ON THE FALK INVARIANT OF SIGNED GRAPHIC ARRANGEMENTS

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Abstract. The fundamental group of the complement of a hyperplane arrangement in a complex vector space is an important topological invariant. The third rank of successive quotients in the lower central series of the fundamental group was called *Falk invariant* of the arrangement since Falk gave the first formula and asked to give a combinatorial interpretation. In this article, we give a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a B_2 as sub-arrangement.

1. Introduction

A **hyperplane** H in $\mathbb{C}^\ell$ is an affine subspace of dimension $\ell - 1$. A finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of hyperplanes is called a **hyperplane arrangement**. If $\bigcap_{i=1}^n H_i \neq \emptyset$, then $\mathcal{A}$ is called **central**. In this paper, we only consider central arrangements and assume that all the hyperplanes contain the origin. For more details on hyperplane arrangements, see [5].

Let $M := \mathbb{C}^\ell \setminus_{H \in \mathcal{A}} H$ be the complement of the arrangement $\mathcal{A}$. It is known that the cohomology ring $H^*(M)$ is completely determined by $L(\mathcal{A})$ the lattice of intersection of $\mathcal{A}$. Similarly to this result, there are several conjectures concerning the relationship between M and $L(\mathcal{A})$. To study such problems, Falk introduced in [1] a multiplicative invariant, called **global invariant**, of the Orlik-Solomon algebra of $\mathcal{A}$. The invariant is now known as the (3^{rd}) **Falk invariant** and it is denoted by ϕ_3. In [2], Falk posed as an open problem to give a combinatorial interpretation of ϕ_3.

Several authors already studied this invariant. In [6], Schenck and Suciu studied the lower central series of arrangements and described a formula for the Falk invariant in the case of graphic arrangements. In [3], the authors gave a formula for ϕ_3 in the case of simple sign graphic arrangements. In the preprint [4], the authors extended the previous result for sign graphic arrangements coming from graphs without loops. This article is devoted to extend these results further and to describe

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a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a B_2 as sub-arrangement. Our result gives a partial answer to the question posed by Falk in [2].

2. Preliminares on Orlik-Solomon algebras

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. Let $E^1 = \bigoplus_{j=1}^n \mathbb{C}e_j$ be the free module generated by $e_1, e_2, \dots, e_n$, where e_i is a symbol corresponding to the hyperplane H_i. Let $E = \bigwedge E^1$ be the exterior algebra over $\mathbb{C}$. The algebra E is graded via $E = \bigoplus_{p=0}^n E^p$, where $E^p = \bigwedge^p E^1$. The $\mathbb{C}$-module E^p is free and has the distinguished basis consisting of monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$, where $S = \{i_1, \dots, i_p\}$ is running through all the subsets of $\{1, \dots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$. The graded algebra E is a commutative DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions $\partial e_i = 1$ for all $i = 1, \dots, n$ and the graded Leibniz formula. Then for every $S \subseteq \{1, \dots, n\}$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j},$$

where S_j is the complement in S to its j-th element.

For every $S \subseteq \{1, \dots, n\}$, put $\cap S = \bigcap_{i \in S} H_i$ (possibly $\cap S = \emptyset$). The set of all intersections $L(\mathcal{A}) := \{\cap S \mid S \subseteq \{1, \dots, n\}\}$ is called the **intersection poset of** $\mathcal{A}$. The subset $S \subseteq \{1, \dots, n\}$ is called **dependent** if $\cap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i \mid i \in S\}$ with $H_i = \alpha_i^{-1}(0)$, is linearly dependent.

Definition 2.1. The **Orlik-Solomon ideal** of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by

(1) all e_S with $\cap S = \emptyset$,
(2) all ∂e_S with S dependent.

The algebra $A := A^\bullet(\mathcal{A}) = E/I(\mathcal{A})$ is called the **Orlik-Solomon algebra** of $\mathcal{A}$.

Clearly I is a homogeneous ideal of E and $I^p = I \cap E^p$ whence A is a graded algebra and we can write $A = \bigoplus_{p \geq 0} A^p$, where $A^p = E^p/I^p$. If $\mathcal{A}$ is central, then for any $S \subseteq \mathcal{A}$, we have $\cap S \neq \emptyset$. Therefore, the Orlik-Solomon ideal is generated by the elements of type (2) from Definition 2.1. In this case, the map ∂ induces a well-defined differential $\partial \colon A^\bullet(\mathcal{A}) \longrightarrow A^{\bullet-1}(\mathcal{A})$.

