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Spectral analysis of an abstract pair interaction model
(抽象的な対相互作用モデルのスペクトル解析)

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Abstract

We consider an abstract pair-interaction model in quantum field theory with a coupling constant $\lambda \in \mathbb{R}$ and analyze the Hamiltonian $H(\lambda)$ of the model. In the massive case, there exist constants $\lambda_{\mathrm{c}} < 0$ and $\lambda_{\mathrm{c},0} < \lambda_{\mathrm{c}}$ such that, for each $\lambda \in (\lambda_{\mathrm{c},0}, \lambda_{\mathrm{c}}) \cup (\lambda_{\mathrm{c}}, \infty)$, $H(\lambda)$ is diagonalized by a proper Bogoliubov transformation, so that the spectrum of $H(\lambda)$ is explicitly identified, where the spectrum of $H(\lambda)$ for $\lambda > \lambda_{\mathrm{c}}$ is different from that for $\lambda \in (\lambda_{\mathrm{c},0}, \lambda_{\mathrm{c}})$. As for the case $\lambda < \lambda_{\mathrm{c},0}$, we show that $H(\lambda)$ is unbounded from above and below. In the massless case, λ_{c} coincides with $\lambda_{\mathrm{c},0}$.

Contents

1 Introduction

This thesis is based on the joint work [6]. We consider an abstract pair-interaction model in quantum field theory. The Hamiltonian of the model is of the form

$$H(\lambda) := \mathrm{d}\Gamma_{\mathrm{b}}(T) + \frac{\lambda}{2}\Phi_{\mathrm{s}}(g)^2$$

acting in the boson Fock space $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ over a Hilbert space $\mathscr{H}$ (see Subsection 2.1), where T is a self-adjoint operator on $\mathscr{H}$, $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ is the second quantization operator of T, $\Phi_{\mathrm{s}}(g)$ is the Segal field operator with test vector g in $\mathscr{H}$ (see Subsection 2.1) and $\lambda \in \mathbb{R}$ is a coupling constant. A model of this type is called a ϕ^2-model.

There have been many studies on massive or massless ϕ^2-models in concrete forms or abstract forms (see, e.g., [4, 8, 9, 11, 12, 16]). In [11] and [16], the (essential) self-adjointness of the Hamiltonian of a ϕ^2-model is proved in the case where $\lambda > 0$ or $|\lambda|$ is sufficiently small. In [11], the existence of a ground state of a ϕ^2-model also is shown in the case where the quantum field under consideration is massive and $\lambda > 0$.

It is a well known that Hamiltonians with linear and/or quadratic interactions in quantum fields may be analyzed by the method of Bogoliubov transformations (see, e.g., [1, 2, 3, 4, 7, 8, 10, 12]). A typical Bogoliubov transformation is constructed from bounded linear operators U, V and a conjugation operator J on $\mathscr{H}$ satisfying the following equations:

$$\begin{cases} U^*U - V^*V &= I, \\ U_J^*V - V_J^*U &= 0, \\ UU^* - V_JV_J^* &= I, \\ UV^* - V_JU_J^* &= 0, \end{cases} \tag{1.1}$$

where $A_J := JAJ$ and A^* is the adjoint of a densely defined linear operator A. It is well known that there is a unitary operator $\mathbb{U}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ which implements the Bogoliubov transformation in question if and only if V is Hilbert-Schmidt [7, 13, 14, 15]. Moreover, it is shown that, under the condition that V is Hilbert-Schmidt and suitable additional conditions, the Hamiltonian under consideration is unitarily equivalent via $\mathbb{U}$ to a second quantization operator up to a constant addition. For example, the Pauli-Fierz model with dipole approximation, which can be regarded as a kind of ϕ^2-model, is analyzed by this method in [10].

Recently, a general quadratic form Hamiltonian with a coupling constant $\lambda \in \mathbb{R}$ has been analyzed in [12] and it is shown that, in the case of a massive quantum field, under suitable conditions, the Hamiltonian is diagonalized by a Bogoliubov transformation. In [8], the sufficient condition formulated in [12] to obtain the result just mentioned has been extended. The spectrum of the standard pair-interaction model in physics, which is a concrete realization of the abstract pair-interaction model, is formally known [9] in the case where $\lambda > \lambda_{\mathrm{c},0}$ and $\lambda \neq \lambda_{\mathrm{c}}$ for the constants λ_{c} and $\lambda_{\mathrm{c},0}$ which satisfy $\lambda_{\mathrm{c},0} < \lambda_{\mathrm{c}}$. The paper [4] gives a rigorous proof for that in the framework of the boson Fock space theory over $\mathscr{H} = L^2(\mathbb{R}^d)$ for any $d \in \mathbb{N}$ and $\lambda > \lambda_{\mathrm{c}}$.

One of the motivations for the present work is to extend the theory developed in [4] with $\mathscr{H} = L^2(\mathbb{R}^d)$ to the theory with $\mathscr{H}$ being an abstract Hilbert space including the case where $\lambda < \lambda_{\mathrm{c}}$. It is a well known fact (see [9]) that the spectral properties of the standard pair-interaction model may depend on whether $\lambda > \lambda_{\mathrm{c}}$ or $\lambda < \lambda_{\mathrm{c}}$. Hence it is important to clarify this aspect mathematically. Therefore we analyze our model also for the region $\lambda < \lambda_{\mathrm{c}}$. We show that, in the massive case with $\lambda \in (\lambda_{\mathrm{c},0}, \lambda_{\mathrm{c}})$ also, the method of Bogoliubov transformations can be applied to prove that the Hamiltonian $H(\lambda)$ is unitarily equivalent to a second quantization operator up to a constant addition. Then we see that the spectrum of $H(\lambda)$ for $\lambda \in (\lambda_{\mathrm{c},0}, \lambda_{\mathrm{c}})$ is different from that for $\lambda > \lambda_{\mathrm{c}}$. In the massless case, $\lambda_{\mathrm{c},0}$ coincides with λ_0.

The main results of the present paper include the following (1)–(3) (see Theorem 2.8 for more details): (1) Identification of the spectra of $H(\lambda)$ for $\lambda > \lambda_{\mathrm{c}}$. (2) Identification of the spectra of $H(\lambda)$ for $\lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}$ it is only in the massive case; in the massless case, $\lambda_{\mathrm{c},0} = \lambda_{\mathrm{c}}$). In this case, bound states different from the ground state appear. (3) Unboundedness of $H(\lambda)$ from above and below for $\lambda < \lambda_{\mathrm{c},0}$.

The outline of this paper is as follows. In Section 2, we define our model and recall a fundamental fact in a general theory of Bogoliubov transformations. We prove the (essential) self-adjointness of $H(\lambda)$ (Theorem 2.3). Then we state the main theorem of this paper (Theorem 3.6). In Section 3, we construct the operators U and V which are used to define the Bogoliubov transformation we need. In Section 4, we show that U and V satisfy (1.1) and V is Hilbert-Schmidt. In Section 5, we prove Theorem 2.8 (1) and calculate the ground state energy of $H(\lambda)$ in the case $\lambda > \lambda_c$. In Section 6, we prove Theorem 2.8 (2). In Section 7, we prove Theorem 2.8 (3). In Section 8, we consider a slightly generalized Hamiltonian which is of the form $H(\eta,\lambda) := H(\lambda) + \eta\Phi_S(f)$ for $\eta \in \mathbb{R}$ and $f \in \mathscr{H}$. Applying the methods and results in the preceding sections, we analyze $H(\eta,\lambda)$ and identify the spectra of it. In Appendix, we state some basic facts in the theory of boson Fock space.

2 Preliminaries

2.1 The abstract boson Fock Space

Let $\mathscr{H}$ be a Hilbert space over the complex field $\mathbb{C}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$. The inner product is linear in the second variable and anti-linear in the first one. The symbol $\|\cdot\|_{\mathscr{H}}$ denotes the norm associated with it. We omit $\mathscr{H}$ in $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$, respectively if there is no danger of confusion. For each non-negative integer $n = 0,1,2,\ldots,$ $\otimes_{\mathrm{s}}^n\mathscr{H}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathscr{H}$ with convention $\otimes_{\mathrm{s}}^0\mathscr{H} := \mathbb{C}$. Then

$$\mathscr{F}_{\mathrm{b}}(\mathscr{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathscr{H}$$

is called the boson Fock space over $\mathscr{H}$. For a dense subspace $\mathscr{D}$ in $\mathscr{H}$, $\hat{\otimes}_{\mathrm{s}}^n\mathscr{D}$ denotes the algebraic n-fold symmetric tensor product of $\mathscr{D}$ with $\hat{\otimes}_{\mathrm{s}}^0\mathscr{H} := \mathbb{C}$. Then

$$\mathscr{F}_{\mathrm{b,fin}}(\mathscr{D}) := \hat{\oplus}_{n=0}^{\infty}\hat{\otimes}_{\mathrm{s}}^n\mathscr{D}$$

is a dense subspace of $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, where $\hat{\oplus}_{n=0}^{\infty}\mathscr{D}_n$ stands for the algebraic direct sum of subspace $\mathscr{D}_n \subset \otimes_{\mathrm{s}}^n\mathscr{H}, n = 0,1,2,\ldots$. The finite particle vector subspace

$$\mathscr{F}_{\mathrm{b},0}(\mathscr{H}) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \;\middle|\; \begin{array}{c} \psi^{(n)} \in \otimes_{\mathrm{s}}^n\mathscr{H},\ n \geq 0, \text{there is an integer } n_0 \in \mathbb{N} \\ \text{such that } \psi^{(n)} = 0, \text{ for all } n \geq n_0 \end{array} \right\}$$

satisfies $\mathscr{F}_{\mathrm{b,fin}}(\mathscr{D}) \subset \mathscr{F}_{\mathrm{b},0}(\mathscr{H}) \subset \mathscr{F}_{\mathrm{b}}(\mathscr{H})$, in particular, it is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. For a linear operator T on a Hilbert space, the domain of T will be denoted by $D(T)$.

