Hypergeometric Series with Gamma Product Formula*

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Abstract

We consider non-terminating Gauss hypergeometric series with one free parameter. Using various properties of hypergeometric functions we obtain some necessary conditions of arithmetic flavor for such series to admit gamma product formulas.

1 Introduction

Given a data $\lambda = (p, q, r; a, b; x) \in \mathbb{C}^5 \times \mathbb{D}$, we consider an entire meromorphic function

$$f(w; \lambda) := {}_2F_1(pw + a,\ qw + b;\ rw;\ x), \tag{1}$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ and ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric series.

We say that $f(w; \lambda)$ admits a *gamma product formula* (GPF), if there exist a rational function $S(w) \in \mathbb{C}(w)$; a constant $d \in \mathbb{C}^{\times}$; two integers $m, n \in \mathbb{Z}_{\geq 0}$; m numbers $u_1, \dots, u_m \in \mathbb{C}$; and n numbers $v_1, \dots, v_n \in \mathbb{C}$ such that

$$f(w; \lambda) = S(w) \cdot d^w \cdot \frac{\Gamma(w + u_1) \cdots \Gamma(w + u_m)}{\Gamma(w + v_1) \cdots \Gamma(w + v_n)}, \tag{2}$$

where $\Gamma(w)$ is the Euler gamma function. We are interested in the following.

Problem I Find a data $\lambda = (p, q, r; a, b; x)$ for which $f(w; \lambda)$ admits a GPF.

An abundance of solutions can be found in Apagodu and Zeilberger [2], Bailey [3], Brychkov [4], Ebisu [7], Ekhad [8], Erdélyi [9], Gessel and Stanton [10], Gosper [11], Goursat [13], Karlsson [15], Koepf [16], Maier [17], Vidunas [18] etc. To illustrate what this problem is all about, we present some examples of solutions at the end of the Introduction (see Tables 1 and 2).

Problem I has a close relative. We say that $f(w; \lambda)$ is *of closed form* if

$$\frac{f(w + 1; \lambda)}{f(w; \lambda)} =: R(w; \lambda) \in \mathbb{C}(w) \ \ : \ \text{a rational function of } w. \tag{3}$$

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Problem II Find a data $\lambda = (p, q, r; a, b; x)$ for which $f(w; \lambda)$ is of closed form.

Any solution to Problem I is a solution to Problem II. Indeed, by the recursion formula for the gamma function $\Gamma(w+1) = w\,\Gamma(w)$, the condition (2) implies (3) with

$$R(w; \lambda) = \frac{S(w+1)}{S(w)} \cdot d \cdot \frac{(w+u_1)\cdots(w+u_m)}{(w+v_1)\cdots(w+v_n)}.$$

It is natural to ask when a solution to Problem II leads back to a solution to Problem I.

A data $\lambda = (p, q, r; a, b; x)$ is said to be *integral* if its principal part $\boldsymbol{p} := (p, q, r) \in \mathbb{Z}^3$. There is a method of finding integral solutions to Problem II due to Ebisu [5, 6, 7], which we call *the method of contiguous relations*. It relies on the fifteen contiguous relations of Gauss (see e.g. Andrews et. al. [1, §2.5]). Composing a series of contiguous relations yields

$$\begin{aligned} {}_2F_1(\alpha+p, \beta+q; \gamma+r; z) = r(\alpha, \beta; \gamma; z)\, {}_2F_1(\alpha, \beta; \gamma; z) \\ + q(\alpha, \beta; \gamma; z)\, {}_2F_1(\alpha+1, \beta+1; \gamma+1; z), \end{aligned} \tag{4}$$

where $q(\alpha, \beta; \gamma; z)$ and $r(\alpha, \beta; \gamma; z)$ are rational functions of $(\alpha, \beta; \gamma; z)$ depending uniquely on $\boldsymbol{p}$. Vidunas [19] and Ebisu [5] showed how to compute $q(\alpha, \beta; \gamma; z)$ and $r(\alpha, \beta; \gamma; z)$ rapidly and efficiently. Given a data $\lambda = (p, q, r; a, b; x)$ we put

$$\tilde{f}(w; \lambda) := {}_2F_1(pw+a+1,\ qw+b+1;\ rw+1;\ x). \tag{5}$$

When λ is integral, substituting $(\alpha, \beta; \gamma; z) = (pw+a, qw+b; rw; x)$ into (4) yields

$$f(w+1; \lambda) = R(w; \lambda)\, f(w; \lambda) + Q(w; \lambda)\, \tilde{f}(w; \lambda), \tag{6}$$

where $Q(w; \lambda)$ and $R(w; \lambda)$ are rational functions of w depending uniquely on λ. If λ is a data such that $Q(w; \lambda)$ vanishes in $\mathbb{C}(w)$, then three-term relation (6) reduces to a two-term one (3) so that λ is a solution to Problem II. Such a solution is said to *come from contiguous relations*. It is interesting to ask when an integral solution comes from contiguous relations.

The hypergeometric series enjoys well-known symmetries (see [1, Theorem 2.2.5]):

$$\begin{aligned} {}_2F_1(\alpha,\ \beta;\ \gamma;\ z) &= {}_2F_1(\beta,\ \alpha;\ \gamma;\ z) && \text{(trivial)}, && \text{(7a)} \\ &= (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ z) && \text{(Euler)}, && \text{(7b)} \\ &= (1-z)^{-\alpha}{}_2F_1(\alpha,\ \gamma-\beta;\ \gamma;\ z/(z-1)) && \text{(Pfaff)}, && \text{(7c)} \\ &= (1-z)^{-\beta}{}_2F_1(\gamma-\alpha,\ \beta;\ \gamma;\ z/(z-1)) && \text{(Pfaff)}. && \text{(7d)} \end{aligned}$$

Compatible with Problems I and II, they induce symmetries on their solutions:

$$\begin{aligned} \lambda = (p, q, r; a, b; x) &\mapsto (q, p, r; b, a; x), && \text{(8a)} \\ &\mapsto (r-p,\ r-q,\ r;\ -a,\ -b;\ x), && \text{(8b)} \\ &\mapsto (p,\ r-q,\ r;\ a,\ -b;\ x/(x-1)), && \text{(8c)} \\ &\mapsto (r-p,\ q,\ r;\ -a,\ b;\ x/(x-1)), && \text{(8d)} \end{aligned}$$

which are referred to as *classical symmetries* for these problems.

We are interested in finding necessary conditions for a given data λ to be a solution to Problem I or II. In this article we discuss this issue when λ lies in a real domain

$$p,\ q,\ r \in \mathbb{R}, \quad 0 < p < r \quad \text{or} \quad 0 < q < r; \qquad a,\ b \in \mathbb{R}; \qquad -1 < x < 1.$$

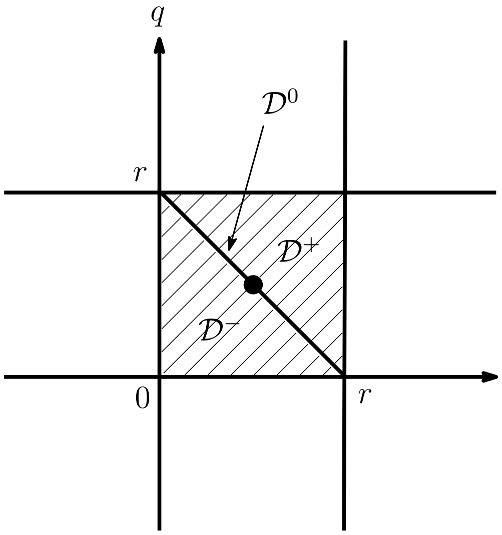

Figure 1: Central square $\mathcal{D}$.

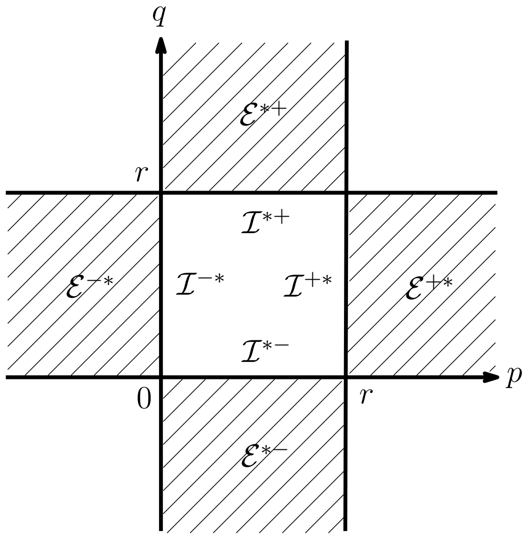

Figure 2: Wings $\mathcal{E}$ and borders $\mathcal{I}$.

By Pfaff's transformation (8c) or (8d) this domain can be reduced to

$$p,\, q,\, r \in \mathbb{R}, \quad 0 < p < r \quad \text{or} \quad 0 < q < r; \qquad a,\, b \in \mathbb{R}; \qquad 0 < x < 1, \tag{9}$$

where the trivial case $x = 0$ is excluded. According to the location of (p, q) relative to r the domain (9) is partitioned into three parts $\mathcal{D}$ (central square), $\mathcal{E}$ (wings) and $\mathcal{I}$ (borders), where $\mathcal{D}$ is further divided into three components $\mathcal{D}^{\pm}$ and $\mathcal{D}^0$ (anti-diagonal) as in Figure 1, while $\mathcal{E}$ resp. $\mathcal{I}$ is decomposed into four components $\mathcal{E}^{\star\star}$ resp. $\mathcal{I}^{\star\star}$ as in Figure 2. For example,

$$\mathcal{D}^- = \{\, p > 0,\, q > 0,\, p + q < r \,\}, \qquad \mathcal{D}^0 = \{\, p > 0,\, q > 0,\, p + q = r \,\},$$
$$\mathcal{E}^{*-} = \{\, 0 < p < r,\, q < 0 \,\}, \qquad \mathcal{I}^{*-} = \{\, 0 < p < r,\, q = 0 \,\}.$$

The trivial symmetry (8a) or Euler transformation (8b) permutes these components in one way or another and we have only to deal with $\mathcal{D}^-$, $\mathcal{D}^0$, $\mathcal{E}^{*-}$ and $\mathcal{I}^{*-}$ up to classical symmetries.

In Table 1 we present nine examples of integral solutions $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ to Problem I, each of which has a gamma product formula of the form:

$$f(w; \lambda) = C \cdot d^w \cdot \frac{\Gamma(w + u_1) \cdots \Gamma(w + u_m)}{\Gamma(w + v_1) \cdots \Gamma(w + v_m)}, \tag{10}$$

where $1 \le m \le r$ and $\{u_1, \dots, u_m\}$ is a subset of $\{\frac{0}{r}, \frac{1}{r}, \dots, \frac{r-1}{r}\}$. The data of λ, d, $u_1, \dots, u_m$ and $v_1, \dots, v_m$ are presented in Table 1, while C can easily be evaluated by putting $w = -a/p$ or $w = -b/q$ into (10). Solutions 1–8 can be found in (1,3,4-4,iii), (1,3,4-4,xx), (1,3,4-3,iii), (1,5,6-1,iii), (1,5,6-1,xx), (2,4,6-4,iii), (3,5,6-1,xvii), (3,5,6-1,ii) of Ebisu [7] up to affine changes of variable w. Solutions 1, 2, 3 also appear in formulas (3.4), (3.5), (3.1) of Karlsson [15], while solution 9 is in Brychkov [4, §8.1, formula (172)], all up to classical symmetries (8) and affine changes of variable w. In Table 2 we present two non-integral solutions in $\mathcal{D}^-$ to Problem I, either of which also has a gamma product formula of the form (10). The (A)-solutions 4 and 5 in Table 1 are the duplications of (B)-solutions 4 and 5 in Table 2, respectively, where the concepts of (A)-solution, (B)-solution and duplication will be introduced in Theorem 2.2.

r	p	q	x	d	a	b	$u_1 \cdots u_m$ $v_1 \cdots v_m$	No.
4	1	1	$\frac{8}{9}$	$\frac{4}{3}$	0	$\frac{1}{4}$	$\frac{2}{4}\ \frac{3}{4}$ $\frac{7}{12}\ \frac{2}{3}$	1
					$\frac{1}{2}$	$\frac{1}{4}$	$0\ \frac{3}{4}$ $\frac{1}{6}\ \frac{7}{12}$	2
					0	$\frac{1}{2}$	$\frac{1}{4}\ \frac{3}{4}$ $\frac{1}{3}\ \frac{2}{3}$	3
6	1	1	$\frac{4}{5}$	$\frac{3^6}{5^4}$	0	$\frac{1}{2}$	$\frac{1}{6}\ \frac{2}{6}\ \frac{4}{6}\ \frac{5}{6}$ $\frac{1}{5}\ \frac{3}{10}\ \frac{7}{10}\ \frac{4}{5}$	4
					$\frac{2}{3}$	$\frac{1}{6}$	$0\ \frac{2}{6}\ \frac{3}{6}\ \frac{5}{6}$ $\frac{1}{15}\ \frac{4}{15}\ \frac{17}{30}\ \frac{23}{30}$	5
	2	2	$\frac{3}{4}(3-\sqrt{3})$	$\frac{3}{2}\sqrt{3}$	0	$\frac{1}{3}$	$\frac{2}{6}\ \frac{5}{6}$ $\frac{5}{12}\ \frac{3}{4}$	6
	3	1	$4(\sqrt{5}-2)$	$\frac{27}{125}(5+2\sqrt{5})$	0	$\frac{1}{6}$	$\frac{3}{6}\ \frac{5}{6}$ $\frac{17}{30}\ \frac{23}{30}$	7
					0	$\frac{1}{2}$	$\frac{1}{6}\ \frac{5}{6}$ $\frac{3}{10}\ \frac{7}{10}$	8
8	4	2	$4(3\sqrt{2}-4)$	$\frac{4}{27}(17+12\sqrt{2})$	0	$\frac{1}{4}$	$\frac{3}{8}\ \frac{7}{8}$ $\frac{11}{24}\ \frac{19}{24}$	9

Table 1: Nine examples of (A)-solutions in $\mathcal{D}^-$.

r	p	q	x	d	a	b	$u_1 \cdots u_m$ $v_1 \cdots v_m$	No.
3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{3^3}{5^2}$	0	$\frac{1}{2}$	$\frac{1}{3}\ \frac{2}{3}$ $\frac{2}{5}\ \frac{3}{5}$	4
					$\frac{2}{3}$	$\frac{1}{6}$	$0\ \frac{2}{3}$ $\frac{2}{15}\ \frac{8}{15}$	5

Table 2: Two examples of (B)-solutions in $\mathcal{D}^-$.

2 Main Results

In this section we state the main results together with an outline of this article.

Theorem 2.1 *In region* $\mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$ *any solution to Problem* II *leads back to a solution to Problem* I *and hence the two problems are equivalent.*

Thus in this region we can speak of a solution without specifying to which problem it is a solution. A solution λ is said to be *elementary* if $f(w;\lambda)$ has at most finitely many poles in $\mathbb{C}_w$.

Theorem 2.2 *Let* $\lambda = (p, q, r; a, b; x) \in \mathcal{D}$.

(1) *Every solution in* $\mathcal{D}^0$ *is elementary, while every solution in* $\mathcal{D}^{\pm}$ *is non-elementary. All elementary solutions are at the center* $\bullet$ *of the square* $\mathcal{D}$ *in Figure* 1*, or more precisely,*

$$p = q = r/2 > 0; \quad a = i, \quad b = j - 1/2, \quad i, j \in \mathbb{Z}; \quad 0 < x < 1, \tag{11}$$

where a *and* b *are exchangeable by symmetry, while* r *and* x *are free. The corresponding* $f(w;\lambda)$ *is a degenerate hypergeometric function with a dihedral monodromy group,*

$$f(w;\lambda) = S_{ij}(rw;x) \cdot \left(\tfrac{1+\sqrt{1-x}}{2}\right)^{1-rw}, \tag{12}$$

due to Vidunas [20, Theorem 3.1], where $S_{ij}(w;x)$ *is a rational function of* w *defined by*

$$S_{ij}(w;x) = (1-x)^{-\frac{i+j}{2}} F_3\Big(i+j,\, j-i;\, 1-i-j,\, 1+i-j;\, w;\, -\tfrac{1-\sqrt{1-x}}{2\sqrt{1-x}},\, \tfrac{1-\sqrt{1-x}}{2}\Big),$$

with $F_3(\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma; u, v)$ *being Appell's hypergeometric series* F_3 *in two variables.*

(2) *Any solution* $\lambda \in \mathcal{D}^{\pm}$ *falls into one of the following two types:*

$$\text{(A)} \quad p,\, q,\, r \in \mathbb{Z}, \quad 0 \neq r - p - q \equiv 0 \mod 2, \qquad \text{(B)} \quad p,\, q \in \tfrac{1}{2} + \mathbb{Z}, \quad r \in \mathbb{Z}.$$

(3) *Any* (A)*-solution in* $\mathcal{D}^{\pm}$ *comes from contiguous relations.*

(4) *If* $\lambda \in \mathcal{D}^{\pm}$ *is a* (B)*-solution then its duplication* $2\lambda := (2p, 2q, 2r; a, b; x) \in \mathcal{D}^{\pm}$ *is an* (A)*-solution, so any* (B)*-solution in* $\mathcal{D}^{\pm}$ *essentially comes from contiguous relations.*

For assertion (2) we give more detailed conditions involving (a, b) in Theorem 6.10. The principal part of any (A)-solution in $\mathcal{D}^-$ must lie in the integer domain

$$D_{\mathrm{A}}^- := \{\, \boldsymbol{p} = (p, q, r) \in \mathbb{Z}^3 \,:\, p > 0,\ q > 0,\ 0 < r - p - q \equiv 0 \mod 2 \,\}. \tag{13}$$

Theorem 2.3 *Let* $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ *be any* (A)*-solution with* $\boldsymbol{p} = (p, q, r) \in D_{\mathrm{A}}^-$.