Let I_k be the k-adic Orlik-Solomon ideal of $\mathcal{A}$ generated by $\sum_{j \leq k} I^j$ in E. It is clear that I_k is a graded ideal and $I_k^p = (I_k)^p = E^p \cap I_k$.

Write $A_k := A^\bullet_k(\mathcal{A}) = E/I_k$ and $A^p_k := (A^\bullet_k(\mathcal{A}))^p = E^p/I^p_k$ which is called k**-adic Orlik-Solomon algebra** by Falk [1].

In this set up, it is now easy to define the Falk invariant.

Definition 2.2. Consider the map d defined by

$$d\colon E^1 \otimes I^2 \to E^3,$$

$$d(a \otimes b) = a \wedge b.$$

Then the **Falk invariant** is defined as

$$\phi_3 := \dim(\ker(d)).$$

In [1] and [2], Falk gave a beautiful formula to compute such invariant.

Theorem 2.3 (Theorem 4.7, [2]). *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. Then*

$$\phi_3 = 2\binom{n+1}{3} - n\dim(A^2) + \dim(A^3_2). \tag{1}$$

Remark 2.4. Since $\dim(A^3_2) = \dim((E/I_2)^3) = \dim(E^3) - \dim(I^3_2)$ and $\dim(E^3) = \binom{n}{3}$, then we obtain

$$\phi_3 = 2\binom{n+1}{3} - n\dim(A^2) + \binom{n}{3} - \dim(I^3_2). \tag{2}$$

Recall that ϕ_3 can also be describe from the lower central series of the fundamental group $\pi(M)$ of the complement M of the arrangement. In particular, if we consider the lower central series as a chain of normal subgroups N_i, for $k \geq 1$, where $N_1 = \pi(M)$ and $N_{k+1} = [N_k, N_1]$, the subgroup generated by commutators of elements in N_k and N_1, then ϕ_3 is the rank of the finitely generated abelian group N_3/N_4. See [6] for more details.

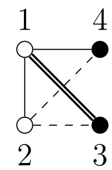
3. Sign graphs

In this section we will recall the main properties of signed graphs. See [7] for a general treatment of such graphs.

Definition 3.1. A **signed graph** is a tuple $G = (V_G, E^+_G, E^-_G, L_G)$, where

- V_G is a finite set called the set of vertices,
- E^+_G is a subset of $\binom{V_G}{2}$ called the set of positive edges,
- E^-_G is a subset of $\binom{V_G}{2}$ called the set of negative edges,
- L_G is a subset of V_G called the set of loops.

Example 3.2. In this article, we illustrate a signed graph as follows:

$G = (V_G, E_G^+, E_G^-, L_G) =$  ,

$$\begin{cases} V_G = \{1,2,3,4\}, \\ E_G^+ = \{\{1,2\},\{1,3\},\{1,4\}\}, \\ E_G^- = \{\{1,3\},\{2,3\},\{2,4\}\}, \\ L_G = \{3,4\}. \end{cases}$$

Let $G^+ = (V_G, E_G^+)$ and $G^- = (V_G, E_G^-)$, then we have an alternative notation $G = (G^+, G^-, L_G)$ for the signed graph G. An unsigned simple graph G may be regarded as a signed graph $G = (G, K_\ell^\circ, \emptyset)$, where K_ℓ° denotes the edgeless graph on ℓ vertices. A signed graph $(G^+, G^-, \emptyset)$ is called **loopless**, which is also denoted by (G^+, G^-). Let E_G denote the edge set $E_G^+ \sqcup E_G^- \sqcup L_G$. For a positive integer ℓ, let $[\ell]$ denote the set $\{1, \dots, \ell\}$. From now on, we suppose that G is a signed graph on vertices $[\ell]$. Let $(x_1, \dots, x_\ell)$ be a basis for the ℓ-dimensional vector space $(\mathbb{C}^\ell)^*$. For $\alpha \in (\mathbb{C}^\ell)^*$, let $\{\alpha = 0\}$ denote the hyperplane $\{v \in \mathbb{C}^\ell \mid \alpha(v) = 0\}$.

