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<thead>
<tr>
<th>Title</th>
<th>A two-dimensional random crystalline algorithm for Gauss curvature flow</th>
</tr>
</thead>
<tbody>
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<td>File Information</td>
<td>AAP34-3.pdf</td>
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A TWO DIMENSIONAL RANDOM CRYSTALLINE ALGORITHM
FOR GAUSS CURVATURE FLOW

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Abstract

We propose and study a random crystalline algorithm (a discrete approximation) of the Gauss curvature flow of smooth simple closed convex curves in \( \mathbb{R}^2 \) as a stepping stone to the full understanding of such a phenomenon as the wearing process of stones on beaches.

Keywords: random crystalline algorithm; Gauss curvature flow; closed curve

AMS 2000 Subject Classification: Primary 60D05
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1. Introduction.

The Gauss curvature flow of closed surfaces in \( \mathbb{R}^3 \) is a mathematical model of the wearing process of stones on beaches (see [3] and also [1], [6] and [11]).

We introduce the definition of the Gauss curvature flow of smooth closed convex hypersurfaces in \( \mathbb{R}^{d+1} \). Let \( \Gamma \) be a smooth closed convex hypersurface in \( \mathbb{R}^{d+1} \) and \( F : S^d \mapsto \mathbb{R}^{d+1} \) be a parametric representation of \( \Gamma \). Then a collection of \( F(\cdot, t) : S^d \mapsto \mathbb{R}^{d+1} \) of smooth closed convex hypersurfaces with parameter \( t \in [0, T) \) for some \( T > 0 \) is called Gauss curvature flow with initial state \( \Gamma \) if the following holds:

\[
\frac{\partial F(s, t)}{\partial t} = -K(s, t)n(s, t) \quad (s \in S^d, 0 < t < T),
\]

\[
F(s, 0) = F(s) \quad (s \in S^d),
\]

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where $K(s,t)$ and $n(s,t)$ denote the Gauss curvature and the unit outward normal vector, respectively, at a point $F(s,t)$ on the hypersurface \( \{ F(s',t)|s' \in S^d \} \). In this paper we assume that the convex set with boundary \( \{ F(s,t)|s \in S^d \} \) is non-increasing in $t$ (see Figure 1).

Suppose that $\Gamma$ is strictly convex. Then there exists the maximum $T^*$ of $T$ for which (1.1)-(1.2) has a unique smooth strictly convex solution and \( \{ F(s,t)|s \in S^d \} \) converges to a point as $t \to T^*$ (see [1], [6] and [11]).

In [8], H. Ishii proposed a discrete time approximation scheme for the Gauss curvature flow. We briefly introduce it. Suppose that we are given the strictly convex set $D$ with smooth boundary $\partial D$ in $\mathbb{R}^{d+1}$ at time $t = 0$. Take $h > 0$ and a function $V: [0, \infty) \to [0, \infty)$. For every $s \in S^d$, let $D_{s,h}$ denote the set which can be obtained by cutting off the volume $V(h)$ from the set $D$ in the direction $-s$ (see Figure 2). Put $D_{0,h} \equiv D$ and $D_{1,h} \equiv \cap_{s \in S^d} D_{s,h}$. Define $D_{n,h}$ inductively in $n$ until $n_h \equiv \max\{k \geq 1\}$ the volume of $D_{k,h}$ is greater than $V(h) + 1$. Let $V(h) \to 0$ as $h \to 0$ in an appropriate rate. Then \( \lim_{h \to 0} n_h h = T_{\max} \), and the flow of $\partial D_{[t/h],h}$ (0 \( \leq t \leq n_h h \)) converges to the Gauss curvature flow in Hausdorff metric uniformly in $t$ on every compact subset of $[0,T^*)$, where $[t/h]$ denotes the integer part of $[t/h]$. Notice that the time variable $t$ is discretized but the space variable $s$ is not in this approximation scheme.

**Remark 1.** Hausdorff metric of compact sets $A$ and $B \in \mathbb{R}^d$ is given by the following:
A crystalline (or a polyhedral) approximation of the curvature flow of convex curves was studied by P. M. Girão and is useful in numerical analysis (see Theorem 1 given below, [4] and also [5] and the references therein). In [4], the space variable \( s \) is discretized but the time variable \( t \) is not. In case when the initial curve is not convex, the results of [4] have been generalized by K. Ishii and M. H. Soner (see [9] and the references therein for further information on this problem). The results of [4] have not been generalized to a class of closed convex hypersurfaces in \( \mathbb{R}^{d+1} \) for \( d \geq 2 \). This is a well-known open problem.

