Finite-Element Solution of Horizontally Polarized Shear Wave Scattering in an Elastic Plate

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Abstract—A method of the solution of scattering of horizontally polarized shear waves in an elastic plate is described. The approach is a combination of the finite-element method and the analytical method. In this approach, both the propagating and evanescent higher order modes are taken into account, and the restriction in the previous work that the number of assumed waveguides modes be equal to the number of boundary finite-element nodal points is removed. This approach is applicable to the frequency range in which the plate waveguide propagates multimodes. Numerical examples on the scattering by a step discontinuity, a wedge-shaped crack, and a metal strip are given.

I. INTRODUCTION

SCATTERING of guided waves by discontinuities in an elastic plate is a basic problem in nondestructive evaluation and also has important implications with regard to electrical signal-processing functions, and the scattering of horizontally polarized shear (SH) waves has been investigated extensively [1]-[5]. Recently, the numerical methods based on the finite-element method (FEM) have been developed for the analysis of the scattering of SH waves by arbitrarily shaped discontinuities in an elastic plate. Sabbagh et al. [6] have analyzed the scattering of SH waves by a step discontinuity joining two plates using a combined method of the FEM and circuit theory. In their approach it is assumed that the plate waveguide propagates a single mode only, and all the higher order modes in the waveguide are neglected. Abduljabbar et al. [7] have analyzed the scattering of SH waves by an infinitely thin crack normal to the surface of a plate using a combined method of the FEM and analytical technique. In their approach the higher-order modes are taken into account, but the stresses along the boundary separating the interior finite-element region from the exterior regular region, in which the scattered field is expanded in a superposition of a finite number of normal modes, are assumed to be piecewise uniform. This combined method requires that the number of the nodal points along the interface boundary between the interior and the exterior regions should be equal to that of the normal modes in the waveguide. Koshiba et al. [8] have analyzed the scattering of Lamb waves in an elastic plate using the combined finite-element and analytical technique. In [8], only the fundamental Lamb wave is taken into account, and all the higher order modes in the waveguide are neglected. Edmunds et al. [9] have used the least-squares collocation approach to solve fracture mechanics problems, removing the restriction that the number of terms in a stress function expansion (analogous to the number of waveguide modes in the present problem) be equal to the number of boundary collocation points.

In this paper, an improved formulation of the combined finite-element and analytical technique is described for the analysis of the scattering of SH waves by arbitrarily shaped discontinuities in an elastic plate. In this approach both the propagating and the evanescent higher order modes are taken into account and the stresses along the interface boundary separating the interior finite-element region from the exterior region are discretized by utilizing the shape functions [7], [8], [10]-[14] which are used for a standard procedure in the FEM. Also, the number of the nodal points along the interface boundary and that of the normal modes are independent of each other and the nodal points can be arbitrarily spaced along the interface boundary. This approach is applicable to the frequency range in which the plate waveguide propagates multimodes. To show the validity and usefulness of this formulation, computed results are given for a step discontinuity, a wedge-shaped crack, and a metal strip. Accuracy of the solution is investigated in detail.

II. BASIC EQUATIONS

We consider the plate waveguide junction as shown in Fig. 1, where the boundaries \( \Gamma_1 \) and \( \Gamma_2 \) connect the discontinuity region to the plate waveguides 1 and 2, respectively, the boundaries \( \Gamma_3 \) and \( \Gamma_4 \) are the stress-free surface, and the region \( \Omega \) with the boundaries \( \Gamma_1 \) to \( \Gamma_4 \) completely encloses the discontinuities.

Assuming that there is no variation in the z-direction, we have the following basic equations for SH waves [1]:

\[
\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} - j \omega \rho v_z = 0 \tag{1}
\]

and

\[
-j \omega T_{xx} = \mu \frac{\partial v_z}{\partial x} \tag{2a}
\]

\[
-j \omega T_{xy} = \mu \frac{\partial v_z}{\partial y} \tag{2b}
\]
where \( v_z \) is the \( z \)-component of the particle velocity, \( T_{zz} \) and \( T_{zy} \) are the stresses, \( \omega \) is the angular frequency, \( \rho \) is the mass density, and \( \mu \) is the Lamé constant.

