eta \subset U_{\tilde\gamma_0} := \{x \in \overline{\Omega} \mid |d_{\partial\Omega}| < \tilde\gamma_0\} \subsetneq U_2$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $U_{\frac{\tilde\gamma_0}{2}} := \{x \in \overline{\Omega} \mid |d_{\partial\Omega}| < 2^{-1}\tilde\gamma_0\} \subsetneq U_{\tilde\gamma_0}$, and

$|\nabla\eta| < \infty$ on $\overline{\Omega}$, we obtain the required test function satisfying $0 < \phi_{\gamma_0} \leq 2$ in Ω, $\phi_{\gamma_0} = 1$ uniformly on $\partial\Omega$, and $\operatorname{spt}\phi_{\gamma_0} \subset U_{\tilde{\gamma}_0}$. Then, from (6.81), we can have the following calculation;

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \phi_{\gamma_0}\, d\mu_t^j\right) &\leq 3D\sup_\Omega |\nabla^2\phi_{\gamma_0}| - \int_{\partial\Omega\cap\operatorname{spt}\eta} \varepsilon_j\, \sigma\, \eta \left(\frac{\partial u^{j,\sigma}}{\partial\nu}\right)^2 d\mathcal{H}^{n-1} \\
&\quad - \int_{\partial\Omega} \varepsilon_j (\nabla\eta\cdot\nabla u^{j,\sigma})(\nabla u^{j,\sigma}\cdot\nu)\, d\mathcal{H}^{n-1} - \int_{\partial\Omega\cap\operatorname{spt}\eta} \varepsilon_j\, \eta(\nabla\tilde{d}_{\partial\Omega}\cdot\nabla u^{j,\sigma})(\nabla u^{j,\sigma}\cdot\nu)\, d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial\Omega} (\nabla\eta\cdot\nu)\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial\Omega\cap\operatorname{spt}\eta} \eta(\nabla\tilde{d}_{\partial\Omega}\cdot\nu)\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1} \\
&\leq C_3(\gamma_0) + \int_{\partial\Omega\cap U_{\tilde{\gamma}_0}} \varepsilon_j\, \eta \left(\frac{\partial u^{j,\sigma}}{\partial\nu}\right)^2 (1-\sigma)\, d\mathcal{H}^{n-1} \\
&\quad - \int_{\partial\Omega\cap(U_2\setminus U_{\frac{\tilde{\gamma}_0}{2}})} \varepsilon_j(\nabla\eta\cdot\nabla u^{j,\sigma})(\nabla u^{j,\sigma}\cdot\nu)\, d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial\Omega\cap(U_2\setminus U_{\frac{\tilde{\gamma}_0}{2}})} (\nabla\eta\cdot\nu)\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1} \\
&\quad - \int_{\partial\Omega\cap U_{\tilde{\gamma}_0}} \eta\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1},
\end{aligned} \tag{6.82}$$

where $C_3(\gamma_0) := 3D\sup_\Omega|\nabla^2\phi_{\gamma_0}| < \infty$. Note that, from the definition of $U_{\tilde{\gamma}_0}$, we have $\partial\Omega\cap U_2\setminus U_{\tilde{\gamma}_0} = \emptyset$ and thus $\partial\Omega\cap\operatorname{spt}\eta = \partial\Omega\cap U_{\tilde{\gamma}_0} (= \Gamma_2)$. Therefore, we may obtain

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \phi_{\gamma_0}\, d\mu_t^j\right) &\leq C_3(\gamma_0) - \int_{\partial\Omega\cap U_{\frac{\tilde{\gamma}_0}{2}}} \left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1} \\
&\leq C_3(\gamma_0) - \int_{\Gamma_2} \left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1}.
\end{aligned} \tag{6.83}$$

Therefore, integrating the both sides of (6.83) over $[0,\, s_0]$, we have that

$$\begin{aligned}
\sup_{\varepsilon>0}\int_0^{s_0}\int_{\Gamma_2}\left(\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2} + \frac{W(u^{j,\sigma})}{\varepsilon_j}\right) d\mathcal{H}^{n-1}dt &\leq C_3(\gamma_0)\, s_0 + 2\sup_{j\in\mathbb{N}}\mu_0^j(\Omega\cap\operatorname{spt}\phi_{\gamma_0}) \\
&\leq C_3(\gamma_0)\, s_0 + 2\sup_{j\in\mathbb{N}}\mu_0^j(\Omega\cap\{x \mid \operatorname{dist}(x,\,\partial\Omega) < \gamma_0\}) \\
&< C_3(\gamma_0)\, s_0 + 2\delta_0,
\end{aligned} \tag{6.84}$$

Thirdly, we will derive another important estimate on the boundry to show (6.77). Taking the test function ϕ_{γ_0} in the above and calculating the time derivative of $\int_\Omega \phi_{\gamma_0}\, d\mu_t^j(x)$, we have the following:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \phi_{\gamma_0}\, d\mu_t^j(x) &= -\int_\Omega \varepsilon_j\phi_{\gamma_0}(\partial_t u^{j,\sigma})^2\, dx - \int_{\partial\Omega}\frac{\varepsilon_j}{\sigma}\phi_{\gamma_0}(\partial_t u^{j,\sigma})^2\, d\mathcal{H}^{n-1} \\
&\quad - \int_\Omega \varepsilon_j\partial_t u^{j,\sigma}\nabla u^{j,\sigma}\cdot\nabla\phi_{\gamma_0}\, dx \\
&\leq -\int_\Omega \varepsilon_j\,\phi_{\gamma_0}\left(\partial_t u^{j,\sigma} + \frac{\nabla u^{j,\sigma}\cdot\nabla\phi_{\gamma_0}}{2\phi_{\gamma_0}}\right)^2 dx + \int_\Omega \frac{|\nabla\phi_{\gamma_0}|^2}{2\phi_{\gamma_0}}\,\frac{\varepsilon_j|\nabla u^{j,\sigma}|^2}{2}\, dx \\
&\quad - \int_{\partial\Omega\cap U_{\tilde{\gamma}_0}} \frac{\varepsilon_j}{\sigma}\,\eta\,(\partial_t u^{j,\sigma})^2\, d\mathcal{H}^{n-1} \\
&\leq \sup_\Omega|\nabla^2\phi_{\gamma_0}|\,\mu_t^j(\Omega) - \int_{\Gamma_2}\frac{\varepsilon_j}{\sigma}(\partial_t u^{j,\sigma})^2\, d\mathcal{H}^{n-1},
\end{aligned} \tag{6.85}$$

where we used the inequality

$$\sup_A \frac{|\nabla f|^2}{2f} \leq \sup_A |\nabla^2 f| \tag{6.86}$$

for any bounded open set $A \subset \mathbb{R}^n$ and $f \in C_c^2(\overline{A})$ with $f > 0$ on $\overline{A}$ (see [16] for the original inequality). Thus, integrating over $[0,\, s_0]$ in the both sides of (6.85), we have

$$\int_0^{s_0}\int_{\Gamma_2} \frac{\varepsilon_j}{\sigma}(\partial_t u^{j,\,\sigma})^2\, d\mathcal{H}^{n-1}\, dt \le \frac{1}{3}C_3(\gamma_0)\, s_0 + 2\sup_{j\in\mathbb{N}} \mu_0^j(\Omega \cap \operatorname{spt}\phi_{\gamma_0}) < C_3(\gamma_0)\, s_0 + 2\delta_0. \tag{6.87}$$