For a densely defined closable operator T on $\mathscr{H}$, let $T_{\mathrm{b}}^{(n)}$ be the densely defined closed operator on $\otimes_{\mathrm{s}}^n\mathscr{H}$ defined by

$$T_{\mathrm{b}}^{(n)} := \begin{cases} \overline{\displaystyle\sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \overbrace{T}^{j-\mathrm{th}} \otimes I \otimes \cdots \otimes I \restriction \hat{\otimes}_{\mathrm{s}}^n D(T)}, & n \geq 1, \\ 0, & n = 0, \end{cases}$$

where I denotes the identity operator on $\mathscr{H}$, $\overline{A}$ denotes the closure of a closable operator A and $A \restriction \mathcal{M}$ denotes the restriction of a linear operator A on a subspace $\mathcal{M}$. The operator

$$\mathrm{d}\Gamma_{\mathrm{b}}(T) := \oplus_{n=0}^{\infty} T_{\mathrm{b}}^{(n)}$$

is called the second quantization operator of T. If T is self-adjoint or non-negative, then so is $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. For each $f \in \mathscr{H}$, there exists a unique densely defined closed operator $A(f)$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that its adjoint $A(f)^*$ is given as follows:

$$D(A(f)^*) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \;\middle|\; \sum_{n=1}^{\infty} n \left\| S_n \left(f \otimes \psi^{(n-1)} \right) \right\|^2 < \infty \right\},$$

$$(A(f)^*\psi)^{(n)} = \sqrt{n} S_n (f \otimes \psi^{(n-1)}),\ n \in \mathbb{N}, \quad (A(f)^*\psi)^{(0)} = 0 \text{ for } \psi \in D(A(f)^*),$$

where S_n is the symmetrization operator on the n-fold tensor product $\otimes^n \mathscr{H}$ of $\mathscr{H}$. The operator $A(f)$ (resp. $A(f)^*$) is called the annihilation (resp. creation) operator with test vector f. We have

$$\mathscr{F}_{\mathrm{b},0}(\mathscr{H}) \subset D(A(f)) \cap D(A(f)^*)$$

for all $f \in \mathscr{H}$ and $A(f)$ and $A(f)^*$ leave $\mathscr{F}_{\mathrm{b},0}(\mathscr{H})$ invariant. Moreover, they satisfy the following commutation relations:

$$[A(f), A(g)^*] = \langle f, g \rangle, \quad [A(f), A(g)] = 0, \quad [A(f)^*, A(g)^*] = 0, \quad \text{for all } f, g \in \mathscr{H} \tag{2.1}$$

on $\mathscr{F}_{\mathrm{b},0}(\mathscr{H})$, where $[A, B] := AB - BA$ is the commutator of linear operators A and B. The relation (2.1) is called the canonical commutation relations (CCR) over $\mathscr{H}$. The symmetric operator

$$\Phi_{\mathrm{s}}(f) := \frac{1}{\sqrt{2}}(A(f) + A(f)^*),\ f \in \mathscr{H}$$

is called the Segal field operator with test vector f. We write its closure by the same symbol.

2.2 Bogoliubov Transformation

In this subsection, we define a Bogoliubov transformation and recall an important theorem about it. For a conjugation J on $\mathscr{H}$ (i.e., J is an anti-linear operator on $\mathscr{H}$ satisfying $\|Jf\| = \|f\|$ for all $f \in \mathscr{H}$ and $J^2 = I$) and a linear operator A on $\mathscr{H}$, we define

$$A_J := JAJ.$$

Definition 2.1. *Let U and V be bounded linear operators on $\mathscr{H}$ and J be a conjugation on $\mathscr{H}$. For each $f \in \mathscr{H}$, let a linear operator $B(f)$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ be given by*

$$B(f) := A(Uf) + A(JVf)^*.$$

Then the correspondence $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is called a Bogoliubov transformation.

By $\mathscr{F}_{\mathrm{b},0}(\mathscr{H}) \subset D(B(f))$, the adjoint $B(f)^*$ exists and the equation $B(f)^* = A(Uf)^* + A(JVf)$ holds on $\mathscr{F}_{\mathrm{b},0}(\mathscr{H})$ for each $f \in \mathscr{H}$. If the equations

$$U^*U - V^*V = I, \quad U_J^*V - V_J^*U = 0$$

hold, then the Bogoliubov transformation preserves CCR, i.e., it holds that

$$[B(f), B(g)^*] = \langle f, g \rangle, [B(f), B(g)] = 0, [B(f)^*, B(g)^*] = 0, \text{ for all } f, g \in \mathscr{H},$$

on $\mathscr{F}_{\mathrm{b},0}(\mathscr{H})$. The following theorem is well known (see [14, 15]):

Theorem 2.2. *Let $\mathscr{H}$ be separable and the operators U and V satisfy* (1.1)*. Then there exists a unitary operator $\mathbb{U}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that*

$$\mathbb{U}\overline{B(f)}\mathbb{U}^{-1} = A(f), \quad f \in \mathscr{H}$$

if and only if V is Hilbert-Schmidt.

2.3 Hamiltonians

For a self-adjoint operator T on $\mathscr{H}$, constants $\lambda, \eta \in \mathbb{R}$ which are called coupling constants, and vectors $f, g \in \mathscr{H}$, we define Hamiltonians $H(\lambda)$ and $H(\eta, \lambda)$ by

$$H(\lambda) := \mathrm{d}\Gamma_{\mathrm{b}}(T) + \frac{\lambda}{2}\Phi_{\mathrm{s}}(g)^2, \quad H(\eta, \lambda) := H(\lambda) + \eta\Phi_{\mathrm{s}}(f).$$

If $g = 0$, then $H(\lambda)$ and $H(\eta, \lambda)$ are well-known operators. Thus, we always assume that $g \neq 0$ in the present paper. If $g \in D(T^{-1/2})$, let the constant be defined by

$$\lambda_{\mathrm{c},0} := -\|T^{-1/2}g\|^{-2}.$$

Theorem 2.3. *Suppose that T is an injective, non-negative, self-adjoint operator on $\mathscr{H}$. Let $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T)$. Then the following* (1)*-*(3) *hold:*

(1) *Let*

$$\lambda_T(g) := \|T^{-1/2}g\|^{-1}(\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1} \tag{2.2}$$

and $|\lambda| < \lambda_T(g)$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ and essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.

(2) *Let $|\lambda| \geq \lambda_T(g)$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ for all $\eta \in \mathbb{R}$. Moreover, if $\lambda \geq \lambda_T(g)$, then $H(\eta, \lambda)$ is self-adjoint.*

(3) *Let* $f \in D(T^{1/2})$*. Then* $\overline{H(\lambda_{c,0})}$ *is bounded from below. Moreover, if* $\lambda > \lambda_{c,0}$*, then* $\overline{H(\eta,\lambda)}$ *is also bounded from below for all* $\eta \in \mathbb{R}$ *and* $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) = D((\overline{H(\eta,\lambda)} + M)^{1/2})$ *for all constant* $M \geq 0$ *satisfying* $\overline{H(\eta,\lambda)} + M \geq 0$.

Proof. (1) For any $\lambda \in \mathbb{R}$, by using (2.1), (10.1), (10.2) and [5, Theorem 5.18.], there are constants $a, b \geq 0$ such that for all $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$\left\| \frac{\lambda}{2}\Phi_{\mathrm{s}}(g)^2\psi \right\| \leq \frac{|\lambda|}{4}\left(a\|\mathrm{d}\Gamma_{\mathrm{b}}(T)\psi\| + b\|\psi\|\right).$$

In particular, we can choose a and b which satisfy $a|\lambda|/4 < 1$ if $|\lambda| < \lambda_T(g)$. We remark that, to obtain the factor $\lambda_T(g)$, we need to deform terms $\|A(g)^{*2}\psi\|^2$, $\|A(g)^*A(g)\psi\|^2$ and $\|A(g)^2\psi\|^2$ coming from $\|\Phi_{\mathrm{s}}(g)^2\psi\|^2$ ($\psi \in \mathscr{F}_{\mathrm{b},0}(\mathscr{H})$) to $\|A(g)A(g)^*\psi\|^2+$ a marginal term respectively. Thus, for $|\lambda| < \lambda_T(g)$, by the Kato-Rellich theorem, $H(\lambda)$ is self-adjoint. It is well known that $\Phi_{\mathrm{s}}(f)$ is infinitesimally small with respect to $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. Hence, by the Kato-Rellich theorem, for $|\lambda| < \lambda_T(g)$, $H(\eta,\lambda)$ is self-adjoint.