Definition 3.3. Given a signed graph G, let $\mathcal{A}(G)$ be the hyperplane arrangement in $\mathbb{C}^\ell$ consisting of the following hyperplane

$$\{x_i - x_j = 0\} \text{ for } \{i,j\} \in E_G^+,$$

$$\{x_i + x_j = 0\} \text{ for } \{i,j\} \in E_G^-,$$

$$\{x_i = 0\} \text{ for } i \in L_G.$$

We will call $\mathcal{A}(G)$ the **signed graphic arrangement** associated to the signed graph G.

Given a signed graph it is natural to introduce the following function.

Definition 3.4. Given a sign graph $G = (V_G, E_G^+, E_G^-, L_G)$, the **sign function** of G is the function $sgn\colon E_G^+ \cup E_G^- \cup L_G \to \{+,-\}$ defined by

$$sgn(e) = \begin{cases} + & \text{if } e \in E_G^+, \\ - & \text{if } e \in E_G^- \cup L_G. \end{cases}$$

We can naturally extend the previous definition to path in G

Definition 3.5. Given $P = e_1 e_2 \cdots e_k$ a path in G, the **sign** of P is $sgn(P) = \prod_{i=1}^k sgn(e_i)$.

Definition 3.6. A cycle C in a sign graph G is called **balanced** if $sgn(C) = +$.

Given a sign graph G and a function $\sigma\colon V_G \to \{+,-\}$, we can define a new sign graph G' that has the same underlying graph as G but with a different sign function. In particular, if $e = \{i,j\} \in E_G$ then $sgn_{G'}(e) = \sigma(i)sgn_G(e)\sigma(j)$.

Definition 3.7. In the previous construction, we will call G' the **switching of G by** σ and we will denote it by G^σ. In this case, σ is called a **switching function for** G.

Definition 3.8. Given two sign graph G_1 and G_2 with the same underlying graph, we will say they are **switching equivalent** and write $G_1 \backsim G_2$, if there exists a switching function σ such that $G_2 = G_1^\sigma$.

Proposition 3.9 (Proposition 3.2, [7])**.** *Two signed graphs with the same underlying graph are switching equivalent if and only if they have the same list of balanced circles.*

Proposition 3.10 (Corollary 5.4, [7])**.** *Two signed graphs with the same underlying graph are switching equivalent if and only if they define the same matroid.*

Using the previous results, we obtain

Corollary 3.11. *Let G_1 and G_2 be two signed graph with the same underlying graph. If $G_1 \backsim G_2$, then $\phi_3(\mathcal{A}(G_1)) = \phi_3(\mathcal{A}(G_2))$.*

In this paper taking inspiration from graph theory and the study of hyperplane arrangements, we denote by K_ℓ a complete graph with ℓ vertices and all edges being positive, i.e. $K_\ell = (K_\ell, K_\ell^\circ, \emptyset)$, by D_ℓ a complete sign graph with ℓ vertices and no loops, i.e. $D_\ell = (K_\ell, K_\ell, \emptyset)$, and by B_ℓ a sign complete graph with ℓ vertices and a full set of loops, i.e. $B_\ell = (K_\ell, K_\ell, [\ell])$. Moreover, we denote by K_ℓ^ℓ a complete graph with ℓ vertices, all edges being positive and a full set of loops, i.e. $K_\ell^\ell = (K_\ell, K_\ell^\circ, [\ell])$, by D_ℓ^1 a complete sign graph with ℓ vertices and one loop, i.e. $D_\ell^1 = (K_\ell, K_\ell, \{1\})$ and by $G_\circ$ the signed graph in Figure 1. Furthermore, if G is a signed graph we denote by $\overline{G}$ a signed graph switching equivalent to G for some switching function σ.

4. Main Theorem

In this section we describe how to compute the Falk invariant ϕ_3 for $\mathcal{A}(G)$, a signed graphic arrangement associated to a signed graph G that do not have a subgraph isomorphic to B_2. In the remaining of the paper, to fix the notation we will suppose G is a graph on ℓ vertices having n edges, and we will label only the edges as elements of $[n] := \{1,\ldots,n\}$.

