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<thead>
<tr>
<th>Title</th>
<th>A two-dimensional random crystalline algorithm for Gauss curvature flow</th>
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</thead>
<tbody>
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<td>File Information</td>
<td>AAP34-3.pdf</td>
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A TWO DIMENSIONAL RANDOM CRISTALLINE ALGORITHM
FOR GAUSS CURVATURE FLOW

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Abstract

We propose and study a random crystalline algorithm (a discrete approximation) of the Gauss curvature flow of smooth simple closed convex curves in $\mathbb{R}^2$ as a stepping stone to the full understanding of such a phenomenon as the wearing process of stones on beaches.

Keywords: random crystalline algorithm; Gauss curvature flow; closed curve

AMS 2000 Subject Classification: Primary 60D05
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1. Introduction.

The Gauss curvature flow of closed surfaces in $\mathbb{R}^3$ is a mathematical model of the wearing process of stones on beaches (see [3] and also [1], [6] and [11]).

We introduce the definition of the Gauss curvature flow of smooth closed convex hypersurfaces in $\mathbb{R}^{d+1}$. Let $\Gamma$ be a smooth closed convex hypersurface in $\mathbb{R}^{d+1}$ and $F : S^d \rightarrow \mathbb{R}^{d+1}$ be a parametric representation of $\Gamma$. Then a collection of $F(\cdot, t) : S^d \rightarrow \mathbb{R}^{d+1}$ of smooth closed convex hypersurfaces with parameter $t \in [0, T)$ for some $T > 0$ is called Gauss curvature flow with initial state $\Gamma$ if the following holds:

\[
\begin{align*}
\frac{\partial F(s, t)}{\partial t} &= -K(s, t)n(s, t) \quad (s \in S^d, 0 < t < T), \\
F(s, 0) &= F(s) \quad (s \in S^d),
\end{align*}
\]

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where $K(s, t)$ and $n(s, t)$ denote the Gauss curvature and the unit outward normal vector, respectively, at a point $F(s, t)$ on the hypersurface $\{F(s', t)|s' \in S^d\}$. In this paper we assume that the convex set with boundary $\{F(s, t)|s \in S^d\}$ is non-increasing in $t$ (see Figure 1).

Suppose that $\Gamma$ is strictly convex. Then there exists the maximum $T^*$ of $T$ for which (1.1)-(1.2) has a unique smooth strictly convex solution and $\{F(s, t)|s \in S^d\}$ converges to a point as $t \uparrow T^*$ (see [1], [6] and [11]).

In [8], H. Ishii proposed a discrete time approximation scheme for the Gauss curvature flow. We briefly introduce it. Suppose that we are given the strictly convex set $D$ with smooth boundary $\partial D$ in $\mathbb{R}^{d+1}$ at time $t = 0$. Take $h > 0$ and a function $V : [0, \infty) \rightarrow [0, \infty)$. For every $s \in S^d$, let $D_{s, h}$ denote the set which can be obtained by cutting off the volume $V(h)$ from the set $D$ in the direction $-s$ (see Figure 2). Put $D_{0, h} \equiv D$ and $D_{1, h} \equiv \cap_{s \in S^d} D_{s, h}$. Define $D_{n, h}$ inductively in $n$ until $n_h \equiv \max\{k \geq 1\}$ the volume of $D_{k, h}$ is greater than $V(h) + 1$. Let $V(h) \rightarrow 0$ as $h \rightarrow 0$ in an appropriate rate. Then $\lim_{h \rightarrow 0} n_h h = T_{max}$, and the flow of $\partial D_{[t/h], h} (0 \leq t \leq n_h h)$ converges to the Gauss curvature flow in Hausdorff metric uniformly in $t$ on every compact subset of $[0, T^*)$, where $[t/h]$ denotes the integer part of $[t/h]$. Notice that the time variable $t$ is discretized but the space variable $s$ is not in this approximation scheme.

**Remark 1.** Hausdorff metric of compact sets $A$ and $B \in \mathbb{R}^d$ is given by the following:
\begin{equation}
    d_H(A, B) \equiv \max_{p \in A} \max_{q \in B} \text{dist}(p, B), \max_{q \in B} \text{dist}(q, A)).
\end{equation}

A crystalline (or a polyhedral) approximation of the curvature flow of convex curves was studied by P. M. Girão and is useful in numerical analysis (see Theorem 1 given below, [4] and also [5] and the references therein). In [4], the space variable $s$ is discretized but the time variable $t$ is not. In case when the initial curve is not convex, the results of [4] have been generalized by K. Ishii and M. H. Soner (see [9] and the references therein for further information on this problem). The results of [4] have not been generalized to a class of closed convex hypersurfaces in $\mathbb{R}^{d+1}$ for $d \geq 2$. This is a well-known open problem.