Now, we will calculate the time derivative of $\int_{\Gamma_2} w^j\, d\mathcal{H}^{n-1}$ as follows; from Schwarz inequality, we have, for any $t \in [0,\, s_0]$,

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_2} w^j\, d\mathcal{H}^{n-1} &= \int_{\Gamma_2} \sqrt{2W(u^j)}\,(\partial_t u^j)\, d\mathcal{H}^{n-1} \\ &\le \left(\int_{\Gamma_2} 2\frac{W(u^j)}{\varepsilon_j}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \left(\int_{\Gamma_2} \varepsilon_j(\partial_t u^j)^2\, \mathcal{H}^{n-1}\right)^{\frac{1}{2}}.\end{aligned} \tag{6.88}$$

Then, integrating over $[0,\, s]$, where $s \in [0,\, s_0]$ is arbitrary, in the both sides of (6.88), we obtain

$$\begin{aligned}\left|\int_{\Gamma_2} w^j(s)\, d\mathcal{H}^{n-1} - \int_{\Gamma_2} w_0^j\, d\mathcal{H}^{n-1}\right| &\le \int_0^{s_0}\int_{\Gamma_2} \frac{W(u^j)}{\varepsilon_j}\, d\mathcal{H}^{n-1}\, dt + \int_0^{s_0}\int_{\Gamma_2} \varepsilon_j(\partial_t u^j)^2\, d\mathcal{H}^{n-1}\, dt \\ &< 2C_3(\gamma_0)\, s_0 + 4\delta_0,\end{aligned} \tag{6.89}$$

and thus, for any $s \in [0,\, s_0]$,

$$\left|\int_{\Gamma_2} w^j(s)\, \mathcal{H}^{n-1}\right| < 2C_3(\gamma_0)\, s_0 + 4\delta_0 + \left|\int_{\Gamma_2} w_0^j\, \mathcal{H}^{n-1}\right|, \tag{6.90}$$

where we used (6.84) and (6.87) in the above. Therefore, for any $< t_1 < t_2 < s_0$, integrating over $s \in [t_1,\, t_2]$ in (6.90) and using the first assumption (6.78), we conclude that

$$\begin{aligned}\limsup_{j\to\infty}\int_{t_1}^{t_2}\left|\int_{\Gamma_2} w^j(s)\, \mathcal{H}^{n-1}\right| ds &\le \left(2C_3(\gamma_0)\, s_0 + 4\delta_0 + \limsup_{j\to\infty}\left|\int_{\Gamma_2} w_0^j\, \mathcal{H}^{n-1}\right|\right)(t_2 - t_1) \\ &\le (2C_3(\gamma_0)\, s_0 + 4\delta_0 + c_0)\,(t_2 - t_1) \\ &\le \left(\frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2) - c_0 - 5\delta_0 + 4\delta_0 + c_0\right)(t_2 - t_1) \\ &= \left(\frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2) - \delta_0\right)(t_2 - t_1) < \frac{2}{3}\mathcal{H}^{n-1}(\Gamma_2)\,(t_2 - t_1).\end{aligned} \tag{6.91}$$

This completes the proof of (6.77). □

6.2.3 First variations of associated varifolds and proof of Lemma 4.16 (dynamic boundary conditions)

In Subsection 6.2.1, we have already proved the existence of the convergent subsequence $\{\mu_t^{\varepsilon_j',\,\sigma}\}_{\varepsilon>0}$ such that it is independent of $t \in [0,\, \infty)$. Then, in this subsection, we mainly discuss the first variation of the varifold associated with $\mu_t^{\varepsilon_j',\,\sigma}$ and we will show the proof of Lemma 4.16 as we discussed the similar topics in Subsection 6.1.3. Note that we assume that "General assumptions" and "Vanishing hypothesis for the discrepancy measure" in this subsection are valid.

First of all, we associate a varifold for each $\mu_t^{\varepsilon_j',\,\sigma}$ as follows:

Definition 6.11. Let $\{u^{\varepsilon,\,\sigma}\}_{\varepsilon>0}$ be a family of the solutions of the equation (1.10) and $\mu_t^{\varepsilon,\,\sigma}$ be as in (1.12). Then for $\psi \in C_c(G_{n-1}(\overline{\Omega}))$ and any $t \ge 0$, define

$$V_t^{\varepsilon,\,\sigma}(\psi) := \int_{\Omega\cap\{|\nabla u^{\varepsilon,\,\sigma}(\cdot,\,t)|\neq 0\}} \psi(x,\, \mathbf{I} - \mathbf{a}^{\varepsilon,\,\sigma}\otimes\mathbf{a}^{\varepsilon,\,\sigma})\, d\mu_t^{\varepsilon,\,\sigma}(x), \tag{6.92}$$

where $\mathbf{a}^{\varepsilon,\,\sigma} := \frac{\nabla u^{\varepsilon,\,\sigma}}{|\nabla u^{\varepsilon,\,\sigma}|}$.

From the definition, we may obtain $\|V_t^{\varepsilon,\,\sigma}\| = \mu_t^{\varepsilon,\,\sigma}\lfloor_{\{|\nabla u^{\varepsilon,\,\sigma}(\cdot,\,t)|\neq 0\}}$, hence, by considering the first variation of $V_t^{\varepsilon,\,\sigma}$, we may derive the same formula as (6.27) in Lemma 6.3.