**Remark 2.** Let \( \Gamma \) be a smooth simple closed convex curve on \( \mathbb{R}^2 \). Fix a point \( x_0 \) on \( \Gamma \). For any \( x \in \Gamma \), let \( s(x) \) be the length of the curve which connects \( x_0 \) and \( x \) on \( \Gamma \) clockwise. Then one can parametrize \( x \in \Gamma \) by \( s(x) \). Let \( p_1(s(x)) \) and \( p_2(s(x)) \) denote, respectively, the clockwise unit tangent vector and the unit outward normal vector at \( x \) on \( \Gamma \). Then the Gauss curvature \( K(s(x))(\in \mathbb{R}) \) at \( x \) on \( \Gamma \) satisfies the following:

\[
\frac{dp_1(s(x))}{ds} = -K(s(x))p_2(s(x)),
\]
\[
\frac{dp_2(s(x))}{ds} = K(s(x))p_1(s(x)).
\]
We refer to [4] since it plays a crucial role in this paper. First of all we introduce one of the conventions in this paper. Every convex polygon with \( n \) sides \((n\text{-polygon for short})\) has outward normals \( N_{n,i} \equiv (\cos(2\pi i/n), \sin(2\pi i/n)) \) \((i = 0, \ldots, n - 1)\). By the \( i \) th side of the \( n \)-polygon we denote the side with the outward normal \( N_{n,i} \).

Take a smooth simple closed convex curve \( \Gamma \) on \( \mathbb{R}^2 \). For \( n \geq 5 \), let \( \Gamma_n \) denote the \( n \)-polygon of which the \( i \) th side is tangent to \( \Gamma \) (see Figure 3). Let \( \{\Gamma_n(t)\}_{0 \leq t < T_n^*} \) be the flow of \( n \)-polygons which can be defined as follows, where \( T_n^* \) denotes the extinction time of \( \Gamma_n(\cdot) \).

\[
\Gamma_n(0) = \Gamma_n,
\]

and for \( t \in [0, T_n^*) \), the inward normal velocity \( V_{n,i}(t) \) of the \( i \) th side of \( \Gamma_n(t) \) is given by the following:

\[
V_{n,i}(t) = 2 \frac{\tan(\pi/n)}{\ell_{n,i}(t)},
\]

where \( \ell_{n,i}(t) \) denotes the length of the \( i \) th side of \( \Gamma_n(t) \) (see Figure 4). It is known that there exists the Gauss curvature flow \( \{\Gamma(t)\}_{0 \leq t < T} \) on \( \mathbb{R}^2 \), with \( \Gamma(0) = \Gamma \), where \( T^* \) denotes the extinction time of \( \Gamma(t) \) (see [4]). Let \( \Omega_{\ell,n}(t) \) and \( \Omega(t) \subset \mathbb{R}^2 \) be the closed convex sets such that \( \partial \Omega_{\ell,n}(t) = \Gamma_n(t) \) and \( \partial \Omega(t) = \Gamma(t) \), and such that \( \Omega_{\ell,n}(t) \subset \Omega_{\ell,n}(s) \) and \( \Omega(t) \subset \Omega(s) \) if \( 0 \leq s \leq t \).

Then the following holds.
Figure 4: Motion of the i th side of $\Gamma_n(t)$

**Theorem 1.** (see [4]). As $t \uparrow T^*$, $\Omega(t)$ converges in Hausdorff metric to a point or a segment, $\lim_{n \to \infty} T^*_n = T^*$, and for any $t \in [0, T^*)$,

$$\lim_{n \to \infty} \sup_{0 \leq s \leq t} d_H(\Omega_{t,n}(s), \Omega(s)) = 0. \quad (1.5)$$

Since the wearing process of stones on beaches is random, we would like to construct a stochastic model instead of a deterministic one such as Theorem 1.

In this paper we introduce the flow of random $n$-polygons with outward normals $N_{n,i}$ ($i = 0, \cdots, n - 1$) and show that it converges in probability to the Gauss curvature flow of smooth simple closed convex curves on $\mathbb{R}^2$ as $n \to \infty$ in Hausdorff metric uniformly in $t$ on every compact subset of $[0, T^*)$ (see Theorem 2 in section 2).

In the proof we approximate the random $n$-polygon by $\Gamma_n(t)$ at time $t$ and use Theorem 1.

We use the word “Gauss” even for the curvature flow in $\mathbb{R}^2$ since a part of our idea that the volume is cut off from the stone is originally from the deterministic model of the Gauss curvature flow (see [8]).

In section 2 we introduce our random model and state our result which will be proved in section 4. Technical lemmas will be stated and proved in section 3.

