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On loops in the hyperbolic locus of the complex Hénon map and their monodromies

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ABSTRACT

We prove John Hubbard’s conjecture on the topological complexity of the hyperbolic horseshoe locus of the complex Hénon map. In fact, we show that there exist several non-trivial loops in the locus which generate infinitely many mutually different monodromies. Furthermore, we prove that the dynamics of the real Hénon map is completely determined by the monodromy of the complex Hénon map, providing the parameter of the map is contained in the hyperbolic horseshoe locus.

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1. Introduction

One of the motivations of this work is to give an answer to the conjecture of John Hubbard on the topology of hyperbolic horseshoe locus of the complex Hénon map

\[ H_{a,c} : C^2 \rightarrow C^2 : \left( \begin{array}{c} x \\ y \end{array} \right) \mapsto \left( \begin{array}{c} x^2 + c - ay \\ x \end{array} \right). \]

Here \( a \) and \( c \) are complex parameters.

Below we describe the conjecture following a formulation given by Bedford and Smillie [1].

Let us define

\[ K_{a,c}^C := \{ p \in C^2 : \{ H^n_{a,c}(p) \}_{n \in \mathbb{Z}} \text{ is bounded} \}, \quad K_{a,c}^R := K_{a,c}^C \cap \mathbb{R}^2. \]

The set \( K_{a,c}^C \) is compact and invariant with respect to \( H_{a,c} \). When the parameters \( a \) and \( c \) are both real, the real plane \( \mathbb{R}^2 \subset C^2 \) is invariant and hence so is \( K_{a,c}^R \). In this case, we call \( H_{a,c} |_{\mathbb{R}^2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) the real Hénon map.

Our primary interest is on the structure of the parameter space, especially on the topology of the set of parameter values on which complex and/or real Hénon maps become a uniformly hyperbolic horseshoe. More precisely, we study the following sets:

\[ \mathcal{H}^C := \{ (a, c) \in C^2 : H_{a,c} | K_{a,c}^C \text{ is a hyperbolic full horseshoe} \}, \]

\[ \mathcal{H}^R := \{ (a, c) \in \mathbb{R}^2 : H_{a,c} | K_{a,c}^R \text{ is a hyperbolic full horseshoe} \}. \]

Here we mean by a hyperbolic full horseshoe a uniformly hyperbolic invariant set which is topologically conjugate to the full shift map \( \sigma \) defined on \( \Sigma_2 = \{ 0, 1 \}^\mathbb{Z} \), the space of bi-infinite sequences of two symbols.

A classical result of Devaney and Nitecki [2] claims that if \( (a, c) \) is in

\[ \text{DN} := \{ (a, c) \in \mathbb{R}^2 : c < -(5 + 2\sqrt{5}) (|a| + 1)^2/4, \ a \neq 0 \} \]

then \( K_{a,c}^R \) is a hyperbolic full horseshoe. Thus \( \text{DN} \subset \mathcal{H}^R \) holds. They also showed that the set

\[ \text{EMP} := \{ (a, c) \in \mathbb{R}^2 : c > (a + 1)^2/4 \} \]
It is easy to see that $\rho$ defines a group homomorphism
$$\rho : \pi_1(\mathcal{H}_0^c, (a_0, c_0)) \to \text{Aut}(\Sigma_2)$$
where $\text{Aut}(\Sigma_2)$ is the group of the automorphisms of $\Sigma_2$. Recall that an automorphism of $\Sigma_2$ is a homeomorphism of $\Sigma_2$ which commutes with the shift map $\sigma$ [5]. We call $\rho$ the monodromy homomorphism and denote its image by $\Gamma$.

For example, let $\gamma_0$ be a loop in $\mathcal{H}_0^c$ based at $(a_0, c_0)$ which is homotopic to the generator of $\pi_1(\text{HOV})$. It is then shown [1] that $\rho(\gamma_0)$ is an involution which interchanges the symbols 0 and 1. Namely, $\rho(\gamma_0)(s) = 1 - s$ for all $s \in [0, 1]^2$.

The monodromy homomorphism was originally defined for polynomial maps of a single complex variable. In this case, since the map does not have the inverse, the target space of the monodromy homomorphism is the automorphism group of one-sided shift space of $d$-symbols, where $d$ is the degree of the polynomial.

When $d = 2$, this group is isomorphic to $\mathbb{Z}_2$ and the monodromy homomorphism is shown to be surjective since it maps the generator of $\pi_1(\mathbb{C} \setminus \text{[the Mandelbrot set]})$ to $1 \in \mathbb{Z}_2$. The monodromy homomorphism is also surjective even when $d > 2$, although the proof is much harder than the case $d = 2$ because the automorphism group becomes much more complicated [6].

