ON THE FALK INVARIANT OF SIGNED GRAPHIC ARRANGEMENTS

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Abstract. The fundamental group of the complement of a hyperplane arrangement in a complex vector space is an important topological invariant. The third rank of successive quotients in the lower central series of the fundamental group was called Falk invariant of the arrangement since Falk gave the first formula and asked to give a combinatorial interpretation. In this article, we give a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a $B_2$ as sub-arrangement.

1. Introduction

A hyperplane $H$ in $\mathbb{C}^\ell$ is an affine subspace of dimension $\ell - 1$. A finite collection $\mathcal{A} = \{H_1, \ldots, H_n\}$ of hyperplanes is called a hyperplane arrangement. If $\bigcap_{i=1}^n H_i \neq \emptyset$, then $\mathcal{A}$ is called central. In this paper, we only consider central arrangements and assume that all the hyperplanes contain the origin. For more details on hyperplane arrangements, see [5].

Let $M := \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ be the complement of the arrangement $\mathcal{A}$. It is known that the cohomology ring $H^\ast(M)$ is completely determined by $L(\mathcal{A})$ the lattice of intersection of $\mathcal{A}$. Similarly to this result, there are several conjectures concerning the relationship between $M$ and $L(\mathcal{A})$. To study such problems, Falk introduced in [1] a multiplicative invariant, called global invariant, of the Orlik-Solomon algebra of $\mathcal{A}$. The invariant is now known as the (3rd) Falk invariant and it is denoted by $\phi_3$. In [2], Falk posed as an open problem to give a combinatorial interpretation of $\phi_3$.

Several authors already studied this invariant. In [6], Schenck and Suciu studied the lower central series of arrangements and described a formula for the Falk invariant in the case of graphic arrangements. In [3], the authors gave a formula for $\phi_3$ in the case of simple sign graphic arrangements. In the preprint [4], the authors extended the previous result for sign graphic arrangements coming from graphs without loops. This article is devoted to extend these results further and to describe
a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a $B_2$ as sub-arrangement. Our result gives a partial answer to the question posed by Falk in [2].

2. Preliminaries on Orlik-Solomon algebras

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. Let $E^1 = \bigoplus_{j=1}^n \mathbb{C}e_j$ be the free module generated by $e_1, e_2, \ldots, e_n$, where $e_i$ is a symbol corresponding to the hyperplane $H_i$. Let $E = \bigwedge E^1$ be the exterior algebra over $\mathbb{C}$. The algebra $E$ is graded via $E = \bigoplus_{p=0}^n E^p$, where $E^p = \bigwedge^p E^1$. The $\mathbb{C}$-module $E^p$ is free and has the distinguished basis consisting of monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$, where $S = \{i_1, \ldots, i_p\}$ is running through all the subsets of $\{1, \ldots, n\}$ of cardinality $p$ and $i_1 < i_2 < \cdots < i_p$. The graded algebra $E$ is a commutative DGA with respect to the differential $\partial$ of degree $-1$ uniquely defined by the conditions $\partial e_i = 1$ for all $i = 1, \ldots, n$ and the graded Leibniz formula. Then for every $S \subseteq \{1, \ldots, n\}$ of cardinality $p$

$$\partial e_S = \sum_{j=1}^p (-1)^{j-1} e_{S_j},$$

where $S_j$ is the complement in $S$ to its $j$-th element.

For every $S \subseteq \{1, \ldots, n\}$, put $\cap S = \bigcap_{i \in S} H_i$ (possibly $\cap S = \emptyset$). The set of all intersections $L(\mathcal{A}) := \{\cap S \mid S \subseteq \{1, \ldots, n\}\}$ is called the intersection poset of $\mathcal{A}$. The subset $S \subseteq \{1, \ldots, n\}$ is called dependent if $\cap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i \mid i \in S\}$ with $H_i = \alpha_i^{-1}(0)$, is linearly dependent.

**Definition 2.1.** The Orlik-Solomon ideal of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of $E$ generated by

1. all $e_S$ with $\cap S = \emptyset$,
2. all $\partial e_S$ with $S$ dependent.

The algebra $A := A^*(\mathcal{A}) = E/I(\mathcal{A})$ is called the Orlik-Solomon algebra of $\mathcal{A}$.