(1) *Let* $Y(z) = Y(z;\boldsymbol{p}) \in \mathbb{Z}[z]$ *be defined via the expansion*

$$Z_{\pm}(z) = X(z) \pm Y(z)\sqrt{\varDelta}, \tag{14}$$

where $\varDelta = \varDelta(z) = \varDelta(z;\boldsymbol{p})$ *and* $Z_{\pm}(z) = Z_{\pm}(z;\boldsymbol{p})$ *are given by*

$$\varDelta(z) := (p-q)^2 z^2 - 2\{(p+q)r - 2pq\}z + r^2, \tag{15}$$

$$Z_{\pm}(z) := \{r + (p-q)z \pm \sqrt{\varDelta}\}^p \{r - (p-q)z \pm \sqrt{\varDelta}\}^q \{(2r-p-q)z - r \mp \sqrt{\varDelta}\}^{r-p-q}. \tag{16}$$

Then x *must be an algebraic number as a root in* $0 < z < 1$ *of the algebraic equation*

$$Y(z;\boldsymbol{p}) = 0. \tag{17}$$

(2) *There exists a positive constant $C > 0$ such that a gamma product formula*

$$f(w;\lambda) = C \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(w + v_i\right)} \tag{18}$$

holds true, where the number d is given by

$$d = \frac{r^r}{\sqrt{p^p q^q (r-p)^{r-p} (r-q)^{r-q} x^r (1-x)^{p+q-r}}}, \tag{19}$$

while $v_1, \dots, v_r$ are such numbers that sum up to

$$v_1 + \cdots + v_r = (r-1)/2, \tag{20}$$

and that admit a division relation in $\mathbb{C}[w]$,

$$\prod_{i=1}^{r} (w + v_i) \;\Big|\; \prod_{i=1}^{p-1} \left(w + \tfrac{i+a}{p}\right) \prod_{i=1}^{q-1} \left(w + \tfrac{i+b}{q}\right) \prod_{j=0}^{r-p-1} \left(w + \tfrac{j-a}{r-p}\right) \prod_{j=0}^{r-q-1} \left(w + \tfrac{j-b}{r-q}\right). \tag{21}$$

Theorems 2.2 and 2.3 are illustrated by the examples presented in Tables 1 and 2. We conjecture that a and b must be rational and the solutions with a prescribed $\boldsymbol{p}$ are finite in cardinality. As for $\mathcal{I}^{*-}$ and $\mathcal{E}^{*-}$ we have the following results corresponding to Theorem 2.2.

Theorem 2.4 *Let $\lambda = (p, q = 0, r; a, b; x) \in \mathcal{I}^{*-}$.*

(1) *λ is an elementary solution if and only if $b \in \mathbb{Z}_{\le 0}$, in which case the hypergeometric series that defines $f(w;\lambda)$ is terminating, so $f(w;\lambda)$ itself is a rational function of w.*

(2) *For any non-elementary solution $\lambda \in \mathcal{I}^{*-}$ we have p, $r \in \mathbb{Z}$, $p \equiv r \mod 2$ and $b = 1/2$.*

(3) *Any non-elementary solution in $\mathcal{I}^{*-}$ comes from contiguous relations.*

Theorem 2.5 *Let $\lambda = (p, q, r; a, b; x) \in \mathcal{E}^{*-}$.*

(1) *Every solution in $\mathcal{E}^{*-}$ is non-elementary.*

(2) *For any solution $\lambda \in \mathcal{E}^{*-}$ we must have p, $r \in \mathbb{Z}$ and $p \equiv r \mod 2$.*

(3) *Any solution $\lambda \in \mathcal{E}^{*-}$ with $q \in \mathbb{Z}$ comes from contiguous relations.*

It is not known whether $\mathcal{E}^{*-}$ contains any solution with a non-integral or irrational q. We have a plan to obtain results on $\mathcal{I}^{*-}$ and $\mathcal{E}^{*-}$ corresponding to Theorem 2.3. The main results in this section are presented in a manner suitable for citations in forthcoming papers, e.g. [14]. They can readily be established by combining or rearranging the theorems and propositions to be proved in the main body of this article (§3–§10).

3 Saddle Point Method for Euler's Integral

In this section $f(w;\lambda)$ is just the function defined by formula (1), that is, λ may or may not be a solution to Problem I or II. When $\lambda = (p, q, r; a, b; x) \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$, that is,

$$p,\ q,\ r \in \mathbb{R}, \quad 0 < p < r \ \text{ and } \ q < r; \qquad a,\ b \in \mathbb{R}; \qquad 0 < x < 1, \tag{22}$$

we study the asymptotic behavior of $f(w;\lambda)$ as $w \to \infty$ on a right half-plane in $\mathbb{C}_w$. Euler's integral representation for the hypergeometric function (see e.g. [1, Theorem 2.2.1]) allows us to write $f(w;\lambda) = \psi(w) f_1(w)$, where $\psi(w)$ and $f_1(w)$ are given by

$$\psi(w) = \frac{\Gamma(rw)}{\Gamma(pw+a)\Gamma((r-p)w-a)}, \tag{23}$$

$$f_1(w) = \int_0^1 t^{pw+a-1}(1-t)^{(r-p)w-a-1}(1-xt)^{-qw-b}dt. \tag{24}$$

The improper integral in (24) converges if $p\,\mathrm{Re}(w) + a > 0$ and $(r-p)\,\mathrm{Re}(w) - a > 0$. By assumption (22) this condition is fulfilled on the right half-plane $\mathrm{Re}(w) \ge R_1$, provided

$$R_1 > \max\{-a/p,\ a/(r-p)\}\ (\ge 0). \tag{25}$$

The gamma factor $\psi(w)$ can be estimated by Stirling's formula, which states that $\Gamma(t) \sim \sqrt{2\pi}\, e^{-t}\, t^{t-1/2}$ as $t \to \infty$ uniformly on every proper subsector of the sector $|\arg(t)| < \pi$, where $* \sim **$ indicates that the ratio of $*$ and $**$ tends to 1 as $t \to \infty$. It is convenient to note a slightly generalized version of Stirling's formula: for any $\alpha > 0$ and $\beta \in \mathbb{C}$,

$$\Gamma(\alpha t + \beta) \sim \sqrt{2\pi}\, \alpha^{\beta - 1/2}\, (\alpha/e)^{\alpha t}\, t^{\alpha t + \beta - 1/2} \qquad \text{as} \quad t \to \infty, \tag{26}$$

which is valid on the same sector as above and is easily derived from the original formula.

Lemma 3.1 *The function $\psi(w)$ in* (23) *is holomorphic and admits a uniform estimate*

$$\psi(w) \sim A_1 \cdot B_1^w \cdot w^{1/2},$$

on the right half-plane $\mathrm{Re}(w) > 0$, *where A_1 and B_1 are given by*

$$A_1 = \frac{1}{\sqrt{2\pi}} \cdot \frac{(r-p)^{a+1/2}}{p^{a-1/2}\, r^{1/2}}, \qquad B_1 = \frac{r^r}{p^p\,(r-p)^{r-p}}.$$

Proof. The poles of $\psi(w)$ are contained in the arithmetic progression $\{-j/r\}_{j=0}^{\infty}$ and so $\psi(w)$ is holomorphic on $\mathrm{Re}(w) > 0$. By Stirling's formula (26) we have

$$\begin{aligned}\psi(w) &\sim \frac{\sqrt{2\pi}\, r^{-1/2}\, (r/e)^{rw}\, w^{rw-1/2}}{\sqrt{2\pi}\, p^{a-1/2}\, (p/e)^{pw}\, w^{pw+a-1/2} \cdot \sqrt{2\pi}\, (r-p)^{-a-1/2}\, ((r-p)/e)^{(r-p)w}\, w^{(r-p)w-a-1/2}} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{(r-p)^{a+1/2}}{p^{a-1/2}\, r^{1/2}} \cdot \left(\frac{r^r}{p^p\,(r-p)^{r-p}}\right)^w \cdot w^{1/2} = A_1 \cdot B_1^w \cdot w^{1/2},\end{aligned}$$

as $w \to \infty$ uniformly on $\mathrm{Re}(w) > 0$. □

The integral in formula (24) can be rewritten

$$f_1(w) = \int_0^1 \Phi(t)^w \, \eta(t) \, dt = \int_0^1 e^{-w\phi(t)} \, \eta(t) \, dt, \tag{27}$$

where $\Phi(t)$, $\phi(t)$ and $\eta(t)$ are defined by

$$\begin{aligned} \Phi(t) &= t^p (1-t)^{r-p} (1-xt)^{-q}, \qquad \phi(t) = -\log \Phi(t), \\ \eta(t) &= t^{a-1} (1-t)^{-a-1} (1-xt)^{-b}. \end{aligned} \tag{28}$$

We apply the saddle point method to estimate the integral (27). Observe that

$$\phi'(t) = \frac{\phi_1(t)}{t(1-t)(1-xt)}, \qquad \phi_1(t) := -(r-q)xt^2 + \{(p-q)x + r\}t - p,$$

where $\phi_1(t)$ is a concave quadratic function by assumption (22). The roots of quadratic equation $\phi_1(t) = 0$ are the *saddle points* for the the integral (27). We remark that the discriminant of $\phi_1(t)$ is just $\Delta(x)$ in formula (15). Since $\phi_1(0) = -p < 0$ and $\phi_1(1) = (r-p)(1-x) > 0$, there is a unique saddle point t_0 in the interval $0 < t < 1$. Note that $\phi_1'(t_0) > 0$ and hence

$$\phi''(t_0) = \frac{\phi_1'(t_0)}{t_0(1-t_0)(1-xt_0)} = \frac{-2(r-q)xt_0 + (p-q)x + r}{t_0(1-t_0)(1-xt_0)} > 0,$$

because t_0 lies strictly to the left of the axis of symmetry for the parabola $\phi_1(t)$.

Lemma 3.2 *The function $f_1(w)$ in* (24) *is holomorphic and admits a uniform estimate*

$$f_1(w) \sim \sqrt{2\pi} \, \frac{\eta(t_0)}{\sqrt{\phi''(t_0)}} \, \Phi(t_0)^w \, w^{-1/2} \qquad \text{as} \quad w \to \infty, \tag{29}$$

on the right half-plane $\mathrm{Re}(w) \ge R_1$, *where R_1 is any number satisfying condition* (25).

Proof. The function $f_1(w)$ is holomorphic on $\mathrm{Re}(w) \ge R_1$ by the convergence condition for the improper integral (24) mentioned above. Asymptotic formula (29) is obtained by the standard saddle point method, so only an outline of its derivation will be included below. Suppose that $\arg w = 0$ for simplicity. Then the path of integration is just the real interval $0 < t < 1$ as taken in (27), where the phase function $\phi(t)$ attains its minimum at $t = t_0$ so that the vicinity of this point has the greatest contribution to the integral (27). Observing that

$$\phi(t) = \phi(t_0) + \frac{1}{2}\phi''(t_0)(t-t_0)^2 + O((t-t_0)^3), \quad \eta(t) = \eta(t_0) + O(t-t_0), \quad \text{as} \quad t \to t_0,$$

we have for any sufficiently small positive number $\varepsilon > 0$,

$$\begin{aligned} f_1(w) &\sim \int_{t_0-\varepsilon}^{t_0+\varepsilon} e^{-w\{\phi(t_0) + \frac{1}{2}\phi''(t_0)(t-t_0)^2\}} \, \eta(t) \, dt \sim \Phi(t_0)^w \int_{t_0-\varepsilon}^{t_0+\varepsilon} e^{-\frac{1}{2}w\phi''(t_0)(t-t_0)^2} \, \eta(t_0) \, dt \\ &\sim \eta(t_0) \, \Phi(t_0)^w \int_{-\infty}^{\infty} e^{-\frac{1}{2}w\phi''(t_0)t^2} \, dt = \frac{\eta(t_0)}{\sqrt{\frac{1}{2}w\phi''(t_0)}} \, \Phi(t_0)^w \int_{-\infty}^{\infty} e^{-t^2} dt, \end{aligned}$$

from which formula (29) follows, where we made a change of variable $t\sqrt{\frac{1}{2}w\phi''(t_0)} \mapsto t$ to obtain the last equality. This argument carries over for a general complex variable w on the right half-plane $\mathrm{Re}(w) \ge R_1$ if the path of integration is deformed as in Figure 3. □

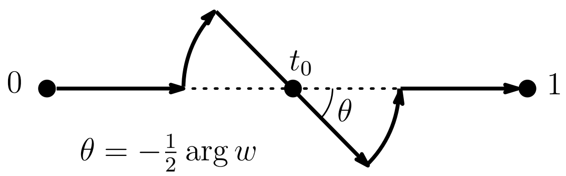

Figure 3: A path of steepest descent in the t-plane.

Proposition 3.3 *The function $f(w;\lambda)$ is holomorphic and admits a uniform estimate*

$$f(w;\lambda) \sim A \cdot B^w \qquad \textit{as} \quad w \to \infty, \tag{30}$$

on the right half-plane $\mathrm{Re}(w) \geq R_1$*, where A and B are given by*

$$A = \frac{(r-p)^{a+1/2}}{p^{a-1/2}\, r^{1/2}} \cdot \frac{\eta(t_0)}{\sqrt{\phi''(t_0)}}, \qquad B = \frac{r^r}{p^p\,(r-p)^{r-p}} \cdot \Phi(t_0). \tag{31}$$

Proof. This proposition follows immediately from Lemmas 3.1 and 3.2. □

4 Poles and Their Residues

Also in this section $f(w;\lambda)$ is just the function defined by formula (1), which may or may not be a solution to Problem I or II, while condition (22) is retained. We discuss the pole structure of the function $f(w;\lambda)$. Any pole of it is simple and must lie in the arithmetic progression

$$W := \{w_j := -j/r\}_{j=0}^{\infty}, \tag{32}$$

but $f(w;\lambda)$ may be holomorphic at some points of (32). In order to know whether a given point w_j is actually a pole or not, we need to calculate the residue of $f(w;\lambda)$ at $w = w_j$.

Lemma 4.1 *The residue of $f(w;\lambda)$ at $w = w_j$ admits a hypergeometric expression*

$$\operatorname*{Res}_{w=w_j} f(w;\lambda) = C_j \cdot {}_2F_1(a_j,\, b_j;\, j+2;\, x) \qquad (j = 0, 1, 2, \dots), \tag{33}$$

where $a_j := pw_j + j + a + 1$, $b_j := qw_j + j + b + 1$ and

$$C_j := \frac{(-1)^j}{r} \cdot \frac{(pw_j + a)_{j+1}\,(qw_j + b)_{j+1}}{j!\,(j+1)!}\, x^{j+1}, \qquad (z)_j := \frac{\Gamma(z+j)}{\Gamma(z)}. \tag{34}$$

Proof. Let j and k be nonnegative integers. At the point $w = w_j$ the k-th summand of the hypergeometric series $f(w;\lambda) = {}_2F_1(pw+a, qw+b; rw; x)$ has residue

$$\operatorname*{Res}_{w=w_j} \frac{(pw+a)_k (qw+b)_k}{(rw)_k\, k!} x^k = \begin{cases} 0 & (k \leq j), \\ \dfrac{1}{r} \cdot \dfrac{(-1)^j}{j!} \cdot \dfrac{(pw_j+a)_k (qw_j+b)_k}{(k-j-1)!\, k!}\, x^k & (k \geq j+1). \end{cases}$$

Sum of these numbers over $k \ge j+1$ gives the residue of $f(w;\lambda)$ at $w = w_j$. Putting $k = i+j+1$,

$$\begin{aligned}
\operatorname*{Res}_{w=w_j} f(w;\lambda) &= \frac{x^{j+1}}{r} \cdot \frac{(-1)^j}{j!} \sum_{i=0}^{\infty} \frac{(pw_j+a)_{i+j+1}\,(qw_j+b)_{i+j+1}}{(i+j+1)!\,i!}\, x^i \\
&= \frac{x^{j+1}}{r} \cdot \frac{(-1)^j}{j!} \sum_{i=0}^{\infty} \frac{(pw_j+a)_{j+1}(pw_j+a+j+1)_i \cdot (qw_j+b)_{j+1}(qw_j+b+j+1)_i}{(j+1)!\,(j+2)_i\,i!}\, x^i \\
&= C_j \sum_{i=0}^{\infty} \frac{(a_j)_i\,(b_j)_i}{(j+2)_i\,i!}\, x^i = C_j \cdot {}_2F_1(a_j,\, b_j;\, j+2;\, x),
\end{aligned}$$

where $(t)_{i+j+1} = (t)_{j+1}\,(t+j+1)_i$ is used in the second equality. □

For every sufficiently large integer j, Lemma 4.1 reduces it to an elementary arithmetic to know whether $f(w;\lambda)$ is holomorphic or has a pole at $w = w_j$.

