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## ON THE FALK INVARIANT OF SIGNED GRAPHIC ARRANGEMENTS

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ABSTRACT. The fundamental group of the complement of a hyperplane arrangement in a complex vector space is an important topological invariant. The third rank of successive quotients in the lower central series of the fundamental group was called *Falk invariant* of the arrangement since Falk gave the first formula and asked to give a combinatorial interpretation. In this article, we give a combinatorial for the Falk invariant of a signed graphic arrangement that do not have a  $B_2$  as sub-arrangement.

## 1. INTRODUCTION

A hyperplane H in  $\mathbb{C}^{\ell}$  is an affine subspace of dimension  $\ell - 1$ . A finite collection  $\mathcal{A} = \{H_1, \ldots, H_n\}$  of hyperplanes is called a hyperplane arrangement. If  $\bigcap_{i=1}^n H_i \neq \emptyset$ , then  $\mathcal{A}$  is called **central**. In this paper, we only consider central arrangements and assume that all the hyperplanes contain the origin. For more details on hyperplane arrangements, see [5].

Let  $M := \mathbb{C}^{\ell} \setminus_{H \in \mathcal{A}} H$  be the complement of the arrangement  $\mathcal{A}$ . It is known that the cohomology ring  $H^*(M)$  is completely determined by  $L(\mathcal{A})$  the lattice of intersection of  $\mathcal{A}$ . Similarly to this result, there are several conjectures concerning the relationship between M and  $L(\mathcal{A})$ . To study such problems, Falk introduced in [1] a multiplicative invariant, called **global invariant**, of the Orlik-Solomon algebra of  $\mathcal{A}$ . The invariant is now known as the  $(3^{rd})$  **Falk invariant** and it is denoted by  $\phi_3$ . In [2], Falk posed as an open problem to give a combinatorial interpretation of  $\phi_3$ .

Several authors already studied this invariant. In [6], Schenck and Suciu studied the lower central series of arrangements and described a formula for the Falk invariant in the case of graphic arrangements. In [3], the authors gave a formula for  $\phi_3$  in the case of simple sign graphic arrangements. In the preprint [4], the authors extended the previous result for sign graphic arrangements coming from graphs without loops. This article is devoted to extend these results further and to describe

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a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a  $B_2$  as sub-arrangement. Our result gives a partial answer to the question posed by Falk in [2].

## 2. Preliminares on Orlik-Solomon Algebras

Let  $\mathcal{A} = \{H_1, \ldots, H_n\}$  be an arrangement of hyperplanes in  $\mathbb{C}^{\ell}$ . Let  $E^1 = \bigoplus_{j=1}^n \mathbb{C}e_j$  be the free module generated by  $e_1, e_2, \ldots, e_n$ , where  $e_i$  is a symbol corresponding to the hyperplane  $H_i$ . Let  $E = \bigwedge E^1$  be the exterior algebra over  $\mathbb{C}$ . The algebra E is graded via  $E = \bigoplus_{p=0}^n E^p$ , where  $E^p = \bigwedge^p E^1$ . The  $\mathbb{C}$ -module  $E^p$  is free and has the distinguished basis consisting of monomials  $e_S := e_{i_1} \land \cdots \land e_{i_p}$ , where  $S = \{i_1, \ldots, i_p\}$  is running through all the subsets of  $\{1, \ldots, n\}$  of cardinality p and  $i_1 < i_2 < \cdots < i_p$ . The graded algebra E is a commutative DGA with respect to the differential  $\partial$  of degree -1 uniquely defined by the conditions  $\partial e_i = 1$  for all  $i = 1, \ldots, n$  and the graded Leibniz formula. Then for every  $S \subseteq \{1, \ldots, n\}$  of cardinality p

$$\partial e_S = \sum_{j=1}^p (-1)^{j-1} e_{S_j},$$

where  $S_j$  is the complement in S to its *j*-th element.