(2) Firstly, we show that, for any $f \in D(T^{1/2})$ and $\eta, \lambda \in \mathbb{R}$, $H(\eta,\lambda)$ is essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. By (10.1), (10.2) and [5, Theorem 5.18.], we can see that there exists $a > 0$ such that $\|H(\eta,\lambda)\psi\| \leq a\|(\mathrm{d}\Gamma_{\mathrm{b}}(T) + I)\psi\|$ for all $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$. Let $f \in D(T)$. Then by (2.1) and (10.3), for any $\psi \in \mathscr{F}_{\mathrm{b,fin}}(D(T))$, we have

$$\begin{aligned}&\langle H(\eta,\lambda)\psi, (\mathrm{d}\Gamma_{\mathrm{b}}(T) + I)\psi\rangle - \langle(\mathrm{d}\Gamma_{\mathrm{b}}(T) + I)\psi, H(\eta,\lambda)\psi\rangle \\ =&\frac{\lambda}{\sqrt{2}}\left(\langle\Phi_{\mathrm{s}}(g)\psi, A(Tg)\psi\rangle - \langle A(Tg)\psi, \Phi_{\mathrm{s}}(g)\psi\rangle\right) + \frac{\eta}{\sqrt{2}}(\langle\psi, A(Tf)\psi\rangle - \langle A(Tf)\psi,\psi\rangle).\end{aligned}$$

Thus, by (10.1) and (10.2), we obtain

$$|\langle H(\eta,\lambda)\psi, (\mathrm{d}\Gamma_{\mathrm{b}}(T) + I)\psi\rangle - \langle(\mathrm{d}\Gamma_{\mathrm{b}}(T) + I)\psi, H(\eta,\lambda)\psi\rangle| \leq C\|(\mathrm{d}\Gamma_{\mathrm{b}}(T) + I)^{1/2}\psi\|^2, \tag{2.3}$$

where $C := \left\{|\lambda|\|T^{1/2}g\|(\|g\| + 2\|T^{-1/2}g\|) + \sqrt{2}|\eta|\|T^{1/2}f\|\right\}$. By a limiting argument, using the fact that $\mathscr{F}_{\mathrm{b,fin}}(D(T))$ is a core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ and $\mathrm{d}\Gamma_{\mathrm{b}}(T)$-boundedness of $\Phi_{\mathrm{s}}(g)^2$, we can show that for $f \in D(T^{1/2})$ and $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$, (2.3) holds. Thus, by the Nelson commutator theorem, for all $\eta, \lambda \in \mathbb{R}$, $H(\eta,\lambda)$ is essentially self-adjoint and $\overline{H(\eta,\lambda)}$ is essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. The equation $\overline{H(\eta,\lambda) \restriction \mathscr{D}} = \overline{\overline{H(\eta,\lambda)} \restriction \mathscr{D}}$ holds for any core $\mathscr{D}$ of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. Hence $H(\eta,\lambda)$ is essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ for all $\eta, \lambda \in \mathbb{R}$. Next we show that, if $\lambda > \|T^{-1/2}g\|^{-1}(\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1}$, then $H(\eta,\lambda)$ is self-adjoint. We can show that, for $\lambda > 0$ and any $0 < \varepsilon < 1$, there is a constant $c_\varepsilon > 0$ such that

$$(1-\varepsilon)\|\mathrm{d}\Gamma_{\mathrm{b}}(T)\psi\|^2 + \left\|\frac{\lambda}{2}\Phi_{\mathrm{s}}(g)^2\psi\right\|^2 \leq \|H(\eta,\lambda)\psi\|^2 + c_\varepsilon\|\psi\|^2, \quad \psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)).$$

Hence $H(\eta, \lambda)$ is closed. In particular, it is self-adjoint.

(3) From the fact that $\Phi_s(f)$ is infinitesimally small with respect to $d\Gamma_b(T)$, for any $\varepsilon > 0$, $\varepsilon d\Gamma_b(T) + \eta\Phi_s(f)$ is bounded from below. By (10.1), for any $\varepsilon > 0$ and $\psi \in D(d\Gamma_b(T)^{1/2})$,

$$| \langle \psi, A(f)\psi \rangle | \leq \|T^{-1/2}f\| \left(\varepsilon \|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{1}{4\varepsilon}\|\psi\|^2 \right).$$

Hence if the assertion follows for $\eta = 0$, then so is for all η. Thus we show that the assertion follows for $\eta = 0$. If $\lambda > 0$, then clearly $H(\lambda) \geq 0$. Let $\lambda < 0$. By (10.1) and (10.2), for any $\psi \in D(d\Gamma_b(T)^{1/2})$, it follows that

$$\|\Phi_s(g)\psi\|^2 \leq 2\|T^{-1/2}g\|^2 \|d\Gamma_b(T)^{1/2}\psi\|^2 + \|g\|^2\|\psi\|^2.$$

Thus for any $\psi \in D(d\Gamma_b(T))$,

$$\begin{aligned} \langle \psi, H(\lambda)\psi \rangle &= \|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{\lambda}{2}\|\Phi_s(g)\psi\|^2 \\ &\geq (1 + \lambda\|T^{-1/2}g\|^2)\|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{\lambda}{2}\|g\|^2\|\psi\|^2. \end{aligned} \tag{2.4}$$

Hence $H(\lambda)$ is bounded from below if $\lambda \geq \lambda_{c,0}$.

Let $\lambda \geq \lambda_{c,0}$ and $M \geq 0$ be a constant satisfying $H(\lambda) + M \geq 0$. Then for any $\psi \in D(d\Gamma_b(T)) = D(H(\lambda))$,

$$\|(\overline{H(\lambda)} + M)^{1/2}\psi\|^2 \leq (1 + |\lambda|\|T^{-1/2}g\|^2)\|d\Gamma_b(T)^{1/2}\psi\|^2 + \left(\frac{|\lambda|}{2}\|g\|^2 + M \right) \|\psi\|^2. \tag{2.5}$$

By the fact that $D(d\Gamma_b(T))$ is a core of $d\Gamma_b(T)^{1/2}$, we have $D(d\Gamma_b(T)^{1/2}) \subset D((\overline{H(\lambda)} + M)^{1/2})$ and (2.5) holds on $D(d\Gamma_b(T)^{1/2})$.

In the case $\lambda > 0$, the fact that $\Phi_s(g)^2$ is non-negative implies that $\|H(\lambda)^{1/2}\psi\| \geq \|d\Gamma_b(T)^{1/2}\psi\|$ holds for any $\psi \in D(d\Gamma_b(T))$. In the case $0 > \lambda > \lambda_{c,0}$,

$$\|d\Gamma_b(T)^{1/2}\psi\|^2 \leq \frac{1}{1 + \lambda\|T^{-1/2}g\|^2} \left\{ \|(\overline{H(\lambda)} + M)^{1/2}\psi\|^2 - \left(\frac{\lambda}{2}\|g\|^2 + M \right) \|\psi\|^2 \right\}$$

holds for any $\psi \in D(d\Gamma_b(T))$ by (2.4). Hence for $\lambda > \lambda_{c,0}$ there is a constant $a, b \geq 0$ such that for any $\psi \in D(d\Gamma_b(T))$,

$$\|d\Gamma_b(T)^{1/2}\psi\| \leq a\|(\overline{H(\lambda)} + M)^{1/2}\psi\| + b\|\psi\|. \tag{2.6}$$

By a functional calculus, $D(d\Gamma_b(T))$ is a core of $(\overline{H(\lambda)} + M)^{1/2}$. This fact and (2.6) imply that $D((\overline{H(\lambda)} + M)^{1/2}) \subset D(d\Gamma_b(T)^{1/2})$ and (2.6) holds on $D((\overline{H(\lambda)} + M)^{1/2})$.

□

Remark 2.4. By [3, Lemma 13-15], if $\mathscr{H}$ is separable, then Theorem 2.3 takes the following forms:
Let $\mathscr{H}$ be separable, T be a non-negative, injective self-adjoint operator, $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$. Then the following (1)-(3) hold:

(1) Let $\lambda > \lambda_{\mathrm{c},0}$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ and essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.

(2) Let $\lambda \leq \lambda_{\mathrm{c},0}$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ for all $\eta \in \mathbb{R}$. In particular, if $\eta = 0$ and $\lambda = \lambda_{\mathrm{c},0}$, then $H(\lambda_{\mathrm{c},0}) = H(0, \lambda_{\mathrm{c},0})$ is bounded from below.

(3) Let $\lambda > \lambda_{\mathrm{c},0}$. Then $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) = D((H(\eta, \lambda) + M)^{1/2})$ for all constant $M \geq 0$ satisfying $H(\eta, \lambda) + M \geq 0$.

3 The Main Theorem

3.1 Assumptions

To prove our main theorem stated later (Theorem 3.6), we need some assumptions. For a closed operator A, $\sigma(A)$ denotes the spectrum of A. If A is self-adjoint, then $\sigma_{\mathrm{ac}}(A)$ (resp. $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{sc}}(A)$) denotes the absolutely continuous (resp. point, singular continuous) spectrum of A. For a self-adjoint operator A which is bounded from below,

$$E_0(A) := \inf \sigma(A)$$

is called the lowest energy of A. In particular, it is called the ground state energy of A if $E_0(A) \in \sigma_{\mathrm{p}}(A)$. In this case, an eigenvector of A with eigenvalue $E_0(A)$ is called a ground state of A. The ground state is said to be unique if dim Ker$(A - E_0(A)) = 1$. For linear operators A and B, the symbol $A \subset B$ means that $D(A) \subset D(B)$ and $Af = Bf$ for all $f \in D(A)$, i.e., B is an extension of A.