Lemma 6.12. *Let $\{u^{\varepsilon,\sigma}\}_{\varepsilon,\sigma>0}$ and $\mu_t^{\varepsilon,\sigma}$ be as in Definition 6.11. Then, for any $\varepsilon>0$, $\sigma>0$, $t\geq 0$ and all $\boldsymbol{g}\in(C_c^1(\overline{\Omega}))^n$, we have*

$$\begin{aligned}\delta V_t^{\varepsilon,\sigma}(\boldsymbol{g}) &= \int_\Omega (\boldsymbol{g}\cdot\nabla u^{\varepsilon,\sigma})\left(\varepsilon\Delta u^{\varepsilon,\sigma}-\frac{W'(u^{\varepsilon,\sigma})}{\varepsilon}\right)dx+\int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|\neq 0\}}\nabla\boldsymbol{g}:(\boldsymbol{a}^{\varepsilon,\sigma}\otimes\boldsymbol{a}^{\varepsilon,\sigma})\,d\xi_t^{\varepsilon,\sigma}\\ &\quad+\int_{\partial\Omega}(\boldsymbol{g}\cdot\nu)\left(\frac{\varepsilon|\nabla u^{\varepsilon,\sigma}|^2}{2}+\frac{W(u^{\varepsilon,\sigma})}{\varepsilon}\right)d\mathcal{H}^{n-1}-\int_{\partial\Omega}\varepsilon(\boldsymbol{g}\cdot\nabla u^{\varepsilon,\sigma})\frac{\partial u^{\varepsilon,\sigma}}{\partial\nu}\,d\mathcal{H}^{n-1}\\ &\quad-\int_{\Omega\cap\{|\nabla u^{\varepsilon,\sigma}|=0\}}\nabla\boldsymbol{g}:\boldsymbol{I}\frac{W(u^{\varepsilon,,\sigma})}{\varepsilon}\,dx. \end{aligned}\tag{6.93}$$

The proof of this lemma is the same as that of Lemma 6.3, and thus we do not repeat it again.

Proposition 6.13. *Let $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ be such that Lemma 4.11 and 4.13 hold and let $\{u^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$ satisfy the equations* (1.10)*. Then for a.e. $t\geq 0$,*

$$\begin{aligned}\liminf_{j\to\infty}\Bigg(\int_\Omega|\nabla u^{\varepsilon_j',\sigma}|\left|\varepsilon_j'\Delta u^{\varepsilon_j',\sigma}-\frac{W'(u^{\varepsilon_j',\sigma})}{\varepsilon_j'}\right|dx+\int_{\partial\Omega}\left(\frac{\varepsilon_j'|\nabla u^{\varepsilon_j',\sigma}|^2}{2}+\frac{W(u^{\varepsilon_j',\sigma})}{\varepsilon'}\right)d\mathcal{H}^{n-1}&\\ +\int_{\partial\Omega}\varepsilon_j'|\nabla u^{\varepsilon_j',\sigma}|\left|\frac{\partial u^{\varepsilon_j',\sigma}}{\partial\nu}\right|d\mathcal{H}^{n-1}\Bigg)<\infty.&\end{aligned}\tag{6.94}$$

Recalling that the parameter σ is 1, the proof of this proposition is basically the same as that of Proposition 6.5. Hence we do not write the proof here again.

Next we show that the limit measure μ_t^σ is actually $(n-1)$-rectifiable on $\overline{\Omega}$ for a.e. $t\geq 0$ and a proper subsequence od the associated varifolds $\{V_t^{\varepsilon,\sigma}\}_{\varepsilon>0}$ converges uniquely to the varifold V_t associated with μ_t^σ as $\varepsilon\downarrow 0$.

Proposition 6.14. *For a.e. $t\geq 0$, μ_t^σ is $(n-1)$-rectifiable on $\overline{\Omega}$ and any convergent subsequence $\{V^{\varepsilon_j'',\sigma}\}_{j\in\mathbb{N}}$ of $\{V_t^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$, where $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ is such that Lemma 4.11 and 4.13 hold, converges to the unique $(n-1)$-rectifiable varifold V_t associated with μ_t. Moreover, we have*

$$\|\delta V_t^\sigma\|(\overline{\Omega})<\infty,\quad \int_0^T\|\delta V_t^\sigma\|(\overline{\Omega})\,dt<\infty\tag{6.95}$$

for a.e. $t\geq 0$ and any $T>0$ respectively.

Note that we may also prove this proposition in the same manner as we show in the proof of Propostion 6.6. Thus, we do not write the proof again here.

Finally, considering all the claims shown in Proposition 6.13 and 6.14, we may conclude that Lemma 4.16 is valid.

6.2.4 Proof of Theorem 4.17 (dynamic boundary conditions)

In this section, we will prove Theorem 4.17, that is, the existence of the singular limits of the Allen-Cahn equations described in (1.10) by taking $\varepsilon\to 0$ with fixed $\sigma\in[1,\infty)$. Before proving Theorem 4.17, as a preparation, we will show three propositions. First of all, we show that the first variation in an integral form $\int_0^\infty\delta V_t^{\varepsilon_j',\sigma}\,dt$ converges to $\int_0^\infty\delta V_t^\sigma\,dt$ locally in time as $\varepsilon\to 0$, where the subsequence $\{V_t^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$ has the limit varifold V_t^σ.

Note that, through this subsection, we assume that "Generalized assumptions" and "Vanishing hypothesis for the discrepancy measure" in Subsection 4.2.1 hold. Moreover, we only consider the subsequence $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ such that Lemma 4.11 and 4.13 hold.

Proposition 6.15. *Let $\{V_t^{\varepsilon_j',\sigma}\}_{j\in\mathbb{N}}$ be a family of associated varifolds with $\mu_t^{\varepsilon_j',\sigma}$ satisfying Proposition 6.13 and 6.14. Then we have*

$$\lim_{j\to\infty}\int_0^T\delta V_t^{\varepsilon_j',\sigma}(\boldsymbol{g})\,dt=\int_0^T\delta V_t^\sigma(\boldsymbol{g})\,dt\tag{6.96}$$

for all $T>0$ and all $\boldsymbol{g}\in(C^1(\overline{\Omega}\times[0,\infty)))^n$ with $g(\cdot,t)\in(C_c^1(\overline{\Omega}))^n$.

The proof of this porposition can be done in the same way as Proposition 6.8 and hence we do not write the proof here.

