Brudno’s theorem for $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts

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Abstract

We generalize Brudno’s theorem of 1-dimensional shift dynamical system to $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts. That is to say, in $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshift, the Kolmogorov-Sinai entropy is equivalent to the Kolmogorov complexity density almost everywhere for an ergodic shift-invariant measure.

Keywords. Brudno’s theorem, Kolmogorov-Sinai entropy, Kolmogorov complexity, Shannon-McMillan-Breiman theorem, Subshifts, $Z^d$-action, Universally typical sets

1 Introduction

In a topological dynamical system, A. A. Brudno defined a complexity of the trajectory of a point in the space by using the notion of Kolmogorov complexity, and showed the relationship between this quantity and the Kolmogorov-Sinai entropy [2]. As a preliminary step, Brudno considered the 1-dimensional shift dynamical system and showed that, for an ergodic shift-invariant measure, the Kolmogorov complexity density is equal to the Kolmogorov-Sinai entropy almost everywhere [2, Theorem 1.1].

A partial approach to generalize this theorem to a $d$-dimensional case is found in [10]. S. G. Simpson showed that, in $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts, there exists a point such that its Kolmogorov complexity density is coincident with the topological entropy [10]. Examining Simpson’s proof, we see that what he showed substantively is that the Kolmogorov complexity density is equal to the Kolmogorov-Sinai entropy almost everywhere only for a measure of maximal entropy.

The purpose of this paper is to generalize the Brudno’s theorem of the $\mathbb{Z}_+^1$-action shift dynamical system to $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts. The main theorem is the following:

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Theorem 3.1 If $\mu \in EM(S, \varsigma)$, then

$$\mathcal{K}(\omega) = h_c(\mu), \quad \mu\text{-a.e. } \omega \in S.$$  

Here $S$ denotes $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshift, $\varsigma$ denotes the shift action on $S$, $EM(S, \varsigma)$ denotes the set of all ergodic shift-invariant measures on the topological dynamical system $(S, \mathcal{G})$, $\mathcal{K}(\omega)$ denotes the Kolmogorov complexity density of $\omega$, and $h_c(\mu)$ denotes the Kolmogorov-Sinai entropy of the measure preserving dynamical system $(S, \mathfrak{B}(S), \mu, \varsigma)$. We give the rigorous definition of these terms in Section 2.

In Section 2, we introduce some basic mathematical notions in ergodic theory, Kolmogorov complexity and shift dynamical systems. We used [4, 7, 9, 11] as main references for this section. Using these basic notions, we define the Kolmogorov complexity density of each point of $\Sigma_{\mathbb{Z}^d}$ (or $\Sigma_{\mathbb{Z}_+^d}$) naturally. In Section 3, we prove the main theorem and give some examples. The proof essentially uses Shannon-McMillan-Breiman theorem and universally typical sets.

2 Some Mathematical Preliminaries

We first give quick reviews for some mathematical results related to the main theorem. We will not give proofs of theorems, see e.g. [4, 8]. We write $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. For an arbitrary fixed $d \in \mathbb{N}$, we set $G := \mathbb{Z}^d$ or $G := \mathbb{Z}_+^d$. For all $n \in \mathbb{N}$, let $\Lambda_n := \{g = (g_i)_{i=1}^d \in G : \forall i \in \{1, \ldots, d\}, |g_i| < n\}$. Then we have

$$|\Lambda_n| = \begin{cases} (2n-1)^d & (G = \mathbb{Z}^d), \\ n^d & (G = \mathbb{Z}_+^d), \end{cases}$$

where we denote by $|A|$ the cardinality of a set $A$.

