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A TWO DIMENSIONAL RANDOM CRYSTALLINE ALGORITHM FOR GAUSS CURVATURE FLOW

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Abstract

We propose and study a random crystalline algorithm (a discrete approximation) of the Gauss curvature flow of smooth simple closed convex curves in $\mathbf{R}^2$ as a stepping stone to the full understanding of such a phenomenon as the wearing process of stones on beaches.

Keywords: random crystalline algorithm; Gauss curvature flow; closed curve

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1. Introduction.

The Gauss curvature flow of closed surfaces in $\mathbf{R}^3$ is a mathematical model of the wearing process of stones on beaches (see [3] and also [1], [6] and [11]).

We introduce the definition of the Gauss curvature flow of smooth closed convex hypersurfaces in $\mathbf{R}^{d+1}$. Let Γ be a smooth closed convex hypersurface in $\mathbf{R}^{d+1}$ and $F : \mathbf{S}^d \mapsto \mathbf{R}^{d+1}$ be a parametric representation of Γ. Then a collection of $F(\cdot, t) : \mathbf{S}^d \mapsto \mathbf{R}^{d+1}$ of smooth closed convex hypersurfaces with parameter $t \in [0, T)$ for some $T > 0$ is called Gauss curvature flow with initial state Γ if the following holds:

$$\frac{\partial F(s,t)}{\partial t} = -K(s,t)n(s,t) \quad (s \in \mathbf{S}^d, 0 < t < T), \tag{1.1}$$

$$F(s,0) = F(s) \quad (s \in \mathbf{S}^d), \tag{1.2}$$

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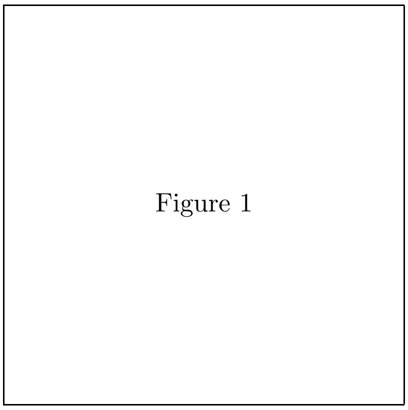

Figure 1: Motion of $F(\cdot, t)$ at $F(s, t)$ in $\mathbf{R}^2$

where $K(s,t)$ and $n(s,t)$ denote the Gauss curvature and the unit outward normal vector, respectively, at a point $F(s,t)$ on the hypersurface $\{F(s',t)|s' \in \mathbf{S}^d\}$. In this paper we assume that the convex set with boundary $\{F(s,t)|s \in \mathbf{S}^d\}$ is non-increasing in t (see Figure 1).

Suppose that Γ is strictly convex. Then there exists the maximum T^* of T for which (1.1)-(1.2) has a unique smooth strictly convex solution and $\{F(s,t)|s \in \mathbf{S}^d\}$ converges to a point as $t \uparrow T^*$ (see [1], [6] and [11]).

In [8], H. Ishii proposed a discrete time approximation scheme for the Gauss curvature flow. We briefly introduce it. Suppose that we are given the strictly convex set D with smooth boundary ∂D in $\mathbf{R}^{d+1}$ at time $t = 0$. Take $h > 0$ and a function $V : [0, \infty) \mapsto [0, \infty)$. For every $s \in \mathbf{S}^d$, let $D_{s,h}$ denote the set which can be obtained by cutting off the volume $V(h)$ from the set D in the direction $-s$ (see Figure 2). Put $\mathbf{D}_{0,h} \equiv D$ and $\mathbf{D}_{1,h} \equiv \cap_{s \in \mathbf{S}^d} D_{s,h}$. Define $\mathbf{D}_{n,h}$ inductively in n until $n_h \equiv \max\{k \geq 1|$ the volume of $\mathbf{D}_{k,h}$ is greater than $V(h)\}+1$. Let $V(h) \to 0$ as $h \to 0$ in an appropriate rate. Then $\lim_{h \to 0} n_h h = T_{max}$, and the flow of $\partial \mathbf{D}_{[t/h],h}$ $(0 \leq t \leq n_h h)$ converges to the Gauss curvature flow in Hausdorff metric uniformly in t on every compact subset of $[0, T^*)$, where $[t/h]$ denotes the integer part of $[t/h]$. Notice that the time variable t is discretized but the space variable s is not in this approximation scheme.

Remark 1. Hausdorff metric of compact sets A and $B \in \mathbf{R}^d$ is given by the following:

Figure 2

Figure 2: D and $D_{s,h}$ in $\mathbf{R}^2$

$$d_H(A,B) \equiv \max(\max_{p\in A} dist(p,B), \max_{q\in B} dist(q,A)). \tag{1.3}$$

A crystalline (or a polyhedral) approximation of the curvature flow of convex curves was studied by P. M. Girão and is useful in numerical analysis (see Theorem 1 given below, [4] and also [5] and the references therein). In [4], the space variable s is discretized but the time variable t is not. In case when the initial curve is not convex, the results of [4] have been generalized by K. Ishii and M. H. Soner (see [9] and the references therein for further information on this problem). The results of [4] have not been generalized to a class of closed convex hypersurfaces in $\mathbf{R}^{d+1}$ for $d \geq 2$. This is a well-known open problem.