Proposition 6.16. *Let $\{V_t^\sigma\}_{t\geq 0}$ be as in Lemma 4.16 and Suppose that α^σ and v_b^σ are followed from Lemma 4.13 and $\delta V_t^\sigma\lfloor_{\partial\Omega}^T$ and $\mathcal{S}_{\alpha^\sigma,\, v_b^\sigma}$ are as in Definition 2.12. Then we obtain*

$$\left\| \int_0^\infty \delta V_t^\sigma \lfloor_\Omega \, dt + \int_0^\infty \delta V_t^\sigma \lfloor_{\partial\Omega}^T \, dt + \sigma^{-1} \mathcal{S}_{\alpha^\sigma,\, v_b^\sigma} \right\| \ll \|V_t^\sigma\| \otimes \mathcal{L}_t^1 \quad \text{on } \overline{\Omega} \times [0,\, \infty). \tag{6.97}$$

Thus, we may obtain the existence of the modified generalized mean curvature vector $\widetilde{\boldsymbol{H}}_V^\sigma$ (see Definition 3.4) and, moreover, $\boldsymbol{H}_V^\sigma$ belongs to $(L^2(\|V_t^\sigma\| \otimes \mathcal{L}_t^1,\, \overline{\Omega} \times [0,\, \infty)))^n$ and we also obtain (4.33) and the inequality

$$\int_{t_1}^{t_2} \int_{\overline{\Omega}} \phi |\boldsymbol{H}_V^\sigma|^2 \, d\|V_t^\sigma\| \, dt \leq \liminf_{j\to\infty} \int_{t_1}^{t_2} \int_\Omega \varepsilon_j' \phi (\partial_t u^{\varepsilon_j',\, \sigma})^2 \, dx \, dt \tag{6.98}$$

for all $\phi \in C_c(\overline{\Omega} \times [0,\, \infty))$ with $\phi \geq 0$ and any $0 < t_1 \leq t_2 < \infty$.

Proof. First of all, we show the absolute continuity

$$\|\delta V_t^\sigma \lfloor_\Omega\| \ll \|V_t^\sigma\| \quad \text{in } \Omega \text{ for a.e. } t \geq 0. \tag{6.99}$$

To do this , we take any time $t \geq 0$ such that (6.94) and the vanishing of the discrepancy measure $\{\xi_t^{\varepsilon_j',\, \sigma}\}_{j\in\mathbb{N}}$ hold. Let $\{V_t^{\varepsilon_j',\, \sigma}\}_{j\in\mathbb{N}}$ be a subsequence converging to V_t^σ. Then, from (5.1), (6.27) and the vanishing of $\xi_t^{\varepsilon_j',\, \sigma}$, we have (6.62) in Propostition 6.9 for all $\mathbf{g} \in (C_c(\Omega))^n$. Thus, by taking the supremum with respect to $\mathbf{g}$, we have (6.99). From Riesz theorem and Radon-Nikodym theorem, we have that, for a.e. $t \geq 0$, there exists $\mathbf{H}_V^\sigma\lfloor_\Omega(\cdot,\, t) \in (L^1_{loc}(\|V_t\|,\, \Omega))^n$ such that

$$(\delta V_t^\sigma \lfloor_\Omega)\,(\mathbf{g}) = -\int_\Omega \mathbf{H}_V^\sigma \lfloor_\Omega(\cdot,\, t) \cdot \mathbf{g} \, d\|V_t^\sigma\| \tag{6.100}$$

for all $\mathbf{g} \in (C_c(\Omega))^n$ and, moreover, we have that $\mathbf{H}_V^\sigma\lfloor_\Omega \in (L^2(\|V_t\| \otimes [0,\, \infty),\, \Omega \times [0,\, \infty)))^n$ is valid.

Now, given arbitrary $\delta > 0$, we can take the function $\nu^\delta \in (C^1(\overline{\Omega}))^n$ such that $\nu^\delta\lfloor_{\partial\Omega} = \nu$, $|\nu^\delta| \leq 1$ and $\operatorname{spt} \nu^\delta \subset \Omega_\delta := \{x \in \overline{\Omega} \mid \operatorname{dist}(x,\, \partial\Omega) < \delta\}$. This function can be simply constructed by using the signed distance function d from $\partial\Omega$ and extending d smoothly onto $\overline{\Omega}$. Then for any $\mathbf{g} \in (C_c^1(\overline{\Omega} \times [0,\, T])^n$, setting $\mathbf{g}^\delta := \mathbf{g} - (\mathbf{g} \cdot \nu^\delta)\nu^\delta$, we have $\mathbf{g}^\delta(\cdot,\, t) \cdot \nu = 0$ on $\partial\Omega$ and $\delta V_t^\sigma \lfloor_{\partial\Omega}^T(\mathbf{g}) = \delta V_t^\sigma \lfloor_{\partial\Omega}(\mathbf{g}^\delta)$. Now let $U \subset\subset \overline{\Omega} \times [0,\, \infty)$ be an open set and $\mathbf{g} \in (C_c^1(U))^n$ be any test function such that $|\mathbf{g}| \leq 1$. In order to prove (6.97), we need to compute the following:

$$\begin{aligned}
&\left(\int_0^\infty \delta V_t^\sigma \lfloor_{\partial\Omega}^T \, dt + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega \, dt + \sigma^{-1} \mathcal{S}_{\alpha^\sigma,\, \mathbf{v}_b^\sigma} \right)(\mathbf{g}) \\
&= \int_0^\infty \delta V_t^\sigma \lfloor_{\partial\Omega}(\mathbf{g}^\delta) \, dt + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega(\mathbf{g}^\delta) \, dt + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \, dt \\
&\qquad + \sigma^{-1} \int_{\partial\Omega \times [0,\, \infty)} \mathbf{g} \cdot \mathbf{v}_b^\sigma \, d\alpha^\sigma \\
&= \lim_{j\to\infty} \left(\int_0^\infty \delta V_t^{\varepsilon_j',\, \sigma}(\mathbf{g}^\delta) \, dt - \sigma^{-1} \int_0^\infty \int_{\partial\Omega} \mathbf{g}^\delta \cdot (\varepsilon_j' \partial_t u^{\varepsilon_j',\, \sigma} \nabla u^{\varepsilon_j',\, \sigma}) \, d\mathcal{H}^{n-1} \, dt \right) \\
&\qquad + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \, dt \\
&= \lim_{j\to\infty} \int_0^\infty \int_\Omega (\mathbf{g}^\delta \cdot \nabla u^{\varepsilon_j',\, \sigma}) \varepsilon_j' (\partial_t u^{\varepsilon_j',\, \sigma}) \, dx \, dt + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \, dt. \\
&\leq D^{\frac{1}{2}} \left(\int_0^\infty \int_{\overline{\Omega}} |\mathbf{g}|^2 \, d\|V_t^\sigma\| \, dt \right)^{\frac{1}{2}} + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \, dt.
\end{aligned} \tag{6.101}$$