## 2. Main result.

We first introduce our random model.
Figure 5: The isogonal trapezoid with the height \( h_n(x) \)

Let \( \{T(n)\}_{n \geq 1} \) be an increasing sequence of positive real numbers and put

\[
\theta_n = \frac{2\pi}{n}.
\]

(2.1)

For \( x > 0 \) and \( n \geq 1 \), put

\[
h_n(x) = \frac{\tan \theta_n \{-x + (x^2 + 4(\cot \theta_n)\theta_n/T(n))^{1/2}\}}{2}.
\]

(2.2)

**Remark 3.** \( h_n(x) \) is the height of the isogonal trapezoid, with the area \( \theta_n/T(n) \), of which the lengths of upper and lower sides are \( x \) and \( x + 2(\cot \theta_n)h_n(x) \) respectively (see Figure 5). In particular,

\[
(x + (\cot \theta_n)h_n(x))h_n(x) = \frac{\theta_n}{T(n)}.
\]

(2.3)

For \( n \geq 5 \), we consider the Markov process \( \{X_{n,i}(t)\}_{i=0}^{n-1} \) on \( \mathbb{R}^n \) such that

\[
(X_{n,i}(0))_{i=0}^{n-1} = (\ell_{n,i}(0))_{i=0}^{n-1} \quad \text{(see (1.4))}
\]

and of which the generator is given by the following: for a bounded Borel measurable function \( f : \mathbb{R}^n \mapsto \mathbb{R} \) and \( x = (x_i)_{i=0}^{n-1} \in \mathbb{R}^n \),

\[
L f(x) = \frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \sum_{i=0}^{n-1} I_{\{y|_{\min(y_{i-1},y_{i+1})} \sin \theta_n > h_n(y_i)\}}(x)
\]

\[
\times [f(x + 2(\cot \theta_n)h_n(x)e_{n,i})e_{n,i} = \frac{h_n(x_i)}{\sin \theta_n}(e_{n,i} + e_{n,i+1})] - f(x)
\]

(2.4)
Random crystalline algorithm for curvature flow

(see [2, Chap. 4, section 2]). Here \(I_A(x)\) and \(\{e_{n,k}\}_{k=0}^{n-1}\) denote the indicator function of the set \(A\) and the standard normal base in \(\mathbb{R}^n\) respectively, and we put \(e_{n,n+k} = e_{n,k}\) and \(y_{n+k} = y_k\) (\(k = -1, 0\)).

It is easy to see that one can construct the flow of random closed convex sets \(\{\Omega_{X,n}(t)\}_{t \geq 0}\) in \(\mathbb{R}^2\), surrounded by \(n\)-polygons, such that \(\Omega_{X,n}(0) = \Omega_{\ell,n}(0)\), and that \(\Omega_{X,n}(t) \subset \Omega_{X,n}(s)\) if \(s \leq t\), and that the length of the \(i\)th side of \(\partial \Omega_{X,n}(t)\) is equal to \(X_{n,i}(t)\).

We discuss the meaning of our model.

For \(n \geq 5\), put

\[
\sigma_{n,i} \equiv \left\{ \begin{array}{ll}
0 & \text{if } i = 0, \\
\inf\{t > \sigma_{n,i-1} | \sum_{k=0}^{n-1} |X_{n,k}(t) - X_{n,k}(t^-)| > 0\} & \text{if } i \geq 1,
\end{array} \right.
\]

where \(X_{n,k}(t^-) \equiv \lim_{s \to t^-} X_{n,k}(s)\), and where we consider the right hand side as infinity if the set over which the infimum is taken is empty. Then

\[
P(\sigma_{n,i} < \sigma_{n,i+1} \text{ for all } i \text{ for which } \sigma_{n,i} < \infty) = 1.
\]

Put

\[
A_n \equiv \{j \in \{0, \ldots, n-1\} | \min(X_{n,j-1}(0), X_{n,j+1}(0)) \sin \theta_n > h_n(X_{n,j}(0))\}.
\]

If the set \(A_n\) is not empty, then \(\sigma_{n,1}\) is exponentially distributed with parameter 
\([\#A_n \cdot T(n) \tan(\theta_n/2)]/\theta_n\) (see [2, p. 163]), where we put \(j = n + j\) for \(j = -1\) and \(0\), and where \(#A_n\) denotes the cardinal number of the set \(A_n\). For any \(k \in A_n\), the probability that the isogonal trapezoid with the area \(\theta_n/T(n)\) is cut off from \(\Omega_{X,n}(0)\) in the direction \(-N_{n,k} = \{-\cos(2\pi k/n), -\sin(2\pi k/n)\}\) at time \(t = \sigma_{n,1}\) is equal to \((#A_n)^{-1}\) (see Figure 6).

If the set \(A_n\) is empty, then \(\sigma_{n,1} = \infty\) and \(X_{n,k}(0) = X_{n,k}(t)\) for all \(k = 0, \ldots, n-1\) and all \(t \geq 0\) a.s..

The following also holds a.s.: \(\{\Omega_{X,n}(t)\}\) continues to change the shape in a similar manner to above at times \(t = \sigma_{n,i}\) which is finite; \(\sigma_{n,i}\) is infinite if \(i\) is greater than \((\text{the area of } \Omega_{X,n}(0))/T(n)^{-1}\theta_n\); \(\Omega_{X,n}(t)\) is an \(n\)-polygon for all \(t \geq 0\).