**Remark 2.** Let $\Gamma$ be a smooth simple closed convex curve on $\mathbb{R}^2$. Fix a point $x_0$ on $\Gamma$. For any $x \in \Gamma$, let $s(x)$ be the length of the curve which connects $x_0$ and $x$ on $\Gamma$ clockwise. Then one can parametrize $x \in \Gamma$ by $s(x)$. Let $p_1(s(x))$ and $p_2(s(x))$ denote, respectively, the clockwise unit tangent vector and the unit outward normal vector at $x$ on $\Gamma$. Then the Gauss curvature $K(s(x))(\in \mathbb{R})$ at $x$ on $\Gamma$ satisfies the following:

\[
    \frac{dp_1(s(x))}{ds} = -K(s(x))p_2(s(x)),
\]
\[
    \frac{dp_2(s(x))}{ds} = K(s(x))p_1(s(x)).
\]
Figure 3: $\Gamma$ and $\Gamma_0$

We refer to [4] since it plays a crucial role in this paper. First of all we introduce one of the conventions in this paper. Every convex polygon with $n$ sides ($n$-polygon for short) has outward normals $N_{n,i} \equiv (\cos(2\pi i/n), \sin(2\pi i/n))$ ($i = 0, \cdots, n - 1$). By the $i$ th side of the $n$-polygon we denote the side with the outward normal $N_{n,i}$.

Take a smooth simple closed convex curve $\Gamma$ on $\mathbb{R}^2$. For $n \geq 5$, let $\Gamma_n$ denote the $n$-polygon of which the $i$ th side is tangent to $\Gamma$ (see Figure 3). Let $\{\Gamma_n(t)\}_{0 \leq t < T^*_n}$ be the flow of $n$-polygons which can be defined as follows, where $T^*_n$ denotes the extinction time of $\Gamma_n(\cdot)$.

$$\Gamma_n(0) = \Gamma_n,$$

and for $t \in [0, T^*_n)$, the inward normal velocity $V_{n,i}(t)$ of the $i$ th side of $\Gamma_n(t)$ is given by the following:

$$V_{n,i}(t) = 2\tan(\pi/n) \frac{\ell_{n,i}(t)}{\ell_{i,n}(t)},\quad (1.4)$$

where $\ell_{n,i}(t)$ denotes the length of the $i$ th side of $\Gamma_n(t)$ (see Figure 4). It is known that there exists the Gauss curvature flow $\{\Gamma(t)\}_{0 \leq t < T^*}$ on $\mathbb{R}^2$, with $\Gamma(0) = \Gamma$, where $T^*$ denotes the extinction time of $\Gamma(t)$ (see [4]). Let $\Omega_{\ell,n}(t)$ and $\Omega(t) \subset \mathbb{R}^2$ be the closed convex sets such that $\partial\Omega_{\ell,n}(t) = \Gamma_n(t)$ and $\partial\Omega(t) = \Gamma(t)$, and such that $\Omega_{\ell,n}(t) \subset \Omega_{\ell,n}(s)$ and $\Omega(t) \subset \Omega(s)$ if $0 \leq s \leq t$.

Then the following holds.
Figure 4: Motion of the $i$ th side of $\Gamma_n(t)$

**Theorem 1.** (see [4]). As $t \uparrow T^*$, $\Omega(t)$ converges in Hausdorff metric to a point or a segment. $\lim_{n \to \infty} T_n^* = T^*$, and for any $t \in [0, T^*)$,

$$
\lim_{n \to \infty} \sup_{0 \leq s \leq t} d_H(\Omega_{t,n}(s), \Omega(s)) = 0. 
$$

Since the wearing process of stones on beaches is random, we would like to construct a stochastic model instead of a deterministic one such as Theorem 1.

In this paper we introduce the flow of random $n$-polygons with outward normals $N_{n,i}$ ($i = 0, \cdots, n-1$) and show that it converges in probability to the Gauss curvature flow of smooth simple closed convex curves on $\mathbb{R}^2$ as $n \to \infty$ in Hausdorff metric uniformly in $t$ on every compact subset of $[0, T^*)$ (see Theorem 2 in section 2).

In the proof we approximate the random $n$-polygon by $\Gamma_n(t)$ at time $t$ and use Theorem 1.

We use the word “Gauss” even for the curvature flow in $\mathbb{R}^2$ since a part of our idea that the volume is cut off from the stone is originally from the deterministic model of the Gauss curvature flow (see [8]).

In section 2 we introduce our random model and state our result which will be proved in section 4. Technical lemmas will be stated and proved in section 3.

