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# On Closability of Differential Operators Acting in Vector-valued $L^2$ -Functions in Fock Spaces

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Abstract: Differential operators acting in a general class of  $L^2$ -functions in Fock spaces with values in tensor products of separable Hilbert spaces are considered. Then, we prove that those operators are closable.

Keywords: Fock space, integration by parts formula, Q-space representation, Fréchet differentiation.

MSC-class: 47B38

## I. INTRODUCTION

Infinite dimensional exterior differential operators on Fock spaces were introduced in [5,6,8]. The theory developed there is based on the action of annihilation operators and creation operators on Fock spaces in polynomials of Gaussian random variables and independent of both Malliavin calculus [9] and differential calculus in Banach spaces using Fréchet derivatives. In [3], the author studied an extension and complexification of a part of the theory by which one can treat the action of exterior differential operators on a general class of functionals including polynomials in Boson Fock spaces. On the other hand, the author studied semi-classical asymptotics of partition functions of an abstract Bose field model using the notion of Fréchet differentiation and utility of applying it to an analysis in infinite dimensional spaces. Then, it is natural to study relations between the notion of differentiation in [5,6,8] and that of Fréchet differentiation. In [4], the author showed they coincide for a general class of functionals in the sense of functional directional derivatives and derived integration by parts formulae with respect to such derivatives and functionals. The purpose of this paper is to introduce differential operators acting in a general class of  $L^2$ -functions in Fock spaces with values in tensor products of separable Hilbert spaces and to prove closability of those operators, applying results in [4]. Moreover, integration by parts formulae with respect

to those differential operators are also derived. Considering antisymmetrization of these results, they could be applied for foundations on a comprehensive infinite dimensional analysis on boson-fermion Fock spaces which treat general classes of differential forms concluding those with polynomial coefficients.

The outline of this paper is as follows. In Section II, we review the notion of functional directional differential operators acting in the Boson Fock space in the Q-space representation with Hilbert scales of strictly positive self-adjoint operators and integration by parts formulae with respect to that differential operators. In Section III, we introduce the notion of differentiability which is proper to an analysis of functions with values in tensor products of Hilbert spaces. In Section IV, we introduce differential operators acting in a general class of vector-valued functions and prove closability of those operators.

## II. PRELIMINARIES

Let  $\mathcal{H}$  be a real separable Hilbert space, and  $A$  be a strictly positive self-adjoint operator acting in  $\mathcal{H}$ . We denote by  $\{\mathcal{H}_s(A)\}_{s \in \mathbb{R}}$  the Hilbert scale associated with  $A$ . For all  $s \in \mathbb{R}$ ,  $\mathcal{H}_s(A)'$ , the dual space of  $\mathcal{H}_s(A)$ , can be canonically identified with  $\mathcal{H}_{-s}(A)$ .

We denote by  $\mathcal{I}_1(\mathcal{H})$  the ideal of the trace class operators on  $\mathcal{H}$ . Let  $\gamma > 0$  be fixed. We assume the following.

*Assumption I.*  $A^{-\gamma} \in \mathcal{I}_1(\mathcal{H})$ .

We define  $\mathcal{E}$  by

$$\mathcal{E} := \mathcal{H}_\gamma(A).$$

Then, we have

$$\mathcal{E}' = \mathcal{H}_{-\gamma}(A).$$

Let  $X$  be a set. In what follows, we denote a mapping  $F : \mathcal{E}' \longrightarrow X$ ;  $\phi \longmapsto F(\phi)$  by  $F(\phi)$  simply. Under Assumption I, the embedding mapping of  $\mathcal{H}$  into  $\mathcal{E}'$  is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure  $\mu$  on  $(\mathcal{E}', \mathcal{B})$  such that the Borel field  $\mathcal{B}$  is generated by  $\{\phi(f) | f \in \mathcal{E}\}$  and

$$\int_{\mathcal{E}'} e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|_{\mathcal{H}}^2/4}, \quad f \in \mathcal{E},$$

where  $\|\cdot\|_{\mathcal{H}}$  denotes the norm of  $\mathcal{H}$ .