Remark 2. Let Γ be a smooth simple closed convex curve on $\mathbf{R}^2$. Fix a point x_0 on Γ. For any $x \in \Gamma$, let $s(x)$ be the length of the curve which connects x_0 and x on Γ clockwise. Then one can parametrize $x \in \Gamma$ by $s(x)$. Let $p_1(s(x))$ and $p_2(s(x))$ denote, respectively, the clockwise unit tangent vector and the unit outward normal vector at x on Γ. Then the Gauss curvature $K(s(x))(\in \mathbf{R})$ at x on Γ satisfies the following:

$$\begin{aligned} \frac{dp_1(s(x))}{ds} &= -K(s(x))p_2(s(x)), \\ \frac{dp_2(s(x))}{ds} &= K(s(x))p_1(s(x)). \end{aligned}$$

Figure 3

Figure 3: Γ and Γ_6

We refer to [4] since it plays a crucial role in this paper. First of all we introduce one of the conventions in this paper. Every convex polygon with n sides (n-polygon for short) has outward normals $N_{n,i} \equiv (\cos(2\pi i/n), \sin(2\pi i/n))$ $(i = 0, \cdots, n-1)$. By the i th side of the n-polygon we denote the side with the outward normal $N_{n,i}$.

Take a smooth simple closed convex curve Γ on $\mathbf{R}^2$. For $n \geq 5$, let Γ_n denote the n-polygon of which the i th side is tangent to Γ (see Figure 3). Let $\{\Gamma_n(t)\}_{0 \leq t < T_n^*}$ be the flow of n-polygons which can be defined as follows, where T_n^* denotes the extinction time of $\Gamma_n(\cdot)$.

$$\Gamma_n(0) = \Gamma_n,$$

and for $t \in [0, T_n^*)$, the inward normal velocity $V_{n,i}(t)$ of the i th side of $\Gamma_n(t)$ is given by the following:

$$V_{n,i}(t) = 2\frac{\tan(\pi/n)}{\ell_{n,i}(t)}, \tag{1.4}$$

where $\ell_{n,i}(t)$ denotes the length of the i th side of $\Gamma_n(t)$ (see Figure 4). It is known that there exists the Gauss curvature flow $\{\Gamma(t)\}_{0 \leq t < T^*}$ on $\mathbf{R}^2$, with $\Gamma(0) = \Gamma$, where T^* denotes the extinction time of $\Gamma(t)$ (see [4]). Let $\Omega_{\ell,n}(t)$ and $\Omega(t) (\subset \mathbf{R}^2)$ be the closed convex sets such that $\partial\Omega_{\ell,n}(t) = \Gamma_n(t)$ and $\partial\Omega(t) = \Gamma(t)$, and such that $\Omega_{\ell,n}(t) \subset \Omega_{\ell,n}(s)$ and $\Omega(t) \subset \Omega(s)$ if $0 \leq s \leq t$.

Then the following holds.

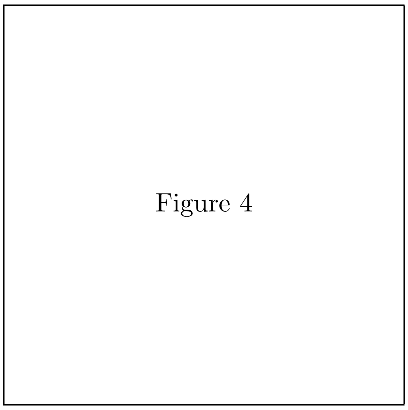

Figure 4: Motion of the i th side of $\Gamma_n(t)$

Theorem 1. *(see [4]). As $t \uparrow T^*$, $\Omega(t)$ converges in Hausdorff metric to a point or a segment.* $\lim_{n\to\infty} T_n^* = T^*$, *and for any $t \in [0, T^*)$,*

$$\lim_{n\to\infty} \sup_{0\le s\le t} d_H(\Omega_{\ell,n}(s), \Omega(s)) = 0. \tag{1.5}$$

Since the wearing process of stones on beaches is random, we would like to construct a stochastic model instead of a deterministic one such as Theorem 1.

In this paper we introduce the flow of random n-polygons with outward normals $N_{n,i}$ $(i = 0, \cdots, n-1)$ and show that it converges in probability to the Gauss curvature flow of smooth simple closed convex curves on $\mathbf{R}^2$ as $n \to \infty$ in Hausdorff metric uniformly in t on every compact subset of $[0, T^*)$ (see Theorem 2 in section 2).

In the proof we approximate the random n-polygon by $\Gamma_n(t)$ at time t and use Theorem 1.

We use the word "Gauss" even for the curvature flow in $\mathbf{R}^2$ since a part of our idea that the volume is cut off from the stone is originally from the deterministic model of the Gauss curvature flow (see [8]).

In section 2 we introduce our random model and state our result which will be proved in section 4. Technical lemmas will be stated and proved in section 3.

2. Main result.

We first introduce our random model.