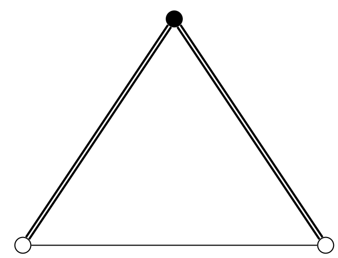


FIGURE 1. The sign graph $G_\circ$

The goal of this section is to prove the following theorem.

Theorem 4.1. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2 as subgraph, we have*

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_\circ) + 5d_{3,1}, \tag{3}$$

where k_l denotes the number of subgraph of G isomorphic to a $\overline{K_l}$, d_l denotes the number of subgraph of G isomorphic to D_l but not contained in D_l^1, $d_{l,1}$ denotes the number of subgraph of G isomorphic to D_l^1, $k_{l,l}$ denotes the number of subgraph of G isomorphic to a $\overline{K_l^l}$ and $g_\circ$ denotes the number of subgraph of G isomorphic to a $\overline{G_\circ}$ but not contained in D_l^1.

Remark 4.2. Theorem 4.1 is a generalization of the previously known results for graphic arrangements [6] and for simple signed graphic arrangements [3]. In fact, in both cases, we have graphs whose subgraphs can only be isomorphic to a $\overline{K_l}$, and hence we obtain that for these cases $\phi_3 = 2(k_3 + k_4)$.

In order to compute ϕ_3, we need firstly to identify the ordered 3-tuple S in $\{1, \dots, n\}$ that are dependent. Clearly, we have the following

Lemma 4.3. *$S = (i_1, i_2, i_3)$ is dependent if and only if i_1, i_2, i_3 correspond to the edges of a subgraph of G that is isomorphic to a $\overline{K_3}$, or a D_2^1 or a $\overline{K_2^2}$.*

With an abuse of notation, we will call a dependent 3-tuple S a **triangle**. Moreover, we will write

$$\mathcal{C}_3 := \{e_S \in E \mid S \text{ is a triangle}\}$$

which is a subset of E as a vector space over $\mathbb{C}$.

Remark 4.4. Notice that the triangles are exactly the balanced 3-cycles together with the subgraphs isomorphic to a $\overline{K_2^2}$ or a D_2^1. In particular, If G_1 and G_2 are two signed graph with the same underlying graph such that $G_1 \backsim G_2$, then $\mathcal{C}_3(G_1) = \mathcal{C}_3(G_2)$.

Since $e_ie_je_k = -e_je_ie_k$, it is clear that the dimension of the vector space $\mathcal{C}_3$ is $k_3 + d_{2,1} + k_{2,2}$.

Lemma 4.5. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2 as subgraph, we have*

$$\dim(A^2) = \binom{n}{2} - k_3 - d_{2,1} - k_{2,2}.$$

Proof. By definition $A = E/I$, hence

$$\dim(A^2) = \dim(E^2) - \dim(I^2) = \binom{n}{2} - \dim(I^2).$$

Since $I^2 = \text{span}\{\partial e_{ijk} \mid e_{ijk} \in \mathcal{C}_3\}$, then $\dim(I^2) = k_3 + d_{2,1} + k_{2,2}$, and the thesis follows. □

Using Theorem 2.3 and Remark 2.4, to prove Theorem 4.1 we just need to describe $\dim(I^3_2)$. To do so, let us consider C'_3 a basis of $\mathcal{C}_3$. Then each element of C'_3 is in a one-to-one correspondence of the subgraph of G isomorphic to a $\overline{K_3}$, or a D^1_2 or a $\overline{K^2_2}$. Define then

$$C_3 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C'_3, t \in \{i, j, k\}\},$$

and

$$F_3 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}\}.$$

By construction $I^3_2 = I^2 \cdot E^1 = \text{span}\{e_t\partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n]\}$, and hence

$$I^3_2 = \text{span}(C_3) + \text{span}(F_3).$$

Lemma 4.6. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2 as subgraph, we have*

$$I^3_2 = \text{span}(C_3) \oplus \text{span}(F_3).$$

Proof. Since G do not contain a B_2 as subgraph, any two triangles shares at most one element. This then gives us that $\text{span}(C_3)\cap\text{span}(F_3) = \emptyset$. □

Remark 4.7. Notice that if we allow G to have subgraphs isomorphic to B_2, then the previous lemma is not true anymore.