The following is our main result.
Figure 6: The change of the \( k \) th side of \( \Omega_{X,n}(0) \)

**Theorem 2.** Suppose that \( \Gamma \) is a smooth simple closed convex curve on \( \mathbb{R}^2 \) and that the following holds:

\[
\lim_{n \to \infty} T(n)n^{-5} = \infty. \quad (2.5)
\]

Then for any \( t \in [0,T^*) \) and any \( \eta > 0 \),

\[
\lim_{n \to \infty} P\left( \sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega(s)) < \eta \right) = 1. \quad (2.6)
\]

**Remark 4.** (2.5) implies that \( \theta_n/T(n) \sim o(n^{-6}) \) (as \( n \to \infty \)), where \( \theta_n/T(n) \) is the area of the isogonal trapezoid which is cut off from an \( n \)-polygon in our model.

Consider a convex stone which rotates randomly on a beach where waves are even. Our result suggests that the time evolution of the surface of such a stone can be considered as Gauss curvature flow.

### 3. Lemmas.

In this section we state and prove lemmas which will be used in the next section.

For \( n \geq 1 \) and \( i = 0, \cdots, n - 1 \), put
\[ D_{n,i}(t) = \sum_{0<s \leq t} h_n(X_{n,i}(s-))I_{(X_{n,i}(s-),\infty)}(X_{n,i}(s)) \quad (t \geq 0), \quad (3.1) \]
\[ d_{n,i}(t) = \int_{0}^{t} \frac{2\tan(\theta_n/2)}{\ell_{n,i}(s)}ds \quad (0 \leq t < T^*_n). \quad (3.2) \]

**Remark 5.** \( D_{n,i}(t) \) is the distance between the straight line which includes the \( i \) th side of \( \Omega_{X,n}(t) \) and that which includes the \( i \) th side of \( \Omega_{X,n}(0) \). \( d_{n,i}(t) \) is also the distance between the straight line which includes the \( i \) th side of \( \Omega_{\ell,n}(t) \) and that which includes the \( i \) th side of \( \Omega_{\ell,n}(0) \).

Put the intersection point of the \( \theta \) th and the first sides of \( \Omega_{\ell,n}(0) \) at the origin. Then the coordinate of the intersection point of the \( i \) th and the \((i+1)\) th sides of \( \Omega_{X,n}(t) \) and \( \Omega_{\ell,n}(t) \) can be written as follows, respectively: for \( t \geq 0 \),

\[ Y_{n,0}(t) = (-D_{n,0}(t), D_{n,0}(t) \cot \theta_n - D_{n,1}(t)/\sin \theta_n) \quad \text{if } i = 0, \quad (3.3) \]
\[ Y_{n,i}(t) = Y_{n,0}(t) + \sum_{k=1}^{i} X_{n,k}(t)(-\sin(k\theta_n), \cos(k\theta_n)) \quad \text{if } i = 1, \cdots, n-1. \quad (3.4) \]

and for \( t \in [0,T^*_n) \)

\[ y_{n,0}(t) = (-d_{n,0}(t), d_{n,0}(t) \cot \theta_n - d_{n,1}(t)/\sin \theta_n) \quad \text{if } i = 0, \quad (3.5) \]
\[ y_{n,i}(t) = y_{n,0}(t) + \sum_{k=1}^{i} \ell_{n,k}(t)(-\sin(k\theta_n), \cos(k\theta_n)) \quad \text{if } i = 1, \cdots, n-1. \quad (3.6) \]

**Remark 6.** \( X_{n,i}(t) = |Y_{n,i}(t) - Y_{n,i-1}(t)| \) for \( t \geq 0 \) and \( \ell_{n,i}(t) = |y_{n,i}(t) - y_{n,i-1}(t)| \) for \( t \in [0,T^*_n) \), where we put \((Y_{n,i}(t), y_{n,i}(t)) = (Y_{n,n+i}(t), y_{n,n+i}(t)) \) for \( i = -1, 0 \).

The time evolution of \( \{y_{n,i}(t)\}_{0 \leq t < T^*_n} \) \((n \geq 5, i = 0, \cdots, n-1) \) can be given by the following.

**Lemma 1.** For \( n \geq 5, i = 0, \cdots, n-1, \) and \( s \in (0,T^*_n) \),

\[ \frac{dy_{n,i}(s)}{ds} = \frac{(\sin(i\theta_n), -\cos(i\theta_n))}{\ell_{n,i+1}(s)\cos^2(\theta_n/2)} - \frac{(\sin((i+1)\theta_n), -\cos((i+1)\theta_n))}{\ell_{n,i}(s)\cos^2(\theta_n/2)}, \quad (3.7) \]

where we put \( \ell_{n,n}(s) = \ell_{n,0}(s) \).
Proof. It is known that \( \{ \ell_{n,i}(t) \}_{i=0}^{n-1} \) satisfies the following (see [4]):

\[
\frac{d\ell_{n,i}(t)}{dt} = \left( \frac{2 \cos \theta_n}{\ell_{n,i}(t)} - \frac{1}{\ell_{n,i+1}(t)} - \frac{1}{\ell_{n,i-1}(t)} \cos^2(\theta_n/2) \right) \cos(\theta_n/2),
\]

where we put \( \ell_{n,k}(t) = \ell_{n,k}(t) \) \((k = -1, 0)\).