Definition 3.1. *Let T be a self-adjoint operator on $\mathscr{H}$ and $\{E(B) \mid B \in \mathbf{B}^1\}$ be the spectral measure associated with T on the Borel field $\mathbf{B}^1$ on $\mathbb{R}$. The operator T is called purely absolutely continuous if, for each $f \in \mathscr{H}$, the measure $\|E(\cdot)f\|^2$ on $\mathbf{B}^1$ is absolutely continuous with respect to the one-dimensional Lebesgue measure.*

Definition 3.2. *For a purely absolutely continuous self-adjoint operator* T *and vectors* $f, g \in \mathscr{H}$, $\psi_{g,f}$ *denotes the Radon-Nikodym derivative of the finite complex Borel measure* $\langle g, E(\cdot)f\rangle$ *on* $\mathbf{B}^1$. *In particular, we set* $\psi_g := \psi_{g,g}$.

Assumption 3.3. (1) *The operator* T *is a non-negative, purely absolutely continuous self-adjoint operator.*

(2) *The fixed vector* $g \in \mathscr{H}$ *satisfies* $g \in D(\hat{T}^{-1/2}) \cap D(T^{1/2})$ *and* $Jg = g$, *where* $\hat{T} := T - E_0$, $E_0 := E_0(T)$ *and* J *is a conjugation on* $\mathscr{H}$ *satisfying* $JD(T) \subset D(T)$ *and* $JT\psi = TJ\psi$ *for any* $\psi \in D(T)$ (*i.e.,* $JT \subset TJ$),

(3) $\sup_{E_0 < x} x^{\pm 1}\psi_g(x) < \infty$ *and* $\psi_g(x) > 0$ *for all* $x \in (E_0, \infty)$,

(4) $\psi_g \in C([E_0, \infty)) \cap C^1((E_0, \infty))$ *and* $\lim_{x \downarrow E_0} x^{-1}\psi_g'(x) = 0 = \lim_{x \to \infty} x^{-1}\psi_g'(x)$.

Remark 3.4. The operator T is injective since it is a purely absolutely continuous self-adjoint operator. Since T has no eigenvector, the inverse of $\hat{T}$ exists. Assumption 3.3 (2) implies that $T_J = T$. In general, for a self-adjoint operator A and a conjugation J, we can choose a vector $f \in D(A)$ satisfying $Jf = f$ if $A_J = A$. Thus the vector g in Assumption 3.3 (2) exists. By Assumption 3.3 (3), one can easily show that $\sup_{x \in \sigma(T)} \psi_g(x) < \infty$ and, for each $f \in \mathscr{H}$, the functions $\psi_{g,f}$ and $\psi_{T^{\pm 1/2}g,f}$ are in $L^2(\mathbb{R})$ and the maps : $f \mapsto \psi_{g,f}, \psi_{T^{\pm 1/2}g,f}$ are bounded. Actually, for any $h \in \mathscr{H}$ and $B \in \mathbf{B}^1$, the following inequality holds

$$| \langle E(B)h, f\rangle |^2 \leq \|E(B)h\|^2 \|E(B)f\|^2$$

by Schwarz's inequality. Thus we obtain $|\psi_{h,f}(\mu)|^2 \leq \psi_h(\mu)\psi_f(\mu)$ for almost all $\mu \in \mathbb{R}$ with respect to the Lebesgue measure. Hence, by Assumption 3.3 (3), we have the boundedness of the mappings. Moreover, we see that for any $F \in L^2(\mathbb{R})$, $g \in D(F(T))$, where $F(T)$ denotes the operator defined by $F(T) := \int_{\mathbb{R}} F(\mu)dE(\mu)$. In particular, g is in $D(\psi_{g,f}(T))$ for any $f \in \mathscr{H}$.

Lemma 3.5. *Let* T *be a self-adjoint operator such that* $JT \subset TJ$. *Then*

(1) $E(B)_J = E(B)$, *for all* $B \in \mathbf{B}^1$.

(2) *Let* F *be a Borel measurable function on* $\mathbb{R}$. *Then* $F(T)_J = F^*(T)$, *where* F^* *is complex conjugation of* F.

Proof. These are proved by using the spectral theorem. □

3.2 The Main Theorem

In this subsection, we state the main theorem of the present paper. Let λ_{c} be a constant defined by

$$\lambda_{\mathrm{c}} := -\left(\int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2}\, d\|E(\mu)g\|^2\right)^{-1} < 0.$$

Then, by a functional calculus, we obtain $\lambda_{\mathrm{c},0} \le \lambda_{\mathrm{c}}$, and $\lambda_{\mathrm{c},0} = \lambda_{\mathrm{c}}$ if and only if $E_0 = 0$.

Theorem 3.6. *Let $\mathscr{H}$ be separable. Then the following* (1)-(3) *hold:*

(1) *Let T and g satisfy Assumption 3.3. If $\lambda > \lambda_{\mathrm{c}}$, then there are a unitary operator $\mathbb{U}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ and a constant $E_{\mathrm{g}} \in \mathbb{R}$ such that*

$$\mathbb{U}H(\lambda)\mathbb{U}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(T) + E_{\mathrm{g}}. \tag{3.1}$$

In particular, $\mathbb{U}^{-1}\Omega_0$ is the unique ground state of $H(\lambda)$, where $\Omega_0 := (1, 0, 0, \ldots) \in \mathscr{F}_{\mathrm{b}}(\mathscr{H})$ is the Fock vacuum, and

$$\sigma(H(\lambda)) = \{E_{\mathrm{g}}\} \cup [E_0 + E_{\mathrm{g}}, \infty), \tag{3.2}$$

$$\sigma_{\mathrm{ac}}(H(\lambda)) = [E_0 + E_{\mathrm{g}}, \infty),\ \ \sigma_{\mathrm{p}}(H(\lambda)) = \{E_{\mathrm{g}}\},\ \ \sigma_{\mathrm{sc}}(H(\lambda)) = \emptyset. \tag{3.3}$$

(2) *Let T and g satisfy Assumption 3.3 and $E_0 > 0$. If $\lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}$, then there exist a unitary operator $\mathbb{V}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, an injective non-negative self-adjoint operator ξ on $\mathscr{H}$ and a constant $E_{\mathrm{b}} \ge 0$ such that ξ has a ground state and*

$$\mathbb{V}H(\lambda)\mathbb{V}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(\xi) + E_{\mathrm{g}} - E_{\mathrm{b}}.$$

In particular, $\mathbb{V}^{-1}\Omega_0$ is the unique ground state of $H(\lambda)$, and

$$\begin{aligned}
&\sigma(H(\lambda)) = \cup_{n=0}^{\infty}\{n\beta + E_{\mathrm{g}} - E_{\mathrm{b}}\} \cup [E_0 + E_{\mathrm{g}} - E_{\mathrm{b}}, \infty),\\
&\sigma_{\mathrm{ac}}(H(\lambda)) = [E_0 + E_{\mathrm{g}} - E_{\mathrm{b}}, \infty),\\
&\sigma_{\mathrm{p}}(H(\lambda)) = \cup_{n=0}^{\infty}\{n\beta + E_{\mathrm{g}} - E_{\mathrm{b}}\},\ \ \sigma_{\mathrm{sc}}(H(\lambda)) = \emptyset,
\end{aligned}$$

where $\beta > 0$ is the discrete ground state energy of ξ.

(3) *Let T be a non-negative, injective self-adjoint operator. If $g \in D(T^{-1/2})$ and $\lambda < \lambda_{\mathrm{c},0}$, then $H(\lambda)$ is unbounded from above and below.*

Example 3.7. A concrete realization of the abstract model is given as follows (see [9, Chapter 12]):

$$\mathscr{H} \leftrightarrow L^2(\mathbb{R}^d),\quad T \leftrightarrow \omega,\quad g \leftrightarrow \frac{\hat{\rho}}{\sqrt{\omega}},$$

where ω is the multiplication operator associated with the function $\omega(k) := \sqrt{|k|^2 + m^2}, k \in \mathbb{R}^d$ for a fixed $m \geq 0$ and $\hat{\rho}$ is the Fourier transform of a function $\rho \in L^2(\mathbb{R}^d)$ satisfying $\hat{\rho}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Assume that $\hat{\rho}$ is rotation invariant, i.e., there exists a function v on $[0, \infty)$ such that $\hat{\rho}(k) = v(|k|)$ for all $k \in \mathbb{R}^d$. Then we have $\psi_g(s) = |S^{d-1}|\ \omega_1^{-1}(s)^{d-2}\ |v(\omega_1^{-1}(s))|^2$ for $s > m$, where $|S^{d-1}|$ is the surface area of the $(d-1)$-dimensional unite sphere with convention $|S^0| = 2\pi$ and $\omega_1(r) = \sqrt{r^2 + m^2}, r \geq 0$. Set $\psi_g(m) := 0$. Hence, with J being the complex conjugation, the following conditions (2)'-(4)' imply that the present model satisfies Assumption 3.3:

(2)' $\hat{\rho}(k)^* = \hat{\rho}(k)$ and

$$\hat{\rho} \in L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} \frac{|\hat{\rho}(k)|^2}{|k|^2} dk < \infty.$$

(3)' $\hat{\rho}$ is rotation invariant. $\sup_{k \in \mathbb{R}^d} \omega(k)^{\pm 1/2} |k|^{(d-2)/2} |\hat{\rho}(k)| < \infty$. $\hat{\rho}(k) > 0$, for all $k \neq 0$.