Lemma 4.2 *There exists a positive integer j_0 such that for any integer $j \ge j_0$ the function $f(w;\lambda)$ is holomorphic at $w = w_j$ if and only if either condition* (35a) *or* (35b) *below holds:*

$$r(a+i) = pj \qquad \textit{for some} \quad i \in \{0, \dots, j\}, \tag{35a}$$

$$r(b+i) = qj \qquad \textit{for some} \quad i \in \{0, \dots, j\}. \tag{35b}$$

Proof. Observe that $a_j = \{(r-p)j + r(a+1)\}/r$ and $b_j = \{(r-q)j + r(b+1)\}/r$, where $r-p$ and $r-q$ are positive by assumption (22). Take an integer j_0 so that

$$j_0 > \max\left\{-\frac{r(a+1)}{r-p},\ -\frac{r(b+1)}{r-q},\ 0\right\}.$$

Then a_j and b_j are positive for every $j \ge j_0$ so that $(a_j)_i$ and $(b_j)_i$ are also positive for every $i \ge 1$. Since $0 < x < 1$ by assumption (22), we have ${}_2F_1(a_j, b_j; j+2; x) \ge {}_2F_1(a_j, b_j; j+2; 0) = 1$. Thus formula (33) tells us that $\operatorname*{Res}_{w=w_j} f(w;\lambda) = 0$, that is, $f(w;\lambda)$ is holomorphic at $w = w_j$ if and only if $C_j = 0$. In view of definition (34) this condition is equivalent to $(pw_j+a)_{j+1}\,(qw_j+b)_{j+1} = 0$, which in turn holds true exactly when either condition (35a) or (35b) is satisfied. □

The following theorem enumerates all elementary solutions in domain (22).

Theorem 4.3 *For any $\lambda \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ the function $f(w;\lambda)$ has at most a finite number of poles in $\mathbb{C}_w$ if and only if λ satisfies either condition* (11) *or*

$$q = 0, \qquad b \in \mathbb{Z}_{\le 0}, \tag{36}$$

in which case λ actually gives an elementary solution to Problems I *and* II. *In particular, in this region elementary solutions can exist only on $\mathcal{D}^0 \cup \mathcal{I}^{*-}$.*

Proof. The proof below is due to the anonymous referee and is simpler than the author's original proof. Suppose that $f(w;\lambda)$ has at most a finite number of poles. By Lemma 4.2 there exists an integer $j_1 \ge j_0$ such that any integer $j \ge j_1$ satisfies either condition (35a) or (35b), where j_0 is the positive integer mentioned in Lemma 4.2.

Claim 1. For any $j \ge j_1$, if j satisfies condition (35a) then $j+1$ must satisfy condition (35b).

Indeed, if both j and $j+1$ satisfy (35a) then we have $r(a+i) = pj$ and $r(a+i') = p(j+1)$ for some $i \in \{0, \dots, j\}$ and $i' \in \{0, \dots, j+1\}$. Taking their difference yields $r(i'-i) = p$ and hence $p/r = i' - i \in \mathbb{Z}$, which is impossible since $0 < p < r$ and so $0 < p/r < 1$. Hence $j+1$ cannot satisfy (35a) and thus must satisfy (35b) instead.

Claim 2. Condition (35b) is satisfied by infinitely many $j \ge j_1$. Moreover we must have $q \ge 0$.

Suppose the contrary that those $j \ge j_1$ which satisfy (35b) are finite in cardinality. There then exists an integer $j_2 \ge j_1$ such that any $j \ge j_2$ satisfies (35a) but not (35b). This is impossible by Claim 1, so (35b) must be satisfied by infinitely many $j \ge j_1$. Let J be the infinite set of such j's. For each $j \in J$ condition (35b) yields $qj = r(b+i) \ge rb$, that is, $q \ge rb/j$, since $r > 0$, $j \ge j_1 > 0$ and $i \ge 0$. Letting $j \to \infty$ in J, we have $q \ge 0$.

Claim 3. If $q = 0$ then $b \in \mathbb{Z}_{\le 0}$, in which case we are in condition (36).

By Claim 2 condition (35b) is satisfied by an integer $j_3 \ge j_1$. Since $q = 0$ and $r > 0$, we have $r(b+i) = 0$ for some $i \in \{0, \dots, j_3\}$ and hence $b = -i \in \mathbb{Z}_{\le 0}$.

Claim 4. When $q > 0$, if $j \ge j_1$ satisfies (35b) then $j+1$ must satisfy (35a).

This claim can be proved in the same manner as Claim 1.

Claim 5. When $q > 0$, we have $p = q = r/2$ and either $a \in \mathbb{Z}$, $b \in \frac{1}{2} + \mathbb{Z}$ or $a \in \frac{1}{2} + \mathbb{Z}$, $b \in \mathbb{Z}$, in which case we are in condition (11).

By Claims 1, 2, 4, there exists an integer $k \ge j_1$ such that k and $k+2$ satisfy (35a) while $k+1$ and $k+3$ satisfy (35b). Thus there exist integers i_0, i_1, i_2, $i_3 \in \mathbb{Z}$ such that

$$r(a+i_0) = pk, \qquad r(a+i_2) = p(k+2), \tag{37a}$$

$$r(b+i_1) = q(k+1), \qquad r(b+i_3) = q(k+3), \tag{37b}$$

Taking the differences in (37a) and (37b) yields $r(i_2 - i_0) = 2p$ and $r(i_3 - i_1) = 2q$, that is, $2p/r = i_2 - i_0 \in \mathbb{Z}$ and $2q/r = i_3 - i_1 \in \mathbb{Z}$, respectively. But, since $0 < 2p/r < 2$ and $0 < 2q/r < 2$, we must have $2p/r = 2q/r = 1$. From the first equations in (37a) and (37b),

$$a = \frac{2p}{r} \cdot \frac{k}{2} - i_0 = \frac{k}{2} - i_0, \qquad b = \frac{2q}{r} \cdot \frac{k+1}{2} - i_1 = \frac{k+1}{2} - i_1.$$

If k is an even integer then $a \in \mathbb{Z}$ and $b \in \frac{1}{2} + \mathbb{Z}$, while if k is an odd integer then $a \in \frac{1}{2} + \mathbb{Z}$ and $b \in \mathbb{Z}$. This proves Claim 5.

The "only if" part of the theorem is now established by Claims 3 and 5. Conversely, if λ satisfies condition (11) then $f(w; \lambda)$ becomes a dihedral function (12) due to Vidunas [20, Theorem 3.1], while if λ satisfies (36) then $f(w; \lambda)$ is clearly a rational function of w. In either case $f(w; \lambda)$ has at most a finite number of poles, which proves the"if" part of the theorem. □

5 Gamma Product Formula

Given a nontrivial rational function $R(w) \in \mathbb{C}(w)$, we consider a representation of the form

$$R(w) = \frac{S(w+1)}{S(w)} \cdot d \cdot \frac{P(w)}{Q(w)}, \tag{38a}$$

$$P(w) = (w+u_1) \cdots (w+u_m), \quad Q(w) = (w+v_1) \cdots (w+v_n), \tag{38b}$$

with a rational function $S(w) \in \mathbb{C}(w)$, $d \in \mathbb{C}^\times$ and u_i, $v_j \in \mathbb{C}$, where if $R(w)$ is real then $S(w)$, d, $P(w)$, $Q(w)$ should also be real; such an expression is always feasible (but not unique).

Let $\lambda \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ be a solution to Problem II and write $R(w;\lambda)$ in the form (38). Consider an entire meromorphic function defined by

$$g(w) := S(w) \cdot d^w \cdot \frac{\Gamma(w+u_1)\cdots\Gamma(w+u_m)}{\Gamma(w+v_1)\cdots\Gamma(w+v_n)}. \tag{39}$$

Put $u := u_1 + \cdots + u_m$ and $v := v_1 + \cdots + v_n$; they are real because $R(w;\lambda)$ is real.

Lemma 5.1 *There exists a constant $R_2 \in \mathbb{R}$ such that on the right half-plane $\mathrm{Re}(w) \ge R_2$ the function $g(w)$ is holomorphic, nowhere vanishing, and admits a uniform estimate*

$$g(w) \sim S_0 \cdot (2\pi)^{(m-n)/2} \left(de^{n-m}\right)^w \cdot w^{-(m-n)/2+u-v+s_0} \cdot e^{(m-n)w\log w},$$

where $S_0 \in \mathbb{R}^\times$ and $s_0 \in \mathbb{Z}$ are defined by the asymptotics $S(w) \sim S_0 w^{s_0}$ as $w \to \infty$.

Proof. Take a number $R_2 \in \mathbb{R}$ in such a manner that all the points $-u_1, \dots, -u_m; -v_1, \dots, -v_n$ as well as all the zeros and poles of $S(w)$ are strictly to the left of the vertical line $\mathrm{Re}(w) = R_2$. Then it is clear from the locations of its poles and zeros that $g(w)$ is holomorphic and non-vanishing on the half-plane $\mathrm{Re}(w) \ge R_2$. By Stirling's formula (26), we have

$$\begin{aligned} g(w) = S(w) \cdot d^w \frac{\prod_{i=1}^m \Gamma(w+u_i)}{\prod_{j=1}^n \Gamma(w+v_j)} &\sim S_0 w^{s_0} \cdot d^w \frac{\prod_{i=1}^m \sqrt{2\pi}\, e^{-w}\, w^{w+u_i-1/2}}{\prod_{j=1}^n \sqrt{2\pi}\, e^{-w}\, w^{w+v_j-1/2}} \\ &= S_0 w^{s_0} \cdot d^w \cdot (2\pi)^{(m-n)/2}\, e^{(n-m)w}\, w^{(m-n)(w-1/2)+u-v} \\ &= S_0 \cdot (2\pi)^{(m-n)/2} \cdot (de^{n-m})^w \cdot w^{-(m-n)/2+u-v+s_0} \cdot w^{(m-n)w} \\ &= S_0 \cdot (2\pi)^{(m-n)/2} \cdot (de^{n-m})^w \cdot w^{-(m-n)/2+u-v+s_0} \cdot e^{(m-n)w\log w}. \end{aligned}$$

uniformly on the right-half plane $\mathrm{Re}(w) \ge R_2$. □

Observe that $g(w)$ satisfies the same recurrence relation (3) as the function $f(w;\lambda)$. So it is natural to compare $f(w;\lambda)$ with $g(w)$ or in other words to think of the ratio

$$h(w) := f(w;\lambda)/g(w). \tag{40}$$

It is clear that $h(w)$ is an entire meromorphic function that does not vanish identically.

Lemma 5.2 *$h(w)$ is an entire holomorphic function which is periodic of period one. For any $R_3 > \max\{R_1, R_2, 1\}$ there exists a constant $A_2 > 0$ such that*

$$|h(w)| \le A_2 \cdot K^{\mathrm{Re}(w)} \cdot |w|^{(n-m)\{\mathrm{Re}(w)-1/2\}+v-u-s_0} \cdot e^{-(n-m)\arg(w)\cdot\mathrm{Im}(w)}, \tag{41}$$

on $\mathrm{Re}(w) \ge R_3$, where $K := e^{m-n}B/d$ with B being the positive constant in (31).

Proof. Since $f(w;\lambda)$ and $g(w)$ satisfy the same recurrence relation (3), their ratio $h(w)$ must be a periodic function of period one. From Proposition 3.3 and Lemma 5.1 the function $h(w)$ has no poles on $\mathrm{Re}(w) \ge R_3$ and so holomorphic there. The periodicity then implies that $h(w)$ must be holomorphic on the entire complex plane. In view of $\left|e^{w\log w}\right| = e^{\mathrm{Re}(w\log w)} = e^{\mathrm{Re}(w)\cdot\log|w| - \mathrm{Im}(w)\cdot\arg(w)} = |w|^{\mathrm{Re}(w)} e^{-\mathrm{Im}(w)\cdot\arg(w)}$, Lemma 5.1 implies that

$$|g(w)| \sim |S_0| \cdot (2\pi)^{(m-n)/2} \left(de^{n-m}\right)^{\mathrm{Re}(w)} \cdot |w|^{(m-n)\{\mathrm{Re}(w)-1/2\}+u-v+s_0} \cdot e^{-(m-n)\mathrm{Im}(w)\cdot\arg(w)},$$

uniformly on $\mathrm{Re}(w) \geq R_3$. Since $g(w)$ has no zero there, there is a constant $A_3 > 0$ such that

$$|g(w)| \geq A_3 \cdot \left(de^{n-m}\right)^{\mathrm{Re}(w)} \cdot |w|^{(m-n)\{\mathrm{Re}(w)-1/2\}+u-v+s_0} \cdot e^{-(m-n)\mathrm{Im}(w)\cdot\arg(w)},$$

on $\mathrm{Re}(w) \geq R_3$. On the other hand, by Proposition 3.3 there is a constant $A_4 > 0$ such that $|f(w;\lambda)| \leq A_4 \cdot B^{\mathrm{Re}(w)}$ on $\mathrm{Re}(w) \geq R_3$. Thus (41) holds true with constant $A_2 := A_4/A_3$. □

Lemma 5.3 *$h(w)$ must be a nonzero constant and $m = n$ in formulas* (38b) *and* (39)*.*

Proof. First we show $m \leq n$. Suppose the contrary $m > n$. Estimate (41) with real w reads $|h(w)| \leq A_2 \cdot K^w \cdot w^{-(m-n)(w-1/2)+v-u-s_0}$ for every $w \geq R_3$. Fix any $w \in \mathbb{R}$ and take a positive integer k_0 such that $w + k_0 \geq R_3$. Since $h(w)$ is periodic of period one, for any integer $k \geq k_0$,

$$\begin{aligned}|h(w)| = |h(w+k)| &\leq A_2 \cdot K^{w+k} \cdot (w+k)^{-(m-n)(w+k-1/2)+v-u-s_0} \\ &= A_2 \cdot K^w \cdot (1+w/k)^{\rho} \cdot (1+w/k)^{-(m-n)k} \cdot k^{\rho} \cdot (K/k^{m-n})^k \\ &\sim A_2 \cdot K^w \cdot e^{-(m-n)w} \cdot k^{\rho} \cdot (K/k^{m-n})^k \quad \text{as} \quad k \to +\infty,\end{aligned}$$

where $\rho := -(m-n)(w-1/2) + v - u - s_0$. Since we are assuming that $m - n > 0$ there exists an integer $k_1 \geq k_0$ such that $0 < K/k_1^{m-n} \leq 1/2$. Then there exists a constant $A_5 > A_2 \cdot K^w \cdot e^{-(m-n)w}$ such that $|h(w)| \leq A_5 \cdot k^{\rho} \cdot 2^{-k}$ for every $k \geq k_1$. Letting $k \to +\infty$ we have $h(w) = 0$ for every $w \in \mathbb{R}$. By the unicity theorem for holomorphic functions, $h(w)$ must vanish identically. But this is absurd because $h(w)$ is nontrivial and thus we have proved $m \leq n$.

Next we show that $h(w)$ is a nonzero constant. We make use of estimate (41) on the strip $R_3 \leq \mathrm{Re}(w) \leq R_3 + 1$, where we recall $R_3 > 1$. On this strip $K^{\mathrm{Re}(w)}$ are bounded while $|w|^{(n-m)\{\mathrm{Re}(w)-1/2\}+v-u-s_0} \leq |w|^{\mu} \leq A_6(1 + |\mathrm{Im}(w)|^{\mu})$ for some constant A_6, where μ is a nonnegative number with $\mu \geq (n-m)(R_2 - 1/2) + v - u - s_0$. On the strip, if $|\mathrm{Im}(w)| \geq R_3 + 1$ then $|\arg(w)| \geq \pi/4$ and $\arg(w) \cdot \mathrm{Im}(w) \geq (\pi/4)|\mathrm{Im}(w)|$. So there is a constant A_7 such that

$$|h(w)| \leq A_7(1 + |\mathrm{Im}(w)|^{\mu})e^{-\pi(n-m)|\mathrm{Im}(w)|/4}, \tag{42}$$

for any w on the strip $R_3 \leq \mathrm{Re}(w) \leq R_3 + 1$, provided $|\mathrm{Im}(w)| \geq R_3 + 1$. Estimate (42) remains true on the entire strip if A_7 is chosen sufficiently large. This estimate extends to the entire complex plane, since both sides of (42) are periodic functions of period one. In particular, in view of $m \leq n$, estimate (42) yields $|h(w)| \leq A_7 \cdot (1 + |\mathrm{Im}(w)|^{\mu}) \leq A_7 \cdot (1 + |w|^{\mu})$ for every $w \in \mathbb{C}$. Liouville's theorem then implies that $h(w)$ must be a polynomial. But a polynomial can be a periodic function only when it is a constant. Hence $h(w)$ must be a constant, which is nonzero as $h(w)$ is nontrivial. Finally we shall show $m = n$; we already know $m \leq n$. If $m < n$ then the right-hand side of (42) would tend to zero as $|\mathrm{Im}(w)| \to \infty$. But this contradicts the fact that $h(w)$ is a nonzero constant. Thus we must have $m = n$. □

Theorem 5.4 *Let $\lambda \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ be a solution to Problem* II *and write $R(w;\lambda)$ in the form* (38)*. Then $m = n$ and after multiplying $S(w)$ by a nonzero constant we have*

$$f(w;\lambda) = S(w) \cdot d^w \cdot \frac{\Gamma(w+u_1)\cdots\Gamma(w+u_m)}{\Gamma(w+v_1)\cdots\Gamma(w+v_m)}. \tag{43}$$

In particular λ is a solution to Problem I*. If A and B are the constants defined in* (31) *and $S_0 \in \mathbb{R}^{\times}$ and $s_0 \in \mathbb{Z}$ are defined by the asymptotics $S(w) \sim S_0 w^{s_0}$ as $w \to \infty$, then*

$$S_0 = A, \qquad d = B, \qquad v_1 + \cdots + v_m = u_1 + \cdots + u_m + s_0. \tag{44}$$

Proof. We have $m = n$ by Lemma 5.3. We can multiply $S(w)$ by any nonzero constant without changing the expression (38). So after multiplying $S(w)$ by a suitable constant if necessary, we may put $h(w) \equiv 1$ in Lemma 5.3, then (39) and (40) yield (43). The asymptotic formula in Lemma 5.1 reads $g(w) \sim S_0 \cdot d^w \cdot w^{u-v+s_0}$ as $w \to \infty$ on $\mathrm{Re}(w) \ge R_2$. Compare this with the formula $f(w;\lambda) \sim A \cdot B^w$ in Proposition 3.3. Then $f(w;\lambda) \equiv g(w)$ implies equations (44). □

Remark 5.5 By classical symmetries Theorem 5.4 extends to $\mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$ and gives Theorem 2.1, implying that Problems I and II are equivalent in this domain. Hereafter, always working there, we can and shall refer to a solution without specifying to which problem it is a solution.

