Coefficient stripping in the matricial Nehari problem

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Abstract

This note deals with a matricial Schur function arising from a completely indeterminate Nehari problem. The Schur algorithm is characterized by a unilateral shift for a Nehari sequence.

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1. Introduction

In [12], the authors focused on a class of probability measures on the unit circle relevant to the indeterminate Nehari problems, and established fundamental results on the correspondence between the Nehari sequences and the Verblunsky coefficients, which are also known as the Schur parameters. The aim of this note is to present some matricial extensions of their results, answering an open question posed by the second author [7]. In particular, it will be shown that the Schur algorithm is induced by "coefficient stripping" for a Nehari sequence; the term, quoted from Simon [18], means a unilateral shift defined by dropping the first entry of a sequence.

Let $\mathscr{V}$ be a complex Euclidean space and $\mathscr{M}$ the space of square matrices of corresponding order. Denote by $\mathbf{0}$ the zero matrix and by $\mathbf{1}$ the unit matrix in $\mathscr{M}$. As usual, a^* stands for the Hermitian conjugate of a, and the symbols $a > \mathbf{0}$ and $a \geq \mathbf{0}$ mean that a is Hermitian, positive definite and positive semi-definite, respectively. For $1 \leq p \leq \infty$, let L^p be the standard Lebesgue space on the unit circle $\mathbb{T}$, and H^p the associated Hardy space, which is a closed subspace of L^p composed of functions having natural analytic extensions into the open unit disc $\mathbb{D}$. Also, write $L^p_{\mathscr{M}}$ and $H^p_{\mathscr{M}}$ for the spaces of $\mathscr{M}$-valued functions with entries in L^p and H^p, respectively. See Rosenblum–Rovnyak [17] for the theory of matrix/operator-valued Hardy functions.

A function f in $H^\infty_{\mathscr{M}}$ is called a *Schur function* if $f(z)^* f(z) \leq \mathbf{1}$ (a.e.). In the non-trivial case, it yields a sequence of Schur functions $f_1, f_2, \ldots$ ($f_1 = f$) via the *Schur recurrence formula*

$$f_{n+1} = z^{-1}(\rho_n^R)^{-1}(f_n - \alpha_n)(\mathbf{1} - \alpha_n^* f_n)^{-1}(\rho_n^L)^* \tag{1.1}$$

with the *Schur parameters* $\alpha_n = f_n(0)$ and subordinate matrices ρ_n^L, ρ_n^R obeying

$$(\rho_n^L)^* \rho_n^L = \mathbf{1} - \alpha_n^* \alpha_n, \qquad \rho_n^R (\rho_n^R)^* = \mathbf{1} - \alpha_n \alpha_n^*.$$

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Here, ρ_n^L, ρ_n^R are unique up to constant unitary factors, and usually chosen so that $\rho_n^L > \mathbf{0}, \rho_n^R > \mathbf{0}$. On the other hand, a Schur function f is associated with a *measure* μ, defined on $\mathbb{T}$ and taking values in the positive semi-definite matrices in $\mathscr{M}$, via the Herglotz formula

$$(\mathbf{1} + zf(z))(\mathbf{1} - zf(z))^{-1} = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu \qquad (z \in \mathbb{D}),$$

and this association $f \leftrightarrow \mu$ is a one-to-one correspondence between the set of Schur functions and the set of measures on $\mathbb{T}$ normalized so $\mu(\mathbb{T}) = \mathbf{1}$. For such a measure μ, one may define the $\mathscr{M}$-valued orthogonal polynomials with respect to the $\mathscr{M}$-valued "inner products"

$$\langle\langle \varphi, \psi \rangle\rangle_L = \int_{\mathbb{T}} \varphi\, d\mu\, \psi^*, \qquad \langle\langle \varphi, \psi \rangle\rangle_R = \int_{\mathbb{T}} \varphi^* d\mu\, \psi.$$

The Geronimus theorem states that, in the non-trivial case, the orthonormal polynomials

$$\begin{aligned} \varphi_n^L &= \kappa_n^L z^n + \text{lower order}, \qquad \langle\langle \varphi_m^L, \varphi_n^L \rangle\rangle_L = \delta_{mn}\mathbf{1}, \qquad \kappa_n^L = \{(\rho_n^L \cdots \rho_2^L \rho_1^L)^*\}^{-1}, \\ \varphi_n^R &= \kappa_n^R z^n + \text{lower order}, \qquad \langle\langle \varphi_m^R, \varphi_n^R \rangle\rangle_R = \delta_{mn}\mathbf{1}, \qquad \kappa_n^R = \{(\rho_1^R \rho_2^R \cdots \rho_n^R)^*\}^{-1}, \end{aligned}$$

obey the *Szegő recurrence formula*

$$z\varphi_n^L = (\rho_{n+1}^L)^* \varphi_{n+1}^L + \alpha_{n+1}^* (\varphi_n^R)^\dagger, \qquad z\varphi_n^R = \varphi_{n+1}^R (\rho_{n+1}^R)^* + (\varphi_n^L)^\dagger \alpha_{n+1}^*,$$

where $\varphi^\dagger$ is the reversed polynomial of φ, defined by $\varphi^\dagger(z) = z^n \varphi(1/\bar{z})^*$ if $\deg(\varphi) = n$. In this case, ρ_n^L, ρ_n^R are sometimes chosen so that $\kappa_n^L > \mathbf{0}$, $\kappa_n^R > \mathbf{0}$. See Damanik–Pushnitski–Simon [8] for details and background, and also Simon [18, 19] for further information.