III. MATHEMATICAL FORMULATION

A. Finite-Element Approach

Dividing the region \( \Omega \) into a number of second-order triangular elements with six nodal points as shown in Fig. 1, the particle velocity \( v_z \) within each element may be written as

\[
v_z = [N]^T \{v_z\}_e
\]  

where \( \{v_z\}_e \) is the particle velocity vector corresponding to the nodal points within each element, \( \{N\} \) is the shape function vector \([7], [8], [10] - [14]\), and \( \Position{\cdot} \) denotes a transpose, a column vector, and a row vector, respectively. The advantage of an arbitrary triangular shape in approximating to any boundary shape has been demonstrated in \([10]\). Also, the usefulness of the second-order triangular element has been discussed in \([12]\) and \([13]\).

Using a Galerkin procedure on (1) and integrating by parts, we obtain

\[
\int_\Omega \left[ (\partial \{N\}/\partial x) T_{zx} + (\partial \{N\}/\partial y) T_{zy} + j\omega \rho \{N\} v_z \right] dx \, dy - \int_\Omega \{N\} T_{zn} \, d\Gamma = \{0\}
\]  

where the first and second integrations on the left side are carried over the element subdomain \( \Omega_e \) and the contour \( \Gamma_e \) of \( \Omega_e \), respectively, \( \{0\} \) is a null vector, and

\[
T_{zn} = T_{zn} n_x + T_{zy} n_y.
\]  

Here \( n_x \) and \( n_y \) are the \( x \) and \( y \) components of an outward-normal unit vector to \( \Gamma_e \), respectively.

Noting that \( T_{zn} \) is continuous across \( \Gamma_e \) and \( T_{zn} = 0 \) on \( \Gamma_3 \) and \( \Gamma_4 \), from (2) to (4) the following global matrix equation is derived

\[
[A] \{v_z\} = \sum_{i=1}^2 \int_\Omega \{N\}_i \{N\}_i^T \{0\} \, dy^{(i)}
\]  

where

\[
[A] = \sum_{i=1}^2 \sum_{e} \int_\Omega \left[ \mu_i (\partial \{N\}/\partial x) \right] \left( \partial \{N\}^T/\partial x \right) \right. \\
+ \mu_i (\partial \{N\}/\partial y) \left( \partial \{N\}^T/\partial y \right) \\
- \omega^2 \rho_i \{N\} \{N\}^T \right] \, dx \, dy.
\]  

Here \( \{v_z\}_e \) is the nodal particle velocity vector, \( \Sigma_\epsilon \) and \( \Sigma'_\epsilon \) extend over all different elements and the elements related to \( \Gamma_i \) (\( i = 1, 2 \)), respectively, \( T_{zx,i}(x^{(i)} = 0, y^{(i)}) \) is the stress on \( \Gamma_i \), and \( \{N\}_i \) is the shape function vector on \( \Gamma_i \), namely \( \{N\}_i = \{N(x^{(i)} = 0, y^{(i)})\} \).

Using the shape function vector \( \{N\}_i \), the particle velocity \( v_{z,i} \) and the stress \( T_{zx,i} \) on \( \Gamma_i \) may be discretized as follows:

\[
v_{z,i}(x^{(i)} = 0, y^{(i)}) = \{N\}_i^T \{v_{z,i}\}_e
\]  

\[
T_{zx,i}(x^{(i)} = 0, y^{(i)}) = \{N\}_i^T \{T_{zx,i}\}_e
\]  

where \( \{v_{z,i}\}_e \) and \( \{T_{zx,i}\}_e \) are the particle velocity vector and the stress vector corresponding to the nodal points within each element related to \( \Gamma_i \), respectively.