Note that, in (6.101), we used Cauchy-Schwarz inequality and (5.1). From the definitions of ν^δ and $\mathbf{g}^\delta$, we have

$$\begin{aligned}
\left| \int_0^\infty \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \, dt \right| &= \left| \int_0^\infty \int_\Omega \mathbf{H}_V^\sigma \lfloor_\Omega \cdot (\mathbf{g} - \mathbf{g}^\delta) \, d\|V_t\| \, dt \right| \\
&\leq \int_0^\infty \int_{\Omega_\delta \setminus \partial\Omega} |\mathbf{H}_V^\sigma \lfloor_\Omega| \, d\|V_t\| \, dt \to 0
\end{aligned} \tag{6.102}$$

as $\delta \to 0$. From dominated convergence theorem, we obtain

$$\left(\int_0^\infty \delta V_t^\sigma \lfloor_{\partial\Omega}^T dt + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega dt + \sigma^{-1}\mathcal{S}_{\alpha^\sigma, \mathbf{v}_b^\sigma}\right)(\mathbf{g}) \le D^{\frac12}\left(\int_0^\infty \int_{\overline{\Omega}} |\mathbf{g}|^2 \, d\|V_t^\sigma\| \, dt\right)^{\frac12}$$
$$\le D^{\frac12}\left(\|V_t^\sigma\| \otimes \mathcal{L}_t^1(U)\right)^{\frac12} \tag{6.103}$$

Therefore, taking the supremum with respect to $\mathbf{g}$, we obtain the absolute continuity (6.97) from the arbitrariness of $U \subset \overline{\Omega} \times [0, \infty)$ is arbitrary. From this, it follows that there exists a $\|V_t\| \otimes \mathcal{L}_t^1$-integrable vector-valued function $\widetilde{\mathbf{H}}_V^\sigma$ such that

$$\int_0^\infty \delta V_t^\sigma \lfloor_{\partial\Omega}^T dt + \int_0^\infty \delta V_t^\sigma \lfloor_\Omega dt + \sigma^{-1}\mathcal{S}_{\alpha^\sigma, \mathbf{v}_b^\sigma} = -\widetilde{\mathbf{H}}_V^\sigma \, \|V_t\| \otimes \mathcal{L}_t^1 \quad \text{on } \overline{\Omega} \times [0, \infty), \tag{6.104}$$

Next we will show that the Radon-Nikodym derivative $\widetilde{\mathbf{H}}_V^\sigma$ belongs to $(L^2(\|V_t^\sigma\| \otimes \mathcal{L}_t^1, \overline{\Omega} \times [0,\infty)))^n$. To prove this, we again use the approximation shownin (6.64) in the proof of Proposition 6.9 as follows: let $\mathbf{g} \in (C_c^1(\overline{\Omega} \times [0, \infty))^n$ with $\|\mathbf{g}\|_{L^2(\|V_t^\sigma\| \otimes \mathcal{L}_t^1, \overline{\Omega} \times [0, \infty))} \le 1$. Then from (6.101) and (6.102), we may compute as follows:

$$\begin{aligned}
\int_0^\infty \int_{\overline{\Omega}} \widetilde{\mathbf{H}}_V^\sigma \cdot \mathbf{g} \, d\|V_t^\sigma\| \, dt &\le \liminf_{j\to\infty}\left(\int_0^\infty \int_\Omega \varepsilon_j'(\partial_t u^{\varepsilon_j', \sigma}))^2 \, dx \, dt\right)^{\frac12}\left(\int_0^\infty \int_{\overline{\Omega}} |\mathbf{g}|^2 \, d\|V_t^\sigma\| \, dt\right)^{\frac12} \\
&\qquad - \int_0^T \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \, dt \\
&\le D^{\frac12} - \int_0^T \delta V_t^\sigma \lfloor_\Omega(\mathbf{g} - \mathbf{g}^\delta) \\
&\to D^{\frac12} \quad \text{as } \delta \to 0.
\end{aligned} \tag{6.105}$$

Here we used the estimate (5.1). Hence, from (6.64), we have

$$\int_0^\infty \int_{\overline{\Omega}} |\widetilde{\mathbf{H}}_V^\sigma|^2 \, d\|V_t^\sigma\| \, dt \le D. \tag{6.106}$$

Therefore we may conclude that $\widetilde{\mathbf{H}}_V^\sigma$ belongs to $(L^2(\|V_t^\sigma\| \otimes \mathcal{L}_t^1, \overline{\Omega} \times [0,\infty)))^n$ and thus we conclude that $\widetilde{\mathbf{H}}_V^\sigma$ is actually the modified generalized mean curvature vector.

Finally we need to prove (6.98) for all $\phi \in C_c(\overline{\Omega} \times [0, \infty))$ with $\phi \ge 0$. To prove this, we may carry out the approximation argument which is stated by Ilmanen [15]. We can apply it to our problem because the associated varifold V_t^σ is $(n-1)$-rectifiable on $\overline{\Omega}$ for a.e. $t \ge 0$. Therefore, Proposition 6.16 follows. ☐

Considering all the claims in Proposition 6.15 and 6.16, we obtain the absolute continuity (3.11) and the estimate (4.33).

Now we prove Brakke's inequality (3.13) and this completes the proof of Theorem 4.17. First of all, we show that the ε-approximated velocity vector converges to the modified generalized mean curvature vector $\widetilde{\mathbf{H}}_V^\sigma$ up to the boundary.

Proposition 6.17. *Let $\{V_t^{\varepsilon_j', \sigma}\}_{j\in\mathbb{N}}$ be as in Proposition 6.13 and 6.14 for a.e. $t \ge 0$ and let $\widetilde{\boldsymbol{H}}_V^\sigma$ be as in Proposition 6.16. Then we have*

$$\lim_{j\to\infty} \int_0^\infty \int_\Omega (\boldsymbol{g} \cdot \nabla u^{\varepsilon_j', \sigma})\varepsilon_j'(\partial_t u^{\varepsilon_j', \sigma}) \, dx \, dt = -\int_0^\infty \int_{\overline{\Omega}} \boldsymbol{g} \cdot \widetilde{\boldsymbol{H}}_V^\sigma \, d\|V_t^\sigma\| \, dt \tag{6.107}$$

for all $\boldsymbol{g} \in (C_c^1(\overline{\Omega} \times [0, \infty)))^n$ with $\boldsymbol{g}(\cdot, t) \cdot \nu = 0$ on $\partial\Omega$.

