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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on July 30 through August 1 in 2001 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broard spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium almost 25 years ago. Professor Kôji Kubota and Professor Rentaro Agemi who made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

Y. Giga and T. Ozawa

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Programme

第２６回偏微分方程式論札幌シンポジウム

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代表者　　儀我 美一, 小澤 徹
Organizers: Y. Giga and T. Ozawa

記

1. 日時　2001年7月30日（月）～8月1日（水）
2. 場所　北海道大学大学院 理学研究科 5号館 大講義室
　　および 202号室 (数学教室の南向かい)
3. 講演

July 30, 2001 (月)

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Area and Dirichlet integrals

11:00-12:00　M. A. Herrero (Madrid)
The growth of vascular systems : a reaction - diffusion approach

14:00-14:30　*

14:30-15:00　中島 徹 (T. Nakajima), 東北大理院 (Tohoku U.)
Behavior of harmonic maps into spheres around their singular points

15:30-16:00　福泉 麗佳 (R. Fukuizumi), 東北大理院 (Tohoku U.)
Stability of standing waves for nonlinear Schrödinger equations with potentials

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Blow-up rate for semilinear heat equations with subcritical nonlinearity

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* この時間は講演者を囲んで自由な質問の時間とする予定です。
* indicates discussion time. Lecturers in each session are invited to stay in the coffee-tea room during discussion time.

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例年と会場が異なりますのでご注意下さい。

Some results on sequences of maps with equibounded energies

Mariano Giaquinta, Scuola Normale Superiore, Pisa

July 9, 2001

Let $\mathcal{X}$, $\mathcal{Y}$ be two oriented Riemannian manifolds respectively of dimensions $n, m \geq 2$. We shall assume that $\mathcal{Y}$ is compact and without boundary and that its integral 2-homology group $\mathcal{H}_2(\mathcal{Y})$ has no torsion, so that $\mathcal{H}_2(\mathcal{Y}, \mathbb{Z}) = \{\sum_{s=1}^{\bar{s}} n_s[\gamma]_s\}$, $\gamma_1, \dots \gamma_{\bar{s}}$ being integral cycles and $\mathcal{H}_2(\mathcal{Y}, \mathbb{R}) = \mathcal{H}_2(\mathcal{Y}, \mathbb{Z}) \otimes \mathbb{R}$, and for future use we denote by $\omega^1, \dots, \omega^{\bar{s}}$ the harmonic forms such that

$$\int_{\gamma_s} \omega^r = ([\gamma_s]_{\mathbb{R}} | [\omega^r]) = \delta^r_s.$$

$\mathcal{D}_{n,2}$-currents. Every differential n-form $\omega \in \mathcal{D}^n(\mathcal{X} \times \mathcal{Y})$ splits as a sum $\omega = \sum_{k=0}^{\underline{n}} \omega^{(k)}$, $\underline{n} := \min(n, m)$, where the $\omega^{(k)}$'s are the n-forms which contain exactly k differentials in the vertical $\mathcal{Y}$ variables. We denote by $\mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})$ the subspace of $\mathcal{D}^n(\mathcal{X} \times \mathcal{Y})$ of n-forms of the type $\omega = \sum_{k=0}^{2} \omega^{(k)}$, and by $\mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ the dual space of $\mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})$. Every $(n, 2)$-*current* $T \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ splits as $T = \sum_{k=1}^{2} T_{(k)}$, where $T_{(k)}(\omega) := T(\omega^{(k)})$.

$\mathcal{D}$-norm. For $\omega \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ we set

$$||\omega|| := \max \left\{ \sup_{x,y} \frac{|\omega^{(0)}(x,y)|}{1 + |y|^2}, \int_{\mathcal{X}} \sup_{y} |\omega^{(1)}(x,y)|^2 \, d\,\mathrm{vol}_{\mathcal{X}}, \right.$$
$$\left. \int_{\mathcal{X}} \sup_{y} |\omega^{(2)}(x,y)| \, d\,\mathrm{vol}_{\mathcal{X}} \right\},$$

$$||T||_{\mathcal{D}} := \sup \Big\{ T(\omega) \,\Big|\, \omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y}), \ ||\omega||_{\mathcal{D}} \leq 1 \Big\}.$$

It is easily seen that $||T||_{\mathcal{D}}$ is a norm on $\{T \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y}) \,\big|\, ||T||_{\mathcal{D}} < \infty\}$. Let $\{T_k\} \subset \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$; we say that $\{T_k\}$ $\mathcal{D}$-converges to T, $T_k \stackrel{\mathcal{D}}{\rightharpoonup} T$, if $T_k(\omega) \to T(\omega)$ $\forall \omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})$ and $\sup_k ||T_k||_{\mathcal{D}} < \infty$. Clearly $|| \ ||_{\mathcal{D}}$ is $\mathcal{D}$-weakly lower semicontinuous, in particular $T \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ and $||T||_{\mathcal{D}} < \infty$ if $T_k \stackrel{\mathcal{D}}{\rightharpoonup} T$; moreover every sequence $\{T_k\} \subset \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ has a subsequence which $\mathcal{D}$-converges to some $T \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$, $||T||_{\mathcal{D}} < \infty$.

Boundaries. The exterior differential d splits into a horizontal and a vertical differential $d = d_x + d_y$. Since $d_x\omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})$ whenever $\omega \in \mathcal{D}^{n-1,2}(\mathcal{X} \times \mathcal{Y})$ (differentiation in the x variables does not change the number of vertical differentials), $\partial_x T(\omega) := T(d_x\omega)$ defines a boundary operator $\partial_x : \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{D}_{n-1,2}(\mathcal{X} \times \mathcal{Y})$. Instead $\partial_y T$ makes sense only as element of the dual space of

$$\mathcal{Z}^{n,2}(\mathcal{X} \times \mathcal{Y}) := \left\{\omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y}) \,\Big|\, d_y\omega^{(2)} = 0\right\},$$

in fact, if $\omega \in \mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$, $d_y\omega$ belongs to $\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$ if and only if $d_y\omega^{(2)} = 0$.

Boundary data. Let Ω be a bounded domain in $\mathcal{X}$ and let $\widetilde{\Omega}$ be a domain with $\widetilde{\Omega} \supset\supset \Omega$. We prescribe the boundary data by fixing a (smooth) map $\varphi : \widetilde{\Omega} \setminus \overline{\Omega} \to \mathcal{Y}$ and setting

$$W^{1,2}_{\varphi}(\widetilde{\Omega}; \mathcal{Y}) := \left\{u \in W^{1,2}(\widetilde{\Omega}, \mathcal{Y}) \,\Big|\, u = \varphi \text{ on } \widetilde{\Omega} \setminus \overline{\Omega}\right\},$$

and we say that T agrees with φ on the boundary of Ω if $T \in \mathcal{D}^{n,2}(\widetilde{\Omega} \times \mathcal{Y})$ and $T = G_\varphi$ on $(\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathcal{Y}$, G_φ being the current in $\mathcal{D}_{n,2}(\widetilde{\Omega} \times \mathcal{Y})$ integration on the graph of φ.

$\mathcal{D}$-graphs. In order to study weak limits of sequences of maps with equibounded Dirichlet energies, minimization of the Dirichlet integral and concentration phenomena, compare [5], we introduced in [4] the following class of $(n, 2)$-currents

$$\begin{aligned}\mathcal{D}\text{-graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y}) := \Big\{T \in \mathcal{D}_{n,2}(\widetilde{\Omega} \times \mathcal{Y}) \,\Big|\, &||T||_{\mathcal{D}} < \infty, \text{ there exist} \\ &u_T \in W^{1,2}_\varphi(\widetilde{\Omega}, \mathcal{Y}) \text{ and } S_T \in \mathcal{D}_{n,2}(\widetilde{\Omega} \times \mathcal{Y}) \text{ with} \\ &\text{spt } S_T \subset \overline{\Omega} \times \mathcal{Y} \text{ such that } T = G_{u_T} + S_T,\ S_{T(0)} = S_{T(1)} = 0, \\ &\text{and } \partial_x T = 0,\ \partial_y T = 0 \text{ on } \mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y})\Big\}\end{aligned}$$

We proved in [4], see also [5],

1. for $T \in \mathcal{D}\text{-graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ the decomposition $T = G_{u_T} + S_T$ is unique,
2. $\mathcal{D}$-weak limits of graphs of smooth maps $u_k : \widetilde{\Omega} \to \mathcal{Y}$, $u_k = \varphi$ on $\widetilde{\Omega} \setminus \overline{\Omega}$, with equibounded Dirichlet energies belong to $\mathcal{D}\text{-graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$,
3. if $T \in \mathcal{D}\text{-graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$, then $\partial_x S_T = 0$ on $\mathcal{D}^{n-1,2}(\widetilde{\Omega} \times \mathcal{Y})$, and
$$\partial_y S_T = 0 \qquad \text{on } \mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y}),$$
4. $||G_{u_T}||_{\mathcal{D}} = ||u_T||_{W^{1,2}} \leq ||T||_{\mathcal{D}}$, consequently $||S_T||_{\mathcal{D}} \leq 2||T||_{\mathcal{D}}$,
5. $\mathcal{D}\text{-graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ if $\mathcal{D}$-weakly closed.

A simple structure theorem. Let $T \in \mathcal{D}-\text{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$. For any closed 2-form ω in $\mathcal{Y}$ define the $(n-2)$-currents in $\widetilde{\Omega}$ by setting

$$\mathbb{D}(T,\omega)(\zeta) := G_{u_T}(\zeta \wedge \omega), \qquad \mathbb{L}(T,\omega) := S_T(\zeta \wedge \omega).$$

We say in [4], [5] that $\mathbb{D}(T,\omega)$ and $\mathbb{L}(T,\omega)$ have finite mass, spt $(\mathbb{L}(T,\omega)) \subset \overline{\Omega}$, $\partial(\mathbb{D}(T,\omega)+\mathbb{L}(T,\omega)) = 0$ in $\widetilde{\Omega}$, moreover $\mathbb{L}(T,\omega)$ depends only on the cohomology class of ω. Finally, setting for $s = 1, \dots, \bar{s}$

$$\mathbb{D}_s(T) := \mathbb{D}(T,\omega^s), \qquad \mathbb{L}_s(T) := \mathbb{L}(T,\omega^s),$$

we have, see [4] [5],

Proposition 1 *Let $T \in \mathcal{D}-\text{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$. Then*

$$T - G_{u_T} - \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s = 0 \qquad \text{on } \mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y}).$$

In general however

$$S_{T,sing} := T - G_{u_T} - \sum_{s=1}^{\bar{s}} \mathbb{L}_s \times \gamma_s,$$

though completely vertical and null on $\mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y})$, is non zero, see [4].

The 2-dimensional case. In $n = \dim \mathcal{X} = 2$, obviously $\mathcal{D}_{n,2}(\widetilde{\Omega} \times \mathcal{Y}) = \mathcal{D}_2(\widetilde{\Omega} \times \mathcal{Y})$ and $\partial T = \partial_x T + \partial_y T$ is the usual boundary of currents. Consequently $\mathcal{D}$-limits T of smooth graphs G_{u_k} are integer multiplicity rectifiable currents $T \in \mathcal{R}_2(\widetilde{\Omega} \times \mathcal{Y})$, and $\mathcal{D}-\text{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y}) \cap \mathcal{R}_2(\widetilde{\Omega} \times \mathcal{Y})$ is *$\mathcal{D}$-weakly closed.*

It is proved in [4] [5] that every T in $\mathcal{D}-\text{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y}) \cap \mathcal{R}_2(\widetilde{\Omega} \times \mathcal{Y})$ decomposes as

$$T = G_{u_T} + \sum_{i=1}^{k} [[x_i]] \times C_i + R_{T,sing} \tag{1}$$

where $x_i \in \overline{\Omega}$, C_i are integral 2-cycles with non trivial homology and $R_{T,sing}$ is a completely vertical, homologically trivial integer multiplicity rectifiable current supported on a set S not containing $\{x_i\} \times \mathcal{Y}$, $i = 1, \dots, k$. Moreover

$$T = G_{u_T} + \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s + S_{T,sing} \tag{2}$$

with

$$\mathbb{L}_s(T) = \sum_{i=1}^{k} n_{is}[[x_i]], \qquad C_i = \sum_{s=1}^{\bar{s}} n_{is}\gamma_s + S_{T,sing} \llcorner \Big(\cup \{x_i\} \times \mathcal{Y} \Big), \; n_{is} \in \mathbb{Z}.$$

If moreover T is in the sequential weak closure of smooth graphs, then the homology class of each C_i contains a Lipschitz image of S^2, shortly *C_i is of spherical type.*

Definition 2 *Let* $\dim \mathcal{X} = 2$. *We denote by* $\mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ *the class of integer multiplicity rectifiable currents* $T \in \mathcal{D}\text{-}\mathrm{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ *which decomposes as in* (1) *where the* C_i*'s are of spherical type.*

Notice that $\mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ is $\mathcal{D}$-closed and contains the $\mathcal{D}$-limits of sequences of smooth graphs.

The n-dimensional case. We assume $\mathcal{X} = \mathbb{R}^n$, $n \geq 3$, $\widetilde{\Omega} \subset \mathbb{R}^n$, and $\mathcal{Y}$ isometrically embedded in $\mathbb{R}^N$. Denote by $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ the orthogonal projection on the first factor. Let P be an oriented two plane in $\mathbb{R}^n$, and $P_t := P + \sum_{i=1}^{n-2} t_i \nu_i$ the family of oriented two planes parallel to P, $t = (t_1, \dots, t_{n-2}) \in \mathbb{R}^{n+N}$, $\mathrm{span}(\nu_1, \dots, \nu_{n-2})$ being the orthogonal subspace to P. Similarly to the case of normal current, for every $T \in \mathcal{D}_{n,2}(\widetilde{\Omega} \times \mathcal{Y})$ with $||T||_{\mathcal{D}} < \infty$, the *slice of* T over $\pi^{-1}(P_t)$ is a well defined current $T \llcorner \pi^{-1}(P_t)$ in $\mathcal{D}_2(((\widetilde{\Omega} \cap P_t) \times \mathcal{Y})$ for $\mathcal{H}^{n-2}$-a.e. t with finite $\mathcal{D}$-norm; moreover, whenever $T_k \stackrel{\mathcal{D}}{\rightharpoonup} T$, for $\mathcal{H}^{n-2}$-a.e. t passing to a subsequence, we have $T_k \llcorner \pi^{-1}(P_t) \stackrel{\mathcal{D}}{\rightharpoonup} T \llcorner \pi^{-1}(P_t)$; finally, for $\mathcal{H}^{n-2}$-a.e. t,

$$\partial_x(T \llcorner \pi^{-1}(P_t)) = \partial_x T \llcorner \pi^{-1}(P_t), \qquad \partial_y(T \llcorner \pi^{-1}(P_t)) = \partial_y T \llcorner \pi^{-1}(P_t).$$

If moreover $T \in \mathcal{D}\text{-}\mathrm{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$, $T = G_{u_T} + \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s$ in $\mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y})$, then for $\mathcal{H}^{n-2}$-a.e. t the current $T \llcorner \pi^{-1}(P_t)$ is in $\mathcal{D}\text{-}\mathrm{graph}_\varphi((\widetilde{\Omega} \cap P_t) \times \mathcal{Y})$ and

$$G_{u_T} \llcorner \pi^{-1}(P_t) = G_{u_{T|P_t}}, \qquad \mathbb{L}_s(T \llcorner \pi^{-1}(P_t)) = \mathbb{L}_s \llcorner P_t.$$

In any dimension $n \geq 3$ we now set

Definition 3 *We say that* $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ *if* $T \in \mathcal{D}\text{-}\mathrm{graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ *and for any 2-plane* $P \in \mathbb{R}^n$ *and for* $\mathcal{H}^{n-2}$-a.e. t *the two dimensional current* $T \llcorner \pi^{-1}(P_t)$ *belongs to* $\mathrm{cart}^{2,1}_\varphi((\widetilde{\Omega} \cap P_t) \times \mathcal{Y})$.

¿From the previous calculations we easily infer

Theorem 4 $\mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ *is wealkly closed.*

We also have, compare [10], as consequence of the rectifiability criterion of B. White [9]

Theorem 5 *Let* $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$, $T = G_{u_T} + \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s$ *in* $\mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y})$. *Then* $\mathbb{L}_s(T)$, $s = 1, \dots, \bar{s}$, *are i.m. rectifiable* $(n-2)$*-currents.*

We can also see that every $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ decomposes as

$$T = G_{u_T} + \sum_{q \in \mathcal{H}_2(\mathcal{Y})} \mathbb{L}_q \times R_q \qquad \text{on } \mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y})$$

where $\mathbb{L}_q$ is an i.m. rectifiable $(n-2)$-current with pairwise disjoint $\mathcal{L}_q$'s, and $R_q \in q$ is of spherical type.