For every  $S \subseteq \{1, \ldots, n\}$ , put  $\cap S = \bigcap_{i \in S} H_i$  (possibly  $\cap S = \emptyset$ ). The set of all intersections  $L(\mathcal{A}) := \{\cap S \mid S \subseteq \{1, \ldots, n\}\}$  is called the **intersection poset of**  $\mathcal{A}$ . The subset  $S \subseteq \{1, \ldots, n\}$  is called **dependent** if  $\cap S \neq \emptyset$  and the set of linear polynomials  $\{\alpha_i \mid i \in S\}$ with  $H_i = \alpha_i^{-1}(0)$ , is linearly dependent.

**Definition 2.1.** The **Orlik-Solomon ideal** of  $\mathcal{A}$  is the ideal  $I = I(\mathcal{A})$  of E generated by

(1) all  $e_S$  with  $\cap S = \emptyset$ ,

(2) all  $\partial e_S$  with S dependent.

The algebra  $A := A^{\bullet}(A) = E/I(A)$  is called the **Orlik-Solomon** algebra of A.

Clearly I is a homogeneous ideal of E and  $I^p = I \cap E^p$  whence A is a graded algebra and we can write  $A = \bigoplus_{p \ge 0} A^p$ , where  $A^p = E^p/I^p$ . If  $\mathcal{A}$  is central, then for any  $S \subseteq \mathcal{A}$ , we have  $\cap S \neq \emptyset$ . Therefore, the Orlik-Solomon ideal is generated by the elements of type (2) from Definition 2.1. In this case, the map  $\partial$  induces a well-defined differential  $\partial \colon A^{\bullet}(\mathcal{A}) \longrightarrow A^{\bullet-1}(\mathcal{A}).$ 

Let  $I_k$  be the k-adic Orlik-Solomon ideal of  $\mathcal{A}$  generated by  $\sum_{j \leq k} I^j$ in E. It is clear that  $I_k$  is a graded ideal and  $I_k^p = (I_k)^p = E^p \cap I_k$ . Write  $A_k := A_k^{\bullet}(\mathcal{A}) = E/I_k$  and  $A_k^p := (A_k^{\bullet}(\mathcal{A}))^p = E^p/I_k^p$  which is called *k*-adic Orlik-Solomon algebra by Falk [1].

In this set up, it is now easy to define the Falk invariant.

**Definition 2.2.** Consider the map d defined by

$$d\colon E^1\otimes I^2\to E^3,$$

$$d(a\otimes b)=a\wedge b.$$

Then the **Falk invariant** is defined as

$$\phi_3 := \dim(\ker(d)).$$

In [1] and [2], Falk gave a beautiful formula to compute such invariant.

**Theorem 2.3** (Theorem 4.7, [2]). Let  $A = \{H_1, ..., H_n\}$  be an arrangement of hyperplanes in  $\mathbb{C}^{\ell}$ . Then

(1) 
$$\phi_3 = 2\binom{n+1}{3} - n\dim(A^2) + \dim(A_2^3).$$

**Remark 2.4.** Since  $\dim(A_2^3) = \dim((E/I_2)^3) = \dim(E^3) - \dim(I_2^3)$ and dim $(E^3) = \binom{n}{3}$ , then we obtain

(2) 
$$\phi_3 = 2\binom{n+1}{3} - n\dim(A^2) + \binom{n}{3} - \dim(I_2^3).$$

Recall that  $\phi_3$  can also be describe from the lower central series of the fundamental group  $\pi(M)$  of the complement M of the arrangement. In particular, if we consider the lower central series as a chain of normal subgroups  $N_i$ , for  $k \ge 1$ , where  $N_1 = \pi(M)$  and  $N_{k+1} = [N_k, N_1]$ , the subgroup generated by commutators of elements in  $N_k$  and  $N_1$ , then  $\phi_3$  is the rank of the finitely generated abelian group  $N_3/N_4$ . See [6] for more details.

#### 3. SIGN GRAPHS

In this section we will recall the main properties of signed graphs. See [7] for a general treatment of such graphs.