We denote by  $\mathcal{H}_{\mathbb{C}}$  the complexification of  $\mathcal{H}$  and define a complex Hilbert space

$\mathcal{F}_b(\mathcal{H}_\mathbb{C})$ , which is called the Boson Fock space over  $\mathcal{H}_\mathbb{C}$ , by

$$\begin{aligned}\mathcal{F}_b(\mathcal{H}_\mathbb{C}) &:= \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{H}_\mathbb{C} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \mid \psi^{(n)} \in \bigotimes_s^n \mathcal{H}_\mathbb{C}, n \geq 0, \sum_{n=0}^{\infty} \|\psi^{(n)}\|_{\bigotimes_s^n \mathcal{H}_\mathbb{C}}^2 < \infty \right\},\end{aligned}$$

where  $\bigotimes_s^n \mathcal{H}_\mathbb{C}$  is the  $n$ -fold symmetric tensor product of  $\mathcal{H}_\mathbb{C}$  with  $\bigotimes_s^0 \mathcal{H}_\mathbb{C} := \mathbb{C}$ . For all  $f \in \mathcal{H}_\mathbb{C}$ , we denote by  $a(f)$  the Boson annihilation operator in  $\mathcal{F}_b(\mathcal{H}_\mathbb{C})$  with test vector  $f$ , which is the unique densely defined closed linear operator on  $\mathcal{F}_b(\mathcal{H}_\mathbb{C})$  such that its adjoint  $a(f)^*$  takes the following form :

$$(a(f)^*\psi)^{(0)} = 0, \quad (a(f)^*\psi)^{(n)} = \sqrt{n}S_n(f \otimes \psi^{(n-1)}), \quad n \geq 1, \quad \psi \in D(a(f)^*),$$

where  $S_n$  denotes the symmetrization operator (symmetrizer) on  $\bigotimes^n \mathcal{H}_\mathbb{C}$ . The operator  $a(f)^*$  is called the creation operator with test vector  $f$ .

For all  $f_1, \dots, f_n \in \mathcal{H}$ , we denote by  $:\phi(f_1) \cdots \phi(f_n):$  the Wick product of the random variables  $\phi(f_1), \dots, \phi(f_n)$ , which obeys the following recursion relations :

$$\begin{aligned}:\phi(f_1) &:= \phi(f_1), \\ :\phi(f_1) \cdots \phi(f_n) : &= \phi(f_1) : \phi(f_2) \cdots \phi(f_n) : \\ &\quad - \frac{1}{2} \sum_{j=2}^n \langle f_1, f_j \rangle : \phi(f_2) \cdots \widehat{\phi(f_j)} \cdots \phi(f_n) : \quad n \geq 2,\end{aligned}$$

where  $\widehat{\phi(f_j)}$  indicates the omission of  $\phi(f_j)$ . We define a vector  $\Omega \in \mathcal{F}_b(\mathcal{H}_\mathbb{C})$  by

$$\Omega := \{1, 0, 0, \dots\},$$

which is called the Fock vacuum in  $\mathcal{F}_b(\mathcal{H}_\mathbb{C})$ . The following fact is well known (e.g. [7,10]):

**THEOREM 2.1.** *There exists a unique unitary operator  $U$  from  $\mathcal{F}_b(\mathcal{H}_\mathbb{C})$  to  $L^2(\mathcal{E}^l, d\mu)$  such that*

$$\begin{aligned}U\Omega &= 1, \\ U(a(f_1)^* \cdots a(f_n)^*\Omega) &= (\sqrt{2})^n : \phi(f_1) \cdots \phi(f_n) : , \quad f_1, \dots, f_n \in \mathcal{H}\end{aligned}$$

For all  $f \in \mathcal{H}$ , we define  $\pi(f)$  by

$$\pi(f) := U \left( \frac{i}{\sqrt{2}} \overline{(a(f)^* - a(f))} \right) U^{-1}.$$

Then, for all  $f \in \mathcal{H}$ ,  $\pi(f)$  is self-adjoint. For all  $f \in \mathcal{H}$ , we define  $D_f$ , called the functional directional derivative along  $f$ , by

$$D_f := i\pi(f) + \phi(f).$$

Then,  $D_f$  is closable [4,7], and we have the following theorem.

**THEOREM 2.2.** [4] *Let  $g \in \mathcal{H}$ . Then, for all  $F, G \in D(D_g)$ ,*

$$\int_{\mathcal{E}'} (D_g F)(\phi) G(\phi) d\mu(\phi) = \int_{\mathcal{E}'} F(\phi) (2\phi(g)G(\phi) - (D_g G)(\phi)) d\mu(\phi). \quad (2.1)$$

We introduce a linear subspace of  $L^2(\mathcal{E}', d\mu)$ .