Representation (38) is said to be *canonical* if $P(w)$ and $Q(w+j)$ are coprime in $\mathbb{C}[w]$ for every $j \in \mathbb{Z}$. Gosper [12] considered a similar situation where $P(w)$ and $Q(w+j)$ were coprime for every $j \in \mathbb{Z}_{\ge 0}$ with $S(w)$ being a polynomial instead of a rational function.

Lemma 5.6 *Any rational function $R(w) \in \mathbb{C}(w)$ admits a canonical representation* (38)*. If $R(w) \in \mathbb{R}(w)$ then d, $P(w)$, $Q(w)$ and $S(w)$ can be taken to be real.*

Proof. Start with the reduced representation $R(w) = d\,P(w)/Q(w)$, where $P(w), Q(w) \in \mathbb{C}[w]$ are monic and coprime. We work inductively on the degree of $P(w)$. If $P(w)$ and $Q(w+j)$ are coprime for every $j \in \mathbb{Z}$ then we are done. Otherwise, $P(w)$ has a root α such that $\alpha + j$ is a root of $Q(w)$ for some nonzero $j \in \mathbb{Z}$. Obviously $(w-\alpha)/(w-\alpha-j) = S_1(w+1)/S_1(w)$, where $S_1(w) := (w-\alpha-j)_j$ if $j > 0$ and $S_1(w) := 1/(w-\alpha)_{|j|}$ if $j < 0$. Thus we can rewrite

$$R(w) = \frac{S_1(w+1)}{S_1(w)}\,R_1(w) \quad \text{with} \quad R_1(w) = d\,\frac{P_1(w)}{Q_1(w)}, \quad \deg P_1(w) = \deg P(w) - 1.$$

We then proceed with $R_1(w)$. The case of $\mathbb{R}(w)$ can be dealt with by using “irreducible factors” (affine or irreducible quadratic polynomials) over $\mathbb{R}$ instead of $w - \alpha$. □

Proposition 5.7 *Let $\lambda \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ be a non-elementary solution and write $R(w;\lambda)$ in a canonical form* (38)*. Then we must have $r \in \mathbb{Z}$, $1 \le m = n \le r$ and there exist $s_1, \dots, s_m \in \mathbb{Z}$ mutually distinct modulo r such that the numbers $u_1, \dots, u_m$ in* (38b) *are represented as*

$$u_i = s_i/r \qquad (i = 1, \dots, m). \tag{45}$$

Proof. Let W_{pole} be the set of all poles of $f(w;\lambda)$. It is an infinite set since $f(w;\lambda)$ is assumed to be non-elementary. In gamma product formula (43) the poles of $\Gamma(w+u_1)\cdots\Gamma(w+u_m)$ and those of $\Gamma(w+v_1)\cdots\Gamma(w+v_m)$ constitute two families of arithmetic progressions:

$$U_i := \{-u_i - k\}_{k=0}^{\infty} \qquad (i = 1, \dots, m), \tag{46a}$$

$$V_j := \{\,-v_j - k\,\}_{k=0}^{\infty} \qquad (j = 1, \dots, m), \tag{46b}$$

respectively. Since representation (38) is canonical, U_i and V_j are disjoint for every $i, j = 1, \dots, m$, and hence W_{pole} is commensurable to the union $\bigcup_{i=1}^m U_i$, which is disjoint because all poles of $f(w;\lambda)$ are simple so that $u_i - u_j \notin \mathbb{Z}$ for every $i \ne j$, that is,

$$W_{\mathrm{pole}} \overset{\circ}{=} \coprod_{i=1}^{m} U_i. \tag{47}$$

Thus when expression (38) is canonical, λ is non-elementary if and only if $m \geq 1$.

Take $i = 1$ and k to be sufficiently large in the arithmetic progression (46a). Formula (43) then shows that $w = -u_1 - k$ and $w = -u_1 - k - 1$ are poles of $f(w;\lambda)$, so they must lie in the arithmetic progression (32). Thus there exist $j_1,\, j_2 \in \mathbb{Z}_{\geq 0}$ with $j_1 < j_2$ such that $-u_1 - k = -j_1/r$ and $-u_1 - k - 1 = -j_2/r$. Their difference gives $1 = (j_2 - j_1)/r$, which shows that $r = j_2 - j_1$ must be a positive integer. In a similar manner for each $i = 1, \dots, m$ there exists an integer k_i such that $w = -u_i - k_i$ is a pole of $f(w;\lambda)$. So it must lie in the arithmetic progression (32), namely, we can write $-u_i - k_i = -j_i/r$ for some $j_i \in \mathbb{Z}$. If we put $s_i := j_i + rk_i$ then formula (45) holds. Note that $s_1, \dots, s_m$ are mutually distinct modulo r, because $U_i, \dots, U_m$ are disjoint. □

6 Asymptotics of the Residues

Throughout this section $\lambda = (p, q, r; a, b, x) \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ is a *non-elementary* solution with $R(w;\lambda)$ being in a *canonical form* (38). By formula (47) the poles of $f(w;\lambda)$ are commensurable to the disjoint union of m arithmetic progressions U_i $(i = 1, \dots, m)$. In view of formulas (32) and (45) the general term of U_i is represented as $-u_i - k = -(rk + s_i)/r = w_j$ with $j = rk + s_i$. In this situation formula (33) reads

$$\mathrm{Res}_k^{(i)} = C_k^{(i)} \cdot F_k^{(i)}, \tag{48}$$

where using the notation of Lemma 4.1 we put

$$\mathrm{Res}_k^{(i)} := \operatorname*{Res}_{w=w_j} f(w;\lambda), \quad C_k^{(i)} := C_j, \quad F_k^{(i)} := {}_2F_1(a_j,\, b_j;\, j+2;\, x).$$

We study the asymptotic behavior of $\mathrm{Res}_k^{(i)}$ as $k \to \infty$ for a fixed $i = 1, \dots, m$.

Lemma 6.1 *Let B and t_0 be the same constants as in* (31) *and put*

$$\xi(t) := t^{-2p/r-1}(1-t)^{2p/r-3}(1-xt)^{2q/r-1}, \tag{49}$$

$$\widetilde{A} := \frac{p^{a+2p/r-1/2}}{(r-p)^{a+2p/r-3/2} \cdot r^{1/2}} \cdot \frac{\xi(t_0)}{\eta(t_0)\sqrt{\phi''(t_0)}}, \tag{50}$$

where $\phi(t)$ and $\eta(t)$ are defined by (28). *Then for each $i = 1, \dots, m$, we have*

$$F_k^{(i)} \sim \{(1-x)^{(p+q-r)u_i-a-b}\widetilde{A}B^{u_i+2/r}\} \cdot \{(1-x)^{p+q-r}B\}^k \qquad \text{as} \quad k \to \infty. \tag{51}$$

Proof. Euler's transformation (7b) and the definitions of a_j and b_j in Lemma 4.1 yield

$$\begin{aligned} F_k^{(i)} &= (1-x)^{j+2-a_j-b_j}\, {}_2F_1(j+2-a_j,\, j+2-b_j;\, j+2;\, x) \quad \text{with} \quad j = rk + s_i \\ &= (1-x)^{(p+q-r)(k+u_i)-a-b}\, {}_2F_1(p\, w_k^{(i)} + \widetilde{a},\, q\, w_k^{(i)} + \widetilde{b};\, r\, w_k^{(i)};\, x), \end{aligned}$$

where $\widetilde{a} := 1 - a - 2p/r$, $\widetilde{b} := 1 - b - 2q/r$ and $w_k^{(i)} := k + u_i + 2/r$. Asymptotic behavior of $F(p\, w_k^{(i)} + \widetilde{a},\, q\, w_k^{(i)} + \widetilde{b},\, r\, w_k^{(i)};\, x)$ can be extracted from that of $f(w;\lambda) = {}_2F_1(pw+a, qw+b, rw; x)$

in formula (30) by substituting $a \mapsto \widetilde{a}$, $b \mapsto \widetilde{b}$, $w \mapsto w_k^{(i)}$, where p, q, r, x and so $\Phi(t)$, $\phi(t)$, t_0, B in formula (31) are left unchanged. This substitution replaces $\eta(t)$ with

$$\widetilde{\eta}(t) := t^{-a-2p/r}(1-t)^{a+2p/r-2}(1-xt)^{b+2q/r-1} = \frac{t^{-1-2p/r}(1-t)^{2p/r-3}(1-xt)^{2q/r-1}}{t^{a-1}(1-t)^{-a-1}(1-xt)^{-b}} = \frac{\xi(t)}{\eta(t)},$$

where $\xi(t)$ is defined by (49), which in turn induces the change of constant $A \mapsto \widetilde{A}$ in (50). Proposition 3.3 then yields $F_k^{(i)} \sim (1-x)^{(p+q-r)(k+u_i)-a-b} \cdot \widetilde{A} \cdot B^{k+u_i+2/r}$ as $k \to \infty$. After a rearrangement, it just gives the desired formula (51). □

We proceed to investigating $C_k^{(i)}$. Substituting $j = rk + s_i$ into formula (34) yields

$$C_k^{(i)} = \frac{(-1)^{rk+s_i}}{r} \cdot \frac{(-(pk+\alpha_i))_{rk+s_i+1}(-(qk+\beta_i))_{rk+s_i+1}}{(rk+s_i)!\,(rk+s_i+1)!}\, x^{rk+s_i+1}, \tag{52}$$

where α_i and β_i are defined in terms of u_i in (45) by

$$\alpha_i := pu_i - a, \qquad \beta_i := qu_i - b \qquad (i = 1, \dots, m). \tag{53}$$

We study the asymptotic behavior of $C_k^{(i)}$ as $k \to \infty$ by dividing condition (22) into three cases.

Lemma 6.2 *For each $i = 1, \dots, m$, according to the value of q we have*

$$C_k^{(i)} \sim D_1^{(i)} \cdot E_1^k \cdot (-1)^{rk+s_i} \cdot \sin\pi(pk+\alpha_i) \cdot \sin\pi(qk+\beta_i) \qquad (0 < q < r), \tag{54a}$$

$$C_k^{(i)} \sim D_2^{(i)} \cdot E_2^k \cdot k^{b-1/2} \cdot (-1)^{rk+s_i+1} \cdot \sin\pi(pk+\alpha_i) \qquad (q = 0 < r), \tag{54b}$$

$$C_k^{(i)} \sim D_3^{(i)} \cdot E_3^k \cdot (-1)^{rk+s_i+1} \cdot \sin\pi(pk+\alpha_i) \qquad (q < 0 < r), \tag{54c}$$

as $k \to \infty$, where $D_\nu^{(i)}$ and E_ν, $\nu = 1, 2, 3$, are constants defined by

$$D_1^{(i)} := \frac{2\,p^{\alpha_i+1/2}(r-p)^{s_i-\alpha_i+1/2}q^{\beta_i+1/2}(r-q)^{s_i-\beta_i+1/2}}{\pi\, r^{2s_i+3}}\, x^{s_i+1}, \qquad E_1 := \frac{p^p(r-p)^{r-p}q^q(r-q)^{r-q}}{r^{2r}}\, x^r, \tag{55a}$$

$$D_2^{(i)} := \sqrt{\frac{2}{\pi}} \cdot \frac{p^{\alpha_i+1/2}(r-p)^{s_i-\alpha_i+1/2}}{\Gamma(b)\, r^{s_i-b+5/2}}\, x^{s_i+1}, \qquad E_2 := \frac{p^p(r-p)^{r-p}}{r^r}\, x^r, \tag{55b}$$

$$D_3^{(i)} := \frac{p^{\alpha_i+1/2}(r-p)^{s_i-\alpha_i+1/2}|q|^{\beta_i+1/2}(r-q)^{s_i-\beta_i+1/2}}{\pi\, r^{2s_i+3}}\, x^{s_i+1}, \qquad E_3 := \frac{p^p(r-p)^{r-p}|q|^q(r-q)^{r-q}}{r^{2r}}\, x^r. \tag{55c}$$

Proof. For $t \in \mathbb{R}$ we denote by $[t]$ the largest integer not exceeding t and by $\{t\} := t - [t]$ the fractional part of t. It follows from $0 < p < r$ that $[pk+\alpha_i]+1$ and $rk+s_i-[pk+\alpha_i]$ are positive integers for every sufficiently large integer k. Since $pk+\alpha_i = [pk+\alpha_i] + \{pk+\alpha_i\}$,

$$\begin{aligned}
(-(pk+\alpha_i))_{rk+s_i+1} &= \overbrace{(-(pk+\alpha_i))(1-(pk+\alpha_i))\cdots(-\{pk+\alpha_i\})}^{[pk+\alpha_i]+1} \\
&\quad \times \overbrace{(1-\{pk+\alpha_i\})(2-\{pk+\alpha_i\})\cdots(rk+s_i-(pk+\alpha_i))}^{rk+s_i-[pk+\alpha_i]} \\
&= (-1)^{[pk+\alpha_i]+1}(\{pk+\alpha_i\})_{[pk+\alpha_i]+1}\,(1-\{pk+\alpha_i\})_{rk+s_i-[pk+\alpha_i]} \\
&= (-1)^{[pk+\alpha_i]+1}\frac{\Gamma(pk+\alpha_i+1)}{\Gamma(\{pk+\alpha_i\})} \cdot \frac{\Gamma((r-p)k+s_i-\alpha_i+1)}{\Gamma(1-\{pk+\alpha_i\})} \\
&= -\frac{\sin\pi(pk+\alpha_i)}{\pi}\,\Gamma(pk+\alpha_i+1)\,\Gamma((r-p)k+s_i-\alpha_i+1),
\end{aligned}$$

by the recursion and reflection formulas for the gamma function. By Stirling's formula (26),

$$\frac{\Gamma(pk+\alpha_i+1)\Gamma((r-p)k+s_i-\alpha_i+1)}{\Gamma(rk+s_i+3/2)} \sim \sqrt{2\pi}\,\frac{p^{\alpha_i+1/2}(r-p)^{s_i-\alpha_i+1/2}}{r^{s_i+1}}\left(\frac{p^p(r-p)^{r-p}}{r^r}\right)^k$$

as $k \to \infty$. Using this asymptotic formula in the above equation we have

$$\frac{(-(pk+\alpha_i))_{rk+s_i+1}}{\Gamma(rk+s_i+3/2)} \sim -\sqrt{\tfrac{2}{\pi}}\cdot \tfrac{p^{\alpha_i+1/2}(r-p)^{s_i-\alpha_i+1/2}}{r^{s_i+1}}\left(\tfrac{p^p(r-p)^{r-p}}{r^r}\right)^k \sin\pi(pk+\alpha_i) \qquad (56)$$

as $k \to \infty$. Exactly in the same manner, if $0 < q < r$ then we have as $k \to \infty$,

$$\frac{(-(qk+\beta_i))_{rk+s_i+1}}{\Gamma(rk+s_i+3/2)} \sim -\sqrt{\tfrac{2}{\pi}}\cdot \tfrac{q^{\beta_i+1/2}(r-q)^{s_i-\beta_i+1/2}}{r^{s_i+1}}\left(\tfrac{q^q(r-q)^{r-q}}{r^r}\right)^k \sin\pi(qk+\beta_i). \qquad (57)$$

Next we consider the case $q \le 0 < r$. For every sufficiently large integer k,

$$\frac{(-(qk+\beta_i))_{rk+s_i+1}}{\Gamma(rk+s_i+3/2)} = \begin{cases} \dfrac{\Gamma(b+rk+s_i+1)}{\Gamma(b)\Gamma(rk+s_i+3/2)} & (q=0<r), \\ \dfrac{\Gamma((r-q)k+s_i-\beta_i+1)}{\Gamma(|q|k-\beta_i)\Gamma(rk+s_i+3/2)} & (q<0<r). \end{cases}$$

Applying Stirling's formula (26) to the right-hand side above we have as $k \to \infty$,

$$\frac{(-(qk+\beta_i))_{rk+s_i+1}}{\Gamma(rk+s_i+3/2)} \sim \begin{cases} (rk)^{b-1/2}/\Gamma(b) & (q=0<r), \\ \tfrac{1}{\sqrt{2\pi}}\cdot\tfrac{|q|^{\beta_i+1/2}(r-q)^{s_i-\beta_i+1/2}}{r^{s_i+1}}\left(\tfrac{|q|^q(r-q)^{r-q}}{r^r}\right)^k & (q<0<r), \end{cases} \qquad (58)$$

Notice that $(rk+s_i)!\,(rk+s_i+1)! \sim \Gamma(rk+s_i+3/2)^2$ as $k \to \infty$ by Stirling's formula (26). Thus substituting formulas (56) and (57) into (52) yields formula (54a). Similarly substituting formulas (56) and (58) into (52) yields formulas (54b) and (54c). □

Remark 6.3 When $q = 0$, we have $b \notin \mathbb{Z}_{\le 0}$ since elementary solutions (36) are excluded from our consideration. Thus the constant $D_2^{(i)}$ in (55b) is nonzero.

