Synthesis of Cauer-Equivalent Circuit Based on Model Order Reduction Considering Nonlinear Magnetic Property

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The Cauer-equivalent circuit of electric apparatus is synthesized by applying the model order reduction (MOR) to the finite element equations. In this method, the admittance function of a given electric apparatus is expressed by a rational polynomial, from which the Cauer-equivalent circuit is directly synthesized. Magnetic saturation in the magnetic core is considered by introducing nonlinearity in the primal inductance of the circuit. The synthesized circuit is shown to express the input-output properties of inductors and induction heating devices in good accuracy.

Index Terms— Cauer circuit, model order reduction, Padé via Lanczos, finite element method.

I. INTRODUCTION

LECTRIC APPARATUS such as inductor, transformers and motors are often expressed by means of equivalent circuits in the design of driving and control circuits. The conventional equivalent circuit modeling, however, has difficulties in treatment of magnetic nonlinearity, computation of eddy current and hysteresis losses, and expression of characteristics over a wide frequency range.

The Cauer-equivalent circuit of a steel sheet has been derived from the analytical solution to the quasi-static Maxwell equations [1]–[3]. The synthesized Cauer circuit does not only express the frequency property of the steel plate over wide frequency range in good accuracy, but also can treat saturated cores. The winding coils have also been modeled by the Cauer circuit on the basis of analytical approach [4] (see also references in [4]). These methods can be, however, applied only to electric apparatus with simple geometry which can be analyzed with analytical methods.

The authors have proposed the synthesis of the Foster-equivalent circuit from the finite element (FE) equation of a given electric apparatus based on the model order reduction (MOR) [5]–[8]. This method can synthesize the equivalent circuits of electric apparatuses with arbitrary geometry and also accurately express wide-range frequency characteristic provided that there is no magnetic nonlinearity.

In this study, we extend the above MOR-based method to consider magnetic saturation in the magnetic cores. The proposed method synthesizes the Cauer circuit from the rational polynomials derived from the quasi-static Maxwell equations using Padé approximation via the Lanczos processes (PVL) based-MOR [9]. We apply this method to synthesize the Cauer-equivalent circuit of an inductor used in a DC-DC converter as well as an induction heating device to evaluate the validity and performance of the proposed method.

II. FINITE ELEMENT METHOD

Let us consider the Maxwell equations represented in the Laplace domain for quasi-static electromagnetic fields

\[ \text{rot} \sigma \text{rot} A + s \sigma (A + \text{grad} \phi) = i \sum_{k} j_{0k} \quad (1a) \]

\[ \text{div} [\sigma (A + \text{grad} \phi)] = 0 \quad (1b) \]

where \( A, \phi, \nu, \sigma, i, \) and \( j_{0k} \) are magnetic vector and scalar potentials, magnetic reluctivity, conductivity, current and unit current density, respectively. Applying the weighted residual method in conjunction with Gelerkin method to (1), we obtain

\[ \sum_{i} A_{i} \int \text{rot} N_{i} \cdot \text{rot} N_{j} d\Omega + s \sigma \sum_{i} A_{i} \int N_{i} \cdot N_{j} d\Omega \]

\[ + s \sigma \sum_{k} \phi_{k} \int N_{i} \cdot \text{grad} N_{k} d\Omega = i \sum_{k} \int N_{i} \cdot j_{0k} d\Omega \quad (2a) \]

\[ s \sigma \sum_{i} A_{i} \int N_{i} \cdot \text{grad} N_{a} d\Omega + s \sigma \sum_{k} \phi_{k} \int \text{grad} N_{a} \cdot \text{grad} N_{k} d\Omega = 0 \quad (2b) \]

where \( N_{i} \) and \( N_{j} \) are the vector and the scalar interpolation functions, respectively. To determine the current \( i \), we couple (2) with the circuit equation given by

\[ v_{in} = \sum_{j} \Delta v_{j} + i \sum_{j} R_{j} \quad (3) \]

where \( v_{in} \), \( \Delta v_{j} \), and \( R_{j} \) are the input voltage, voltage drop along the \( j \)-th coil and its resistance, respectively. Note that \( \Delta v_{j} \) is a function of \( A \) and \( \phi \). By solving (2) and (3), we obtain the admittance of the equivalent circuit at any frequency. This computation can be effectively performed using MOR technique which will be described below.
III. SYNTHESIS OF CAUER-EQUIVALENT CIRCUIT

In the present method, we use PVL-based MOR [9] to compute the admittance function of a given apparatus, which is expressed by a rational polynomial function of frequency.