(3.7) can be proved inductively in \( i \), by (3.2), (3.5)-(3.6) and by the following:

\[
\begin{align*}
\sin((i-1)\theta_n) + \sin((i+1)\theta_n) &= 2 \cos \theta_n \sin(i\theta_n), \\
\cos((i-1)\theta_n) + \cos((i+1)\theta_n) &= 2 \cos \theta_n \cos(i\theta_n).
\end{align*}
\]

Before we state and prove the following lemma, we give some notation. Put for \( \delta \in (0,T^*_n) \),

\[
C_n(\delta) = n \min \{ \ell_{n,k}(s) \mid 0 \leq k \leq n-1, 0 \leq s \leq T^*_n - \delta \},
\]

\[
\tau_{n,\delta} = \inf \{ t > 0 \mid C_n(\delta)/(2n) \geq \min \{ X_{n,k}(t) \mid 0 \leq k \leq n-1 \} \}.
\]

For any \( f \in C^2_0(\mathbb{R}^{2n}; \mathbb{R}) \) and \( y = (y_i)_{i=0}^{2n-1} \in \mathbb{R}^{2n} \), put

\[
\hat{L} f(y) = \frac{T(n) \tan(\theta_n/2)}{\theta_n/2} \sum_{i=0}^{n-1} \left( f(y) + \frac{h_n(\{(y_{2i} - y_{2(i-1)})^2 + |y_{2i+1} - y_{2i-1}|^2 \}^{1/2})}{\sin \theta_n} \times (\sin((i-1)\theta_n)\mathbf{e}_{2n,2(i-1)} - \cos((i-1)\theta_n)\mathbf{e}_{2n,2i-1}) \right)
\]

\[
- |\sin((i+1)\theta_n)\mathbf{e}_{2n,2i} + \cos((i+1)\theta_n)\mathbf{e}_{2n,2i+1})| f(y) \}
\]

(see (2.4) for the convention of the notation).

Remark 7. For \( (y_{2i}, y_{2i+1}) \in \mathbb{R}^2 \) \((i = 0, \cdots, n-1)\), put

\[
x_i \equiv \{(y_{2i} - y_{2(i-1)}^2 + (y_{2i+1} - y_{2i-1})^2 \}^{1/2},
\]

where we put \( (y_{2i}, y_{2i+1}) = (y_{2(n+i)}, y_{2(n+i)+1}) \) for \( i = -1, 0 \). If

\[
\min(x_{i+1}, x_{i-1}) \sin \theta_n > h_n(x_i),
\]

\[
(y_{2i}, y_{2i+1}) - (y_{2(i-1)}, y_{2i-1}) = x_i(- \sin(i\theta_n), \cos(i\theta_n))
\]
for all $i = 0, \ldots, n-1$, then for any $g \in C^2_0(\mathbb{R}^n; \mathbb{R})$,

$$
\hat{L}g(((y_{2i} - y_{2(i-1)})^2 + (y_{2i+1} - y_{2i-1})^2)^{1/2})_{i=0}^{n-1} = Lg((x_i)_{i=0}^{n-1}).
$$

Put also $Y_n(t) = (Y_{n,k}(t))_{k=0}^{n-1}$. Then the time evolution of $\{Y_n(t)\}_{0 \leq t}$ for sufficiently large $n$ can be given by the following.

**Lemma 2.** Suppose that (2.5) holds. Then for any $\delta \in (0, T^*)$, there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$ and any $f \in C^2_0(\mathbb{R}^{2n}; \mathbb{R})$, $\delta$ is less than $T_n^*$ and

$$
f(Y_n(\min(t, \tau_{n,\delta}))) = f(Y_n(0)) + \int_0^{\min(t, \tau_{n,\delta})} \hat{L}f(Y_n(s))ds + \mathcal{M}_t^{f(Y_n)}(\min(t, \tau_{n,\delta}))
$$

for $t \geq 0$, P.a.s., where $\mathcal{M}_t^{f(Y_n)}(t)$ denotes a purely discontinuous martingale part of $f(Y_n(t))$.

**Proof.** Take $n_0 \in \mathbb{N}$ such that $\delta < T_n^*$ for any $n \geq n_0$, which is possible from Theorem 1. First we show that there exists $n_1 \geq n_0$ such that for any $n \geq n_1$ and $k = 0, \ldots, n - 1$

$$
\min(X_{n,k-1}(t), X_{n,k+1}(t)) \sin \theta_n > h_n(X_{n,k}(t)) \quad \text{for } t \in [0, \tau_{n,\delta}) \text{ P.a.s.,} \quad (3.13)
$$

where we put $X_{n,i}(t) = X_{n,n+i}(t)$ for $i = -1, 0$.