Clearly $I$ is a homogeneous ideal of $E$ and $I^p = I \cap E^p$ whence $A$ is a graded algebra and we can write $A = \bigoplus_{p \geq 0} A^p$, where $A^p = E^p/I^p$. If $\mathcal{A}$ is central, then for any $S \subseteq \mathcal{A}$, we have $\cap S \neq \emptyset$. Therefore, the Orlik-Solomon ideal is generated by the elements of type (2) from Definition 2.1. In this case, the map $\partial$ induces a well-defined differential $\partial : A^*(\mathcal{A}) \rightarrow A^{*-1}(\mathcal{A})$.

Let $I_k$ be the $k$-adic Orlik-Solomon ideal of $\mathcal{A}$ generated by $\sum_{j \leq k} I^j$ in $E$. It is clear that $I_k$ is a graded ideal and $I_k^p = (I_k)^p = E^p \cap I_k$. 

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Write $A_k := A_k^*(A) = E/I_k$ and $A_k^p := (A_k^*(A))^p = E^p/I_k^p$ which is called $k$-adic Orlik-Solomon algebra by Falk [1].

In this set up, it is now easy to define the Falk invariant.

**Definition 2.2.** Consider the map $d$ defined by
\[ d: E^1 \otimes I^2 \rightarrow E^3, \]
\[ d(a \otimes b) = a \land b. \]

Then the **Falk invariant** is defined as
\[ \phi_3 := \dim(\ker(d)). \]

In [1] and [2], Falk gave a beautiful formula to compute such invariant.

**Theorem 2.3** (Theorem 4.7, [2]). Let $A = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^l$. Then
\[
(1) \quad \phi_3 = 2\left(\frac{n+1}{3}\right) - n \dim(A^2) + \dim(A_2^3).
\]

**Remark 2.4.** Since $\dim(A_3^3) = \dim((E/I_2)^3) = \dim(E^3) - \dim(I_2^3)$ and $\dim(E^3) = \binom{n}{3}$, then we obtain
\[
(2) \quad \phi_3 = 2\left(\frac{n+1}{3}\right) - n \dim(A^2) + \binom{n}{3} - \dim(I_2^3).
\]

Recall that $\phi_3$ can also be describe from the lower central series of the fundamental group $\pi(M)$ of the complement $M$ of the arrangement. In particular, if we consider the lower central series as a chain of normal subgroups $N_i$, for $k \geq 1$, where $N_1 = \pi(M)$ and $N_{k+1} = [N_k, N_1]$, the subgroup generated by commutators of elements in $N_k$ and $N_1$, then $\phi_3$ is the rank of the finitely generated abelian group $N_3/N_1$. See [6] for more details.

### 3. Sign graphs

In this section we will recall the main properties of signed graphs. See [7] for a general treatment of such graphs.

**Definition 3.1.** A **signed graph** is a tuple $G = (V_G, E_G^+, E_G^-, L_G)$, where
- $V_G$ is a finite set called the set of vertices,
- $E_G^+$ is a subset of $\binom{V_G}{2}$ called the set of positive edges,
- $E_G^-$ is a subset of $\binom{V_G}{2}$ called the set of negative edges,
- $L_G$ is a subset of $V_G$ called the set of loops.
Example 3.2. In this article, we illustrate a signed graph as follows:

\[
G = (V_G, E^+_G, E^-_G, L_G) = \begin{align*}
V_G &= \{1, 2, 3, 4\}, \\
E^+_G &= \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}, \\
E^-_G &= \{\{1, 3\}, \{2, 3\}, \{2, 4\}\}, \\
L_G &= \{3, 4\}.
\end{align*}
\]

Let \(G^+ = (V_G, E^+_G)\) and \(G^- = (V_G, E^-_G)\), then we have an alternative notation \(G = (G^+, G^-, L_G)\) for the signed graph \(G\). An unsigned simple graph \(G\) may be regarded as a signed graph \(G = (G, K_\ell \circ \ell, \emptyset)\), where \(K_\ell \circ \ell\) denotes the edgeless graph on \(\ell\) vertices. A signed graph \((G^+, G^-, \emptyset)\) is called loopless, which is also denoted by \((G^+, G^-)\). Let \(E_G\) denote the edge set \(E^+_G \cup E^-_G \cup L_G\). For a positive integer \(\ell\), let \([\ell]\) denote the set \(\{1, \ldots, \ell\}\). From now on, we suppose that \(G\) is a signed graph on vertices \([\ell]\). Let \((x_1, \ldots, x_\ell)\) be a basis for the \(\ell\)-dimensional vector space \((\mathbb{C}^\ell)^*\). For \(\alpha \in (\mathbb{C}^\ell)^*\), let \(\{\alpha = 0\}\) denote the hyperplane \(\{v \in \mathbb{C}^\ell \mid \alpha(v) = 0\}\).