Hubbard conjectured that the surjectivity also holds in the case of the complex Hénon map, with the only exception being $\sigma$.

**Conjecture 2 (Hubbard).** The image $\Gamma$ of the monodromy homomorphism and the shift map $\sigma$ generate $\text{Aut}(\Sigma_2)$.

Here we remark that the structure of $\text{Aut}(\Sigma_2)$ is quite complicated [7]: it contains every finite group; furthermore, it contains the direct sum of any countable collection of finite groups; and it also contains the direct sum of countably many copies of $\mathbb{Z}$. Therefore, the conjecture implies, provided it is true, that the topological structure of $\mathcal{H}_c$ is very rich, in contrast to the one-dimensional case where the fundamental group of $\mathbb{C} \setminus \text{[the Mandelbrot set]}$ is simply $\mathbb{Z}$.

Let us state the main results of the paper now.

First, we claim that **Conjecture 1** is true.

**Theorem 1.** There exist parameter values of type-3. In fact, if $(a, c)$ is in one of the following sets:
$$I_p := \{1\} \times [-5.46875, -5.3125],$$
$$I_q := \{0.25\} \times [-2.296875, -2.21875],$$
$$I_r := \{-1\} \times [-5.671875, -4.4375],$$
$$I_s := \{-0.375\} \times [-2.15625, -1.8125]$$

then $(a, c)$ is of type 3.

**Toward Conjecture 2,** we obtain the following partial result.

**Theorem 2.** The order of the group $\Gamma$ is infinite. In particular, it contains an element of infinite order.

Apart from the theoretical interest, the monodromy theory of complex Hénon map can contribute to the understanding of the real Hénon map.

Let $(a, c) \in \mathcal{H}_c \cap \mathbb{R}^2$. If $(a, c)$ is of type-1 or 2, then by definition $K_{a,c}^\mathbb{R}$ is a full horseshoe, or empty. Suppose $(a, c)$ is of type-3. We then ask what is $K_{a,c}^\mathbb{R}$ in this case. By definition, $K_{a,c}^\mathbb{R}$ is a proper subset of $K_{a,c}^\mathbb{C} \subseteq \Sigma_2$. The uniform hyperbolicity implies the existence of a Markov partition for $K_{a,c}^\mathbb{R}$, and therefore, $K_{a,c}^\mathbb{R}$ must be topologically conjugate to some subshift of finite type. The following theorem reveals that $K_{a,c}^\mathbb{R}$ is actually a subshift of $\Sigma_2$ which is realized as the fixed point set of the monodromy of a loop passing through $(a, c)$. 

---
Theorem 3. For any \((a, c) \in \mathcal{H}_Q^C \cap \mathbb{R}^2\), there exists a loop \(\gamma : [0, 1] \to \mathcal{H}_Q^C\) with \(\gamma(1/2) = (a, c)\) such that \(\rho(\gamma)\) is an involution and \(H_{a,c} : K_{a,c}^R \to K_{a,c}^R\) is topologically conjugate to
\[
\sigma |_{\text{Fix}(\rho(\gamma))} : \text{Fix}(\rho(\gamma)) \to \text{Fix}(\rho(\gamma)).
\]

In fact, it suffices to set \(\gamma : \alpha^{-1} \cdot \alpha\), where \(\alpha\) is an arbitrary path in \(\mathcal{H}_Q^C\) connecting a point in \(D\) to \((a, c)\). Here \(\alpha\) denotes the complex conjugate of \(\alpha\). The conjugacy is given by the restriction of \(h_{1/2}\) to \(K_{a,c}^R\). Namely, the following diagram commutes;
\[
\begin{array}{ccc}
K_{a,c}^R & \xrightarrow{h_{1/2}} & K_{a,c}^R \\
\downarrow \Phi & & \downarrow \Phi \\
\text{Fix}(\rho(\gamma)) & \xrightarrow{\sigma} & \text{Fix}(\rho(\gamma)).
\end{array}
\]

As an application of Theorem 3, we obtain the following.

Theorem 4. Let \((a, c) \in I_p\). The real Hénon map \(H_{a,c} : K_{a,c}^R \to K_{a,c}^R\) is topologically conjugate to the subshift of \(\Sigma_2\) with two forbidden blocks \(0010100\) and \(0011100\). Similarly, \(\mathcal{H}_Q^C\) is conjugate to the subshift of \(\Sigma_2\) defined by the following forbidden blocks:

\[
\begin{align*}
10100 & \text{ and } 10110 \text{ for } (a, c) \in I_{p}^c, \\
1010 & \text{ and } 1011 \text{ for } (a, c) \in I_{p}^c. \\
\end{align*}
\]

Notice that \(I_p\) contains \((a, c) = (1, -5.4)\), the parameter studied by Davis, Mackay and Sannami [8]. The subshift for \((a, c) \in I_p\) given in Theorem 4 is equivalent to that observed by them. Thus, we can say that their observation is now rigorously verified. We also remark that this theorem is closely related to the so-called “pruning front” theory [5-8]. Theorem 3 implies that “primary pruned regions”, or, “missing blocks” of \(K_{a,c}^R\) is nothing else but the region where the generating partitions are interchanged along \(\gamma\).