Substituting (8) and (9) into (6), we obtain

\[
[A] \begin{bmatrix} \{v_{z,1}\}_e \\ \{v_{z,2}\}_e \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \end{bmatrix}
\]  

where

\[
A = \begin{bmatrix} [B]_1 & [B]_2 \\ [B]_1 & [B]_2 \end{bmatrix}
\]  

and

\[
[B]_i = j\omega \sum_{e} \int_\Omega \{N\}_i \{N\}_i^T \, dy^{(i)}, \quad i = 1, 2.
\]  

Here the components of the \( \{v_{z,i}\} \) vector are the values of \( v_z \) at the nodal points in \( \Omega \) except \( \Gamma_1 \) and \( \Gamma_2 \), the components of the \( \{v_{z,1}\}_i \) and \( \{T_{zx,1}\}_i \) vectors are the values of \( v_z \) and \( T_{zx} \) at the nodal points on \( \Gamma_i \), respectively, and \( \{0\} \) is a zero matrix.

B. Analytical Approach

The internal fields in the waveguide \( i = 1, 2 \) in Fig. 1 are represented by the sum of the normal modes as \([1], [2], [15]\)

\[
v_{z,i}(x^{(i)}, y^{(i)}) = \sum_n \left[ a_{in} \exp(-j\beta_{in} x^{(i)}) + b_{in} \exp(j\beta_{in} x^{(i)}) \right] f_m(y^{(i)})
\]  

\[\text{(15a)}\]
\[-T_{zx,i}(x^{(i)}, y^{(i)}) = \sum_{n} Y_{in} \{ a_{in} \exp \left(-j\beta_{in} x^{(i)} \right) \\
- b_{in} \exp \left(j\beta_{in} x^{(i)} \right) \}] g_{in}(y^{(i)}) \quad (15b)\]

where
\[
\beta_{in} = \sqrt{k_{n}^{2} - \left(\frac{n \pi}{d_{i}}\right)^{2}} \quad (16)
\]
\[
Y_{in} = \mu_{i} \beta_{in}/\omega \quad (17)
\]
\[
f_{in}(y^{(i)}) = g_{in}(y^{(i)}) = \sqrt{I_{n}/d_{i}} \cos \left(\frac{n \pi y^{(i)}}{d_{i}}\right) \quad (18)
\]
\[
k_{n} = \omega \sqrt{\rho_{i}/\mu_{i}} \quad (19)
\]
and
\[
l_{i} = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0. \end{cases} \quad (20)
\]

Here \(a_{in}\) and \(b_{in}\) are the amplitudes of the \(n\)th mode \((n = 0, 1, 2, \cdots)\) in the waveguide \(i\) propagating (or decaying) toward the \(+x^{(i)}\) and \(-x^{(i)}\) directions, respectively.

Now, assuming that the \(m\)th mode \((m = 0, 1, 2, \cdots)\) of unit amplitude is incident from the left of waveguide 1 in Fig. 1, the amplitudes of normal modes may be written as
\[
b_{1n} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \quad (21a)
\]
\[
b_{2n} = 0. \quad (21b)
\]

Considering (15) and (20), \(v_{z,i}\) and \(T_{zx,i}\) on \(\Gamma_{i}\) may be expressed as
\[
v_{z,i}(x^{(i)} = 0, y^{(i)}) = \delta_{ii} 2 f_{im}(y^{(i)}) \]
\[+ \int_{y_{0}^{(i)}}^{y_{0}^{(i)}} \sum_{n} \left(1/Y_{in}\right) f_{in}(y^{(i)}) f_{in}(y_{0}^{(i)}) \]
\[- T_{zx,i}(x^{(i)} = 0, y^{(i)}) dy_{0}^{(i)} \]
\[-T_{zx,i}(x^{(i)} = 0, y^{(i)}) = \delta_{ii} 2 Y_{1m} g_{1m}(y^{(i)}) \]
\[+ \int_{y_{0}^{(i)}}^{y_{0}^{(i)}} \sum_{n} Y_{in} g_{in}(y^{(i)}) g_{in}(y_{0}^{(i)}) \]
\[- v_{z,i}(x^{(i)} = 0, y^{(i)}) dy_{0}^{(i)} \]

where \(\delta_{ii}\) is the Kronecker \(\delta\).