Figure 5

Figure 5: The isogonal trapezoid with the height $h_n(x)$

Let $\{T(n)\}_{n\geq 1}$ be an increasing sequence of positive real numbers and put

$$\theta_n = \frac{2\pi}{n}. \tag{2.1}$$

For $x > 0$ and $n \geq 1$, put

$$h_n(x) = \frac{\tan\theta_n\{-x + (x^2 + 4(\cot\theta_n)\theta_n/T(n))^{1/2}\}}{2}. \tag{2.2}$$

Remark 3. $h_n(x)$ is the height of the isogonal trapezoid, with the area $\theta_n/T(n)$, of which the lengths of upper and lower sides are x and $x + 2(\cot\theta_n)h_n(x)$ respectively (see Figure 5). In particular,

$$(x + (\cot\theta_n)h_n(x))h_n(x) = \frac{\theta_n}{T(n)}. \tag{2.3}$$

For $n \geq 5$, we consider the Markov process $\{(X_{n,i}(t))_{i=0}^{n-1}\}_{t\geq 0}$ on $\mathbf{R}^n$ such that $(X_{n,i}(0))_{i=0}^{n-1} = (\ell_{n,i}(0))_{i=0}^{n-1}$ (see (1.4)) and of which the generator is given by the following: for a bounded Borel measurable function $f : \mathbf{R}^n \mapsto \mathbf{R}$ and $x = (x_i)_{i=0}^{n-1} \in \mathbf{R}^n$,

$$\begin{aligned} Lf(x) &= \frac{T(n)\tan(\theta_n/2)}{\theta_n/2}\sum_{i=0}^{n-1} I_{\{y \mid \min(y_{i-1},y_{i+1})\sin\theta_n > h_n(y_i)\}}(x) \\ &\quad \times[f(x + 2(\cot\theta_n)h_n(x_i)\mathbf{e}_{n,i} - \frac{h_n(x_i)}{\sin\theta_n}(\mathbf{e}_{n,i-1} + \mathbf{e}_{n,i+1})) - f(x)] \end{aligned} \tag{2.4}$$

(see [2, Chap. 4, section 2]). Here $I_A(x)$ and $\{\mathbf{e}_{n,k}\}_{k=0}^{n-1}$ denote the indicator function of the set A and the standard normal base in $\mathbf{R}^n$ respectively, and we put $\mathbf{e}_{n,n+k} = \mathbf{e}_{n,k}$ and $y_{n+k} = y_k$ $(k = -1, 0)$.

It is easy to see that one can construct the flow of random closed convex sets $\{\Omega_{X,n}(t)\}_{t\geq 0}$ in $\mathbf{R}^2$, surrounded by n-polygons, such that $\Omega_{X,n}(0) = \Omega_{\ell,n}(0)$, and that $\Omega_{X,n}(t) \subset \Omega_{X,n}(s)$ if $s \leq t$, and that the length of the i th side of $\partial\Omega_{X,n}(t)$ is equal to $X_{n,i}(t)$.

We discuss the meaning of our model.

For $n \geq 5$, put

$$\sigma_{n,i} \equiv \begin{cases} 0 & \text{if } i = 0, \\ \inf\{t > \sigma_{n,i-1} | \sum_{k=0}^{n-1} |X_{n,k}(t) - X_{n,k}(t-)| > 0\} & \text{if } i \geq 1, \end{cases}$$

where $X_{n,k}(t-) \equiv \lim_{s\uparrow t} X_{n,k}(s)$, and where we consider the right hand side as infinity if the set over which the infimum is taken is empty. Then

$$P(\sigma_{n,i} < \sigma_{n,i+1} \quad \text{for all } i \text{ for which } \sigma_{n,i} < \infty) = 1.$$

Put

$$A_n \equiv \{j \in \{0, \cdots, n-1\} | \min(X_{n,j-1}(0), X_{n,j+1}(0)) \sin\theta_n > h_n(X_{n,j}(0))\}.$$

If the set A_n is not empty, then $\sigma_{n,1}$ is exponentially distributed with parameter $[\#A_n \cdot T(n)\tan(\theta_n/2)]/(\theta_n/2)$ (see [2, p. 163]), where we put $j = n+j$ for $j = -1$ and 0, and where $\#A_n$ denotes the cardinal number of the set A_n. For any $k \in A_n$, the probability that the isogonal trapezoid with the area $\theta_n/T(n)$ is cut off from $\Omega_{X,n}(0)$ in the direction $-N_{n,k} = (-\cos(2\pi k/n), -\sin(2\pi k/n))$ at time $t = \sigma_{n,1}$ is equal to $(\#A_n)^{-1}$ (see Figure 6).

If the set A_n is empty, then $\sigma_{n,1} = \infty$ and $X_{n,k}(0) = X_{n,k}(t)$ for all $k = 0, \cdots, n-1$ and all $t \geq 0$ a.s..

The following also holds a.s.: $\{\Omega_{X,n}(t)\}$ continues to change the shape in a similar manner to above at times $t = \sigma_{n,i}$ which is finite; $\sigma_{n,i}$ is infinite if i is greater than (the area of $\Omega_{X,n}(0))/(T(n)^{-1}\theta_n)$; $\Omega_{X,n}(t)$ is an n-polygon for all $t \geq 0$.

The following is our main result.