**Definition 3.3.** Given a signed graph \(G\), let \(\mathcal{A}(G)\) be the hyperplane arrangement in \(\mathbb{C}^\ell\) consisting of the following hyperplane

- \(\{x_i - x_j = 0\}\) for \((i, j) \in E^+_G\),
- \(\{x_i + x_j = 0\}\) for \((i, j) \in E^-_G\),
- \(\{x_i = 0\}\) for \(i \in L_G\).

We will call \(\mathcal{A}(G)\) the signed graphic arrangement associated to the signed graph \(G\).

Given a signed graph it is natural to introduce the following function.

**Definition 3.4.** Given a sign graph \(G = (V_G, E^+_G, E^-_G, L_G)\), the sign function of \(G\) is the function \(\text{sgn}: E^+_G \cup E^-_G \cup L_G \to \{+, -\}\) defined by \(\text{sgn}(e) = \begin{cases} + & \text{if } e \in E^+_G, \\ - & \text{if } e \in E^-_G \cup L_G. \end{cases}\)

We can naturally extend the previous definition to path in \(G\)

**Definition 3.5.** Given \(P = e_1 e_2 \cdots e_k\) a path in \(G\), the sign of \(P\) is \(\text{sgn}(P) = \prod_{i=1}^k \text{sgn}(e_i)\).

**Definition 3.6.** A cycle \(C\) in a sign graph \(G\) is called balanced if \(\text{sgn}(C) = +\).
Given a sign graph \( G \) and a function \( \sigma : V_G \to \{+,-\} \), we can define a new sign graph \( G' \) that has the same underlying graph as \( G \) but with a different sign function. In particular, if \( e = \{i,j\} \in E_G \) then \( \text{sgn}_{G'}(e) = \sigma(i) \text{sgn}_G(e) \sigma(j) \).

**Definition 3.7.** In the previous construction, we will call \( G' \) the **switching** of \( G \) by \( \sigma \) and we will denote it by \( G^\sigma \). In this case, \( \sigma \) is called a **switching function** for \( G \).

**Definition 3.8.** Given two sign graph \( G_1 \) and \( G_2 \) with the same underlying graph, we will say they are **switching equivalent** and write \( G_1 \sim G_2 \), if there exists a switching function \( \sigma \) such that \( G_2 = G_1^\sigma \).

**Proposition 3.9** (Proposition 3.2, [7]). Two signed graphs with the same underlying graph are switching equivalent if and only if they have the same list of balanced circles.

**Proposition 3.10** (Corollary 5.4, [7]). Two signed graphs with the same underlying graph are switching equivalent if and only if they define the same matroid.

Using the previous results, we obtain

**Corollary 3.11.** Let \( G_1 \) and \( G_2 \) be two signed graph with the same underlying graph. If \( G_1 \sim G_2 \), then \( \phi_3(\mathcal{A}(G_1)) = \phi_3(\mathcal{A}(G_2)) \).

In this section we describe how to compute the Falk invariant \( \phi_3 \) for \( \mathcal{A}(G) \), a signed graphic arrangement associated to a signed graph \( G \) that do not have a subgraph isomorphic to \( B_2 \). In the remaining of the paper, to fix the notation we will suppose \( G \) is a graph on \( \ell \) vertices having \( n \) edges, and we will label only the edges as elements of \( [n] := \{1,\ldots,n\} \).
The goal of this section is to prove the following theorem.

**Theorem 4.1.** For a signed graphic arrangement associated to a signed graph \( G \) not containing a subgraph isomorphic to \( B_2 \) as subgraph, we have

\[
\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_o) + 5d_{3,1},
\]

where \( k_l \) denotes the number of subgraph of \( G \) isomorphic to a \( K_l \), \( d_l \) denotes the number of subgraph of \( G \) isomorphic to \( D_l \) but not contained in \( D^1_l \), \( d_{l,1} \) denotes the number of subgraph of \( G \) isomorphic to a \( K^l_l \) and \( g_o \) denotes the number of subgraph of \( G \) isomorphic to a \( G^l_o \) but not contained in \( D^1_l \).