(4)' $v \in C^1([0, \infty))$ and

$$\lim_{|k| \to 0} |k|^{d-4} \hat{\rho}(k) \{(d-2)\hat{\rho}(k) + 2|k| v'(|k|)\} = 0,$$
$$\lim_{|k| \to \infty} |k|^{d-4} \hat{\rho}(k) \{(d-2)\hat{\rho}(k) + 2|k| v'(|k|)\} = 0.$$

We can show that ψ_g is right continuous at m by $\int_{\mathbb{R}^d} |\hat{\rho}(k)|^2 |k|^{-2} dk < \infty$ and $v \in C^1([0, \infty))$. Thus, $\psi_g \in C([m, \infty))$. For example, one can easily check that the function

$$\hat{\rho}(k) := \exp\left(-\frac{1}{|k|^2} - |k|^2\right),\ k \in \mathbb{R}^d \backslash \{0\},\ \hat{\rho}(0) := 0$$

satisfies the above conditions (2)'-(4)'.

4 Definitions and properties of some functions and operators

In this section, we introduce some functions and operators. We assume that $\mathscr{H}$ is separable and Assumption 3.3 from this section to Section 6.

4.1 Functions D and $D_\pm$

Lemma 4.1. *Let $D : \mathbb{C} \backslash (0, \infty) \to \mathbb{C}$ be the function*

$$D(z) := 1 + \lambda \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2 - z} d\|E(\mu) g\|^2, \quad z \in \mathbb{C} \backslash (0, \infty).$$

Then D is well-defined and analytic in $\mathbb{C}\backslash[0,\infty)$. Moreover, the following hold :

(1) *For all $\lambda > \lambda_c$, $D(z)$ has no zeros in $\mathbb{C}\backslash[0,\infty)$.*

(2) *For all $\lambda < \lambda_c$, $D(z)$ has a unique simple zero in the negative real axis $(-\infty, 0)$.*

Proof. If $\mathrm{Im} z \neq 0$ (resp. $\mathrm{Re} z < 0$), then for any $n \in \mathbb{N}$,

$$\int_{[E_0,\infty)} \left| \frac{\mu}{(\mu^2 - E_0^2 - z)^n} \right| d\|E(\mu)g\|^2 \leq c^{-n} \|T^{1/2} g\|^2 < \infty,$$

where c is $|\mathrm{Im} z|$ (resp. $|\mathrm{Re} z|$). If $z = 0$, then

$$\int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 \leq \|\hat{T}^{-1/2} g\|^2 < \infty.$$

Thus, by using the Lebesgue dominated convergence theorem, D is well-defined and analytic in $\mathbb{C}\backslash[0,\infty)$.

(1) If $\lambda = 0$, then $D(z) = 1$ for all $z \in \mathbb{C}\backslash(0,\infty)$, so it has no zeros. Let $\lambda \neq 0$ and $z = x + iy \in \mathbb{C}\backslash(0,\infty)$. Then we see that

$$\mathrm{Im}\ D(z) = y\lambda \int_{[E_0,\infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2 + y^2} d\|E(\mu)g\|^2.$$

Thus $\mathrm{Im}\ D(z) = 0$ is equivalent to $y = 0$. Therefore $D(z) = 0$ if and only if $D(x) = 0$. Let $y = 0$. In the case $\lambda > 0$, one has $D(x) > 0$ for all $x \in (-\infty, 0]$. Thus D has no zeros. Next, we consider the case $\lambda < 0$. We have for $x < 0$,

$$D'(x) = \lambda \int_{[E_0,\infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2} d\|E(\mu)g\|^2 < 0.$$

Thus D is monotone decreasing in $(-\infty, 0)$. If $\lambda > \lambda_c$, then $D(0) > 0$. Hence D has no zeros.

(2) Let $\lambda < \lambda_c$. We can see that

$$D(0) = 1 + \lambda \int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 = 1 - \frac{\lambda}{\lambda_c} < 0.$$

By the Lebesgue dominated convergence theorem, $D(x) \to 1$ as $x \to -\infty$. Since D is monotone decreasing in $(-\infty, 0)$, D has a unique simple zero in $(-\infty, 0)$.

□

Let

$$\phi_g(x) := \psi_g(\sqrt{x})\chi_{[E_0^2,\infty)}(x),\ x \in \mathbb{R},$$

where χ_B is the characteristic function of $B \in \mathbf{B}^1$.

Lemma 4.2. *The following hold :*

(1) *The function ϕ_g satisfies $\phi_g \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sup_{x\in\mathbb{R}} |\phi_g'(x)| < \infty$.*

(2) *Let*

$$A_\varepsilon^{(1)}(x) := \frac{x}{\pi(x^2+\varepsilon^2)},\ A_\varepsilon^{(2)}(x) := \frac{\varepsilon}{\pi(x^2+\varepsilon^2)},\ x \in \mathbb{R},\ \varepsilon > 0$$

be the conjugate poisson kernel and the poisson kernel respectively and $f * h$ denote the convolution of functions f and h. Let

$$(H_\varepsilon f)(s) := \frac{1}{\pi}\int_{|x-s|\geq\varepsilon} \frac{f(x)}{s-x}dx,\ (Hf)(s) := \lim_{\varepsilon\downarrow 0}(H_\varepsilon f)(s),\ s \in \mathbb{R},\ \varepsilon > 0,$$

where Hf is called the Hilbert transform of f. Then for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon\downarrow 0}\left(A_\varepsilon^{(1)} * \phi_g\right)(x) = (H\phi_g)(x), \quad \lim_{\varepsilon\downarrow 0}\left(A_\varepsilon^{(2)} * \phi_g\right)(x) = \phi_g(x),$$

hold uniformly in x.

Proof. For any $h > 0$, by Assumption 3.3 (1), (4) and the mean value theorem, there exists $\theta \in (E_0 + h/2, E_0 + h)$ such that

$$\int_{E_0+h/2}^{E_0+h} \frac{\psi_g(\mu)}{\mu - E_0}d\mu = \frac{h}{2}\frac{\psi_g(\theta)}{\theta - E_0}.$$

This fact and $\theta < E_0 + h$ imply that

$$\|E([E_0, E_0+h])\hat{T}^{-1/2}g\|^2 = \int_{[E_0,E_0+h]} \frac{\psi_g(\mu)}{\mu - E_0}d\mu > \frac{\psi_g(\theta)}{2}. \tag{4.1}$$

By taking the limit $h \downarrow 0$ and Assumption 3.3 (1), the left hand side of (4.1) tends to zero. Thus we obtain $\lim_{h\to E_0+0}\psi_g(h) = 0$. This fact and $\psi_g \in C([E_0,\infty))$ imply that $\psi_g(E_0) = 0$. Since ψ_g is the Radon-Nikodym derivative of $\|E(\cdot)g\|^2$ and $E_0 \leq T$, we have $\psi_g(x) = 0$ for $x < E_0$. Thus $\phi_g \in C(\mathbb{R})$. By the differentiability of ψ_g, we obtain $\phi_g'(x) = \psi_g'(\sqrt{x})/(2\sqrt{x})$ for $x > E_0^2$ and $\phi_g'(x) = 0$ for $x < E_0^2$. Thus, ϕ_g' is continuous on $(-\infty, E_0^2) \cup (E_0^2, \infty)$. Since $\phi_g'(x) = 0$ for $x < E_0^2$ and $\lim_{h\to 0+0}(E_0+h)^{-1}\psi_g(E_0+h) = 0$, we have $\lim_{h\to 0}\phi_g'(E_0^2+h) = 0$. By this fact and the l'Hôpital theorem, we obtain $\lim_{h\to 0+0}(\phi_g(E_0^2+h) - \phi_g(E_0^2))/h = 0$. We have $\lim_{h\to 0-0}(\phi_g(E_0^2+h)-\phi_g(E_0^2))/h = 0$ since $\phi_g(x) = 0$ for $x < E_0^2$. Thus ϕ_g is continuous at E_0^2. Hence $\phi_g \in C^1(\mathbb{R})$. By the fact that $\psi_g'(x) = 0$ for $x < E_0$ and Assumption 3.3 (4) imply that $\phi_g \in C^1(\mathbb{R})$ and $\phi_g'(E_0^2) = 0$. By Assumption 3.3 (2) and a change of variable, we have $\phi_g \in L^1(\mathbb{R})$. We obtain $\phi_g \in L^2(\mathbb{R})$ by Assumption 3.3 (3) and a change of variable. The inequality $\sup_{x\in\mathbb{R}} |\phi_g'(x)| < \infty$ is given by Assumption 3.3 (4). The assertion (1) holds. Next we consider the assertion (2). By (1), in particular, ϕ_g is bounded and uniformly continuous. Thus it is easy to see that $A_\varepsilon^{(2)} * \phi_g$ converges uniformly to ϕ_g. Moreover, by (1), Hölder's inequality, the mean value theorem and a similar estimate to the proof of [17, Theorem 92.], we can show that $(A_\varepsilon^{(1)} * \phi_g)(x) - (H_\varepsilon\phi_g)(x)$ tends to 0 uniformly in x as $\varepsilon \downarrow 0$. Hence the assertion (2) holds. $\square$

Detailed studies of the Hilbert transform are given in [17].