## 2. Main result.

We first introduce our random model.
Figure 5

Figure 5: The isogonal trapezoid with the height $h_n(x)$

Let $\{T(n)\}_{n \geq 1}$ be an increasing sequence of positive real numbers and put

$$\theta_n = \frac{2\pi}{n}. \quad (2.1)$$

For $x > 0$ and $n \geq 1$, put

$$h_n(x) = \frac{\tan \theta_n \{-x + (x^2 + 4\cot \theta_n/T(n))^{1/2}\}}{2}. \quad (2.2)$$

**Remark 3.** $h_n(x)$ is the height of the isogonal trapezoid, with the area $\theta_n/T(n)$, of which the lengths of upper and lower sides are $x$ and $x + 2\cot \theta_n h_n(x)$ respectively (see Figure 5). In particular,

$$(x + (\cot \theta_n)h_n(x))h_n(x) = \frac{\theta_n}{T(n)}. \quad (2.3)$$

For $n \geq 5$, we consider the Markov process $\{(X_{n,i}(t))_{i=0}^{n-1}, t \geq 0\}$ on $\mathbb{R}^n$ such that $(X_{n,i}(0))_{i=0}^{n-1} = (\ell_{n,i}(0))_{i=0}^{n-1}$ (see (1.4)) and of which the generator is given by the following: for a bounded Borel measurable function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $x = (x_i)_{i=0}^{n-1} \in \mathbb{R}^n$,

$$Lf(x) = \frac{T(n) \tan(\theta_n/2)}{\theta_n/2} \sum_{i=0}^{n-1} I_{\{y_i \leq \min(y_{i-1}, y_{i+1}) \sin \theta_n > h_n(y_i)\}}(x)$$

$$\times [f(x + 2(\cot \theta_n)h_n(x_i) e_{n,i} - \frac{h_n(x_i)}{\sin \theta_n}(e_{n,i-1} + e_{n,i+1})) - f(x)]$$
Random crystalline algorithm for curvature flow

(see [2, Chap. 4, section 2]). Here $I_A(x)$ and $\{e_{n,k}\}_{k=0}^{n-1}$ denote the indicator function of the set $A$ and the standard normal base in $\mathbb{R}^n$ respectively, and we put $e_{n,n+k} = e_{n,k}$ and $y_{n+k} = y_k$ ($k = -1, 0$).

It is easy to see that one can construct the flow of random closed convex sets $\{\Omega_{X,n}(t)\}_{t \geq 0}$ in $\mathbb{R}^2$, surrounded by $n$-polygons, such that $\Omega_{X,n}(0) = \Omega_{\ell,n}(0)$, and that $\Omega_{X,n}(t) \subset \Omega_{X,n}(s)$ if $s \leq t$, and that the length of the $i$-th side of $\partial \Omega_{X,n}(t)$ is equal to $X_{n,i}(t)$.

We discuss the meaning of our model.

For $n \geq 5$, put

$$
\sigma_{n,i} \equiv \begin{cases} 
0 & \text{if } i = 0, \\
\inf \{t > \sigma_{n,i-1} | \sum_{k=1}^{n-1} |X_{n,k}(t) - X_{n,k}(t^-)| > 0 \} & \text{if } i \geq 1,
\end{cases}
$$

where $X_{n,k}(t^-) \equiv \lim_{s \uparrow t} X_{n,k}(s)$, and where we consider the right hand side as infinity if the set over which the infimum is taken is empty. Then

$$
P(\sigma_{n,i} < \sigma_{n,i+1} \text{ for all } i \text{ for which } \sigma_{n,i} < \infty) = 1.
$$

Put

$$
A_n \equiv \{ j \in \{0, \cdots, n-1\} | \min(X_{n,j-1}(0), X_{n,j+1}(0)) \sin \theta_n > h_n(X_{n,j}(0)) \}.
$$

If the set $A_n$ is not empty, then $\sigma_{n,1}$ is exponentially distributed with parameter $[\#A_n \cdot T(n) \tan(\theta_n/2)]/(\theta_n/2)$ (see [2, p. 163]), where we put $j = n+j$ for $j = -1$ and 0, and where $\#A_n$ denotes the cardinal number of the set $A_n$. For any $k \in A_n$, the probability that the isogonal trapezoid with the area $\theta_n/T(n)$ is cut off from $\Omega_{X,n}(0)$ in the direction $-N_{n,k} = (\cos(2\pi k/n), \sin(2\pi k/n))$ at time $t = \sigma_{n,1}$ is equal to $(\#A_n)^{-1}$ (see Figure 6).

If the set $A_n$ is empty, then $\sigma_{n,1} = \infty$ and $X_{n,k}(0) = X_{n,k}(t)$ for all $k = 0, \cdots, n-1$ and all $t \geq 0 \text{ a.s.}

The following also holds a.s.: $\{\Omega_{X,n}(t)\}$ continues to change the shape in a similar manner to above at times $t = \sigma_{n,i}$ which is finite; $\sigma_{n,i}$ is infinite if $i$ is greater than (the area of $\Omega_{X,n}(0))/(T(n)^{-1} \theta_n)$; $\Omega_{X,n}(t)$ is an $n$-polygon for all $t \geq 0$.