**Definition 3.1.** A signed graph is a tuple  $G = (V_G, E_G^+, E_G^-, L_G)$ , where

- $V_G$  is a finite set called the set of vertices,
- $E_G^+$  is a subset of  $\binom{V_G}{2}$  called the set of positive edges,
- E<sub>G</sub><sup>-</sup> is a subset of (<sup>2</sup>/<sub>Q</sub>) called the set of negative edges,
  L<sub>G</sub> is a subset of V<sub>G</sub> called the set of loops.

**Example 3.2.** In this article, we illustrate a signed graph as follows:

$$G = (V_G, E_G^+, E_G^-, L_G) = \begin{cases} 1 & 4 \\ 0 & - & - \\ 2 & 3 \end{cases}, \qquad \begin{cases} V_G = \{1, 2, 3, 4\}, \\ E_G^+ = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}, \\ E_G^- = \{\{1, 3\}, \{2, 3\}, \{2, 4\}\}, \\ L_G = \{3, 4\}. \end{cases}$$

Let  $G^+ = (V_G, E_G^+)$  and  $G^- = (V_G, E_G^-)$ , then we have an alternative notation  $G = (G^+, G^-, L_G)$  for the signed graph G. An unsigned simple graph G may be regarded as a signed graph  $G = (G, K_{\ell}^{\circ}, \emptyset)$ , where  $K_{\ell}^{\circ}$ denotes the edgeless graph on  $\ell$  vertices. A signed graph  $(G^+, G^-, \emptyset)$ is called **loopless**, which is also denoted by  $(G^+, G^-)$ . Let  $E_G$  denote the edge set  $E_G^+ \sqcup E_G^- \sqcup L_G$ . For a positive integer  $\ell$ , let  $[\ell]$  denote the set  $\{1, \ldots, \ell\}$ . From now on, we suppose that G is a signed graph on vertices  $[\ell]$ . Let  $(x_1, \ldots, x_\ell)$  be a basis for the  $\ell$ -dimensional vector space  $(\mathbb{C}^{\ell})^*$ . For  $\alpha \in (\mathbb{C}^{\ell})^*$ , let  $\{\alpha = 0\}$  denote the hyperplane  $\{v \in \mathbb{C}^{\ell} \mid \alpha(v) = 0\}$ .

**Definition 3.3.** Given a signed graph G, let  $\mathcal{A}(G)$  be the hyperplane arrangement in  $\mathbb{C}^{\ell}$  consisting of the following hyperplane

$$\{x_i - x_j = 0\} \text{ for } \{i, j\} \in E_G^+, \\ \{x_i + x_j = 0\} \text{ for } \{i, j\} \in E_G^-, \\ \{x_i = 0\} \text{ for } i \in L_G.$$

We will call  $\mathcal{A}(G)$  the signed graphic arrangement associated to the signed graph G.

Given a signed graph it is natural to introduce the following function.

**Definition 3.4.** Given a sign graph  $G = (V_G, E_G^+, E_G^-, L_G)$ , the **sign function** of *G* is the function  $sgn: E_G^+ \cup E_G^- \cup L_G \to \{+, -\}$  defined by

$$sgn(e) = \begin{cases} + & \text{if } e \in E_G^+, \\ - & \text{if } e \in E_G^- \cup L_G. \end{cases}$$

We can naturally extend the previous definition to path in G

**Definition 3.5.** Given  $P = e_1 e_2 \cdots e_k$  a path in G, the sign of P is  $sgn(P) = \prod_{i=1}^k sgn(e_i)$ .

**Definition 3.6.** A cycle C in a sign graph G is called **balanced** if sgn(C) = +.

Given a sign graph G and a function  $\sigma: V_G \to \{+, -\}$ , we can define a new sign graph G' that has the same underlying graph as G but with a different sign function. In particular, if  $e = \{i, j\} \in E_G$  then  $sgn_{G'}(e) = \sigma(i)sgn_G(e)\sigma(j)$ .

**Definition 3.7.** In the previous construction, we will call G' the **switching of** G by  $\sigma$  and we will denote it by  $G^{\sigma}$ . In this case,  $\sigma$  is called a **switching function for** G.

**Definition 3.8.** Given two sign graph  $G_1$  and  $G_2$  with the same underlying graph, we will say they are **switching equivalent** and write  $G_1 \sim G_2$ , if there exists a switching function  $\sigma$  such that  $G_2 = G_1^{\sigma}$ .