The structure of the paper is as follows. We prove the theorems in Section 2, leaving computational algorithms to Sections 3 and 4. In Section 3, we summarize the algorithm for proving uniform hyperbolicity developed by the author [3]. Section 4 is devoted to an algorithm for computing the monodromy homomorphism. In the Appendix, we discuss a method for rigorously counting the number of periodic points, which gives rise to an alternative proof of Theorem 1. Programs for computer assisted proofs are available at the author’s web page (http://www.math.sci.hokudai.ac.jp/~zin/).

2. Proofs

We first prove Theorem 3. We note that the idea of the theorem and techniques used in the proof are similar to Theorem 5.2 of [1]; the difference is that in our formulation the emphasis is put on the relation between real and complex dynamics.

The key is the symmetry of the Hénon map with respect to the complex conjugation \([1, 1]\), by which we mean the equation
\[
\phi \circ H_{a,c} = H_{\bar{a},\bar{c}} \circ \phi \quad \text{where} \quad \phi \text{ is the complex conjugation that maps } z = (x, y) \text{ to } \bar{z} = (\bar{x}, \bar{y}).
\]

Proof of Theorem 3. Let \(\gamma = \alpha^{-1} \cdot \alpha\) be a loop in \(\mathcal{H}_Q^C\) where \(\alpha\) is a path connecting a point in \(D\) to \((a, c)\), and let \(\tilde{\gamma} := \phi \circ \gamma\) be the complex conjugate of it. Note that by construction, we have \(\tilde{\gamma} = \gamma^{-1} \cdot \gamma\).

Take an arbitrary point \(z \in K_{a,c}^C\) and define
\[
s_z := h_{1/2}(z) \in \Sigma_2.
\]

To prove the theorem, it suffices to show that \(\rho(\gamma)(s_z) = s_z\) if and only if \(z \in \mathbb{R}^2\).

We denote the continuation of \(z\) along \(\gamma\) by \(z(\gamma, t)\), where \(z(\gamma, 1/2) = z\). Recall that by construction, \(z(\gamma, t) = (h_t)^{-1}(s_z) \in \mathbb{C}\).

By the continuity of hyperbolic invariant sets, \(z(\gamma, t) : [0, 1] \to \mathbb{C} : t \mapsto z(\gamma, t)\) defines a continuous curve, which is a closed loop if \(z(\gamma, 0) = z(\gamma, 1)\).

From the symmetry of the Hénon map with respect to the complex conjugation it follows that
\[
\bar{z}(\gamma, t) = \bar{z}(\tilde{\gamma}, t).
\]

Here \(\bar{z}(\tilde{\gamma}, t)\) is the continuation of \(\bar{z}\) along \(\tilde{\gamma}\). Since \(\tilde{\gamma} = \gamma^{-1}\), we have \(\bar{z}(\gamma, t) = \bar{z}(\gamma, 1 - t)\). Therefore,
\[
\rho(\gamma)(s_z) = h_1((h_0)^{-1}(s_z)) = h_1(z(\gamma, 0)) = h_1(z(\gamma, 0)) = h_1(\bar{z}(\gamma, 0)) = h_1(\bar{z}(\gamma, 1)) = h_1((h_1)^{-1}(s_z)) = s_z.
\]

The third equality holds because \(K_{a,c}^C \subseteq \mathbb{R}^2\) and hence \(z(\gamma, 0) = z(\gamma, y)\).

Thus, we know that \(\rho(\gamma)(s_z) = s_z\) if and only if \(s_z = s_z\). Since the map \(h_1/2\) is a bijection between \(K_{a,c}^C\) and \(\Sigma_2\), it follows that \(s_z = s_z\) if and only if \(z = \bar{z}\).

The equation \(\rho(\gamma)(s_z) = s_z\) implies that \(\rho(\gamma)\) is an involution, since \(\bar{z} = z\).