The Dirichlet energy in $\mathrm{cart}^{2,1}$**.** Let $T \in \mathcal{D}\mathrm{-graph}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$. Define the $(n,2)$-total variation by

$$||T|| := \sup\left\{ T(\omega) \,\Big|\, ||\omega|| \leq 1,\ \omega \in \mathcal{D}^{n,2}(\widetilde{\Omega} \times \mathcal{Y}) \right\}$$

and the Dirichlet density as the function of $x \in \widetilde{\Omega}$, $y \in \mathcal{Y}$, $\xi \in \Lambda_n \mathbb{R}^{n+N}$ given by

$$F(x,y,\xi) := \sup \left\{ \phi(\xi) \,\Big|\, \phi : \Lambda_n \mathbb{R}^{n+N} \to \mathbb{R} \text{ linear },\ \phi(M(G)) \leq \frac{1}{2}|G|^2 \right. \\ \left. \text{for all linear maps } G : T_x\widetilde{\Omega} \to T_y\mathcal{Y} \right\}$$

The Dirichlet integral extends then to $\mathcal{D}$-graphs, $T = G_{u_T} + S_T$, compare [5], as

$$\mathcal{D}(T) := \int F(x, y, \overrightarrow{T})\, d||T||,$$

$\overrightarrow{T}$ being the Radon-Nikodym derivative $dT/d||T||$, and one has

$$\mathcal{D}(T) = \frac{1}{2} \int_{\widetilde{\Omega}} |Du_T|^2\, dx + \int F(x, y, \overrightarrow{S}_T)\, d||S_T||.$$

We do not have any explicit formula for the second term on the right hand side as in general S_T is not a product current even for T in $\mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$. However for energy minimizing currents we do have an explicit formula.

Theorem 6 *Let $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$ be a minimizer or a minimizer in its homology class*

$$[T] := \left\{ R \in \mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y}) \,\Big|\, R = T \text{ on } \mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y}) \right\}$$

for the Dirichlet integral. Write T as

$$T = G_{u_T} + \sum_{q \in \mathcal{H}_2(\mathcal{Y})} \mathbb{L}_q \times R_q \qquad \text{on } \mathcal{Z}^{n,2}(\widetilde{\Omega} \times \mathcal{Y}). \qquad R_q \in q,$$

and set

$$T^H := G_{u_T} + \sum_{q \in \mathcal{H}_2(\mathcal{Y})} \mathbb{L}_q \times S_q,$$

S_q being the least mass integral cycle in q. Then $T^H \in \mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega} \times \mathcal{Y})$, $T^H \in [T]$ and $\mathcal{D}(T^H) = \mathcal{D}(T)$; moreover

$$\mathcal{D}(T) = \frac{1}{2} \int |Du_T|^2\, dx + \sum_{q \in \mathcal{H}_2(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\mathbf{M}(S_q). \tag{3}$$

Partial regularity. Finally the explicit formula (3) for the energy of minimizers allows us to prove now a partial regularity theorem for u_T, compare [10]: u_T is smooth in $\widetilde{\Omega}$ except on a closed set of Hausdorff dimension not greater than $n-2$.

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THE GROWTH OF VASCULAR SYSTEMS :
A REACTION – DIFFUSION APPROACH.

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1.- INTRODUCTION : ON FORM AND STRUCTURE .

A problem that has fascinated mankind since the early stages of scientific thought is that of the formation of large structures from their simpler components. Roughly speaking, the main underlying question can be stated as follows: Is it possible to predict the geometric structure of a complex aggregate , and its subsequent dynamical evolution, starting from the knowledge of its constitutive elements and the type of interactions existing among them?.

In principle, in any given situation there might be many theoretically possible structures compatible with the data corresponding to their elementary components . Of these , Nature has long been suspected of usually making a restricted choice of admissible (or actually observable) ones . For instance, as early as in 1611, Johannes Kepler suggested that the hexagonal form of snowflakes should be due to the fact that their structure is ¨particularly efficient in some sense ¨, and that such form could not explained merely by mathematical arguments , but should instead be obtained from a knowledge of the physical laws governing phase transitions, or as the author said , in the ¨chemists´studies of the form of the salts ¨ (cf. (K)) .The subsequent development of science has led to the consideration of many situations where the interplay between structure, form and function poses deep and challenging questions, as illustrated , for instance, by modern studies on the dynamics of protein folding.

In the following we shall consider two examples of models describing the generation of a complex structure under given physical laws. The first one is a simple case of polymer formation in a dilute solution . The second one deals with the evolution of a filamentary structure under the competing effects of autocatalysis, lateral inhibition and attraction from the surrounding medium.

2.- AN EARLY AGGREGATION MODEL.

Perhaps the simplest situation for which a mathematical aggregation model has been derived is that described by Smoluchowski´s theory of colloid nucleation (see (S) , (C)) . This author proposed a kinetic mechanism for the formation of large chains consisting of individual particles (say, monomers), under the assumptions that one is given an initial soup consisting of colloidal particles dissolved in a solvent , and that :

I) Solute particles undergo random (Brownian) motion,

II) When an electrolyte is added to the solution, each colloidal particle is surrounded by a ¨sphere of influence ¨ of a certain radius R,

III) When particles come within a distance R from each other, they stick together to form a larger unit .

Double, triple... particles then follow their respective Brownian motions, at a rate consequent to their increased size.

In such a way, Smoluchowski proposed a system of infinitely coupled and nonlinear differential equations, whose subsequent analysis (and that of related problems, as for instance reaction – diffusion systems derived when subsequent cluster diffusion is taken into account) has led to a number of interesting results (and open questions) , both at the physical and mathematical level, see for instance (HVW).

3.- REACTION – DIFFUSION DRIVEN VASCULAR NETWOKS.

In the example just considered, the aggregation mechanism consists just in the intermolecular atraction among particles which are otherwise subject to random motion. When more complicated situations are considered, various transport and reaction mechanisms may play a role in determining the final configuration of the aggregate . We shall briefly discuss below a particularly simple approach to an exceedingly complex subject.

It is well known that, in the course of their evolution, higher organisms rapidly grow to a size where passive diffusion becomes inadequate to supply tissues with oxygen, water , nutrients and information. Nature has found a way to solve such puzzling question, which consists in the invention of complex- shaped organs made up of long – branching filaments that eventually yield highly ramified networks that efficiently expand into the surrounding organic matrix. Typical examples of such organs are , among many others, the blood vessels, the trachaeae of insects and the nervous system of vertebrates

From a biological point of view, the genetic programs that direct the formation of the tree- like branching structure of some animal organs (for instance, the Drosophila fly tracheal system) have begun to be elucidated only recently (see (MK) for a survey). On the other hand, much effort is being currently devoted to understanding a related problem : angiogenesis . This last may be shortly described as the unfolding of the system of blood vessels, both under normal and pathological conditions (see (Y) for a recent review).

Even in a simple setting (corresponding, for instance, to the airways of Drosphila) , the problem is a challenging one . Indeed , there are hundred to millions of branches in any such organ, and a huge amount of patterning information is required to configure the whole network. In particular, at each branch, the following instructions have to be codified :

I) Where the branch buds, and the direction in which it grows,

II) The size and shape of each branch,

III) When and where along the branch a new generation of (secondary) branches has to appear.

The question of whether insight on the nature and function of such complex organs can be gained from mathematics is far from being settled . In this lecture, I shall briefly describe a system of differential equations that was proposed by H. Meinhardt (see (M)) , as a first attempt to model such type of situations . To derive it, a number of assumptions are made. For instance, only reaction and diffusion are assumed to govern the unfolding of the net , and this is assumed to be the consequence of the interplay of just three substances , termed as activator, inhibitor and growth factor. Futhermore, also the geometry of the net is drastically restricted, by considering only two – dimensional evolution. Even in such simplified setting, the system thus obtained leads to a number of interesting mathematical questions, some of which are to be found in recent work with D. Andreucci and J. J. L. Velázquez (AHV) , and will be shortly described in the lecture.

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Behavior of harmonic maps into spheres around their isolated singular points

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1 Definitions and known results

Let Ω^n be a bounded domain in the Euclidean space $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ be the $(n-1)$-dimensional sphere. We define the Sobolev class $W^{1,2}(\Omega, \mathbb{S}^{n-1})$ by

$$W^{1,2}(\Omega, \mathbb{S}^{n-1}) = \{u \in W^{1,2}(\Omega, \mathbb{S}^{n-1}) |\ |u| = 1 \text{ (a.e.)}\}$$

We consider the energy functional

$$\mathbf{E}(u) = \int_\Omega |\nabla u|^2\, dx$$

over $W^{1,2}(\Omega, \mathbb{S}^{n-1})$. Here we treat the minimum points and critical points of $\mathbf{E}$.

Definition 1 (energy minimizing map, harmonic mao)
Suppose $u \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$.
(1) We say that u is an *energy minimizing map* if

$$\mathbf{E}(u) \leq \mathbf{E}(v)$$

holds for any $v \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$ with $u - v \in W_0^{1,2}(\Omega, \mathbb{R}^n)$.
(2) We say that u is a *harmonic map* if u is a weak solution of Euler-Lagrange equation of $\mathbf{E}$

$$\left.\frac{d}{dt}\mathbf{E}(u_t)\right|_{t=0} = 0$$

Here, $u_t = \dfrac{u + t\phi}{|u + t\phi|}$, $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$.

The energy minimizing map is a generalization of the harmonic function, but in general this is not continuous. We call a discontinuous point of the energy minimizing map a singular point. The typhical example of discontinuous energy minimizing maps is $x/|x| \in W^{1,2}(\mathbb{B}^n, \mathbb{S}^{n-1})$ $(n \geq 3)$. This map is energy minimizing and has a singular point 0. We are interested in the behavior of the energy minimizing maps around their isolated singular points. In this talk, we treat only inteiror singular points, so we always assume that the energy minimizing maps are continuous near the boundary. In the case $n = 3$, Brezis-Coron-Lieb proved the following theorem.

Theorem 1 (Brezis-Coron-Lieb)
If $u \in W^{1,2}(\Omega^3, \mathbb{S}^2)$ is an energy minimizing map and $p \in \Omega$ is an isolated singular point of u, then we have

$$\deg(u, p) = \pm 1$$

Moreover u behaves like $A\dfrac{x-p}{|x-p|}$ for some $A \in O(3)$ around p.

But if $n = 3$, we have no results to determine the degree of non-minimum harmonic maps.

2 Results

Here, we treat the case $n = 4$. We have the following.

Theorem 2 (N)
If $u \in W^{1,2}(\Omega^4, \mathbb{S}^3)$ is an energy minimizing map and $p \in \Omega$ is an isolated singular point of u, then we have

$$\deg(u, p) = 0 \text{ or } \pm 1$$

Moreover, if $\deg(u, p) = \pm 1$, u behaves like $A\dfrac{x-p}{|x-p|}$ for some $A \in O(4)$ around p.

So, if we consider the energy minimizing maps, there may be little differences between $n = 3$ and $n = 4$. But if we consider the non-minimum harmonic maps, the results are quite different. In fact, in the case n=4, we can prove the similar results of non-minimum harmonic maps if we impose some conditions.

Definition 2 (stationary harmonic map)
Let $u \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$ be a harmonic map. We call u is a stationary harmonic map if

$$\left.\frac{d}{dt}\right|_{t=0} \mathbf{E}(u(x + t\eta(x))) = 0.$$

for $\forall \eta \in C_0^\infty(\Omega, \mathbb{R}^n)$.

The Energy minimizing maps are always stationary harmonic maps, and by definitions, stationary harmonic maps are aloways harmonic maps. But the converse is not true.

Next we consider the condition of the second variation of $\mathbf{E}$. We define the second variation as follows.

Definition 3 (second variation)
Let $u \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$ be a harmonic map. We define the second variation of $\mathbf{E}$ at u by

$$\delta_u^2 \mathbf{E}(\phi) = \left.\frac{d^2}{dt^2}\mathbf{E}(u_t)\right|_{t=0}$$

$\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$ with $u \cdot \phi = 0$.

Definition 4 (weakly stable)
Let $u \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$ be a harmonic map. u is called weakly stable if

$$\delta_u^2 \mathbf{E}(\phi) \geq 0 \quad \text{for } \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^n) \text{ with } u \cdot \phi = 0$$

Now, we have the following result.

Theorem 3 (N)
If $u \in W^{1,2}(\Omega^4, \mathbb{S}^3)$ is stable stationary harmonic map and $p \in \Omega$ is an isolated singular point of u, then we have

$$\deg(u, p) = 0 \text{ or } \pm 1$$

Moreover, if $\deg(u,p) = \pm 1$, u behaves like $A\dfrac{x-p}{|x-p|}$ for some $A \in O(4)$ around p.

Stability of Standing Waves for Nonlinear Schrödinger Equations with Potentials

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1. Introduction

We consider the stability of standing wave solutions for the nonlinear Schrödinger equations with a real valued potential $V(x)$:

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{NLS}$$

where $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$.

By a standing wave, we mean a solution of (NLS) of the form

$$u_\omega(t,x) = e^{i\omega t}\phi_\omega(x),$$

where $\omega \in \mathbb{R}$, and $\phi_\omega(x)$ is a ground state of

$$-\Delta\phi + V(x)\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n. \tag{SP}$$

The main purpose of this talk is to show that when $-\Delta + V(x)$ has the first simple eigenvalue λ_1, under some suitable assumptions on $V(x)$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (NLS) is stable for ω with $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$ and for all $1 < p < 2^* - 1$.

In what follows, we consider hamonic potentials

$$V(x) = \sum_{j=1}^{n} x_j^2$$

for the sake of simplicity. We will remark on more general potentials later.

Equation (NLS) with a harmonic potential is known as a model to describe the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap (see, e.g., [14]).

For (NLS), the Cauchy problem in the energy space

$$\Sigma = \{v \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx < \infty\},$$

$$(v,w)_\Sigma := \mathrm{Re}\int_{\mathbb{R}^n} (v(x)\overline{w(x)} + \nabla v(x)\cdot\overline{\nabla w(x)} + V(x)v(x)\overline{w(x)})dx$$

was studied by Oh [12]. Here, we regard Σ as a real Hilbert space consisting of complex valued functions. It was proved in [12] that for any $u_0 \in \Sigma$ there exist $T > 0$ and a unique solution $u(t) \in C([0,T], \Sigma)$ of (NLS) with $u(0) = u_0$, and the conservation of energy and charge holds:

$$E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad t \in [0,T],$$

where E is the energy functional defined on Σ by

$$E(v) := \frac{1}{2}||\nabla v||_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \frac{1}{p+1}||v||_{L^{p+1}}^{p+1}.$$

We say that $\phi(x)$ is a ground state of (SP) if $\phi(x)$ is a nontrivial solution of (SP) and $S_\omega(\phi) \leq S_\omega(v)$ for any nontrivial solution v of (SP), where

$$S_\omega(v) := \frac{1}{2}||\nabla v||_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(x)|v(x)|^2 dx + \frac{\omega}{2}||v||_{L^2}^2 - \frac{1}{p+1}||v||_{L^{p+1}}^{p+1}.$$

The existence of a ground state in Σ of (SP) is proved for $\omega \in (-\lambda_1, \infty)$ by the standard variational argument, where

$$\lambda_1 := \inf\{||\nabla v||_{L^2}^2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx : \ ||v||_{L^2} = 1, \ v \in \Sigma\}. \tag{E}$$

The uniqueness of the positive radial solution in Σ of (SP) was proved for $\omega \in (-\lambda_1, \infty)$ (see, e.g., [8], [9]). We note that if $\phi_\omega(x)$ is a solution of (SP), then $e^{i\omega t}\phi_\omega(x)$ is a solution of (NLS).

We recall some known results. First, we consider the case $V(x) \equiv 0$. For any $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of

$$-\Delta\psi + \omega\psi - |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^n \tag{NLS0}$$

in $H^1(\mathbb{R}^n)$ (see [11] for the uniqueness), and the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (NLS) with $V(x) \equiv 0$ is stable for any $\omega > 0$ if $p < 1+4/n$, and unstable for any $\omega > 0$ if $p \geq 1+4/n$ (see [1, 2, 15]). Therefore, we see that $p = 1+4/n$ is the critical power for the stability and instability of standing waves for (NLS0).

Meanwhile, in the presence of $V(x)$, it was showed in [4] that there is a sequence of $\{\omega_k\}$ approaching $-\lambda_1$, for which the standing wave solutions $e^{i\omega_k t}\phi_{\omega_k}$ of (NLS) are stable even if $p \geq 1 + 4/n$ (see also [13]). Also, we proved in [5], [6] that under suitable assumptions on $V(x)$ and $p < 1 + 4/n$ (resp. $p > 1 + 4/n$), the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (NLS) is stable (resp. unstable) for sufficiently large $\omega > 0$.