The following is our main result.
Figure 6: The change of the $k$ th side of $\Omega_{X,n}(0)$

**Theorem 2.** Suppose that $\Gamma$ is a smooth simple closed convex curve on $\mathbb{R}^2$ and that the following holds:

$$\lim_{n \to \infty} T(n)n^{-5} = \infty. \quad (2.5)$$

Then for any $t \in [0, T^*)$ and any $\eta > 0$,

$$\lim_{n \to \infty} P\left( \sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega(s)) < \eta \right) = 1. \quad (2.6)$$

**Remark 4.** $(2.5)$ implies that $\theta_n/T(n) \sim o(n^{-6}) \ (as \ n \to \infty)$, where $\theta_n/T(n)$ is the area of the isogonal trapezoid which is cut off from an $n$-polygon in our model.

Consider a convex stone which rotates randomly on a beach where waves are even. Our result suggests that the time evolution of the surface of such a stone can be considered as Gauss curvature flow.

### 3. Lemmas.

In this section we state and prove lemmas which will be used in the next section.

For $n \geq 1$ and $i = 0, \cdots, n - 1$, put
\begin{align}
D_{n,i}(t) &= \sum_{0<s\leq t} h_n(X_{n,i}(s-))I(X_{n,i}(s-),\infty)(X_{n,i}(s)) \quad (t \geq 0), \quad (3.1) \\
n_{n,i}(t) &= \int_0^t \frac{2 \tan(\theta_n/2)}{\ell_{n,i}(s)} ds \quad (0 \leq t < T^*). \quad (3.2)
\end{align}

**Remark 5.** $D_{n,i}(t)$ is the distance between the straight line which includes the $i$ th side of $\Omega_{X,n}(t)$ and that which includes the $i$ th side of $\Omega_{X,n}(0)$. $d_{n,i}(t)$ is also the distance between the straight line which includes the $i$ th side of $\Omega_{t,n}(t)$ and that which includes the $i$ th side of $\Omega_{t,n}(0)$.

Put the intersection point of the $\theta$ th and the first sides of $\Omega_{t,n}(0)$ at the origin. Then the coordinate of the intersection point of the $i$ th and the $(i+1)$ th sides of $\Omega_{X,n}(t)$ and $\Omega_{t,n}(t)$ can be written as follows, respectively: for $t \geq 0$,

\begin{align}
Y_{n,0}(t) &= (-D_{n,0}(t), D_{n,0}(t) \cot \theta_n - D_{n,1}(t)/\sin \theta_n) \quad \text{if } i = 0, \quad (3.3) \\
Y_{n,i}(t) &= Y_{n,0}(t) + \sum_{k=1}^i X_{n,k}(t)(-\sin(k\theta_n), \cos(k\theta_n)) \quad \text{if } i = 1, \cdots, n-1, \quad (3.4)
\end{align}

and for $t \in [0, T^*)$

\begin{align}
y_{n,0}(t) &= (-d_{n,0}(t), d_{n,0}(t) \cot \theta_n - d_{n,1}(t)/\sin \theta_n) \quad \text{if } i = 0, \quad (3.5) \\
y_{n,i}(t) &= y_{n,0}(t) + \sum_{k=1}^i \ell_{n,k}(t)(-\sin(k\theta_n), \cos(k\theta_n)) \quad \text{if } i = 1, \cdots, n-1. \quad (3.6)
\end{align}

**Remark 6.** $X_{n,i}(t) = |Y_{n,i}(t) - Y_{n,i-1}(t)|$ for $t \geq 0$ and $\ell_{n,i}(t) = |y_{n,i}(t) - y_{n,i-1}(t)|$ for $t \in [0, T^*)$, where we put $(Y_{n,i}(t), y_{n,i}(t)) = (Y_{n,n+i}(t), y_{n,n+i}(t))$ for $i = -1, 0$.

The time evolution of $\{y_{n,i}(t)\}_{0 \leq t < T^*}$ ($n \geq 5, i = 0, \cdots, n-1$) can be given by the following.