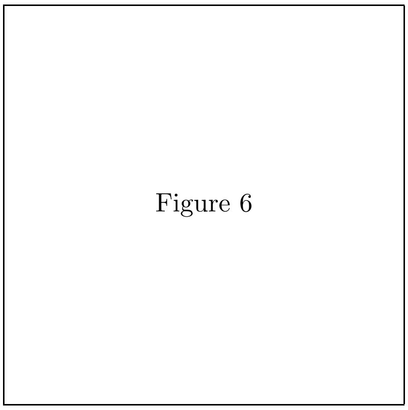

Figure 6: The change of the k th side of $\Omega_{X,n}(0)$

Theorem 2. *Suppose that* Γ *is a smooth simple closed convex curve on* $\mathbf{R}^2$ *and that the following holds:*

$$\lim_{n\to\infty} T(n)n^{-5} = \infty. \tag{2.5}$$

Then for any $t \in [0, T^*)$ *and any* $\eta > 0$*,*

$$\lim_{n\to\infty} P(\sup_{0\le s\le t} d_H(\Omega_{X,n}(s), \Omega(s)) < \eta) = 1. \tag{2.6}$$

Remark 4. (2.5) implies that $\theta_n/T(n) \sim o(n^{-6})$ (as $n \to \infty$), where $\theta_n/T(n)$ is the area of the isogonal trapezoid which is cut off from an n-polygon in our model.

Consider a convex stone which rotates randomly on a beach where waves are even. Our result suggests that the time evolution of the surface of such a stone can be considered as Gauss curvature flow.

3. Lemmas.

In this section we state and prove lemmas which will be used in the next section.

For $n \ge 1$ and $i = 0, \cdots, n-1$, put

$$D_{n,i}(t) = \sum_{0<s\le t} h_n(X_{n,i}(s-))I_{(X_{n,i}(s-),\infty)}(X_{n,i}(s)) \quad (t\ge 0), \tag{3.1}$$

$$d_{n,i}(t) = \int_0^t \frac{2\tan(\theta_n/2)}{\ell_{n,i}(s)}ds \quad (0\le t<T_n^*). \tag{3.2}$$

Remark 5. $D_{n,i}(t)$ is the distance between the straight line which includes the i th side of $\Omega_{X,n}(t)$ and that which includes the i th side of $\Omega_{X,n}(0)$. $d_{n,i}(t)$ is also the distance between the straight line which includes the i th side of $\Omega_{\ell,n}(t)$ and that which includes the i th side of $\Omega_{\ell,n}(0)$.

Put the intersection point of the *0* th and the first sides of $\Omega_{\ell,n}(0)$ at the origin. Then the coordinate of the intersection point of the i th and the *(i+1)* th sides of $\Omega_{X,n}(t)$ and $\Omega_{\ell,n}(t)$ can be written as follows, respectively: for $t\ge 0$,

$$Y_{n,0}(t) = (-D_{n,0}(t), D_{n,0}(t)\cot\theta_n - D_{n,1}(t)/\sin\theta_n) \quad \text{if } i=0, \tag{3.3}$$

$$Y_{n,i}(t) = Y_{n,0}(t) + \sum_{k=1}^{i} X_{n,k}(t)(-\sin(k\theta_n),\cos(k\theta_n)) \quad \text{if } i=1,\cdots,n-1, \tag{3.4}$$

and for $t\in[0,T_n^*)$

$$y_{n,0}(t) = (-d_{n,0}(t), d_{n,0}(t)\cot\theta_n - d_{n,1}(t)/\sin\theta_n) \quad \text{if } i=0, \tag{3.5}$$

$$y_{n,i}(t) = y_{n,0}(t) + \sum_{k=1}^{i} \ell_{n,k}(t)(-\sin(k\theta_n),\cos(k\theta_n)) \quad \text{if } i=1,\cdots,n-1. \tag{3.6}$$

Remark 6. $X_{n,i}(t) = |Y_{n,i}(t) - Y_{n,i-1}(t)|$ for $t\ge 0$ and $\ell_{n,i}(t) = |y_{n,i}(t) - y_{n,i-1}(t)|$ for $t\in[0,T_n^*)$, where we put $(Y_{n,i}(t), y_{n,i}(t)) = (Y_{n,n+i}(t), y_{n,n+i}(t))$ for $i=-1, 0$.

The time evolution of $\{y_{n,i}(t)\}_{0\le t<T_n^*}$ ($n\ge 5$, $i=0,\cdots,n-1$) can be given by the following.

Lemma 1. *For $n\ge 5$, $i=0,\cdots,n-1$, and $s\in(0,T_n^*)$,*

$$\frac{dy_{n,i}(s)}{ds} = \frac{(\sin(i\theta_n), -\cos(i\theta_n))}{\ell_{n,i+1}(s)\cos^2(\theta_n/2)} - \frac{(\sin((i+1)\theta_n), -\cos((i+1)\theta_n))}{\ell_{n,i}(s)\cos^2(\theta_n/2)}, \tag{3.7}$$

where we put $\ell_{n,n}(s) = \ell_{n,0}(s)$.