By (14) of [4], $C_n(\delta)^{-1}_{k=n_0}$ defined in (3.11) is bounded. Therefore there exists $n_1 \geq n_0$ such that for any $n \geq n_1$

$$
\frac{C_n(\delta)}{2n} \sin \theta_n > \frac{2n}{C_n(\delta) T(n)} \frac{\theta_n}{T(n)}
$$

by (2.5). Hence, for $k, i = 0, \ldots, n - 1$ and $t \in [0, \tau_{n,\delta})$

$$
X_{n,k}(t) \sin \theta_n > \frac{C_n(\delta)}{2n} \sin \theta_n > \frac{2n}{C_n(\delta) T(n)} \frac{\theta_n}{T(n)} > \frac{1}{X_{n,i}(t)} \frac{\theta_n}{T(n)} > h_n(X_{n,i}(t)) \text{ a.s.}
$$

by (2.3), which implies (3.13).

By (3.3)-(3.4), we have the following:
\[
Y_n(s) = -x_1 \sum_{k=0}^{n-1} e_{2n,2k} + (x_1 \cot \theta_n - x_3/\sin \theta_n) \sum_{k=0}^{n-1} e_{2n,2k+1} \\
+ \sum_{i=1}^{n-1} x_{2i} \left[-(\sin(i\theta_n)) \sum_{k=i}^{n-1} e_{2n,2k} + (\cos(i\theta_n)) \sum_{k=i}^{n-1} e_{2n,2k+1}\right],
\]

with \(x_{2i} = X_{n,i}(s)\) and \(x_{2i+1} = D_{n,i}(s)\) \((i = 0, \cdots, n - 1)\). Therefore, from (2.4), (3.1), (3.9)-(3.10) and (3.13), by the Itô formula (see [9]), the proof is over (see Remark 5).

The following lemma plays a crucial role when we approximate \(\Omega_{\mathcal{E},n}\) by \(\Omega_{X,n}\).

**Lemma 3.** Suppose that (2.5) holds. Then for any \(\delta \in (0,T^*)\),

\[
\lim_{n \to \infty} n^2 E\left[ \sup_{0 \leq t \leq \min(T^*_n-\delta, \tau_{n,\delta})} \sum_{i=0}^{n-1} |Y_{n,i}(t) - y_{n,i}(t)|^2 \right] = 0. 
\tag{3.14}
\]

**Proof.** For \(n_1 \in \mathbb{N}\) in Lemma 2, there exists a positive constant \(C\) such that the following which will be proved later holds: for any \(n \geq n_1\) and \(t \in [0,T^*_n - \delta]\),

\[
E\left[ \sup_{0 \leq s \leq \min(t, \tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 \right] 
\leq CT(n)^{-1}n^3 + C \int_0^t E\left[ \sup_{0 \leq u \leq \min(s, \tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 \right] ds. 
\tag{3.15}
\]

This implies (3.14), by Gronwall’s inequality, from (2.5).

We prove (3.15) to complete the proof. For any \(n \geq n_1\), by Lemmas 1 and 2, the following holds: for \(t \in [0, \min(T^*_n - \delta, \tau_{n,\delta})]\),
\[
\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2
\]
\[
= \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s),
\]
\[
\left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \sin(k\theta_n), -\cos(k\theta_n))
\]
\[
+ \left( \frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) (-\sin((k+1)\theta_n),\cos((k+1)\theta_n)) > ds
\]
\[
+ \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(t) - Y_{n,k}(t),
\]
\[
\left[ -\frac{T(n)}{\theta_n} h_n(X_{n,k+1}(s)) + \frac{1}{X_{n,k+1}(s)} \right] \sin(k\theta_n), -\cos(k\theta_n))
\]
\[
+ \left[ \frac{T(n)}{\theta_n} h_n(X_{n,k}(s)) - \frac{1}{X_{n,k}(s)} \right] (-\sin((k+1)\theta_n),\cos((k+1)\theta_n)) > ds
\]
\[
+ 2\frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \sum_{k=0}^{n-1} \int_0^t \left| \frac{h_n(X_{n,k}(s))}{\sin \theta_n} \right|^2 ds + M(t),
\]

where \(M(t)\) denotes a purely discontinuous martingale part of \(\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2\).