By the previous lemma, we can write

$$\dim(I^3_2) = \dim(\text{span}(C_3)) + \dim(\text{span}(F_3)) = k_3 + d_{2,1} + k_{2,2} + \dim(\text{span}(F_3)).$$

To prove our main result we need to be able to compute $\dim(\operatorname{span}(F_3))$.
To do so, consider the following sets

$F_3^1 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n]\backslash\{i,j,k\}, i,j,k$ are not in the same $\overline{K_4}, D_3, \overline{G_\circ}, D_3^1, \overline{K_3^3}\}$,

$F_3^2 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $\overline{K_4}\}$,

$F_3^3 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n]\backslash\{i,j,k\}, i,j,k$ are in the same D_3 but not same $D_3^1\}$,

$F_3^4 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n]\backslash\{i,j,k\}, i,j,k$ are in the same $\overline{G_\circ}$ but not same $D_3^1\}$,

$F_3^5 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $D_3^1\}$,

$F_3^6 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $\overline{K_3^3}\}$,

Lemma 4.8. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2, we have*

$$\operatorname{span}(F_3) = \bigoplus_{i=1}^{6} \operatorname{span}(F_3^i).$$

Proof. There is an evident direct summand decomposition

$$\operatorname{span}(F_3) = \bigoplus_{X\in L_3(\mathcal{A})} \operatorname{span}\{e_t\partial e_{ijk} \mid H_t \cap H_i \cap H_j \cap H_k = X\},$$

where $\{i,j,k\}$ is a triangle and $L_3(\mathcal{A})$ is the set of rank three flats of the lattice of intersections.

Then the result follows from recognizing F_3^2, F_3^3, F_3^4, F_3^5, F_3^6 as particular groups of summands of the above direct sum, corresponding to particular types of rank three flats. Specifically, F_3^2 corresponds to the $X = H_t \cap H_i \cap H_j \cap H_k$ where t,i,j,k are in the same $\overline{K_4}$, F_3^3 corresponds to the X where t,i,j,k are in the same D_3 but not same D_3^1, F_3^4 corresponds to the X where t,i,j,k are in the same $\overline{G_\circ}$ but not same D_3^1, F_3^5 corresponds to the X where t,i,j,k are in the same D_3^1, and F_3^6 corresponds to the X where t,i,j,k are in the same $\overline{K_3^3}$. Finally, F_3^1 consists of the rest of the summands. □

We now proceed to computing the dimensions of F_3^i for $i = 1,\dots,6$, beginning with two examples which illustrate the general idea.

Example 4.9. We consider the dimension of $\operatorname{span}(F_3)$ for the sign graphic arrangement A_3 associated to the graph $G_\circ$ (see Figure 2).

In this situation we have $E^+ = \{1,2,3\}, E^- = \{4,5\}$ and $L = \{6\}$. Then the number of the elements in F_3 is 12, listed as follows.

$$e_4\partial e_{123} = e_{234} - e_{134} + e_{124}, e_5\partial e_{123} = e_{235} - e_{135} + e_{125},$$

$$e_6\partial e_{123} = e_{236} - e_{136} + e_{126}, e_1\partial e_{345} = e_{145} - e_{135} + e_{134},$$

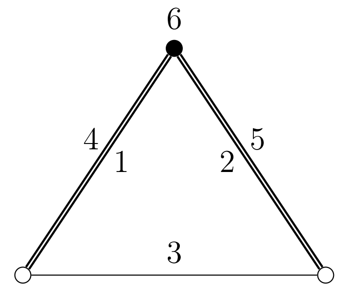


FIGURE 2. The sign graph $G_\circ$

$$e_2\partial e_{345} = e_{245} - e_{235} + e_{234}, e_6\partial e_{345} = e_{456} - e_{356} + e_{346},$$
$$e_2\partial e_{146} = e_{246} + e_{126} - e_{124}, e_3\partial e_{146} = e_{346} + e_{136} - e_{136},$$
$$e_5\partial e_{146} = -e_{456} + e_{156} + e_{145}, e_1\partial e_{256} = e_{156} - e_{126} + e_{125},$$
$$e_3\partial e_{256} = e_{356} + e_{236} - e_{235}, e_4\partial e_{256} = e_{456} + e_{246} - e_{245}.$$

Then an easy computation shows that in this case $\dim(\operatorname{span}(F_3)) = 10$.