**Remark 4.2.** Theorem 4.1 is a generalization of the previously known results for graphic arrangements \([6]\) and for simple signed graphic arrangements \([3]\). In fact, in both cases, we have graphs whose subgraphs can only be isomorphic to a \( K_3 \), and hence we obtain that for these cases \( \phi_3 = 2(k_3 + k_4) \).

In order to compute \( \phi_3 \), we need firstly to identify the ordered 3-tuple \( S \) in \( \{1, \ldots, n\} \) that are dependent. Clearly, we have the following

**Lemma 4.3.** \( S = (i_1, i_2, i_3) \) is dependent if and only if \( i_1, i_2, i_3 \) correspond to the edges of a subgraph of \( G \) that is isomorphic to a \( K_3 \), or a \( D^1_1 \) or a \( K^2_2 \).

With an abuse of notation, we will call a dependent 3-tuple \( S \) a **triangle**. Moreover, we will write

\[
\mathcal{C}_3 := \{ e_S \in E \mid S \text{ is a triangle} \}
\]

which is a subset of \( E \) as a vector space over \( \mathbb{C} \).

**Remark 4.4.** Notice that the triangles are exactly the balanced 3-cycles together with the subgraphs isomorphic to a \( K^2_2 \) or a \( D^1_1 \). In
particular, if \( G_1 \) and \( G_2 \) are two signed graphs with the same underlying graph such that \( G_1 \sim G_2 \), then \( C_3(G_1) = C_3(G_2) \).

Since \( e_i e_j e_k = -e_j e_i e_k \), it is clear that the dimension of the vector space \( C_3 \) is \( k_3 + d_{2,1} + k_{2,2} \).

**Lemma 4.5.** For a signed graphic arrangement associated to a signed graph \( G \) not containing a subgraph isomorphic to \( B_2 \) as subgraph, we have

\[
\dim(A^2) = \binom{n}{2} - k_3 - d_{2,1} - k_{2,2}.
\]

**Proof.** By definition \( A = E/I \), hence

\[
\dim(A^2) = \dim(E^2) - \dim(I^2) = \binom{n}{2} - \dim(I^2).
\]

Since \( I^2 = \text{span}\{\partial e_{ijk} | e_{ijk} \in C_3\} \), then \( \dim(I^2) = k_3 + d_{2,1} + k_{2,2} \), and the thesis follows. \( \square \)

Using Theorem \ref{thm:2.3} and Remark \ref{remark:2.4} to prove Theorem \ref{thm:4.1} we just need to describe \( \dim(I^3_2) \). To do so, let us consider \( C'_3 \) a basis of \( C_3 \).

Then each element of \( C'_3 \) is in a one-to-one correspondence of the subgraph of \( G \) isomorphic to a \( K_3 \), or a \( D_1 \) or a \( \overline{K}_2 \). Define then

\[
C_3 := \{ e_t \partial e_{ijk} | e_{ijk} \in C'_3, t \in \{i,j,k\} \},
\]

and

\[
F_3 := \{ e_t \partial e_{ijk} | e_{ijk} \in C'_3, t \in \{i,j,k\} \}.
\]

By construction \( I^3_2 = I^2 \cdot E^1 = \text{span}\{e_t \partial e_{ijk} | e_{ijk} \in C'_3, t \in [n] \setminus \{i,j,k\}\} \), and hence

\[
I^3_2 = \text{span}(C_3) + \text{span}(F_3).
\]

**Lemma 4.6.** For a signed graphic arrangement associated to a signed graph \( G \) not containing a subgraph isomorphic to \( B_2 \) as subgraph, we have

\[
I^3_2 = \text{span}(C_3) \oplus \text{span}(F_3).
\]

**Proof.** Since \( G \) do not contain a \( B_2 \) as subgraph, any two triangles shares at most one element. This then gives us that \( \text{span}(C_3) \cap \text{span}(F_3) = \emptyset \). \( \square \)

**Remark 4.7.** Notice that if we allow \( G \) to have subgraphs isomorphic to \( B_2 \), then the previous lemma is not true anymore.