**Lemma 1.** For $n \geq 5$, $i = 0, \cdots, n-1$, and $s \in (0, T^*)$,

\begin{align}
\frac{dy_{n,i}(s)}{ds} &= \frac{\sin(i\theta_n) - \cos(i\theta_n)}{\ell_{n,i+1}(s) \cos^2(\theta_n/2)} - \frac{\sin((i+1)\theta_n) - \cos((i+1)\theta_n)}{\ell_{n,i}(s) \cos^2(\theta_n/2)}, \quad (3.7)
\end{align}

where we put $\ell_{n,n}(s) = \ell_{n,0}(s)$. 
Proof. It is known that \( \{ \ell_n(i, t) \}_{i=0}^{n-1} \) satisfies the following (see [4]):

\[
\frac{d\ell_n(i, t)}{dt} = \left( \frac{2\cos \theta_n}{\ell_n(i, t)} - \frac{1}{\ell_{n, i+1}(t)} - \frac{1}{\ell_{n, i-1}(t)} \right) \cos^2(\theta_n/2),
\]

where we put \( \ell_{n,i+k}(t) = \ell_{n,k}(t) \) \((k = -1, 0)\).

(3.7) can be proved inductively in \( i \), by (3.2), (3.5)-(3.6) and by the following:

\[
\sin((i - 1)\theta_n) + \sin((i + 1)\theta_n) = 2\cos \theta_n \sin(i\theta_n), \quad (3.9)
\]

\[
\cos((i - 1)\theta_n) + \cos((i + 1)\theta_n) = 2\cos \theta_n \cos(i\theta_n). \quad (3.10)
\]

Before we state and prove the following lemma, we give some notation. Put for \( \delta \in (0, T_n^\ast) \),

\[
C_n(\delta) = \inf \{ k \in \mathbb{N} \mid \ell_n(k, t) \geq \delta \}, \quad (3.11)
\]

\[
\tau_{n, \delta} = \inf \{ t > 0 \mid f_n(\delta) \geq \min \{ X_{n,k}(t) \mid 0 \leq k \leq n - 1 \} \}. \quad (3.12)
\]

For any \( f \in C^2_0(\mathbb{R}^{2n}; \mathbb{R}) \) and \( y = (y_i)_{i=0}^{2n-1} \in \mathbb{R}^{2n} \), put

\[
\hat{L}f(y) = \frac{T(n) \tan(\theta_n/2)}{\theta_n/2} \sum_{i=0}^{n-1} \left\{ f(y + h_n(y_i, y_{i+1}, y_{i+2})) \right\}
\]

\[
\times \left[ \sin((i - 1)\theta_n) \right] \left[ \sin((i + 1)\theta_n) \right] \left[ \cos((i - 1)\theta_n) \right] \left[ \cos((i + 1)\theta_n) \right]
\]

(see (2.4) for the convention of the notation).

Remark 7. For \( (y_{2i}, y_{2i+1}) \in \mathbb{R}^2 \) \((i = 0, \cdots, n - 1)\), put

\[
x_i \equiv \{(y_{2i} - y_{2(i+1)})^2 + (y_{2i+1} - y_{2i-1})^2 \}^{1/2},
\]

where we put \( (y_{2i}, y_{2i+1}) = (y_{2(n+i)}, y_{2(n+i)+1}) \) for \( i = -1, 0 \). If

\[
\min(x_{i+1}, x_{i-1}) \sin \theta_n > h_n(x_i),
\]

\[
(y_{2i}, y_{2i+1}) - (y_{2(i-1)}, y_{2i-1}) = x_i(-\sin(i\theta_n), \cos(i\theta_n))
\]
for all $i = 0, \ldots, n-1$, then for any $g \in C^2_b(\mathbb{R}^n; \mathbb{R})$,

$$
\hat{L}g\left(\left(\frac{(y_{2i} - y_{2(i-1)})^2 + (y_{2i+1} - y_{2i-1})^2}{2}\right)^{n/2}\right) = Lg(\frac{(x_i)^{n-1}}{2}).
$$

Put also $Y_n(t) = (Y_{n,k}(t))_{k=0}^{n-1}$. Then the time evolution of $\{Y_n(t)\}_{0 \leq t}$ for sufficiently large $n$ can be given by the following.

**Lemma 2.** Suppose that (2.5) holds. Then for any $\delta \in (0, T^*)$, there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$ and any $f \in C^2_b(\mathbb{R}^{2n}; \mathbb{R})$, $\delta$ is less than $T^*_n$ and

$$
f(Y_n(\min(t, \tau_{n,\delta}))) = f(Y_n(0)) + \int_0^{\min(t, \tau_{n,\delta})} \hat{L}f(Y_n(s))ds + M^{[f(Y_n)]}(\min(t, \tau_{n,\delta}))
$$

for $t \geq 0$, P-a.s., where $M^{[f(Y_n)]}(t)$ denotes a purely discontinuous martingale part of $f(Y_n(t))$.