Using (8) and (9), (22) and (23) can be discretized as follows:
\[
\{v_{z}\}_i = \delta_{ii} \{f\}_1 - [F]_i \{T_{zx}\}_i \quad (24)
\]
\[-\{T_{zx}\}_i = \delta_{ii} \{g\}_1 + [G]_i \{v_{z}\}_i \quad (25)\]

where
\[
\{f\}_1 = 2 \{f_{m}\} \quad (26)
\]
\[
\{g\}_1 = -2 Y_{1m} \{g_{m}\} \quad (27)
\]

\[\begin{bmatrix} [F] \end{bmatrix}_i = \sum_{n} (1/Y_{in}) \{ f_{n}\}_i \sum_{e} f_{n}(y_{0}^{(i)}) \]
\[
\cdot \{N(x^{(i)} = 0, y_{0}^{(i)})\} \quad (28)
\]

and
\[\begin{bmatrix} [G] \end{bmatrix}_i = \sum_{n} Y_{in} \{ g_{n}\}_i \sum_{e} g_{in}(y_{0}^{(i)}) \]
\[
\cdot \{N(x^{(i)} = 0, y_{0}^{(i)})\} \quad (29)
\]

Here the components of the \(\{ f_{n}\}_i\) and \(\{ g_{n}\}_i\) vectors are the values of \(f_{in}(y^{(i)})\) and \(g_{in}(y^{(i)})\) at the nodal points on \(\Gamma_{i}\), respectively.

C. Combination of Finite-Element and Analytical Relations

Using (11) and (12), from (10) and (24) we obtain the following final matrix equation:
\[
\begin{bmatrix} [A] & [0] \\ [0] & [B] \end{bmatrix} \begin{bmatrix} \{v_{z}\}_i \\ \{v_{z}\}_B \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{f\} \end{bmatrix} \quad (30)
\]

Similarly, from (10) and (25) we obtain the following final matrix equation:
\[
\begin{bmatrix} [A] & [0] \\ [0] & [B] \end{bmatrix} \begin{bmatrix} \{v_{z}\}_i \\ \{v_{z}\}_B \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{g\} \end{bmatrix} \quad (31)
\]

where
\[
\{f\} = \begin{bmatrix} \{f\}_1 \end{bmatrix} \quad (32)
\]
\[
\{g\} = \begin{bmatrix} \{g\}_1 \end{bmatrix} \quad (33)
\]

\[\begin{bmatrix} [F] \end{bmatrix} = \begin{bmatrix} \{F\}_1 \\ \{F\}_2 \end{bmatrix} \quad (34)
\]
\[\begin{bmatrix} [G] \end{bmatrix} = \begin{bmatrix} \{G\}_1 \\ \{G\}_2 \end{bmatrix} \quad (35)
\]

and \([1]\) is a unit matrix.

The values of \(v_{z}\) at the nodal points on \(\Gamma_{i}\) \((i = 1, 2)\), namely \(\{v_{z}\}_i\) are computed from (30) or (31), and then by using (8), \(v_{z,i}\) on \(\Gamma_{i}\) is calculated from these values.

The solutions \(v_{z,i}\) allow the determination of the reflection coefficient \(R_{mn}\) and the transmission coefficient \(T_{mn}\) of the \(n\)th propagating mode for the \(m\)th incident mode as fol-
The condition of power conservation may be written as
\[
\sum_{n=0}^{n_i-1} |R_{mn}|^2 + \sum_{n=0}^{n_i-1} |T_{mn}|^2 = P_{\text{total}} = 1
\]  
where \(n_i\) is the number of the propagating modes in the waveguide \(i\).