**Proposition 3.9** (Proposition 3.2, [7]). Two signed graphs with the same underlying graph are switching equivalent if and only if they have the same list of balanced circles.

**Proposition 3.10** (Corollary 5.4, [7]). Two signed graphs with the same underlying graph are switching equivalent if and only if they define the same matroid.

Using the previous results, we obtain

**Corollary 3.11.** Let  $G_1$  and  $G_2$  be two signed graph with the same underlying graph. If  $G_1 \sim G_2$ , then  $\phi_3(\mathcal{A}(G_1)) = \phi_3(\mathcal{A}(G_2))$ .

In this paper taking inspiration from graph theory and the study of hyperplane arrangements, we denote by  $K_{\ell}$  a complete graph with  $\ell$ vertices and all edges being positive, i.e.  $K_{\ell} = (K_{\ell}, K_{\ell}^{\circ}, \emptyset)$ , by  $D_{\ell}$  a complete sign graph with  $\ell$  vertices and no loops, i.e.  $D_{\ell} = (K_{\ell}, K_{\ell}, \emptyset)$ , and by  $B_{\ell}$  a sign complete graph with  $\ell$  vertices and a full set of loops, i.e.  $B_{\ell} = (K_{\ell}, K_{\ell}, [\ell])$ . Moreover, we denote by  $K_{\ell}^{\ell}$  a complete graph with  $\ell$  vertices, all edges being positive and a full set of loops, i.e.  $K_{\ell}^{\ell} = (K_{\ell}, K_{\ell}^{\circ}, [\ell])$ , by  $D_{\ell}^{1}$  a complete sign graph with  $\ell$  vertices and one loop, i.e.  $D_{\ell}^{1} = (K_{\ell}, K_{\ell}, \{1\})$  and by  $G_{\circ}$  the signed graph in Figure 1. Furthermore, if G is a signed graph we denote by  $\overline{G}$  a signed graph switching equivalent to G for some switching function  $\sigma$ .

### 4. MAIN THEOREM

In this section we describe how to compute the Falk invariant  $\phi_3$  for  $\mathcal{A}(G)$ , a signed graphic arrangement associated to a signed graph G that do not have a subgraph isomorphic to  $B_2$ . In the remaining of the paper, to fix the notation we will suppose G is a graph on  $\ell$  vertices having n edges, and we will label only the edges as elements of  $[n] := \{1, \ldots, n\}$ .



FIGURE 1. The sign graph  $G_{\circ}$ 

The goal of this section is to prove the following theorem.

**Theorem 4.1.** For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to  $B_2$  as subgraph, we have

(3) 
$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_\circ) + 5d_{3,1},$$

where  $k_l$  denotes the number of subgraph of G isomorphic to a  $\overline{K_l}$ ,  $d_l$ denotes the number of subgraph of G isomorphic to  $D_l$  but not contained in  $D_l^1$ ,  $d_{l,1}$  denotes the number of subgraph of G isomorphic to  $D_l^1$ ,  $k_{l,l}$ denotes the number of subgraph of G isomorphic to a  $\overline{K_l^l}$  and  $g_\circ$  denotes the number of subgraph of G isomorphic to a  $\overline{G_\circ}$  but not contained in  $D_l^1$ .

**Remark 4.2.** Theorem 4.1 is a generalization of the previously known results for graphic arrangements [6] and for simple signed graphic arrangements [3]. In fact, in both cases, we have graphs whose subgraphs can only be isomorphic to a  $\overline{K_l}$ , and hence we obtain that for these cases  $\phi_3 = 2(k_3 + k_4)$ .

In order to compute  $\phi_3$ , we need firstly to identify the ordered 3-tuple S in  $\{1, \ldots, n\}$  that are dependent. Clearly, we have the following

**Lemma 4.3.**  $S = (i_1, i_2, i_3)$  is dependent if and only if  $i_1, i_2, i_3$  correspond to the edges of a subgraph of G that is isomorphic to a  $\overline{K_3}$ , or a  $D_2^1$  or a  $\overline{K_2^2}$ .