Let g be a function in $H^1_{\mathscr{M}}$ having invertible values (a.e.). It admits the polar decompositions $g = u(g^*g)^{1/2} = (gg^*)^{1/2}u$, where u is the unitary factor, and the allied factorization $g = g_L g_R$ with a pair of functions g_L, g_R in $H^2_{\mathscr{M}}$ satisfying $g_L^* g_L = g_R g_R^*$. Then g is called *rigid* if the functions in $H^1_{\mathscr{M}}$ sharing with it the same unitary factor u are of the form $g_L k g_R$ for a constant matrix $k > \mathbf{0}$. Let m be the normalized Lebesgue measure on $\mathbb{T}$, and write $d\mu = w\,dm + d\mu_s$, where μ_s is the singular part. If the *Szegő condition* $\log\det(w) \in L^1$ is fulfilled, there is a unique pair of outer functions h_L, h_R in $H^2_{\mathscr{M}}$, called *Szegő functions*, such that

$$w = h_L^* h_L = h_R h_R^*, \qquad h_L(0) > \mathbf{0}, \qquad h_R(0) > \mathbf{0}.$$

This note is mainly concerned with a measure μ such that

$$\mu_s = 0, \qquad \log\det(w) \in L^1, \qquad h_L h_R \text{ is rigid}, \tag{1.2}$$

which goes back to Levinson–McKean [15]. See Kasahara–Inoue–Pourahmadi [14] for a general concept of $\mathscr{M}$-valued rigid functions and its application to $\mathscr{V}$-valued stationary processes.

Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a sequence of matrices in $\mathscr{M}$. The problem of finding the functions in the unit ball of $L^\infty_{\mathscr{M}}$ with γ as negatively-indexed Fourier coefficients is called the *Nehari problem* after Nehari [16]; the Nehari theorem states that a solution exists if and only if an infinite block Hankel matrix $(\gamma_{i+j-1})_{i,j=1}^\infty$ acts as a contraction on the ℓ^2-space of $\mathscr{V}$-valued sequences. In the so-called *completely indeterminate* case (see Section 3 below), the problem was fully solved by Adamjan [1], extending the work of Adamjan–Arov–Krein [2], as follows: There is a unique Schur function f which corresponds to a measure μ obeying (1.2), and the solutions ϕ are parametrized by Schur functions ξ in such a way that

$$\phi = (h_L^*)^{-1}h_R + h_L(\mathbf{1} - zf)\{\xi(\mathbf{1} - zf\xi)^{-1} - (\mathbf{1} - zf)^{-1}\}(\mathbf{1} - zf)h_R.$$

See also Arov [3], Arov–Fritzsche–Kirstein [6] and Arov–Dym [4] for relevant results, and Arov–Dym [5] for a textbook account on the Nehari problem.

A sequence γ will be called a *Nehari sequence* if it gives rise to a completely indeterminate Nehari problem. Adamjan's result defines a one-to-one correspondence $\gamma \leftrightarrow f$ between the set of Nehari sequences and the set of Schur functions restricted by the condition (1.2) for its μ. As will be shown later, if $(\gamma_1, \gamma_2, \ldots)$ is a Nehari sequence, $(\gamma_2, \gamma_3, \ldots)$ is also a Nehari sequence. Hence, a sequence of Schur functions $f_1, f_2, \ldots$ can be derived from γ by coefficient stripping, namely, via $(\gamma_n, \gamma_{n+1}, \ldots) \leftrightarrow f_n$. They enter the Schur algorithm in the following way.

Theorem 1.1. *Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a Nehari sequence, with associated Szegő functions h_L, h_R. Then the Schur functions $f_1, f_2, \ldots$ obtained by coefficient stripping satisfy the Schur recurrence formula* (1.1)*, where ρ_n^L, ρ_n^R are determined by the condition*

$$\kappa_n^L h_L(0) > \mathbf{0}, \qquad h_R(0)\kappa_n^R > \mathbf{0}. \tag{1.3}$$

From the viewpoint of coefficient stripping for the Schur parameters, the above relation may be regarded as a correspondence $(\gamma_n, \gamma_{n+1}, \ldots) \leftrightarrow (\alpha_n, \alpha_{n+1}, \ldots)$. The condition (1.3) should be compared with the standard choices

$$\rho_n^L > \mathbf{0}, \quad \rho_n^R > \mathbf{0}; \qquad\qquad \kappa_n^L > \mathbf{0}, \quad \kappa_n^R > \mathbf{0}.$$

Notice that (1.3) is not a choice but an outcome from coefficient stripping for a Nehari sequence; however, the correspondence $\gamma \leftrightarrow f$ depends on a choice $h_L(0) > \mathbf{0}$, $h_R(0) > \mathbf{0}$. In the language of orthogonal polynomials, (1.3) means that

$$\langle\langle(\varphi_n^L)^\dagger, h_L^{-1}\rangle\rangle_R > \mathbf{0}, \qquad \langle\langle h_R^{-1}, (\varphi_n^R)^\dagger\rangle\rangle_L > \mathbf{0},$$

which might be viewed as a natural choice; h_L^{-1}, h_R^{-1} are the "limits" of $(\varphi_n^L)^\dagger, (\varphi_n^R)^\dagger$ as $n \to \infty$.

The following is a fundamental result on the inheritance of property (1.2) under coefficient stripping for the Schur parameters $(\alpha_1, \alpha_2, \ldots)$. Note that ρ_n^L, ρ_n^R can be freely chosen here.

Theorem 1.2. *Let $f_1, f_2, \ldots$ be Schur functions obeying the Schur recurrence formula* (1.1)*. Then either all of them correspond to measures satisfying* (1.2)*, or none of them do.*

After some preparation in Section 2, the above theorems will be established in Section 3. In Appendix, a few simple examples will be given in order to illustrate the correspondence

$$\gamma = (\gamma_1, \gamma_2, \ldots) \quad \leftrightarrow \quad \alpha = (\alpha_1, \alpha_2, \ldots).$$

The latter is interpreted as the *partial autocorrelation function* in the finite prediction problem for a $\mathscr{V}$-valued stationary process, and a Nehari sequence plays a crucial role there if the spectral measure satisfies (1.2). In particular, α_n can be expressed in terms of $(\gamma_n, \gamma_{n+1}, \ldots)$ and also $h_L(\mathbf{0})$, $h_R(\mathbf{0})$, subject to $\kappa_n^L > \mathbf{0}$, $\kappa_n^R > \mathbf{0}$, see Inoue–Kasahara–Pourahmadi [11]. Recently, the authors [13] proved Baxter's theorem which asserts that γ is summable if and only if so is α. These results mostly answer an open question posed by the second author [7], while an important problem remains open: Strong Szegő theorem with a Nehari sequence.