Proof. Let $\mathbf{g} \in (C_c^1(\overline{\Omega} \times [0, \infty)))^n$ be such that $\mathbf{g}(\cdot, t) \cdot \nu = 0$ on $\partial\Omega$. We have already shown that $V_t^{\varepsilon_j', \sigma}$ converges to V_t^σ as $j \to \infty$ for a.e. $t \ge 0$ and

$$\lim_{j\to\infty} \int_0^\infty \delta V_t^{\varepsilon_j', \sigma} \, dt(\mathbf{g}) = \int_0^\infty \delta V_t \, dt(\mathbf{g}) \tag{6.108}$$

for any $\mathbf{g} \in (C_c^1(\overline{\Omega} \times [0, \infty)))^n$. Furthermore, from the choice of $\mathbf{g}$, the third term of the left-hand side in (6.93) vanishes if we substitute $\mathbf{g}$ in (6.93), and we have $\mathbf{g} = \mathbf{g} - (\mathbf{g} \cdot \nu)\nu$ on $\partial\Omega \times [0, \infty)$. Therefore, from (6.101), (6.104), Proposition 6.14, and 6.16, we obtain (6.107). ☐

Second, we will show the following inequality.

Proposition 6.18. *Let $\boldsymbol{v}_b^\sigma$ and α^σ be as in Lemma 4.13. Then we have*

$$\int_0^\infty \int_{\partial\Omega} \phi|\boldsymbol{v}_b^\sigma|^2 \, d\alpha^\sigma \le \liminf_{j\to\infty} \int_0^\infty \int_{\partial\Omega} \varepsilon_j' \phi(\partial_t u^{\varepsilon_j',\sigma})^2 \, d\mathcal{H}^{n-1} \, dt \tag{6.109}$$

for all $\phi \in C_c^1(\overline{\Omega} \times [0,\infty))$ with $\phi \ge 0$.

Proof. Let ϕ be in $C_c^1(\overline{\Omega} \times [0,\infty))$ with $\phi \ge 0$. Since α^σ is a Radon measure on $\partial\Omega \times [0,\infty)$, $(C_c(\partial\Omega \times [0,\infty)))^n$ is dense in $(L^2(\alpha, \partial\Omega \times [0,\infty)))^n$ and thus we can choose the sequence $\{\mathbf{g}_m^\sigma\}_{m\in\mathbb{N}}$ of $(C_c(\partial\Omega \times [0,\infty)))^n$ approximating $\mathbf{v}_b^\sigma$ in L^2-norm. Here the sequence $\{\phi\mathbf{g}_m^\sigma\}_{m\in\mathbb{N}}$ is also a subset of $(C_c(\partial\Omega\times[0,\infty)))^n$ and then, from (5.1), $\sigma = 1$ and Cauchy-Schwarz inequality, we can compute as follows:

$$\begin{aligned}
\left(-\int_{\partial\Omega\times[0,\,\infty)} \phi\mathbf{g}_m^\sigma \cdot \mathbf{v}_b^\sigma \, d\alpha^\sigma\right)^2 &= \lim_{j\to\infty} \left(\int_{\partial\Omega\times[0,\,\infty)} \phi\mathbf{g}_m^\sigma \cdot \mathbf{v}_b^{\varepsilon_j',\,\sigma} \, d\alpha^{\varepsilon_j',\,\sigma}\right)^2 \\
&= \left(\int_{\partial\Omega\times[0,\,\infty)} \phi|\mathbf{g}_m^\sigma|^2 \, d\alpha^\sigma\right) \\
&\qquad \times \left(\liminf_{j\to\infty} \int_{\partial\Omega\times[0,\,\infty)} \varepsilon_j' \phi(\partial_t u^{\varepsilon_j',\,\sigma})^2 \, d\mathcal{H}^{n-1} \, dt\right)
\end{aligned} \tag{6.110}$$

for all $m \in \mathbb{N}$. In (6.110), we used the convergence of $\{\alpha^{\varepsilon_j',\,\sigma}\}_{j\in\mathbb{N}}$ in the sense of Radon measures. Then, from the definition of $\{\mathbf{g}_m^\sigma\}_{m\in\mathbb{N}}$, we can show

$$\left(-\int_{\partial\Omega\times[0,\,\infty)} \phi\mathbf{g}_m^\sigma \cdot \mathbf{v}_b^\sigma \, d\alpha^\sigma\right)^2 \xrightarrow[m\to\infty]{} \left(\int_{\partial\Omega\times[0,\,\infty)} \phi|\mathbf{v}_b^\sigma|^2 \, d\alpha^\sigma\right)^2 \tag{6.111}$$

$$\int_{\partial\Omega\times[0,\,\infty)} \phi|\mathbf{g}_m^\sigma|^2 \, d\alpha^\sigma \xrightarrow[m\to\infty]{} \int_{\partial\Omega\times[0,\,\infty)} \phi|\mathbf{v}_b^\sigma|^2 \, d\alpha^\sigma. \tag{6.112}$$

Therefore by substituting (6.111) and (6.112) into (6.110), we obtain (6.109). ☐

Finally, considering all of the above arguments, we can prove Brakke's inequality (3.13) and Theorem 4.17.

Proof of Brakke's inequality. In Lemma 4.11 and 4.13, we can take the same subsequence $\{\varepsilon_j'\}_{j\in\mathbb{N}}$ such that $\mu_t^{\varepsilon_j',\,\sigma}$ converges to μ_t^σ on $\overline{\Omega}$ for all $t \ge 0$ and $\alpha^{\varepsilon_j',\,\sigma}$ converges to α^σ on $\partial\Omega \times [0,\,\infty)$, and thus it is sufficient to consider such a subsequence in the following. Let $\phi \in C_c^1(\overline{\Omega} \times [0,\,\infty))$ be such that $\phi \ge 0$ and $\nabla\phi(\cdot,\,t) \cdot \nu = 0$ for any $t \ge 0$. For any $0 \le t_1 < t_2 < \infty$, from (5.6) in Proposition 5.2 and the notation $f^{\varepsilon_j',\,\sigma} := -\varepsilon_j' \Delta u^{\varepsilon_j',\,\sigma} + \frac{W'(u^{\varepsilon_j',\,\sigma})}{\varepsilon_j'}$, we have that

$$\begin{aligned}
\mu^{\varepsilon_j',\,\sigma}(\phi)\Big|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2} \left(\int_\Omega -\frac{1}{\varepsilon_j'}\phi(f^{\varepsilon_j',\,\sigma})^2 + f^{\varepsilon_j',\,\sigma}\nabla\phi \cdot \nabla u^{\varepsilon_j',\,\sigma} \, dx + \int_\Omega \partial_t\phi \, d\mu_t^{\varepsilon_j',\,\sigma}\right) dt \\
&\quad - \frac{1}{\sigma} \int_{t_1}^{t_2} \int_{\partial\Omega} \phi\varepsilon_j'(\partial_t u^{\varepsilon_j',\,\sigma})^2 \, d\mathcal{H}^{n-1} \, dt
\end{aligned} \tag{6.113}$$