Since \(\{C_k^{-1}(\delta)\}_{k \geq n_1}\) is bounded by (14) of [4], we only have to show the following (3.16)-(3.19) to complete the proof: for \(t \in [0, \min(T_n - \delta, \tau_n, \delta)]\),

\[
\sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s),\]
\[
\left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \sin(k\theta_n), -\cos(k\theta_n))
\]
\[
+ \left( \frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) (-\sin((k+1)\theta_n),\cos((k+1)\theta_n)) > ds
\]
\[
\leq \frac{4n^2 \sin^2 \theta_n}{C_n(\delta)^2 \cos \theta_n \cos^2(\theta_n/2)} \int_0^t \sup_{0 \leq u \leq \min(s, \tau_n, \delta)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds,
\]
\[
\frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(t) - Y_{n,k}(s), \quad (3.17)
\]
\[
- \frac{T(n)}{\theta_n} h_n(X_{n,k+1}(s)) + \frac{1}{X_{n,k+1}(s)} \right] (\sin(k \theta_n), -\cos(k \theta_n)) \\
+ \left[ \frac{T(n)}{\theta_n} h_n(X_{n,k}(s)) - \frac{1}{X_{n,k}(s)} \right] (\sin((k+1) \theta_n), -\cos((k+1) \theta_n)) > ds \\
\leq 2 \int_0^t \sup_{0 \leq u \leq \min(s, \tau_n, s)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds + 2nt \frac{8n^3 \theta_n}{T(n)C_n(\delta)^2 \sin \theta_n}^2, \\
\]
\[
2\frac{T(n) \tan(\theta_n/2)}{\theta_n/2} \sum_{k=0}^{n-1} \int_0^t \left| \frac{h_n(X_{n,k}(s))}{\sin \theta_n} \right|^2 ds \leq \frac{8n^3 \theta_n}{T(n)C_n(\delta)^2 \cos^2(\theta_n/2) \sin \theta_n}, \quad (3.18)
\]
and for \( t \in [0, T_n - \delta] \),
\[
\left\{ E\left[ \sup_{0 \leq s \leq \min(t, \tau_n, s)} |M(s)|^2 \right] \right\}^{1/2} \leq \frac{32n^2 \theta_n}{T(n)C_n(\delta)^2 \sin \theta_n \cos^2(\theta_n/2)} + \frac{6n^3 \theta_n^2}{T(n)C_n(\delta) \sin \theta_n)^2} \\
+ 3 \int_0^t E\left[ \sup_{0 \leq u \leq \min(s, \tau_n, s)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 \right] ds. \quad (3.19)
\]

We first prove (3.16). By (3.4) and (3.6), for \( k = 0, \cdots, n-1 \) and \( s \in [0, T_n] \),
\[
y_{n,k+1}(s) - Y_{n,k+1}(s) = y_{n,k}(s) - Y_{n,k}(s) + (\ell_{n,k+1}(s) - X_{n,k+1}(s))(-\sin((k+1) \theta_n), \cos((k+1) \theta_n)). \quad (3.20)
\]
Hence for \( s \in [0, T_n] \),
\[
\sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) (\sin(k\theta_n), -\cos(k\theta_n)) + \left( \frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) (-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > \\
= \sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) > \\
\times \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \\
+ \sum_{k=0}^{n-1} (\cos\theta_n)(\ell_{n,k}(s) - X_{n,k}(s)) \left( \frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right).
\]

This together with (3.11)-(3.12) and the following implies (3.16) : for \( s \in [0, T_n] \),

\[
< y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) > \\
\times \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \\
\leq \frac{|y_{n,k}(s) - Y_{n,k}(s)|^2}{\ell_{n,k+1}(s)X_{n,k+1}(s)} \cos\theta_n + \frac{(\ell_{n,k+1}(s) - X_{n,k+1}(s))^2}{\ell_{n,k+1}(s)X_{n,k+1}(s)} \cos\theta_n.
\]

(3.17) can be proved by (3.12) and by the following: for \( x > 0 \),

\[
\frac{1}{x} - \frac{T(n)}{\theta_n} \frac{h_n(x)}{\theta_n} = \frac{\theta_n \cot\theta_n}{T(n)x^3} \left( \frac{2}{1 + (1 + 4x^{-2}T(n)^{-1}\theta_n \cot\theta_n)^{1/2}} \right)^2,
\]

since \( \cos\theta_n < \cos^2(\theta_n/2) \).

(3.18) is true, since \( h_n(x) < \theta_n(T(n)x)^{-1} \) by (2.3).

Finally we prove (3.19). For \( t \in [0, T_n - \delta] \),

\[
E[ \sup_{0 \leq s \leq \min(t, \tau_{n,s})} |M(s)|^2 ] \\
\leq 4 \frac{T(n) \tan(\theta_{n}/2)}{\theta_{n}/2} \sum_{i=0}^{n-1} E[ \int_0^{\min(t, \tau_{n,s})} \left( -2 \frac{h_n(X_{n,i}(s))}{\sin\theta_n} \times < y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) > \\
+ < y_{n,i}(s) - Y_{n,i}(s), (\sin((i+1)\theta_n), \cos((i+1)\theta_n)) > \\
+ 2 \frac{h_n(X_{n,i}(s))}{\sin\theta_n} \right)^2 ds ].
\]
For $s \in [0, \min(t, \tau_n)]$ and $i = 0, \cdots, n - 1$, by (3.12),