2. γ-generating matrices

In this section, we prepare some basic matters on the γ-generating matrices, which are useful for studying completely indeterminate Nehari problems. Details and proofs omitted here can be found in Arov–Dym [5] and Dubovoj–Fritzsche–Kirstein [9].

Let $\mathscr{V}$ be a complex Euclidean space and $\mathscr{M}$ the space of square matrices of corresponding order, in which a matrix a is assigned the Euclidean norm $\|a\|_{\mathscr{M}}$ as a bounded linear operator $x \mapsto ax$ on $\mathscr{V}$. The following three conditions are equivalent:

$$\|a\|_{\mathscr{M}} \leq 1; \qquad a^* a \leq \mathbf{1}; \qquad aa^* \leq \mathbf{1}.$$

For $1 \leq p \leq \infty$, let L^p be the standard Lebesgue space on the unit circle $\mathbb{T}$, and H^p the associated Hardy space, which is a closed subspace of L^p composed of functions having analytic extensions into the open unit disc $\mathbb{D}$. Also, let N^+ be the Smirnov class, which is an algebra of all quotients ξ/η with functions ξ, η in H^∞, where η is outer. These three kinds of spaces meet in

$$H^p = L^p \cap N^+.$$

Let $L^p_{\mathscr{M}}, H^p_{\mathscr{M}}$ and $N^+_{\mathscr{M}}$ denote the spaces of $\mathscr{M}$-valued functions with entries in L^p, H^p and N^+, respectively. By introducing an appropriate norm, $L^p_{\mathscr{M}}$ becomes a Banach space with $H^p_{\mathscr{M}}$ a closed subspace. As for $L^\infty_{\mathscr{M}}$, set

$$\|f\|_{L^\infty_{\mathscr{M}}} = \operatorname{ess\,sup}\{\|f(z)\|_{\mathscr{M}} \mid z \in \mathbb{T}\}.$$

Let $S_{\mathscr{M}}$ be the set of Schur functions, in other words, the unit ball of $H^\infty_{\mathscr{M}}$. For a function f in $S_{\mathscr{M}}$, the following three conditions are equivalent:

$$\log(1 - \|f\|_{\mathscr{M}}) \in L^1; \qquad \log\det(\mathbf{1} - f^* f) \in L^1; \qquad \log\det(\mathbf{1} - f f^*) \in L^1.$$

Recall that the Herglotz formula defines a one-to-one correspondence $f \leftrightarrow \mu$ between $S_{\mathscr{M}}$ and the set of measures on $\mathbb{T}$ with $\mu(\mathbb{T}) = \mathbf{1}$. With $d\mu = w dm + d\mu_s$ as before, the Szegő condition $\log\det(w) \in L^1$ is equivalent to one (hence, all) of the three conditions just mentioned. For this reason, $\log(1 - \|f\|_{\mathscr{M}}) \in L^1$ will also be called the Szegő condition.

A 2×2 block matrix A with entries in $\mathscr{M}$ is called *J-unitary* if $A^* J A = J$, where

$$J = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}.$$

It brings the fractional linear transformation T_A defined by

$$T_A(x) = (ax + b)(cx + d)^{-1} \quad \text{with} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which acts as a bijection from the unit ball of $\mathscr{M}$ to itself; $cx + d$ is invertible if x lies in the ball. If A and B are J-unitary, AB is also J-unitary, and $T_{AB} = T_A T_B$. Notice that A is J-unitary if and only if A^* is so. These basic matters can be found in [5, Sections 2.2 and 2.3].

According to Arov [3], a *γ-generating matrix* $\mathfrak{A}$ is a matrix-valued function on $\mathbb{T}$ of the form

$$\mathfrak{A} = \begin{pmatrix} a_L^* & b_L^* \\ b_R & a_R \end{pmatrix},$$

where a_L, a_R, b_L, b_R are functions in $N^+_{\mathscr{M}}$, a_L, a_R are outer, and $\mathfrak{A}$ has J-unitary values (a.e), so

$$a_L^* a_L - b_L^* b_L = \mathbf{1}, \qquad a_R a_R^* - b_R b_R^* = \mathbf{1}, \qquad b_L a_L^{-1} = a_R^{-1} b_R.$$

Put $\chi = -b_L a_L^{-1} = -a_R^{-1} b_R$. Then the functions a_L^{-1}, a_R^{-1} and χ lie in $S_{\mathscr{M}}$, in view of

$$\mathbf{1} - \chi^* \chi = (a_L^{-1})^* (a_L^{-1}), \qquad \mathbf{1} - \chi \chi^* = (a_R^{-1})(a_R^{-1})^*.$$

Also, since a_L and a_R are outer, χ satisfies $\log(1 - \|\chi\|_{\mathscr{M}}) \in L^1$. Such a function χ can be traced back to a γ-generating matrix $\mathfrak{A}$, which is unique up to a constant unitary block-diagonal left factor depending on the choice of a_L and a_R. A γ-generating matrix $\mathfrak{A}$ is called *normalized* if $a_L(0) > \mathbf{0}$, $a_R(0) > \mathbf{0}$, $b_L(0) = \mathbf{0}$ and $b_R(0) = \mathbf{0}$. Every γ-generating matrix can be normalized by multiplying by an appropriate constant J-unitary matrix on the right. The important point here is that all the functions in $T_{\mathfrak{A}}(S_{\mathscr{M}})$ have common negatively-indexed Fourier coefficients. Indeed, the difference of two functions in $T_{\mathfrak{A}}(S_{\mathscr{M}})$ is analytic: For any Schur functions ξ, η,

$$T_{\mathfrak{A}}(\xi) - T_{\mathfrak{A}}(\eta) = a_L^{-1} \{\xi(\mathbf{1} - \chi\xi)^{-1} - \eta(\mathbf{1} - \chi\eta)^{-1}\} a_R^{-1}.$$