2. Main Result

Definition. We say that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (NLS) is stable in Σ if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $u_0 \in \Sigma$ satisfies

$$\inf_{\theta\in\mathbb{R}} ||u_0 - e^{i\theta}\phi_\omega||_\Sigma < \delta,$$

then the solution $u(t)$ of (NLS) with $u(0) = u_0$ exists for all $t \geq 0$ and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}} ||u(t) - e^{i\theta}\phi_\omega||_\Sigma < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in Σ.

Our main result in this talk is the following.

Theorem 1. Let $1 < p < 2^* - 1$. There exists $\omega^* > -\lambda_1$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (NLS) is stable in Σ for any $\omega \in (-\lambda_1, \omega^*)$.

By the general theory in Grillakis, Shatah and Strauss [7], under some assumptions on the spectrum of a linearized operator, the standing wave solution $e^{i\omega_1 t}\phi_{\omega_1}(x)$ is stable (resp. unstable) if the function $\|\phi_\omega\|_2^2$ is strictly increasing (resp. decreasing) at $\omega = \omega_1$. In the case $V(x) \equiv 0$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, it is easy to check the increase and decrease of $\|\psi_\omega\|_2^2$. However, it seems difficult to check this property of $\|\phi_\omega\|_2^2$ for general $V(x)$. So, for the proof of Theorem 1, we use the following sufficient condition for stability.

Proposition 1. ([7, 10]) If there exists $\delta > 0$ such that

$$\langle S''_\omega(\phi_\omega)v, v\rangle \geq \delta\|v\|_\Sigma^2$$

for any $v \in \Sigma$ with $\mathrm{Re}(v, \phi_\omega)_{L^2} = 0$ and $\mathrm{Re}(v, i\phi_\omega)_{L^2} = 0$, then the standing wave solution $e^{i\omega t}\phi_\omega$ is stable in Σ.

The condition $\mathrm{Re}(v, \phi_\omega)_{L^2} = 0$ corresponds to the conservation of charge. $\mathrm{Re}(v, i\phi_\omega)_{L^2} = 0$ is related to the gauge invariance. Proposition 1 means that if the action $S_\omega(u)$ is minimized at $u = \phi_\omega$, for u with $\|u\|_{L^2} = \|\phi_\omega\|_{L^2}$, then the standing wave solution is stable. For a bounded potential $V(x)$, Rose and Weinstein [13] claimed that when $-\Delta + V(x)$ has the first eigenvalue λ_1, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (NLS) is stable for ω such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$ (see also [4]). To verify the stability condition of Grillakis, Shatah and Strauss theory [7], they investigated the behavior of $\|\phi_\omega\|_{L^2}$ by the bifurcation theory, but it seems that there would remain a possibility of oscillations of $\|\phi_\omega\|_{L^2}$ and the extraction of a sequence $\{\omega_k\}$ would be required. Theorem 1 gives an improvement of those results of Rose and Weinstein [13] and the author [4] since the standing wave solution $e^{i\omega t}\phi_\omega$ of (NLS) is stable for any ω close to $-\lambda_1$ without extracting a sequence $\{\omega_k\}$.

3. Outline of proof for Theorem 1

For any $v \in \Sigma$, let $v_1(x) = \mathrm{Re}\, v(x)$ and $v_2(x) = \mathrm{Im}\, v(x)$. We explicitly write $\langle S''_\omega(\phi_\omega)v, v\rangle$ to obtain

$$\begin{aligned}
&\langle S''_\omega(\phi_\omega)v, v\rangle = \langle L_{1,\omega}v_1, v_1\rangle + \langle L_{2,\omega}v_2, v_2\rangle,\\
&\langle L_{1,\omega}v_1, v_1\rangle = \int_{\mathbb{R}^n} \left(|\nabla v_1(x)|^2 + \{V(x) + \omega - p\phi_\omega^{p-1}(x)\}|v_1(x)|^2\right) dx,\\
&\langle L_{2,\omega}v_2, v_2\rangle = \int_{\mathbb{R}^n} \left(|\nabla v_2(x)|^2 + \{V(x) + \omega - \phi_\omega^{p-1}(x)\}|v_2(x)|^2\right) dx,\\
&\mathrm{Re}(v, \phi_\omega)_{L^2} = (v_1, \phi_\omega)_{L^2}, \quad \mathrm{Re}(v, i\phi_\omega)_{L^2} = (v_2, \phi_\omega)_{L^2}.
\end{aligned}$$

Since $L_{2,\omega}\phi_\omega = 0$ and $\phi_\omega > 0$, it follows that there exists $\delta_2 > 0$ such that

$$\langle L_{2,\omega}v_2, v_2\rangle \geq \delta_2\|v_2\|_\Sigma^2$$

for any $v_2 \in \Sigma$ satisfying $(v_2, \phi_\omega)_{L^2} = 0$. Therefore, it suffices to show the following Lemma.

Lemma 1. Let $1 < p < 2^* - 1$. There exists $\omega^* > -\lambda_1$ such that for any $\omega \in (-\lambda_1, \omega^*)$ there exists $\delta > 0$ such that

$$\langle L_{1,\omega} v_1, v_1 \rangle \geq \delta \|v_1\|_\Sigma^2$$

for any $v_1 \in \Sigma$ with $(v_1, \phi_\omega)_{L^2} = 0$.

To prove Lemma 1, we follow the idea of Esteban and Strauss [3]. When $\omega \to -\lambda_1$, if the effect of the nonlinear term $|\phi|^{p-1}\phi$ in (SP) would disappear, we could have the linear equation

$$-\Delta\phi + V(x)\phi = \lambda_1 \phi, \quad x \in \mathbb{R}^n, \tag{LSP}$$

which has the first simple eigenfunction $\Phi(x)$ with $\|\Phi\|_{L^2} = 1$, corresponding to λ_1. So, we expect that $\phi_\omega(x)/\|\phi_\omega\|_{L^2}$ may converge to the solution $\Phi(x)$ as $\omega \to -\lambda_1$ in some sense. Since the standing wave solution of (LSP) is stable in Σ, we also expect that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (NLS) may be stable in Σ when ω is close to $-\lambda_1$, for all $1 < p < 2^* - 1$.

If Lemma 1 were false, there would be a sequence $\{\omega_j\} \to -\lambda_1$ and functions $\{v_j\} \subset \Sigma$ such that

$$\begin{aligned} &\lim_{j\to\infty} \langle L_{1,\omega_j} v_j, v_j \rangle \leq 0, \\ &\|v_j\|_\Sigma^2 = 1, \\ &(v_j, \phi_{\omega_j})_{L^2} = 0. \end{aligned}$$

We consider the passage to the limit as $\{\omega_j\} \to -\lambda_1$ and derive a contradiction from the following Lemma.

Lemma 2. Let $\hat{\phi}_\omega(x) := \phi_\omega(x)/\|\phi_\omega\|_{L^2}$. Then, we have

$$\lim_{\omega\to-\lambda_1} \|\hat{\phi}_\omega - \Phi\|_\Sigma = 0.$$

4. More general potentials

We conclude this report with some remarks for more general potentials. Theorem 1 holds for $V(x)$ satisfying (I) and (II).

(I) There exist $V_1(x)$ and $V_2(x)$ such that $V(x) = V_1(x) + V_2(x)$ satisfying the following:
(V1) $V_1(x) \in C^2(\mathbb{R}^n)$ and there exist positive constants m and C such that $0 \leq V_1(x) \leq C(1 + |x|^m)$ on $\mathbb{R}^n$.
(V2) There exists q such that $q \geq 1$, $q > n/2$ and $V_2(x) \in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$.

(II)

(V3) λ_1 is the simple eigenvalue and all minimizing sequences of (E) are relatively compact in $L^2(\mathbb{R}^n)$.

(V4) There exists $\delta_1 > \lambda_1$ such that

$$\langle(-\Delta + V(x))v, v\rangle \geq \delta_1 \|v\|_{L^2}^2$$

for all $v \in X$ with $\mathrm{Re}(\Phi, v)_{L^2} = 0$, where Φ is the eigenfunction corresponding to λ_1 with $\|\Phi\|_{L^2} = 1$.

Here,

$$X := \{v \in H^1(\mathbb{R}^n, \mathbb{C}) : V_1(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}$$

with the inner product

$$(v, w)_X := \mathrm{Re} \int_{\mathbb{R}^n} (v(x)\overline{w(x)} + \nabla v(x) \cdot \overline{\nabla w(x)} + V_1(x)v(x)\overline{w(x)})dx.$$

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Blow up rate for semilinear heat equation with subcritical nonlinearity

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1 Problem and Main Results

This is ajoint work Y.Giga and S.Matsui. The aim of this paper is to prove a blowup rate estimate for solutions of the semilinear heat equation of form:

$$\text{(P)} \quad \begin{cases} u_t &= \Delta u + |u|^{p-1}u, \quad \text{in } \mathbf{R}^n \times (0,\ T), \\ u(0,\ x) &= u_0(x), \qquad\quad\ \text{in } \mathbf{R}^n, \end{cases}$$

for subcritical power $p \in (1,\ \frac{n+2}{n-2})$.

Main theorem

Let u be a solution with blowup time T. Then

$$(GR) \quad \|u(t)\ ;\ L^\infty(\mathbf{R}^n)\| \leq C(T-t)^{-\frac{1}{p-1}},$$

with some constant independent of time t, provided that $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$ or $n = 1,\ 2$ and arbitrary $p > 1$.

2 Known Results

1. Subcritical exponent i.e. $1 < p < \frac{n+2}{n-2}$
   [5] (GR) holds when $1 < p < \frac{3n+8}{3n-4}$ or $u_0 \geq 0$.

2. Critical exponent i.e. $p = \frac{n+2}{n-2}$
   [4] (GR) does not hold when $n = 3, 4, 5, 6$.

3. General exponent i.e. $p > 1$ and on a bounded, convex domain
   [3] (GR) holds when $\Delta u_0 + |u_0|^{p-1}u_0 \geq 0$ so that $u_t \geq 0$.

4. Supercritical exponent i.e. $p > \frac{n+2}{n-2}$
   [6] (GR) does not hold when $n \geq 11,\ p > \frac{n-2\sqrt{n-1}}{n-4-2\sqrt{n-1}}$

3 Uniform bound for rescaled equation

Following the method developed by [5], we use rescaled variables,

$$w_a(y,s) = (T-t)^{\beta} u(a + y\sqrt{t-T}, t),\ \ \beta = \frac{1}{p-1},\ \ s = -\log(T-t).$$

$$\text{Growth rate estimate} \Longleftrightarrow \sup_{a\in\mathbf{R}^n} |w_a(0,\ s)| < \infty$$

We shall prove a uniform bound for w independent of a.
The bounded $\overline{\lim}_{s\to\infty} |w_a(y,\ s)| < \infty$ is not enough for our purpose. We need uniformity in $a \in \mathbf{R}^n$. We shall adjust Quittner's approach(a kind of bootstrap argument) to our rescaled equation.

4 Local energy

Let φ be a cutoff function defined by

$$\varphi \in C_0^{\infty}(\mathbf{R}^n),\ \ 0 \le \varphi \le 1 \text{ and } \varphi(y) = \begin{cases} 1 & y \in B_R \\ 0 & y \in B_{2R} \end{cases}.$$

B_R is open ball of radius R centerd at the origin. We set $\rho(y) = \exp(-\frac{|y|^2}{4})$. Using φ, we localize the equation of w in [5] into the following form:

$$\text{(LWRP)} \qquad \rho(\varphi w)_s - \nabla\cdot(\rho\nabla(\varphi w)) + \nabla\cdot(\rho w \nabla\varphi) + \rho\nabla\varphi\cdot\nabla w + \beta\varphi w\rho - \varphi|w|^{p-1}w\rho = 0$$

We define two types of *local energies* as follows,

$$E_{\varphi}[w](s) = \tfrac{1}{2}\int(|\nabla(\varphi w)|^2 + (\beta\varphi^2 - |\nabla\varphi|^2)|w|^2)\rho dy - \tfrac{1}{p+1}\int\varphi^2|w|^{p+1}\rho dy$$
$$\mathcal{E}_{\varphi}[w](s) = \tfrac{1}{2}\int\varphi^2(|\nabla w|^2 + \beta|w|^2)\rho dy - \tfrac{1}{p+1}\int\varphi^2|w|^{p+1}\rho dy.$$

We drive several estimates involved local energies. Let L^2_ρ be the weighted L^2 space with weight ρ.
Let *global energy* $E[w](s)$ denote $E_\varphi[w](s)$ when $\varphi \equiv 1$.
Lemma 1 (Local energy estimate).
(1)$\frac{1}{2}\frac{d}{ds}\int|\varphi w|^2\rho dy = -2E_\varphi[w](s) + \frac{p-1}{p+1}\int\varphi^2|w|^{p+1}\rho dy$,
(2)$\forall t \ge s_0,\ \exists K_1 = K_1(E[w](s_0), n, p) > 0$ s.t. $\int_t^{t+1}\mathcal{E}_\varphi[w](s)ds \le K_1$,
(3)$\exists K_2 = K_2(E[w](s_0), n, p, |\nabla\varphi|) > 0$ s.t. $\frac{d}{ds}\mathcal{E}_\varphi[w](s) \le K_2(1 + \|w_s(s)\ ;\ L^2_\rho(\mathbf{R}^n)\|)$,
(4)$\forall t \ge s_0 + 1,\ \mathcal{E}_\varphi[w](t) \le K_1 + K_2$.
idea of the proof
Multiplying (LWRP) with φw, integration by parts we obtain (1). The estimate (2) follows easily from global energy. Multiplying the divergence form of w by $(\varphi w)_s$, integration by parts, and using Hölder's inequality we obtain (3). Using the next elementary lemma, (4)

follows from (2)and (3).

Lemma 2.
Assume that $f \in C^1(s_0,\ \infty)$, $m \in L^1_{loc}(s_0,\ \infty)$ and $m \geq 0$. Let F and Ibe

$$F \stackrel{\text{def}}{=} \sup_{t>s_0} \int_t^{t+1} f(s)ds < \infty,$$
$$I \stackrel{\text{def}}{=} \sup_{t>s_0} \int_t^{t+1} m(s)ds < \infty.$$

Assume that $f'(t) \leq m(t) \quad (t > s_0)$.
Then $f(t) \leq F + I \quad (t \geq s_0 + 1)$ holds.

5 Integral estimate

The following *integral estimate* plays a crucial role to show (GR) for subcritical exponent.
Lemma.
$\forall q \geq 2\ \exists R > 0\ \exists C > 0\ (I)\ \int_t^{t+1} \|w\ ;\ W^{1,2}(B_R)\|^{2q} ds \leq C$ for all $t \geq s_0 + 1$.
idea of the proof
We use an inductive argument as in [7]. The estimate (I) holds for $q = 2$ by Lemma in §4. By an interpolation theorem and local energies estimate we get

$$(U)\ \sup_{t \geq s_0+1} \|w(t)\ ;\ L^{\lambda}(B_R)\| \leq C$$

for $\lambda = \lambda(2)$. Then an interpolatin thorem, local energies estimate(Lemma1) and L^p-L^q estimate for equation of φw then we get (I) for better $\tilde{q}$ by taking R smaller. We repeat this procedure to get (U) and (I). Finally for sufficiently large q so that $\lambda(q)$ in (U) is close to $\frac{2n}{n-2}$(but less this number). Then using standard regularity theorem for parabolic equation yields a uniform bound for $\sup_{t \geq s_0+1} \|w(t)\ ;\ W^{1,2}(B_R)\|$ provided that $p < \frac{n+2}{n-2}$. Once $\sup_{t \geq s_0+1} \|w(t)\ ;\ W^{1,2}(B_R)\|$ is estimated, we obtain $\sup_{t \geq s_0+1} \|w(t)\ ;\ L^{\infty}(B_R)\| \leq C$ with C depending only on $\mathcal{E}_{\varphi}(w_0), |\nabla\varphi|$. Thus we have (GR).

6 Related problem

There is a related problem. For example [7] prove that all global solutions of

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u, & \text{in } \Omega \times (0,\ \infty), \\ u &= 0, & \text{in } \partial\Omega \times (0,\ \infty), \\ u(0,\ x) &= u_0(x), & \text{in } \Omega, \end{cases}$$

has a priori bound $\sup_{t \geq 0} \|u(t)\ ;\ L^{\infty}(\Omega)\| \leq C$ with C depending only on $\|u_0\ ;\ L^{\infty}(\Omega)\|$ provided that p is subcritical. This improves the earlier result of [5] where $u_0 \geq 0$ is assumed. T. Cazenave and P.-L. Lions proved such an estimate for $1 < p < \frac{3n+8}{3n-4}$.