2.1 Ergodic theory

Let $(X, \mathfrak{B}, \mu, \mathcal{T})$ be a measure preserving dynamical system (m.p.d.s.), namely, $(X, \mathfrak{B}, \mu)$ be a probability space and $\mathcal{T} = (T^g)_{g \in G}$ be a measurable $\mu$-invariant action of $G$ on $X$. A set $A \in \mathfrak{B}$ is said to be $\mathcal{T}$-invariant mod $\mu$ if and only if $\mu(T^{-g}A \triangle A) = 0$ holds for all $g \in G$, where $\triangle$ denotes the symmetric difference. We write $\mathcal{J}_\mu(\mathcal{T}) := \{A \in \mathfrak{B} : A$ is $\mathcal{T}$-invariant mod $\mu\}$. If $\mu(A) = 0$ or $\mu(A) = 1$ for all $A \in \mathcal{J}_\mu(\mathcal{T})$, then the m.p.d.s. $(X, \mathfrak{B}, \mu, \mathcal{T})$ is said to be ergodic. A family of measurable sets $\alpha = \{A_i\}_{i \in I}$ is called a $\mu$-partition of $X$ if $\mu(A_i \cap A_j) = 0$ ($i \neq j$), $\mu\left(X \setminus \bigcup_{i \in I} A_i\right) = 0$ and $\mu(A_i) > 0$ ($\forall i \in I$). Let $\alpha$ be a $\mu$-partition of $X$. The information of $\alpha$ is the function $I_\alpha$ on $X$ defined by $I_\alpha(x) := -\sum_{A \in \alpha} (\log_2 \mu(A)) \cdot 1_A(x)$ ($\forall x \in X$). The entropy of $\alpha$ is defined by the average information, i.e., $H_\mu(\alpha) := \int_X I_\alpha \, d\mu = \sum_{A \in \alpha} \varphi(\mu(A))$ where $\varphi(t) := -t \log_2 t$. From Kolmogorov complexity’s point of view, we choose the binary logarithm $\log_2$ instead of $\log_e$. Let $\beta$ be another $\mu$-partition. The common refinement of $\alpha$ and $\beta$ is $\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta, \mu(A \cap B) > 0\}$. We set $T^{-g} \alpha := \{T^{-g}A : A \in \alpha\}$ for each $g \in G$ and $\alpha^A := \alpha \vee \beta$.
Theorem 2.1 (Shannon-McMillan-Breiman) Let $(X, \mathcal{B}, \mu, \mathcal{T})$ be an ergodic m.p.d.s. and $\alpha$ be a $\mu$-partition of $X$ with $H_\mu(\alpha) < \infty$. Then

$$h(\mu, \alpha, \mathcal{T}) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} I_{\alpha^\Lambda_n} \quad \text{in } L^1(X, \mu).$$

Moreover, if $\alpha$ is finite, then this convergence holds also for $\mu$-a.s. $x \in X$.

The Kolmogorov-Sinai entropy of the m.p.d.s $(X, \mathcal{B}, \mu, \mathcal{T})$ is defined by

$$h_T(\mu) := \sup\{h(\mu, \alpha, \mathcal{T}) : \alpha \text{ is a } \mu\text{-partition with } H_\mu(\alpha) < \infty\}.$$

We denote by $\alpha^G$ the $\sigma$-algebra generated by all $T^{-g}\alpha$, $g \in G$. A $\mu$-partition $\alpha$ is called a $\mu$-generator if $\alpha^G = \mathcal{B}$ mod $\mu$, where this equation means that $\forall A \in \mathcal{B}, \exists B \in \alpha^G, \mu(A \Delta B) = 0$.

Theorem 2.2 (Kolmogorov-Sinai) Let $(X, \mathcal{B}, \mu, \mathcal{T})$ be a m.p.d.s. and $\alpha$ be a $\mu$-generator such that $H(\alpha) < \infty$. Then $h_T(\mu) = h(\mu, \alpha, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological dynamical system (t.d.s.), namely, $X$ be a compact metrizable space and $\mathcal{T} = (T^g)_{g \in G}$ be a continuous action of $G$ on $X$. In this setting we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra of $X$. We denote by $M(X)$ the set of all probability measures on the Borel measurable space $(X, \mathcal{B}(X))$, by $M(X, \mathcal{T})$ the set of all $\mathcal{T}$-invariant probability measures on $(X, \mathcal{B}(X))$ and by $EM(X, \mathcal{T})$ the set of all ergodic members in $M(X, \mathcal{T})$, respectively.

### 2.2 Kolmogorov complexity

Let $\Sigma$ be a finite set and $|\Sigma| \geq 2$. Without loss of generality, we set $\Sigma := \{0, 1, \cdots, N\}$ where $N \in \mathbb{N}$. We define the set of all finite strings over $\Sigma$ as

$$\Sigma^* := \bigcup_{n=0}^{\infty} \Sigma^n = \{\lambda, 0, 1, \cdots, N, 00, 01, \cdots, 0N, 10, \cdots, 1N, \cdots, NN, 000, \cdots\},$$