Proof. It is known that $\{\ell_{n,i}(t)\}_{i=0}^{n-1}$ satisfies the following (see [4]):

$$\frac{d\ell_{n,i}(t)}{dt} = \left(\frac{2\cos\theta_n}{\ell_{n,i}(t)} - \frac{1}{\ell_{n,i+1}(t)} - \frac{1}{\ell_{n,i-1}(t)}\right)\frac{1}{\cos^2(\theta_n/2)}. \tag{3.8}$$

where we put $\ell_{n,n+k}(t) = \ell_{n,k}(t)$ $(k = -1, 0)$.

(3.7) can be proved inductively in i, by (3.2), (3.5)-(3.6) and by the following:

$$\sin((i-1)\theta_n) + \sin((i+1)\theta_n) = 2\cos\theta_n \sin(i\theta_n), \tag{3.9}$$

$$\cos((i-1)\theta_n) + \cos((i+1)\theta_n) = 2\cos\theta_n \cos(i\theta_n). \tag{3.10}$$

Before we state and prove the following lemma, we give some notation. Put for $\delta \in (0, T_n^*)$,

$$C_n(\delta) = n\min\{\ell_{n,k}(s) | 0 \le k \le n-1, 0 \le s \le T_n^* - \delta\}, \tag{3.11}$$

$$\tau_{n,\delta} = \inf\{t > 0 | C_n(\delta)/(2n) \ge \min\{X_{n,k}(t); 0 \le k \le n-1\}\}. \tag{3.12}$$

For any $f \in C_o^2(\mathbf{R}^{2n}; \mathbf{R})$ and $y = (y_i)_{i=0}^{2n-1} \in \mathbf{R}^{2n}$, put

$$\begin{aligned}\tilde{L}f(y) = \frac{T(n)\tan(\theta_n/2)}{\theta_n/2}\sum_{i=0}^{n-1}\{f(y + \frac{h_n(\{|y_{2i} - y_{2(i-1)}|^2 + |y_{2i+1} - y_{2i-1}|^2\}^{1/2})}{\sin\theta_n} \\ \times([\sin((i-1)\theta_n)]\mathbf{e}_{2n,2(i-1)} - [\cos((i-1)\theta_n)]\mathbf{e}_{2n,2i-1} \\ -[\sin((i+1)\theta_n)]\mathbf{e}_{2n,2i} + [\cos((i+1)\theta_n)]\mathbf{e}_{2n,2i+1})) - f(y)\}\end{aligned}$$

(see (2.4) for the convention of the notation).

Remark 7. For $(y_{2i}, y_{2i+1}) \in \mathbf{R}^2$ $(i = 0, \cdots, n-1)$, put

$$x_i \equiv \{(y_{2i} - y_{2(i-1)})^2 + (y_{2i+1} - y_{2i-1})^2\}^{1/2},$$

where we put $(y_{2i}, y_{2i+1}) = (y_{2(n+i)}, y_{2(n+i)+1})$ for $i = -1, 0$. If

$$\min(x_{i+1}, x_{i-1})\sin\theta_n > h_n(x_i),$$

$$(y_{2i}, y_{2i+1}) - (y_{2(i-1)}, y_{2i-1}) = x_i(-\sin(i\theta_n), \cos(i\theta_n))$$

for all $i = 0, \cdots, n-1$, then for any $g \in C_o^2(\mathbf{R}^n; \mathbf{R})$,

$$\tilde{L}g((\{(y_{2i} - y_{2(i-1)})^2 + (y_{2i+1} - y_{2i-1})^2\}^{1/2})_{i=0}^{n-1}) = Lg((x_i)_{i=0}^{n-1}).$$

Put also $\mathbf{Y}_n(t) = (Y_{n,k}(t))_{k=0}^{n-1}$. Then the time evolution of $\{\mathbf{Y}_n(t)\}_{0 \le t}$ for sufficiently large n can be given by the following.

Lemma 2. *Suppose that (2.5) holds. Then for any $\delta \in (0, T^*)$, there exists $n_1 \in \mathbf{N}$ such that for any $n \ge n_1$ and any $f \in C_o^2(\mathbf{R}^{2n}; \mathbf{R})$, δ is less than T_n^* and*

$$f(\mathbf{Y}_n(\min(t, \tau_{n,\delta}))) = f(\mathbf{Y}_n(0)) + \int_0^{\min(t, \tau_{n,\delta})} \tilde{L}f(\mathbf{Y}_n(s))ds + M^{[f(\mathbf{Y}_n)]}(\min(t, \tau_{n,\delta}))$$

for $t \ge 0$, P-a.s., where $M^{[f(\mathbf{Y}_n)]}(t)$ denotes a purely discontinuous martingale part of $f(\mathbf{Y}_n(t))$.

Proof. Take $n_0 \in \mathbf{N}$ such that $\delta < T_n^*$ for any $n \ge n_0$, which is possible from Theorem 1. First we show that there exists $n_1 \ge n_0$ such that for any $n \ge n_1$ and $k = 0, \cdots, n-1$

$$\min(X_{n,k-1}(t), X_{n,k+1}(t)) \sin\theta_n > h_n(X_{n,k}(t)) \quad \text{for } t \in [0, \tau_{n,\delta}) \text{ P-a.s.,} \tag{3.13}$$

where we put $X_{n,i}(t) = X_{n,n+i}(t)$ for $i = -1$, 0.

