Discussion Paper, Series A, No.2018-331

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October 2018

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On the use of the simplex method for a type of allocation problems

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Abstract: In this study we discuss the use of the simplex method to solve allocation problems whose flow matrices are doubly stochastic. Although these problems can be solved via a 0-1 integer programming method, H.W. Kuhn [4] suggested the use of linear programming in addition to the Hungarian method.
Specifically, we use the Birkhoff's theorem to prove that the simplex method enables solving these problems. We also provide insights how to obtain a partition that includes a particular unit.

Key words: allocation problems, Hall's theorem, Birkhoff's theorem, simplex method

1. Introduction

The type of allocation problems in which flow matrices are doubly stochastic can be solved via 0-1 integer programming, which, however, is generally not solvable in polynomial time. To address this issue, Kuhn [4] proposed the Hungarian algorithm which can be recently computed in $O(n^3)$ time, and also suggested the use of the simplex method.

In our Proposal 2033 in *Mathematics Magazine* [7], instances could be formulated as allocation problems for which the Hungarian method may not be effective, since the non-zero elements in the coefficient matrix are the same.

In this paper, we examine the use of the simplex method in this type of problems by using Birkhoff's theorem. Specifically, we provide the proof that solutions to these problems can be obtained by the simplex method, which is also easy to use and usually attains a solution efficiently. We also consider a modified problem to obtain a partition including a particular unit.

The remainder of the paper is organized as follows. Section 2 describes the kind of allocation problems, object of the study. Section 3 illustrates the simplex method, while Section 4 concludes.

2. Allocation problems

Let $G = (V, E)$ be a graph, where V is a vertex set and E is an edge set. We consider the following allocation problem:

Problem There are kn units comprising n kinds of goods, and the same k $(1 \leq k < n)$ units are available for each of them. After randomization, the kn units are divided into n groups. Is it possible to obtain k partitions each of which consists of n different goods by choosing one goods from each group?

Proposal 2033 [7] is an application of the Problem to a deck of cards $(n = 13, k = 4)$.

We let $G = (S + T, E)$ be a bipartite graph (which admits multiple edges) with bipartition $\{S,\ T\}$. In this case, S includes n groups, while T comprises n different goods. We assign e_{ij}, $i \in S$, $j \in T$ if there is a goods j in group i. We notice that G

is k-regular, that is, every vertex $v \in G$ has degree $d_G(v) = k$.

We use the following lemma of independent interest (see also [2]), which we prove here for convenience sake.

Lemma 1 (cf. [2, Corollary 2.1.3]) *If G is k-regular $(k \geq 1)$ and bipartite, then G has a perfect matching.*

Proof Summing up the number of edges,

$$|E| = \sum_{s \in S} d_G(s) = \sum_{t \in T} d_G(t).$$

As G is k-regular and bipartite, then

$$|E| = k|S| = k|T|,$$

namely,

$$|S| = |T|.$$

We denote $N(X) \subset T$ as the neighbors of $X \subset S$.

Let

$$N = \{|(x, y) \in E| \mid x \in X, y \in N(X)\}$$

Then,

$$N = k|X| \quad \text{for } X \subset S$$

and

$$N \leq k|N(X)| \quad \text{for } N(X) \subset T.$$

Therefore,

$$|X| \leq |N(X)|.$$

Hence, as per Hall's theorem [3], G has a perfect matching. □

Thus, we give an affirmative answer to the Problem.

Theorem 2 *In the Problem setting, there are k disjoint perfect matchings in $G = (S + T, E)$.*

Proof We apply Lemma 1, and recursively get and delete the resulting perfect matching k times. □

We now consider how to solve the Problem in specific instances.

3. Solution methods

By adding a source node s to the left of S, and a sink node t to the right of T, and by setting capacities of all arcs C_{ij} to 1, we can consider the resulting network $N = (G, s, t, C)$. Then, since the Problem can be regarded as a maximum flow problem, we can resort to such as a variety of Ford-Fulkerson method to solve specific examples. However, we will not use network algorithms because they are not easy to implement for people who have not specialized in networks. We will focus on optimization methods in this paper.