Now we discuss Theorem 1. We first define the sets in which we will find non-trivial loops. Let
\[
L_p = \{1\} \times \{\mathbb{C} \setminus \text{white regions in Fig. 2}\},
\]
\[
L_q = \{0.25\} \times \{\mathbb{C} \setminus \text{white regions in Fig. 3}\},
\]
\[
L_r = \{-1\} \times \{\mathbb{C} \setminus \text{white regions in Fig. 4}\},
\]
\[
L_s = \{-0.375\} \times \{\mathbb{C} \setminus \text{white regions in Fig. 5}\}
\]

and \(L := L_p \cup L_q \cup L_r \cup L_s \subseteq \mathbb{C}^2\). To be precise, these regions are defined by a finite number of closed rectangles. The complete list of these rectangles is available at the author’s web page.

Lemma 5. If \((a, c) \in L\) then \(H_{a,c}\) is uniformly hyperbolic on its chain recurrent set \(\mathcal{R}(H_{a,c})\).

The proof of this lemma is computer assisted. We leave it to Section 3.

Recall that the hyperbolicity of the chain recurrent set implies the \(\mathcal{R}\)-structural stability [10, Corollary 8.2.4]. Therefore, it follows from Lemma 5 that no bifurcation occurs in \(\mathcal{R}(H_{a,c})\) as long as \((a, c) \in L\). Since \(L \cap D\) have non-empty intersection and \(K_{a,c}^C = K_{a,c}^R\) is a hyperbolic full horseshoe on \(D\), we know that \(\mathcal{R}(H_{a,c})\) is also a hyperbolic full horseshoe for all \((a, c) \in L\). However, this observation is not sufficient for our purpose because \(\mathcal{R}(H_{a,c})\) and \(K_{a,c}^C\) do not necessarily coincide. To conclude the hyperbolicity of \(K_{a,c}^C\), we need to show that these sets are equal in the horseshoe locus, as follows.

Corollary 6. If \((a, c) \in L\) then \(H_{a,c} \mid K_{a,c}^C\) is a hyperbolic full horseshoe, that is, \(L \subseteq \mathcal{H}_Q^C\).

Proof of Corollary 6. Let
\[
K_{a,c}^C := \{p \in \mathbb{C}^2 : \{H_{a,c}^n(p)\}_{n \in \mathbb{N}} \text{ is bounded}\},
\]
\[
K_{a,c}^R := \{p \in \mathbb{C}^2 : \{H_{a,c}^n(p)\}_{n \in \mathbb{N}} \text{ is bounded}\},
\]
and \(I_{a,c}^+ := \partial K_{a,c}^C\). Define \(J_{a,c}^+ := K_{a,c}^C \cap I_{a,c}^+\). Then \(K_{a,c}^C = K_{a,c}^R \cup K_{a,c}^C\) and we have \(J_{a,c}^+ \subseteq \mathcal{R}(H_{a,c}) \subseteq K_{a,c}^C\) [11, Proposition 9.2.6, Theorem 9.2.7]. Suppose \((a, c) \in L\). Since \(\mathcal{R}(H_{a,c})\) is a full horseshoe,
The shaded region is contained in \( \mathcal{H}^c \cap \{ a = 1 \} \).

Fig. 2. The shaded region is contained in \( \mathcal{H}^c \cap \{ a = 1 \} \).

The shaded region is contained in \( \mathcal{H}^c \cap \{ a = 0.25 \} \).

Fig. 3. The shaded region is contained in \( \mathcal{H}^c \cap \{ a = 0.25 \} \).

The shaded region is contained in \( \mathcal{H}^c \cap \{ a = -1 \} \).

Fig. 4. The shaded region is contained in \( \mathcal{H}^c \cap \{ a = -1 \} \).

The automorphism \( \gamma \) the base point of \( \beta \) where \( \gamma \) by setting \( \rho(\gamma) = \bar{\rho}^{-1} \cdot \omega \) where \( \omega : [0, 1] \to \text{HOV} \cup L_p \) is a path from \( (1, -10) \) to the basepoint of \( \beta_p \). Choose the parameterization of \( \gamma_p \), so that \( \gamma_p(1/2) \in L_p \) and \( \gamma_p = \gamma_p^{-1} \) hold. Similarly, we define loops \( \gamma_p \), \( \gamma_r \), and \( \gamma_i \) based at \( (1, -10) \) turning around the smaller islands in \( L_q, L_r, \) and \( L_s \), respectively.

**Proposition 7.** The automorphism \( \rho(\gamma_p) \) interchanges the words \( 00101000 \) and 00111000 contained in \( s \in \Sigma_2 \). Namely,

\[
(\rho(\gamma_p)(s)) = \begin{cases} 
0 & \text{if } S_{i} \cdot \ldots \cdot S_{i+3} = 00111000 \\
1 & \text{if } S_{i} \cdot \ldots \cdot S_{i+3} = 00010100 \\
s_i & \text{otherwise.}
\end{cases}
\]

Similarly, \( \rho(\gamma_r) \) interchanges 101000 and 111000, \( \rho(\gamma_r) \) interchanges 10010 and 10110, and \( \rho(\gamma_i) \) interchanges 0010 and 0110.
The proof of Proposition 7 is also computer assisted. An algorithm for this will be discussed in Section 4.