where $\Sigma^0 = \{\lambda\}$ and $\lambda$ denote the empty string. The length of $x \in \Sigma^*$ is denoted by $l(x)$. For all $x, y \in \Sigma^*$, we call $x$ a prefix of $y$ if there exists $z \in \Sigma^*$ such that $y = xz$. A set $A \subset \Sigma^*$ is said to be prefix-free if, for all $x \in A$, the elements of $A \setminus \{x\}$ are not prefixes of $x$. Let $\mathcal{D}$ be a subset of $\{0, 1\}^*$ and let $f$ be a function from $\mathcal{D}$ to $\Sigma^*$. If $\mathcal{D} \subseteq \{0, 1\}^*$, we call such a function $f$ a partial function and write $f : \{0, 1\}^* \rightsquigarrow \Sigma^*$, and if $\mathcal{D} = \{0, 1\}^*$ then we call $f$ a total function. A partial function $\phi : \{0, 1\}^* \rightsquigarrow \Sigma^*$ is said to be partial recursive if and only if there exists a Turing machine $M$ such that $\phi$ is computed by $M$, i.e., for all $x \in \{0, 1\}^*$, $M$ on input $x$ halts if and only if $x \in \text{dom}(\phi)$, in that case, $M$
outputs $\phi(x)$. Moreover, if $\text{dom}(\phi)$ is prefix-free, then we call $\phi$ a \textit{partial recursive prefix function}. Let $\phi : \{0, 1\}^* \leadsto \Sigma^*$ be a partial recursive prefix function. For all $x \in \Sigma^*$, the \textit{complexity} of $x$ with respect to $\phi$ is defined by

$$K_\phi(x) := \begin{cases} \min\{l(p) : p \in \phi^{-1}(x)\}, & (\phi^{-1}(x) \neq \emptyset), \\ \infty, & (\phi^{-1}(x) = \emptyset). \end{cases}$$

A partial recursive prefix function $\phi : \{0, 1\}^* \leadsto \Sigma^*$ is said to be \textit{additively optimal} if for all partial recursive prefix functions $\psi : \{0, 1\}^* \leadsto \Sigma^*$, there exists a constant $c_{\phi, \psi} \in \mathbb{R}$ such that for all $x \in \Sigma^*$, $K_\phi(x) \leq K_\psi(x) + c_{\phi, \psi}$. We fix such a function $\phi$ and define the \textit{prefix Kolmogorov complexity} of $x \in \Sigma^*$ by $K(x) := K_\phi(x)$.

### 2.3 Shift dynamical system

Let $\Sigma := \{0, 1, \cdots, N\} (N \in \mathbb{N})$ and we set $\Omega := \Sigma^G$. By Tychonoff’s theorem, $\Omega$ endowed with the product topology of the discrete topology on $\Sigma$ is a compact topological space. For all $n \in \mathbb{N}$ and for all $s \in \Sigma^{\Lambda_n}$, we define the \textit{cylinder set} of $s$ by $[s] := \{\omega \in \Omega : \omega \upharpoonright \Lambda_n = s\}$. We set

$$\Sigma^{\Lambda_n} := \bigcup_{n=0}^\infty \Sigma^{\Lambda_n}$$

where $\Sigma^{\Lambda_0} := \{\lambda\}$ and write $[V] := \bigcup_{s \in V} [s]$ for all $V \subset \Sigma^{\Lambda_\ast}$. Let $\sigma^g : \Omega \rightarrow \Omega$ denote the \textit{shift} by $g \in G$, i.e., $(\sigma^g \omega)_i := \omega_{i+g}$ for all $\omega = (\omega_i)_{i \in G}$, and we write $\sigma := (\sigma^g)_{g \in G}$. Since $\sigma$ is a continuous action of $G$ on $\Omega$, $(\Omega, \sigma)$ is a t.d.s.. Note that for all $\mu \in M(\Omega, \sigma)$, the partition $\{[s]\}_{s \in \Sigma^{\Lambda_1}}$ is a $\mu$-generator. A nonempty subset $S \subset \Omega$ is called a \textit{subshift} if and only if $S$ is shift-invariant and closed. If $S \subset \Omega$ is a subshift, then $(S, \sigma \upharpoonright S)$ is a t.d.s., where $\sigma \upharpoonright S := (\sigma^g \upharpoonright S)_{g \in G}$. We fix an arbitrary bijective computable function $f : \mathbb{Z}_+ \rightarrow G$ such that for all $n \in \mathbb{N}$, $f\{(0, 1, \cdots, |\Lambda_n| - 1)\} = \Lambda_n$ and define $\mathcal{G} : \Sigma^{\Lambda_n} \rightarrow \Sigma^\ast$ as follows:

$$\mathcal{G}(s) := \begin{cases} s f(0) \cdots s f(|\Lambda_n| - 1), & s = (s_g)_{g \in \Lambda_n} \in \Sigma^{\Lambda_n} (n \in \mathbb{N}), \\ \lambda, & s = \lambda. \end{cases}$$

We define the prefix Kolmogorov complexity of $s \in \Sigma^{\Lambda_\ast}$ by $K(s) := K(\mathcal{G}(s))$.