We let x_{ij} $(0 \leq x_{ij} \leq 1)$ be a flow between $i \in S$ and $j \in T$ $(|S| = |T| = n)$, and $N(i)$ be the neighbors of i.
We can now formulate instances of the Problem as the following 0-1 integer programming problem:

Problem I (PI)

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{i=1}^{n} \sum_{j \in N(i)} x_{ij} \\ \text{subject to} \quad & \textstyle\sum_{j \in N(i)} x_{ij} = 1, \quad i = 1, \dots, n \\ & \textstyle\sum_{i \in \{i | j \in N(i)\}} x_{ij} = 1, \quad j = 1, \dots, n \\ & x_{ij} = 0 \ \text{or} \ 1. \end{aligned} \tag{1}$$

In (PI), we only have to consider kn variables since $\sum_{i=1}^{n} |N(i)| = kn$. Nonetheless, there may be multiple edges. We are able to solve (PI) via 0-1 programming method, since Lemma 1 guarantees the existence of solutions. However, it may be intractable as n becomes large, since 0-1 integer programming problems are generally NP-hard.

Considering that the coefficient matrix of the constraints is doubly stochastic matrix into account, the following result (Birkhoff's theorem, 1946) holds.

Lemma 3 [6, Corollary 18.1a] *Any doubly stochastic matrix is expressed as a convex combination of permutation matrices.* □

In light of Lemma 3, we consider the relationship between (PI) and its linear relaxation problem:

Problem L (PL)

$$
\begin{aligned}
&\text{maximize} \quad && \textstyle\sum_{i=1}^{n} \sum_{j \in N(i)} x_{ij} \\
&\text{subject to} \quad && \textstyle\sum_{j \in N(i)} x_{ij} = 1, \quad i = 1, \dots, n \\
& && \textstyle\sum_{i \in \{i | j \in N(i)\}} x_{ij} = 1, \quad j = 1, \dots, n \\
& && x_{ij} \geq 0.
\end{aligned} \tag{2}
$$

We can now restrict the linear programming method to the simplex method, since $x_{ij} = \frac{1}{k}$ for $j \in N(i)$, $x_{ij} = 0$ for $j \notin N(i)$ is a trivial solution to (PL). Let the matrix $A = (a_{ij})$, where

$$a_{ij} = 1, \ j \in N(i), \quad i = 1, \dots, n$$

$$a_{ij} = 0, \ j \notin N(i), \quad i = 1, \dots, n,$$

and $X = (x_{ij})$.

Theorem 4 *The simplex method applied to (PL) solves (PI).*

Proof The simplex method is valid since, by Lemma 1, there are feasible points in both (PI) and (PL). Let

$$x = \left(x_{11_1}, \dots, x_{11_k}, x_{22_1}, \dots, x_{22_k}, \dots \dots \dots, x_{nn_1}, \dots, x_{nn_k}\right) \in \mathbb{R}^{kn}, \tag{3}$$

where $x_{ii_l} \in x_{ij}$, $j \in N(i)$, be a solution to (PI). By Lemma 1, there exists an $n \times n$ permutation matrix

$$P^l = \{(p_{ij}) \mid p_{ii_l} = 1 \text{ for } \exists i_l \in N(i),\ p_{ij} = 0 \text{ for } j \neq i_l, j = 1, \dots, n;\ i = 1, \dots, n\},$$

which is a non-zero subset of A. We let $x^l \in \mathbb{Z}^{kn}$ be

$$x^l = \{x \mid x_{ii_l} = 1,\ x_{ij} = 0 \text{ for } j \neq i_l, j \in N(i);\ i = 1, \dots, n\},$$

which is the solution to (PI) where $A = P^l$.

Then, all feasible points $x^{l_1}, \dots, x^{l_N}$ $(N \gg 0)$ are determined by all possible permutation matrices $P^{l_1}, \dots, P^{l_N}$. Thus, when we let

$$X = \{ X \mid X = \textstyle\sum_{i=1}^{N} \lambda_i P^{l_i}, \quad \lambda_i \geq 0, \ \sum_{i=1}^{N} \lambda_i = 1 \},$$

it becomes trivial that $x^{l_1}, \dots, x^{l_N}$ are all basic feasible points, since

$$\{ x \mid x = \textstyle\sum_{i=1}^{N} \lambda_i x^{l_i}, \quad \lambda_i \geq 0, \ \sum_{i=1}^{N} \lambda_i = 1 \}$$

exists in the constraint region of (PL).