Our argument is somewhat a localized version of Quittner's inductive argument. Because of localization we have to take care of extra term caused by cutting off. To control these terms we need local energies estimates. Quittner's inductive argument is the method as follows. Using (I) and an interpolation theorem, we obtain (U) for $\lambda < \lambda(q) \stackrel{\text{def}}{=} \max\{\frac{2n(q+1)}{q(n-2)+n}, \frac{(p+1)q+2}{q+1}\}$. Using a regularity theorem for parabolic equation in [1], (U) implies a bound for $\|(t)\ ;\ W^{1,2}(\Omega)\|$(and, consequently, for $\|u(t)\ ;\ C^1(\Omega)\|$) provided $p < p(q) = 1 + \frac{2\lambda(q)}{n}$. Then $p(q) \to \frac{n+2}{n-2}$ as $q \to \infty$.

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Free energy and variational structures in the system of self-interacting particles

Takashi Suzuki

In this talk, stationary solutions to the system of chemotaxis and that of self -interacting particles are studied from the dynamical point of view. We have two variational structures for that problem, provided with the duality. They are connected by the Lyapunov function, and consequently their spectral and dynamical equivalences are obtained. We also describe some applications. Namely, our purpose is to show the dynamical equivalence between dual variations in the system of self-interacting particles, and is to describe its applications. This is the joint work with T. Senba of Miyazaki University.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $V = V(x) > 0$ be a smooth function of $x \in \overline{\Omega}$. Given a constant $\tau \geq 0$ and a self-adjoint operator $A > 0$ in $L^2(\Omega)$ with the compact resolvent, we study the system

$$\begin{array}{lll} u_t = \nabla \cdot (\nabla u - u\nabla (v + \log V)) & \text{in} & \Omega \times (0,T) \\ \frac{\partial}{\partial \nu}u - u\frac{\partial}{\partial \nu}(v + \log V) = 0 & \text{on} & \partial\Omega \times (0,T) \\ \tau\frac{d}{dt}v + Av = u & \text{for} & t \in (0,T), \end{array} \tag{1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are unknown functions of $(x,t) \in \overline{\Omega} \times [0,T)$. The initial value is provided with

$$u|_{t=0} = u_0(x) \geq 0 \qquad \text{in} \quad \Omega, \tag{2}$$

and if $\tau > 0$, then

$$v|_{t=0} = v_0(x) \geq 0 \qquad \text{in} \quad \Omega \tag{3}$$

is added.

System (1) is found in several areas of natural sciences. There is a form in biology proposed by Keller and Segel [15] and Nanjundiah [23]. There,

$\tau > 0$, and A is the differential operator $-\Delta + a$ with the homogeneous Neumann boundary condition, and $a > 0$ is a constant. This case with $\tau = 0$ was studied by Nagai [20] and Senba and Suzuki [26].

In the other system of Jäger and Luckhaus [13], $\tau = 0$ and A is equal to $-\Delta$ with the homogeneous Neumann boundary condition under the constraint $\int_\Omega \cdot = 0$. That is, $Av = u$ if and only if

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in} \quad \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \qquad \text{and} \qquad \int_\Omega v = 0. \tag{4}$$

One may take A to be $-\Delta$ with the homogeneous Dirichlet boundary condition, in which case it forms a modified system of Diaz and Nagai [9]. This case of $\tau > 0$ has also a biological sense. Those systems of Keller-Segel and its modifications describe chemotactic feature of cellular slime molds sensitive to the gradient of some chemical substances secreted by themselves, and $V = V(x)$ acts as an environment function. Actually, the micro-scopic derivation was done by Alt [2] from the biased random walk.

System (1) arises also in statistical mechanics. That is, the adiabatic limit of the Fokker-Planck equation, describing the motion of mean field of self-interacting particles subject to the attractive inner force. In this case, $\tau = 0$ is always assumed, A^{-1} is an integral operator with the kernel expressing the potential of the interaction, and $\log V(x)$ is that of the outer force. See Bavaud [5] and Wolansky [33], [34] for details. If the third equation is replaced by $-Av = u$, then it indicates that the interaction is repulsive. In that case, system is dissipative, and describes, for instance, the motion of electrons inside the semi-conductor. We do not treat those dissipative systems here, and just refer to the monograph Bank [4].

Under reasonable assumptions to A, unique solvability of (1) with (2) and (3) locally in time is obtained by the methods of Yagi [35] and Biler [6]. If $T_{\max} > 0$ denotes the existence time of the solution, the first two equations of (1) guarantee the non-negativity of $u = u(x,t)$, and therefore

$$\|u(t)\|_1 = \|u_0\|_1 \qquad (t \in [0, T_{\max})) \tag{5}$$

follows from $\int_\Omega u_t = 0$. Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm on Ω for $p \in [1,\infty]$. In the following, we shall write $\|\cdot\|$ for $\|\cdot\|_2$ and $(\ ,\ )$ denotes the L^2 inner product:

$$\|v\| = \left(\int_\Omega v^2\right)^{1/2}, \qquad (v,w) = \int_\Omega vw.$$

Long time behavior of the solution is controlled by the Lyapunov function, given as

$$\mathcal{W}(u,v) = \int_\Omega (u\log u - u\log V - uv) + \frac{1}{2}\left\|A^{1/2}v\right\|^2.$$

In fact, writing the first equation of (1) as

$$u_t = \nabla\cdot u\nabla\left(\log u - v - \log V\right),$$

we get

$$\int_\Omega u_t\left(\log u - v - \log V\right) = -\int_\Omega u\left|\nabla\left(\log u - v - \log V\right)\right|^2$$

from the second equation. Here, the left-hand side is equal to

$$\frac{d}{dt}\int_\Omega (u\log u - uv - u\log V) + (u, v_t)$$

and the third equation gives

$$(u, v_t) = \tau\left\|v_t\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|A^{1/2}v\right\|^2.$$

Thus we obtain

$$\frac{d}{dt}\mathcal{W}(u,v) + \tau\left\|v_t\right\|^2 + \int_\Omega u\left|\nabla\left(\log u - v - \log V\right)\right|^2 = 0 \tag{6}$$

for $t\in[0,T_{\max})$, and hence $\mathcal{W}\left(u(t),v(t)\right)$ is a non-increasing function of t.

Now, we introduce the stationary problem to (1), following the idea of Childress and Percus [8]. In fact, if $u_0(x)\not\equiv 0$, then the strong maximum principle guarantees $u(x,t)>0$ for $(x,t)\in\overline{\Omega}\times(0,T_{\max})$. Therefore, in the case of $\frac{d}{dt}W\left(u(t),v(t)\right)=0$ for some $t\in(0,T_{\max})$, one gets

$$\log u - v - \log V = \text{constant} \qquad \text{in} \quad \Omega$$

for $u=u(t)$ and $v=v(t)$ from the third term of the left-hand side of (6). In use of (5), this implies

$$u = \lambda V e^v \Big/ \int_\Omega V e^v \tag{7}$$

for $\lambda = \|u_0\|_1$. If $\tau > 0$, (6) implies also $v_t(t) = 0$. In any case we have $u = Av$ for $u = u(t)$ and $v = v(t)$, and therefore

$$v \in \mathrm{Dom}(A) \qquad \text{and} \qquad Av = \lambda V e^v / \int_\Omega V e^v \tag{8}$$

follow, where $\mathrm{Dom}(A)$ denotes the domain of A. If $v = v(x)$ solves (8) conversely, then (u, v) with $u = u(x)$ defined by (7) is a stationary solution to (1) satisfying (5). In this way the stationary problem for (1) with $\|u_0\|_1 = \lambda$ is formulated as (8), which is equivalent to

$$\log u - A^{-1}u - \log V = \text{constant}, \qquad \|u\|_1 = \lambda \tag{9}$$

in terms of u. Furthermore, the non-stationary solution $(u, v) = (u(t), v(t))$ always satisfies

$$\frac{d}{dt}\mathcal{W}(u(t), v(t)) < 0$$

for $t \in [0, T_{\max})$.

Henceforth, we call the cases $\tau > 0$ and $\tau = 0$ the *full* and *simplified* systems, respectively. In the simplified system, we always have $v = A^{-1}u$ and the Lyapunov function is reduced to

$$\mathcal{F}(u) = \mathcal{W}(u, A^{-1}u) = \int_\Omega (u \log u - u \log V) - \frac{1}{2}(A^{-1}u, u).$$

This is nothing but the free energy in the terminology of thermodynamics, and therefore, naturally induces a variational problem describing the equilibrium state of (1). That is, to find a critical point u of $\mathcal{F}$ on

$$\mathcal{P}_\lambda = \{u : \text{measurable} \mid u \geq 0 \text{ a.e.}, \quad \|u\|_1 = \lambda\}. \tag{10}$$

Actually, this variational problem is equivalent to (9) as we shall see.

On other hand, problem (8) has a variational structure of its own. In fact, it is not hard to see, at least formally, that $v = v(x)$ is a solution to (8) if and only if it is a critical point of

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\left\|A^{1/2}v\right\|^2 - \lambda \log\left(\int_\Omega V e^v\right).$$

Thus, we have two structures of variation to the stationary problem of (1).

In this talk, first, we provide an existence and uniqueness theorem for (1) locally in time and also the well-known criterion of the blowup of the solution. Then, we describe the key fact that those varitaional structures stated above are equivalent up to the Morse indices. This fact is known to some extent concerning the stability in the modified Diaz-Nagai system ([33], [29]), but in that section, a general theory will be presented. Now, such a spectral equivalence implies the dynamical equivalence. Actually, the simplified system posseses precise features such as stable, unstable, and center manifolds of stationary solutions, with the dimension of the unstable manifold equal to the Morse index, and so forth. Even in the full system, they are equivalent in the stability. Finally, we turn to concrete systems, and examine existence, uniqueness, and stability of stationary solutions.

We have the identity

$$\mathcal{W}\left(\lambda Ve^v / \int_\Omega Ve^v, v\right) = \mathcal{J}_\lambda(v) + \lambda \log \lambda. \tag{11}$$

Therefore, $\mathcal{F}(u)$ and $\mathcal{J}_\lambda(v)+\lambda\log\lambda$ are nothing but the restrictions of $\mathcal{W}(u,v)$ to

$$\mathcal{M} = \left\{(u,v) \mid v = A^{-1}u,\ \|u\|_1 = \lambda\right\}$$

and

$$\mathcal{N} = \left\{(u,v) \mid u = \lambda Ve^v / \int_\Omega Ve^v\right\},$$

respectively. Furthermore, the intersection of those manifolds coincides with the set of stationary solutions. However, $\mathcal{M}$ and $\mathcal{N}$ meet transversally, and the spectral equivalence described above follows from the algebraic properties of $\mathcal{W}(u,v)$, that is, $\mathcal{W}_v = 0$ and $\mathcal{W}_u = 0$ on $\mathcal{M} \times T\mathcal{M}$ and $\mathcal{N} \times T\mathcal{N}$, respectively. Neverthless, that spectral equivalence is reasonable, because $\mathcal{F}(u)$, $\mathcal{W}(u,v)$, and $\mathcal{J}_\lambda(v) + \lambda\log\lambda$ are regarded as the free energies for the system (1) with $\tau = 0$, $0 < \tau < +\infty$, and $\tau = +\infty$, respectively.

Henceforth, we adopt the following notations, where $a > 0$ is a constant and $\Gamma(x)$ is the fundamental solution of $-\Delta$, with ω_n being the volume of the n-dimensional unit ball:

$$\Gamma(x) = \begin{cases} \frac{1}{2}|x| & (n=1) \\ \frac{1}{2\pi}\log\frac{1}{|x|} & (n=2) \\ \frac{1}{\omega_n(n-2)}|x|^{2-n} & (n \geq 3). \end{cases}$$

That is, we say that system (1) is (N), (JL), (DN), and (W), if $A = -\Delta + a$ with $\frac{\partial}{\partial\nu}\cdot\Big|_{\partial\Omega} = 0$, $Av = u$ if and only if (4), $A = -\Delta$ with $\cdot|_{\partial\Omega} = 0$, and $A^{-1}v = \int_\Omega \Gamma(\cdot - y)v(y)dy$, respectively.

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A variational problem for the Helfrich functional related to the shape of red blood cells

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1 Physical background

We discuss a variational problem that models shapes of red blood cell. About 30 years ago some models of cell are proposed by several physicists ([2, 4, 5]). They considered that the shape is determined by some variational structure of the bending energy of the cell membrane.

One of them is the *spontaneous curvature* model. We regard the cell as an oriented closed smooth surface embedded in $\mathbb{R}^3$, denoted by Σ. We denote its mean curvature by h. The sign of mean curvature is positive if the surface is convex. $A(\Sigma)$ and $V(\Sigma)$ mean the area and enclosed volume:

$$A(\Sigma) = \int_\Sigma dS, \quad V(\Sigma) = -\frac{1}{3}\int_\Sigma \boldsymbol{n}\cdot\boldsymbol{p}\,dS.$$

Here $\boldsymbol{p}$ and $\boldsymbol{n}$ are respectively the position vector and the inner unit normal vector at $\boldsymbol{p}$ on Σ. We introduce the bending energy defined by

$$W(\Sigma) = \int_\Sigma (h - c_0)^2 dS,$$

where c_0 is a given constant, not necessarily positive, called the *spontaneous curvature*.

Let A_0 and V_0 be given constant. Then it is considered that a critical surface, in particular a minimizer, of the bending energy under the prescribed area A_0 and volume V_0 is the shape of the red blood cell. Of course A_0 and V_0 must satisfy the isoperimetric inequality. The spontaneous curvature c_0 determined by the structure of the cell membrane.

We formulate the problem mathematically. For a smooth function ϕ on a surface Σ and $t \in \mathbb{R}$, we shift Σ to the normal direction with length $t\phi$. If

$|t|$ is sufficiently small, then we get a surface, denoted by Σ_t. This is called the normal variation. We denote the first and second variations by δ and δ^2 respectively, for a functional F on Σ, that is, $\delta F(\Sigma)$ and $\delta^2 F(\Sigma)$ are the first and second derivatives with respect to t at $t = 0$ of $F(\Sigma_t)$. Our problem is a variational one with two constraints. The theory of Lagrange multipliers gives the Euler-Lagrange equation

$$\delta W(\Sigma) + \lambda_1 \delta A(\Sigma) + \lambda_2 \delta V(\Sigma) = 0. \tag{1.1}$$

Here λ_1 and λ_2 are Lagrange multipliers. This is equivalent to the Euler-Lagrange equation of the functional

$$H(\Sigma) = W(\Sigma) + \lambda_1 A(\Sigma) + \lambda_2 V(\Sigma)$$

without constraints. It is sometimes called the *Helfrich* functional, who is one of the proposers of this model ([5]).

There ia another models about the red blood cells. It is called the *bilayer-coupled* model. This is the same variational problem but with different meaning of physical parameters λ_1 and λ_2 (see [4]).

2 Known results

By use of differential geometry, (1.1) is reduced to a second order elliptic equation

$$\Delta_g h + 2h(h^2 - k) + 2c_0 k - 2c_0^2 h - 2\lambda_1 h - \lambda_2 = 0 \tag{2.1}$$

for the mean curvature h. Here k is the Gauss curvature and Δ_g is the Laplace-Beltrami operator of Σ with the induced metric g from $\mathbb{R}^3$. The metric g is determined from the embedding of Σ, and therefore unknown. Hence this equation is quasi-linear, not semi-linear.

Several results are known about this problem. If $c_0 = \lambda_1 = \lambda_2 = 0$, then our functional is the Willmore functional, which has a long history in differential geometry. Spheres are critical point of the functional, which is the unique global minimizer. The Clifford torus is a critical point of the Willmore functional with nontrivial topology. Willmore conjectured that the Clifford torus is a global minimizer among the surfaces of genus 1 (the *Willmore conjecture*, see [12]), which has not been solved yet. Simon showed the existence of minimizers with the prescribed genus of the surface ([11]). There are many other works about the Willmore function, but we omit them here.

Spheres are critical points of the Helfrich functional for any c_0. That is, spheres satisfy the Euler-Lagrange equation for any c_0 and suitable λ_1 and λ_2, since h and k is constant. This is a direct calculation.

We have a lot of discussions of critical points the Helfrich functional other than spheres. Most of them are based on formal calculations or experiments, however, we would like to point out some of them.

The normal red blood cell has a biconcave shape when at rest in the plasma. This is not the only shape of the cell. Adding distilled water to the plasma, we can observe various shapes. At first cell loses the biconcavity, gradually shapes an oblate ellipsoid, and finally, spherical. These are determined by the excess of osmotic pressure between interior and exterior of cell. The physical constant λ_2 corresponds to the excess of osmotic pressure. This shape transformation implies the existence of "a bifurcating family" of shapes of cell from the sphere with the bifurcation parameter of λ_2.

Jenkins [6] showed numerically the existence of families of solutions bifurcating from spheres when $c_0 = 0$. The solutions are surfaces of revolution. Subsequently Peterson [10] and Ou-Yang and Helfrich [9] investigated the stability and instability of surfaces of mode 2 by formal computation. We shall explain the meaning of "mode" in the next section. Unfortunately these results seemed to be based on formal calculations, and the rigorous proofs were expected.