Now we are prepared to prove Theorem 1.

Proof of Theorem 1. Since $\text{Fix}(\rho(y_{2}))$ is a non-empty proper subset of $\Sigma_2$, Theorem 3 implies that $y_{2}(1/2) \in I_p$ is of type-3. By considering loops homotopic to $y_{2}$, we can show that all $(a, c) \in I_p$ are also of type-3. Proofs for other intervals are the same. \[\square\]

Theorem 2 immediately follows from the following proposition.

Proposition 8. The order of $\psi = \rho(y_{1}) \cdot \rho(y_{2})$ is infinite.

The proof below is due to G. A. Hedlund [13, Theorem 20.1].

Proof. For non-negative integer $p$, we define elements of $\Sigma_2$ named $x^{(2p)}$ and $x^{(2p+1)}$ by

$$x^{(2p)} = \cdots 0101010101101(10)^p 11111 \cdots,$$

$$x^{(2p+1)} = \cdots 0101010111101(10)^p 10000 0 \cdots.$$

We then look at the orbit of $x = x^{(0)}$ under the map $\psi$. A simple calculation shows that

$$x = \cdots 01010101011111 \cdots = x^{(0)},$$

$$\psi(x) = \cdots 1010101011101100000 \cdots = x^{(1)},$$

$$\psi^2(x) = \cdots 01010101111011011111 \cdots = x^{(3)}.$$  

By induction, it follows that $\psi^n(x^{(0)}) = x^{(m)}$. Since $x^{(n)} \neq x^{(m)}$ if $n \neq m$, this implies that the order of $\psi$ is infinite. \[\square\]

Theorem 4 is a direct consequence of Theorem 3 and Proposition 7.

3. Hyperbolicity

We recall an algorithm for proving the uniform hyperbolicity of chain recurrent sets developed by the author [3]. We also refer the reader to the work of Suzanne Lynch Hruska [14, 15] for another algorithm.

Let $f$ be a diffeomorphism on a manifold $M$ and $\Lambda$ a compact invariant set of $f$. We denote by $T\Lambda$ the restriction of the tangent bundle $TM$ to $\Lambda$.

Definition 2. We say that $f$ is uniformly hyperbolic on $\Lambda$ if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two $Tf$-invariant subbundles and there exist constants $c > 0$ and $0 < \lambda < 1$ such that $\|Tf^n_{E^s}(x)\| < c\lambda^n$ and $\|Tf^{-n}_{E^u}(x)\| < c\lambda^n$ hold for all $n \geq 0$. Here $\| \cdot \|$ denotes a metric on $M$.

In general, proving the uniform hyperbolicity of $f$ according to this usual definition is quite difficult, because we must control two parameters $c$ and $\lambda$ at the same time, and further, we also need to construct a metric on $M$ adapted to the hyperbolic splitting.

To avoid this difficulty, we introduce a weaker notion of hyperbolicity called "quasi-hyperbolicity". We consider $Tf : T\Lambda \to T\Lambda$, the restriction of $Tf$ to $T\Lambda$, as a dynamical system. An orbit of $Tf$ is said to be trivial if it is contained in the image of the zero section.

Definition 3. We say that $f$ is quasi-hyperbolic on $\Lambda$ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

It is easy to see that uniform hyperbolicity implies quasi-hyperbolicity. The converse is not true in general. However, when $f|_{\Lambda}$ is chain recurrent, these two notions of hyperbolicity are equivalent.

Theorem 9 ([16, 17]). Assume that $f|_{\Lambda}$ is chain recurrent, that is, $\mathcal{R}(f|_{\Lambda}) = \Lambda$. Then $f$ is uniformly hyperbolic on $\Lambda$ if and only if $f$ is quasi-hyperbolic on $\Lambda$.

The definition of quasi-hyperbolicity can be rephrased in terms of isolating neighborhoods as follows. Recall that a compact set $N$ is an isolating neighborhood with respect to $f$ if the maximal invariant set

$$\text{Inv}(N, f) := \{ x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z} \}$$

is contained in int $N$, the interior of $N$. An invariant set $S$ of $f$ is said to be isolated if there is an isolating neighborhood $N$ such that $\text{Inv}(N, f) = S$.