With an abuse of notation, we will call a dependent 3-tuple S a **triangle**. Moreover, we will write

 $\mathcal{C}_3 := \{ e_S \in E \mid S \text{ is a triangle} \}$ 

which is a subset of E as a vector space over  $\mathbb{C}$ .

**Remark 4.4.** Notice that the triangles are exactly the balanced 3cycles together with the subgraphs isomorphic to a  $\overline{K_2^2}$  or a  $D_2^1$ . In particular, If  $G_1$  and  $G_2$  are two signed graph with the same underlying graph such that  $G_1 \sim G_2$ , then  $\mathcal{C}_3(G_1) = \mathcal{C}_3(G_2)$ .

Since  $e_i e_j e_k = -e_j e_i e_k$ , it is clear that the dimension of the vector space  $C_3$  is  $k_3 + d_{2,1} + k_{2,2}$ .

**Lemma 4.5.** For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to  $B_2$  as subgraph, we have

$$\dim(A^2) = \binom{n}{2} - k_3 - d_{2,1} - k_{2,2}.$$

*Proof.* By definition A = E/I, hence

$$\dim(A^2) = \dim(E^2) - \dim(I^2) = \binom{n}{2} - \dim(I^2).$$

Since  $I^2 = \text{span}\{\partial e_{ijk} \mid e_{ijk} \in \mathcal{C}_3\}$ , then  $\dim(I^2) = k_3 + d_{2,1} + k_{2,2}$ , and the thesis follows.  $\Box$ 

Using Theorem 2.3 and Remark 2.4, to prove Theorem 4.1 we just need to describe dim $(I_2^3)$ . To do so, let us consider  $C'_3$  a basis of  $C_3$ . Then each element of  $C'_3$  is in a one-to-one correspondence of the subgraph of G isomorphic to a  $\overline{K_3}$ , or a  $D_2^1$  or a  $\overline{K_2^2}$ . Define then

$$C_3 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in \{i, j, k\} \},\$$

and

$$F_3 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\} \}.$$

By construction  $I_2^3 = I^2 \cdot E^1 = \text{span}\{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n]\}$ , and hence

$$I_2^3 = \operatorname{span}(C_3) + \operatorname{span}(F_3).$$

**Lemma 4.6.** For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to  $B_2$  as subgraph, we have

$$I_2^3 = \operatorname{span}(C_3) \oplus \operatorname{span}(F_3).$$

*Proof.* Since G do not contain a  $B_2$  as subgraph, any two triangles shares at most one element. This then gives us that  $\operatorname{span}(C_3) \cap \operatorname{span}(F_3) = \emptyset$ .  $\Box$ 

**Remark 4.7.** Notice that if we allow G to have subgraphs isomorphic to  $B_2$ , then the previous lemma is not true anymore.

By the previous lemma, we can write

 $\dim(I_2^3) = \dim(\operatorname{span}(C_3)) + \dim(\operatorname{span}(F_3)) = k_3 + d_{2,1} + k_{2,2} + \dim(\operatorname{span}(F_3)).$ 

To prove our main result we need to be able to compute  $\dim(\text{span}(F_3))$ . To do so, consider the following sets

$$\begin{split} F_3^1 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are not in the same } \overline{K_4}, D_3, \overline{G_\circ}, D_3^1, K_3^3\}, \\ F_3^2 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \overline{K_4}\}, \\ F_3^3 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } D_3 \text{ but not same } D_3^1\}, \\ F_3^4 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \overline{G_\circ} \text{ but not same } D_3^1\}, \\ F_3^5 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } D_3^1\}, \\ F_3^6 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \overline{K_3^3}\}, \end{split}$$

**Lemma 4.8.** For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to  $B_2$ , we have

$$\operatorname{span}(F_3) = \bigoplus_{i=1}^{6} \operatorname{span}(F_3^i).$$

*Proof.* There is an evident direct summand decomposition

$$\operatorname{span}(F_3) = \bigoplus_{X \in L_3(\mathcal{A})} \operatorname{span}\{e_t \partial e_{ijk} \mid H_t \cap H_i \cap H_j \cap H_k = X\},$$

where  $\{i, j, k\}$  is a triangle and  $L_3(\mathcal{A})$  is the set of rank three flats of the lattice of intersections.