Since $\mu_t^{\varepsilon_j',\,\sigma}$ converges to $\mu_t^\sigma = \|V_t\|$ on $\overline{\Omega}$ for all $t \ge 0$, the left hand side of (6.113) converges to that of (3.13) and so does the third term of the right hand side of (6.113). Hence, combinig (6.98), (6.107) and (6.109) with (6.113) and taking $j \to \infty$, we obtain

$$\begin{aligned}
\int_{\overline{\Omega}} \phi \, d\|V_t^\sigma\|\Big|_{t=t_1}^{t_2} &\le -\liminf_{j\to\infty} \left(\int_{t_1}^{t_2} \int_\Omega \varepsilon_j'(\partial_t u^{\varepsilon_j',\,\sigma})^2 \, dx \, dt\right) + \int_{t_1}^{t_2} \int_{\overline{\Omega}} \nabla\phi \cdot \widetilde{\mathbf{H}}_V^\sigma \, d\|V_t^\sigma\| \, dt \\
&\quad + \int_{t_1}^{t_2} \int_{\overline{\Omega}} \partial_t\phi \, d\|V_t\| \, dt - \liminf_{j\to\infty} \frac{1}{\sigma} \int_{t_1}^{t_2} \int_{\partial\Omega} \varepsilon_j' \, \phi \, (\partial_t u^{\varepsilon_j',\,\sigma_j'})^2 \, d\mathcal{H}^{n-1} \, dt \\
&\le \int_{t_1}^{t_2} \int_{\overline{\Omega}} (-\phi|\widetilde{\mathbf{H}}_V^\sigma|^2 + \nabla\phi \cdot \widetilde{\mathbf{H}}_V^\sigma + \partial_t\phi) \, d\|V_t^\sigma\| \, dt - \frac{1}{\sigma} \int_{\partial\Omega\times[t_1,\,t_2]} \phi|\mathbf{v}_b^\sigma|^2 \, d\alpha^\sigma.
\end{aligned} \tag{6.114}$$

This completes the proof of Brakke's inequality. Hence we obtain Theorem 4.17. ☐

7 Appendix

7.1 Appendix A

In this appendix, we show the proposition that $u^{\varepsilon,\sigma}$ can be estimated by 1 in $\Omega \times (0, T]$ under some assumptions. Before stating the proposition and proving this, we fix the notations as follows;

$n \in \mathbb{N}$ with $n \geq 1$, $T > 0$

$\Omega \subset \mathbb{R}^n$: a bounded domain with smooth boundary

$Q_T := \Omega \times (0, T)$, $Q_T^* := \Omega \times (0, T]$, and $\partial_p Q_T := \overline{Q_T} \setminus Q_T^*$

$\gamma := \frac{4}{\varepsilon^2}$

Now we are prepared for statintg the claim.

Proposition 7.1. *Assume that* $\sup_\Omega |u_0^{\varepsilon,\sigma}| \leq 1$ *for any* $\varepsilon, \sigma > 0$. *Then,* $\sup_{\Omega\times(0,\infty)} |u^{\varepsilon,\sigma}| \leq 1$ *holds for any* $\varepsilon, \sigma > 0$.

Proof. It is sufficient to prove the estimate for the time interval $(0, T]$ for any $T > 0$. Defining a function w by $w(x, t) := ((u^{\varepsilon,\sigma}(x, t))^2 - 1)\, e^{-\gamma t}$ for any $(x, t) \in Q_T$, we may have

$$\begin{aligned}
\partial_t w - \Delta w &= -\gamma\, e^{-\gamma t}((u^{\varepsilon,\sigma})^2 - 1) + 2u^{\varepsilon,\sigma}\,\partial_t u^{\varepsilon,\sigma}\, e^{-\gamma t} - 2u^{\varepsilon\,\sigma}\,\Delta u^{\varepsilon,,\sigma}\, e^{\gamma t} - 2e^{-\gamma t}|\nabla u^{\varepsilon,\sigma}|^2 \\
&= -\gamma w + 2u^{\varepsilon,\sigma}\, e^{-\gamma t}(\partial_t u^{\varepsilon,\sigma} - \Delta u^{\varepsilon,\sigma}) - 2e^{-\gamma t}|\nabla u^{\varepsilon,\sigma}|^2 \\
&= -\gamma w + 2u^{\varepsilon,\sigma}\, e^{-\gamma t}\left(-\frac{1}{\varepsilon^2} W'(u^{\varepsilon,\sigma})\right) - 2e^{-\gamma t}|\nabla u^{\varepsilon,\sigma}|^2 \\
&= -\gamma w + \frac{4}{\varepsilon^2} e^{-\gamma t}(u^{\varepsilon,\sigma})^2(1 - (u^{\varepsilon,\sigma})^2) - 2e^{-\gamma t}|\nabla u^{\varepsilon,\sigma}|^2 \\
&= -\gamma w - \gamma w(1 + e^{\gamma t} w) - 2e^{-\gamma t}|\nabla u^{\varepsilon,\sigma}|^2 \\
&= -2\gamma w - \gamma e^{\gamma t} w^2 - 2e^{-\gamma t}|\nabla u^{\varepsilon,\sigma}|^2 \leq -2\gamma w \quad \text{in } \Omega \times (0, T].
\end{aligned} \tag{7.1}$$