**Lemma 2.3** For all $n, k \in \mathbb{N}$, $|\{s \in \Sigma^{\Lambda_n} : K(s) < k\}| < 2^k$.

**Proof.** By [3, Theorem 7.2.4], we have for all $n, k \in \mathbb{N}$,

$$|\{s \in \Sigma^{\Lambda_n} : K(s) < k\}| = |\{\mathcal{G}(s) \in \Sigma^{|\Lambda_n|} : K(\mathcal{G}(s)) < k\}| \leq |\{x \in \Sigma^\ast : K(x) < k\}| < 2^k.$$

The \textit{upper and lower Kolmogorov complexity density} of $\omega \in \Omega$ are defined by

$$\overline{\mathcal{K}}(\omega) := \limsup_{n \to \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}, \quad \underline{\mathcal{K}}(\omega) := \liminf_{n \to \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}.$$
If $\overline{K}(\omega) = K(\omega)$, we simply denote them by $K(\omega)$. The quantities $\overline{K}(\omega)$ and $K(\omega)$ are independent of the choice of additively optimal partial recursive prefix function $\phi$ and $\mathcal{G}$, and uniquely defined.

Lemma 2.4 The functions $\overline{K}, K : \Omega \to \mathbb{R}$ are measurable.

Proof. Let us show that $\overline{K}$ is measurable. For all $x \in \mathbb{R}$, we have

$$\overline{K}^{-1}((-\infty, x)) = \left\{ \omega \in \Omega : \limsup_{n \to \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x \right\}$$

$$= \bigcup_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ \omega \in \Omega : \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x - \frac{1}{k} \right\}.$$ 

Here

$$\left\{ \omega : \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x - \frac{1}{k} \right\} = \left\{ \bigcup_{l=0}^{|\Lambda_n|(x-\frac{1}{k})^{-1}} \{ \omega : K(\omega \upharpoonright \Lambda_n) = l \} \right\}, \quad x - \frac{1}{k} > 0,$$

$$\{ \omega : \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x - \frac{1}{k} \} = \{ \emptyset \}, \quad x - \frac{1}{k} \leq 0.$$

Since the set $\{ \omega \in \Omega : K(\omega \upharpoonright \Lambda_n) = l \}$ is cylinder, then the set $\overline{K}^{-1}((-\infty, x))$ is measurable. Hence the function $\overline{K}$ is measurable. The proof for $K$ is similar. \qed

Remark 2.5 Let $C$ be a plain Kolmogorov complexity (that is not conditioned on prefix function). By [7, Example 3.1.4], we have for all $s \in \Sigma^\Lambda$,

$$C(\mathcal{G}(s)) \leq K(\mathcal{G}(s)) \leq C(\mathcal{G}(s)) + 2 \log C(\mathcal{G}(s)).$$

It means that $C$ and $K$ are asymptotically equal. Then we may use $C$ to define $\overline{K}, \overline{K}$.

3 Relation between KS entropy and Kolmogorov complexity

Let $d \in \mathbb{N}$, $G = \mathbb{Z}^d$ or $G = \mathbb{Z}^d_+$, $\Sigma = \{0, 1, \cdots, N\}$ ($N \in \mathbb{N}$) and $S \subset \Omega$ ($:= \Sigma^G$) be a subshift. Other notations are the same as before. We set $\varsigma := \sigma \upharpoonright S$. Note that $(S, \varsigma)$ is a t.d.s.. We now state the main result.