By the previous lemma, we can write

\[
\dim(I^3_2) = \dim(\text{span}(C_3)) + \dim(\text{span}(F_3)) = k_3 + d_{2,1} + k_{2,2} + \dim(\text{span}(F_3)).
\]
To prove our main result we need to be able to compute $\dim(\text{span}(F_3))$. To do so, consider the following sets

\[
F_3^1 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \backslash \{i, j, k\}, i, j, k \text{ are not in the same } \overline{K}_4, D_3, \overline{G}_o, D_3^1, \overline{K}_3^3 \},
\]

\[
F_3^2 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \backslash \{i, j, k\}, i, j, k \text{ are in the same } \overline{K}_4 \},
\]

\[
F_3^3 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \backslash \{i, j, k\}, i, j, k \text{ are in the same } D_3 \text{ but not same } D_3^1 \},
\]

\[
F_3^4 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \backslash \{i, j, k\}, i, j, k \text{ are in the same } \overline{G}_o \text{ but not same } D_3^1 \},
\]

\[
F_3^5 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \backslash \{i, j, k\}, i, j, k \text{ are in the same } D_3^1 \},
\]

\[
F_3^6 := \{ e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \backslash \{i, j, k\}, i, j, k \text{ are in the same } \overline{K}_3^3 \},
\]

**Lemma 4.8.** For a signed graphic arrangement associated to a signed graph $G$ not containing a subgraph isomorphic to $B_2$, we have

\[
\text{span}(F_3) = \bigoplus_{i=1}^{6} \text{span}(F_3^i).
\]

**Proof.** There is an evident direct summand decomposition

\[
\text{span}(F_3) = \bigoplus_{X \in L_3(A)} \text{span}\{ e_t \partial e_{ijk} \mid H_t \cap H_i \cap H_j \cap H_k = X \},
\]

where $\{i, j, k\}$ is a triangle and $L_3(A)$ is the set of rank three flats of the lattice of intersections.

Then the result follows from recognizing $F_3^1$, $F_3^2$, $F_3^3$, $F_3^4$, $F_3^5$, $F_3^6$ as particular groups of summands of the above direct sum, corresponding to particular types of rank three flats. Specifically, $F_3^2$ corresponds to the $X = H_t \cap H_i \cap H_j \cap H_k$ where $t, i, j, k$ are in the same $\overline{K}_4$, $F_3^3$ corresponds to the $X$ where $t, i, j, k$ are in the same $D_3$ but not same $D_3^1$, $F_3^4$ corresponds to the $X$ where $t, i, j, k$ are in the same $\overline{G}_o$ but not same $D_3^1$, $F_3^5$ corresponds to the $X$ where $t, i, j, k$ are in the same $D_3^1$, and $F_3^6$ corresponds to the $X$ where $t, i, j, k$ are in the same $\overline{K}_3^3$. Finally, $F_3^1$ consists of the rest of the summands. \(\square\)

We now proceed to computing the dimensions of $F_3^i$ for $i = 1, \ldots, 6$, beginning with two examples which illustrate the general idea.

**Example 4.9.** We consider the dimension of $\text{span}(F_3)$ for the sign graphic arrangement $A_3$ associated to the graph $G_o$ (see Figure 2).

In this situation we have $E^+ = \{1, 2, 3\}$, $E^- = \{4, 5\}$ and $L = \{6\}$. Then the number of the elements in $F_3$ is 12, listed as follows.

\[
e_4 \partial e_{123} = e_{234} - e_{134} + e_{124},
e_5 \partial e_{123} = e_{235} - e_{135} + e_{125},
e_6 \partial e_{123} = e_{236} - e_{136} + e_{126},
e_1 \partial e_{345} = e_{145} - e_{135} + e_{134},
\]
Figure 2. The sign graph $G_o$

$$e_2 \partial e_{345} = e_{245} - e_{235} + e_{234}, e_3 \partial e_{245} = e_{456} - e_{356} + e_{346},$$
$$e_2 \partial e_{146} = e_{146} + e_{126} - e_{124}, e_3 \partial e_{146} = e_{156} - e_{136} + e_{135}, e_3 \partial e_{156} = - e_{145} + e_{156} + e_{145},$$
$$e_5 \partial e_{146} = - e_{145} + e_{156} + e_{145}, e_4 \partial e_{256} = e_{156} - e_{126} + e_{125},$$
$$e_5 \partial e_{256} = e_{156} + e_{256} - e_{235}, e_4 \partial e_{256} = e_{456} + e_{246} - e_{245}.$$

Then an easy computation shows that in this case $\dim(\text{span}(F_3)) = 10$.

