Hyperscaling for oriented percolation in $1 + 1$ space-time dimensions

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Abstract

Consider nearest-neighbor oriented percolation in $d + 1$ space-time dimensions. Let ρ, η, ν be the critical exponents for the survival probability up to time t, the expected number of vertices at time t connected from the space-time origin, and the gyration radius of those vertices, respectively. We prove that the hyperscaling inequality $d\nu \geq \eta + 2\rho$, which holds for all $d \geq 1$ and is a strict inequality above the upper-critical dimension 4, becomes an equality for $d = 1$, i.e., $\nu = \eta + 2\rho$, provided existence of at least two among ρ, η, ν. The key to the proof is the recent result on the critical box-crossing property by Duminil-Copin, Tassion and Teixeira [6].

1 Introduction and the main results

Oriented percolation is a time-oriented model of percolation. It is also considered as a discrete-time model for the spread of an infectious disease, known as the contact process or the SIS model. Since it became known to exhibit a phase transition and critical behavior, there have been intensive researches in both theory and applications in various fields. Recently, a possible association to the laminar-turbulent flow transition was reported in [20].

Consider the following nearest-neighbor bond oriented percolation on the space-time lattice $\mathbb{L}^d \equiv \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ : \|x\|_1 + t \text{ is even}\}$. A pair of vertices $[(x, s), (y, t)\rangle$ is called a bond if $\|x - y\|_1 = 1$ and $t = s + 1$. Each bond $[(x, t), (y, t + 1)\rangle$ is either occupied with probability $p \in [0, 1]$ or vacant with probability $1 - p$, independently of the other bonds. Let $\mathbb{P}_p$ be the associated probability measure. We say that $(x, s) \in \mathbb{L}^d$ is connected to $(y, t) \in \mathbb{L}^d$, denoted by $(x, s) \longrightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is a sequence of occupied bonds $\{[(v_j, j), (v_{j+1}, j + 1)\rangle\}_{j=s}^{t-1}$ from $v_s = x$ to $v_t = y$. We

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simply write $(x,s) \longrightarrow t$ for the event $\bigcup_y \{(x,s) \longrightarrow (y,t)\}$, and $s \longrightarrow (y,t)$ for the event $\bigcup_x \{(x,s) \longrightarrow (y,t)\}$.

The major quantities we are interested in are the following. The first quantity is the survival probability up to time t, defined as

$$\theta_t = \mathbb{P}_p\big((o,0) \longrightarrow t\big), \tag{1.1}$$

where, and in the rest of the paper, the p-dependence is suppressed for lighter notation. Since $\{\theta_t\}_{t\in\mathbb{N}}$ is a decreasing sequence of increasing and continuous functions in p, the limit $\theta_\infty \equiv \lim_{t\uparrow\infty} \theta_t$ is nondecreasing and right-continuous in p. Let

$$p_{\mathrm{c}} = \inf\{p \in [0,1] : \theta_\infty > 0\}. \tag{1.2}$$

It is proven in [8] that θ_∞ is also left-continuous in p. In particular, $\theta_\infty = 0$ at $p = p_{\mathrm{c}}$, which has not been proven yet for unoriented percolation in full generality.

The second and third quantities are the expected number of vertices at time t connected from the origin $(o,0)$ and the gyration radius of those vertices, defined as

$$\chi_t = \sum_x \tau(x,t), \qquad \xi_t = \bigg(\frac{1}{\chi_t}\sum_x |x|^2 \tau(x,t)\bigg)^{1/2}, \tag{1.3}$$

where $\tau(x,t)$ is the two-point function:

$$\tau(x,t) = \mathbb{P}_p\big((o,0) \longrightarrow (x,t)\big). \tag{1.4}$$

It is first proven in [1, 12], and recently reproved in a much simpler way in [5], that the critical point is unique in the sense that

$$p_{\mathrm{c}} = \sup\bigg\{p \in [0,1] : \sum_{t=0}^{\infty} \chi_t < \infty\bigg\}. \tag{1.5}$$

The sum $\sum_t \chi_t$ is often called the susceptibility.