Note that the linearity of $Tf$ in fibers of $TM$ implies that if there exists a non-trivial bounded orbit of $Tf : T\Lambda \to T\Lambda$, then any neighborhood of the image of the zero-section must contain a non-trivial bounded orbit. Therefore, the definition of quasi-hyperbolicity is equivalent to saying that the image of the zero section of $T\Lambda$ is an isolated invariant set with respect to $Tf : T\Lambda \to T\Lambda$. To confirm that $\Lambda$ is quasi-hyperbolic, in fact, it suffices to find an isolating neighborhood containing the image of the zero section.

Proposition 10 ([3, Proposition 2.5]). Assume that $N \subset T\Lambda$ is an isolating neighborhood with respect to $Tf : T\Lambda \to T\Lambda$ and $N$ contains the image of the zero-section of $T\Lambda$. Then $\Lambda$ is quasi-hyperbolic.

Next, we check that the hypothesis of Theorem 9 is satisfied in the case of the complex Hénon map. Let us define

$$R(a, c) := \frac{1}{2}(1 + |a| + \sqrt{(1 + |a|)^2 + 4|c|}),$$

$$S(a, c) := \{(x, y) \in \mathbb{C}^2 : |x| \leq R(a, c), |y| \leq R(a, c)\}.$$  

Then the following holds as in the case of the real Hénon map [3, Lemma 4.1].

Lemma 11. The chain recurrent set $\mathcal{R}(H_{a, c})$ is contained in $S(a, c)$. Furthermore, $H_{a, c}$ restricted to $\mathcal{R}(H_{a, c})$ is chain recurrent.

To prove Lemma 5, therefore, it suffices to show that $\mathcal{R}(H_{a, c})$ is quasi-hyperbolic for $(a, c) \in L$. By Proposition 10, all we have to do is to find an isolating neighborhood that contains the image of the zero-section of $T \mathcal{R}(H_{a, c})$. More precisely, it is enough to find $N \subset TM$ such that

$$\mathcal{R}(H_{a, c}) \subset N \quad \text{and} \quad \text{Inv}(N, TH_{a, c}) \subset \text{int } N$$

hold. Here we identify $\mathcal{R}(H_{a, c})$ and its image by the zero-section of TM. Since there are algorithms [3, Proposition 3.3] that efficiently compute rigorous outer approximations of $\mathcal{R}(H_{a, c})$ and $\text{Inv}(N, TH_{a, c})$, these conditions can be checked on computers rigorously.
In practice, we fix the parameter $a$ to $+1$ (or $0.25, -0.375, -1$) and regard $(H_{1,c})$ as a parameterized family with a single complex parameter $c \in \mathbb{C}$. In the parameter plane, we define

$$C := \{ c \in \mathbb{C} : |\text{Im } c| \leq 8 \text{ and } |\text{Re } c| \leq 8 \}.$$ 

If $c \not\in C$ then $(1, c) \in \text{HOV}$, and thus we do not need to check the hyperbolicity for such $c$. Furthermore, our computation can be restricted to the case when $|\text{Im } c| \geq 0$ because $H_{1,c}$ and $H_{1,-c}$ are conjugate via $\phi$ and hence the hyperbolicity of these two maps is equivalent.

Finally, we perform Algorithm 3.6 of [3] for the family $(H_{1,c})$ with the initial parameter set $C \cap \{ |\text{Im } c| \geq 0 \}$. The algorithm inductively subdivide the initial parameter set and outputs a list of parameter cubes on which the quasi-hyperbolicity is verified. This proves the quasi-hyperbolicity of $R(H_{1,c})$ for $(1, c) \in L_\gamma$. The quasi-hyperbolicity for $L_\eta, L_r$ and $L_s$ is also obtained by applications of the same algorithm.

### 4. Monodromy

In this section, we develop an algorithm for computing the monodromy homomorphism $\rho$.

Let $\gamma : [0, 1] \to K^C_{\gamma(0)}$ be a loop based at $\gamma(0) = \gamma(1) = (a_0, c_0) \in \text{DN}$. Since $\rho(\gamma)$ is defined in terms of conjugacies $h_t = h_t^{\gamma} : K^C_{\gamma(t)} \to \Sigma_2$ along $\gamma$, we first discuss how to compute them.