**Proof.** Take $n_0 \in \mathbb{N}$ such that $\delta < T^*_n$ for any $n \geq n_0$, which is possible from Theorem 1. First we show that there exists $n_1 \geq n_0$ such that for any $n \geq n_1$ and $k = 0, \ldots, n-1$

$$
\min(X_{n,k-1}(t), X_{n,k+1}(t)) \sin \theta_n > h_n(X_{n,k}(t)) \quad \text{for } t \in [0, \tau_{n,\delta}] \text{ P-a.s.,} \quad (3.13)
$$

where we put $X_{n,i}(t) = X_{n,n+i}(t)$ for $i = -1, 0$.

By (14) of [4], $(C_n(\delta)^{-1})_{k=n_0}^{\infty}$ defined in (3.11) is bounded. Therefore there exists $n_1 \geq n_0$ such that for any $n \geq n_1$

$$
\frac{C_n(\delta)}{2n} \sin \theta_n > \frac{2n}{C_n(\delta) T(n)} \frac{\theta_n}{T(n)}
$$

by (2.5). Hence, for $k, i = 0, \ldots, n-1$ and $t \in [0, \tau_{n,\delta}]$

$$
X_{n,k}(t) \sin \theta_n > \frac{C_n(\delta)}{2n} \sin \theta_n > \frac{2n}{C_n(\delta) T(n)} \frac{\theta_n}{T(n)} > \frac{1}{X_{n,i}(t)} \frac{\theta_n}{T(n)} > h_n(X_{n,i}(t)) \quad \text{a.s.}
$$

by (2.3), which implies (3.13).

By (3.3)-(3.4), we have the following:
\[
Y_n(s) = -x_1 \sum_{k=0}^{n-1} e_{2n,2k} + \left( x_1 \cot \theta_n - x_3 / \sin \theta_n \right) \sum_{k=0}^{n-1} e_{2n,2k+1} + \sum_{i=1}^{n-1} x_{2i} [-\sin(i \theta_n)] \sum_{k=i}^{n-1} e_{2n,2k} + (\cos(i \theta_n)) \sum_{k=i}^{n-1} e_{2n,2k+1},
\]

with \(x_{2i} = X_{n,i}(s)\) and \(x_{2i+1} = D_{n,i}(s)\) \((i = 0, \cdots, n-1)\). Therefore, from (2.4), (3.1), (3.9)-(3.10) and (3.13), by the Itô formula (see [9]), the proof is over (see Remark 5).

The following lemma plays a crucial role when we approximate \(\Omega_{\epsilon,n}\) by \(\Omega_{X,n}\).

**Lemma 3.** Suppose that (2.5) holds. Then for any \(\delta \in (0, T^*)\),

\[
\lim_{n \to \infty} n^2 E\left[ \sup_{0 \leq t \leq \min(T_n^* - \delta, \tau_n, s)} \sum_{i=0}^{n-1} |Y_{n,i}(t) - y_{n,i}(t)|^2 \right] = 0. \quad (3.14)
\]

**Proof.** For \(n_1 \in \mathbb{N}\) in Lemma 2, there exists a positive constant \(C\) such that the following which will be proved later holds: for any \(n \geq n_1\) and \(t \in [0, T_n^* - \delta]\),

\[
E[ \sup_{0 \leq s \leq \min(t, \tau_n, s)} \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 ] \leq C T(n)^{-1} n^3 + C \int_0^t E[ \sup_{0 \leq u \leq \min(s, \tau_n, s)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ] ds.
\]

This implies (3.14), by Gronwall’s inequality, from (2.5).

We prove (3.15) to complete the proof. For any \(n \geq n_1\), by Lemmas 1 and 2, the following holds: for \(t \in [0, \min(T_n^* - \delta, \tau_n, s)]\),
\[
\begin{align*}
\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2 &= \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t <y_{n,k}(s) - Y_{n,k}(s), \\
&\quad \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \left( \sin(k\theta_n), -\cos(k\theta_n) \right) \right) \left( -\sin((k+1)\theta_n), \cos((k+1)\theta_n) \right) > ds \\
+ \frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t <y_{n,k}(t) - Y_{n,k}(s), \\
&\quad \left[ -\frac{T(n)}{\theta_n} h_n(X_{n,k+1}(s)) + \frac{1}{X_{n,k+1}(s)} \right] \left( \sin(k\theta_n), -\cos(k\theta_n) \right) \\
+ \left[ \frac{T(n)}{\theta_n} h_n(X_{n,k}(s)) - \frac{1}{X_{n,k}(s)} \right] \left( \sin((k+1)\theta_n), -\cos((k+1)\theta_n) \right) > ds \\
+ 2 \frac{T(n) \tan(\theta_n/2)}{\theta_n/2} \sum_{k=0}^{n-1} \int_0^t \frac{h_n(X_{n,k}(s))}{\sin\theta_n}^2 ds + M(t),
\end{align*}
\]

where \( M(t) \) denotes a purely discontinuous martingale part of \( \sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2 \).