Lemma 6.4 *For each $i = 1, \dots, m$, according to the value of q we have*

$$\mathrm{Res}_k^{(i)} \sim D_4^{(i)}\cdot E_4^k\cdot(-1)^{rk+s_i}\cdot\sin\pi(pk+\alpha_i)\cdot\sin\pi(qk+\beta_i) \qquad (0<q<r), \qquad (59a)$$

$$\mathrm{Res}_k^{(i)} \sim D_5^{(i)}\cdot E_5^k\cdot k^{b-1/2}\cdot(-1)^{rk+s_i+1}\cdot\sin\pi(pk+\alpha_i) \qquad (q=0<r), \qquad (59b)$$

$$\mathrm{Res}_k^{(i)} \sim D_6^{(i)}\cdot E_6^k\cdot(-1)^{rk+s_i+1}\cdot\sin\pi(pk+\alpha_i) \qquad (q<0<r), \qquad (59c)$$

as $k \to \infty$, where $D_\nu^{(i)}$ and E_ν, $\nu = 4,5,6$, are constants defined by

$$D_\nu^{(i)} := (1-x)^{(p+q-r)u_i-a-b}\widetilde{A}B^{u_i+2/r}D_{\nu-3}^{(i)}, \quad E_\nu := (1-x)^{p+q-r}B\cdot E_{\nu-3} \quad (\nu=4,5,6). \quad (60)$$

Proof. In view of (48) this lemma is proved by putting Lemmas 6.1 and 6.2 together. □

Lemma 6.5 *In the circumstances of Theorem* 5.4*, we have for each* $i = 1, \dots, m$*,*

$$\mathrm{Res}_k^{(i)} \sim K^{(i)} \cdot B^{-k} \quad \textit{as} \quad k \to \infty, \qquad K^{(i)} := \frac{(-1)^{s_0} A}{\pi B^{u_i}} \cdot \frac{\prod_{j=1}^{m} \sin \pi (v_j - u_i)}{\prod_{j=1}^{*m} \sin \pi (u_j - u_i)}, \tag{61}$$

where $\prod_{j=1}^{*m}$ *denotes the product taken over all* $j = 1, \dots, m$ *but* $j = i$*.*

Proof. Applying the reflection formula for the gamma function to formula (43) yields

$$f(w; \lambda) = S(w) \cdot B^w \cdot \frac{\prod_{j=1}^{m} \Gamma(1 - v_j - w)}{\prod_{j=1}^{m} \Gamma(1 - u_j - w)} \cdot \frac{\prod_{j=1}^{m} \sin \pi (w + v_j)}{\prod_{j=1}^{m} \sin \pi (w + u_j)},$$

where $d = B$ in (44) is also used. Taking its residue at $w = -k - u_i$ gives

$$\begin{aligned} \mathrm{Res}_k^{(i)} &= S(-(k + u_i)) \cdot B^{-k-u_i} \frac{\prod_{j=1}^{m} \Gamma(k + u_i - v_j + 1)}{\prod_{j=1}^{m} \Gamma(k + u_i - u_j + 1)} \\ &\quad \times \frac{\prod_{j=1}^{m} \sin \pi (v_j - u_i - k)}{\prod_{j=1}^{*m} \sin \pi (u_j - u_i - k)} \lim_{w \to -k-u_i} \frac{w + k + u_i}{\sin \pi (w + u_i)} \\ &= \frac{1}{\pi} S(-(k + u_i)) B^{-k-u_i} \frac{\prod_{j=1}^{m} \Gamma(k + u_i - v_j + 1)}{\prod_{j=1}^{m} \Gamma(k + u_i - u_j + 1)} \cdot \frac{\prod_{j=1}^{m} \sin \pi (v_j - u_i)}{\prod_{j=1}^{*m} \sin \pi (u_j - u_i)}. \end{aligned}$$

By (44) we have $S(-(k + u_i)) \sim S_0(-k)^{s_0} = A(-1)^{s_0} k^{s_0}$ and by Stirling's formula (26),

$$\frac{\prod_{j=1}^{m} \Gamma(k + u_i - v_j + 1)}{\prod_{j=1}^{m} \Gamma(k + u_i - u_j + 1)} \sim k^{u-v} = k^{-s_0} \qquad \text{as} \quad k \to \infty.$$

Substituting these asymptotic formulas into the above equation we get formula (61). □

It follows from (61) that $K^{(i)} \neq 0$ for $i = 1, \dots, m$, because $A > 0$ and $B > 0$ by (31) and $v_j - u_i \notin \mathbb{Z}$, $i, j = 1, \dots, m$, as (38) is assumed to be canonical. Since asymptotic formulas (59) and (61) must be equivalent, taking the ratio of them gives the following.

Proposition 6.6 *We have asymptotic formulas*

$$D_7^{(i)} \cdot E_7^k \cdot (-1)^{rk+s_i} \cdot \sin \pi (pk + \alpha_i) \cdot \sin \pi (qk + \beta_i) \to 1 \quad (0 < q < r), \tag{62a}$$

$$D_8^{(i)} \cdot E_8^k \cdot k^{b-1/2} \cdot (-1)^{rk+s_i+1} \cdot \sin \pi (pk + \alpha_i) \to 1 \quad (q = 0 < r), \tag{62b}$$

$$D_9^{(i)} \cdot E_9^k \cdot (-1)^{rk+s_i+1} \cdot \sin \pi (pk + \alpha_i) \to 1 \quad (q < 0 < r), \tag{62c}$$

as $k \to \infty$*, where* $D_\nu^{(i)}$ *and* E_ν*,* $\nu = 7, 8, 9$*,* $i = 1, \dots, m$*, are constants defined by*

$$D_\nu^{(i)} := D_{\nu-3}^{(i)} / K^{(i)}, \qquad E_\nu := B \cdot E_{\nu-3}. \qquad (\nu = 7, 8, 9). \tag{63}$$

To extract important information from Proposition 6.6 we need a couple of lemmas.

Lemma 6.7 *Let* $w_1, \dots, w_m \in \mathbb{C}^\times$ *be mutually distinct,* $c_1, \dots, c_m \in \mathbb{C}$ *and* $\delta \in \mathbb{R}$*. If*

$$k^\delta \sum_{i=1}^{m} c_i \, w_i^k \to c \in \mathbb{C}^\times \qquad \textit{as} \quad k \to \infty, \tag{64}$$

then $\delta = 1$ *and there exists a unique index* $\mu \in \{1, \dots, m\}$ *such that* $w_\mu = 1$*,* $c_\mu = c$ *and for any index* $i = 1, \dots, m$ *with* $i \neq \mu$ *either* $|w_i| < 1$ *or* $c_i = 0$ *holds.*

Proof. Replacing k with $k+j-1$ in (64) and using $(k+j-1)^{\delta}/k^{\delta} \to 1$ as $k \to \infty$, we have

$$k^{\delta} \sum_{i=1}^{m} c_i \, w_i^{k+j-1} \to c \in \mathbb{C}^{\times} \qquad \text{as} \quad k \to \infty \qquad (j = 1, \dots, m).$$

Since the Vandermonde matrix $\left(w_i^{j-1}\right)_{i,j=1}^{m}$ is invertible, this implies that for each $i = 1, \dots, m$, the sequence $k^{\delta} \, c_i \, w_i^k$ has a limit $\gamma_i \in \mathbb{C}$ as $k \to \infty$, which must satisfy $w_i \gamma_i = \gamma_i$. Since $c \neq 0$ in (64), there exists an index $\mu \in \{1, \dots, m\}$ such that $\gamma_{\mu} \neq 0$ and hence $w_{\mu} = 1$, $c_{\mu} \neq 0$ and so $\delta = 0$. As $w_1, \dots, w_m$ are distinct, such an index μ is unique and $\gamma_i = 0$ for any index i other than μ. Then $c_{\mu} = c$ and either $|w_i| < 1$ or $c_i = 0$ holds for every index $i \neq \mu$. $\square$

Lemma 6.8 *Let $r \in \mathbb{Z}$, $E > 0$ and p, q, α, $\beta \in \mathbb{R}$. If the sequence*

$$\sigma_k := (-1)^{rk} \cdot E^k \cdot \sin \pi(pk + \alpha) \cdot \sin \pi(qk + \beta) \text{ has a nonzero limit as } k \to \infty,$$

then $E = 1$ and there exists a dichotomy:

(A) *p, $q \in \mathbb{Z}$, $p + q - r$ must be even, α, $\beta \in \mathbb{R} \setminus \mathbb{Z}$,*

(B) *p, $q \in \frac{1}{2} + \mathbb{Z}$, $\alpha - (-1)^{p+q-r} \beta \in \frac{1}{2} + \mathbb{Z}$, α, $\beta \in \mathbb{R} \setminus \mathbb{Z}$.*

Proof. If we put $z_1 = E \, e^{\mathrm{i}\pi(p+q-r)}$, $z_2 = E \, e^{-\mathrm{i}\pi(p+q-r)}$, $z_3 = E \, e^{\mathrm{i}\pi(p-q-r)}$, $z_4 = E \, e^{-\mathrm{i}\pi(p-q-r)}$, $b_1 = -e^{\mathrm{i}\pi(\alpha+\beta)}$, $b_2 = -e^{-\mathrm{i}\pi(\alpha+\beta)}$, $b_3 = e^{\mathrm{i}\pi(\alpha-\beta)}$, $b_4 = e^{-\mathrm{i}\pi(\alpha-\beta)}$ with $\mathrm{i} := \sqrt{-1}$, then

$$4\sigma_k = b_1 \, z_1^k + b_2 \, z_2^k + b_3 \, z_3^k + b_4 \, z_4^k = c_1 \, w_1^k + \cdots + c_m \, w_m^k, \tag{65}$$

where $w_1, \dots, w_m$ are the mutually distinct members of z_1, z_2, z_3, z_4 and c_{μ} is the sum of those b_j's with $z_j = w_{\mu}$. Since sequence (65) satisfies condition (64) with $\delta = 0$, we have $z_j = 1$ for some $j \in \{1, 2, 3, 4\}$ by Lemma 6.7. Then $E = 1$ and $z_1 z_2 = z_3 z_4 = 1$, thus $z_1 = z_2 = 1$ or $z_3 = z_4 = 1$. First, if $z_1 = z_2 = z_3 = z_4 = 1$, then $p + q - r \in 2\mathbb{Z}$, $p - q - r \in 2\mathbb{Z}$ and $4\sigma_k = b_1 + b_2 + b_3 + b_4 = 4 \sin \pi\alpha \cdot \sin \pi\beta \neq 0$, which falls into case (A). Secondly, if $z_1 = z_2 = 1$ but $z_3 \neq 1$, then $4\sigma_k = (b_1 + b_2) + b_3 \, z_3^k + b_4 \, z_4^k$ with $b_3 b_4 \neq 0$ and $|z_3| = |z_4| = 1$, which together with Lemma 6.7 and $z_3 z_4 = 1$ forces $z_3 = z_4 = -1$ and $4\sigma_k = (b_1 + b_2) + (b_3 + b_4) \cdot (-1)^k$, so that $b_1 + b_2 = -2 \cos \pi(\alpha + \beta) \neq 0$ and $b_3 + b_4 = 2 \cos \pi(\alpha - \beta) = 0$, and hence $p + q - r \in 2\mathbb{Z}$, $p - q - r \in 2\mathbb{Z} + 1$, $\alpha + \beta \notin 1/2 + \mathbb{Z}$ and $\alpha - \beta \in 1/2 + \mathbb{Z}$, which falls into case (B). Finally, if $z_3 = z_4 = 1$ but $z_1 \neq 1$, then a similar reasoning shows that $p + q - r \in 2\mathbb{Z} + 1$, $p - q - r \in 2\mathbb{Z}$, $\alpha + \beta \in 1/2 + \mathbb{Z}$ and $\alpha - \beta \notin 1/2 + \mathbb{Z}$, which again falls into case (B). $\square$

Lemma 6.9 *Let $r \in \mathbb{Z}$, $E > 0$ and δ, p, $\alpha \in \mathbb{R}$. If the sequence*

$$\tau_k := (-1)^{rk} \cdot E^k \cdot k^{\delta} \cdot \sin \pi(pk + \alpha) \text{ has a nonzero limit as } k \to \infty,$$

then $E = 1$, $\delta = 0$, $p \in \mathbb{Z}$, $p - r$ must be even, and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

Proof. If we put $z_1 = E \, e^{\mathrm{i}\pi(p-r)}$, $z_2 = E \, e^{-\mathrm{i}\pi(p-r)}$, $b_1 = e^{\mathrm{i}\pi\alpha}$, $b_2 = -e^{-\mathrm{i}\pi\alpha}$, then $2\mathrm{i}\tau_k = k^{\delta}(b_1 \, z_1^k + b_2 \, z_2^k)$, where z_1 and z_2 may or may not be equal. In either case Lemma 6.7 implies that $\delta = 1$ and at least one of z_1 and z_2 must be 1, which in turn forces $E = 1$, $p - r \in 2\mathbb{Z}$ and hence $p \in \mathbb{Z}$, $z_1 = z_2 = 1$ and $2\mathrm{i}\tau_k = b_1 + b_2 = 2\mathrm{i} \sin \pi\alpha \neq 0$, that is, $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. $\square$

The author's original proofs of Lemmas 6.8 and 6.9 are more cumbersome. The present proofs based on Lemma 6.7 are due to Hiroyuki Ochiai and the anonymous referee, to whom the author is very grateful.

Theorem 6.10 *Let $\lambda = (p, q, r; a, b; x) \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ be a non-elementary solution and α_i, β_i $(i = 1, \dots, m)$ be the numbers defined in (53). Then r must be a positive integer and*

$$d = \frac{r^r}{\sqrt{p^p \cdot |q|^q \cdot (r-p)^{r-p} \cdot (r-q)^{r-q} \cdot x^r \cdot (1-x)^{p+q-r}}}, \tag{66}$$

where $|q|^q := 1$ when $q = 0$; this convention is reasonable since $|q|^q \to 1$ as $q \to 0$.

(1) *If $\lambda \in \mathcal{D}$ then there is a dichotomy:*

(A) *p, $q \in \mathbb{Z}$, $p + q - r$ must be even, α_i, $\beta_i \in \mathbb{R} \setminus \mathbb{Z}$ $(i = 1, \dots, m)$,*

(B) *p, $q \in \frac{1}{2} + \mathbb{Z}$, $\alpha_i - (-1)^{p+q-r} \beta_i \in \frac{1}{2} + \mathbb{Z}$, α_i, $\beta_i \in \mathbb{R} \setminus \mathbb{Z}$ $(i = 1, \dots, m)$.*

(2) *If $\lambda \in \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$ then $p \in \mathbb{Z}$, $p - r$ must be even, $\alpha_i \in \mathbb{R} \setminus \mathbb{Z}$ $(i = 1, \dots, m)$.*

(3) *If moreover $\lambda \in \mathcal{I}^{*-}$, that is, $q = 0$, then we must have $b = 1/2$.*

Proof. Applying a part of Lemma 6.8 or 6.9 to Proposition 6.6 yields $E_\nu = 1$ $(\nu = 7, 8, 9)$ in (62) and $b = 1/2$ in (62b). Combining $E_\nu = 1$ with (63), (60) and (55) in this order we observe that B^2 is equal to the square of the right-hand side of (66). Formula (66) then follows from $d = B$ in (44) and $B > 0$ in (31). The remaining assertions of the theorem are also obtained by applying Lemma 6.8 or 6.9 to Proposition 6.6. □

Remark 6.11 If $\lambda = (p, q, r; a, b; x) \in \mathcal{D}$ is a non-elementary (B)-solution, then its *duplication* $2\lambda := (2p, 2q, 2r; a, b; x)$ is an (A)-solution with $R(w; 2\lambda) = R(2w; \lambda) \cdot R(2w+1; \lambda)$.

7 Rational Independence

Finding when an integral solution comes from contiguous relations relies on the following.