Let us recall the definition of $h_t$. Define

\[
K^0_0 := \{(x, y) \in K^C_{\gamma(0)} : \text{Re } y \leq 0\}, \\
K^1_0 := \{(x, y) \in K^C_{\gamma(0)} : \text{Re } y \geq 0\}.
\]

By the argument of Devaney and Nitecki [2], we have $K^0_0 \cap K^1_0 = \emptyset$ and the partition $K^0_0 = K^0_1 \cup K^1_1$ induces a topological conjugacy $h_0$. The continuation of this partition along $\gamma$ is defined by

\[
\gamma \in K^0_{\gamma(t)} : \text{the continuation of } z \text{ along } \gamma \text{ at } t = 0 \text{ is in } K^0_0), \\
K^1_{\gamma(t)} := \{z \in K^1_{\gamma(t)} : \text{the continuation of } z \text{ along } \gamma \text{ at } t = 0 \text{ is in } K^1_0\}.
\]

The conjugacy $h_t$ is, by definition, the symbolic coding with respect to this partition. Namely,

\[
(h_t(z))_i := \begin{cases} 0 & \text{if } h_t^{\gamma(1)}(z) \in K^0_0 \\ 1 & \text{if } h_t^{\gamma(1)}(z) \in K^1_0 \end{cases}.
\]

To determine this conjugacy, however, we do not need to compute $K^0_0$ and $K^1_0$ exactly. It suffices to have rigorous outer approximations of them. That is, if $N^0_0$ and $N^1_0$ are disjoint subsets of $\mathbb{C}^2$ such that $K^0_0 \subset N^0_0$ and $K^1_0 \subset N^1_0$ hold for all $t \in [0, 1]$, then $k_t : K^C_{\gamma(t)} \to \Sigma_2$ defined by

\[
(k_t(z))_i := \begin{cases} 0 & \text{if } h_t^{\gamma(1)}(z) \in N^0_0 \\ 1 & \text{if } h_t^{\gamma(1)}(z) \in N^1_0 \end{cases}
\]

is identical to $h_t$.

Here is an algorithm to construct such $N^0_0$ and $N^1_0$.

**Step 1.** Subdivide the interval $[0, 1]$ into $n$ closed intervals $I_1, I_2, \ldots, I_n$ of equal length.

**Step 2.** Using interval arithmetic, we compute a cubical set $N_k$ for each $1 \leq k \leq n$ such that $K^0_{\gamma(t)} \subset N_k$ rigorously holds for all $(a, c) \in \gamma(\tilde{t}_k)$. Define $N_t := N_k$ for $t \in I_k$.

**Step 3.** Consider the set

\[
N := \bigcup_{t \in [0, 1]} \{t \times N_t \subset [0, 1] \times \mathbb{C}^2\}.
\]

Let $N^0_2$ and $N^1_2$ be the unions of the components of $N$ which intersect with $[0, 1] \times \{\text{Re } y \leq 0\}$ and $[0, 1] \times \{\text{Re } y \geq 0\}$, respectively. If $N^0_0 \cap N^1_0 = \emptyset$, define $N^0_0 = N_t \cap N^0_0$ and $N^1_0 = N_t \cap N^1_0$ then stop. If this is not the case, we refine the subdivision of $[0, 1]$ and the grid of $\mathbb{C}^2$, and then go back to step 1.

Applying the algorithm above to the loop $\gamma_q$, we obtain Figs. 7 and 8. The interval $[0, 1]$ is decomposed into $n = 2^8$ sub-intervals, and the size of the grid for $\mathbb{C}^2$ is $2^{-8}$ in each direction. The lightly and darkly shaded regions in Fig. 7 are $N^0_0$ and $N^1_0$. Similarly, Fig. 8 illustrates $N^0_1$ and $N^1_1$. Notice that two figures differ only in four blocks on the left hand side: two blocks of each of $N^0_0$ and $N^1_0$ are interchanged. Using rigorous interval arithmetic, these blocks are identified as blocks corresponding to the symbol sequences $10.100$ and $11.100$.

We execute the same computation also for loops $\gamma_p$, $\gamma_s$, and $\gamma_t$. This yields Fig. 9, which shows a schematic picture of the change along these loops. Notice that “head” and “tail” labels in the figure indicate the symbolic coding according to the initial partitions $N^0_0$ and $N^1_0$, illustrated in the central square.

Now, we can compute the image of $\gamma = \gamma_p$ (or $\gamma_q$, $\gamma_s$, $\gamma_t$) by $\rho$ as follows: Choose a symbol sequence $s = (s_k)_{k \in \mathbb{Z}} \in \Sigma_2$. Then $z := h_0(s)$ is located in the central square of Fig. 9. By definition, $\rho(\gamma(s))$ is the symbolic coding of the same point $z$, but with respect to the partition on the top left corner of Fig. 9. Since two partitions differ only on blocks $001.100$ and $001.110$, it follows that $\rho(\gamma(s)) \neq s$ if and only if $H^t_{\gamma(0)}(s)$ is contained in these blocks.
blocks. Namely,

\[
(\rho(\gamma(s)))_i = \begin{cases} 
0 & \text{if } s_i - 3s_i - 2s_i - 1s_i + 1s_i + 3s_i + 3 = 0011100 \\
1 & \text{if } s_i - 3s_i - 2s_i - 1s_i + 1s_i + 3s_i + 3 = 0010100 \\
s_i & \text{otherwise.}
\end{cases}
\]

Similarly we can compute \(\rho(\gamma_q), \rho(\gamma_r), \text{ and } \rho(\gamma_s)\). This proves Proposition 7.