Since \( \{C_k^{-1}(\delta)\}_{k \geq n} \) is bounded by (14) of [4], we only have to show the following (3.16)-(3.19) to complete the proof: for \( t \in [0, \min(T_n - \delta, \tau_n, \delta)] \),

\[
\begin{align*}
\frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t <y_{n,k}(s) - Y_{n,k}(s), \\
&\quad \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \left( \sin(k\theta_n), -\cos(k\theta_n) \right) \right) \left( -\sin((k+1)\theta_n), \cos((k+1)\theta_n) \right) > ds \\
&\leq \frac{4n^2 \sin^2 \theta_n}{C_n(\delta)^2 \cos \theta_n \cos^2(\theta_n/2)} \int_0^t \sup_{0 \leq u \leq \min(s, \tau_n, \delta)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds,
\end{align*}
\]
\[
\frac{2}{\cos^2(\theta_n/2)} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(t) - Y_{n,k}(s), \quad (3.17)
\]
\[
\left[ -\frac{T(n)}{\theta_n} h_n(X_{n,k+1}(s)) + \frac{1}{X_{n,k+1}(s)} \right] (\sin(k\theta_n), -\cos(k\theta_n))
+ \left[ \frac{T(n)}{\theta_n} h_n(X_{n,k}(s)) - \frac{1}{X_{n,k}(s)} \right] (\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > ds
\leq 2 \int_0^t \sup_{0 \leq u \leq \min(s, \tau_n, a)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds + 2nt \frac{8n^3\theta_n}{T(n)C_n(\delta)^2 \sin \theta_n},
\]

\[
2\frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \sum_{k=0}^{n-1} \int_0^t \left| \frac{h_n(X_{n,k}(s))}{\sin \theta_n} \right|^2 ds \leq \frac{8n^3\theta_n}{T(n)C_n(\delta)^2 \cos^2(\theta_n/2) \sin \theta_n},
\]

and for \( t \in [0, T_n^* - \delta] \),

\[
\{ E[ \sup_{0 \leq s \leq \min(t, \tau_n, a)} |M(s)|^2 ] \}^{1/2} \leq 32n^2\theta_n 
\frac{6n^3\theta_n^2}{T(n)C_n(\delta)^2 \sin \theta_n \cos^2(\theta_n/2)} + \frac{6n^3\theta_n^2}{T(n)C_n(\delta)^2 \sin \theta_n^2} + 3 \int_0^t E[ \sup_{0 \leq u \leq \min(s, \tau_n, a)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ] ds.
\]

We first prove (3.16). By (3.4) and (3.6), for \( k = 0, \cdots, n-1 \) and \( s \in [0, T_n^*) \),

\[
y_{n,k+1}(s) - Y_{n,k+1}(s)
= y_{n,k}(s) - Y_{n,k}(s) + (\ell_{n,k+1}(s) - X_{n,k+1}(s))(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)).
\]

Hence for \( s \in [0, T_n^*), \)
\[ \sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) (\sin(k\theta_n) - \cos(k\theta_n)) \]
\[ + \left( \frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) (\sin((k+1)\theta_n) - \cos((k+1)\theta_n)) > \]
\[ = \sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) > \]
\[ \times \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \]
\[ + \sum_{k=0}^{n-1} (\cos\theta_n)(\ell_{n,k}(s) - X_{n,k}(s)) \left( \frac{1}{\ell_{n,k}(s)} - \frac{1}{X_{n,k}(s)} \right) . \]

This together with (3.11)-(3.12) and the following implies (3.16) : for \( s \in [0, T_n^*] \),
\[ < y_{n,k}(s) - Y_{n,k}(s), -2(\sin\theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) > \]
\[ \times \left( \frac{1}{\ell_{n,k+1}(s)} - \frac{1}{X_{n,k+1}(s)} \right) \]
\[ \leq \frac{|y_{n,k}(s) - Y_{n,k}(s)|^2 \sin^2\theta_n}{\ell_{n,k+1}(s)X_{n,k+1}(s) \cos\theta_n} + \frac{(\ell_{n,k+1}(s) - X_{n,k+1}(s))^2 \cos\theta_n}{\ell_{n,k+1}(s)X_{n,k+1}(s)}. \]

(3.17) can be proved by (3.12) and by the following: for \( x > 0 \),
\[ \frac{1}{x} - \frac{T(n)}{\theta_n} h_n(x) = \frac{\theta_n \cot\theta_n}{T(n)x^3} \left( 1 + \frac{2}{1 + 4x^{-2}T(n)^{-1} \theta_n \cot\theta_n} \right)^{1/2} , \]
since \( \cos\theta_n < \cos^2(\theta_n/2) \).

(3.18) is true, since \( h_n(x) < \theta_n(T(n)x)^{-1} \) by (2.3).