Example 4.10. We consider the dimension of $\operatorname{span}(F_3)$ for the sign graphic arrangement associated to the graph D_3^1 (see Figure 3).

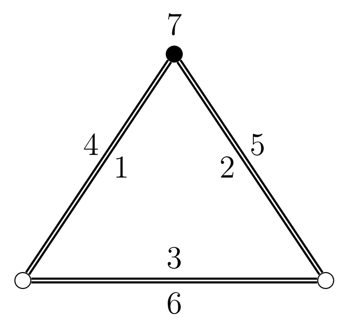


FIGURE 3. The sign graph D_3^1

In this situation we have $E^+ = \{1, 2, 3\}, E^- = \{4, 5, 6\}$ and $L = \{7\}$. Then the number of the elements in F_3 is 24, listed as follows.

$$e_4\partial e_{123} = e_{124} - e_{134} + e_{234}, e_5\partial e_{123} = e_{125} - e_{135} + e_{235},$$
$$e_6\partial e_{123} = e_{126} - e_{136} + e_{236}, e_7\partial e_{123} = e_{127} - e_{137} + e_{237},$$
$$e_2\partial e_{156} = -e_{125} + e_{126} + e_{256}, e_3\partial e_{156} = -e_{135} + e_{136} + e_{356},$$
$$e_4\partial e_{156} = -e_{145} + e_{146} + e_{456}, e_7\partial e_{156} = e_{157} - e_{167} + e_{567},$$
$$e_1\partial e_{246} = e_{124} - e_{126} + e_{146}, e_3\partial e_{246} = -e_{234} + e_{236} + e_{346},$$
$$e_5\partial e_{246} = e_{245} + e_{256} - e_{456}, e_7\partial e_{246} = e_{247} - e_{267} + e_{467},$$
$$e_1\partial e_{345} = e_{134} - e_{135} + e_{145}, e_2\partial e_{345} = e_{234} - e_{235} + e_{245},$$

$$e_6\partial e_{345} = e_{346} - e_{356} + e_{456}, e_7\partial e_{345} = e_{347} - e_{357} + e_{457},$$

$$e_2\partial e_{147} = -e_{124} + e_{127} + e_{247}, e_3\partial e_{147} = -e_{134} + e_{137} + e_{347},$$

$$e_5\partial e_{147} = e_{145} + e_{157} - e_{457}, e_6\partial e_{147} = e_{146} + e_{167} - e_{467},$$

$$e_1\partial e_{257} = e_{125} - e_{127} + e_{157}, e_3\partial e_{257} = -e_{235} + e_{237} + e_{357},$$

$$e_4\partial e_{257} = -e_{245} + e_{247} + e_{457}, e_6\partial e_{257} = e_{256} + e_{267} - e_{567}.$$

Then an easy computation shows that in this case $\dim(\mathrm{span}(F_3)) = 19$.

Remark 4.11. Similarly to the previous examples, we can compute $\dim(\mathrm{span}(F_3))$ directly for several sign graph. In particular, if we consider D_3, K_4 and K_3^3, then $\dim(\mathrm{span}(F_3)) = 10$.

Lemma 4.12. $\dim(\mathrm{span}(F_3^1)) = (n-3)(k_3+d_{2,1}+k_{3,3})-12k_4-12d_3-12g_\circ-12k_{3,3}-24d_{3,1}$.

Proof. The result follows from the equality $|F_3^1| = |F_3| - (\sum_{i=2}^6 |F_3^i|)$, and noticing that the elements of F_3^1 are independent and they form a basis for $\mathrm{span}(F_3^1)$. □

Lemma 4.13. $\dim(\mathrm{span}(F_3^2)) = 10k_4$*,* $\dim(\mathrm{span}(F_3^3)) = 10d_3$*,* $\dim(\mathrm{span}(F_3^4)) = 10g_\circ$*,* $\dim(\mathrm{span}(F_3^5)) = 19d_{3,1}$ *and* $\dim(\mathrm{span}(F_3^6)) = 10k_{3,3}$*.*