Hence, the simplex method applied to (PL) solves (PI), since it terminates at a basic optimal point [5, Theorem 13.4]. □

Example 1.　A deck of cards ($n = 13,\ k = 4$)

Group	Cards
1	7♣　K♢　J♣　Q♢
2	8♣　9♣　6♣　Q♠
3	7♢　8♢　9♢　5♣
4	J♠　9♡　K♠　3♢
5	8♡　A♠　6♢　2♠
6	Q♡　3♣　5♢　5♠
7	A♡　2♣　6♠　7♠
8	K♣　4♡　5♡　Q♣
9	4♣　10♠　6♡　7♡
10	K♡　2♡　3♡　A♢
11	10♡　2♢　J♡　8♠
12	4♠　10♣　A♣　3♠
13	10♢　9♠　J♢　4♢

The experiments are implemented on a laptop, using FORTRAN to code the simplex method. At first, there are 52 variables and 26 constraints. The solutions are as follows:

$k = 4$ (initial): 54 iterations
(K♢, 9♣, 8♢, J♠, 2♠, Q♡, 6♠, 5♡, 7♡, 3♡, 10♡, A♣, 4♢),
$k = 3$ (delete the previous solution): 32 iterations
(J♣, 8♣, 7♢, 3♢, 6♢, 5♠, A♡, Q♣, 10♠, K♡, 2♢, 4♠, 9♠),
$k = 2$ (delete the previous solution): 28 iterations
(7♣, Q♠, 9♢, K♠, 8♡, 5♢, 2♣, 4♡, 6♡, A♢, J♡, 3♠, 10♢),
$k = 1$ (the remainder)
(Q♢, 6♣, 5♣, 9♡, A♠, 3♣, 7♠, K♣, 4♣, 2♡, 8♠, 10♣, J♢).

We should note that the objective function value of (1) is n provided that there is a feasible x in (3). Therefore, the simplex method terminates in phase I, by which we may generally expect high efficiency.

It is also worth noticing that we can select a particular unit (e.g., A♠) in a partition, since the existence of such a partition is guaranteed by Theorem 2. Let c_{ij} be the coefficients of x_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n$, *i.e.*, $c_{i_0 j_0} = 1 + \varepsilon \quad (\varepsilon > 0)$, $\exists j_0 \in N(i_0)$; $c_{ij} = 1$, $j \neq j_0$, $j \in N(i)$, $i = 1, \dots, n$.
Consider the following problem:

Problem $\mathbf{L_c}$ ($\mathbf{PL_c}$)

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{i=1}^{n} \sum_{j \in N(i)} c_{ij} x_{ij} \\ \text{subject to} \quad & \textstyle\sum_{j \in N(i)} x_{ij} = 1, \quad i = 1, \dots, n \\ & \textstyle\sum_{i \in \{i | j \in N(i)\}} x_{ij} = 1, \quad j = 1, \dots, n \\ & x_{ij} \geq 0. \end{aligned} \tag{4}$$

If we set (i_0, j_0) so as to correspond to the particular unit, the solution of (4) becomes the desired partition, as formalized in the following theorem.

Theorem 5 *The simplex method applied to (PL_c) determines a partition that includes the particular unit.*

Proof The simplex method is valid since, as per Theorem 2, there are feasible points in both (PI) and ($\mathrm{PL_c}$). In this case, the simplex method terminates in either phase I or phase II, when the objective function value $n + \varepsilon$ is attained. □

In general transportation problems, *i.e.*, when $c_{ij} \geq 0$ are arbitrary and $|S| = |T| = n$ but there are no multiple edges, the number of variables is n^2, which tends to be much larger than kn. This may cause difficulties in using the simplex method. Note that, for the considered Problem (kn variables), the simplex method is efficient, and we can select an arbitrary unit in the solution.

4. Conclusion

In this study, we derived the result concerning allocation problems which can be modeled by bipartite graphs, which might lead to unexpected results. Specifically, we provided the proof that the simplex method solves these problems by using Birkhoff's theorem. The elementary numerical result we presented shows the validity and efficiency of the method. Moreover, we considered the modified problem to apply the simplex method to obtain a partition that includes a particular unit.

References

[1] Cormen, T.H., Leiserson, C.E., Rivest, R.L., and Stein, C., *Introduction to Algorithms – 3rd ed.*, The MIT Press, 2009.

[2] Diestel, R., *Graph Theory – 4th ed.*, Springer, Heidelberg, 2010.

[3] Hall, P., “On representatives of subsets,” *Journal of the London Mathematical Society* 10 (1935), 26-30.

[4[Kuhn, H.W., “On the origin of the Hungarian methods,” in *History of Mathematical Programming*, Lenstra, J.K. et al. Eds., Elsevier, 1991, 77-81.

[5] Nocedal, J. and Wright, S.J., *Numerical Optimization – 2nd ed.*, Springer, Berlin, 2006.

[6] Schrijver, A., *Combinatorial Optimization*, Springer, Berlin, 2003.

[7] Tanaka, Y., Proposal 2033, *Mathematical Magazine* 90 (5) (2017), 383-384.