The author jointly with Takagi succeeded in giving a rigorous proof of existence of solutions of mode n. We can also analyze the stability and instability of solutions of general modes. The solutions are surfaces of revolution, however, it is to be emphasized that we include variations which are not rotationally symmetric in the stability question. Our results in this note have already announced in [7] without precise proofs. The full paper [8] is now in preparation.

The solutions above are "near" the sphere. Recently Au and Wan [1] showed the existence of a solution with the biconcave shape "far from" the sphere. The cross section of the surface is convex firstly, turns to a concave shape, and blows down with finite radius. It is uncertain that Au-Wans's solutions are the same ones as those of Jenkins. The stability/instability of Au-Wan's solution is also uncertain.

3 Critical points of near spheres

In the rest of this note we devote ourselves to explain the results of [8]. Let Σ be a surface of revolution:

$$\Sigma = \{\boldsymbol{p} = (r(s)\cos\theta, r(s)\sin\theta, z(s)) \mid 0 \le \theta < 2\pi,\ 0 \le s \le \bar{s}\}.$$

Here r and z are unknown functions, and s is the arch-length parameter of generating curve. The range of s is unknown, that is, the problem is the free boundary problem. This is one of the difficulties. To avoid this, we introduce a new coordinate. By scaling we may assume the area is 4π. A new coordinate ζ is defined by

$$s \mapsto \zeta = \int_0^s r(s)\, ds - 1 \in \{\zeta \mid -1 \leq \zeta \leq 1\}.$$

We may call this "the area-wise coordinate" in the following sense. The surface generates the curve parametrized by s. Let consider the segment of curve with the arch-length parameter between 0 and s, and the surface patch generating the segment. The area of the patch varies from 0 to 4π. Normalizing the range to the interval $[-1, 1]$, we get ζ. Changing an unknown function r to ρ, where ρ is the square of r. And put $\lambda = -2\lambda_1$ and $\mu = \lambda_2$. Then we reduce our problem (2.1) to

$$\begin{cases} (\rho h')' + 2(h - c_0)\left\{(h - z')^2 + c_0 h\right\} + \lambda h - \mu = 0, \\ \frac{1}{2}\rho'' - (h - z')^2 + h^2 = 0, \\ (\rho z')' - \rho' h = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } -1 < \zeta < 1, \\ \sqrt{\rho} h' = \rho = \sqrt{\rho} z' = 0 \quad \text{at} \quad \zeta = \pm 1, \\ \rho' = \mp 2 \quad \text{at} \quad \zeta = \pm 1. \end{cases}$$

The first equation is the Euler-Lagrange equation. The second and third ones are the relations between ρ, z and h. The boundary conditions except the last one mean that the surface closes smoothly. The last condition comes from the normalization of area, or reduction of free boundary condition.

This system, however, is overdetermined as a system of second order ordinary differential equations. The normalization of area gives an extra condition. We would like to construct bifurcation solutions from the unit sphere S^2, but we cannot apply the standard bifurcation theory. Instead we consider the system of equations

$$\begin{cases} (\rho h')' + 2(h - c_0)\left\{(h - z')^2 + c_0 h\right\} + \lambda h - \mu + \nu_1 \rho' = 0, \\ \dfrac{1}{2}\rho'' - (h - z')^2 + h^2 + \dfrac{\nu_2}{4}\rho' = 0, \\ (\rho z')' - \rho' h + \dfrac{\nu_2}{2}\rho z' = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } -1 < \zeta < 1 \end{cases}$$

with the same boundary conditions. The new system contains new parameters ν_1 and ν_2, and therefore it is not overdetermined. Furthermore we can show that if there exists a solution, then ν_1 and ν_2 are zero. Hence the solution satisfies the original system. Conversely solutions of the original system solve the new system putting ν_1 and ν_2 zero. Consequently two systems are equivalent. This fact can be shown by using translation invariance of functional. Of course we can show that by analytic argument, but we need length calculations.

The unit sphere S^2 corresponds to

$$h \equiv 1,\ \rho = 1 - \zeta^2,\ z = \zeta,\ \lambda = -2c_0 + 2c_0^2 + \mu.$$

We denote $C^2(-1,1)$, $L^2(-1,1)$, and $H^k(-1,1)$ simply by C^2, L^2, and H^k. Let P_n be the Legendre polynomial of order n, and

$\mathcal{D} =$ the graph closure in L^2 of

$$\left\{ u \in C^2 \cap L^2 \,\middle|\, \lim_{\zeta \to \pm 1} \sqrt{1-\zeta^2}\frac{du}{d\zeta} = 0,\ \frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{du}{d\zeta}\right\} \in L^2 \right\},$$

$$\mathcal{D}_0^1 = \left\{ u \in \mathcal{D} \,\middle|\, \int_{-1}^{1} u\, d\zeta = 0,\ \frac{du}{d\zeta} \in \mathcal{D} \right\}.$$

H_0^2 is the completion of the space of smooth functions with compact support in H^2 topology. As an application of Crandall-Rabinowitz' theorem [3] to the new system, we have the existence theorem.

Theorem 3.1 *Let n be an integer greater than 1. Then we have families of solutions $\Sigma_n(\varepsilon) = (h(\varepsilon), \rho(\varepsilon), z(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon)) \in \mathcal{D} \times (\{1-\zeta^2\} + H_0^2) \times \mathcal{D}_0^1 \times \mathbb{R} \times \mathbb{R}$:*

$$\begin{cases} h = 1 + \varepsilon P_n + \mathcal{O}(\varepsilon^2), \\ \rho = 1 - \zeta^2 - \dfrac{4\varepsilon(1-\zeta^2)^2 P_n''}{(n-1)n(n+1)(n+2)} + \mathcal{O}(\varepsilon^2), \\ z' = 1 + \dfrac{2\varepsilon(\zeta P_n' - P_n)}{(n-1)(n+2)} + \mathcal{O}(\varepsilon^2), \\ \lambda = n(n+1) - 4c_0 + 2c_0^2 + \mathcal{O}(\varepsilon)(= -2c_0 + 2c_0^2 + \mu + \mathcal{O}(\varepsilon)), \\ \mu = n(n+1) - 2c_0 + \mathcal{O}(\varepsilon), \\ (\nu_1 = \nu_2 = 0) \end{cases}$$

for sufficiently small $|\varepsilon|$, say $|\varepsilon| < \varepsilon_1$. The mapping from ε to the solution $\Sigma_n(\varepsilon)$ is analytic from $\mathbb{R}$ *to the above class.*

Note that the part of order 1 is the unit sphere. Therefore these are families of critical points bifurcating from S^2. We call the solution of Theorem 3.1 that of *mode n*.

The surfaces obtained in Theorem 3.1 are critical points of $W(\cdot)$ under the prescribed area $A_0 = 4\pi$ and volume V_0. Next we would like to discuss the result of stability of them as critical points of the constraint variational problem. As usual we define the Nullity and the Index of critical points. That is, Nullity is the multiplicity of zero eigenfunction of the quadratic form associated with the second variation, and Index is the number of negative eigenvalues. Then we have the lower bound of Index and Nullity.

Theorem 3.2 *For the solution of mode n it holds that* $\mathrm{Index}(\Sigma_n(\varepsilon)) \geq (n-2)(n+2)$, *and* $\mathrm{Nullity}(\Sigma_n(\varepsilon)) \geq 5$ *provided $|\varepsilon| > 0$ is sufficiently small.*

The lower bound 5 of Nullity comes form the rigid motion. Since the surface is rotationally symmetric, the rotation around the axis of symmetry generates the tangential variation but not the normal variation. Therefore the space of normal variations coming from infinitesimal rigid motions of is a 5-dimensional space, not 6-dimensional. The lower bound of Index shows that the surfaces of mode n is unstable if n is greater than 2.

The theorem giving below is the more precise bounds in case of even n. Let γ be

$$\gamma = c_0(3n^4 + 6n^3 - 3n^2 - 6n + 8) + 3n^4 + 6n^3 - 7n^2 - 10n,$$

and let σ be the sign of $\varepsilon \times \gamma$. P_n^m is the associate Legendre functions of the first kind. $E_{n,+}$, $E_{n,-}$, and $E_{n,0}$ are the spaces defined by

$$\begin{aligned} E_{n,+} &= \mathrm{span}\{P_n^m \cos m\theta,\ P_n^m \sin m\theta \,|\, 2 \leq m \leq n,\ S_n^m = 1\}, \\ E_{n,-} &= \mathrm{span}\{P_n^m \cos m\theta,\ P_n^m \sin m\theta \,|\, 2 \leq m \leq n,\ S_n^m = -1\}, \\ E_{n,0} &= \mathrm{span}\{P_n^m \cos m\theta,\ P_n^m \sin m\theta \,|\, 2 \leq m \leq n,\ S_n^m = 0\}. \end{aligned}$$

Here

$$A_n^m = \int_{-1}^{1} P_n \, (P_n^m)^2 \, d\zeta,$$

and

$$S_n^m = \mathrm{sgn}\left\{ \frac{(n+m)!}{(n-m)!} A_n^0 - 2A_n^m \right\}.$$

Then dimensions of these spaces give the lower and upper bound of Nullity and Index.

The second term in the right-hand side does not appear when the variation is linear. The Euler-Lagrange equation, and the constraints of the area and volume yield

$$\delta W(\Sigma)[\psi''(0)] = -\lambda_1 \delta A(\Sigma)[\psi''(0)] - \lambda_2 \delta V(\Sigma)[\psi''(0)],$$

$$-\delta A(\Sigma)[\psi''(0)] = \delta^2 A(\Sigma)[\psi'(0)], \quad -\delta V(\Sigma)[\psi''(0)] = \delta^2 V(\Sigma)[\psi'(0)]$$

respectively. Combining these, we get our formula.

Next proposition says the admissibility of variations.

Proposition 3.2 (Admissibility of test function) *The variation* $\Sigma \to \Sigma(\psi(t)) = \{\boldsymbol{p} + \psi(t)\boldsymbol{n} \,|\, \boldsymbol{p} \in \Sigma\}$ *preserves the area and volume, if and only if*

$$\int_{S^2} \psi'(0)\, dS = \int_{S^2} h\psi'(0)\, dS = 0.$$

This means that the first variation of area and volume vanish. Therefore the necessity is clear. The sufficiency is not trivial. We show this by use of the implicit function theorem. The class of admissible variations is not linear space, but manifold. The condition of Proposition 3.2 determines the tangent space of the manifold.

Put $\psi'(0) = \phi$. Now we define the quadratic from II associated with the second variation by

$$\begin{aligned}
\mathrm{II}[\phi,\phi] &= \delta^2 W(\Sigma)[\phi] + \lambda_1 \delta^2 A(\Sigma)[\phi] + \lambda_2 \delta^2 V(\Sigma)[\phi] \\
&= 2\int_{S^2} \Bigg(\left[\frac{1}{2}\Delta_g (h^2 - k) + (4h^2 - k)(h^2 - k) + \frac{1}{\sqrt{g}} \left\{ \sqrt{g} h^{ij} (h)_i \right\}_j \right] \phi^2 \\
&\quad - h h^{ij} \phi_i \phi_j - \frac{1}{2}(h^2 - 2k)|\nabla_g \phi|^2 + \frac{1}{4}(\Delta_g \phi)^2 - 2c_0 \left(-\frac{1}{2} h^{ij} \phi_i \phi_j + h|\nabla_g \phi|^2 \right) \\
&\quad + \left(c_0^2 - \frac{\lambda}{2} \right) \left(k\phi^2 + \frac{1}{2}|\nabla_g \phi|^2 \right) + \mu h \phi^2 \Bigg)\, dS.
\end{aligned}$$

Here Δ_g and ∇_g are the Laplacian and the gradient on the surface with the induced metric. g_{ij} and h_{ij} are the first and second fundamental forms.

Inserting the expansion of solution $h = 1 + \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$ etc. in the formula above, we get

$$\mathrm{II}[\phi,\phi] = \mathrm{II}_0[\phi,\phi] + \varepsilon \mathrm{II}_1[\phi,\phi] + \cdots,$$

where

$$\mathrm{II}_0[\phi,\psi] = \int_{S^2} \left\{ \frac{1}{2}(\Delta_0 \phi)(\Delta_0 \psi) - \frac{n^2 + n + 2}{2} \nabla_0 \phi \cdot \nabla_0 \psi + n(n+1)\phi\psi \right\} dS,$$

Theorem 3.3 *Let n be even. Then there exists $\varepsilon_2 = \varepsilon_2(n, c_0) > 0$ such that for $0 < |\varepsilon| < \varepsilon_2$*

$$\begin{aligned}(n-2)(n+2) + \dim E_{n,-\sigma} &\leq \operatorname{Index}(\Sigma_n(\varepsilon)) \\ &\leq (n-2)(n+2) + \dim E_{n,-\sigma} + \dim E_{n,0}\end{aligned}$$

and

$$5 \leq \operatorname{Nullity}(\Sigma_n(\varepsilon)) \leq 5 + \dim E_{n,0},$$

hold provided $\gamma \neq 0$.

Note that if $E_{n,0} = \emptyset$, then estimates are optimal (here we interpret $\dim \emptyset = 0$). When $n \leq 6$ and even, it can be shown that $E_{n,0} = \emptyset$ by direct calculation. When $8 \leq n \leq 30$ and even, we have $E_{n,0} = \emptyset$ with help of computer.

Furthermore when n is 2, 4, or 6, we can give $E_{n,\pm}$ explicitly. Consequently we can give the exact value of Index and Nullity. In particular, if $\varepsilon(5c_0 + 3) > 0$, then the solution of mode 2 is stable. All other solutions except $5c_0 + 3 = 0$ are unstable.

The result on the stability and instability in mode 2 coincides with formal results of Peterson [10] and Ou-Yang-Helfrich [9]. The result for higher modes is completely new. Note that we include variations which are not rotationally symmetric in the study of the stability and instability.

To show Theorems 3.2 and 3.3 we must check the sign of the second variation. In the following we sketch the proof of Theorem 3.3. Since the problem is the variational one with constraints, we must restrict the variations to those which satisfies the constraints. We call such variations *admissible.* Hence we need the second variation formula for admissible variations, and the necessary and sufficient condition of admissibility.

The first proposition gives the second variation formula of the bending energy under constraints.

Proposition 3.1 (The second variation formula) *Let $\Sigma \to \Sigma(\psi(t)) = \{\boldsymbol{p} + \psi(t)\boldsymbol{n} \,|\, \boldsymbol{p} \in \Sigma\}$ be a variation preserving the area and volume. Then we have*

$$\frac{d^2}{dt^2} W(\Sigma(\psi(t)))\bigg|_{t=0} = \delta^2 W(\Sigma)[\psi'(0)] + \lambda_1 \delta^2 A(\Sigma)[\psi'(0)] + \lambda_2 \delta^2 V(\Sigma)[\psi'(0)].$$

This formula is derived in the following way. If the variation is linear, then it does not satisfy the constraints. Therefore we must consider the nonlinear variations. If the variation is nonlinear, then the second derivative is

$$\frac{d^2}{dt^2} W(\Sigma(\psi(t)))\bigg|_{t=0} = \delta^2 W(\Sigma)[\psi'(0)] + \delta W(\Sigma)[\psi''(0)].$$

$$\mathrm{II}_1[\phi,\psi] = \int_{S^2} \Bigg([\{4c_0 - 2n(n+1)\}\, h_1 + \mu_1]\, \varphi\psi$$

$$- \left[\frac{n^2+n+2}{2}\rho_1 + \rho_0 \left\{ 4h_1 + (c_0-1)\frac{dz_1}{d\zeta} + \frac{\mu_1}{2} \right\} \right] \frac{\partial \varphi}{\partial \zeta}\frac{\partial \psi}{\partial \zeta}$$

$$+ \left[\frac{n^2+n+2}{2}\frac{\rho_1}{\rho_0^2} + \frac{1}{\rho_0} \left\{ -4c_0 h_1 + 2(c_0-1)\frac{dz_1}{d\zeta} - \frac{\mu_1}{2} \right\} \right] \frac{\partial \varphi}{\partial \theta}\frac{\partial \psi}{\partial \theta}$$

$$+ \frac{1}{2}(\Delta_0\varphi) \left\{ \frac{\partial}{\partial \zeta}\left(\rho_1 \frac{\partial \psi}{\partial \zeta}\right) - \frac{\rho_1}{\rho_0^2}\frac{\partial^2 \psi}{\partial \theta^2} \right\} + \frac{1}{2}(\Delta_0\psi) \left\{ \frac{\partial}{\partial \zeta}\left(\rho_1 \frac{\partial \varphi}{\partial \zeta}\right) - \frac{\rho_1}{\rho_0^2}\frac{\partial^2 \varphi}{\partial \theta^2} \right\} \Bigg)\, dS.$$

Using the expansion of solution in Theorem 3.1, we compute the sign on II_0 and II_1 for admissible function ϕ.