**DEFINITION 2.3.** [4] *Let  $g \in \mathcal{H}$  and  $F$  be a measurable function on  $\mathcal{E}'$ . We say that  $F \in \mathcal{D}_g$  if  $F$  is Fréchet differentiable and for all  $\delta > 0$ ,*

$$\begin{aligned} \sup_{|t| \leq \delta} |F(\phi + tg)| e^{\delta|\phi(g)|}, \quad \sup_{|t| \leq \delta} |F(\phi + tg)| \phi(g) e^{\delta|\phi(g)|} &\in L^2(\mathcal{E}', d\mu), \\ \sup_{|t| \leq \delta} |F'(\phi + tg)(g)| e^{\delta|\phi(g)|} &\in L^2(\mathcal{E}', d\mu). \end{aligned}$$

Then, We have the following theorems.

**THEOREM 2.4.** [4] *Let  $g \in \mathcal{H}$ . Then,  $\mathcal{D}_g \subset D(\pi(g))$ . Moreover, for all  $F \in \mathcal{D}_g$ ,*

$$(D_g F)(\phi) = F'(\phi)(g), \quad \mu - a.e. \phi \in \mathcal{E}'. \quad (2.2)$$

**THEOREM 2.5.** [4] *Let  $g \in \mathcal{H}$ . Then, for all  $F, G \in \mathcal{D}_g$ ,*

$$\int_{\mathcal{E}'} F'(\phi)(g) G(\phi) d\mu(\phi) = \int_{\mathcal{E}'} F(\phi) (2\phi(g)G(\phi) - G'(\phi)(g)) d\mu(\phi). \quad (2.3)$$

### III. NOTION OF STRONGLY DIFFERENTIABILITY

For all  $p \in \mathbb{N}$ , we denote by  $\hat{\otimes}^p \mathcal{H}$  the  $p$ -fold algebraic tensor product of  $\mathcal{H}$ , by  $B(\mathcal{H}^p)$  the Banach space of all continuous  $p$ - multilinear forms on  $\mathcal{H}$  and by  $\mathcal{L}(\mathcal{H})$  the Banach space of all continuous linear operators on  $\mathcal{H}$ .

**PROPOSITION 3.1.** *Let  $\iota$  be the embedding from  $\hat{\otimes}^p \mathcal{H}$  to  $B(\mathcal{H}^p)$ . Then,  $\iota$  is continuous and  $\|\iota\| \leq 1$ .*

*Proof.* We prove the proposition in the case that  $p = 2$ . Let  $L \in \hat{\otimes}^2 \mathcal{H}$ . Then, there exists orthonormal systems  $\{f_j\}_{j=1}^l$ ,  $\{g_k\}_{k=1}^m$  and real numbers  $\{a_{j,k}\}_{j,k}$  such that

$$L = \sum_{j,k} a_{j,k} f_j \otimes g_k.$$

We remark that in the sense of Banach spaces,

$$B(\mathcal{H}^2) \simeq \mathcal{L}(\mathcal{H}).$$

For all  $h \in \mathcal{H}$ ,

$$\begin{aligned} |L(h)|^2 &= \left| \sum_k \langle h, \sum_j a_{j,k} f_j \rangle g_k \right|^2 \\ &= \sum_k \left| \langle h, \sum_j a_{j,k} f_j \rangle \right|^2 \\ &\leq \|h\|^2 \sum_k \sum_j |a_{j,k}|^2 \\ &= \|h\|^2 \|L\|_{\otimes^2 \mathcal{H}}^2. \end{aligned}$$

Hence, we have

$$\|L\|_{B(\mathcal{H}^2)} \leq \|L\|_{\otimes^2 \mathcal{H}}.$$

Then, we have the conclusion.  $\square$

**PROPOSITION 3.2.** *There exists a unique injective continuous linear operator  $\tilde{\iota}$  from  $\otimes^p \mathcal{H}$  to  $B(\mathcal{H}^p)$  such that*

$$\tilde{\iota}|_{\otimes^p \mathcal{H}} = \iota, \quad \|\tilde{\iota}\| \leq 1.$$

*Proof.* By Proposition 3.1 and the fact that  $\hat{\otimes}^p \mathcal{H}$  is dense in  $\otimes^p \mathcal{H}$ , we have existence and uniqueness of  $\tilde{\iota}$ .