Theorem 3.1 If $\mu \in EM(S, \varsigma)$, then

$$K(\omega) = h_\varsigma(\mu), \quad \mu \text{-a.e.}\omega \in S. \quad (3.1)$$

Remark 3.2 Brudno’s original result is on the case $G = \mathbb{Z}_+$ only [2]. In the case $G = \mathbb{Z}^d$ or $G = \mathbb{Z}^d_+$, Simpson showed that if $\mu$ is a measure of maximal entropy, then (3.1) holds [10]. Our theorem is a generalization of them.
It is sufficient to prove the theorem for the case $S = \Omega$. Because, if $\mu \in EM(\Omega, \sigma)$ and $\mu(S) = 1$, then $\mu \upharpoonright S \in EM(S, \zeta)$ and $h_\sigma(\mu \upharpoonright S) = h_\sigma(\mu)$ hold where $\mu \upharpoonright S$ denotes the restriction of $\mu$ to $S$. So we prove the theorem about full shift $(\Omega, \sigma)$.

**Theorem 3.3 (\(\mu\)-typical sets)** Let $\mu \in EM(\Omega, \sigma)$. For all $\epsilon > 0$ and $n \in \mathbb{N}$, we set

\[
\mathcal{T}_\epsilon(n) := \left\{ s \in \Sigma^{\Lambda_n} : 2^{-|\Lambda_n|(h_\sigma(\mu)+\epsilon)} < \mu([s]) < 2^{-|\Lambda_n|(h_\sigma(\mu)-\epsilon)} \right\},
\]

\[
\mathfrak{T}_\epsilon := \lim_{n \to \infty} [\mathcal{T}_\epsilon(n)].
\]

Then the following holds:

\[
\mu(\mathfrak{T}_\epsilon) = \lim_{n \to \infty} \mu([\mathcal{T}_\epsilon(n)]) = 1 \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log_2 |\mathcal{T}_\epsilon(n)|}{|\Lambda_n|} \leq h_\sigma(\mu) + \epsilon.
\]

**Proof.** It follows from Shannon-McMillan-Breiman theorem (Theorem 2.1). \hfill \Box

**Lemma 3.4** If $\mu \in EM(\Omega, \sigma)$, then

\[
\mathcal{K}(\omega) \geq h_\sigma(\mu), \quad \mu\text{-a.e.} \omega \in \Omega. \quad (3.2)
\]

**Proof.** If $h_\sigma(\mu) = 0$, then (3.2) is obvious. Let $h_\sigma(\mu) > 0$ and fix an arbitrary $k \in \mathbb{N}$ such that $\frac{1}{k} < h_\sigma(\mu)$. For all $n \in \mathbb{N}$, we set $D_{n,k} := \left\{ s \in \Sigma^{\Lambda_n} : \frac{\mathcal{K}(s)}{|\Lambda_n|} \leq h_\sigma(\mu) - \frac{1}{k} \right\}$. By Lemma 2.3, we have

\[
|D_{n,k}| \leq 2^{|\Lambda_n|(h_\sigma(\mu)-\frac{1}{k})+1}. \quad (3.3)
\]

We fix an arbitrary $\epsilon \in \left(0, \frac{1}{k}\right)$ and set $\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}$ as in Theorem 3.3. Then, by Theorem 3.3, we have for $\mu$-a.e. $\omega \in \Omega$,

\[
\exists N_\omega \in \mathbb{N}, \forall n \geq N_\omega, \omega \in [\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}]. \quad (3.4)
\]

On the other hand, by (3.3) and the definition of $\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}$, we have

\[
\mu([D_{n,k}] \cap [\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}]) = \mu \left( \bigcup_{s \in D_{n,k} \cap [\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}]} [s] \right) \leq \sum_{s \in D_{n,k} \cap [\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}]} \mu([s]) \leq 2^{|\Lambda_n|(h_\sigma(\mu)-\frac{1}{k})+1} \cdot 2^{-|\Lambda_n|(h_\sigma(\mu)-\frac{1}{k}+\epsilon)} = 2^{-|\Lambda_n|\epsilon+1}.
\]

Hence $\sum_{n=1}^{\infty} \mu([D_{n,k}] \cap [\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}]) < \infty$ holds. Therefore, by the Borel-Cantelli lemma, for $\mu$-a.e. $\omega \in \Omega$,

\[
\exists N_\omega' \in \mathbb{N}, \forall n \geq N_\omega', \omega \notin [D_{n,k}] \cap [\mathcal{T}_{\frac{1}{k}-\epsilon}^{(n)}]. \quad (3.5)
\]
By (3.4) and (3.5), for \( \mu \)-a.e. \( \omega \in \Omega \), we have

\[
\exists N''_n \in \mathbb{N}, \forall n \geq N''_n, \omega \notin [D_{n,k}].
\]