Proposition 3.3 *Let $\mathcal{A}$ and $\mathcal{R}$ be spaces of admissible variations and rigid motions respectively. $E_{n,+}$, $E_{n,-}$, and $E_{n,0}$ are as before. There exist linear spaces $E_{n,1}$ and $E_{n,2}$ such that*

$$E_{n,1} \simeq \mathrm{span}\{P_\ell, P_\ell^m \cos m\theta,\ P_\ell^m \sin m\theta \,|\, 2 \le \ell \le n,\ 1 \le m \le \ell\},$$

$$E_{n,2} \simeq \mathrm{span}\{P_\ell, P_\ell^m \cos m\theta,\ P_\ell^m \sin m\theta \,|\, \ell \ge n+1,\ 1 \le m \le \ell\},$$

and the direct sum decomposition

$$\mathcal{A} = E_{n,-\sigma} \oplus E_{n,1} \oplus E_{n,0} \oplus \mathcal{R} \oplus E_{n,\sigma} \oplus E_{n,2}.$$

Furthermore it holds that

$$\mathrm{II}_0 = 0,\ \mathrm{II}_1 < 0 \quad on \quad E_{n,-\sigma}, \quad \mathrm{II}_0 < 0 \quad on \quad E_{n,1},$$

$$\mathrm{II}_0 = \mathrm{II}_1 = 0 \quad on \quad E_{n,0}, \quad \mathrm{II} = 0 \quad on \quad \mathcal{R},$$

$$\mathrm{II}_0 = 0,\ \mathrm{II}_1 > 0 \quad on \quad E_{n,\sigma}, \quad \mathrm{II}_0 > 0 \quad on \quad E_{n,2}.$$

Here $\mathrm{II}_1 > 0$ on $E_{n,-\sigma}$ means that II_1 is positive definite there. Other notation should be understood similarly.

Since the decomposition is not orthogonal one with respect to eigenspaces, we must estimate the cross terms carefully to see the sigunature of quadratic forms. After these estimations we finally obtain the lower and upper bounds:

$$\dim(E_{n,-\sigma} \oplus E_{n,1}) \le \mathrm{Index}(\Sigma_n(\varepsilon)) \le \mathrm{codim}\,(\mathcal{R} \oplus E_{n,\sigma} \oplus E_{n,2})$$

$$= \dim(E_{n,-\sigma} \oplus E_{n,1} \oplus E_{n,0}),$$

$$\dim \mathcal{R} \le \mathrm{Nullity}(\Sigma_n(\varepsilon)) \le \dim(E_{n,0} \oplus \mathcal{R}).$$

By direct calculations we have $\dim E_{n,1} = (n-2)(n+2)$ and $\dim \mathcal{R} = 5$.

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A Level Set Method for Computing Discontinuous Solutions of a Class of Hamilton-Jacobi Equations*

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We introduce two types of finite difference methods to compute the L-solution and the proper viscosity solution recently proposed by the second author for semi-discontinuous solutions to a class of Hamilton-Jacobi quations. By regarding the graph of the solution as the zero level curve of a continuous function in one dimension higher, we can treat the corresponding level set equation using the viscosity theory introduced by Crandall and Lions. However, we need to pay special attention both analytically and numerically to prevent the zero level curve from overturning so that it can be interpreted as the graph of a function. We demonstrate our Lax-Friedrichs type numerical methods for computing the L-solution using its original level set formulation. In addition, we couple our numerical methods with a singular diffusive term which is essential to computing solutions to a more general class of HJ equations that includes scalar conservation laws. With this singular viscosity, our numerical methods do not require the divergence structure of equations and do apply to more general equations developing shocks other than conservation laws. These numerical methods are generalized to higher order accuracy using WENO Local Lax-Friedrichs methods in space and high order strong stability preserving Runge-Kutta method in time. We verify that our numerical solutions approximate the proper viscosity solutions. Since the solution of scalar conservation law equations can be constructed using existing numerical techniques, we use it to verify that our numerical solution approximates the entropy solution. Examples in 1- and 2-dimensions are presented.

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Growth of spirals on the surface of the crystal

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We consider the growth of spirals on the surface of crystals. We propose a "level set model" of the motion of the spiral growth, which is a degenerate parabolic type. So, we need to consider a notion of weak solutions. We shall prove the existence and the uniqueness of the solution for our level set model in the sense of viscosity solutions.

The theory of spiral crystal growth was proposed by F. C. Frank in 1948(see [BCF]). In his theory spiral crystal growth is taken place by the screw dislocation. If a screw dislocation terminates in the exposed surface of a crystal, there is a permanently exposed "cliff" of atoms, say the "step". The step can grow perpetually "up a spiral staircase". To see the surface from above, we can see spirals drawn by exposed "edge of the step".

We postulate that curves corresponding to steps are moving under the

$$V = C - \kappa,$$

where V is the normal verocity of the curve, κ is the curvature of the curve, and C is a positive constant. We also assign the Neumann boundary condition on the end of curves touching the boundary of the surface of the crystal.

The model stated above is for a single spiral, say the "geometric model". However, it is not enough to handle other situation so that there are two or more screw dislocations on the surface of the crystal and they may touch each other. We would like to handle such a situation, we adjust the model including this situation.

There are two methods to make a model including this situation. One is the Allen-Cahn equation model, and the other is a level set method for geometric model. First one is studied by R. Kobayashi([K]), K.-I. Nakamura and T. Ogiwara([NO]), A. Karma and M.Plapp([KP]), and so on. About the second one, which we call "level set model", P. Smereka reported the result of numerical computations([Sm]). Since a level set method treats curves directly, we use a level set method. Since Smereka's one is limited to the situation where the strength of all dislocation is the same, we arrange the model taking variety of strength and direction of spirals into account.

Let Ω be a bounded domain in $\mathbb{R}^2$, which denotes the surface of the crystal. Assume that the form of a screw dislocation is a circle whith finite size. Let $a_1, a_2, \ldots, a_n \in \Omega$ denote the position of the center of n times screw dislocations. Let ρ_i denote the radius of i-th screw dislocation, which is taken so that i-th screw dislocation does not touch the boundary $\partial\Omega$ and any j-th screw dislocation($i \neq j$), i.e. suppose that any ρ_j satisfy

$$\overline{B_{\rho_j}(a_j)} \subset \Omega \quad \text{for } j = 1, 2, \ldots, n,$$
$$\overline{B_{\rho_i}(a_i)} \cap \overline{B_{\rho_j}(a_j)} = \emptyset \quad \text{for } i \neq j,$$

where $B_{\rho_j}(a_j)$ is a open disc of radius ρ_j centered at a_j, and $\overline{B_{\rho_j}(a_j)}$ denotes its closure. We denote by W the complement of all screw dislocations in Ω, i.e.

$$W := \Omega \backslash (\bigcup_{j=1}^{n} \overline{B_{\rho_j}(a_j)}).$$

We denote by Γ_t the curve corresponding to steps on $\overline{W}$, assume that the shape of Γ_t may change as t develops. Geometric model is of form:

$$\begin{aligned} V &= C - \kappa \quad \text{on } \Gamma_t, \\ \partial W &\perp \Gamma_t. \end{aligned}$$

We derive our level set model by using the level set method to geometric model. In conventional method([CGG], [ES]), Γ_t is defined by the zero-level set of an auxiliary unknown function u in W. But this is not good to handle our situation since we cannot distinguish the direction of moving steps only using u in this method. We introduce a "sheet structure function"(proposed by R.Kobayashi([K])) θ to overcome this difficulty. The function θ is defined by

$$\theta(x) = \sum_{j=1}^{n} m_j \arg(x - a_j),$$

where m_j is an integer such that $|m_j|$ denotes the height of steps and the sign of m_j denotes the direction of steps. We remark that θ is a multi-valued function. We set

$$\Gamma_t = \left\{ x \in \overline{W}; u(t,x) - \theta(x) = 0 \right\}.$$

The definition of Γ_t stated above is the most significant feature of our works.

By the definition of Γ_t, we see

$$V = \frac{1}{|\nabla(u-\theta)|}\frac{\partial u}{\partial t}, \quad \kappa = \operatorname{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|}.$$

Here we calculate derivatives of θ on a fixed branch. Since $\nabla\theta$ is single-valued, V and κ are well-defined. Now we obtain the level set model consisting with geometric model of form:

$$\frac{\partial u}{\partial t} - |\nabla(u-\theta)| \left(\operatorname{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C \right) = 0 \quad \text{in} \quad (0,T) \times W, \tag{1}$$

$$\langle \vec{\nu}(x), \nabla(u-\theta) \rangle = 0 \quad \text{on} \quad (0,T) \times \partial W, \tag{2}$$

where $\vec{\nu}$ denotes the outer unit normal vector field of ∂W and $\langle \cdot, \cdot \rangle$ is the inner product of $\mathbb{R}^2$. Since the equation (1) is degenerate parabolic, we need to consider the solution of these equations in weak sense. We consider the solution in viscosity sense since the solution in distribution sense is not good for these equation.

Our main results are summarized in the following theorems. For a function $f\colon \mathbb{R}^k \supset D \longrightarrow \mathbb{R}$, let $f^*, f_*\colon \overline{D} \longrightarrow \mathbb{R}\cup\{\pm\infty\}$ respectively be an upper and a lower semicontinuous envelope of f defined by the following:

$$f^*(z) = \lim_{r\downarrow 0} \sup\{f(\omega); |z-\omega| < r\}, \quad f_*(z) = \lim_{r\downarrow 0} \inf\{f(\omega); |z-\omega| < r\}.$$

Theorem 1 (Comparison Principle) *Let Ω be a bounded domain in $\mathbb{R}^2$. Assume that $\partial\Omega$ is C^2. Let u and v respectively be a viscosity sub- and supersolution of (1)-(2) in $(0,T)\times\overline{W}$ for $T>0$. If*

$$u^*(0,x) \le v_*(0,x) \quad \textit{for } x \in \overline{W}$$

holds, then

$$u^*(t,x) \le v_*(t,x) \quad \textit{for } (t,x) \in (0,T)\times\overline{W}$$

holds.

Theorem 2 (Existence and Uniqueness) *Assume that $\partial\Omega$ is C^2. For a given $u_0 \in C(\overline{W})$, there exists a unique global viscosity solution $u \in C([0,\infty)\times\overline{W})$ with the initial condition*

$$u\big|_{t=0} = u_0 \quad \textit{on } \overline{W}.$$

To overcome the difficulty caused from the presence of the boundary([Sa], [GS]), an important step to prove theorems is to use "exterior ball condition" which reads that there exists a positive constant C_0 such that the inequality

$$\langle \vec{\nu}(x), x-y\rangle \ge -C_0|x-y|^2 \quad \text{for } x\in\partial W,\ y\in\overline{W}$$

holds. We have a new difficulty coming from the multi-valued function θ. To treat θ and to overcome the difficulty, we use different methods on the proofs of Theorem 1 and 2.

To prove Theorem 1, we make double variables as usual. We consider the function

$$\Phi(t,x,y) = u^*(t,x) - \theta(x) - (v_*(t,y) - \theta(y)) - \Psi(t,x,y)$$

and its maximum point to apply the Crandall-Ishii's lemma, where Ψ is a test function taken so that its value takes large when x and y is not close. We use exterior ball condition to define Ψ. The choice of Ψ is the most crucial step to overcome the difficulty arising from the boundary condition.

However Φ does not have a maximum value because Φ is a multi-valued function. So we need to decide the domain of Φ so that we can consider it a single-valued function and Φ takes a maximum value in its domain. To overcome this difficulty, we consider the "covering space".

If we get the comparison principle, it is easy to see the uniqueness and continuity of viscosity solution. So we have only to construct a viscosity solution of (1)-(2) to prove Theorem 2.

For the proof of Theorem 2, we use the Perron's method proposed by H.Ishii([I]) to construct a viscosity solution of (1)-(2). In the Perron's method, we construct a subsolution $f\colon [0,T)\times\overline{W} \longrightarrow \mathbb{R}$ and supersolution $g\colon [0,T)\times\overline{W} \longrightarrow \mathbb{R}$ of (1)-(2) satisfying the initial condition

$$f(0,x) = g(0,x) = u_0(x) \quad \text{for } x \in \overline{W}.$$

f and g are constructed symmetrically, we shall mention the construction of viscosity supersolution only.

The sheet structure function θ is a single-valued function on some small neighborhood of a fixed $y \in \overline{W}$. Let $\delta > 0$ be small enough. We set $U_\delta(y) = B_\delta(y) \cap \overline{W}$. For $\epsilon > 0$ and $y \in \overline{W}$, we define the function $v_{\epsilon,y}\colon (0,T) \times U_\delta(y) \longrightarrow \mathbb{R}$ of form:

$$v_{\epsilon,y}(t,x) = Bt + A_\epsilon e^{\varphi(x)} |x-y|^2 + 2\epsilon + \theta(x) - \theta(y),$$

where A_ϵ is a positive constant satisfying

$$|u_0(x_1) - u_0(x_2)| < \epsilon + A_\epsilon e^{-C_0} |x_1 - x_2|^2 \quad \text{for } x_1, x_2 \in \overline{W},$$

depending on $\epsilon > 0$, and $\varphi \in C^2(\overline{W})$ satisfying

$$-C_0 \leq \varphi \leq 0 \ \text{ on } \overline{W}, \quad \nabla\varphi = 2C_0\vec{\nu} \ \text{ on } \partial W.$$

If $B > 0$ takes large enough, $v_{\epsilon,y}$ is a supersolution of (1)-(2) on $(0,T) \times U_\delta(y)$. Since $v_{\epsilon,y}$ is defined on the small neighborhood of $y \in \overline{W}$, we extend $v_{\epsilon,y}$ to $\overline{W}$ to use the "Invariance Lemma"(see [Sa]). By taking infimum of supersolutions with respect to $\epsilon > 0$ and $y \in \overline{W}$, we construct the desired supersolution.

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Global well-posedness for Schrödinger equations with derivative

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The work was done with the joint study with J. Colliander[1], M. Keel[2], G. Staffilani[3] and T. Tao[4].

1. Introduction

In this note, we shall consider the global well-posedness for the Cauchy problem of 1D derivative nonlinear Schrödinger equations

$$\begin{cases} iu_t + u_{xx} = i\lambda(|u|^2u)_x, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. The equation in (1) is a model of the propagation of circularly polarized Alfvèn waves in magnetized plasma with a constant external magnetic field [15, 19, 20] (see also [11] for the Alfvèn waves).

Let us call our concern to the well-posedness of the Cauchy problem, which means the existence, the uniqueness, the persistency property and the continuous dependence on data. In particular, we refer to the local well-posedness for the result on a certain positive interval, while the global well-posedness for the result over any time intervals. Our main proposition seeks to obtain the global well-posedness for data in a class as large as possible.

2. Known results

Many results are known for the Cauchy problem in the energy space H^1. In [12, 13, 14, 21], it was proved that the Cauchy problem (1) is locally well-posed for

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data u_0 in H^1. This result can be combined with energy conservation to show the global existence of solutions for small data $u_0 \in H^1$.

Let us consider data u_0 in the Sobolev spaces H^s of low order. In [22], the best local well-posedness result was obtained in H^s for $s \geq \frac{1}{2}$, which is sharp in the sense that the data map fails to be uniformly C^3 or C^0 for $s < \frac{1}{2}$ (Bourgain [3], Biagioni-Linares [1], Takaoka [23]). The paper [23] proved the above local solution to be global for $s > \frac{32}{33}$ assuming the smallness condition on data, where the proof uses again the conservation low of H^1 together with a frequency decompositions of Cauchy data (initiated by Bourgain [4, 5]).

3. Results

In this note, we improve the existence result further. Our main result is the following theorem.

Theorem 1. *The Cauchy problem (1) is globally well-posed in H^s for $s > \frac{2}{3}$, assuming $\|u_0\|_{L^2} < \sqrt{2\pi/|\lambda|}$.*

We impose the repulsive smallness condition on data to force the energy positive (used also in [12, 13, 14, 21, 23] via the sharp Gagliardo-Nirenberg inequality [24]). Note that the L^2 norm is conserved in the evolution, in addition, which is invariant under the scaling for (1).

The restriction $s > \frac{2}{3}$ is possibly not sharp, and might be improved by using the *correction term* strategy of [8]. In fact, one may reasonably conjecture that one could extend the global well-posedness result to match the local result at $s > \frac{1}{2}$. We will show, or we are solving this problem in the subsequent paper, with the discovered *correction term*.