Proposition 7.1 *If $\lambda \in \mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$ is a non-elementary solution, then $f(w; \lambda)$ and $\tilde{f}(w; \lambda)$ in (1) and (5) are linearly independent over the rational function field $\mathbb{C}(w)$.*

Proof. If γ is not an integer then the Gauss hypergeometric equation admits

$$u_1 := {}_2F_1(\alpha, \beta; \gamma; z), \quad u_2 := z^{1-\gamma} {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z)$$

as a fundamental set of local solutions around $z = 0$, whose Wronskian is given by

$$W := u_1 u_2' - u_1' u_2 = (1-\gamma) z^{-\gamma} (1-z)^{\gamma - \alpha - \beta - 1},$$

where $u' = du/dz$. From Erdélyi [9, Chapter II, §2.8, formulas (20) and (22)] we have

$$u_1' = \frac{\alpha\beta}{\gamma} {}_2F_1(\alpha+1, \beta+1; \gamma+1; z), \quad u_2' = (1-\gamma) z^{-\gamma} {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 1 - \gamma; z).$$

Substituting these into the Wronskian formula above we have

$$\begin{aligned} (1-z)^{\gamma-\alpha-\beta-1} &= {}_2F_1(\alpha, \beta; \gamma; z)\, {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 1 - \gamma; z) + \\ &\frac{\alpha\beta z}{\gamma(\gamma-1)} {}_2F_1(\alpha+1, \beta+1; \gamma+1; z)\, {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z). \end{aligned} \tag{67}$$

By symmetries we may assume $\lambda = (p, q, r; a, b; x) \in \mathcal{D} \cup \mathcal{I}^{*-} \cup \mathcal{E}^{*-}$. Theorem 5.4 and Proposition 5.7 then imply that $f(w; \lambda)$ has a GPF (43) such that $m \geq 1$ and $u_i - v_j \notin \mathbb{Z}$, $i, j = 1, \dots, m$. If $f(w; \lambda)$ and $\tilde{f}(w; \lambda)$ were linearly dependent over $\mathbb{C}(w)$, then there would be a rational function $T(w)$ such that $\tilde{f}(w; \lambda) = T(w) f(w; \lambda)$. Putting $(\alpha, \beta; \gamma; z) = (pw + a, qw + b; rw; x)$ into (67) yields $f(w; \lambda) f_1(w) = (1 - x)^{(r-p-q)w-a-b-1}$, where $f_1(w)$ is defined by

$$
\begin{aligned}
f_1(w) &:= {}_2F_1((p - r)w + a + 1,\ (q - r)w + b + 1;\ 1 - rw;\ x) \\
&\quad + \frac{(pw + a)(qw + b)x}{rw(rw - 1)} \cdot T(w) \cdot {}_2F_1((p - r)w + a + 1,\ (q - r)w + b + 1;\ 2 - rw;\ x).
\end{aligned}
$$

Take a number $R_5 < 0$ so that all poles of $T(w)$ and $S(w)$ are in the right half-plane $\mathrm{Re}(z) > R_5$, where $S(w)$ is the rational function in (43). Since r is positive, $f_1(w)$ is holomorphic on the left half-plane $\mathrm{Re}(z) < R_5$. Choose a positive integer j so that $-j - v_1 < R_5$. Then $f(w; \lambda)$ has a zero at $w = -j - v_1$ while $f_1(w)$ is holomorphic at this point. Therefore, $0 = f(-j - v_1; \lambda) f_1(-j - v_1) = (1 - x)^{-(r-p-q)(j+v_1)-a-b-1} \neq 0$, which is a contradiction. $\square$

Theorem 7.2 *Any non-elementary integral solution in $\mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$ comes from contiguous relations.*

Proof. For such a solution λ there exists a rational function $\widetilde{R}(w)$ such that $f(w + 1; \lambda) = \widetilde{R}(w) f(w; \lambda)$. Subtracting this from three-term relation (6) yields a linear relation $\{R(w; \lambda) - \widetilde{R}(w)\} f(w; \lambda) + Q(w; \lambda) \tilde{f}(w; \lambda) = 0$ over the field $\mathbb{C}(w)$. By Proposition 7.1 one must have $R(w; \lambda) - \widetilde{R}(w) = Q(w; \lambda) \equiv 0$ in $\mathbb{C}(w)$, so that three-term relation (6) reduces to a two-term one (3). Thus the solution λ comes from contiguous relations. $\square$

8 Contiguous Matrices

It is convenient to recast contiguous relations into a matrix form by putting

$$
\boldsymbol{F}(\boldsymbol{a}) := {}^t({}_2F_1(\boldsymbol{a}; z),\ {}_2F_1(\boldsymbol{a} + \boldsymbol{1}; z)), \qquad \boldsymbol{a} := (a_1, a_2, a_3) = (\alpha, \beta, \gamma), \quad \boldsymbol{1} := (1, 1, 1),
$$

and $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0)$, $\boldsymbol{e}_3 = (0, 0, 1)$. From formulas in Erdélyi [9, Chapter II, §2.8] we observe that the contiguous relation raising parameter a_i by one can be written

$$
\boldsymbol{F}(\boldsymbol{a} + \boldsymbol{e}_i) = A_i(\boldsymbol{a})\, \boldsymbol{F}(\boldsymbol{a}) \qquad (i = 1, 2, 3), \tag{68}
$$

where the matrix $A_i(\boldsymbol{a})$ is given in Table 3 together with its determinant $\det A_i(\boldsymbol{a})$. As the compatibility conditions for three relations (68) one has the commutation relations:

$$
A_i(\boldsymbol{a} + \boldsymbol{e}_j)\, A_j(\boldsymbol{a}) = A_j(\boldsymbol{a} + \boldsymbol{e}_i)\, A_i(\boldsymbol{a}) \qquad (i, j = 1, 2, 3). \tag{69}
$$

Given a lattice point $\boldsymbol{p} = (p_1, p_2, p_3) = (p, q, r) \in \mathbb{Z}^3_{\geq 0}$, a lattice path in $\mathbb{Z}^3_{\geq 0}$ from $\boldsymbol{0} = (0, 0, 0)$ to $\boldsymbol{p}$ can be represented by a sequence $i = (i_1, \dots, i_k)$ of indices $1, 2, 3$ such that $\boldsymbol{p} = \boldsymbol{e}_{i_1} + \cdots + \boldsymbol{e}_{i_k}$ where $k = p + q + r$. By compatibility conditions (69) the matrix product

$$
A(\boldsymbol{a}; \boldsymbol{p}) := A_{i_k}(\boldsymbol{a} + \boldsymbol{e}_{i_1} + \cdots + \boldsymbol{e}_{i_{k-1}}) \cdots A_{i_3}(\boldsymbol{a} + \boldsymbol{e}_{i_1} + \boldsymbol{e}_{i_2})\, A_{i_2}(\boldsymbol{a} + \boldsymbol{e}_{i_1})\, A_{i_1}(\boldsymbol{a})
$$

is independent of the path $i = (i_1, \dots, i_k)$, that is, depends only on the initial point $\boldsymbol{a}$ and the terminal point $\boldsymbol{a} + \boldsymbol{p}$. The matrix version of three-term relation (4) is given by

$$
\boldsymbol{F}(\boldsymbol{a} + \boldsymbol{p}) = A(\boldsymbol{a}; \boldsymbol{p})\, \boldsymbol{F}(\boldsymbol{a}). \tag{70}
$$

$$A_1(\boldsymbol{a}) := \begin{pmatrix} 1 & \frac{\beta z}{\gamma} \\ -\frac{\gamma}{(\alpha+1)(z-1)} & \frac{\gamma-\alpha-1-\beta z}{(\alpha+1)(z-1)} \end{pmatrix} \qquad \det A_1(\boldsymbol{a}) = \frac{\gamma-\alpha-1}{(\alpha+1)(z-1)}$$

$$A_2(\boldsymbol{a}) := \begin{pmatrix} 1 & \frac{\alpha z}{\gamma} \\ -\frac{\gamma}{(\beta+1)(z-1)} & \frac{\gamma-\beta-1-\alpha z}{(\beta+1)(z-1)} \end{pmatrix} \qquad \det A_2(\boldsymbol{a}) = \frac{\gamma-\beta-1}{(\beta+1)(z-1)}$$

$$A_3(\boldsymbol{a}) := \begin{pmatrix} \frac{\gamma(\gamma-\alpha-\beta)}{(\gamma-\alpha)(\gamma-\beta)} & -\frac{\alpha\beta(z-1)}{(\gamma-\alpha)(\gamma-\beta)} \\ \frac{\gamma(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)z} & \frac{\gamma(\gamma+1)(z-1)}{(\gamma-\alpha)(\gamma-\beta)z} \end{pmatrix} \qquad \det A_3(\boldsymbol{a}) = \frac{\gamma(\gamma+1)(z-1)}{(\gamma-\alpha)(\gamma-\beta)z}$$

Table 3: Contiguous matrices and their determinants.

Lemma 8.1 *If $1 \le p \le r$ and $1 \le q \le r$ then $A(\boldsymbol{a};\boldsymbol{p})$ admits a representation*

$$A(\boldsymbol{a};\boldsymbol{p}) = \frac{1}{(\gamma-\alpha)_{r-p}\,(\gamma-\beta)_{r-q}} \begin{pmatrix} \frac{(\gamma)_r\,\phi_{11}^{(r-2)}}{(\alpha+1)_{p-1}\,(\beta+1)_{q-1}} & \frac{(\gamma+1)_{r-1}\,\phi_{12}^{(r-1)}}{(\alpha+1)_{p-1}\,(\beta+1)_{q-1}} \\ \frac{(\gamma)_{r+1}\,\phi_{21}^{(r-1)}}{(\alpha+1)_{p}\,(\beta+1)_{q}} & \frac{(\gamma+1)_{r}\,\phi_{22}^{(r)}}{(\alpha+1)_{p}\,(\beta+1)_{q}} \end{pmatrix}, \tag{71}$$

where $\phi_{11}^{(-1)} = 0$ and $\phi_{ij}^{(k)} = \phi_{ij}^{(k)}(\boldsymbol{a};\boldsymbol{p})$ stands for a polynomial of degree at most k in $\boldsymbol{a} = (\alpha,\beta,\gamma)$ with coefficients in the ring $\mathbb{Z}[z^{\pm1}, 1/(z-1)]$. The determinant of $A(\boldsymbol{a};\boldsymbol{p})$ is given by

$$\det A(\boldsymbol{a};\boldsymbol{p}) = \frac{z^{-r}(z-1)^{r-p-q}\cdot(\gamma)_r\,(\gamma+1)_r}{(\alpha+1)_p\,(\beta+1)_q\,(\gamma-\alpha)_{r-p}\,(\gamma-\beta)_{r-q}}. \tag{72}$$

Proof. Formula (71) is proved by induction on r, where the main claim is the assertion about the degrees of $\phi_{ij}^{(r)}$, $i,j = 1,2$, in $\boldsymbol{a} = (\alpha,\beta,\gamma)$. A direct check shows that it is true for $r = 1$, that is, for $\boldsymbol{p} = \boldsymbol{1}$. Assuming the assertion is true for r we show it for $r+1$, that is, for $(p,q,r+1)$ with $1 \le p \le r+1$ and $1 \le q \le r+1$, where symmetry allows us to assume $p \le q$. There are three cases to deal with: (i) $p \le q \le r$; (ii) $p < q = r+1$; and (iii) $p = q = r+1$. In case (i) the relation $A(\boldsymbol{a};\boldsymbol{p}+\boldsymbol{e}_3) = A_3(\boldsymbol{a}+\boldsymbol{p})\,A(\boldsymbol{a};\boldsymbol{p})$ with $\boldsymbol{p} = (p,q,r)$ leads to the recurrence

$$\begin{pmatrix} \phi_{11}^{(r-1)} & \phi_{12}^{(r)} \\ \phi_{21}^{(r)} & \phi_{22}^{(r+1)} \end{pmatrix} = \begin{pmatrix} \gamma-\alpha-\beta+r-p-q & 1-z \\ \frac{(\alpha+p)(\beta+q)}{z} & \frac{(\gamma+r)(z-1)}{z} \end{pmatrix} \begin{pmatrix} \phi_{11}^{(r-2)} & \phi_{12}^{(r-1)} \\ \phi_{21}^{(r-1)} & \phi_{22}^{(r)} \end{pmatrix},$$

by which the assertion for $r+1$ follows from induction hypothesis. In case (ii) the relation $A(\boldsymbol{a};\boldsymbol{p}+\boldsymbol{e}_2+\boldsymbol{e}_3) = A_2(\boldsymbol{a}+\boldsymbol{p}+\boldsymbol{e}_3)\,A_3(\boldsymbol{a}+\boldsymbol{p})\,A(\boldsymbol{a};\boldsymbol{p})$ with $\boldsymbol{p} = (p,r,r)$ leads to

$$\begin{pmatrix} \phi_{11}^{(r-1)} & \phi_{12}^{(r)} \\ \phi_{21}^{(r)} & \phi_{22}^{(r+1)} \end{pmatrix} = \begin{pmatrix} \beta+r & z-1 \\ -\frac{(\alpha+p)(\beta+r)}{z} & \frac{\gamma+r-(\alpha+p)z}{z} \end{pmatrix} \begin{pmatrix} \phi_{11}^{(r-2)} & \phi_{12}^{(r-1)} \\ \phi_{21}^{(r-1)} & \phi_{22}^{(r)} \end{pmatrix},$$

by which the assertion follows from induction hypothesis. Finally, in case (iii) the relation $A(\boldsymbol{a};\boldsymbol{p}+\boldsymbol{1}) = A_1(\boldsymbol{a}+\boldsymbol{p}+\boldsymbol{e}_2+\boldsymbol{e}_3)\,A_2(\boldsymbol{a}+\boldsymbol{p}+\boldsymbol{e}_3)\,A_3(\boldsymbol{a}+\boldsymbol{p})\,A(\boldsymbol{a};\boldsymbol{p})$ with $\boldsymbol{p} = (r,r,r)$ yields

$$\begin{pmatrix} \phi_{11}^{(r-1)} & \phi_{12}^{(r)} \\ \phi_{21}^{(r)} & \phi_{22}^{(r+1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{(\alpha+r)(\beta+r)}{z(1-z)} & \frac{\gamma+r-(\alpha+\beta+2r+1)z}{z(z-1)} \end{pmatrix} \begin{pmatrix} \phi_{11}^{(r-2)} & \phi_{12}^{(r-1)} \\ \phi_{21}^{(r-1)} & \phi_{22}^{(r)} \end{pmatrix},$$

by which the assertion follows and the induction completes itself.

Determinant formula (72) is obtained by taking the determinant of matrix products

$$\begin{aligned} A(\boldsymbol{a};\boldsymbol{p}) = {} & A_1(\alpha+p-1,\beta+q;\gamma+r)\cdots A_1(\alpha+1,\beta+q;\gamma+r)\,A_1(\alpha,\beta+q;\gamma+r) \\ & \cdot A_2(\alpha,\beta+q-1;\gamma+r)\cdots A_2(\alpha,\beta+1;\gamma+r)\,A_2(\alpha,\beta;\gamma+r) \\ & \cdot A_3(\alpha,\beta;\gamma+r-1)\cdots A_3(\alpha,\beta;\gamma+1)\,A_3(\alpha,\beta;\gamma), \end{aligned} \tag{73}$$

and by using determinant formulas in Table 3. □

Formula (70) leads to a matrix version of three-term relation (6):

$$\boldsymbol{f}(w+1;\lambda) = A(w;\lambda)\boldsymbol{f}(w;\lambda), \qquad \boldsymbol{f}(w;\lambda) := {}^t(f(w;\lambda), \tilde{f}(w;\lambda)), \tag{74}$$

for an integral data $\lambda = (p,q,r;a,b;x)$, where $A(w;\lambda)$ is the matrix $A(\boldsymbol{a};\boldsymbol{p})$ evaluated at $\boldsymbol{a} = \boldsymbol{\alpha}(w) := (pw+a, qw+b; rw)$ and $z = x$.

Lemma 8.2 *If $1 \le p \le r$ and $1 \le q \le r$ then $A(w;\lambda)$ admits a representation*

$$\begin{aligned} A(w;\lambda) = {} & \frac{1}{((r-p)w-a)_{r-p}\,((r-q)w-b)_{r-q}} \\ & \times \begin{pmatrix} \frac{(rw)_r\,\phi_{11}^{(r-2)}(w)}{(pw+a+1)_{p-1}\,(qw+b+1)_{q-1}} & \frac{(rw+1)_{r-1}\,\phi_{12}^{(r-1)}(w)}{(pw+a+1)_{p-1}\,(qw+b+1)_{q-1}} \\ \frac{(rw)_{r+1}\,\phi_{21}^{(r-1)}(w)}{(pw+a+1)_p\,(qw+b+1)_q} & \frac{(rw+1)_r\,\phi_{22}^{(r)}(w)}{(pw+a+1)_p\,(qw+b+1)_q} \end{pmatrix}, \end{aligned} \tag{75}$$

where $\phi_{11}^{(-1)}(w) = 0$ and $\phi_{ij}^{(k)}(w)$ is a polynomial of degree at most k in w. Moreover,

$$\det A(w;\lambda) = \frac{x^{-r}(x-1)^{r-p-q}\cdot(rw)_r\,(rw+1)_r}{(pw+a+1)_p\,(qw+b+1)_q\,((r-p)w-a)_{r-p}\,((r-q)w-b)_{r-q}}. \tag{76}$$

Proof. Substitute $\boldsymbol{a} = \boldsymbol{\alpha}(w)$ and $z = x$ into (71) and (72). □

Remark 8.3 Notice that $R(w;\lambda) = A_{11}(w;\lambda)$ and $Q(w;\lambda) = A_{12}(w;\lambda)$ in formula (6), where $A_{ij}(w;\lambda)$ is the (i,j)-th entry of $A(w;\lambda)$. Thus an integral solution $\lambda = (p,q,r;a,b;x)$ comes from contiguous relations exactly when $A_{12}(w;\lambda)$ or equivalently $\phi_{12}^{(r-1)}(w)$ vanishes in $\mathbb{C}(w)$. If this is the case, taking the determinant of (75) and comparing the result with (76) we find

$$\phi_{11}^{(r-2)}(w)\cdot\phi_{22}^{(r)}(w) = x^{-r}(x-1)^{r-p-q}\cdot(pw+a+1)_{p-1}(qw+b+1)_{q-1}((r-p)w-a)_{r-p}((r-q)w-b)_{r-q}. \tag{77}$$

This implies $\deg\phi_{11}^{(r-2)} = r-2$ and $\deg\phi_{22}^{(r)} = r$, since $\deg\phi_{11}^{(r-2)} \le r-2$ and $\deg\phi_{22}^{(r)} \le r$ while the right side of (77) is of degree $2r-2$. Using (77) in $R(w;\lambda) = A_{11}(w;\lambda)$ yields

$$R(w;\lambda) = x^{-r}(x-1)^{r-p-q}\cdot\frac{(rw)_r}{\phi_{22}^{(r)}(w)}. \tag{78}$$

9 Principal Parts of Contiguous Matrices

Given an integral data $\lambda = (p, q, r; a, b; x)$, the *principal part* of $A_i(w; \lambda)$ is defined by

$$B_i = B_i(\lambda) := \lim_{w \to \infty} A_i(w; \lambda) \qquad (i = 1, 2, 3).$$

A little calculation using Table 3 shows that

$$B_1 = \begin{pmatrix} 1 & \frac{qx}{r} \\ -\frac{r}{p(x-1)} & \frac{r-p-qx}{p(x-1)} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & \frac{px}{r} \\ -\frac{r}{q(x-1)} & \frac{r-q-px}{q(x-1)} \end{pmatrix}, \quad B_3 = \begin{pmatrix} \frac{r(r-p-q)}{(r-p)(r-q)} & -\frac{pq(x-1)}{(r-p)(r-q)} \\ \frac{r^2}{(r-p)(r-q)x} & \frac{r^2(x-1)}{(r-p)(r-q)x} \end{pmatrix}.$$

Compatibility condition (69) implies that B_1, B_2, B_3 are commutative to each other.