Finally we prove (3.19). For \( t \in [0, T_n^* - \delta] \),
\[ E \left[ \sup_{0 \leq s \leq \min(t, \tau_n,s)} |M(s)|^2 \right] \]
\[ \leq 4E[|M(\min(t, \tau_n,s))|^2] \] (by Doob’s inequality)
\[ = 4T(n) \tan(\theta_n/2) \frac{n^{-1}}{\theta_n/2} \sum_{i=0}^{n-1} E[\int_0^{\min(t, \tau_n,s)} (-2 \frac{h_n(X_{n,i}(s))}{\sin\theta_n}) \times \left( y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) > \]
\[ + y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) > \]
\[ + 2 \left( \frac{h_n(X_{n,i}(s))}{\sin\theta_n} \right)^2 ds \].
For $s \in [0, \min(t, \tau_n, \delta))$ and $i = 0, \ldots, n - 1$, by (3.12),

$$-2 \frac{h_n(X_{n,i}(s))}{\sin \theta_n} \left< y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i - 1)\theta_n), -\cos((i - 1)\theta_n)) \right> + \left< y_{n,i}(s) - Y_{n,i}(s), (\sin((i + 1)\theta_n), \cos((i + 1)\theta_n)) \right> + 2 \frac{h_n(X_{n,i}(s))}{\sin \theta_n} \right)^2$$

$$\leq 4 \left| \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right|^2 \left( |y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \right) + \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right|^2$$

since $h_n(x) < \theta_n(T(n)x)^{-1}$ by (2.3).

Use the inequality $(xy)^{1/2} \leq (x + y)/2$ $(x, y > 0)$ for

$$x = 4 \frac{T(n)\tan(\theta_n/2)}{\theta_n/2} \times 4 \left| \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right|^2,$$

$$y = \sum_{i=0}^{n-1} E \left[ \int_0^{\min(t, \tau_n, s)} \left( |y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| + \frac{2n\theta_n}{T(n)C_n(\delta) \sin \theta_n} \right) ds \right].$$

Use also the inequality $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ for $x = |y_{n,i-1}(s) - Y_{n,i-1}(s)|$, $y = |y_{n,i}(s) - Y_{n,i}(s)|$ and $z = |(2n\theta_n)/(T(n)C_n(\delta) \sin \theta_n)|$. Then we obtain (3.19).

4. Proof of Main Result.

In this section we prove Theorem 2 by making use of lemmas given in section 3.

Proof of Theorem 2. For any $t \in (0, T^*)$ and any $\eta > 0$, take $n_2 \in \mathbb{N}$ such that for any $n \geq n_2$

$$t < T_n^* - (T^* - t)/2,$$

$$\sup_{0 \leq s \leq t} d_H(\Omega_{\ell, n}(s), \Omega(s)) < \eta/2,$$

which is possible Theorem 1. Put $\delta = (T^* - t)/2$. Then for any $n \geq n_2$,
\[
\begin{align*}
P(\sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega(s)) \geq \eta) & \leq P(\sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2) \\
& \leq P(\sup_{0 \leq s \leq T_n^*-\delta} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2) \\
& \leq P(\tau_{n,\delta} < T_n^*-\delta) + P(\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2).
\end{align*}
\]

The first probability on the last part of (4.1) can be shown to converge to zero as \(n \to \infty\) as follows: by Chebychev's inequality,

\[
P(\tau_{n,\delta} < T_n^*-\delta) \leq P(\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |X_{n,k}(s) - \ell_{n,k}(s)| \geq C_n(\delta)/(2n))
\]

\[
\leq P(\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)| \geq C_n(\delta)/(4n))
\]

\[
\leq (4nC_n(\delta))^{-1} E[\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|^2]
\]

\[
\to 0, \text{ as } n \to \infty \text{ by Lemma 3,}
\]

since

\[
|X_{n,k}(s) - \ell_{n,k}(s)| \leq |Y_{n,k}(s) - y_{n,k}(s)| + |Y_{n,k-1}(s) - y_{n,k-1}(s)|
\]

by (3.20) and since \(\lim_{k \to \infty} C_k(\delta)^{-1}\) is finite by (14) of [4].

The second probability on the last part of (4.1) can be shown to converge to zero as \(n \to \infty\) as follows: by Chebychev’s inequality,

\[
P(\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2) \leq P(\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)| \geq \eta/2)
\]

\[
\leq (\eta/2)^{-2} E[\sup_{0 \leq s \leq \min(T_n^*-\delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|^2]
\]

\[
\to 0, \text{ as } n \to \infty \text{ by Lemma 3,}
\]
since
\[ d_H(\Omega_{X,n}(s), \Omega_{T,n}(s)) \leq \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|. \]

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