4. Preliminaries

First of all, we may assume $\lambda = 1$ by the scaling of u by $\lambda^{-\frac{1}{2}}u$. We define the non-linear map $\mathcal{G} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as

$$\mathcal{G}f(x) = e^{-i\int_{-\infty}^{x} |f(y)|^2\, dy} f(x).$$

For $s \geq 0$, this transformation is relevantly functioning in H^s, besides this is a bicontinuous map from H^s to itself. Via $w = \mathcal{G}u$ and $w_0 = \mathcal{G}u_0$, we have formally that the Cauchy problem (1) can be equivalent to

$$\begin{cases} iw_t + w_{xx} = -iw^2\overline{w}_x - \frac{1}{2}|w|^4 w, \\ w(0) = w_0. \end{cases} \tag{2}$$

The Cauchy problem (2) is interesting through the removal of the derivative term $|u|^2 u_x$ from the equation of (1). Successfully, this is relevant to the local well-posedness in H^s and the global well-posedness in H^1. We shall prove

Proposition 2. *The Cauchy problem (2) is globally well-posedness in H^s for $s > \frac{2}{3}$, assuming $\|w_0\|_{L^2} < \sqrt{2\pi}$.*

Therefrom the standard argument with $u = \mathcal{G}^{-1} w$, $u_0 = \mathcal{G}^{-1} w_0$ correspondingly exhibits Theorem 1.

5. Out line for the proof of Proposition 2

One may expect the extension of the local solution to be global by making the iteration process for local well-posedness. However iteration method can not by itself yield the global well-posedness. Traditionally, the proof of global well-posedness relies on providing the a priori estimate for solutions, besides the local well-posedness result. We know the conservation law is of use in the a priori estimate of solutions. The H^1 conservation law is actually employed to extend the local solutions at infinitely. If there was the conserved estimate for solutions in H^s, $\frac{1}{2} \le s < 1$, we would immediately show Proposition 2.

The proof of Proposition 2 (Theorem 1) is based on the " *I-method* " used by the authors in other nonlinear wave equations. We mention the out line of the strategy briefly. Let E_N be a new energy for solutions in H^s depending on a parameter $N \gg 1$ and take the rescaling. We prove again the local well-posedness result in the space associated to E_N on time intervals of length $\delta \sim 1$. Finally, we perform the iteration on the time intervals to derive the a priori estimate of solutions with rescaling. How is it that our argument is successfully ? The increment of the energy E_N is very small with polynomial order t^α (the corresponding order inspired by the proof of local well-posedness is exponential $e^{\alpha t}$). More precisely, the energy E_N is *almost conserved* for the solutions of (2).

To clear the essential points of the proof, we shall fix $\frac{2}{3} < s < 1$ here.

5.1. **Local well-posedness.** Now let $m(\xi)$ be an even C^∞ monotone multiplier as

$$m(\xi) = \begin{cases} 1, & \text{if } |\xi| < N, \\ (|\xi|/N)^{s-1}, & \text{if } |\xi| > 2N. \end{cases}$$

We define the multiplier operator I from H^s to H^1 such that $\widehat{Iw}(\xi) = m(\xi)\widehat{w}(\xi)$ which salvages the smooth of order $1-s$, where denote the Fourier transformation with respect to space variable by $\widehat{\cdot}$. We also use the same notion to the space-time Fourier transformation, if it creates no confusion. Our substitute energy is defined by

$$E_N(w) = \int_{\mathbb{R}} |\partial_x Iw|^2\, dx - \frac{1}{2}\mathrm{Im} \int_{\mathbb{R}} Iw\overline{Iw}Iw\overline{Iw}_x\, dx,$$

which will describe the conservation law for H^1 solutions, if $I = identity$ [12, 13, 14, 21]. We present here the modified energy appropriate to H^s theory.

Let define the space $X^{s,b}(\mathbb{R}\times\mathbb{R})$ equipped with the norm (first introduced by Bourgain [2])

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})} = \|(1+|\xi|^2)^{\frac{s}{2}}(1+|\tau-\xi^2|^2)^{\frac{b}{2}}\widehat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}.$$

For the time interval I, we define the restricted space $X^{s,b}(I\times\mathbb{R})$ by

$$\|u\|_{X^{s,b}(I\times\mathbb{R})} = \inf\{\|U\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})} \,:\, U(t) = u(t) \text{ on } I\}.$$

Now we are a position to recall the relevant results on the Cauchy problem (2) in a space comparable to the energy $E_N(w)$.

Lemma 3. *There exist $\delta > 0$ depending only on $\|Iw_0\|_{H^1}$ and the unique solution w of the initial value problem (2) in the time interval $[-\delta,\delta]$ such that*

$$\|Iw\|_{X^{1,\frac{1}{2}+}([-\delta,\delta]\times\mathbb{R})} \le C_{\|Iw_0\|_{H^1}}.$$

Let w be a (global) solution of the Cauchy problem (2). For $\mu > 0$,

$$w^\mu(t,x) = \mu^{-\frac{1}{2}} w(\frac{t}{\mu^2}, \frac{x}{\mu}) \tag{3}$$

is again a solution for (2) with data $w_0^\mu(x) = \mu^{-\frac{1}{2}}w_0(\frac{x}{\mu})$, namely the Cauchy problem (2) is invariant under the rescaling $w \mapsto w^\mu$.

In this section, we shall obtain the global well-posedness for the rescaled Cauchy problem. As a consequence, then the scaling $w^\mu \mapsto w$ is used to prove Proposition 2. Choosing $\mu = N^{\frac{1-s}{s}}$, we have easy $\|Iw_0^\mu\|_{H^1} \lesssim 1$ and $E_N(w_0^\mu) \lesssim 1$, still more $\|Iw_0^\mu\|_{L^2} < \sqrt{2\pi}$. Without loss of generality, it is sufficient to treat the non-negative time, because the case of $t < 0$ is similar. Since our constants in Lemma 3 depend only on the H^1 norm for Iw_0^μ, we have $\delta \sim 1$ and $\|Iw^\mu\|_{X^{1,\frac{1}{2}+}([0,\delta]\times\mathbb{R})} \lesssim 1$ (of course, $\|Iw^\mu(\delta)\|_{H^1} \lesssim 1$ by the Sobolev embedding $H^{\frac{1}{2}+} \hookrightarrow L^\infty$). We prove again the Cauchy problem (2) over the time interval of length δ with data $w^\mu(\delta)$, beyond the time $t = \delta$, and go further. That is possible as far as $\|Iw^\mu(t)\|_{H^1} \lesssim 1$.

5.2. **A priori estimate.** The previous results made no use of the energy $E_N(w)$. We know from [12, Lemma 2.4] and [21, Proposition 3.2] that the energy $E_N(w)$ is positive, more precisely, we have

Lemma 4 ([12], [21]). *Assuming* $\|v\|_{L^2} < \sqrt{2\pi}$, *we have*

$$\|\partial_x v\|_{L^2}^2 \le C_{\|v\|_{L^2}} \left(\int_{\mathbb{R}} |\partial_x v|^2 \, dx - \frac{1}{2} Im \int_{\mathbb{R}} v\bar{v}v\bar{v}_x \, dx \right).$$

It is easy to see that our solutions satisfy $\|Iw^\mu\|_{L^2} < \sqrt{2\pi}$ since by $\|Iw\|_{L^2} \le \|w\|_{L^2}$ and the conservation of L^2 norm (the H^s space is sub-critical). Then by Lemma 4, we have

$$\|\partial_x Iw\|_{L^2}^2 \le C_{\|w_0\|_{L^2}} E_N(w). \tag{4}$$

The next step of argument consists in estimating $E_N(w)$. Our energy $E_N(w)$ is not conserved unfortunately, then there seems to be no hope of its unchanged constant. However we obtain the following estimate, which controls the transition of energy in a frequency modes to be uniform.

Lemma 5. *Let* w *be a solution to (2) over* $[t_1, t_2]$. *Then we have*

$$E_N(w(t_2)) - E_N(w(t_1)) \le N^{-1+} C_{\|Iw\|_{X^{1,\frac{1}{2}+}([t_1,t_2]\times\mathbb{R})}}.$$

The energy transportation in frequency modes occurs from the nonlinear interaction. However the time localized observation allows the low frequency interaction to be still remained in the same frequency mode, which is smooth then might be conserved. On the other hand, Lemma 5 evaluates the energy transition of low-high and high-high modes, which is very small ! Then we can apply Lemma 3 to each time interval as well as observing the increment of energy by Lemma 5, if the solution is approximated by evolving from the Cauchy data restricted to the low frequency mode.

5.3. **Induction.** We are now close to the end of the proof of Proposition 2. Let T be a positive time. The iteration argument combining Lemma 3 with (4) and Lemma 5 implies

$$\|Iw^\mu(\mu^2 T)\|_{H^1} \lesssim 1$$

for $\mu^2 T \ll N^{1-}$. Coming back solutions w with scaling $w^\mu \mapsto w$ and with the inverse operator I^{-1}, we have

$$\|w(T)\|_{H^s} \le C_{\mu,N} \tag{5}$$

for all $T \ll \frac{N^{1-}}{\mu^2}$. For our purpose, we need $\frac{N^{1-}}{\mu^2} \to \infty$, for instance, as N tends to infinity. From previously chosen μ, if $s > \frac{2}{3}$, we obtain the a priori estimate for (5) up to $t = T$ (first fix $T > 0$ large, correspondingly, next choose N large enough), and thus complete the proof of Proposition 2.

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INVERSE CONDUCTIVITY PROBLEM, YARMUKHAMEDOV'S GREEN FUNCTION AND MITTAG-LEFFLER'S FUNCTION

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ABSTRACT. We consider a typical inverse problem related to a simple elliptic partial differential equation in a domain Ω in $\mathbb{R}^n (n = 2,3)$. The problem is to extract information about the discontinuity curve , surface for $n = 2,3$, respectively, of the leading coefficient of the equation from the Dirichlet-to-Neumann map on $\partial\Omega$ or its partial knowledge. In this lecture, we describe a recent progress on this subject.

1. The inverse conductivity problem

Let Ω be a bounded domain in $\mathbb{R}^n (n = 2, 3)$ with smooth boundary. We consider Ω an isotropic electric conductive medium and denote by γ its conductivity. We assume that $\gamma \in L^\infty(\Omega)$ and γ has a positive lower bound. Given $f \in H^{1/2}(\Omega)$ let $u \in H^1(\Omega)$ be the weak solution of the elliptic problem

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial\Omega.$$

Let ν denote the unit outward normal vector field to $\partial\Omega$. The map

$$\Lambda_\gamma : f \longmapsto \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

is called the Dirichlet-to-Neumann map associated with the operator $\nabla \cdot \gamma\nabla$ in Ω. f is a voltage potential on $\partial\Omega$; $\Lambda_\gamma f$ is the electric current density on $\partial\Omega$ that induces f. Calderón [5] raised the question whether Λ_γ uniquely determines γ. For this question and several results we refer the reader to the survey paper Uhlmann [27].

In this talk we are interested in a problem of extracting the singularity of the conductivity from the Dirichlet-to-Neumann map. Let D be an open set with Lipschitz boundary satisfying $\overline{D} \subset \Omega$. We assume that γ takes the form

$$\gamma(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus D, \\ 1 + h(x) & \text{if } x \in D \end{cases}$$

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where $h\,(\neq 0)$ is an essentially bounded function in D. In this talk, for simplicity of description we assume that h is constant. Then ∂D becomes the discontinuity surface ($n=3$), curve ($n=2$) of γ. D represents a defect and may have many connected components (see Fig. 1.1).

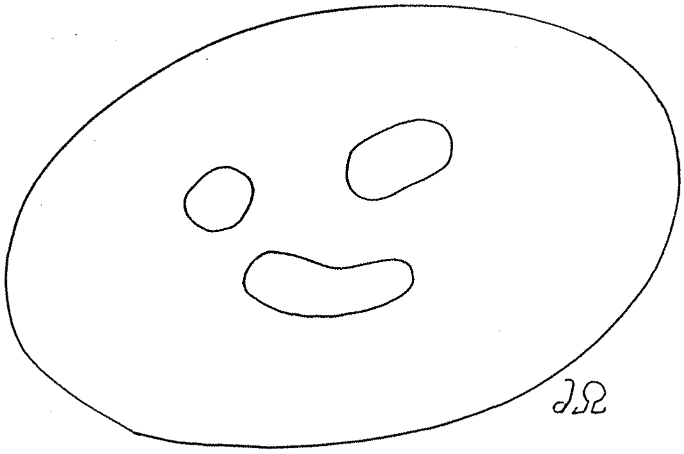

Fig.1.1.

Now the brief description of the problem discussed in this talk is the following.

Problem Find a formula for extracting information about the location of D from the pairs $(\Lambda_\gamma f, f)$ for $f \in \mathcal{D} \subset H^{1/2}(\partial\Omega)$. $\mathcal{D}$ has to be independent of both D and h; $\mathcal{D}$ has to be given explicitly and if possible, it should be as "small" as possible.

The formula is important because it may yield an algorithm for the electrical impedance tomography that has an exact base. The algorithm does not require any solver for the forward problem. Of course, the implementation of the algorithm in the computer itself is another problem.

We refer the reader to the papers [8], [22], [23], [25] and the references therein for the uniqueness resluts.

2. The enclosure method

In [9] we introduced the enclosure method and gave an answer to the problem. The method predicts when a plane ($n=3$), line ($n=2$) with a given normal vector descending from $\partial\Omega$ hits ∂D and yields formulae for drawing a picture of the convex hull of the inclusion. Here we present the result in the three-dimensional case. We denote by h_D the support function of D:

$$h_D(\omega) = \sup_{x\in D} x\cdot\omega,\ \ \omega \in S^2.$$

Given $\omega \in S^2$ choose $\omega^\perp \in S^2$ such that $\omega\cdot\omega^\perp = 0$. Let $\tau > 0$ and $t \in \mathbb{R}$. Define

$$(2.1)\qquad I_{\omega,\omega^\perp}(\tau,t) = e^{-2\tau t}\int_{\partial\Omega}(\Lambda_\gamma - \Lambda_1)(e^{\tau x\cdot(\omega+i\omega^\perp)})\cdot\overline{e^{\tau x\cdot(\omega+i\omega^\perp)}}d\sigma(x).$$

Theorem 2.1([9, 21]).
Assume that ∂D *is* C^2. *We have:*
if $t > h_D(\omega)$, *then*

$$\lim_{\tau \longrightarrow \infty} |I_{\omega,\omega^\perp}(\tau, t)| = 0;$$

if $t < h_D(\omega)$, *then*

$$\lim_{\tau \longrightarrow \infty} |I_{\omega,\omega^\perp}(\tau, t)| = \infty;$$

if $t = h_D(\omega)$, *then*

$$\lim \inf_{\tau \longrightarrow \infty} |I_{\omega,\omega^\perp}(\tau, t)| > 0.$$

Moreover the formula

$$\lim_{\tau \longrightarrow \infty} \frac{\log |I_{\omega,\omega^\perp}(\tau, t)|}{2\tau} = h_D(\omega) - t,$$

is valid.

Therefore to extract $h_D(\omega)$ from the pairs $(\Lambda_\gamma f, f)$ for $f \in \mathcal{D}$ it suffices to take

$$\mathcal{D} = \{e^{\tau x \cdot (\omega + i\omega^\perp)}|_{\partial\Omega} \,|\, \tau > 0\}.$$

An algorithm based on this formula together with the numerical testing is described in [21] (see also [4]).

3. Yarumkhamedov's Green function and the enclosure method in the infinite slab

Yarmukhamedov ([28]) introduced a family of fundamental solutions of the Laplace equation whose members are parametrized by entire functions having some growth property at infinity. In this talk we call them Yarumukhamedov's Green functions. In three-dimensions, his Green functions take the form

$$-2\pi^2 \Phi(x) = \int_0^\infty \mathrm{Im}\,(\frac{K(x_3 + i\sqrt{|x'|^2 + u^2})}{x_3 + i\sqrt{|x'|^2 + u^2}})\frac{du}{\sqrt{|x'|^2 + u^2}}$$

where $x' = (x_1, x_2), |x'|^2 = x_1^2 + x_2^2$ and $K(w)$ is a given entire function of a complex variable w such that

$$K(w) \text{ is real for real } w;$$
$$K(0) = 1;$$
$$\forall R > 0$$
$$\sup_{|\mathrm{Re}\, w| < R} \{|K(w)| + (1 + |\mathrm{Im}\, w|)|K'(w)| + (1 + |\mathrm{Im}\, w|^2)|K''(w)|\} < \infty.$$

He proved that this Φ satisfies $\Delta\Phi + \delta(x) = 0$ in $\mathbb{R}^3$. In [29, 30] He considered the Cauchy problem for the Laplace equation in a domain having special geometry in multi-dimensions. Using his Green functions, he gave explicit formulae for claculating the value of the solution from the Cauchy data on a part of the boundary of the domain. It is a formula of the Carleman type and its prototype goes back to a formula in the complex analysis described in Carleman [6] (see also [1]). In [31] Yarmukhamedov considered the Cauchy problem for the equation $\Delta u - \lambda^2 u = 0$. This is the application to the Cauchy problem for elliptic equations. We believe that there should be other interesting applications for his Green functions. Here we present an application to an inverse problem ([16]). In this section Ω is considered an unbounded domain in $\mathbb{R}^3$ enclosed by two parallel planes. Therefore one can not acces the side. The assumptions on D and h are same as those mentioned in section 1. See Fig. 3.1 below for the geometry.