Lemma 4.3. *For all $s \geq 0$, $D_\pm(s) := \lim_{\varepsilon\downarrow 0} D(s \pm i\varepsilon)$ are uniformly convergent and continuous in $s \geq 0$ with*

$$D_\pm(s) = 1 - \frac{\lambda\pi}{2}(H\phi_g)(E_0^2 + s) \pm i\frac{\lambda\pi}{2}\psi_g\left(\sqrt{E_0^2 + s}\right), \quad s \geq 0. \tag{4.2}$$

Proof. For any $s \geq 0$ and $\varepsilon > 0$, we have by a change of variable

$$D(s \pm i\varepsilon) = 1 - \frac{\lambda\pi}{2}\left(A_\varepsilon^{(1)} * \phi_g\right)(E_0^2 + s) \pm i\frac{\lambda\pi}{2}\left(A_\varepsilon^{(2)} * \phi_g\right)(E_0^2 + s).$$

Thus, by Lemma 4.2, $D_\pm$ converge uniformly in $s \geq 0$ and (4.2) holds. The continuity of $D_\pm$ is due to the uniform convergence. $\square$

Remark 4.4. *For all $\mu \in [E_0, \infty)$, we have*

$$i\pi\lambda\psi_g(\mu) = D_+(\mu^2 - E_0^2) - D_-(\mu^2 - E_0^2). \tag{4.3}$$

Lemma 4.5. *Let $\lambda \neq \lambda_c$, then $\delta := \inf_{s\geq 0} |D_\pm(s)| > 0$.*

Proof. If $\lambda = 0$, then clearly $D_\pm(s) = 1 > 0$ for all $s \in [0, \infty)$. Let $\lambda \neq 0, \lambda_c$. Then $D_\pm(0) = D(0) \neq 0$. Hence, by the continuity of $D_\pm$, $D_\pm$ has no zeros near $s = 0$. For any $\varepsilon > 0$ and $s > E_0^2 + 1$, we have

$$(H_\varepsilon\phi_g)(s) = I_1^{(\varepsilon)}(s) + \sum_{j=2}^{4} I_j(s),$$

$$I_1^{(\varepsilon)}(s) = \int_\varepsilon^1 \frac{\phi_g(s-x) - \phi_g(s+x)}{x}dx, \ I_2(s) = \int_{E_0^2}^{s-1} \frac{\phi_g(x)}{s-x}dx,$$

$$I_3(s) = \int_{s+1}^{2s} \frac{\phi_g(x)}{s-x}dx, \ I_4(s) = \int_{2s}^{\infty} \frac{\phi_g(x)}{s-x}dx.$$

Then, by the Lebesgue dominated convergence theorem, each $I_j(s),\ j = 2, 3, 4$ tends to zero as $s \to \infty$. By the mean value theorem and the property that $\phi_g'(x) \to 0$ as $x \to \infty$, we have $\lim_{s\to\infty} \lim_{\varepsilon\downarrow 0} I_1^{(\varepsilon)}(s) = 0$. Hence we can see that $(H\phi_g)(s) \to 0$ as $s \to \infty$. This fact implies that $\inf_{s_0 \le s} \mathrm{Re} D_\pm(s) > 0$ for a sufficiently large number $s_0 > 0$. In addition, $\mathrm{Im} D_\pm(s)$ are positive for any closed interval included in $(0, \infty)$ by Assumption 3.3 (3) and the continuity of ψ_g. Hence we can see that $\inf_{s\ge 0} |D_\pm(s)| > 0$. □

Remark 4.6. *By Lemmas 4.3 and 4.5, we can see that there are constants $c, d, \varepsilon_0 > 0$ with $0 < c < d$ such that*

$$c \le \left| \frac{D(s \pm i\varepsilon)}{D_\pm(s)} \right| \le d \tag{4.4}$$

for all $s \ge 0, 0 < \varepsilon < \varepsilon_0$.

4.2 Operators $R_\pm$

Through this subsection, we assume $\lambda \neq \lambda_\mathrm{c}$.

Lemma 4.7. *One can define bounded operators $R_\pm$ on $\mathscr{H}$ as follows:*

$$R_\pm f := -\lambda \lim_{\varepsilon\downarrow 0} \int_{[E_0,\infty)} \frac{R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_\pm(\mu'^2 - E_0^2)} d \left\langle T^{1/2} g, E(\mu') f \right\rangle, \quad f \in \mathscr{H},$$

where $R_z(A) := (A - z)^{-1}$ is the resolvent of a linear operator A at $z \in \rho(A)$ (the resolvent set of a linear operator A).

Proof. For a fixed $\varepsilon > 0$ and any $f \in \mathscr{H}$,

$$\int_{[E_0,\infty)} \left\| \frac{R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_\pm(\mu'^2 - E_0^2)} \right\| d\|E(\mu')f\|^2 \le \frac{\|f\|^2 \|T^{1/2} g\|}{\delta\varepsilon} < \infty$$

by Lemma 4.5 and a property of a resolvent. Thus we can define linear operators $R_\pm^{(\varepsilon)}$ on $\mathscr{H}$ by

$$R_\pm^{(\varepsilon)} f := -\lambda \int_{[E_0,\infty)} \frac{R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_\pm(\mu'^2 - E_0^2)} d \left\langle T^{1/2} g, E(\mu') f \right\rangle$$

in the sense of Bochner integral with the polarization identity. For any $h, f \in \mathscr{H}$,

$$\begin{aligned}
&\left\langle h, R_\pm^{(\varepsilon)} f \right\rangle \\
&= -\lambda \int_{[E_0,\infty)} \frac{\left\langle h, R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g \right\rangle}{D_\pm(\mu'^2 - E_0^2)} d \left\langle T^{1/2} g, E(\mu') f \right\rangle \\
&= -\lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \mp i\varepsilon) D_\pm(\mu'^2 - E_0^2)} d \left\langle h, E(\mu) g \right\rangle d \left\langle T^{1/2} g, E(\mu') f \right\rangle,
\end{aligned}$$

where we have used a functional calculus. By change of variables in the Lebesgue-Stieltjes integration, a functional calculus and Fubini's theorem, we have

$$\left\langle h, R_\pm^{(\varepsilon)} f \right\rangle = \frac{\lambda\pi}{2} \int_{[E_0,\infty)} - \left(A_\varepsilon^{(1)} * \phi_{g,f}^\pm\right)(\mu^2)\mu^{1/2} \mp i \left(A_\varepsilon^{(2)} * \phi_{g,f}^\pm\right)(\mu^2)\mu^{1/2} d\left\langle h, E(\mu)g\right\rangle,$$

where $\phi_{g,f}^\pm(x) = \psi_{g,f}(\sqrt{x})x^{-1/4}D_\pm(x - E_0^2)^{-1}\chi_{[E_0^2,\infty)}(x), x \in \mathbb{R}$. We have $\phi_{g,f}^\pm \in L^2(\mathbb{R})$ by Remark 3.4, and the function $\left(A_\varepsilon^{(j)} * \phi_{g,f}^\pm\right)(\mu^2)\mu^{1/2}$ $(\mu \in \mathbb{R})$ is in $L^2(\mathbb{R})$ for each $j = 1, 2$. Thus, by a change of variable, we have

$$\begin{aligned} &\left\| R_\pm^{(\varepsilon)} f - \left(-\frac{\pi\lambda}{2}(H\phi_{g,f}^\pm)(T^2)T^{1/2}g \mp \frac{1}{2}A_\pm f\right)\right\|^2 \\ \le\ &\left(\frac{\lambda\pi}{2}\right)^2 c_g \int_{[E_0^2,\infty)} \left|(A_\varepsilon^{(1)} * \phi_{g,f}^\pm)(x) - (H\phi_{g,f}^\pm)(x)\right|^2 dx \\ &+ \left(\frac{\lambda\pi}{2}\right)^2 c_g \int_{[E_0^2,\infty)} \left|(A_\varepsilon^{(2)} * \phi_{g,f}^\pm)(x) - \phi_{g,f}^\pm(x)\right|^2 dx, \end{aligned}$$

where $c_g := \sup_{x\in[E_0,\infty)} \psi_g(x)$ and the linear operators

$$A_\pm f := i\pi\lambda\psi_{g,f}(T)D_\pm(T^2 - E_0^2)^{-1}g,\ f \in \mathscr{H}$$

are well-defined (see Remark 3.4 and Lemma 4.5). Hence, by $\phi_{g,f}^\pm \in L^2(\mathbb{R})$, we have

$$R_\pm^{(\varepsilon)} f \to -(\pi\lambda/2)(H\phi_{g,f}^\pm)(T^2)T^{1/2}g \mp (1/2)A_\pm f \text{ as } \varepsilon \downarrow 0.$$

Moreover, by change of variables, the isometricity of Hilbert transform and Remark 3.4, we can show that the inequalities

$$\left\|(H\phi_{g,f}^\pm)(T^2)T^{1/2}g\right\| \le \frac{c_g}{\delta}\|f\|, \quad \|A_\pm f\| \le \frac{\pi|\lambda|c_g}{\delta}\|f\|$$

hold for all $f \in \mathscr{H}$. Hence $R_\pm$ are bounded. □

By the definition of the adjoint operator, $R_\pm^* := (R_\pm)^*$ are given as follows: for $f \in \mathscr{H}$,

$$\begin{aligned} R_\pm^{(\varepsilon)*} f &= \lambda \int_{[E_0,\infty)} R_{\mu'^2\pm i\varepsilon}(T^2)D_\mp(T^2 - E_0^2)^{-1}T^{1/2}g\ d\left\langle T^{1/2}g, E(\mu')f\right\rangle, \qquad (4.5) \\ R_\pm^* f &= \lim_{\varepsilon\downarrow 0} R_\pm^{(\varepsilon)*} f. \end{aligned}$$

For a densely defined linear operator A on a Hilbert space, $A^\sharp$ denotes A or A^*.