Example 4.10. We consider the dimension of $\text{span}(F_3)$ for the sign graphic arrangement associated to the graph $D^1_3$ (see Figure 3).

In this situation we have $E^+ = \{1, 2, 3\}, E^- = \{4, 5, 6\}$ and $L = \{7\}$. Then the number of the elements in $F_3$ is 24, listed as follows.

$$e_4 \partial e_{123} = e_{124} - e_{134} + e_{234}, e_5 \partial e_{123} = e_{125} - e_{135} + e_{235},$$
$$e_6 \partial e_{123} = e_{126} - e_{136} + e_{236}, e_7 \partial e_{123} = e_{127} - e_{137} + e_{237},$$
$$e_2 \partial e_{156} = - e_{125} + e_{126} + e_{256}, e_3 \partial e_{156} = - e_{135} + e_{136} + e_{356},$$
$$e_4 \partial e_{156} = - e_{145} + e_{146} + e_{456}, e_7 \partial e_{156} = e_{157} - e_{167} + e_{567},$$
$$e_1 \partial e_{246} = e_{124} - e_{126} + e_{146}, e_3 \partial e_{246} = - e_{234} + e_{236} + e_{346},$$
$$e_5 \partial e_{246} = e_{245} + e_{256} - e_{456}, e_7 \partial e_{246} = e_{247} - e_{267} + e_{467},$$
$$e_1 \partial e_{345} = e_{134} - e_{135} + e_{145}, e_2 \partial e_{345} = e_{234} - e_{235} + e_{245},$$
\[ e_6 \partial e_{345} = e_{346} - e_{356} + e_{456}, \]
\[ e_7 \partial e_{345} = e_{347} - e_{357} + e_{457}, \]
\[ e_2 \partial e_{147} = -e_{124} + e_{127} + e_{247}, \]
\[ e_3 \partial e_{147} = -e_{134} + e_{137} + e_{347}, \]
\[ e_5 \partial e_{147} = e_{145} + e_{157} - e_{457}, \]
\[ e_6 \partial e_{147} = e_{146} + e_{167} - e_{467}, \]
\[ e_1 \partial e_{257} = e_{125} - e_{127} + e_{157}, \]
\[ e_3 \partial e_{257} = -e_{235} + e_{237} + e_{357}, \]
\[ e_4 \partial e_{257} = -e_{245} + e_{247} + e_{457}, \]
\[ e_6 \partial e_{257} = e_{256} + e_{267} - e_{567}. \]

Then an easy computation shows that in this case \( \dim(\text{span}(F_3)) = 19. \)

**Remark 4.11.** Similarly to the previous examples, we can compute \( \dim(\text{span}(F_3)) \) directly for several sign graph. In particular, if we consider \( D_3, K_4 \) and \( K_3^3 \), then \( \dim(\text{span}(F_3)) = 10. \)

**Lemma 4.12.** \( \dim(\text{span}(F_1^3)) = (n-3)(k_3+d_{2,1}+k_{3,3}) - 12k_4 - 12d_3 - 12g_5 - 12k_{3,3} - 24d_{3,1}. \)

*Proof.* The result follows from the equality \( |F_1^3| = |F_3| - (\sum_{i=2}^6 |F_3^i|) \), and noticing that the elements of \( F_1^3 \) are independent and they form a basis for \( \text{span}(F_1^3) \). \( \square \)

**Lemma 4.13.** \( \dim(\text{span}(F_2^3)) = 10k_4, \dim(\text{span}(F_3^3)) = 10d_3, \dim(\text{span}(F_4^3)) = 10g_5, \dim(\text{span}(F_5^3)) = 19d_{3,1} \) and \( \dim(\text{span}(F_6^3)) = 10k_{3,3}. \)

*Proof.* Assume that in the sign graph \( G \) there are exactly \( g_5 = p \) distinct subgraphs isomorphic to a \( G_5, G_1, \ldots, G_p \), none of which is a subgraph of a graph isomorphic to \( D_3^4 \). Consider

\[ F_{3,i}^4 : = \{ e_t \partial e_{ijk} \mid e_{ijk} \in C_3^t, t \in [n] \setminus \{ i,j,k \}, i,j,k \in G_i \}. \]