Since \( \omega \notin [D_{n,k}] \) means \( \frac{K(\omega|\Lambda_n)}{|\Lambda_n|} > h_\sigma(\mu) - \frac{1}{k} \), we have for all \( k \geq \lceil \frac{1}{h_\sigma(\mu)} \rceil + 1 \),

\[
K(\omega) \geq h_\sigma(\mu) - \frac{1}{k}, \quad \mu\text{-a.e. } \omega \in \Omega.
\] (3.6)

Hence

\[
\mu(\{\omega : K(\omega) < h_\sigma(\mu)\}) = \mu\left(\bigcup_{k=[\frac{1}{h_\sigma(\mu)}]+1}^{\infty} \{\omega : K(\omega) < h_\sigma(\mu) - \frac{1}{k}\}\right) \\
\leq \sum_{k=[\frac{1}{h_\sigma(\mu)}]+1}^{\infty} \mu\left(\{\omega : K(\omega) < h_\sigma(\mu) - \frac{1}{k}\}\right) = 0.
\]

Therefore (3.2) holds.

The following theorem plays a key role to prove the inverse direction.

**Theorem 3.5 (Universally typical sets)** For all rational number \( h_0 \in (0, \log_2 |\Sigma|) \), there exists a sequence of subsets \( \{\mathcal{U}_n^{(n)} \subset \Sigma^{\Lambda_n}\}_n \) such that the following conditions hold:

1. For all \( \mu \in EM(\Omega, \sigma) \) with \( h_\sigma(\mu) < h_0 \),

\[
\mu(\mathcal{U}_n^{(n)}) = \lim_{n \to \infty} \mu([\mathcal{U}_n^{(n)}]) = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\log_2 |\mathcal{U}_n^{(n)}|}{|\Lambda_n|} = h_0
\]

hold where \( \mathcal{U}_n := \liminf_{n \to \infty} [\mathcal{U}_n^{(n)}] \).

2. The sequence of subsets \( \{\mathcal{U}_n^{(n)} \subset \Sigma^{\Lambda_n}\}_n \) is computable.

**Proof.** See [5, Theorem 3.1] and its proof.

**Lemma 3.6** If \( \mu \in EM(\Omega, \sigma) \), then

\[
\overline{K}(\omega) \leq h_\sigma(\mu), \quad \mu\text{-a.e. } \omega \in \Omega.
\] (3.7)

**Proof.** Let \( n \in \mathbb{N} \) and \( h_0 \in (h_\sigma(\mu), \log_2 |\Sigma|) \) be a rational number. We set \( \mathcal{U}_n^{(n)} \) and \( \mathcal{U}_n^{(n)} \) as in Theorem 3.5. By its definition, for all \( \omega \in \mathcal{U}_n^{(n)} \), we have

\[
\exists N \in \mathbb{N}, \forall n > N, \omega \in [\mathcal{U}_n^{(n)}].
\] (3.8)
Note that $\omega \in [\mathcal{U}_{h_0}^{(n)}]$ means $\omega \upharpoonright \Lambda_n \in \mathcal{U}_{h_0}^{(n)}$, and here we can encode each $s \in \mathcal{U}_{h_0}^{(n)}$ into $\lceil \log_2 |\mathcal{U}_{h_0}^{(n)}| \rceil$ bits code. Then the following holds for all $\omega \in \mathcal{U}_{h_0}$:

$$\exists N \in \mathbb{N}, \forall n > N, \frac{C(\mathcal{G}(\omega \upharpoonright \Lambda_n))}{|\Lambda_n|} \leq \frac{\log_2 |\mathcal{U}_{h_0}^{(n)}| + \log_2 n + \text{const.}}{|\Lambda_n|},$$ (3.9)

where $C$ be a plain Kolmogorov complexity. As previously stated in Remark 2.5, $C$ and $K$ are asymptotically equal. Then by Theorem 3.5 and (3.9), we have $\overline{\mathcal{K}}(\omega) \leq h_0$ for all $\omega \in \mathcal{U}_{h_0}$, namely, $\mathcal{U}_{h_0} \subset \{ \omega \in \Omega : \overline{\mathcal{K}}(\omega) \leq h_0 \}$. Let $h_{0,k} \in (h_{\sigma}(\mu), \log_2 |\Sigma|)$ ($k \in \mathbb{N}$) be rational numbers with $\lim_{k \to \infty} h_{0,k} = h_{\sigma}(\mu)$. Then we have

$$\mu(\{ \omega : \overline{\mathcal{K}}(\omega) > h_{\sigma}(\mu) \}) = \mu \left( \bigcup_{k=1}^{\infty} \{ \omega : \overline{\mathcal{K}}(\omega) > h_{0,k} \} \right) \leq \sum_{k=1}^{\infty} \mu \left( \{ \omega : \overline{\mathcal{K}}(\omega) > h_{0,k} \} \right) \leq \sum_{k=1}^{\infty} \mu(\mathcal{U}_{h_{0,k}}) = 0.$$

Therefore (3.7) holds. \qed

Theorem 3.1 follows from Lemma 3.4 and Lemma 3.6.