By (14) of [4], $\{C_k(\delta)^{-1}\}_{k=n_0}^{\infty}$ defined in (3.11) is bounded. Therefore there exists $n_1 \ge n_0$ such that for any $n \ge n_1$

$$\frac{C_n(\delta)}{2n} \sin\theta_n > \frac{2n}{C_n(\delta)} \frac{\theta_n}{T(n)}$$

by (2.5). Hence, for k, $i = 0, \cdots, n-1$ and $t \in [0, \tau_{n,\delta})$

$$X_{n,k}(t) \sin\theta_n > \frac{C_n(\delta)}{2n} \sin\theta_n > \frac{2n}{C_n(\delta)} \frac{\theta_n}{T(n)} > \frac{1}{X_{n,i}(t)} \frac{\theta_n}{T(n)} > h_n(X_{n,i}(t)) \quad \text{a.s.}$$

by (2.3), which implies (3.13).

By (3.3)-(3.4), we have the following:

$$\mathbf{Y}_n(s) = -x_1 \sum_{k=0}^{n-1} \mathbf{e}_{2n,2k} + (x_1 \cot\theta_n - x_3/\sin\theta_n) \sum_{k=0}^{n-1} \mathbf{e}_{2n,2k+1} + \sum_{i=1}^{n-1} x_{2i}[-(\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2n,2k} + (\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2n,2k+1}],$$

with $x_{2i} = X_{n,i}(s)$ and $x_{2i+1} = D_{n,i}(s)$ $(i = 0, \cdots, n-1)$. Therefore, from (2.4), (3.1), (3.9)-(3.10) and (3.13), by the Itô formula (see [9]), the proof is over (see Remark 5).

The following lemma plays a crucial role when we approximate $\Omega_{\ell,n}$ by $\Omega_{X,n}$.

Lemma 3. *Suppose that (2.5) holds. Then for any* $\delta \in (0, T^*)$,

$$\lim_{n\to\infty} n^2 E[\sup_{0\le t\le \min(T_n^*-\delta,\tau_{n,\delta})} \sum_{i=0}^{n-1} |Y_{n,i}(t) - y_{n,i}(t)|^2] = 0. \tag{3.14}$$

Proof. For $n_1 \in \mathbf{N}$ in Lemma 2, there exists a positive constant C such that the following which will be proved later holds: for any $n \ge n_1$ and $t \in [0, T_n^* - \delta]$,

$$\begin{aligned} & E[\sup_{0\le s\le \min(t,\tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2] \qquad (3.15) \\ \le\ & CT(n)^{-1} n^3 + C \int_0^t E[\sup_{0\le u\le \min(s,\tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2] ds. \end{aligned}$$

This implies (3.14), by Gronwall's inequality, from (2.5).

We prove (3.15) to complete the proof. For any $n \ge n_1$, by Lemmas 1 and 2, the following holds: for $t \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$
\begin{aligned}
&\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2 \\
= \ & \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), \\
&\qquad \left(\frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) (\sin(k\theta_n), -\cos(k\theta_n)) \\
&\qquad + \left(\frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) (-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > ds \\
&\quad + \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(t) - Y_{n,k}(s), \\
&\qquad \left[-\frac{T(n)}{\theta_n} h_n(X_{n,k+1}(s)) + \frac{1}{X_{n,k+1}(s)} \right] (\sin(k\theta_n), -\cos(k\theta_n)) \\
&\qquad + \left[\frac{T(n)}{\theta_n} h_n(X_{n,k}(s)) - \frac{1}{X_{n,k}(s)} \right] (\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > ds \\
&\quad + 2 \frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \sum_{k=0}^{n-1} \int_0^t \left| \frac{h_n(X_{n,k}(s))}{\sin\theta_n} \right|^2 ds + M(t),
\end{aligned}
$$

where $M(t)$ denotes a purely discontinuous martingale part of $\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2$.

Since $\{C_k^{-1}(\delta)\}_{k \geq n_1}$ is bounded by (14) of [4], we only have to show the following (3.16)-(3.19) to complete the proof: for $t \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$
\begin{aligned}
& \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), \qquad (3.16) \\
&\qquad \left(\frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) (\sin(k\theta_n), -\cos(k\theta_n)) \\
&\qquad + \left(\frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) (-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > ds \\
\leq \ & \frac{4n^2 \sin^2\theta_n}{C_n(\delta)^2 \cos\theta_n \cos^2(\theta_n/2)} \int_0^t \sup_{0 \leq u \leq \min(s, \tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds,
\end{aligned}
$$

$$\frac{2}{\cos^2(\theta_n/2)}\sum_{k=0}^{n-1}\int_0^t < y_{n,k}(t)-Y_{n,k}(s), \tag{3.17}$$
$$\left[-\frac{T(n)}{\theta_n}h_n(X_{n,k+1}(s))+\frac{1}{X_{n,k+1}(s)}\right](\sin(k\theta_n),-\cos(k\theta_n))$$
$$+\left[\frac{T(n)}{\theta_n}h_n(X_{n,k}(s))-\frac{1}{X_{n,k}(s)}\right](\sin((k+1)\theta_n),-\cos((k+1)\theta_n)) > ds$$
$$\le\ 2\int_0^t \sup_{0\le u\le \min(s,\tau_{n,\delta})}\sum_{k=0}^{n-1}|y_{n,k}(u)-Y_{n,k}(u)|^2ds+2nt\left|\frac{8n^3\theta_n}{T(n)C_n(\delta)^3\sin\theta_n}\right|^2,$$