Lemma 9.1 *In formula* (75) *we have*

$$\lim_{w \to \infty} A(w; \lambda) = B_1^p B_2^q B_3^r = c \begin{pmatrix} X(x) - \{r - (p+q)x\} Y(x) & 2(pq/r)\, x(x-1)\, Y(x) \\ -2r\, Y(x) & X(x) + \{r - (p+q)x\} Y(x) \end{pmatrix}, \qquad (79)$$

where $X(z)$, $Y(z) \in \mathbb{Z}[z]$ are the polynomials defined in (14) *and c is the constant*

$$c := \frac{r^r}{2^r p^p q^q (r-p)^{r-p} (r-q)^{r-q} x^r}.$$

Proof. Putting $\boldsymbol{a} = \boldsymbol{\alpha}(w)$ and $z = x$ into (73) and letting $w \to \infty$, we have $\lim_{w \to \infty} A(w; \lambda) = B_1^p B_2^q B_3^r$ by the commutativity of B_1, B_2, B_3. From the explicit formulas for B_i,

$$B_1 B_3 = \begin{pmatrix} \frac{r}{r-q} & \frac{q(x-1)}{r-q} \\ -\frac{r^2}{p(r-q)x} & \frac{r(r-qz)}{p(r-q)x} \end{pmatrix}, \qquad B_2 B_3 = \begin{pmatrix} \frac{r}{r-p} & \frac{p(x-1)}{r-p} \\ -\frac{r^2}{q(r-p)x} & \frac{r(r-px)}{q(r-p)x} \end{pmatrix}.$$

Observe that $B_1 B_3$, $B_2 B_3$ and B_3 are simultaneously diagonalized as

$$T^{-1}(B_1 B_3) T = \frac{r}{2p(r-q)x} \cdot \mathrm{diag}\left\{ r + (p-q)x + \sqrt{\mathit{\Delta}},\ r + (p-q)x - \sqrt{\mathit{\Delta}} \right\},$$
$$T^{-1}(B_2 B_3) T = \frac{r}{2q(r-p)x} \cdot \mathrm{diag}\left\{ r - (p-q)x + \sqrt{\mathit{\Delta}},\ r - (p-q)x - \sqrt{\mathit{\Delta}} \right\},$$
$$T^{-1} B_3 T = \frac{r}{2(r-p)(r-q)x} \cdot \mathrm{diag}\left\{ (2r-p-q)x - r - \sqrt{\mathit{\Delta}},\ (2r-p-q)x - r + \sqrt{\mathit{\Delta}} \right\},$$

where $\mathit{\Delta} = \mathit{\Delta}(x)$ is the quadratic polynomial (15) (evaluated at $z = x$) and

$$T = \begin{pmatrix} \frac{r-(p+q)x-\sqrt{\mathit{\Delta}}}{2r} & \frac{r-(p+q)x+\sqrt{\mathit{\Delta}}}{2r} \\ 1 & 1 \end{pmatrix}.$$

In view of (14) and (16) the matrix $B_1^p B_2^q B_3^r = (B_1 B_3)^p (B_2 B_3)^q B_3^{r-p-q}$ is diagonalized as

$$T^{-1}(B_1^p B_2^q B_3^r) T = c \cdot \mathrm{diag}\left\{ X(x) + Y(x)\sqrt{\mathit{\Delta}},\ X(x) - Y(x)\sqrt{\mathit{\Delta}} \right\}.$$

Then (79) follows from $B_1^p B_2^q B_3^r = c \cdot T \cdot \mathrm{diag}\left\{ X(x) + Y(x)\sqrt{\mathit{\Delta}}, X(x) - Y(x)\sqrt{\mathit{\Delta}} \right\} \cdot T^{-1}$. □

Lemma 9.2 *Let $p,\ q,\ r \in \mathbb{Z}$, $p > 0$, $q > 0$ and $r - p - q \ge 0$. Then $Y(z)$ is a nontrivial polynomial of degree at most $r-1$ resp. $r-p-1$ if $p \neq q$ resp. $p = q$.*

(1) *If $r - p - q = 0$ then $Y(z)$ has no root in $0 \le z < 1$.*

(2) *If $r - p - q$ is a positive even integer then $Y(z)$ has at least one root in $0 < z < 1$.*

Proof. We have $\Delta(0) = r^2 > 0$ and $\Delta(1) = (r-p-q)^2 \ge 0$. If $p \neq q$ then $\Delta(z)$ is a quadratic polynomial with axis of symmetry at $z = 1 + (p-q)^{-2}\{p(r-p) + q(r-q)\} > 1$. If $p = q$ then $\Delta(z)$ is an affine polynomial with slope $-4p(r-p) < 0$. In either case $\Delta = \Delta(z)$ is strictly decreasing and positive in $0 \le z < 1$. We take the branch of $\sqrt{\Delta}$ so that $\sqrt{\Delta} > 0$. Since $Z_+(0) = (-1)^{r-p-q}(2r)^r$, $Z_-(0) = 0$ and hence $Y(0) = (-1)^{r-p-q}(2r)^{r-1} \neq 0$, the polynomial $Y(z)$ is nontrivial. The assertion for the degree of $Y(z)$ is easy to see.

To prove assertion (1) we assume $p + q = r$. Let $0 \le z < 1$ and put $s := p - q \ge 0$. Formulas (15) and (16) yield $\Delta = (1-z)(r^2 - s^2 z) > 0$ and $Z_\pm(z) = (r + sz \pm \sqrt{\Delta})^p (r - sz \pm \sqrt{\Delta})^q$. Since $(r - sz)^2 - \Delta = 4q^2 z \ge 0$, we have $r + sz \ge r - sz \ge \sqrt{\Delta} > 0$, so $r + sz + \sqrt{\Delta} > r + sz - \sqrt{\Delta} \ge 0$ and $r - sz + \sqrt{\Delta} > r - sz - \sqrt{\Delta} \ge 0$. Thus formula (14) yields

$$2\,Y(z)\,\sqrt{\Delta} = (r + sz + \sqrt{\Delta})^p (r - sz + \sqrt{\Delta})^q - (r + sz - \sqrt{\Delta})^p (r - sz - \sqrt{\Delta})^q > 0,$$

which implies $Y(z) > 0$. Therefore $Y(z)$ has no root in $0 \le z < 1$.

To show assertion (2) we assume $r - p - q > 0$. Since $\sqrt{\Delta(0)} = r > 0$, formula (16) gives $Z_+(0) = (2r)^p (2r)^q (-2r)^{r-p-q} = (-1)^{r-p-q}(2r)^r$ and $Z_-(0) = 0^p \cdot 0^q \cdot 0^{r-p-q} = 0$, which are valid even if $r - p - q = 0$. Similarly, since $\sqrt{\Delta(1)} = r - p - q > 0$, formula (16) yields $Z_+(1) = \{2(r-q)\}^p \{2(r-p)\}^q \cdot 0^{r-p-q} = 0$ and $Z_-(1) = (2p)^p (2q)^q \{2(r-p-q)\}^{r-p-q} = 2^r p^p q^q\,(r-p-q)^{r-p-q}$. Thus it follows from (14) that

$$Y(0) = (-1)^{r-p-q}\,(2r)^{r-1}, \qquad Y(1) = -2^{r-1} p^p q^q\,(r-p-q)^{r-p-q-1} < 0.$$

Accordingly, if $r - p - q$ is positive and even, then $Y(0) > 0$ and $Y(1) < 0$, so $Y(z)$ has at least one root in the interval $0 < z < 1$ by the intermediate value theorem. □

Theorem 9.3 *Let $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^- \cup \mathcal{D}^0$ be a non-elementary integral solution. Then x must be a root in $0 < z < 1$ of the algebraic equation $Y(z; \boldsymbol{p}) = 0$ and hence be an algebraic number of degree at most $r-1$ resp. $r-p-1$ if $p \neq q$ resp. $p = q$. Moreover $r - p - q$ must be positive and even. Any solution on $\mathcal{D}^0$, integral or not, must be elementary.*

Proof. Any non-elementary integral solution $\lambda \in \mathcal{D}^- \cup \mathcal{D}^0$ comes from contiguous relations by Theorem 7.2, so $A_{12}(w; \lambda) \equiv 0$ by Remark 8.3. Thus formula (79) yields $0 = \lim_{w \to \infty} A_{12}(w; \lambda) = 2(pq/r)x(x-1)Y(x)$, namely, $Y(x) = 0$ and hence x must be algebraic. The degree bound for x comes from the corresponding statement in Lemma 9.2. Note that $r - p - q$ must be even by Theorem 6.10 and nonnegative by assumption $\lambda \in \mathcal{D}^- \cup \mathcal{D}^0$. Assertion (1) of Lemma 9.2 then rules out the possibility $r - p - q = 0$, which means that there is no non-elementary, integral i.e. (A)-solution on $\mathcal{D}^0$. By Remark 6.11 there is no non-elementary (B)-solution either. □

Remark 9.4 The degree bound for x in Theorem 9.3 is by no means optimal. In fact x is either rational or quadratic for any solution known to the author. As a root of $Y(z; \boldsymbol{p})$ the number x depends only on $\boldsymbol{p} = (p, q, r)$ and so does d by formula (66).

10 Truncated Hypergeometric Products

Emphasizing the dependence on $\boldsymbol{p} = (p, q, r)$ we rewrite three-term relation (4) as

$$ {}_2F_1(\boldsymbol{a} + \boldsymbol{p}; z) = r(\boldsymbol{a}; \boldsymbol{p}; z)\, {}_2F_1(\boldsymbol{a}; z) + q(\boldsymbol{a}; \boldsymbol{p}; z)\, {}_2F_1(\boldsymbol{a} + \boldsymbol{1}; z). \tag{80} $$

Then the matrix $A(\boldsymbol{a}; \boldsymbol{p})$ in formula (70) can be represented as

$$ A(\boldsymbol{a}; \boldsymbol{p}) = \begin{pmatrix} r(\boldsymbol{a}; \boldsymbol{p}; z) & q(\boldsymbol{a}; \boldsymbol{p}; z) \\ r(\boldsymbol{a}; \boldsymbol{p} + \boldsymbol{1}; z) & q(\boldsymbol{a}; \boldsymbol{p} + \boldsymbol{1}; z) \end{pmatrix}. $$

Ebisu [5] made an extensive study of three-term relation (80), whose results enable us to express $q(\boldsymbol{a}; \boldsymbol{p})$ and $r(\boldsymbol{a}; \boldsymbol{p})$ in terms of truncated hypergeometric products. For a nonnegative integer τ we denote by $\langle\, \varphi(z) \,\rangle_\tau := \sum_{j=0}^{\tau} c_j z^j$ the *truncation* at degree τ of a power series $\varphi(z) = \sum_{j=0}^{\infty} c_j z^j$. In what follows truncation is always taken with respect to variable z.

Lemma 10.1 *Let $\boldsymbol{p} = (p, q, r) \in \mathbb{Z}^3$. If $0 \le q \le p$ and $p + q \le r$ then*

$$ q(\boldsymbol{a}; \boldsymbol{p}; z) = z^{1-r}(z-1) \cdot C(\boldsymbol{a}; \boldsymbol{p}) \cdot \langle {}_2F_1(\boldsymbol{a}^*; z) \cdot {}_2F_1(\boldsymbol{v} - \boldsymbol{a}^* - \boldsymbol{p}^*; z) \rangle_{r-q-1}, \tag{81} $$

while if $-1 \le q \le p$ and $p + q \le r - 1$ then

$$ q(\boldsymbol{a}; \boldsymbol{p} + \boldsymbol{1}; z) = z^{-r}(z-1) \cdot C(\boldsymbol{a}; \boldsymbol{p} + \boldsymbol{1}) \cdot \langle {}_2F_1(\boldsymbol{a}^*; z) \cdot {}_2F_1(\boldsymbol{1} - \boldsymbol{a}^* - \boldsymbol{p}^*; z) \rangle_{r-q-1}, \tag{82} $$

where $\boldsymbol{a}^* := (\gamma - \alpha, \gamma - \beta; \gamma)$, $\boldsymbol{p}^* := (r - p, r - q; r)$, $\boldsymbol{v} := (1, 1; 2)$ and

$$ C(\boldsymbol{a}; \boldsymbol{p}) := (-1)^{r-p-q} \, \frac{(\gamma)_{r-1}\, (\gamma+1)_{r-1}}{(\alpha+1)_{p-1}(\beta+1)_{q-1}(\gamma-\alpha)_{r-p}(\gamma-\beta)_{r-q}}. $$

Proof. By case (ii) of Ebisu [5, Proposition 3.4, Theorem 3.7, Remark 3.11], if $p \ge q$ and $r \ge \max\{p+q, 0\}$ then $q(\boldsymbol{a}; \boldsymbol{p}; z) = z^{1-r}(z-1)\, q_0(\boldsymbol{a}; \boldsymbol{p}; z)$, where $q_0(\boldsymbol{a}; \boldsymbol{p}; z)$ is a polynomial of degree at most $r - q - 1$ in z and is explicitly given by

$$ q_0(\boldsymbol{a}; \boldsymbol{p}; z) = C(\boldsymbol{a}; \boldsymbol{p})\, {}_2F_1(\boldsymbol{a}^*; z)\, {}_2F_1(\boldsymbol{v} - \boldsymbol{a}^* - \boldsymbol{p}^*; z) - \frac{\alpha\beta}{\gamma(\gamma-1)}\, z^r\, {}_2F_1(\boldsymbol{v} - \boldsymbol{a}; z)\, {}_2F_1(\boldsymbol{a} + \boldsymbol{p}; z). $$

In particular, if $p \ge q \ge 0$ and $p + q \le r$ then we have $r - q - 1 < r$ so that

$$ q_0(\boldsymbol{a}; \boldsymbol{p}; z) = C(\boldsymbol{a}; \boldsymbol{p})\, \langle {}_2F_1(\boldsymbol{a}^*; z) \cdot {}_2F_1(\boldsymbol{v} - \boldsymbol{a}^* - \boldsymbol{p}^*; z) \rangle_{r-q-1}, \tag{83} $$

which yields (81). Formula (82) follows from the formula (83) with $\boldsymbol{p}$ replaced by $\boldsymbol{p} + \boldsymbol{1}$, which remains true provided $p + 1 \ge q + 1 \ge 0$ and $(p+1) + (q+1) \le r + 1$, that is, $p \ge q \ge -1$ and $p + q \le r - 1$. Here we used the equality $\boldsymbol{v} - \boldsymbol{a}^* - (\boldsymbol{p} + \boldsymbol{1})^* = \boldsymbol{1} - \boldsymbol{a}^* - \boldsymbol{p}^*$. □

Given an integral data $\lambda = (p, q, r; a, b; x)$ with $p \ge 0$, $q \ge 0$ and $p + q \le r$, we define

$$ \begin{aligned} V(w; \lambda) &:= (rw)_{r-1} \, \langle\, {}_2F_1(\boldsymbol{\alpha}^*(w); z) \cdot {}_2F_1(\boldsymbol{v} - \boldsymbol{\alpha}^*(w+1); z) \,\rangle_\tau \big|_{z=x} \\ &= (rw)_{r-1} \, \langle\, (1-z)^{r-p-q} \cdot {}_2F_1(\boldsymbol{\alpha}(w); z) \cdot {}_2F_1(\boldsymbol{v} - \boldsymbol{\alpha}(w+1); z) \,\rangle_\tau \big|_{z=x}, \end{aligned} \tag{84} $$

$$ \begin{aligned} P(w; \lambda) &:= (rw)_{r} \, \langle\, {}_2F_1(\boldsymbol{\alpha}^*(w); z) \cdot {}_2F_1(\boldsymbol{1} - \boldsymbol{\alpha}^*(w+1); z) \,\rangle_\tau \big|_{z=x} \\ &= (rw)_{r} \, \langle\, (1-z)^{r-p-q-1} \cdot {}_2F_1(\boldsymbol{\alpha}(w); z) \cdot {}_2F_1(\boldsymbol{e}_3 - \boldsymbol{\alpha}(w+1); z) \,\rangle_\tau \big|_{z=x}, \end{aligned} \tag{85} $$

where $\boldsymbol{\alpha}(w) := (pw+a,\, qw+b;\, rw)$, $\boldsymbol{\alpha}^*(w) := ((r-p)w-a,\, (r-q)w-b;\, rw)$, $\boldsymbol{v} := (1,1,2)$,

$$\tau := \max\{r-p-1,\, r-q-1\}, \tag{86}$$

and the second equalities in (84) and (85) are due to Euler's transformation (7b). An inspection shows that $V(w;\lambda)$ and $P(w;\lambda)$ are polynomials over $\mathbb{Q}$ in $(w;\lambda) = (w;p,q,r;a,b;x)$ with degrees at most $2r-\min\{p,q\}-2$ and $2r-\min\{p,q\}-1$ in w respectively. By trivial symmetry we may suppose $q \le p$ and hence $\tau = r-q-1$ in (86).