Let  $p = 2$ ,  $L \in \otimes^2 \mathcal{H}$  and  $\{e_n\}_{n \in \mathbb{N}}$  be a CONS of  $\mathcal{H}$ . Then, there exists a unique sequence  $\{a_{n,m}\}$  in  $\mathbb{R}$  such that

$$L = \sum_{n,m} a_{n,m} e_n \otimes e_m, \quad \sum_{n,m} |a_{n,m}|^2 < \infty.$$

Assume that  $\tilde{\iota}(L) = 0$ . Then,

$$\sum_{n,m} a_{n,m} e_n \otimes e_m = 0 \quad \text{in } B(\mathcal{H}^2).$$

Then, for all  $h, k \in \mathcal{H}$ ,

$$\begin{aligned} \left( \sum_{n,m} a_{n,m} e_n \otimes e_m \right) (h, k) &= \sum_{n,m} a_{n,m} (e_n \otimes e_m)(h, k) \\ &= \langle L, h \otimes k \rangle_{\otimes^2 \mathcal{H}} = 0. \end{aligned}$$

Then, we have  $L = 0$ . Hence  $\tilde{\iota}$  is injective.  $\square$

By Proposition 3.2, one can consider  $\otimes^p \mathcal{H} \subset B(\mathcal{H}^p)$ .

Considering the proof of Proposition 3.2, we straightforwardly have the following.

PROPOSITION 3.3. *Let  $p \in \mathbb{N}$ . Then, for all  $L \in \otimes^p \mathcal{H}$ ,*

$$\langle L, \bigotimes_{j=1}^p h_j \rangle_{\otimes^p \mathcal{H}} = L(h_1, \dots, h_p), \quad h_j \in \mathcal{H}, \quad j = 1, \dots, p. \quad (3.4)$$

DEFINITION 3.4. *Let  $p \geq 0$  and  $F$  be a mapping from  $\mathcal{E}'$  to  $\otimes^p \mathcal{H}$ . We say that  $F$  is strongly of class  $C^\infty$  if  $F$  is of class  $C^\infty$  with respect to the topology of the Banach space  $B(\mathcal{H}^p)$  and for all  $l \geq 0$  and  $\phi \in \mathcal{E}'$ ,*

$$F^{(l)}(\phi) \in \otimes^{p+l} \mathcal{H}.$$

DEFINITION 3.5. *Let  $p \geq 0$  and  $F$  be a measurable mapping from  $\mathcal{E}'$  to  $\otimes^p \mathcal{H}$ . We say that  $F \in \mathcal{D}_p$  if  $F$  is differentiable with respect to the topology of the Banach space  $B(\mathcal{H}^p)$  and for all  $\delta > 0$  and  $g \in \mathcal{H}$ ,*

$$\begin{aligned} \sup_{|t| \leq \delta} \|F(\phi + tg)\|_{\otimes^p \mathcal{H}} e^{\delta|\phi(g)|}, \quad \sup_{|t| \leq \delta} \|F(\phi + tg)\|_{\otimes^p \mathcal{H}} \phi(g) e^{\delta|\phi(g)|} \in L^2(\mathcal{E}', d\mu), \\ \sup_{|t| \leq \delta} \|F'(\phi + tg)(g)\|_{\otimes^p \mathcal{H}} e^{\delta|\phi(g)|} \in L^2(\mathcal{E}', d\mu). \end{aligned}$$

DEFINITION 3.6. *Let  $p \geq 0$  and  $F$  be a mapping from  $\mathcal{E}'$  to  $\otimes^p \mathcal{H}$ . We say that  $F \in C_p^\infty$  if  $F$  is strongly of class  $C^\infty$  and for all  $l \geq 0$ ,*

$$F^{(l)} \in \mathcal{D}_{p+l}.$$

#### IV. CLOSABILITY OF DIFFERENTIAL OPERATORS ACTING IN VECTOR-VALUED $L^2$ -FUNCTIONS

We prove the following integration by parts formula.

THEOREM 4.1. *Let  $p \geq 0$  and  $g_j \in \mathcal{H}$ ,  $j = 1, \dots, p+1$ . We set  $\otimes_{j=2}^1 g_j := 1$ . Then, for all  $F \in C_p^\infty$ ,  $G \in C_0^\infty$  and  $g_j \in \mathcal{H}$ ,  $j = 1, \dots, p+1$ ,*

$$\int_{\mathcal{E}'} \langle F'(\phi), G(\phi) \bigotimes_{j=1}^{p+1} g_j \rangle_{\otimes^{p+1} \mathcal{H}} d\mu(\phi) = \int_{\mathcal{E}'} \langle F(\phi), (D_{g_1}^* G)(\phi) \bigotimes_{j=2}^{p+1} g_j \rangle_{\otimes^p \mathcal{H}} d\mu(\phi). \quad (4.5)$$

*Proof.* Let  $p = 0$ . We remark that

$$C_0^\infty \subset \bigcap_{g \in \mathcal{H}} \mathcal{D}_g.$$

Then, by Theorem 2.2 and Theorem 2.5, we have (4.5).