A γ-generating matrix $\mathfrak{A}$ is called *regular* if $T_{\mathfrak{A}}(S_{\mathscr{M}}) = T_{\mathfrak{B}}(S_{\mathscr{M}})$ whenever $T_{\mathfrak{A}}(S_{\mathscr{M}}) \subset T_{\mathfrak{B}}(S_{\mathscr{M}})$ holds for a γ-generating matrix $\mathfrak{B}$ (cf. [3, Theorem 3]). As one might expect, the solution set of the Nehari problem in question can be expressed as $T_{\mathfrak{A}}(S_{\mathscr{M}})$ for some regular γ-generating matrix $\mathfrak{A}$. Moreover, $\mathfrak{A}$ can be normalized without changing its range since $T_{\mathfrak{A}\mathfrak{C}}(S_{\mathscr{M}}) = T_{\mathfrak{A}}(S_{\mathscr{M}})$ holds for every constant J-unitary matrix $\mathfrak{C}$. See [5, Section 7.2] for more information.

It is convenient to parametrize normalized γ-generating matrices as follows.

Lemma 2.1. *Between the normalized γ-generating matrices $\mathfrak{A}$ and the Schur functions f obeying the Szegő condition* $\log(1 - \|f\|_{\mathscr{M}}) \in L^1$*, there is a one-to-one correspondence*

$$\mathfrak{A} = \begin{pmatrix} s_L^* & -\bar{z} t_L^* \\ -z t_R & s_R \end{pmatrix} \quad \leftrightarrow \quad f = t_L s_L^{-1} = s_R^{-1} t_R,$$

via functions s_L, s_R, t_L, t_R in $N^+_{\mathscr{M}}$ such that s_L, s_R are outer, $s_R(0) > \mathbf{0}$, $s_L(0) > \mathbf{0}$, and

$$s_L^* s_L - t_L^* t_L = \mathbf{1}, \qquad s_R s_R^* - t_R t_R^* = \mathbf{1}, \qquad t_L s_L^{-1} = s_R^{-1} t_R.$$

In this case, Szegő functions h_L, h_R of μ corresponding to f can be expressed as

$$h_L = (s_L - z t_L)^{-1}, \qquad h_R = (s_R - z t_R)^{-1}. \tag{2.1}$$

Proof. The correspondence $\mathfrak{A} \leftrightarrow f$ is plain except for the following point: If f obeys the Szegő condition, there are unique outer functions s_L, $s_R \in N^+_{\mathscr{M}}$ with $s_L(0) > \mathbf{0}$, $s_R(0) > \mathbf{0}$ such that

$$s_L s_L^* = (\mathbf{1} - f^* f)^{-1}, \qquad s_R^* s_R = (\mathbf{1} - f f^*)^{-1}$$

(cf. [5, Section 3.16]). As for Szegő functions, notice that both $s_L - z t_L$ and $s_R - z t_R$ are outer because $1 - zf$ is so. Since $w = h_L^* h_L = h_R h_R^*$ imply

$$h_L^* h_L = (\mathbf{1} - \bar{z} f^*)^{-1} (\mathbf{1} - f^* f)(\mathbf{1} - z f)^{-1} = (s_L^* - \bar{z} t_L^*)^{-1} (s_L - z t_L)^{-1},$$
$$h_R h_R^* = (\mathbf{1} - z f)^{-1} (\mathbf{1} - f f^*)(\mathbf{1} - \bar{z} f^*)^{-1} = (s_R - z t_R)^{-1} (s_R^* - \bar{z} t_R^*)^{-1},$$

the last statement follows from the uniqueness of outer functions. □

Accordingly, a normalized γ-generating matrix $\mathfrak{A}$ and a measure μ with the Szegő condition $\log(w) \in L^1$ are associated with each other, via a Schur function f obeying $\log(1 - \|f\|_{\mathscr{M}}) \in L^1$. Recall $d\mu = wdm + d\mu_s$, where μ_s is the singular part. Since $w = h_L^* h_L = h_R h_R^*$, the product $h_L h_R$ admits the polar decompositions $h_L h_R = u\,(h_R^* h_R) = (h_L h_L^*)\,u$ with the unitary factor

$$u = h_L (h_R^*)^{-1} = (h_L^*)^{-1} h_R.$$

Arov–Dym [4, Theorem 5.5] showed that $\mathfrak{A}$ is regular if and only if $\mu_s = 0$ and $\mathrm{index}\{u\} = 0$, which means the following property: If two functions g_L, g_R in $H^2_{\mathscr{M}}$ have invertible values (a.e.) and satisfy $u = (g_L^*)^{-1} g_R$, they are expressed as $g_L = h_L c^*$, $g_R = c h_R$ with an invertible matrix c.

The regularity can also be characterized by rigidity of the product of Szegő functions.

Lemma 2.2. *A normalized γ-generating matrix $\mathfrak{A}$ is regular if and only if μ satisfies* (1.2).