Then the result follows from recognizing  $F_3^2$ ,  $F_3^3$ ,  $F_3^4$ ,  $F_3^5$ ,  $F_3^6$  as particular groups of summands of the above direct sum, corresponding to particular types of rank three flats. Specifically,  $F_3^2$  corresponds to the  $X = H_t \cap H_i \cap H_j \cap H_k$  where t, i, j, k are in the same  $\overline{K_4}$ ,  $F_3^3$  corresponds to the X where t, i, j, k are in the same  $D_3$  but not same  $D_3^1$ ,  $F_3^4$ corresponds to the X where t, i, j, k are in the same  $\overline{G_{\circ}}$  but not same  $D_3^1$ ,  $F_3^5$  corresponds to the X where t, i, j, k are in the same  $D_3^1$ , and  $F_3^6$  corresponds to the X where t, i, j, k are in the same  $\overline{K_3}^3$ . Finally,  $F_3^1$  consists of the rest of the summands.  $\Box$ 

We now proceed to computing the dimensions of  $F_3^i$  for i = 1, ..., 6, beginning with two examples which illustrate the general idea.

**Example 4.9.** We consider the dimension of span( $F_3$ ) for the sign graphic arrangement  $A_3$  associated to the graph  $G_{\circ}$  (see Figure 2).

In this situation we have  $E^+ = \{1, 2, 3\}, E^- = \{4, 5\}$  and  $L = \{6\}$ . Then the number of the elements in  $F_3$  is 12, listed as follows.

$$e_4 \partial e_{123} = e_{234} - e_{134} + e_{124}, e_5 \partial e_{123} = e_{235} - e_{135} + e_{125},$$
  
$$e_6 \partial e_{123} = e_{236} - e_{136} + e_{126}, e_1 \partial e_{345} = e_{145} - e_{135} + e_{134},$$



FIGURE 2. The sign graph  $G_{\circ}$ 

$$\begin{aligned} e_2 \partial e_{345} &= e_{245} - e_{235} + e_{234}, e_6 \partial e_{345} &= e_{456} - e_{356} + e_{346}, \\ e_2 \partial e_{146} &= e_{246} + e_{126} - e_{124}, e_3 \partial e_{146} &= e_{346} + e_{136} - e_{136}, \\ e_5 \partial e_{146} &= -e_{456} + e_{156} + e_{145}, e_1 \partial e_{256} &= e_{156} - e_{126} + e_{125}, \\ e_3 \partial e_{256} &= e_{356} + e_{236} - e_{235}, e_4 \partial e_{256} &= e_{456} + e_{246} - e_{245}. \end{aligned}$$

Then an easy computation shows that in this case  $\dim(\text{span}(F_3)) = 10$ .

**Example 4.10.** We consider the dimension of span( $F_3$ ) for the sign graphic arrangement associated to the graph  $D_3^1$  (see Figure 3).



FIGURE 3. The sign graph  $D_3^1$ 

In this situation we have  $E^+ = \{1, 2, 3\}, E^- = \{4, 5, 6\}$  and  $L = \{7\}$ . Then the number of the elements in  $F_3$  is 24, listed as follows.

$$\begin{split} e_4\partial e_{123} &= e_{124} - e_{134} + e_{234}, e_5\partial e_{123} = e_{125} - e_{135} + e_{235}, \\ e_6\partial e_{123} &= e_{126} - e_{136} + e_{236}, e_7\partial e_{123} = e_{127} - e_{137} + e_{237}, \\ e_2\partial e_{156} &= -e_{125} + e_{126} + e_{256}, e_3\partial e_{156} = -e_{135} + e_{136} + e_{356}, \\ e_4\partial e_{156} &= -e_{145} + e_{146} + e_{456}, e_7\partial e_{156} = e_{157} - e_{167} + e_{567}, \\ e_1\partial e_{246} &= e_{124} - e_{126} + e_{146}, e_3\partial e_{246} = -e_{234} + e_{236} + e_{346}, \\ e_5\partial e_{246} &= e_{245} + e_{256} - e_{456}, e_7\partial e_{246} = e_{247} - e_{267} + e_{467}, \\ e_1\partial e_{345} &= e_{134} - e_{135} + e_{145}, e_2\partial e_{345} = e_{234} - e_{235} + e_{245}, \end{split}$$