IV. COMPUTED RESULTS

First, we consider the symmetric step joining two plates and subdivided one half of the discontinuity region into second-order triangular elements as shown in Fig. 2, where the zeroth mode is incident from the left of the plate with thickness \(d_1, d_1/d_2 = 1.7, 96 (N_E)\) elements are used, and the number of the nodal points \(N_p\) is 233. Table I shows the numbers \(n_1\) and \(n_2\) of the propagating modes in the plates with thicknesses \(d_1\) and \(d_2\), respectively. Table II shows the real and imaginary parts of the reflection coefficient \(R_{00}\) of the zeroth mode at the step, where the computed results are obtained by using (30). Table III shows the energy error \((1 - P_{\text{total}})\), where the computed results are obtained by using (30) and (31). The difference between the values calculated from (30) and the values calculated from (31) is larger than that in the analysis using (30). Therefore, we present below only the results obtained by using (30). It is generally difficult to determine a priori the discretization fineness required to achieve sufficient accuracy. The finite-element method with a suitable mesh refinement strategy [11], [14] may be useful for accurate analysis.

Next, we consider a surface crack in a plate. Results for an infinitely thin crack are given in graphical form by Abduljabbar et al. [7]. Our results agreed perfectly with the results in [7] to the accuracy that can be discerned on the graphs.

Fig. 3(a)–(c) show the magnitude of the reflection coefficient \(R_{mn}\) of a wedge-shaped crack as a function of the wedge apex angle \(\tan \theta\) for the zeroth \((m = 0)\), first \((m = 1)\), and second \((m = 2)\) incident modes, respectively, where \(k_o d = 13.5\) and the reflection coefficient of the mode of order less than \(m\) can be computed from the relation \(R_{mn} = R_{mm} [7]\). The number of the nodal points along the interface boundary corresponding to the boundary \(\Gamma_i (i = 1, 2)\) in Fig. 1 is 41, and 10 normal modes in the waveguide \(i\) are considered in (28). The reflection coefficient is considerably influenced by the wedge apex angle. At an appropriate value of \(\theta\), the conversion coefficient for the fourth reflected mode becomes larger than the conversion coefficients for the other reflected modes.

Lastly, we consider a metal strip on one surface of a fused quartz plate and assume that the zeroth mode is incident [16]. Fig. 4 shows the magnitude of the reflection coefficient \(R_{00}\) of a metal strip as a function of the normalized strip thickness \(h/\lambda\), where \(\lambda\) is the wavelength of the incident wave, \(\lambda = 2 \pi /k_o, d/\lambda = 0.25\), and the normalized strip width \(a/\lambda = 0.25\). The reflectivity of this SH wave is considerably large compared with the Rayleigh-wave reflection from a metal strip on a semi-infinite substrate [17].

\[ R_{mn} = a_{1n} = \sqrt{Y_{1n}/Y_{1m}} \left[ \delta_{mn} + \int_0^{d_1} g_{1n}(z) v_{z,1}(x) dz \right] = 0, y^{(1)} dy^{(1)} \]  
\[ T_{mn} = a_{2n} = \sqrt{Y_{2n}/Y_{1m}} \int_0^{d_2} g_{2n}(y) v_{z,2}(x) dy^{(2)} = 0, y^{(2)} dy^{(2)} \]  
\[ \sum_{n=0}^{n_i-1} |R_{mn}|^2 + \sum_{n=0}^{n_i-1} |T_{mn}|^2 = P_{\text{total}} = 1 \]
### Table II

Real and Imaginary Parts of Reflection Coefficient $R_{00}$ of Zeroth Mode

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<th>$N_M = 30$</th>
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### Table III

Energy Error in the Analysis of Step Discontinuity in Fig. 2

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Fig. 3. Reflection coefficient of wedge-shaped crack for (a) zeroth, (b) first, and (c) second incident modes.
V. CONCLUSION

A method of analysis, based on the finite-element approach and the analytical approach, was developed for the solution of scattering of SH waves in an elastic plate. This method is useful for inhomogeneous discontinuities of arbitrary shape. Numerical examples are presented for a step discontinuity, a wedge-shaped crack, and a metal strip.

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REFERENCES