Proof. It is to be shown that $\mathrm{index}\{u\} = 0$ if and only if $h_L h_R$ is rigid. Let g be a function in $H^1_{\mathscr{M}}$ having invertible values (a.e.). It can be expressed as $g = g_L g_R$, where g_L, g_R lie in $H^2_{\mathscr{M}}$ and obey $g_L^* g_L = g_R g_R^*$ (cf. Helson–Lowdenslager [10, Theorem 10]). Then $(g_L^*)^{-1} g_R$ is its unitary factor. Thus, if $\mathrm{index}\{u\} = 0$ holds, $u = (g_L^*)^{-1} g_R$ makes $g = h_L (c^* c) h_R$ with c invertible, so $h_L h_R$ is rigid. For the converse half, let $u = (g_L^*)^{-1} g_R$. If $h_L h_R$ is rigid, $g_L g_R = h_L k h_R$ for some $k > \mathbf{0}$, whence

$$g_L g_L^* = h_L k h_L^*, \qquad g_R^* g_R = h_R^* k h_R.$$

Further, $g_L g_R$ is also rigid, and g_L, g_R are outer (cf. Kasahara–Inoue–Pourahmadi [14, p. 294]). Hence, $g_L = h_L c_L^*$ and $g_R = c_R h_R$ hold for constants c_L, c_R with $k = c_L^* c_L = c_R^* c_R$, but these lead to $c_L^{-1} c_R = h_L^* u h_R^{-1} = \mathbf{1}$, so $c_L = c_R$, concluding that $\mathrm{index}\{u\} = 0$. □

3. Nehari problem

In this section, we discuss coefficient stripping in a completely indeterminate Nehari problem, and prove Theorems 1.1 and 1.2. See Arov–Dym [5] for a textbook account of the problem.

Let $\gamma = (\gamma_1, \gamma_2, \dots)$ be a sequence of matrices in $\mathscr{M}$. The *Nehari problem* is formulated as the problem of finding the functions in the unit ball of $L^\infty_{\mathscr{M}}$ having γ as negatively-indexed Fourier coefficients, that is, describing the solution set

$$\mathcal{N}(\gamma) = \left\{ \phi \in L^\infty_{\mathscr{M}} \;\middle|\; \|\phi\|_{L^\infty_{\mathscr{M}}} \le 1 \text{ and } \gamma_k = \int_{\mathbb{T}} z^k \phi\, dm \text{ for } k = 1, 2, \dots \right\}.$$

In the solvable case, the mean values of the solutions form a matrix ball, namely,

$$\left\{ \int_{\mathbb{T}} \phi\, dm \;\middle|\; \phi \in \mathcal{N}(\gamma) \right\} = \{c + r_L x r_R \mid x \in \mathscr{M}, \|x\|_{\mathscr{M}} \le 1\}$$

for some matrices c, r_L, r_R in $\mathscr{M}$ with $r_L \ge \mathbf{0}$, $r_R \ge \mathbf{0}$. The problem is called *determinate* if it has a unique solution, so *indeterminate* otherwise, and *completely indeterminate* if $r_L > \mathbf{0}$, $r_R > \mathbf{0}$. Let us call γ a *Nehari sequence* if it provides a completely indeterminate Nehari problem. As the name indicates, a γ-generating matrix $\mathfrak{A}$ actually generates a Nehari sequence γ such that $T_{\mathfrak{A}}(S_{\mathscr{M}}) \subset \mathcal{N}(\gamma)$, that is,

$$\gamma_k = \int_{\mathbb{T}} z^k T_{\mathfrak{A}}(\xi)\, dm \qquad k = 1, 2, \dots,$$

where ξ is a Schur function, and γ does not depend on the choice of ξ (cf. [5, Theorem 7.22]).

A fractional linear parametrization of the solution set of a completely indeterminate Nehari problem was obtained by Adamjan–Arov–Krein [2] in the scalar case, and by Adamjan [1] in the matrix/operator case. To spell it out, for a Nehari sequence γ, there is a unique normalized regular γ-generating matrix $\mathfrak{A}$ such that

$$\mathcal{N}(\gamma) = T_{\mathfrak{A}}(S_{\mathscr{M}}).$$

Notice that, by Lemmas 2.1 and 2.2, γ is associated with a Schur function f, and its measure μ satisfies (1.2). In fact, the fractional linear transformation $T_{\mathfrak{A}}$ was originally derived from

$$T_{\mathfrak{A}}(\xi) = (h_L^*)^{-1}h_R + h_L(\mathbf{1} - zf)\{\xi(\mathbf{1} - zf\xi)^{-1} - (\mathbf{1} - zf)^{-1}\}(\mathbf{1} - zf)h_R, \tag{3.1}$$

where h_L, h_R are Szegő functions of μ. A solution $T_{\mathfrak{A}}(\xi)$ becomes a unitary factor of some rigid function in $H^1_{\mathscr{M}}$ (in other words, $\mathrm{index}\{T_{\mathfrak{A}}(\xi)\} = 0$) if and only if ξ is a constant unitary matrix. As for the matrix ball stated above,

$$c = \int_{\mathbb{T}} (h_L^*)^{-1}h_R dm - (h_L h_R)(0), \qquad r_L = h_L(0), \qquad r_R = h_R(0).$$

To parametrize the interior of the matrix ball, write

$$\mathbb{D}_{\mathscr{M}} = \{\zeta \in \mathscr{M} \mid \|\zeta\|_{\mathscr{M}} < 1\}.$$

A Nehari sequence has the following one-step extension.

Proposition 3.1. *Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a Nehari sequence, f its Schur function, and h_L, h_R the associated Szegő functions. Also, let $\zeta \in \mathbb{D}_{\mathscr{M}}$ and define*

$$\omega_\zeta = \int_{\mathbb{T}} (h_L^*)^{-1}h_R dm - h_L(0)(\mathbf{1} - \zeta)h_R(0).$$

Then $\hat{\gamma} = (\omega_\zeta, \gamma_1, \gamma_2, \ldots)$ is a Nehari sequence, and its Schur function $\hat{f}$ is expressed as

$$\hat{f} = (\rho_R^*)^{-1}(zf - \zeta^*)(\mathbf{1} - \zeta zf)^{-1}\rho_L,$$

where ρ_L, ρ_R are determined by the condition

$$\rho_L\rho_L^* = \mathbf{1} - \zeta\zeta^*, \qquad h_L(0)\rho_L > \mathbf{0}, \qquad \rho_R^*\rho_R = \mathbf{1} - \zeta^*\zeta, \qquad \rho_R h_R(0) > \mathbf{0}.$$