$$2\frac{T(n)\tan(\theta_n/2)}{\theta_n/2}\sum_{k=0}^{n-1}\int_0^t\left|\frac{h_n(X_{n,k}(s))}{\sin\theta_n}\right|^2ds\le\frac{8n^3t\theta_n}{T(n)C_n(\delta)^2\cos^2(\theta_n/2)\sin\theta_n}, \tag{3.18}$$

and for $t\in[0,T_n^*-\delta]$,

$$\{E[\sup_{0\le s\le\min(t,\tau_{n,\delta})}|M(s)|^2]\}^{1/2} \tag{3.19}$$
$$\le\ \frac{32n^2\theta_n}{T(n)C_n(\delta)^2\sin\theta_n\cos^2(\theta_n/2)}+\frac{6n^3t\theta_n^2}{(T(n)C_n(\delta)\sin\theta_n)^2}$$
$$+3\int_0^t E[\sup_{0\le u\le\min(s,\tau_{n,\delta})}\sum_{k=0}^{n-1}|y_{n,k}(u)-Y_{n,k}(u)|^2]ds.$$

We first prove (3.16). By (3.4) and (3.6), for $k=0,\cdots,n-1$ and $s\in[0,T_n^*)$,

$$y_{n,k+1}(s)-Y_{n,k+1}(s) \tag{3.20}$$
$$=\ y_{n,k}(s)-Y_{n,k}(s)+(\ell_{n,k+1}(s)-X_{n,k+1}(s))(-\sin((k+1)\theta_n),\cos((k+1)\theta_n)).$$

Hence for $s\in[0,T_n^*)$,

$$\sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), \left(\frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)}\right)(\sin(k\theta_n), -\cos(k\theta_n))$$
$$+ \left(\frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)}\right)(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) >$$
$$= \sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) >$$
$$\times \left(\frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)}\right)$$
$$+ \sum_{k=0}^{n-1} (\cos\theta_n)(\ell_{n,k}(s) - X_{n,k}(s)) \left(\frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)}\right).$$

This together with (3.11)-(3.12) and the following implies (3.16) : for $s \in [0, T_n^*)$,

$$< y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) >$$
$$\times \left(\frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)}\right)$$
$$\leq \frac{|y_{n,k}(s) - Y_{n,k}(s)|^2 \sin^2\theta_n}{\ell_{n,k+1}(s) X_{n,k+1}(s)\cos\theta_n} + \frac{(\ell_{n,k+1}(s) - X_{n,k+1}(s))^2 \cos\theta_n}{\ell_{n,k+1}(s) X_{n,k+1}(s)}.$$

(3.17) can be proved by (3.12) and by the following: for $x > 0$,

$$\frac{1}{x} - \frac{T(n)}{\theta_n} h_n(x) = \frac{\theta_n \cot\theta_n}{T(n)x^3} \left(\frac{2}{1 + (1 + 4x^{-2}T(n)^{-1}\theta_n \cot\theta_n)^{1/2}}\right)^2,$$

since $\cos\theta_n < \cos^2(\theta_n/2)$.

(3.18) is true, since $h_n(x) < \theta_n (T(n)x)^{-1}$ by (2.3).

Finally we prove (3.19). For $t \in [0, T_n^* - \delta]$,

$$E[\sup_{0 \leq s \leq \min(t,\tau_{n,\delta})} |M(s)|^2]$$
$$\leq 4E[|M(\min(t,\tau_{n,\delta}))|^2] \quad \text{(by Doob's inequality)}$$
$$= 4\frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \sum_{i=0}^{n-1} E[\int_0^{\min(t,\tau_{n,\delta})} (-2\frac{h_n(X_{n,i}(s))}{\sin\theta_n}$$
$$\times[< y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) >$$
$$+ < y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) >]$$
$$+2\left|\frac{h_n(X_{n,i}(s))}{\sin\theta_n}\right|^2)^2 ds].$$

For $s \in [0, \min(t, \tau_{n,\delta}))$ and $i = 0, \cdots, n-1$, by (3.12),

$$
\begin{aligned}
&(-2\frac{h_n(X_{n,i}(s))}{\sin\theta_n}[< y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) > \\
&\quad + < y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) >] + 2\left|\frac{h_n(X_{n,i}(s))}{\sin\theta_n}\right|^2)^2 \\
\leq\ & 4\left|\frac{2n\theta_n}{T(n)C_n(\delta)\sin\theta_n}\right|^2 (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \\
&\quad + \left|\frac{2n\theta_n}{T(n)C_n(\delta)\sin\theta_n}\right|)^2
\end{aligned}
$$

since $h_n(x) < \theta_n(T(n)x)^{-1}$ by (2.3).