Now we briefly summarize the basic properties of those quantities readily obtained from the definition. First we note that, by the Markov property and translation invariance,

$$\theta_{s+t} \ge \theta_s \theta_t, \qquad \chi_{s+t} \le \chi_s \chi_t. \tag{1.6}$$

With the help of the trivial inequality $\theta_t \le \chi_t \le (2t+1)^d \theta_t$, we can conclude that there is a common relaxation time $\zeta \in [0,\infty]$ such that

$$\zeta = \lim_{t\uparrow\infty} \frac{-t}{\log\theta_t} = \sup_{t\in\mathbb{N}} \frac{-t}{\log\theta_t} = \lim_{t\uparrow\infty} \frac{-t}{\log\chi_t} = \inf_{t\in\mathbb{N}} \frac{-t}{\log\chi_t}. \tag{1.7}$$

Using the second and forth equalities, we can say that ζ is bounded away from zero and infinity when $p < p_{\mathrm{c}}$, implying exponential decay of θ_t and χ_t in t in the subcritical regime. This is not the case at the critical point. Moreover, χ_t is nondecreasing in t at $p = p_{\mathrm{c}}$, because, otherwise, there must be a $t_0 \in \mathbb{N}$ such that $\chi_{t_0} < 1$, which together with submultiplicativity implies exponential decay of χ_t and convergence of the susceptibility $\sum_t \chi_t$ at $p = p_c$, such as

$$\sum_{t=0}^{\infty} \chi_t = \sum_{n=0}^{\infty} \sum_{k=0}^{t_0-1} \chi_{nt_0+k} \leq \sum_{n=0}^{\infty} \chi_{t_0}^n \sum_{k=0}^{t_0-1} \chi_k < \infty, \tag{1.8}$$

which is a contradiction to the result in [2]: $\sum_t \chi_t = \infty$ at $p = p_c$.

Let ρ, η, ν be the critical exponents for the above quantities at $p = p_c$: as $t \uparrow \infty$,

$$\theta_t \approx t^{-\rho}, \qquad \chi_t \approx t^{\eta}, \qquad \xi_t \approx t^{\nu}, \tag{1.9}$$

where $f \approx g$ means that $(\log f)/\log g$ goes to 1 in the prescribed limit, allowing corrections of slowly varying functions. In higher dimensions $d \gg 4$ ($d > 4$ is enough for sufficiently spread-out models), the lace expansion converges and the above critical exponents take on their mean-field values $\rho = 1$, $\eta = 0$ and $\nu = 1/2$: the values for branching random walk [3, 4, 9, 10, 13, 14, 18]. In lower dimensions, on the other hand, only numerical values and predictions due to non-rigorous renormalization-group methods are available (see Table 1).

In this paper, we prove the following theorem.

Theorem 1.1. *(i) For any $d \geq 1$, $p \in [0,1]$ and $t \in \mathbb{N}$, we have*

$$\chi_t \leq \frac{4}{3}(4\xi_t + 1)^d \, \theta_{t/2}^2, \tag{1.10}$$

which implies the hyperscaling inequality (assuming existence of ρ, η, ν)

$$d\nu \geq \eta + 2\rho. \tag{1.11}$$

(ii) Let $d = 1$ and $p = p_c$. Then, there is a $K > 0$ such that, for any $t \in \mathbb{N}$,

$$\chi_t \geq K \xi_t \theta_t^2, \tag{1.12}$$

which implies the hyperscaling equality (assuming existence of at least two among ρ, η, ν)

$$\nu = \eta + 2\rho. \tag{1.13}$$

Table 1: Predicted values of the critical exponents in various dimensions (e.g., [15]).

	$d=1$	$d=2$	$d=3$	$d=4-\varepsilon$	$d \geq 4$
ρ	0.159464	0.451	0.73	$1 - \frac{1}{4}\varepsilon - 0.01283\varepsilon^2$	1
η	0.313686	0.230	0.12	$\frac{1}{12}\varepsilon + 0.03751\varepsilon^2$	0
ν	0.632613	0.568	0.526	$\frac{1}{2} + \frac{1}{48}\varepsilon + 0.008171\varepsilon^2$	$\frac{1}{2}$
γ	2.277730	1.60	1.25	$1 + \frac{1}{6}\varepsilon + 0.06683\varepsilon^2$	1
μ	1.733847	1.295	1.105	$1 + \frac{1}{12}\varepsilon + 0.02238\varepsilon^2$	1

Remarks:

1. The inequality (1.10) was first derived in [19]. Since its proof is easy and short, we will show it again for convenience. It was used in [19] to prove two other hyperscaling inequalities that also involve critical exponents defined in the off-critical regime. For example, if the susceptibility $\sum_t \chi_t$ and the relaxation time ζ diverge as $p \uparrow p_\mathrm{c}$ as $(p_\mathrm{c} - p)^{-\gamma}$ and $(p_\mathrm{c} - p)^{-\mu}$ respectively, then, for any $d \geq 1$, we have
$$(d\nu - 2\rho + 1)\mu \geq \gamma. \tag{1.14}$$
If we replace those critical exponents in (1.11) and (1.14) by their mean-field values, then we obtain $d \geq 4$, which is a complement to the aforementioned lace-expansion results. Therefore, the upper-critical dimension d_c for oriented percolation is 4.

2. In general, hyperscaling inequalities are believed to be equalities below and at the model-dependent upper-critical dimension. The values in Table 1 seem to support this belief. The identity (1.13) proves that it is indeed the case for at least $d = 1$. For unoriented percolation, for which $d_\mathrm{c} = 6$, similar results are proven in 2 dimensions by Kesten [11] using the Russo-Seymour-Welsh theorem on the critical box-crossing property [16, 17, 21]. Since the known critical exponents for 2-dimensional unoriented percolation are rational numbers (e.g., $\beta = 5/36$ and $\gamma = 43/18$), it is natural to believe that there must be some balance (i.e., hyperscaling equalities) among those critical exponents. On the other hand, since the values in Table 1 do not seem to be rational numbers, the hyperscaling equality (1.13) is even more surprising.

3. The main reason why the right-hand side of (1.10) is bigger than its left-hand side is due to the inequality
$$\begin{aligned}\tau(x,t) &= \mathbb{P}_p\big((o,0) \longrightarrow (x,t)\big)\\ &\leq \mathbb{P}_p\big((o,0) \longrightarrow t/2,\ t/2 \longrightarrow (x,t)\big) = \theta_{t/2}^2,\end{aligned} \tag{1.15}$$
where, and in the rest of the paper, we do not care much about possibilities of, e.g., $t/2$ not being an integer, since it is easy (but cumbersome) to make the argument rigorous if we introduce floor functions, etc. The last equality in (1.15) is due to reversibility: if we change the direction of each bond and redefine the connectivity in the time-decreasing direction, then we have the identity $\mathbb{P}_p(t/2 \longrightarrow (x,t)) = \theta_{t/2}$.

4. The following theorem on the critical box-crossing property is the key to show the opposite inequality to (1.15):

   Theorem 1.2 (Theorem 1.3 in [6])**.** *Let*
$$V_p(w,t) = \mathbb{P}_p\Big([0,w] \times [0,t] \text{ is crossed vertically}\Big), \tag{1.16}$$
$$H_p(w,t) = \mathbb{P}_p\Big([0,w] \times [0,t] \text{ is crossed from left to right}\Big). \tag{1.17}$$

There exist a constant $\varepsilon \in (0,1)$ and an increasing sequence of integers $\{w_t\}_{t\in\mathbb{N}}$ such that, for all $t \in \mathbb{N}$,

$$\varepsilon \le V_{p_c}(w_t, 3t) \le V_{p_c}(3w_t, t) \le 1-\varepsilon, \tag{1.18}$$

$$\varepsilon \le H_{p_c}(3w_t, t) \le H_{p_c}(w_t, 3t) \le 1-\varepsilon. \tag{1.19}$$

We will also use (1.18)–(1.19) to control an upper bound on $\tau(x,t)$ for $x > jw_t$ that decays exponentially in $j \in \mathbb{N}$ (see Lemma 2.1 below). This is a key element to show that w_t is bounded below by an ε-dependent positive multiple of ξ_t.

5. Applying (1.10) and (1.12) to [19, (5.1)] and its reverse, respectively, we can readily show that the hyperscaling inequality (1.14) also becomes an equality for $d = 1$, i.e.,

$$(\nu - 2\rho + 1)\mu = \gamma. \tag{1.20}$$

6. It is easy to show that the hyperscaling inequality (1.11) holds for other finite-range models of oriented percolation and the contact process. It should not be so difficult to prove Theorem 1.2 for the nearest-neighbor models of oriented site percolation and the contact process, hence the hyperscaling equality (1.13) for $d = 1$. However, it is not so obvious to prove a similar statement to Theorem 1.2 for longer-range models. This may be worth further investigation.