Setting $Q_T^+ := \{(x, t) \in \overline{Q_T} \mid w(x, t) > 0\}$, we have, from the assumption in the above, that $Q_T^+ \subset \overline{\Omega} \times (0, T]$. Here, if either $u^{\varepsilon,,\sigma} \equiv +1$ or $u^{\varepsilon,,\sigma} \equiv -1$, then there is nothing to prove. Hence, we can assume, in the following, that $u^{\varepsilon,,\sigma} \not\equiv \pm 1$ and this implies $Q_T^+ \neq \emptyset$. Now, since Ω is bounded, there exists a point $(x_0, t_0) \in \overline{Q_T}$ such that $\max_{\overline{Q_T}} w = w(x_0, t_0)$. Assume, for a contradiction, that $(x_0, t_0) \in Q_T^+$. From the assumption that $|u_0^{\varepsilon,\sigma}| \leq 1$, we have that $w(\cdot, 0) \leq 0$ and thus, $t_0 \neq 0$. If $x_0 \in \Omega$, then, from the maximality of w and $t_0 \neq 0$, it holds that $\partial_t w(x_0, t_0) \geq 0$ and $\Delta w(x_0, t_0) \leq 0$. Thus we have that

$$0 \leq \partial_t w(x_0, t_0) - \Delta w(x_0, t_0) \leq -2\gamma w < 0 \quad \text{at } (x_0, t_0), \tag{7.2}$$

which is a contradiction. If $x_0 \in \partial\Omega$, then, from the maximality of w at (x_0, t_0), we have that $\nabla w \cdot \nu \geq 0$ at (x_0, t_0), where ν is the outer unit normal of $\partial\Omega$. From the boundary condition in the equations (1.10), we have that

$$\begin{aligned}
0 \leq \partial_t w + \sigma\nabla w \cdot \nu &= -\gamma e^{-\gamma t}((u^{\varepsilon,\sigma})^2 - 1) + 2u^{\varepsilon,\sigma}\partial_t u^{\varepsilon,\sigma}\, e^{-\gamma t} + 2\sigma u^{\varepsilon,\sigma}\, e^{-\gamma t}\nabla u^{\varepsilon,\sigma} \cdot \nu \\
&= -\gamma w + 2u^{\varepsilon,\sigma}\, e^{-\gamma t}(\partial_t u^{\varepsilon,\sigma} + \sigma\nabla u^{\varepsilon,\sigma} \cdot \nu) \\
&= -\gamma w < 0 \quad \text{at } (x_0, t_0),
\end{aligned} \tag{7.3}$$

which is also a contradiction. Therefore, both cases lead us to get a contradiction. Hence, for any $(x, t) \in \overline{Q_T}$, $w(x, t) \leq \max_{\overline{Q_T}} w = w(x_0, t_0) \leq 0$, in other words, $|u^{\varepsilon,\sigma}(x, t)| \leq 1$ holds. ☐

7.2 Appendix B

In this appendix, we will prove Poincaré-Wirtinger inequality on hypersurfaces. This claim was applied to prove the positivity of the limit measure α on $\Gamma_2 \times [t_1, t_2]$ for some non-empty connected component Γ_2 of $\partial\Omega$ and some $0 < t_1 < t_2 < \infty$ in Subsection 6.1.2.

Lemma 7.2. *Let* $N \geq 3$ *and* $M \subset \mathbb{R}^N$ *be a smooth, bounded, and connected hypersurface embedded in* $\mathbb{R}^N$ *without boundaries. Then there exists a constant* $C(N, M) > 0$ *such that, for any* $u \in W^{1,1}(M)$,

$$\|u - u_M\|_{L^1(M)} \leq C(N, M)\|\nabla_M u\|_{L^1(M)} \tag{7.4}$$

and $u_M := (\mathcal{H}^{N-1}(M))^{-1} \int_M u\, d\mathcal{H}^{N-1}$.

Proof. Setting $X := \{u \in W^{1,1}(M) \mid u_M = 0\}$, we have that X is a closed subspace in $W^{1,1}(M)$. Then, it is sufficient to prove that there exists $C(N, M) > 0$ such that $\|u\|_{L^1} \leq C(N, M)\|\nabla_M u\|_{L^1}$ for each $u \in X$. This is because we can easily have that, if $u \in W^{1,1}(M)$, then $u - u_M \in X$ and $\nabla_M(u - u_M) = \nabla_M u$.

We assume, for a contradiction, that, for each $n \in \mathbb{N}$, there exists $u_n \in X$ such that $\|u_n\|_{L^1} > n\|\nabla_M u_n\|_{L^1}$ holds. From the assumption, we may consider that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,1}$ and $\|u_n\|_{W^{1,1}(M)} \geq 1$ for any $n \in \mathbb{N}$. Since $N \geq 3$ and thus $X \subset W^{1,1}(M) \hookrightarrow L^1(M)$ is the compact embedding (see, for instance, [4]) and X is closed in $W^{1,1}(M)$, there exist a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ and $u_\infty \in X$ such that $u_{n_i} \xrightarrow[i\to\infty]{} u_\infty$ in $L^1(M)$. Then, from the assumption, it follows that

$$1 \leq \|u_{n_i}\|_{W^{1,1}} = \|u_{n_i}\|_{L^1} + \|\nabla_M u_{n_i}\|_{L^1} \xrightarrow[i\to\infty]{} \|u_\infty\|_{L^1} + 0, \tag{7.5}$$

and thus $\|u_\infty\|_{L^1} \geq 1$. On the other hand, for any $\phi \in C_c^1(M)$, from the divergence theorem on hypersurfaces and recalling the fact that $u_{n_i} \xrightarrow[i\to\infty]{} u_\infty$ in $L^1(M)$, we have that

$$\begin{aligned} 0 \xleftarrow[i\to\infty]{} \|\phi\|_{C^1(M)} \|\nabla_M u_{n_i}\|_{L^1(M)} &\geq \left| \int_M \phi \nabla_M u_{n_i} \, d\mathcal{H}^{N-1} \right| \\ &= \left| \int_M u_{n_i}(\nabla_M \phi - \phi \mathbf{H}_M) \, d\mathcal{H}^{N-1} \right| \\ &\xrightarrow[i\to\infty]{} \left| \int_M u_\infty(\nabla_M \phi - \phi \mathbf{H}_M) \, d\mathcal{H}^{N-1} \right| \\ &= \left| \int_M \phi \nabla_M u_\infty \, d\mathcal{H}^{N-1} \right|, \end{aligned} \tag{7.6}$$

where $\mathbf{H}_M$ is the mean curvature vector of M which is bounded because M is smooth and compact. This implies $\nabla_M u_\infty = 0$ $\mathcal{H}^{N-1}$-a.e. on M and thus it follows that $\Delta_M u_\infty = 0$ on M in distribution. Then, from Weyl's lemma, we have $u_\infty \in C^\infty(M)$. Since M is connected, u_∞ is in X, and $\nabla_M u_\infty = 0$ $\mathcal{H}^{N-1}$-a.e. on M, we may conclude that $u_\infty = 0$ on M and this contradicts the fact that $\|u_\infty\|_{L^1} \geq 1$. $\square$

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