Proof. Assume that in the sign graph G there are exactly $g_\circ = p$ distinct subgraphs isomorphic to a $\overline{G_\circ}$, $G_1, \dots, G_p$, none of which is a subgraph of a graph isomorphic to D_3^1. Consider

$$F_{3,i}^4 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k \in G_i\}.$$

Since four edges in the graph G can not appear in two distinct $\overline{G_\circ}$ at the same time, then none of the terms of the element $e_t\partial e_{ijk} \in F_{3,i}^2$ appear in the elements of $F_3^4 \setminus F_{3,i}^4$. This shows that

$$\mathrm{span}(F_3^4) = \bigoplus_{i=1}^{p} \mathrm{span}(F_{3,i}^4).$$

By Corollary 3.11 and Example 4.9, we have that $\dim(\mathrm{span}(F_{3,i}^4)) = 10$ for all $i = 1, \dots, p$. This then implies that

$$\dim(\mathrm{span}(F_3^4)) = \sum_{i=1}^{p} \dim(\mathrm{span}(F_{3,i}^4)) = 10g_\circ.$$

Using Remark 4.11 and Example 4.10, the same exact argument used in this case will prove the other equalities. □

Lemma 4.14. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2, we have*

$$\dim(I_2^3) = (n-2)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_\circ - 2k_{3,3} - 5d_{3,1}.$$

Proof. By the previous lemmas

$$\dim(\mathrm{span}(F_3)) = \sum_{i=1}^{6} \dim(\mathrm{span}(F_3^i)) =$$

$$= [(n-3)(k_3 + d_{2,1} + k_{3,3}) - 12k_4 - 12d_3 - 12g_\circ - 12k_{3,3} - 24d_{3,1}] +$$
$$+10k_4 + 10d_3 + 10g_\circ + 10k_{3,3} + 19d_{3,1} =$$
$$(n-3)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_\circ - 2k_{3,3} - 5d_{3,1}.$$

The thesis follows from the equality

$$\dim(I_2^3) = k_3 + d_{2,1} + k_{2,2} + \dim(\mathrm{span}(F_3)).$$

□

Proof of Theorem 4.1. By Remark 2.4 and Lemma 4.5 we have

$$\phi_3 = 2\binom{n+1}{3} - n(\binom{n}{2} - k_3 - d_{2,1} - k_{2,2}) + \binom{n}{3} - \dim(I_2^3).$$

Because $2\binom{n+1}{3} - n\binom{n}{2} + \binom{n}{3} = 0$, then from Lemma 4.14 we obtain

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_\circ) + 5d_{3,1}.$$

□

Let us see how our formula works on a non-trivial example.

Example 4.15. We want to compute ϕ_3 for the arrangement associated to the graph G of Figure 4.

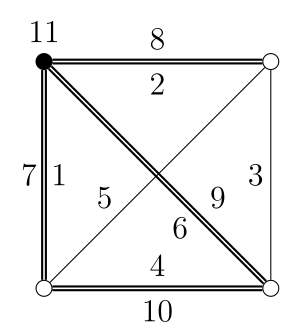


FIGURE 4. The sign graph G

In this situation we have $E^+ = \{1,2,3,4,5,6\}, E^- = \{7,8,9,10\}$ and $L = \{11\}$. In order to compute ϕ_3 with the formula (3), we need to compute the following:

- $k_3 = |\{\{1,2,5\}, \{1,4,6\}, \{2,3,6\}, \{3,4,5\}, \{1,9,10\}, \{6,7,9\}, \{4,7,9\}, \{3,8,9\}, \{5,7,8\}\} = 9$;
- $k_4 = |\{\{1,2,3,4,5,6\}, \{3,4,5,7,8,9\}\}| = 2$;
- $d_3 = 0$;
- $d_{2,1} = |\{\{1,7,11\}, \{6,9,11\}, \{2,8,11\}\}| = 3$;
- $k_{2,2} = 0$;
- $k_{3,3} = 0$;
- $g_\circ = |\{\{1,2,5,7,8,11\}, \{2,3,6,8,9,11\}\}| = 2$;
- $d_{3,1} = |\{\{1,4,6,7,9,10,11\}\}| = 1$.

From formula (3), we obtain

$$\phi_3 = 2(9+2+0+3+0+0+2)+5 = 37.$$

Notice that if we would try to compute the dimension of F_3 directly, we would have to write 96 equations in the e_{ijk}.

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