$$\begin{split} e_{6}\partial e_{345} &= e_{346} - e_{356} + e_{456}, e_{7}\partial e_{345} = e_{347} - e_{357} + e_{457}, \\ e_{2}\partial e_{147} &= -e_{124} + e_{127} + e_{247}, e_{3}\partial e_{147} = -e_{134} + e_{137} + e_{347}, \\ e_{5}\partial e_{147} &= e_{145} + e_{157} - e_{457}, e_{6}\partial e_{147} = e_{146} + e_{167} - e_{467}, \\ e_{1}\partial e_{257} &= e_{125} - e_{127} + e_{157}, e_{3}\partial e_{257} = -e_{235} + e_{237} + e_{357}, \\ e_{4}\partial e_{257} &= -e_{245} + e_{247} + e_{457}, e_{6}\partial e_{257} = e_{256} + e_{267} - e_{567}. \end{split}$$

Then an easy computation shows that in this case  $\dim(\text{span}(F_3)) = 19$ .

**Remark 4.11.** Similarly to the previous examples, we can compute  $\dim(\operatorname{span}(F_3))$  directly for several sign graph. In particular, if we consider  $D_3, K_4$  and  $K_3^3$ , then  $\dim(\operatorname{span}(F_3)) = 10$ .

**Lemma 4.12.** dim $(\text{span}(F_3^1)) = (n-3)(k_3+d_{2,1}+k_{3,3}) - 12k_4 - 12d_3 - 12g_\circ - 12k_{3,3} - 24d_{3,1}.$ 

*Proof.* The result follows from the equality  $|F_3^1| = |F_3| - (\sum_{i=2}^6 |F_3^i|)$ , and noticing that the elements of  $F_3^1$  are independent and they form a basis for span $(F_3^1)$ .  $\Box$ 

**Lemma 4.13.**  $\dim(\operatorname{span}(F_3^2)) = 10k_4$ ,  $\dim(\operatorname{span}(F_3^3)) = 10d_3$ ,  $\dim(\operatorname{span}(F_3^4)) = 10g_\circ$ ,  $\dim(\operatorname{span}(F_3^5)) = 19d_{3,1}$  and  $\dim(\operatorname{span}(F_3^6)) = 10k_{3,3}$ .

*Proof.* Assume that in the sign graph G there are exactly  $g_{\circ} = p$  distinct subgraphs isomorphic to a  $\overline{G_{\circ}}, G_1, \ldots, G_p$ , none of which is a subgraph of a graph isomorphic to  $D_3^1$ . Consider

$$F_{3,i}^4 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \in G_i \}.$$

Since four edges in the graph G can not appear in two distinct  $\overline{G_{\circ}}$  at the same time, then none of the terms of the element  $e_t \partial e_{ijk} \in F_{3,i}^2$  appear in the elements of  $F_3^4 \setminus F_{3,i}^4$ . This shows that

$$\operatorname{span}(F_3^4) = \bigoplus_{i=1}^p \operatorname{span}(F_{3,i}^4).$$

By Corollary 3.11 and Example 4.9, we have that  $\dim(\text{span}(F_{3,i}^4)) = 10$  for all  $i = 1, \ldots, p$ . This then implies that

$$\dim(\operatorname{span}(F_3^4)) = \sum_{i=1}^p \dim(\operatorname{span}(F_{3,i}^4)) = 10g_\circ.$$

Using Remark 4.11 and Example 4.10, the same exact argument used in this case will prove the other equalities.  $\Box$ 