2 Proof of Theorem 1.1

Proof of Theorem 1.1(i). It suffices to prove the inequality (1.10), as the hyperscaling inequality (1.11) immediately follows by using (1.10) at $p = p_c$ (and assuming existence of the three critical exponents). First we note that

$$\chi_t = \frac{1}{\xi_t^2}\sum_x |x|^2\tau(x,t) \ge 4\sum_{x:|x|\ge 2\xi_t}\tau(x,t), \tag{2.1}$$

hence

$$\frac{3}{4}\chi_t \le \sum_{x:|x|\le 2\xi_t}\tau(x,t). \tag{2.2}$$

By (1.15), the right-hand side is further bounded by $(4\xi_t+1)^d\theta_{t/2}^2$. This completes the proof of (1.10). ■

To prove Theorem 1.1(ii), we first assume the following key lemma:

Lemma 2.1. *Let $d = 1$ and $p = p_c$. Let $\varepsilon \in (0,1)$ and w_t be the same as in Theorem 1.2.*

(i) For any $t \in \mathbb{N}$ and any $x \in [-\frac{1}{2}w_t, \frac{1}{2}w_t]$,

$$\tau(x,t) \ge \varepsilon^6\theta_t^2. \tag{2.3}$$

(ii) For any $j,t,x \in \mathbb{N}$ with $j \ge 2$ and $jw_t < x \le (j+1)w_t$,

$$\tau(x,t) \le \varepsilon^{-4}\theta_t^2(1-\varepsilon)^{j-2}. \tag{2.4}$$

Proof of Theorem 1.1(ii) assuming Lemma 2.1. Again, it suffices to prove the inequality (1.12), as the equality (1.13) is a result of the hyperscaling inequality (1.11) for $d = 1$ and the opposite inequality $\nu \le \eta + 2\rho$ that immediately follows from (1.12).

To prove (1.12), we first note that, by (2.3),

$$\chi_t \ge 2 \sum_{x=1}^{\frac{1}{2}w_t} \tau(x,t) \ge \varepsilon^6 w_t \theta_t^2. \tag{2.5}$$

To complete the proof, it suffices to show that w_t is bounded below by a positive multiple of ξ_t. However, by definition,

$$\begin{aligned}\xi_t^2 = 2\sum_{x=1}^{\infty} x^2 \frac{\tau(x,t)}{\chi_t} &= 2\Bigg(\sum_{x=1}^{2w_t} x^2 \frac{\tau(x,t)}{\chi_t} + \sum_{j=2}^{\infty} \sum_{x=jw_t+1}^{(j+1)w_t} x^2 \frac{\tau(x,t)}{\chi_t} \Bigg) \\ &\le 2w_t^2 \Bigg(4 + \sum_{j=2}^{\infty} (j+1)^2 \sum_{x=jw_t+1}^{(j+1)w_t} \frac{\tau(x,t)}{\chi_t} \Bigg). \end{aligned} \tag{2.6}$$

Then, by using (2.4)–(2.5), we obtain

$$\begin{aligned}\xi_t^2 &\overset{(2.5)}{\le} 2w_t^2 \Bigg(4 + \frac{1}{\varepsilon^6 \theta_t^2} \sum_{j=2}^{\infty} (j+1)^2 \max_{jw_t < x \le (j+1)w_t} \tau(x,t) \Bigg) \\ &\overset{(2.4)}{\le} 2w_t^2 \Bigg(4 + \varepsilon^{-10} \sum_{j=2}^{\infty} (j+1)^2 (1-\varepsilon)^{j-2} \Bigg). \end{aligned} \tag{2.7}$$

As a result,

$$\chi_t \ge \underbrace{\frac{\varepsilon^6}{\sqrt{2}} \Bigg(4 + \varepsilon^{-10} \sum_{j=2}^{\infty} (j+1)^2 (1-\varepsilon)^{j-2} \Bigg)^{-1/2}}_{=K} \xi_t \theta_t^2. \tag{2.8}$$

This completes the proof of (1.12). ■

The rest of the paper is devoted to showing Lemma 2.1.