Since four edges in the graph \( G \) can not appear in two distinct \( G_5 \) at the same time, then none of the terms of the element \( e_t \partial e_{ijk} \in F_{3,i}^4 \) appear in the elements of \( F_3^4 \setminus F_{3,i}^4 \). This shows that

\[ \text{span}(F_3^4) = \bigoplus_{i=1}^p \text{span}(F_{3,i}^4). \]

By Corollary 3.11 and Example 4.9, we have that \( \dim(\text{span}(F_{3,i}^4)) = 10 \) for all \( i = 1, \ldots, p \). This then implies that

\[ \dim(\text{span}(F_3^4)) = \sum_{i=1}^p \dim(\text{span}(F_{3,i}^4)) = 10g_5. \]

Using Remark 4.11 and Example 4.10, the same exact argument used in this case will prove the other equalities. \( \square \)
Lemma 4.14. For a signed graphic arrangement associated to a signed graph \( G \) not containing a subgraph isomorphic to \( B_2 \), we have
\[
\dim(I_3^2) = (n - 2)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_0 - 2k_{3,3} - 5d_{3,1}.
\]

Proof. By the previous lemmas
\[
\dim(\text{span}(F_3)) = \sum_{i=1}^{6} \dim(\text{span}(F_3^i)) =
\]
\[
= [(n - 3)(k_3 + d_{2,1} + k_{3,3}) - 12k_4 - 12d_3 - 12g_0 - 12k_{3,3} - 24d_{3,1}] +
\]
\[
+ 10k_4 + 10d_3 + 10g_0 + 10k_{3,3} + 19d_{3,1} =
\]
\[
(n - 3)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_0 - 2k_{3,3} - 5d_{3,1}.
\]
The thesis follows from the equality
\[
\dim(I_3^2) = k_3 + d_{2,1} + k_{2,2} + \dim(\text{span}(F_3)).
\]

\( \square \)

Proof of Theorem 4.1. By Remark 2.4 and Lemma 4.5 we have
\[
\phi_3 = 2\left(\binom{n+1}{3}\right) - n\left(\binom{n}{2} - k_3 - d_{2,1} - k_{2,2}\right) + \binom{n}{3} - \dim(I_2^3).
\]
Because \( 2\binom{n+1}{3} - n\binom{n}{2} + \binom{n}{3} = 0 \), then from Lemma 4.14 we obtain
\[
\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_0) + 5d_{3,1}.
\]

\( \square \)

Let us see how our formula works on a non-trivial example.

Example 4.15. We want to compute \( \phi_3 \) for the arrangement associated to the graph \( G \) of Figure 4.

Figure 4. The sign graph \( G \)
In this situation we have $E^+ = \{1, 2, 3, 4, 5, 6\}, E^- = \{7, 8, 9, 10\}$ and $L = \{11\}$. In order to compute $\phi_3$ with the formula (3), we need to compute the following:

- $k_3 = |\{\{1, 2, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{1, 9, 10\}, \{6, 7, 9\}, \{4, 7, 9\}, \{3, 8, 9\}, \{5, 7, 8\}\}| = 9$;
- $k_4 = |\{\{1, 2, 3, 4, 5, 6\}, \{3, 4, 5, 7, 8, 9\}\}| = 2$;
- $d_3 = 0$;
- $d_{2, 1} = |\{\{1, 7, 11\}, \{6, 9, 11\}, \{2, 8, 11\}\}| = 3$;
- $k_{2, 2} = 0$;
- $k_{3, 3} = 0$;
- $g_6 = |\{\{1, 2, 5, 7, 8, 11\}, \{2, 3, 6, 8, 9, 11\}\}| = 2$;
- $d_{3, 1} = |\{\{1, 4, 6, 7, 9, 10, 11\}\}| = 1$.

From formula (3), we obtain

$$\phi_3 = 2(9 + 2 + 0 + 3 + 0 + 0 + 2) + 5 = 37.$$ 

Notice that if we would try to compute the dimension of $F_3$ directly, we would have to write 96 equations in the $e_{ijk}$.

**Acknowledgements.** The authors thank Professor Yoshinaga for the valuable discussions and the anonymous referee for suggesting a shorter proof for Lemma 4.8. The second authors also thanks Doctor Suyama and Doctor Tsujie for the valuable discussions on signed graphs.

During the preparation of this paper the second author was supported by the MEXT grant for Tenure Tracking system.

**References**