A. PVL-based MOR

To formulate PVL-based MOR, we express (2) and (3) in the state equation as follows:

$$Kx + sNx = bv$$  \hspace{1cm} (5a)

where \(K \in \mathbb{R}^{n \times n}, x, b \in \mathbb{R}^n\), \(n\) is the degree of freedoms (DoFs) in (2) and (3). The output equation is

$$i = l'x$$  \hspace{1cm} (5b)

The transfer function for the system described by (5), which corresponds to the admittance function, is given by

$$Y(s) = l'[I - (s-s_0)A]^{-1}r$$  \hspace{1cm} (6)

where \(A = (K+s_0N)^{-1}N, r = (K+s_0N)^{-1}b\), and \(s_0\) is an expansion point. The eigenvalue decomposition of \(A\) results in

$$Y(s) = l'[I - (s-s_0)SA^{-1}]^{-1}r = f'[I - (s-s_0)A]^{-1}g$$

$$= \sum_{i} \frac{f_i g_i}{1 - (s-s_0)\mu_i}$$  \hspace{1cm} (7)

where \(S, A\) are the matrices composed of the eigenvectors and diagonal matrix composed of the eigenvalues and \(f = SI, g = S' r\).

The eigenvalue decomposition of \(A\) needs, however, long computational time. For this reason, we apply the Lanczos method to (6) to obtain

$$Y(s) = \lim_{q \to \infty} l'q_i' e_i' \left[ I - (s-s_0)T_q \right]^{-1} e_i$$  \hspace{1cm} (8)

where \(e_i = [1, 0, 0, \ldots, 0]^T \in \mathbb{R}^n\), \(T_q \in \mathbb{R}^{q \times q}\) is a tridiagonal matrix and \(q\) is the number of iteration in the Lanczos process. When \(q\) is set much smaller than \(n\), we can easily perform the eigenvalue decomposition of \(T_q\) to obtain

$$Y(s) = \lim_{q \to \infty} l'q_i' e_i' \left[ I - (s-s_0)S_q A_q^{-1} \right]^{-1} e_i = \sum_{j=1}^{q} \frac{l'q_i' y_j}{1 - (s-s_0)\mu_j}$$  \hspace{1cm} (9)

where \(S_q, A_q\) include the eigenvectors and eigenvalues of \(T_q\) and \(y = S_q e_i\) and \(v = S_q^j e_i\). The admittance is now represented as a rational polynomial of \(s\) in (9).

B. Synthesis of Cauer-Equivalent Circuit

The authors have proposed the synthesis method of the Foster-equivalent circuit shown in Fig. 1 via PVL-based MOR [5]. In this method, we can directly and uniquely synthesize the Foster circuit from (9). However, it would be difficult to consider the magnetic saturation in magnetic cores using the Foster circuit. For the Cauer circuit shown in Fig. 2, which is also called continued-fraction Cauer circuit [4], physical interpretation to the each section could be given: \(R_1\) and \(L_1\) correspond to the DC resistance of winding coils and the inductance at low frequencies, while \(R_i\) and \(L_i, i = 2, 3, \ldots\) are inductor and resistor at each section.

\[
\begin{align*}
\nu_{in} & = \frac{d\Phi(i_2-i_1)}{dt} + R_1i_1 \\
0 & = \frac{d\Phi(i_2-i_1)}{dt} + L_2 \frac{d(i_2-i_1)}{dt} + R_2i_2 \\
& \vdots \\
0 & = \frac{d(i_q-i_{q-1})}{dt} + L_q \frac{d(i_q)}{dt} + R_qi_q
\end{align*}
\]

(12)

Fig. 1 Foster circuit.

Fig. 2 Cauer circuit. The current \(i_0\) denotes the external current while \(i_k(k=1,2,\ldots)\) denote the eddy currents. The flux \(\Phi_k\) generated by \(i_{k-1}\) is dominant at low frequencies.

C. Consideration of Nonlinearity in Magnetic Core

As described above, we introduce the magnetic nonlinearity in magnetic cores only for \(L_1\) [1]. When the eddy currents are dominant as in the case discussed in [10], we would have to introduce nonlinearity also in \(L_i, i=2, 3, \ldots\), as discussed in [3]. The circuit equation for the nonlinear Cauer is given by

\[
\begin{align*}
\nu_{in} & = \frac{d\Phi(i_2-i_1)}{dt} + R_1i_1 \\
0 & = \frac{d\Phi(i_2-i_1)}{dt} + L_2 \frac{d(i_2-i_1)}{dt} + R_2i_2 \\
& \vdots \\
0 & = \frac{d(i_q-i_{q-1})}{dt} + L_q \frac{d(i_q)}{dt} + R_qi_q
\end{align*}
\]

(12)
where \( i \) is the loop current in the \( j \)-th stage of the Cauer circuit. Instead of \( L_j \), we use \( \Phi(i) \) to represent the nonlinearity of the magnetic core. Because \( \Phi(i) \) is generated by the external coil current, \( \Phi(i) \) can be determined by magneto- static field analysis. We compute the magnetic flux \( \Phi(i) \) for different coil currents in the preprocessing.