Use the inequality $(xy)^{1/2} \leq (x+y)/2$ (x, $y > 0$) for

$$
x = 4\frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \times 4\left|\frac{2n\theta_n}{T(n)C_n(\delta)\sin\theta_n}\right|^2,
$$

$$
y = \sum_{i=0}^{n-1} E[\int_0^{\min(t,\tau_{n,\delta})} (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| + \left|\frac{2n\theta_n}{T(n)C_n(\delta)\sin\theta_n}\right|)^2 ds].
$$

Use also the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ for $x = |y_{n,i-1}(s) - Y_{n,i-1}(s)|$, $y = |y_{n,i}(s) - Y_{n,i}(s)|$ and $z = |(2n\theta_n)/(T(n)C_n(\delta)\sin\theta_n)|$. Then we obtain (3.19).

4. Proof of Main Result.

In this section we prove Theorem 2 by making use of lemmas given in section 3.

Proof of Theorem 2. For any $t \in (0, T^*)$ and any $\eta > 0$, take $n_2 \in \mathbf{N}$ such that for any $n \geq n_2$

$$
\begin{aligned}
t &< T_n^* - (T^* - t)/2, \\
\sup_{0 \leq s \leq t} d_H(\Omega_{\ell,n}(s), \Omega(s)) &< \eta/2,
\end{aligned}
$$

which is possible Theorem 1. Put $\delta = (T^* - t)/2$. Then for any $n \geq n_2$,

$$\begin{aligned}
& P(\sup_{0\le s\le t} d_H(\Omega_{X,n}(s),\Omega(s)) \ge \eta) \qquad (4.1)\\
\le\ & P(\sup_{0\le s\le t} d_H(\Omega_{X,n}(s),\Omega_{\ell,n}(s)) \ge \eta/2)\\
\le\ & P(\sup_{0\le s\le T_n^*-\delta} d_H(\Omega_{X,n}(s),\Omega_{\ell,n}(s)) \ge \eta/2)\\
\le\ & P(\tau_{n,\delta} < T_n^* - \delta) + P(\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} d_H(\Omega_{X,n}(s),\Omega_{\ell,n}(s)) \ge \eta/2).
\end{aligned}$$

The first probability on the last part of (4.1) can be shown to converge to zero as $n\to\infty$ as follows: by Chebychev's inequality,

$$\begin{aligned}
& P(\tau_{n,\delta} < T_n^* - \delta)\\
\le\ & P(\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} \max_{k=0}^{n-1} |X_{n,k}(s)-\ell_{n,k}(s)| \ge C_n(\delta)/(2n))\\
\le\ & P(\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s)-y_{n,k}(s)| \ge C_n(\delta)/(4n))\\
\le\ & (4nC_n(\delta)^{-1})^2 E[\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s)-y_{n,k}(s)|^2]\\
\to\ & 0,\ \text{as } n\to\infty \text{ by Lemma 3,}
\end{aligned}$$

since

$$|X_{n,k}(s)-\ell_{n,k}(s)| \le |Y_{n,k}(s)-y_{n,k}(s)| + |Y_{n,k-1}(s)-y_{n,k-1}(s)|$$

by (3.20) and since $\limsup_{k\to\infty} C_k(\delta)^{-1}$ is finite by (14) of [4].

The second probability on the last part of (4.1) can be shown to converge to zero as $n\to\infty$ as follows: by Chebychev's inequality,

$$\begin{aligned}
& P(\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} d_H(\Omega_{X,n}(s),\Omega_{\ell,n}(s)) \ge \eta/2)\\
\le\ & P(\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s)-y_{n,k}(s)| \ge \eta/2)\\
\le\ & (\eta/2)^{-2} E[\sup_{0\le s\le \min(T_n^*-\delta,\tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s)-y_{n,k}(s)|^2]\\
\to\ & 0,\ \text{as } n\to\infty \text{ by Lemma 3,}
\end{aligned}$$

since

$$d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \le \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|.$$

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References

[1] Chow, B. (1985). Deforming convex hypersurfaces by the nth root of the Gaussian curvature. *J. Diff. Geometry* **22,** 117–138.

[2] Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes: characterization and convergence*, Jone Wiley & Sons Inc., New York.

[3] Firey, W. J. (1974). Shapes of worn stones. *Mathematika* **21,** 1–11.

[4] Girão, P. M. (1995). Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature. *SIAM J. Numer. Anal.* **32,** 886–899.

[5] Girão, P. M. and Kohn, R. V. (1994). Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature. *Numer. Math.* **67,** 41–70.

[6] Huisken, G. (1984). Flow by mean curvature of convex surfaces into spheres. *J. Diff. Geometry* **20,** 237–266.

[7] Ikeda, N. and Watanabe, S. (1981). *Stochastic differential equations and diffusion processes*, North-Holland/Kodansha, Tokyo.

[8] Ishii, H. (1999). An approximation scheme for Gauss curvature flow and its convergence. preprint.

[9] Ishii, K. and Soner, H. M. (1998). Regularity and convergence of crystalline motion. *SIAM J. Math. Anal.* **30,** 19–37.

[10] Meyer, P. A. (1976). Un cours sur les intégrale stochastiques. In *Séminaire de Prob.* **X**, Proceedings, Strasbourg 1974/75, Lecture Notes in Math. Vol. 511, ed. P. A. Meyer. Springer-Verlag, Berlin, pp. 245–400.

[11] Tso, K. (1985). Deforming a hypersurface by its Gauss-Kronecker curvature. *Comm. Pure Appl. Math.* **38,** 867–882.