**Remark 3.7** In Theorem 3.1, $\mu$ is not necessarily computable. Especially if $\mu$ is a computable measure, then Theorem 3.1 is easily seen by the following way: Let $\nu \in \mathcal{M}(\Sigma^\mathbb{Z}_+)$ be a computable measure such that for all $\omega \in \Omega$, $\mu([\omega \upharpoonright \Lambda_n]) = \nu([\mathcal{G}(\omega \upharpoonright \Lambda_n)])$. By [6, THEOREM 5.1, LEMMA 5.2], if $\mathcal{G}(\omega)(:= \lim_{n \to \infty} \mathcal{G}(\omega \upharpoonright \Lambda_n), \omega \in \Omega)$ is Martin-Löf random with respect to $\nu$, then there exist $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$

$$-\log_2 \nu([\mathcal{G}(\omega \upharpoonright \Lambda_n)]) - c_1 < K(\mathcal{G}(\omega \upharpoonright \Lambda_n)) \leq -\log_2 \nu([\mathcal{G}(\omega \upharpoonright \Lambda_n)]) + 2 \log_2 l(\mathcal{G}(\omega \upharpoonright \Lambda_n)) + c_2.$$

Then we have for $\mu$-a.e. $\omega \in \Omega$

$$\mathcal{K}(\omega) = \lim_{n \to \infty} \frac{-\log_2 \mu([\omega \upharpoonright \Lambda_n])}{|\Lambda_n|} = h_{\sigma}(\mu).$$

The last equality is derived from Shannon-McMillan-Breiman theorem.

**Example 3.8** ($d$-dimensional Bernoulli shifts) Let $(\Omega, \sigma)$ be the $\mathbb{Z}^d$ or $\mathbb{Z}^d_+$ shift space as before. We fix a probability vector $q = (q_i : i \in \Sigma)$ on $\Sigma$ and denote the corresponding Bernoulli measure on $\mathcal{B}(\Omega)$ by $\mu := q^\times G$. Then, by Kolmogorov-Sinai theorem (Theorem 2.2), we can show that $h_{\sigma}(\mu) = \sum_{i \in \Sigma} \varphi(q_i)$. By Theorem 3.1, we have for $\mu$-a.e. $\omega \in \Omega$

$$\mathcal{K}(\omega) = \sum_{i \in \Sigma} \varphi(q_i).$$

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Corollary 3.9 If $\mu \in M(\Omega, \sigma)$, then there exists $K(\omega) = \lim_{n \to \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}$ for $\mu$-a.e. $\omega \in \Omega$ and the following holds:

$$h_\sigma(\mu) = \mu(K) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{s \in \Sigma^{\Lambda_n}} K(s) \mu([s]).$$

Proof. Let $\mu = \int_{EM(\Omega, \sigma)} \nu d\rho(\nu)$ be the ergodic decomposition, where $\rho$ be a probability measure on $EM(\Omega, \sigma)$ (see [4, 9, 11]). By Jacobs’s theorem [11, Theorem 8.4] and Theorem 3.1, we have

$$\mu(K) = \int_{EM(\Omega, \sigma)} \left\{ \int_{\Omega} K(\omega) d\nu(\omega) \right\} d\rho(\nu) = \int_{EM(\Omega, \sigma)} h_\sigma(\nu) d\rho(\nu) = h_\sigma(\mu)$$

and $\mu(K) = h_\sigma(\mu)$ is also the same. Hence for $\mu$-a.e. $\omega \in \Omega$ there exists $K(\omega)$ and $\mu(K) = h_\sigma(\mu)$ holds. On the other hand, by (3.9) and Lebesgue’s convergence theorem, we have

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{s \in \Sigma^{\Lambda_n}} K(s) \mu([s]) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \int_{\Omega} K(\omega \upharpoonright \Lambda_n) d\mu(\omega)$$

$$= \int_{\Omega} \lim_{n \to \infty} \frac{1}{|\Lambda_n|} K(\omega \upharpoonright \Lambda_n) d\mu(\omega) = \mu(K).$$

□

Remark 3.10 In the case $G = \mathbb{Z}_+$, Corollary 3.9 can be found in [1].