**Lemma 4.14.** For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to  $B_2$ , we have

 $\dim(I_2^3) = (n-2)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_\circ - 2k_{3,3} - 5d_{3,1}.$ 

*Proof.* By the previous lemmas

$$\dim(\operatorname{span}(F_3)) = \sum_{i=1}^6 \dim(\operatorname{span}(F_3^i)) =$$

 $= [(n-3)(k_3 + d_{2,1} + k_{3,3}) - 12k_4 - 12d_3 - 12g_\circ - 12k_{3,3} - 24d_{3,1}] + 10k_4 + 10d_3 + 10g_\circ + 10k_{3,3} + 19d_{3,1} =$ 

$$(n-3)(k_3+d_{2,1}+k_{3,3})-2k_4-2d_3-2g_{\circ}-2k_{3,3}-5d_{3,1}$$

The thesis follows from the equality

$$\dim(I_2^3) = k_3 + d_{2,1} + k_{2,2} + \dim(\operatorname{span}(F_3)).$$

Proof of Theorem 4.1. By Remark 2.4 and Lemma 4.5 we have

$$\phi_3 = 2\binom{n+1}{3} - n\binom{n}{2} - k_3 - d_{2,1} - k_{2,2} + \binom{n}{3} - \dim(I_2^3).$$

Because  $2\binom{n+1}{3} - n\binom{n}{2} + \binom{n}{3} = 0$ , then from Lemma 4.14 we obtain

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_\circ) + 5d_{3,1}.$$

Let us see how our formula works on a non-trivial example.

**Example 4.15.** We want to compute  $\phi_3$  for the arrangement associated to the graph G of Figure 4.



FIGURE 4. The sign graph G

In this situation we have  $E^+ = \{1, 2, 3, 4, 5, 6\}, E^- = \{7, 8, 9, 10\}$ and  $L = \{11\}$ . In order to compute  $\phi_3$  with the formula (3), we need to compute the following:

- $k_3 = |\{\{1, 2, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{1, 9, 10\}, \{6, 7, 9\}, \{4, 7, 9\}$  $\{3, 8, 9\}, \{5, 7, 8\}\} = 9;$ •  $k_4 = |\{\{1, 2, 3, 4, 5, 6\}, \{3, 4, 5, 7, 8, 9\}\}| = 2;$
- $d_3 = 0;$
- $d_{2,1} = |\{\{1,7,11\},\{6,9,11\},\{2,8,11\}\}| = 3;$
- $k_{2,2} = 0;$
- $k_{3,3} = 0;$
- $g_{\circ} = |\{\{1, 2, 5, 7, 8, 11\}, \{2, 3, 6, 8, 9, 11\}\}| = 2;$   $d_{3,1} = |\{\{1, 4, 6, 7, 9, 10, 11\}\}| = 1.$

From formula (3), we obtain

$$\phi_3 = 2(9+2+0+3+0+0+2) + 5 = 37.$$

Notice that if we would try to compute the dimension of  $F_3$  directly, we would have to write 96 equations in the  $e_{ijk}$ .

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#### References

- [1] Michael Falk. On the algebra associated with a geometric lattice. Advances in Mathematics, 80(2):152–163, 1990.
- [2] Michael Falk. Combinatorial and algebraic structure in Orlik–Solomon algebras. European Journal of Combinatorics, 22(5):687–698, 2001.
- [3] Qiumin Guo, Weili Guo, Wentao Hu, and Guangfeng Jiang. The global invariant of signed graphic hyperplane arrangements. Graphs and Combinatorics, pages 1-9, 2017.
- [4] Weili Guo, Qiumin Guo, and Guangfeng Jiang. Falk invariants of signed graphic arrangements. In preparation.
- [5] Peter Orlik and Hiroaki Terao. Arrangements of hyperplanes, volume 300. Springer Science & Business Media, 2013.
- [6] Henry Schenck and Alexander Suciu. Lower central series and free resolutions of hyperplane arrangements. Transactions of the American Mathematical Society, 354(9):3409-3433, 2002.
- [7] Thomas Zaslavsky. Signed graphs. Discrete Applied Mathematics, 4(1):47–74, 1982.

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