Lemma 10.2 *Let $\phi_{ij}^{(r)}(w)$ be as in (75). If $\boldsymbol{p} = (p,q,r) \in \mathbb{Z}^3$ satisfies $0 \le q \le p$, $p+q \le r$, then*

$$\phi_{12}^{(r-1)}(w) = (-1)^{r-p-q} \cdot x^{1-r}(x-1) \cdot V(w;\lambda), \tag{87}$$

in particular $V(w,\lambda)$ is of degree at most $r-1$ in w. If moreover $\boldsymbol{p}$ satisfies $p+q \le r-1$, then

$$\phi_{22}^{(r)}(w) = (-1)^{r-p-q-1} \cdot x^{-r}(x-1) \cdot P(w;\lambda), \tag{88}$$

in particular $P(w;\lambda)$ is of degree at most r in w.

Proof. Substituting $\boldsymbol{a} = \boldsymbol{\alpha}(w) := (pw+a, qw+b; rw)$ and $z = x$ into formula (81) we find

$$A_{12}(w;\lambda) = \frac{(-1)^{r-p-q} \cdot x^{1-r}(x-1) \cdot (rw+1)_{r-1} \cdot V(w;\lambda)}{(pw+a+1)_{p-1}(qw+b+1)_{q-1}((r-p)w-a)_{r-p}((r-q)w-b)_{r-q}}. \tag{89}$$

Comparing this with (75) yields formula (87), which shows that $V(w;\lambda)$ is of degree at most $r-1$ in w since so is $\phi_{12}^{(r-1)}(w)$ by Lemma 8.2. Next, formula (82) is compared with (75) to yield formula (88), from which $\deg_w P(w;\lambda) \le r$ also follows. □

Since $V(w;\lambda)$ is of degree at most $r-1$ in w, we can write

$$V(w;\lambda) = \sum_{j=0}^{r-1} V_j(\lambda) w^k, \qquad V_j(\lambda) \in \mathbb{Q}[\lambda]. \tag{90}$$

Theorem 10.3 *Let $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ be an integral data.*

(1) *λ is a solution coming from contiguous relations if and only if $V(w;\lambda)$ vanishes as a polynomial of w, that is, if and only if λ is a simultaneous root of algebraic equations:*

$$V_j(\lambda) = 0 \qquad (j = 0,1,\dots,r-1), \tag{91}$$

where the last equation $V_{r-1}(\lambda) = 0$ is equivalent to $Y(x;\boldsymbol{p}) = 0$ in Theorem 9.3.

(2) *If λ is a non-elementary solution then the polynomial $P(w;\lambda)$ is exactly of degree r and must satisfy the division relation in $\mathbb{C}[w]$,*

$$P(w;\lambda) \,\big|\, (pw+a+1)_{p-1}(qw+b+1)_{q-1}((r-p)w-a)_{r-p}((r-q)w-b)_{r-q}, \tag{92}$$

and the rational function $R(w;\lambda)$ is given by

$$R(w;\lambda) = (1-x)^{r-p-q-1} \cdot \frac{(rw)_r}{P(w;\lambda)}. \tag{93}$$

Proof. Assertion (1). By Remark 8.3, λ is a solution coming from contiguous relations if and only if $\phi_{12}^{(r-1)}(w) \equiv 0$ in $\mathbb{C}(w)$. By formula (87) this is equivalent to $V(w;\lambda) \equiv 0$ in $\mathbb{C}(w)$, which in turn is equivalent to the system (91) in view of (90). Formulas (79) and (89) give

$$\lim_{w\to\infty} A_{12}(w;\lambda) = 2c\,(pq/r)\,x(x-1)\,Y(x) = \frac{(-1)^{r-p-q}\cdot x^{1-r}(x-1)\cdot r^{r-1}\cdot V_{r-1}(\lambda)}{p^{p-1}q^{q-1}(r-p)^{r-p}(r-q)^{r-q}},$$

which yields $Y(x) = (-1)^{r-p-q}2^{r-1}V_{r-1}(\lambda)$. Thus $Y(x) = 0$ is equivalent to $V_{r-1}(\lambda) = 0$.

Assertion (2). Since λ comes from contiguous relations by Theorem 7.2, $\phi_{22}^{(r)}(w)$ is exactly of degree r by Remark 8.3 and so is $P(w;\lambda)$ by formula (88). Division relation (92) follows from (77) and (88). Formula (93) is then obtained by substituting (88) into (78). □

Remark 10.4 A few comments on Theorem 10.3 should be in order at this stage.

(1) If the principal part $\boldsymbol{p} = (p,q,r)$ of a data λ is a priori given then condition (91) gives an *overdetermined system* of algebraic equations over $\mathbb{Q}$ for an unknown $(a,b;x)$.

(2) Let $P(w;\lambda) = \text{const.}\,(w+v_1)\cdots(w+v_r)$, $S(w)$ a constant and $u_i = (i-1)/r$, $i = 1,\dots,r$. Applying Theorem 5.4 to (93) then yields GPF (18) in Theorem 2.3, where the last equality in (44) implies (20) while division relation (92) gives (21). Exactly r among all the $2r-2$ factors on the right side of (21) appear as $w+v_1,\dots,w+v_r$. It is yet to be decided which r should appear. This question seems hard in general, but it has something to do with certain *terminating* hypergeometric sums in §11 and at least

$$w + \tfrac{r-p-1-a}{r-p}; \quad w + \tfrac{r-q-1-b}{r-q}; \quad w + \tfrac{a+p-1}{p} \quad (\text{if } p \ge 2); \quad w + \tfrac{b+q-1}{q} \quad (\text{if } q \ge 2), \qquad (94)$$

must be among $w+v_1,\dots,w+v_r$ (see Remark 11.4). In any case division relation (21) or equivalently (92) provides us with some information about the numbers $v_1,\dots,v_r$. In particular they must be real (rational if so are a and b).

(3) It often occurs in formula (93) that the numerator $(rw)_r$ and denominator $P(w;\lambda)$ have some factors in common which can be canceled to have a reduced expression. Or rather the author knows no example for which such a cancellation does not occur.

11 Terminating Hypergeometric Sums

The content of this section is rather technical, but has a useful application in a forthcoming paper [14]. We show that condition (91) leads to an algebraic system involving *terminating* hypergeometric sums. To this end we employ a *renormalized* hypergeometric polynomial

$$\mathcal{F}_k(\beta;\gamma;z) := (\gamma)_k\cdot {}_2F_1(-k,\beta;\gamma;z) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(\beta)_j(\gamma+j)_{k-j}\,z^j \quad (k\in\mathbb{Z}_{\ge 0}), \qquad (95)$$

which is a polynomial of three variables $(\beta;\gamma;z)$. By evaluating $V(w;\lambda)$ in (84) at

$$w_k^* := \frac{a-k}{r-p} \quad (0\le k\le r-p-1); \qquad w_j := -\frac{a+j}{p} \quad (0\le j\le p-1), \qquad (96)$$

where k and j are integers, the assertion (1) of Theorem 10.3 yields the following.

Proposition 11.1 *The system* (91) *leads to a total of* r *algebraic equations for* $(a,b;x)$,

$$(\gamma_k^*+k)_p\cdot\mathcal{F}_k(\beta_k^*;\gamma_k^*;x)\cdot\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*;x)=0 \qquad (0\le k\le r-p-1), \tag{97a}$$

$$(\gamma_j+j)_{r-p}\cdot\mathcal{F}_j(\beta_j;\gamma_j;x)\cdot\mathcal{F}_{p-1-j}(\tilde{\beta}_j+1;\tilde{\gamma}_j+1;x)=0 \qquad (0\le j\le p-1), \tag{97b}$$

each of which consists of a factorial and two terminating hypergeometric factors, where

$$\beta_k^*:=(r-q)w_k^*-b,\quad \gamma_k^*:=rw_k^*,\quad \tilde{\beta}_k^*:=1-(r-q)(w_k^*+1)+b,\quad \tilde{\gamma}_k^*:=2-r(w_k^*+1)$$

$$\beta_j:=qw_j+b,\quad \gamma_j:=rw_j,\quad \tilde{\beta}_j:=-q(w_j+1)-b,\quad \tilde{\gamma}_j:=1-r(w_j+1).$$

Moreover, the system (97) *leads back to and hence is equivalent to the original system* (91), *if*

$$a\ne\frac{p_1}{r_p}(k+j)-j \quad (0\le {}^\forall k\le r-p-1,\ 0\le {}^\forall j\le p-1), \tag{98}$$

in particular, if $r_pa\notin\mathbb{Z}$ *with* p_1/r_p *being the reduced expression of* $p/r\in\mathbb{Q}$.

Proof. If we substitute $w=w_k^*$ in the first formula of definition (84), then the two hypergeometric series inside the bracket $\langle\cdots\rangle_\tau$ terminate at degrees k and $r-p-1-k$ in z, so their product is of degree at most $k+(r-p-1-k)=r-p-1\le\tau$ by (86). Thus $V(w;\lambda)$ can be evaluated at $w=w_k^*$ without taking truncation. A bit of calculation shows

$$V(w_k^*;\lambda)=(-1)^{r-p-1-k}\cdot(\gamma_k^*+k)_p\cdot\mathcal{F}_k(\beta_k^*;\gamma_k^*;x)\cdot\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*;x).$$

Thus the vanishing $V(w;\lambda)\equiv 0$ yields the $r-p$ equations in (97a). Similarly, if we substitute $w=w_j$ in the second formula of (84) then the two hypergeometric series inside the bracket $\langle\cdots\rangle_\tau$ terminate at degrees j and $p-1-j$ in z, so $(1-z)^{r-p-q}$ times their product is of degree at most $(r-p-q)+j+(p-1-j)=r-q-1\le\tau$. Thus $V(w;\lambda)$ can also be evaluated at $w=w_j$ without taking truncation. After some calculations,

$$V(w_j;\lambda)=(-1)^{p-1-j}\cdot(1-x)^{r-p-q}\cdot(\gamma_j+j)_{r-p}\cdot\mathcal{F}_j(\beta_j;\gamma_j;x)\cdot\mathcal{F}_{p-1-j}(\tilde{\beta}_j+1;\tilde{\gamma}_j+1;x),$$

which together with the vanishing $V(w_j;\lambda)\equiv 0$ leads to the p equations in formula (97b). Note that (98) is the condition that any pair w_k^*, w_j in (96) should be distinct. If this is the case then equations (97) imply that $V(w;\lambda)$, which is a polynomial of degree at most $r-1$ in w, vanishes at distinct r points and hence vanishes identically. This gives equations (91). □

As in the proof of Proposition 11.1 the polynomial $P(w;\lambda)$ in (85) can be evaluated as

$$P(w_k^*;\lambda)=(-1)^{r-p-1-k}\cdot(\gamma_k^*+k)_{p+1}\cdot\mathcal{F}_k(\beta_k^*;\gamma_k^*;x)\cdot\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*-1;x), \tag{99a}$$

$$P(w_j;\lambda)=(-1)^{p-j}\cdot(1-x)^{r-p-q-1}\cdot(\gamma_j+j)_{r-p}\cdot\mathcal{F}_j(\beta_j;\gamma_j;x)\cdot\mathcal{F}_{p-j}(\tilde{\beta}_j;\tilde{\gamma}_j;x). \tag{99b}$$

The question about the factors of $P(w;\lambda)$ in Remark 10.4.(2) can be discussed by comparing (99) with (97) and using the following.

Lemma 11.2 *Let* $k\in\mathbb{N}$, β, $\gamma\in\mathbb{C}$ *and* $x\in\mathbb{C}\setminus\{0,1\}$ *be fixed, while* z *be a symbolic variable.*

(1) $\mathcal{F}_k(\beta;\gamma;z)\equiv 0$ *in* $\mathbb{C}[z]$ *if and only if* $\beta,\gamma\in\mathbb{Z}$ *and* $0\le-\beta\le-\gamma\le k-1$.

(2) *If* $\mathcal{F}_k(\beta;\gamma;x)=\mathcal{F}_{k-1}(\beta+1;\gamma+1;x)=0$ *then* $\mathcal{F}_k(\beta;\gamma;z)\equiv 0$ *in* $\mathbb{C}[z]$.

(3) *If* $\mathcal{F}_k(\beta;\gamma;x) = \mathcal{F}_k(\beta;\gamma-1;x) = 0$ *then* $\mathcal{F}_k(\beta;\gamma;z) \equiv 0$ *in* $\mathbb{C}[z]$.

Proof. By definition (95), $\mathcal{F}_k(\beta;\gamma;z) = 0$ in $\mathbb{C}[z]$ if and only if $(\beta)_j(\gamma+j)_{k-j} = 0$ for every $j = 0,\dots,k$. Putting $j = 0$ there implies $\gamma = -j_0$ with some $j_0 \in \{0,\dots,k-1\}$. Putting $j = j_0+1$ then implies $\beta = -i_0$ with some $i_0 \in \{0,\dots,j_0\}$. These are sufficient to have condition $(\beta)_j(\gamma+j)_{k-j} = 0$ for every $j = 0,\dots,k$, and hence assertion (1) follows.

It follows from Andrews et al. [1, formulas (2.5.1) and (2.5.7)] and definition (95) that

$$\tfrac{d}{dz}\mathcal{F}_k(\beta;\gamma;z) = -k\beta\,\mathcal{F}_{k-1}(\beta+1;\gamma+1;z), \tag{100a}$$

$$z\tfrac{d}{dz}\mathcal{F}_k(\beta;\gamma;z) = (\gamma+k-1)\,\mathcal{F}_k(\beta;\gamma-1;z) - (\gamma-1)\,\mathcal{F}_k(\beta;\gamma;z). \tag{100b}$$

Assumption of assertion (2) and formula (100a) yield a vanishing initial condition $\mathcal{F}_k(\beta;\gamma;z) = \frac{d}{dz}\mathcal{F}_k(\beta;\gamma;z) = 0$ at $z = x$. As a solution to a Gauss hypergeometric equation, which is regular at $z = x$ ($\neq 0,1$), the polynomial $\mathcal{F}_k(\beta;\gamma;z)$ vanishes identically in $\mathbb{C}[z]$. Thus assertion (2) is established. Assertion (3) is proved in a similar manner by using formula (100b). □

Assertion (1) of Lemma 11.2 leads us to think of the following conditions:

$$\tilde{\beta}_k^*,\ \tilde{\gamma}_k^* \in \mathbb{Z}, \qquad 0 \le -\tilde{\beta}_k^* \le -\tilde{\gamma}_k^* \le r-p-k-2, \tag{101a}$$

$$\tilde{\beta}_j,\ \tilde{\gamma}_j \in \mathbb{Z}, \qquad 0 \le -\tilde{\beta}_j \le -\tilde{\gamma}_j \le p-j-1. \tag{101b}$$

Each of (101) is an extremely restrictive condition which in particular implies $r_p a \in \mathbb{Z}$ and $r_q b \in \mathbb{Z}$, if p_1/r_p and q_1/r_q are the reduced expressions of p/r and q/r respectively.

Proposition 11.3 *As to the question in Remark* 10.4.(2),

(1) $(w - w_k^*)|P(w;\lambda)$ *if and only if* $(\gamma_k^*+k)_{p+1}\cdot\mathcal{F}_k(\beta_k^*;\gamma_k^*;x) = 0$*, unless* (101a) *is satisfied;*

(2) $(w - w_j)|P(w;\lambda)$ *if and only if* $(\gamma_j+j)_{r-p}\cdot\mathcal{F}_j(\beta_j;\gamma_j;x) = 0$*, unless* (101b) *is satisfied,*

where the "unless" phrase is not needed for the "if" part.

Proof. The "if" part of assertion (1) follows directly from (99a). To show the "only if" part, suppose that $(w - w_k^*)|P(w;\lambda)$, that is, $P(w_k^*;\lambda) = 0$, but $(\gamma_k^*+k)_{p+1}\cdot\mathcal{F}_k(\beta_k^*;\gamma_k^*;x) \neq 0$. Then (97a) and (99a) imply $\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*;x) = 0$ and $\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*-1;x) = 0$. By assertion (3) of Lemma 11.2, $\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*;z) \equiv 0$ in $\mathbb{C}[z]$. Assertion (1) of the same lemma then yields (101a). Similarly, assertion (2) can be proved by using (97b) and (99b). □

Definitions (84) and (85) are symmetric in (p,a) and (q,b), so we can replace (p,a) with (q,b) to obtain the (q,b)-versions of Propositions 11.1 and 11.3.

Remark 11.4 Each term in (94) appears as a factor of $P(w;\lambda)$. Indeed, if we put $k = r-p-1$ in (97a) then $(\gamma_k^*+k)_p\cdot\mathcal{F}_k(\beta_k^*;\gamma_k^*;x) = 0$ since $\mathcal{F}_{r-p-1-k}(\tilde{\beta}_k^*;\tilde{\gamma}_k^*;x) = \mathcal{F}_0(\tilde{\beta}_k^*;\tilde{\gamma}_k^*;x) = 1$, so the first term in (94) must be a factor of $P(w;\lambda)$ by the "if" part of assertion (1) of Proposition 11.3. For the third term in (94), put $j = p-1$ in (97a) (if $p \ge 2$) and use assertion (2) of the same proposition. As for the second and fourth terms, use the (q,b)-version of it.

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