Proof. Let $\mathfrak{A}$ be a normalized regular γ-generating matrix for γ, so $\mathcal{N}(\gamma) = T_{\mathfrak{A}}(S_{\mathscr{M}})$, and write

$$\mathfrak{A} = \begin{pmatrix} s_L^* & -\bar{z}t_L^* \\ -zt_R & s_R \end{pmatrix}, \qquad \mathfrak{C} = \begin{pmatrix} (\rho_L^*)^{-1} & \zeta\rho_R^{-1} \\ \zeta^*(\rho_L^*)^{-1} & \rho_R^{-1} \end{pmatrix}.$$

It follows from (2.1) that

$$\rho_L^{-1}s_L(0) > \mathbf{0}, \qquad s_R(0)\rho_R^{-1} > \mathbf{0}.$$

Using the product $\mathfrak{A}\mathfrak{C}$, define a normalized γ-generating matrix $\hat{\mathfrak{A}}$ by

$$\mathfrak{A}\mathfrak{C} = \begin{pmatrix} \hat{s}_L^* & -\hat{t}_L^* \\ -\hat{t}_R & \hat{s}_R \end{pmatrix}, \qquad \hat{\mathfrak{A}} = \begin{pmatrix} \hat{s}_L^* & -\bar{z}\hat{t}_L^* \\ -z\hat{t}_R & \hat{s}_R \end{pmatrix},$$

in which

$$\begin{cases} \hat{s}_L = \rho_L^{-1}(s_L - \zeta z t_L) \\ \hat{t}_L = (\rho_R^*)^{-1}(z t_L - \zeta^* s_L), \end{cases} \qquad \begin{cases} \hat{s}_R = (s_R - z t_R \zeta)\rho_R^{-1} \\ \hat{t}_R = (z t_R - s_R \zeta^*)(\rho_L^*)^{-1}. \end{cases} \tag{3.2}$$

Then $zT_{\hat{\mathfrak{A}}}(\mathbf{1}) = T_{\mathfrak{A}\mathfrak{C}}(z\mathbf{1})$. Also, $\xi = T_{\mathfrak{C}}(z\mathbf{1})$ lies in $S_{\mathscr{M}}$ and satisfies $\xi(0) = \zeta$. Hence, by (3.1),

$$\int_{\mathbb{T}} zT_{\hat{\mathfrak{A}}}(\mathbf{1})dm = \int_{\mathbb{T}} T_{\mathfrak{A}}(\xi)dm = \int_{\mathbb{T}} (h_L^*)^{-1}h_R dm - h_L(0)(\mathbf{1} - \xi(0))h_R(0) = \omega_\zeta,$$

and $\mathcal{N}(\gamma) = T_{\mathfrak{A}}(S_{\mathscr{M}})$ implies that, for $k = 1, 2, \ldots$,

$$\int_{\mathbb{T}} z^{k+1} T_{\hat{\mathfrak{A}}}(\mathbf{1})d\mu = \int_{\mathbb{T}} z^k T_{\mathfrak{A}}(\xi)d\mu = \gamma_k.$$

Thus, $T_{\hat{\mathfrak{A}}}(S_{\mathscr{M}}) \subset \mathcal{N}(\hat{\gamma})$, and $\hat{\gamma}$ is a Nehari sequence (cf. [5, Theorem 7.22]). To prove the opposite inclusion, take a solution ϕ from $\mathcal{N}(\hat{\gamma})$. Since $z\phi$ lies in $\mathcal{N}(\gamma)$, there is a function η in $S_{\mathscr{M}}$ such that $z\phi = T_{\mathfrak{A}}(\eta)$, and the value $\eta(0) = \zeta$ is evaluated from

$$\omega_\zeta = \int_{\mathbb{T}} z\phi dm = \int_{\mathbb{T}} T_{\mathfrak{A}}(\eta)dm = \int_{\mathbb{T}} (h_L^*)^{-1}h_R dm - h_L(0)(\mathbf{1} - \eta(0))h_R(0),$$

so that $\check{\eta} = \bar{z}T_{\mathfrak{C}^{-1}}(\eta)$ is a Schur function:

$$\check{\eta} = \bar{z}T_{\mathfrak{C}^{-1}}(\eta) = z^{-1}\rho_L^{-1}(\eta - \zeta)(\mathbf{1} - \zeta^*\eta)^{-1}\rho_R^*.$$

Then $T_{\mathfrak{A}}(\eta) = T_{\mathfrak{A}\mathfrak{C}}(z\check{\eta}) = zT_{\hat{\mathfrak{A}}}(\check{\eta})$, and $\phi = T_{\hat{\mathfrak{A}}}(\check{\eta})$ shows that $\mathcal{N}(\hat{\gamma}) \subset T_{\hat{\mathfrak{A}}}(S_{\mathscr{M}})$. Consequently, $\mathcal{N}(\hat{\gamma}) = T_{\hat{\mathfrak{A}}}(S_{\mathscr{M}})$, and $\hat{\mathfrak{A}}$ is a normalized regular γ-generating matrix for $\hat{\gamma}$. To complete the proof, use Lemma 2.1 to write down $\hat{f} = \hat{t}_L\hat{s}_L^{-1}$ in terms of $f = t_L s_L^{-1}$. □

Theorem 1.1 will be proved using the following basic facts. Recall ω_ζ from Proposition 3.1.