Let  $p \geq 1$ . Then, by Proposition 3.3,

$$\int_{\mathcal{E}'} \langle F'(\phi), G(\phi) \otimes_{j=1}^{p+1} g_j \rangle d\mu(\phi) = \int_{\mathcal{E}'} \langle F'(\phi)(g_1), \otimes_{j=2}^{p+1} g_j \rangle G(\phi) d\mu(\phi).$$

Since

$$\langle F(\phi), \otimes_{j=2}^{p+1} g_j \rangle, G \in \mathcal{D}_{g_1},$$

by Proposition 3.3 and Theorem 2.5, we have

$$\begin{aligned} \int_{\mathcal{E}'} \langle F'(\phi)(g_1), \otimes_{j=2}^{p+1} g_j \rangle G(\phi) d\mu(\phi) &= \int_{\mathcal{E}'} D_{g_1} \langle F(\phi), \otimes_{j=2}^{p+1} g_j \rangle G(\phi) d\mu(\phi) \\ &= \int_{\mathcal{E}'} \langle F(\phi), \otimes_{j=2}^{p+1} g_j \rangle (D_{g_1}^* G)(\phi) d\mu(\phi) \\ &= \int_{\mathcal{E}} \langle F(\phi), (D_{g_1}^* G)(\phi) \otimes_{j=2}^{p+1} g_j \rangle d\mu(\phi). \end{aligned}$$

Then, we have the conclusion.  $\square$

Let  $p \geq 0$ . For all  $F \in C_p^\infty$ , we define  $D_p F \in L^2(\mathcal{E}', d\mu; \otimes^{p+1} \mathcal{H})$  by

$$(D_p F)(\phi) := F'(\phi), \phi \in \mathcal{E}'.$$

We denote by  $\mathcal{P}(\mathbb{R}^n)$  the set of all polynomials of  $n$  variables. We define  $\mathcal{P}(\mathcal{E}')$  by

$$\mathcal{P}(\mathcal{E}') := \{F(\phi(f_1), \dots, \phi(f_n)) \mid F \in \mathcal{P}(\mathbb{R}^n), f_j \in \mathcal{E}, j = 1, \dots, n, n \in \mathbb{N}\}.$$

Then, the algebraic tensor product  $\mathcal{P}(\mathcal{E}') \hat{\otimes} (\hat{\otimes}^p \mathcal{H})$  is dense in  $L^2(\mathcal{E}', d\mu; \otimes^p \mathcal{H})$ . We have the following theorem.

**THEOREM 4.2.** *For all  $p \geq 0$ ,  $D_p$  is densely defined closable linear operator from  $L^2(\mathcal{E}', d\mu; \otimes^p \mathcal{H})$  to  $L^2(\mathcal{E}', d\mu; \otimes^{p+1} \mathcal{H})$ . Moreover, for all  $p \geq 0$ ,  $G \in C_0^\infty$  and  $g_j \in \mathcal{H}$ ,  $j = 1, \dots, p+1$ ,  $G(\phi) \otimes_{j=1}^{p+1} g_j \in D(D_p^*)$  and*

$$D_p^*(G(\phi) \otimes_{j=1}^{p+1} g_j) = (D_{g_1}^* G)(\phi) \otimes_{j=2}^{p+1} g_j \text{ in } L^2\text{-sense.} \quad (4.6)$$

*Proof.* We have

$$\mathcal{P}(\mathcal{E}') \hat{\otimes} (\hat{\otimes}^p \mathcal{H}) \subset C_p^\infty.$$

Hence,  $D_p$  is densely defined.

By Theorem 4.1, one can see

$$C_0^\infty \hat{\otimes} (\hat{\otimes}^{p+1} \mathcal{H}) \subset D(D_p^*)$$



and obtain (4.6). On the other hand, we have

$$\mathcal{P}(\mathcal{E}') \hat{\otimes}^{\otimes p+1} (\hat{\otimes} \mathcal{H}) \subset C_0^\infty \hat{\otimes}^{\otimes p+1} (\hat{\otimes} \mathcal{H}).$$

Then, we see that  $D(D_p^*)$  is dense in  $L^2(\mathcal{E}', d\mu; \hat{\otimes}^{\otimes p+1} \mathcal{H})$ .

Hence,  $D_p$  is closable. □

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