### 5. Discussion

Since the preprint version of this paper first appeared, some application of our results have found.

N. Long [18] studied the fixed point set of involutions on \(\Sigma_2\) and showed that the fixed point set can be decomposed into the disjoint union of “2-cascades”, which is a combinatorial analog of period doubling cascades in the bifurcation theory. Combined with Theorem 3, his result implies that periodic points of the real Hénon map which are missing at a hyperbolic parameter value in \(\mathcal{H}_c\) can be understood as the union of period doubling cascades; this gives yet another explanation why period-doubling cascades are so often observed in bifurcations of dynamical systems [19].

V. Mendoza also used Theorem 3 and our rigorous numerical results to study the pruning front conjecture for certain parameter values of the Hénon map [20]. He combines our results with topological arguments by A. de Carvalho and T. Hall to establish the existence of pruning isotopies.

Finally, we would like to remark that further detailed study of the monodromy representation is ongoing. For example, we can prove that the image \(\Gamma\) of the monodromy representation is contained in the subgroup of inert automorphisms providing there exist infinitely many non-Wieferich prime numbers (it suffices to assume the abc conjecture) [21]. This implies that there are some combinatorial restrictions on the action of the monodromy to the periodic orbits; the geometric meaning of these restrictions will be studied elsewhere.

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### Appendix. Counting periodic orbits

In this appendix, we prove Theorem 1 directly from Corollary 6, without any monodromy argument. Instead of using Theorem 3, we show that the number of periodic points in \(K_{a,c}^n\) is different from that of a full horseshoe. Specifically, we claim that the number of points in \(\text{Fix}(\mathcal{H}_{a,c}^n) \cap \mathbb{R}^2\) is exactly as in Fig. 10.

We use the Conley index theory to prove the claim. The reader not familiar with the Conley index may consult [22,23].

Assume \((a,c)\) is in one of \(I_p, I_q, I_r, \text{ or } I_s\). We remark that the uniform hyperbolicity of \(K_{a,c}^\infty\) implies that the number of periodic points in \(K_{a,c}^\infty\) is constant on these intervals.

First we compute a lower bound for the number of periodic points. We begin with finding periodic points numerically. Since periodic points are of saddle type and hence are numerically unstable, we apply the subdivision algorithm [24] to find them. For each periodic orbit found numerically, we then construct a cubical index pair [22]. The existence of a periodic point in this index pair is then proved by the following Conley index version of Lefschetz fixed point theorem.
Theorem 12 ([22, Theorem 10.102]). Let \((P_1, P_0)\) be an index pair for \(f\) and \(f_k\) the homology index map induced by \(f\). If \(\sum_k (-1)^k \text{tr} f_{-k}^n \neq 0\) then \(\text{Inv}(\text{cl}(P_1 \setminus P_0, f))\) contains a fixed point of \(f^n\).

This theorem assures that there exists at least one periodic orbit in each index pair, and therefore we obtain a lower bound for the number of points in \(\text{Fix}(H^n_{a,c}) \cap \mathbb{R}^2\).

To compute an upper bound, we have two methods.

One is to prove the uniqueness of the periodic orbit in each index pair. As long as the size of the grid used in the subdivision algorithm was fine enough, we can expect that each index pair isolates exactly one periodic orbit of period \(n\). Since periodic points are hyperbolic, uniqueness can be achieved by a Hartman–Grobman type theorem [25, Proposition 4.1].

The other one is to use the fact that the number of fixed points of \(H^n_{a,c} : \mathbb{C}^2 \to \mathbb{C}^2\) is independent of the parameter, in fact it is \(2^n\), counted with multiplicity [26, Theorem 3.1]. In our case, the uniform hyperbolicity implies that the multiplicity is always 1 and hence there are exactly \(2^n\) distinct fixed points in \(\text{Fix}(H^n_{a,c})\). Therefore, if we find \(k\) distinct fixed points of \(H^n_{a,c}\) outside \(\mathbb{R}^2\), then the cardinality of \(\text{Fix}(H^n_{a,c}) \cap \mathbb{R}^2\) must be less than or equal to \(2^n - k\). Again, we can apply Theorem 12 to establish the existence of fixed points in \(\mathbb{C}^2 \setminus \mathbb{R}^2\). This gives an upper bound.

For all cases shown in Fig. 10, the lower and upper bounds obtained by methods above coincide. Thus our claim follows.

References