Lemma 3.2. *Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a Nehari sequence. Then the following hold:*

(i) *$\check{\gamma} = (\gamma_2, \gamma_3, \ldots)$ is a Nehari sequence.*
(ii) *$\hat{\gamma} = (\omega, \gamma_1, \gamma_2, \ldots)$ is a Nehari sequence if and only if ω lies in $\{\omega_\zeta \mid \zeta \in \mathbb{D}_{\mathscr{M}}\}$.*

Proof. Let $\mathfrak{A}$ be a normalized regular γ-generating matrix for γ, so $\mathcal{N}(\gamma) = T_{\mathfrak{A}}(S_{\mathscr{M}})$.
(i) $\check{\gamma}$ is generated by a γ-generating matrix $\mathfrak{B}$ such that $zT_{\mathfrak{A}}(\mathbf{1}) = T_{\mathfrak{B}}(z\mathbf{1})$; it is obtained by

$$\mathfrak{A} = \begin{pmatrix} s_L^* & -\bar{z}t_L^* \\ -zt_R & s_R \end{pmatrix}, \qquad \mathfrak{B} = \begin{pmatrix} s_L^* & -t_L^* \\ -t_R & s_R \end{pmatrix}.$$

(ii) If $\hat{\gamma}$ is a Nehari sequence, it is associated with a normalized regular γ-generating matrix $\hat{\mathfrak{A}}$. Since $zT_{\hat{\mathfrak{A}}}(\mathbf{1})$ lies in $\mathcal{N}(\gamma)$, there is a function ξ in $S_{\mathscr{M}}$ such that $zT_{\hat{\mathfrak{A}}}(\mathbf{1}) = T_{\mathfrak{A}}(\xi)$. By (3.1),

$$\omega = \int_{\mathbb{T}} zT_{\hat{\mathfrak{A}}}(\mathbf{1})dm = \int_{\mathbb{T}} T_{\mathfrak{A}}(\xi)dm = \int_{\mathbb{T}} (h_L^*)^{-1}h_R dm - h_L(0)(\mathbf{1} - \xi(0))h_R(0).$$

Here, ξ is not a constant unitary matrix since $T_{\hat{\mathfrak{A}}}(\mathbf{1}) = (\hat{h}_L^*)^{-1}\hat{h}_R$ shows that $zT_{\hat{\mathfrak{A}}}(\mathbf{1})$ is the unitary factor of a non-rigid function $z\hat{h}_L\hat{h}_R$, where $\hat{h}_L, \hat{h}_R$ are Szegő functions associated with $\hat{\gamma}$. Hence, ω lies in $\{\omega_\zeta \mid \zeta \in \mathbb{D}_{\mathscr{M}}\}$. The other half has been established in the previous assertion. □

Proof of Theorem 1.1. Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a Nehari sequence, and h_L, h_R the associated Szegő functions. By Lemma 3.2 (i), $(\gamma_n, \gamma_{n+1}, \ldots)$ remains a Nehari sequence for every $n = 1, 2, \ldots$. Therefore, each of them has a Schur function f_n and the associated Szegő functions h_n^L, h_n^R. Since $(\gamma_n, \gamma_{n+1}, \ldots)$ is a one-step extension of $(\gamma_{n+1}, \gamma_{n+2}, \ldots)$, by Lemma 3.2 (ii),

$$\gamma_n = \int_{\mathbb{T}} \{(h_{n+1}^L)^*\}^{-1} h_{n+1}^R dm - h_{n+1}^L(0)(\mathbf{1} - \zeta)h_{n+1}^R(0)$$

for some matrix ζ in $\mathbb{D}_{\mathscr{M}}$. Then, by Proposition 3.1,

$$f_n = \{(\rho_n^R)^*\}^{-1}(zf_{n+1} - \zeta^*)(\mathbf{1} - \zeta z f_{n+1})^{-1}\rho_n^L,$$

where ρ_n^L, ρ_n^R are determined by the condition

$$\rho_n^L(\rho_n^L)^* = \mathbf{1} - \zeta\zeta^*, \qquad h_{n+1}^L(0)\rho_n^L > \mathbf{0}, \qquad (\rho_n^R)^*\rho_n^R = \mathbf{1} - \zeta^*\zeta, \qquad \rho_n^R h_{n+1}^R(0) > \mathbf{0},$$

and the parameter $\alpha_n = f_n(0)$ satisfies $\alpha_n = -\{(\rho_n^R)^*\}^{-1}\zeta^*\rho_n^L = -\rho_n^R\zeta^*\{(\rho_n^L)^*\}^{-1}$, whence

$$(\rho_n^L)^*\rho_n^L = \mathbf{1} - \alpha_n^*\alpha_n, \qquad \rho_n^R(\rho_n^R)^* = \mathbf{1} - \alpha_n\alpha_n^*.$$

The above formula is inverted as the Schur recursion (1.1). Also, by (2.1) and (3.2),

$$h_n^L(0) = h_{n+1}^L(0)\rho_n^L, \qquad h_n^R(0) = \rho_n^R h_{n+1}^R(0).$$

Thus, by induction, (1.3) holds. □

Theorem 1.2 will be proved using the following basic fact.

Lemma 3.3. *Let γ be a Nehari sequence, f its Schur function, and h_L, h_R the associated Szegő functions. Also, let u_L, u_R, v_L, v_R be constant unitary matrices such that*

$$u_L h_L(0) v_R > \mathbf{0}, \qquad v_L h_R(0) u_R > \mathbf{0}.$$

Then $\tilde{\gamma} = u_L \gamma u_R$ is a Nehari sequence, and it corresponds to a Schur function $\tilde{f} = v_L f v_R$.

Proof. Let $\mathfrak{A}$ be a normalized regular γ-generating matrix for γ, so $\mathcal{N}(\gamma) = T_{\mathfrak{A}}(S_{\mathscr{M}})$. Write

$$\mathfrak{U} = \begin{pmatrix} u_L & \mathbf{0} \\ \mathbf{0} & u_R^* \end{pmatrix}, \qquad \mathfrak{A} = \begin{pmatrix} s_L^* & -\bar{z}t_L^* \\ -zt_R & s_R \end{pmatrix}, \qquad \mathfrak{V} = \begin{pmatrix} v_R & \mathbf{0} \\ \mathbf{0} & v_L^* \end{pmatrix},$$

and set $\tilde{\mathfrak{A}} = \mathfrak{U}\mathfrak{A}\mathfrak{V}$. Then $T_{\tilde{\mathfrak{A}}}(S_{\mathscr{M}}) = T_{\mathfrak{U}}\mathcal{N}(\gamma) = \mathcal{N}(\tilde{\gamma})$, and (2.1) shows that

$$v_R^* s_L(0) u_L^* > \mathbf{0}, \qquad u_R^* s_R(0) v_L^* > \mathbf{0}.$$

So, $\tilde{\gamma}$ is a Nehari sequence (cf. [5, Theorem 7.22]), and $\tilde{\mathfrak{A}}$ is its normalized regular γ-generating matrix. By Lemma 2.1, $\tilde{\gamma}$ corresponds to $\tilde{f} = v_L f v_R$. □