Lemma 4.8. *The ranges of $R_\pm^\sharp$ are included in $D(T^{-1}) \cap D(T)$ and $R_\pm^\sharp$ map $D(T)$ into $D(T^2)$.*

Proof. For any $f, h \in \mathscr{H}$, we have

$$\langle h, R_\pm f\rangle = \frac{\lambda\pi}{2}\int_{[E_0,\infty)} -\left(H\phi^\pm_{g,f}\right)(\mu^2)\mu^{1/2} \mp i\frac{\psi_{g,f}(\mu)}{D_\pm(\mu^2-E_0^2)} d\langle h, E(\mu)g\rangle . \tag{4.6}$$

By a change of variable, we have

$$(H\phi^\pm_{g,f})(\mu^2) = \left(H\psi^\pm_{T^{-1/2}g,f}\right)(\mu) + \left(H\psi^\pm_{T^{-1/2}g,f}\right)(-\mu),\ \mu\in\mathbb{R}, \tag{4.7}$$

where $\psi^\pm_{h,f}(x) := \psi_{h,f}(x)D_\pm(x^2-E_0)^{-1}\chi_{[E_0,\infty)}(x), x\in\mathbb{R}$ for $h, f \in \mathscr{H}$. Thus we see by Assumption 3.3 (3) and a functional calculus that $\mathrm{Ran}(R_\pm)\subset D(T^{-1})$. The equation

$$\mu\left(H\phi^\pm_{g,f}\right)(\mu^2) = \left(H\psi^\pm_{T^{1/2}g,f}\right)(\mu) - \left(H\psi^\pm_{T^{1/2}g,f}\right)(-\mu),\ \mu\in\mathbb{R}, \tag{4.8}$$

(4.6), Assumption 3.3 (3) and operational calculus imply that $\mathrm{Ran}(R_\pm)\subset D(T)$. For any $f\in D(T)$ and $\mu\in\mathbb{R}$,

$$\mu^2\left(H\phi^\pm_{g,f}\right)(\mu^2) = \left(H\psi^\pm_{T^{1/2}g,Tf}\right)(\mu) + \left(H\psi^\pm_{T^{1/2}g,Tf}\right)(-\mu) + \frac{2}{\pi}\int_{[E_0,\infty)}\psi^\pm_{T^{1/2}g,f}(x)\ dx.$$

Hence $R_\pm f\in D(T^2)$ and the following equation holds for any $h\in\mathscr{H}$,

$$\begin{aligned}\left\langle h, T^2R_\pm f\right\rangle =& \frac{\lambda\pi}{2}\int_{[E_0,\infty)} -\left\{\left(H\psi^\pm_{T^{1/2}g,Tf}\right)(\mu) + \left(H\psi^\pm_{T^{1/2}g,Tf}\right)(-\mu) + \frac{2c}{\pi}\right\}\mu^{1/2} d\langle h, E(\mu)g\rangle \\ &\mp i\frac{\lambda\pi}{2}\int_{[E_0,\infty)}\psi^\pm_{T^{1/2}g,Tf}(\mu)\mu^{1/2}\ d\langle h, E(\mu)g\rangle ,\end{aligned}$$

where $c := \int_{\mathbb{R}}\psi^\pm_{T^{1/2}g,f}(x)dx$. In quite the same manner as in the case of $R_\pm$, we can prove the statement for $R^*_\pm$. □

Lemma 4.9. *The operator equations $(R_\pm)_J = R_\mp$ hold.*

Proof. This follows from Assumption 3.3 (1) and Lemma 3.5. □

Lemma 4.10. *The operator equation $R_- = R_+\gamma + A_-$ holds, where*

$$\gamma := D_+(T^2-E_0^2)D_-(T^2-E_0^2)^{-1}$$

is a bounded operator.

Proof. The first resolvent formula gives that, for any $\mu', \mu''\in\mathbb{R}, \varepsilon>0$,

$$R_{\mu'^2-i\varepsilon}(T^2) - R_{\mu'^2+i\varepsilon}(T^2) = -2i\varepsilon R_{\mu'^2-i\varepsilon}(T^2)R_{\mu'^2+i\varepsilon}(T^2).$$

Then, for any $f \in \mathscr{H}$,

$$
\begin{aligned}
R_-^{(\varepsilon)} f = & -\lambda \int_{[E_0,\infty)} \frac{R_{\mu'^2+i\varepsilon}(T^2)T^{1/2}g}{D_-(\mu'^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle \\
& + 2i\lambda\varepsilon \int_{[E_0,\infty)} \frac{R_{\mu'^2+i\varepsilon}(T^2)R_{\mu'^2-i\varepsilon}(T^2)T^{1/2}g}{D_-(\mu'^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle .
\end{aligned}
$$

Thus, by a change of variable, we have for any $h \in \mathscr{H}$

$$
\begin{aligned}
\left\langle h, R_-^{(\varepsilon)} f \right\rangle = & \left\langle h, R_+^{(\varepsilon)} \gamma f \right\rangle + 2i\lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu)g \right\rangle d\left\langle T^{1/2}g, E(\mu')f \right\rangle \\
& \times \frac{\mu^{1/2}\varepsilon}{\{(\mu^2 - \mu'^2)^2 + \varepsilon^2\} D_-(\mu'^2 - E_0^2)} \\
= & \left\langle h, R_+^{(\varepsilon)} \gamma f \right\rangle + i\pi\lambda \int_{[E_0,\infty)} \left(A_\varepsilon^{(2)} * \phi_{g,f}^- \right)(\mu^2)\mu^{1/2} d\left\langle h, E(\mu)g \right\rangle .
\end{aligned}
$$

By a property of the Poisson kernel, the function $\left(A_\varepsilon^{(2)} * \phi_{g,f}^- \right)(\mu^2)\mu^{1/2}$ ($\mu \in \mathbb{R}$) converges to $\psi_{g,f}(\mu)/D_-(\mu^2 - E_0^2)$ as $\varepsilon \to +0$ in the sense of $L^2(\mathbb{R})$. Hence the continuity of the inner product with $L^2(\mathbb{R})$ implies that

$$
\begin{aligned}
\left\langle h, R_- f \right\rangle &= \left\langle h, R_+ \gamma f \right\rangle + i\pi\lambda \int_{[E_0,\infty)} \frac{\psi_{g,f}(\mu)}{D_-(\mu^2 - E_0^2)} d\left\langle h, E(\mu)g \right\rangle \\
&= \left\langle h, R_+ \gamma f \right\rangle + \left\langle h, A_- f \right\rangle .
\end{aligned}
$$

Since f and h are arbitrary, one obtains the conclusion. □

By the definitions of $A_\pm$, we have

$$
(A_-)^* = -A_+.
$$

Lemma 4.11. *For any Borel measurable function F on $\mathbb{R}$, $A_\pm F(T) \subset F(T)A_\pm$.*

Proof. For any $f \in D(F(T))$, an operational calculus implies that $\psi_{g,F(T)f} = F\psi_{g,f} \in L^2(\mathbb{R})$. This fact imply that $\psi_{g,f}(T)g \in D(F(T))$ and $F(T)\psi_{g,f}(T)g = \psi_{g,F(T)f}(T)g$. Hence $A_\pm f \in D(F(T))$ and $F(T)A_\pm f = A_\pm F(T)f$ by Lemma 4.5. □

Lemma 4.12. *The following operator equations hold:*

$$
A_- R_\pm^* = (\gamma - I)R_\pm^*, \quad A_-(A_-)^* = -A_- - (A_-)^*.
$$

Proof. By applying Lemma 4.11 to the case $F = \chi_B$, one can easily see that $A_\pm E(B) = E(B)A_\pm$ hold for any $B \in \mathbf{B}^1$. For any $f, h \in \mathscr{H}$, we have

$$\begin{aligned}
&\left\langle (A_-)^* h, R_\pm^{(\varepsilon)*} f \right\rangle \\
&= \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{i\pi\lambda^2 \mu^{1/2} \psi_g(\mu)}{(\mu^2 - \mu'^2 \mp i\varepsilon) D_\mp(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g\rangle \, d\left\langle T^{1/2} g, E(\mu')f \right\rangle .
\end{aligned}$$