The synthesis algorithm is summarized as follows:
1. The tri-diagonal matrix \( T_0 \) is generated by the Lanczos process. In this process, the eddy current equation \((5a)\) is repeatedly solved.
2. The rational polynomial \((10)\) is obtained by performing the eigenvalue decomposition of \( T_0 \), and the circuit parameters in the Cauer circuit are obtained by the continued fraction \((11)\).
3. Magnetostatic field is analyzed with FEM to obtain \( \Phi(i) \).

IV. NUMERICAL RESULTS

A. Inductor Model in DC-DC Converter

We synthesize the Cauer-equivalent circuit of the inductor used in the DC-DC converter shown in Fig. 3. The conductivity of the coil is set to \( 5.76 \times 10^7 \) S/m. The magnetic core is assumed to be 50A400. The inductor model is discretized into tetrahedral elements to whose edges 117,548 unknowns are assigned. In PVL-based MOR, we set the number of iteration \( q \) to 5 which corresponds to the number of the stages in the Cauer circuit.

We summarize \( R_0 \), \( L_1 \) obtained by the present method assuming the linear magnetic property in Table I. The computed values of \( R_1 \) and \( L_1 \) correspond to the DC resistance and inductance at low frequency, respectively. The input impedance of the inductor is plotted against frequency in Fig. 4. We can see that the impedance computed from the Cauer circuit is in good agreement with that obtained by FEM.

Next, we analyze the DC-DC converter in which the inductor is modeled by the Cauer circuit. In this converter, \( R_0 \), switching frequency and the duty factor are set to 0.05\( \Omega \), 1MHz and 0.9, respectively. The nonlinear \( \Phi-i \) characteristic of the core material, which is computed from the BH curve of 50A400, shown in Fig. 5 is introduced to \( L_i \). We analyze the time response using conventional FEM, present method and linear Foster-equivalent circuit which is synthesized using the permeability distribution at the driving current \( 2 \). When \( E=5.0V \), \( C_0=1\mu F \) and \( E=1.2V \), \( C_0=100\mu F \), the time response of the current through \( R_0 \) is shown in Fig. 6. It is shown in Fig. 6 that the time responses obtained by FEM and the present method agree well. On the other hand, there are differences between those and the results computed from the Foster circuit. In particular, the differences are rather large in the transients.

B. Induction Heating Model

We consider the induction heating model shown in Fig. 7. The conductivity of the conducting plate to be heated is set to \( 1.25 \times 10^6 \) S/m. The magnetic core is again assumed to be composed of 50A400. The induction heating model is discretized into tetrahedral elements which have 692,797 unknowns at the edges. In PVL-based MOR, \( q \) is set to 5.

In Table II, the values of \( R_1 \) and \( L_1 \) resulted from the present method assuming the linear magnetic property are summarized. The computed values of \( R_1 \) and \( L_1 \) are found to coincide with the DC resistance and inductance computed by static analysis, respectively. In the Cauer circuit of the induction heating model, the higher element \( R_j \) and \( L_j \), \( j=2,3,... \) correspond to the resistance and the inductance relevant to the eddy currents in the metallic plate. Figure 8 shows the frequency dependence of the input impedance of the coil obtained by FEM and the Cauer circuit. They are in good agreement.
We analyze the induction heating model in time-domain considering the magnetic nonlinearity in the magnetic core. By performing the magneto-static field analysis, we compute $\Phi$ - $i$ characteristic of the core, which is plotted in Fig. 9. We introduce the nonlinear characteristic to $L_1$ in the Cauer circuit. In Fig. 10, we plot the time variation of the current to the coil winding. The solutions in the both transient and steady states obtained by FEM agree well with those by the nonlinear Cauer circuit even when the voltage $E$ changes, although the linear circuit has errors in the transients.

The Joule losses in the steady state evaluated by FEM and the Cauer circuit are shown in Fig. 11. From the Cauer circuit, we compute the Joule loss as follows:

$$W_e = \sum_{j=2}^{n} R_j i_j^2$$  \hspace{1cm} (13)

where $i_j$ is the current to $R_j$. As can be seen in Fig. 11, the Joule loss obtained by the Cauer circuit is in good agreement with that obtained by FEM.

V. CONCLUSION

We have proposed a novel method to synthesize the Cauer circuit from the FEM model of electric apparatus using PVL-based MOR. The synthesized Cauer circuit can express the nonlinear magnetic property of magnetic cores. It is shown that the Cauer circuit provides accurate results in both frequency and time domains. We plan to study the property of the physical Cauer circuit which is synthesized from (10).

REFERENCES