Proof of Theorem 1.2. Let f be a Schur function with $\alpha = f(0)$ lying in $\mathbb{D}_{\mathscr{M}}$. Set

$$\check{f} = z^{-1}\rho_R^{-1}(f - \alpha)(\mathbf{1} - \alpha^* f)^{-1}\rho_L^*$$

after taking some matrices ρ_L, ρ_R such that $\rho_L^* \rho_L = \mathbf{1} - \alpha^* \alpha$ and $\rho_R \rho_R^* = \mathbf{1} - \alpha\alpha^*$. It is enough to consider these two Schur functions. Write μ, $\check{\mu}$ for the corresponding measures. First, assume that μ satisfies (1.2). Pick ϱ_L, ϱ_R so that

$$\varrho_L^* \varrho_L = \mathbf{1} - \alpha^* \alpha, \qquad h_L(0) \varrho_L^{-1} > \mathbf{0}, \qquad \varrho_R \varrho_R^* = \mathbf{1} - \alpha_n \alpha_n^*, \qquad \varrho_R^{-1} h_R(0) > \mathbf{0},$$

where h_L, h_R are Szegő functions of μ. Then there are constant unitary matrices v_L, v_R such that

$$\check{f} = v_L \{ z^{-1} \varrho_R^{-1} (f - \alpha)(\mathbf{1} - \alpha^* f)^{-1} \varrho_L^* \} v_R.$$

So, by Theorem 1.1 and Lemmas 2.1, 2.2 and 3.3, $\check{\mu}$ satisfies (1.2). Let us reuse $\varrho_L, \varrho_R, v_L, v_R$ for other constants. Assume that $\check{\mu}$ satisfies (1.2). Also, let $\check{h}_L, \check{h}_R$ be its Szegő functions, and put

$$\zeta = -\rho_L \alpha^* (\rho_R^*)^{-1} = -(\rho_L^*)^{-1} \alpha^* \rho_R,$$

which lies in $\mathbb{D}_{\mathscr{M}}$ and obeys $\rho_L \rho_L^* = \mathbf{1} - \zeta\zeta^*$ and $\rho_R^* \rho_R = \mathbf{1} - \zeta^* \zeta$. Pick ϱ_L, ϱ_R so that

$$\varrho_L \varrho_L^* = \mathbf{1} - \zeta\zeta^*, \qquad \check{h}_L(0) \varrho_L > \mathbf{0}, \qquad \varrho_R^* \varrho_R = \mathbf{1} - \zeta^* \zeta, \qquad \varrho_R \check{h}_R(0) > \mathbf{0}.$$

Then, for some constant unitary matrices v_L, v_R,

$$f = v_L \{ (\varrho_R^*)^{-1} (z\check{f} - \zeta^*)(\mathbf{1} - \zeta z \check{f})^{-1} \varrho_L \} v_R.$$

Hence, by Proposition 3.1 and Lemmas 2.1, 2.2 and 3.3, μ satisfies (1.2). $\square$

Appendix. Examples

Let us write $a_n = -\alpha_n^*$, the *Verblunsky coefficients* in the Szegő recurrence formulas

$$\varphi_{n+1}^L = \{(\rho_{n+1}^L)^*\}^{-1} \{ z\varphi_n^L + a_{n+1} (\varphi_n^R)^\dagger \}, \qquad \varphi_{n+1}^R = \{ z\varphi_n^R + (\varphi_n^L)^\dagger a_{n+1} \} \{(\rho_{n+1}^R)^*\}^{-1},$$

where $\varphi^\dagger$ stands for the reversed polynomials of φ, as before. By repeated use of Proposition 3.1 with a fixed parameter ζ in $\mathbb{D}_{\mathscr{M}}$, from the free case 0), one can construct the following Bernstein–Szegő models 1), 2), 3) of degree 1, 2, 3, respectively, illustrating the correspondence between $a = (a_1, a_2, \ldots)$ and $\gamma = (\gamma_1, \gamma_2, \ldots)$ under the condition (1.3).

0) $f(z) = \mathbf{0}$

$$a = (\mathbf{0}, \mathbf{0}, \ldots) \quad \leftrightarrow \quad \gamma = (\mathbf{0}, \mathbf{0}, \ldots)$$

1) $f(z) = -\zeta^*$

$$a = (\zeta, \mathbf{0}, \mathbf{0}, \ldots) \quad \leftrightarrow \quad \gamma = (\zeta, \mathbf{0}, \mathbf{0}, \ldots)$$

2) $f(z) = -\zeta^* (\mathbf{1} + z\mathbf{1})(\mathbf{1} + z\zeta\zeta^*)^{-1}$

$$a = (\zeta, \zeta, \mathbf{0}, \mathbf{0}, \ldots) \quad \leftrightarrow \quad \gamma = (\zeta - \zeta\zeta^*\zeta, \zeta, \mathbf{0}, \mathbf{0}, \ldots)$$

3) $f(z) = -\zeta^* \{ \mathbf{1} + z(\mathbf{1} + \zeta\zeta^*) + z^2 \mathbf{1} \} (\mathbf{1} + 2z\zeta\zeta^* + z^2 \zeta\zeta^*)^{-1}$

$$a = (\zeta, \zeta, \zeta, \mathbf{0}, \mathbf{0}, \ldots) \quad \leftrightarrow \quad \gamma = (\zeta - 3\zeta\zeta^*\zeta + 2\zeta\zeta^*\zeta\zeta^*\zeta, \zeta - \zeta\zeta^*\zeta, \zeta, \mathbf{0}, \mathbf{0}, \ldots)$$

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