Example 3.11 ($d$-dimensional Ising model) Let $d \in \mathbb{N}$ and $\Sigma := \{+1, -1\}$. Here $+1$ and $-1$ represent “spin up” and “spin down” at the sites of a “lattice gas” on $G := \mathbb{Z}^d$, respectively. Let $\Omega := \Sigma^G$ be a configuration space and $\sigma$ be a shift action of $G$ on $\Omega$. For $d$-dimensional Ising model, the local energy function $\psi : \Omega \to \mathbb{R}$ is defined by

$$\psi(\omega) := -\beta \left( -\sum_{j=1}^{d} (\omega_{0} \delta_{e_j} + \omega_{0} \delta_{-e_j}) - B \omega_{0} \right), \quad \omega \in \Omega,$$

where $0 := (0, \cdots, 0), e_j := (0, \cdots, 1, \cdots, 0) \in G$. Here $-\sum_{j=1}^{d} (\omega_{0} \delta_{e_j} + \omega_{0} \delta_{-e_j})$ represents the interaction between neighboring spins, $-B \omega_{0}$ represents the effect of a magnetic field $B \in \mathbb{R}$ on the spin at site $0$ and $\beta \geq 0$ denote the inverse temperature. Then the pressure of this model is given by

$$p(\psi) = \sup_{\mu \in M(\Omega, \sigma)} \mu(K + \psi).$$

In mathematical point of view, this example is just a replacement of $h_\sigma(\mu)$ by $\mu(K)$, but it shows that the generalization of Brudno’s theorem to $\mathbb{Z}^d$-action (especially $d = 2$ or 3) has a physical background.

By using Brudno’s theorem for multidimensional subshifts, we can construct a universally typical sets of multidimensional data as follows.
Theorem 3.12 (Universally typical sets using Brudno’s theorem) Let $h_0 > 0$ and $n \in \mathbb{N}$. We set
\[ R_{h_0}^{(n)} = \left\{ s \in \Sigma^\Lambda_n : \frac{K(s)}{|A_n|} < h_0 \right\} \quad \text{and} \quad R_h = \liminf_{n \to \infty} [R_{h_0}^{(n)}]. \]
Then for all $\mu \in EM(\Omega, \sigma)$ with $h_\sigma(\mu) < h_0$ the following holds:
\[ \mu(R_h) = \lim_{n \to \infty} \mu([R_{h_0}^{(n)}]) = 1 \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log_2 |R_{h_0}^{(n)}|}{|A_n|} \leq h_0. \]

Proof. For all $\mu \in EM(\Omega, \sigma)$ with $h_\sigma(\mu) < h_0$, by Theorem 3.1, we have
\[
1 = \mu \left( \left\{ \omega \in \Omega : \lim_{n \to \infty} \frac{K(\omega \upharpoonright A_n)}{|A_n|} - h_\sigma(\mu) = 0 \right\} \right)
\leq \mu \left( \bigcap_{\epsilon > 0} \bigcup_{N \in \mathbb{N} \setminus N} \left\{ \omega \in \Omega : h_\sigma(\mu) - \epsilon < \frac{K(\omega \upharpoonright A_n)}{|A_n|} < h_\sigma(\mu) + \epsilon \right\} \right)
\leq \mu \left( \bigcup_{N \in \mathbb{N} \setminus N} \left\{ \omega \in \Omega : \frac{K(\omega \upharpoonright A_n)}{|A_n|} < h_0 \right\} \right) = \mu(R_h) \leq \liminf_{n \to \infty} \mu([R_{h_0}^{(n)}]).
\]
Then $\mu(R_h) = \lim_{n \to \infty} \mu([R_{h_0}^{(n)}]) = 1$ holds. Since $\lim_{n \to \infty} \mu([R_{h_0}^{(n)}]) = 1$, $|R_{h_0}^{(n)}| \neq 0$ holds for sufficiently large $n \in \mathbb{N}$. Therefore, by Lemma 2.3, we have $0 \neq |R_{h_0}^{(n)}| < 2^{h_0|A_n|+1}$ ($n \gg 1$). Hence $\log_2 |R_{h_0}^{(n)}| < h_0 + \frac{1}{|A_n|}$ ($n \gg 1$). This completes the proof.

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