$$
\begin{align*}
&-2 \frac{h_n(X_{n,i}(s))}{\sin \theta_n} < y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i - 1)\theta_n), -\cos((i - 1)\theta_n)) > \\
&\quad + < y_{n,i}(s) - Y_{n,i}(s), (-\sin((i + 1)\theta_n), \cos((i + 1)\theta_n)) > + 2 \left| \frac{h_n(X_{n,i}(s))}{\sin \theta_n} \right|^2 \\
&\leq 4 \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \left[ \left| y_{n,i-1}(s) - Y_{n,i-1}(s) \right| + \left| y_{n,i}(s) - Y_{n,i}(s) \right| \
&\quad + \left| y_{n,i-1}(s) - Y_{n,i-1}(s) \right| \right] \left[ \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right]^2
\end{align*}
$$

since $h_n(x) < \theta_n(T(n)x)^{-1}$ by (2.3).

Use the inequality $(xy)^{1/2} \leq (x + y)/2$ ($x, y > 0$) for

$$
x = 4 \frac{T(n) \tan(\theta_n/2)}{\theta_n/2} \times 4 \left| \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right|^2,
$$

$$
y = \sum_{i=0}^{n-1} E \int_0^{\min(t, \tau_n,s)} \left\{ \left| y_{n,i-1}(s) - Y_{n,i-1}(s) \right| + \left| y_{n,i}(s) - Y_{n,i}(s) \right| + \left| y_{n,i-1}(s) - Y_{n,i-1}(s) \right| \right] \left[ \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right]^2 ds.
$$

Use also the inequality $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ for $x = \left| y_{n,i-1}(s) - Y_{n,i-1}(s) \right|$, $y = \left| y_{n,i}(s) - Y_{n,i}(s) \right|$ and $z = \left| (2n\theta_n)/(T(n)C_n(\delta) \sin \theta_n) \right|$. Then we obtain (3.19).

4. Proof of Main Result.

In this section we prove Theorem 2 by making use of lemmas given in section 3.

Proof of Theorem 2. For any $t \in (0, T^*)$ and any $\eta > 0$, take $n_2 \in \mathbb{N}$ such that for any $n \geq n_2$

$$
t < T_n^* - (T^* - t)/2,
$$

$$
\sup_{0 < s \leq t} d_H(\Omega_{\epsilon,n}(s), \Omega(s)) < \eta/2,
$$

which is possible Theorem 1. Put $\delta = (T^* - t)/2$. Then for any $n \geq n_2$,
\[ P(\sup_{0 \leq s \leq \tau} d_H(\Omega_X, n(s), \Omega(s)) \geq \eta) \leq P(\sup_{0 \leq s \leq \tau} d_H(\Omega_X, n(s), \Omega_{\ell, n}(s)) \geq \eta/2) \leq P(\sup_{0 \leq s \leq T_{n}^* - \delta} d_H(\Omega_X, n(s), \Omega(s)) \geq \eta/2) \leq P(\tau_{n,\delta} < T_{n}^* - \delta) + P(\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} d_H(\Omega_X, n(s), \Omega_{\ell, n}(s)) \geq \eta/2). \]

The first probability on the last part of (4.1) can be shown to converge to zero as \( n \to \infty \) as follows: by Chebychev’s inequality,

\[ P(\tau_{n,\delta} < T_{n}^* - \delta) \leq P(\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |X_{n, k}(s) - \ell_{n, k}(s)| \geq C_n(\delta)/(2n)) \leq P(\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n, k}(s) - y_{n, k}(s)| \geq C_n(\delta)/(4n)) \leq (4nC_n(\delta))^{-1} 2E[\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n, k}(s) - y_{n, k}(s)|^2] \to 0, \text{ as } n \to \infty \text{ by Lemma 3}, \]

since

\[ |X_{n, k}(s) - \ell_{n, k}(s)| \leq |Y_{n, k}(s) - y_{n, k}(s)| + |Y_{n, k-1}(s) - y_{n, k-1}(s)| \]

by (3.20) and since \( \lim_{k \to \infty} C_k(\delta)^{-1} \) is finite by (14) of [4].

The second probability on the last part of (4.1) can be shown to converge to zero as \( n \to \infty \) as follows: by Chebychev’s inequality,

\[ P(\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} d_H(\Omega_X, n(s), \Omega_{\ell, n}(s)) \geq \eta/2) \leq P(\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n, k}(s) - y_{n, k}(s)| \geq \eta/2) \leq (\eta/2)^{-1} E[\sup_{0 \leq s \leq \min(T_{n}^* - \delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n, k}(s) - y_{n, k}(s)|^2] \to 0, \text{ as } n \to \infty \text{ by Lemma 3}, \]
since
\[
d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \leq \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|.
\]

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