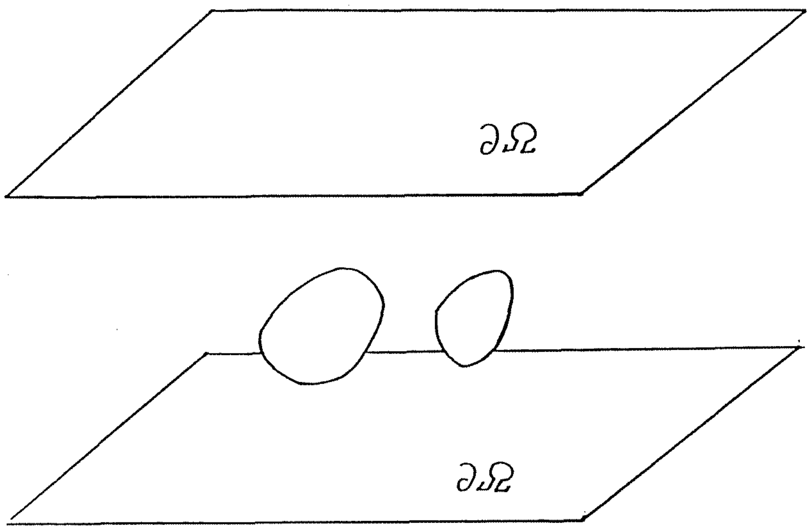

Fig.3.1.

Now choose the system of the Cartesian coordinates in the three-dimensional Euclidian space in such a way that

$$\Omega = \mathbb{R}^2 \times]a, b[$$

where $0 < a < b < \infty$ and $b - a = d$. d stands for the thickness of Ω.

To describe the result clearly we introduce the following notation. Let $R > 0$ and $\epsilon \in]0, a[$. Define

$$\Omega_\epsilon(R) = \{(x', x_3) \,|\, |x'| < R,\ x_3 \in]a - \epsilon, b + \epsilon[\}$$

and

$$\Omega_0(R) = \{(x', x_3) \,|\, |x'| < R,\ x_3 \in]a, b[\}.$$

One has the disjoint decomposition of the boundary:

$$\partial\Omega_\epsilon(R) = \Gamma_\epsilon^+(R) \cup \Gamma_\epsilon^-(R) \cup \Gamma_\epsilon'(R)$$

where

$$\Gamma_\epsilon^+(R) = \{(x', x_3) \,|\, |x'| < R,\ x_3 = b + \epsilon\},$$
$$\Gamma_\epsilon^-(R) = \{(x', x_3) \,|\, |x'| < R,\ x_3 = a - \epsilon\},$$
$$\Gamma_\epsilon'(R) = \{(x', x_3) \,|\, |x'| = R,\ a - \epsilon \le x_3 \le b + \epsilon\}.$$

Let $0 < \delta < 1$. Define

$$\Omega_\epsilon^\delta(R) = \cup_{y \in \Gamma'_\epsilon(R)} \{x \in \Omega_0(R) \,|\, |y_3 - x_3| > \delta |y' - x'|\}.$$

See fig. 3.2 for the geometry.

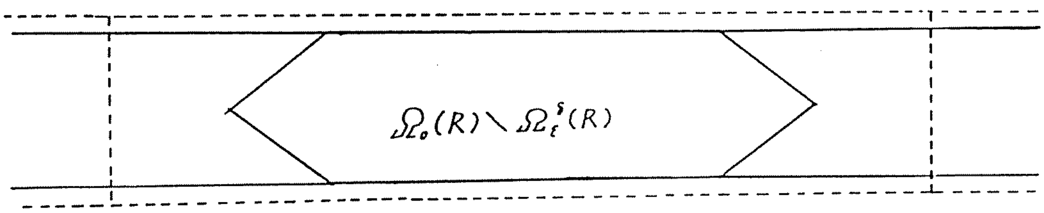

Fig.3.2.

Note that if R is sufficiently large, then $\Omega_0(R) \setminus \Omega_\epsilon^\delta(R) \neq \emptyset$.

We consider a Yarmukhamedov's Green function for special $K(w)$:

$$K(w) = e^{mw^2}.$$

There is no description about this choice in Yarmukhamedov's papers. This function clearly satisfies assumptions on K mentioned above. We write $\Phi(x) = \Phi_m(x)$. $\Phi_m(x)$ becomes

$$\begin{aligned} -2\pi^2 \Phi_m(x) &= \int_0^\infty \operatorname{Im}\left(\frac{e^{m(x_3 + i\sqrt{|x'|^2+u^2})^2}}{x_3 + i\sqrt{|x'|^2+u^2}}\right) \frac{du}{\sqrt{|x'|^2+u^2}} \\ &= e^{m(x_3^2 - |x'|^2)} \int_0^\infty e^{-mu^2} w_m(x,u) \frac{du}{|x|^2+u^2} \end{aligned}$$

where

$$w_m(x,u) = x_3 \frac{\sin 2mx_3\sqrt{|x'|^2+u^2}}{\sqrt{|x'|^2+u^2}} - \cos 2mx_3\sqrt{|x'|^2+u^2}.$$

Using this function, we give an explicit harmonic function in a neighbourhood of $\overline{\Omega}$.

Definition 3.1. Define

$$\begin{aligned} e_m &= e_m(x;\tau,\omega,\omega^\perp) \\ &= \int_{\Gamma_\epsilon^+(R) \cup \Gamma_\epsilon^-(R)} \{\Phi_m(y-x) \frac{\partial}{\partial\nu(y)} e^{\tau y\cdot(\omega + i\omega^\perp)} - e^{\tau y\cdot(\omega+i\omega^\perp)} \frac{\partial}{\partial\nu(y)} \Phi_m(y-x)\} d\sigma(y). \end{aligned}$$

This function is harmonic in $\mathbb{R}^3 \setminus \overline{(\Gamma_\epsilon^+(R) \cup \Gamma_\epsilon^-(R))}$ and thus in a neighbourhood of $\overline{\Omega}$.

e_m has the following important property.

Proposition 3.1.

e_m as $m \longrightarrow \infty$, approximates $e^{\tau x\cdot(\omega+i\omega^{\perp})}$ in $\Omega_0(R)\setminus\Omega_\epsilon^\delta(R)$ and 0 in $\Omega\setminus\Omega_0(R+R')$.

Now take

$$\mathcal{D} = \{e_m(x;\tau,\omega,\omega^{\perp})|_{\partial\Omega}\,|\,m=1,\cdots;\tau>0\}.$$

Theorem 3.1([16]).

Let $R' > d+\epsilon$ and $\eta > 0$. Assume that

$$\overline{D} \subset \Omega_0(R)\setminus\Omega_\epsilon^\delta(R).$$

For each $\omega \in S^2$ one can calculate $h_D(\omega)$ from the pairs $(\Lambda_\gamma f|_{\{|x'|\le R+R'+\eta\}}, f)$ for $f \in \mathcal{D}$.

Note that the results dose not require $\Lambda_\gamma f$ on the whole boundary.

4. Mittag-Leffler's function and a generalization of the enclosure method

One can rewrite (2.1) as

$$I_{\omega,\omega^{\perp}}(\tau,t) = \int_{\partial\Omega}(\Lambda_\gamma - \Lambda_1)(e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})})\cdot\overline{e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})}}d\sigma(x).$$

The function $e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})}$ is harmonic and has the following special property:

if $x\cdot\omega > t$, then $|e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})}| \longrightarrow \infty$ as $\tau \longrightarrow \infty$;

if $x\cdot\omega < t$, then $|e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})}| \longrightarrow 0$ as $\tau \longrightarrow \infty$.

For the proof of theorem 2.1 the harmonicity of $e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})}$ is important.

These suggest that a family of surfaces S_t and harmonic functions $e_\tau(x;t)$ having the following property will play a similar role for $e^{-\tau t}e^{\tau x\cdot(\omega+i\omega^{\perp})}$ and yields a generalization of the enclosure method:

S_t devides the whole space into two parts H_t^+ and H_t^-;

if $x \in H_t^+$, then $|e_\tau(x;t)| \longrightarrow \infty$ as $\tau \longrightarrow \infty$;

if $x \in H_t^-$, then $|e_\tau(x;t)| \longrightarrow 0$ as $\tau \longrightarrow \infty$.

In this section we show that such a generalization is possible if one makes use of Mittag-Leffler's function insted of the exponential function. First let us recall its definition. Let $0 < \alpha \le 1$. The entire function

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad z \in \mathbb{C}$$

is called Mittag-Leffler's function ([2, pp.206-208], [24]). This includes e^z as a special case be cause $E_1(z) = e^z$. If $0 < \alpha < 1$, this function has the following remarkable property:

if $|\arg z| \le \pi\alpha/2$, then

$$E_\alpha(z) \sim \frac{1}{\alpha}e^{z^{1/\alpha}};$$

if $\pi\alpha/2 < |\arg z| \leq \pi$, then

$$E_\alpha(z) \sim -\frac{z^{-1}}{\Gamma(1-\alpha)}.$$

Here we consider only two-dimensional case. Three-dimensional case will be reported eleswhere.

Let $y \in \mathbb{R}^2$ and $\omega \in S^1$. Take $\omega^\perp \in S^1$ such that $\omega \cdot \omega^\perp = 0$. For each $t \in \mathbb{R}$ consider the functions dependeing $\tau > 0$:

$$e_\tau^\alpha(x; y, \omega, \omega^\perp, t) = E_\alpha(\tau\{(x-y)\cdot\omega - t + i(x-y)\cdot\omega^\perp\}).$$

This functions are harmonic. Let $C_{y+t\omega}(\omega, \pi\alpha/2)$ denote the c one about ω of opening angle $\pi\alpha/2$ with vertex at $y + t\omega$ (see Fig. 4.1).

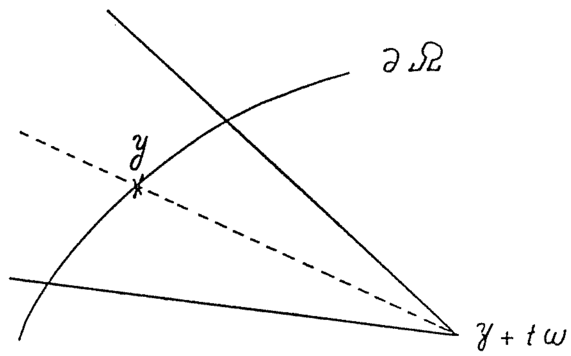

Fig.4.1.

From the property of $E_\alpha(z)$ mentioned above one knows that:

if $x \in C_{y+t\omega}(\omega, \pi\alpha/2) \setminus \partial C_{y+t\omega}(\omega, \pi\alpha/2)$, then $|v_\tau^\alpha(x; y, \omega, t)| \longrightarrow \infty$ as $\tau \longrightarrow \infty$;

if $x \in \mathbb{R}^2 \setminus C_{y+t\omega}(\omega, \pi\alpha/2)$, then $|v_\tau^\alpha(x; y, \omega, t)| \longrightarrow 0$ as $\tau \longrightarrow \infty$.

Define

$$I^\alpha_{(y,\omega)}(\tau, t) = \int_{\partial\Omega} (\Lambda_\gamma - \Lambda_1) f^\alpha(x) \cdot \overline{f^\alpha(x)} d\sigma(x), \ \tau > 0$$

where

$$f^\alpha(x) = e_\tau^\alpha(x; y, \omega, t), x \in \partial\Omega.$$

$I^\alpha_{(y,\omega)}(\tau, t)$ does not depend on the choice of $\omega^\perp$.

Definition 4.1. Given $(y, \omega) \in \partial\Omega \times S^1$ with

$$C_y(\omega, \frac{\pi\alpha}{2}) \subset \mathbb{R}^2 \setminus \Omega \tag{4.1}$$

define

$$h_D^\alpha(y, \omega) = \inf\{t \in]-\infty, 0[\ |\ \forall s \in]t, 0[\ C_{y+s\omega}(\omega, \frac{\pi\alpha}{2}) \subset \mathbb{R}^2 \setminus \overline{D}\}.$$

See Fig. 4.2 below for the meaning of $h_D^\alpha(y,\omega)$.

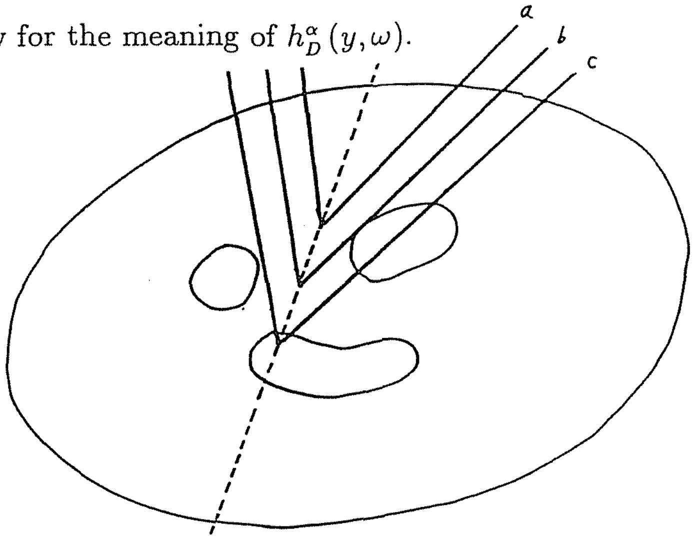

Fig. 4.2. $a:\ t > h_D^\alpha(y,\omega)$, $b:\ t = h_D^\alpha(y,\omega)$, $c:\ t < h_D^\alpha(y,\omega)$.

Theorem 4.1([19]).

Let $0<\alpha<1$. Let $(y,\omega)\in\partial\Omega\times S^1$ satisfy (4.1). We have:

if $t > h_D^\alpha(y,\omega)$, then

$$\lim_{\tau\longrightarrow\infty}|I^\alpha_{(y,\omega)}(\tau,t)| = 0;$$

if $t < h_D^\alpha(y,\omega)$, then

$$\lim_{\tau\longrightarrow\infty}|I^\alpha_{(y,\omega)}(\tau,t)| = \infty;$$

if $t = h_D^\alpha(y,\omega)$, then

$$\lim\inf_{\tau\longrightarrow\infty}|I^\alpha_{(y,\omega)}(\tau,t)| > 0.$$

This gives the characterization of $h_D^\alpha(y,\omega)$:

$$]h_D^\alpha(y,\omega),0[=\{t\in]-\infty,0[\,|\lim_{\tau\longrightarrow\infty}I^\alpha_{(y,\omega)}(\tau,t)=0\}.$$

Therefore to extract $h_D^\alpha(y,\omega)$ from the pairs $(\Lambda_\gamma f, f)$ for $f\in\mathcal{D}$ it suffices to take

$$\mathcal{D}=\{e^\alpha_\tau(\,\cdot\,;y,\omega,t)|_{\partial\Omega}\,|\,\tau>0,-\infty<t<0\}.$$

5. Remarks

Remark 5.1. The enclosure method presented in section 1 can cover the case when the conductivity of Ω takes the form

$$\gamma(x)=\begin{cases}\gamma_0(x) & \text{if } x\in\Omega\setminus D,\\ \gamma_0(x)+h(x) & \text{if } x\in D\end{cases}$$

where γ_0 is a known smooth function having a positive lower bound. Instead of $e^{\tau x\cdot(\omega+i\omega^\perp)}$, just use the exponentially growing solutions for the equation $\nabla\cdot\gamma_0\nabla u=0$ which are constructed by Sylvester-Uhlmann [26]. For the construction Faddeev's Green function [7] played the central role. In [16], using an integral reprentation of Faddeev's Green function obtained by Beals-Coifman [3], we pointed out that Faddeev's Green function is a special member of Yarmukhamedov's Green functions.

Remark 5.2.

This remark is closely related to remak 5.1. It would be interesting to consider whether one can construct a special solution of the equation $\nabla \cdot \gamma_0 \nabla u = 0$ that coincides with $e_\tau^\alpha(x; y, \omega, t)$ when γ_0 is constant and plays the same role.

Remark 5.3.

[18] is a survey paper on the enclosure method.

For other applications of the enclosure method see:

[9,11] (inverse boundary value problems for the Helmholtz equation which is closely related to the inverse obstacle scattering problem);

[10] (inverse source problem for the Helmholtz equation);

[12, 13](inverse conductivity problem with one measurement);

[15](reconstruction from the difference of the voltage potentials between given two points on $\partial\Omega$);

[17] (the Cauchy problem for the stationary Schrödinger equation);

[20] (a numerical testing of an algorithm based on a formula in [12]).

And we refer the reader to [14], introduction of [16] for other methods and their applications.

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