Proof of Lemma 2.1(i). First we note that, for $1 \le x \le \frac{1}{2}w_t$, the event $(o,0) \longrightarrow (x,t)$ occurs if the following four increasing events occur:

- $(o,0) \longrightarrow t$ in $[-w_t, w_t] \times [0,t]$,
- $0 \longrightarrow (x,t)$ in $[x - w_t, x + w_t] \times [0,t]$,
- $[-\frac{3}{2}w_t, \frac{3}{2}w_t] \times [0,t]$ is crossed from left to right,
- $[-\frac{3}{2}w_t, \frac{3}{2}w_t] \times [0,t]$ is crossed from right to left.

The last two events take care of the possibility that the forward cluster from the origin $(o,0)$ and the backward cluster from (x,t) do not collide. Using the FKG inequality (see, e.g., [7]), translation invariance and the reversibility explained below (1.15), we obtain

$$\tau(x,t) \geq \mathbb{P}_p\Big((o,0) \longrightarrow t \text{ in } [-w_t,w_t] \times [0,t]\Big)^2 H_p(3w_t,t)^2. \tag{2.9}$$

We further note that the event $(o,0) \longrightarrow t$ in $[-w_t,w_t] \times [0,t]$ occurs if the following three increasing events occur:

- $(o,0) \longrightarrow t$,
- $[0,w_t] \times [0,t]$ is crossed vertically,
- $[-w_t,0] \times [0,t]$ is crossed vertically.

Again, by the FKG inequality, translation invariance and the monotonicity $V_p(w_t,t) \geq V_p(w_t,3t)$, we obtain

$$\mathbb{P}_p\Big((o,0) \longrightarrow t \text{ in } [-w_t,w_t] \times [0,t]\Big) \geq \theta_t V_p(w_t,3t)^2, \tag{2.10}$$

hence

$$\tau(x,t) \geq \theta_t^2 V_p(w_t,3t)^4 H_p(3w_t,t)^2. \tag{2.11}$$

The inequality (2.3) follows from the above inequality at $p=p_\mathrm{c}$ and (1.18)–(1.19). ■

Proof of Lemma 2.1(ii). Recall that $j \geq 2$ and $jw_t < x \leq (j+1)w_t$. If $(o,0) \longrightarrow (x,t)$, then the following three independent events occur:

- $(o,0)$ is connected to the boundary ∂B_o of the box $B_o \equiv [-w_t,w_t] \times [0,t]$,
- $[w_t,(j-1)w_t] \times [0,t]$ is crossed from left to right,
- (x,t) is connected from the boundary ∂B_x of the box $B_x = [(j-1)w_t,(j+2)w_t] \times [0,t]$.

By this observation and using $H_p((j-2)w_t,t) \leq H_p(w_t,t)^{j-2} \leq H_p(w_t,3t)^{j-2}$, we obtain

$$\tau(x,t) \leq \mathbb{P}_p\Big((o,0) \longrightarrow \partial B_o\Big) H_p(w_t,3t)^{j-2} \mathbb{P}_p\Big(\partial B_x \longrightarrow (x,t)\Big). \tag{2.12}$$

However, by reversibility and monotonicity, we have

$$\mathbb{P}_p\Big(\partial B_x \longrightarrow (x,t)\Big) \leq \mathbb{P}_p\Big((o,0) \longrightarrow \partial B_o\Big). \tag{2.13}$$

Therefore,

$$\tau(x,t) \leq \mathbb{P}_p\Big((o,0) \longrightarrow \partial B_o\Big)^2 H_p(w_t,t)^{j-2}. \tag{2.14}$$

To bound the probability on the right-hand side by θ_t, we borrow the idea in the proof of [6, (4.7)]. First, we note that $(o,0) \longrightarrow t$ if the following three increasing events occur:

- $(o, 0) \longrightarrow \partial B_o$,
- $[0, w_t] \times [0, t]$ is crossed vertically,
- $[-w_t, 0] \times [0, t]$ is crossed vertically.

By the FKG inequality, translation invariance and the monotonicity $V_p(w_t, t) \geq V_p(w_t, 3t)$, we obtain

$$\theta_t \geq \mathbb{P}_p\Big((o, 0) \longrightarrow \partial B_o\Big) V_p(w_t, 3t)^2. \tag{2.15}$$

To summarize the above computations at $p = p_c$, we arrived at

$$\tau(x, t) \leq \left(\frac{\theta_t}{V_{p_c}(w_t, 3t)^2}\right)^2 H_{p_c}(w_t, 3t)^{j-2} \leq \varepsilon^{-4}\theta_t^2(1-\varepsilon)^{j-2}, \tag{2.16}$$

as required. ■

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