Then, since γ and $E(B)$ commute on $\mathscr{H}$ for any $B \in \mathbf{B}^1$, (4.3) gives

$$\begin{aligned}
&\left\langle (A_-)^* h, R_\pm^{(\varepsilon)*} f \right\rangle \\
&= \lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \mp i\varepsilon) D_\mp(\mu^2 - E_0^2)} d\langle h, E(\mu)(\gamma - 1)g\rangle \, d\left\langle T^{1/2} g, E(\mu')f \right\rangle \\
&= \left\langle h, (\gamma - 1) R_\pm^{(\varepsilon)*} f \right\rangle .
\end{aligned}$$

Thus, by a limiting argument, we obtain $A_- R_\pm^* = (\gamma - 1) R_\pm^*$. Moreover, (4.3) and the equation $(A_-)^* = -A_+$ imply that

$$\begin{aligned}
\langle h, A_-(A_-)^* f\rangle &= -(i\pi\lambda)^2 \int_{[E_0,\infty)} \frac{\psi_{g,f}(\mu)\psi_g(\mu)}{D_+(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g\rangle \\
&= -i\pi\lambda \int_{[E_0,\infty)} \frac{(D_+(\mu^2 - E_0^2) - D_-(\mu^2 - E_0^2))\psi_{g,f}(\mu)}{D_+(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g\rangle \\
&= -\langle h, (A_-)^* f + A_- f\rangle .
\end{aligned}$$

Hence the equation $A_-(A_-)^* = -A_- - (A_-)^*$ holds. □

4.3 Operators $\Omega_\pm$

In this subsection we consider the bounded operators

$$\Omega_\pm := I + R_\pm.$$

Let $x_0 < 0$ be the zero of $D(z)$ given in Lemma 4.1 (2) and

$$U_\mathrm{b} := \sqrt{\frac{\lambda}{D'(x_0)}} R_{E_0^2 + x_0}(T^2) T^{1/2} g, \;\; P := \langle U_\mathrm{b}, \cdot \rangle \, U_\mathrm{b}.$$

Then, by functional calculus, we see that $\|U_\mathrm{b}\| = 1, U_\mathrm{b} \in D(T^{-1}) \cap D(T^2)$ and

$$TU_\mathrm{b} = \sqrt{\lambda / D'(x_0)} T^{-1/2} g + (E_0^2 + x_0) T^{-1} U_\mathrm{b}.$$

Hence P is a projection operator.

Lemma 4.13. *Let $\lambda \neq \lambda_c$. Then the following equations hold:*

$$\Omega_\pm^* \Omega_\pm = I, \tag{4.9}$$

$$\Omega_\pm \Omega_\pm^* = I - \theta(\lambda_c - \lambda) P, \tag{4.10}$$

where θ is the Heaviside function:

$$\theta(t) = \begin{cases} 1 & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Remark 4.14. *Lemma 4.13 implies that $\Omega_\pm$ are unitary operators if $\lambda > \lambda_c$ and partial isometries with their final subspace* $\mathrm{Ran}(I - P)$ *if $\lambda < \lambda_c$.*

Proof. (1) We first prove (4.9).

It is sufficient to prove that $R_\pm^* R_\pm = -(R_\pm + R_\pm^*)$ hold. For any $f, h \in \mathscr{H}$ and $\varepsilon > 0$,

$$\left\langle R_\pm^{(\varepsilon)} h, R_\pm^{(\varepsilon)} f \right\rangle = \lambda^2 \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu') T^{1/2} g\right\rangle d\left\langle T^{1/2} g, E(\mu'') f\right\rangle \times \left\langle \frac{R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_\pm(\mu'^2 - E_0^2)}, \frac{R_{\mu''^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_\pm(\mu''^2 - E_0^2)} \right\rangle.$$

By the definition of the function D, we have

$$\lambda \left\langle T^{1/2} g, R_z(T^2) T^{1/2} g\right\rangle = D(z - E_0^2) - 1, \ z \in \mathbb{C} \backslash (E_0^2, \infty).$$

By this formula and the resolvent identity, we obtain

$$\begin{aligned} \left\langle R_\pm^{(\varepsilon)} h, R_\pm^{(\varepsilon)} f \right\rangle &= \lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu') T^{1/2} g\right\rangle d\left\langle T^{1/2} g, E(\mu'') f\right\rangle \\ &\quad \times \frac{D(\mu'^2 - E_0^2 \mp i\varepsilon) - D(\mu''^2 - E_0^2 \pm i\varepsilon)}{(\mu'^2 - \mu''^2 \mp 2i\varepsilon) D_\mp(\mu'^2 - E_0^2) D_\pm(\mu''^2 - E_0^2)}. \\ &= -\left\langle E_\pm^{(\varepsilon)} h, R_\pm^{(2\varepsilon)} f \right\rangle - \left\langle R_\pm^{(2\varepsilon)} h, E_\pm^{(\varepsilon)} f \right\rangle, \end{aligned}$$

where the operators $E_\pm^{(\varepsilon)}$ on $\mathscr{H}$ are given as follows:

$$E_\pm^{(\varepsilon)} := D(T^2 - E_0^2 \pm i\varepsilon) D_\pm(T^2 - E_0^2)^{-1}.$$

The inequality (4.4) implies that $E_\pm^{(\varepsilon)}$ are bounded for all $0 < \varepsilon < \varepsilon_0$. Thus, by the Lebesgue dominated convergence theorem, we have $\text{s-}\lim_{\varepsilon \downarrow 0} E_\pm^{(\varepsilon)} = I$. Hence we obtain $R_\pm^* R_\pm = -(R_\pm + R_\pm^*)$.

(2) We next prove (4.10) for $\lambda \neq \lambda_c$.

It is sufficient to prove that $R_\pm R_\pm^* = -(R_\pm + R_\pm^*) - \theta(\lambda_c - \lambda)P$ holds. For any $f, h \in \mathscr{H}$ and a fixed $\varepsilon > 0$, (4.5) implies

$$\begin{aligned}
&\left\langle R_\pm^{(\varepsilon)*} h, R_\pm^{(\varepsilon)*} f \right\rangle \\
&= \lambda^2 \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu) T^{1/2} g \right\rangle d\left\langle T^{1/2} g, E(\mu') f \right\rangle \\
&\quad \times \left\langle R_{\mu^2 \pm i\varepsilon}(T^2) D_\mp (T^2 - E_0^2)^{-1} T^{1/2} g, R_{\mu'^2 \pm i\varepsilon}(T^2) D_\mp (T^2 - E_0^2)^{-1} T^{1/2} g \right\rangle .
\end{aligned}$$

Then, by operational calculus, we see that

$$\begin{aligned}
&\left\langle R_\pm^{(\varepsilon)*} h, R_\pm^{(\varepsilon)*} f \right\rangle \\
&= \lambda^2 \int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu) T^{1/2} g \right\rangle d\left\langle T^{1/2} g, E(\mu') f \right\rangle d\|E(\mu'')g\|^2 \\
&\quad \times \frac{\mu''}{(\mu''^2 - \mu^2 \pm i\varepsilon)(\mu''^2 - \mu'^2 \mp i\varepsilon) D_\pm(\mu''^2 - E_0^2) D_\mp(\mu''^2 - E_0^2)} \\
&= \lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} J_\varepsilon^\pm(\mu, \mu') d\left\langle h, E(\mu) T^{1/2} g \right\rangle d\left\langle T^{1/2} g, E(\mu') f \right\rangle , \quad (4.11)
\end{aligned}$$

where, for any $\mu, \mu' \in [E_0, \infty)$,

$$\begin{aligned}
&J_\varepsilon^\pm(\mu, \mu') \\
&= \int_{[E_0,\infty)} \frac{\lambda \mu''}{D_\pm(\mu''^2 - E_0^2) D_\mp(\mu''^2 - E_0^2)} \left(\frac{1}{\mu''^2 - \mu^2 \pm i\varepsilon} - \frac{1}{\mu''^2 - \mu'^2 \mp i\varepsilon} \right) d\|E(\mu'')g\|^2.
\end{aligned}$$

Then, by a change of variable and (4.3), one can show that

$$J_\varepsilon^\pm(\mu, \mu') = \lim_{R \to \infty} \frac{1}{2\pi i} I_{\varepsilon,R}^\pm(\mu, \mu'),$$

where, for $R > 0$,

$$I_{\varepsilon,R}^\pm(\mu, \mu') := \int_0^R \left(\frac{1}{D_+(s)} - \frac{1}{D_-(s)} \right) G_{\mu,\mu'}^{\varepsilon,\pm}(s) ds$$

and

$$G_{\mu,\mu'}^{\varepsilon,\pm}(z) := \frac{1}{z - \mu'^2 + E_0^2 \mp i\varepsilon} - \frac{1}{z - \mu^2 + E_0^2 \pm i\varepsilon}, \ z \in \mathbb{C}.$$

For $0 < \eta < \varepsilon$ and $R > 0$, let C_i $(i = 1, 2, 3)$ be the curve given as follows:

$$\begin{aligned}
C_1 &: \ \theta_1(t) = R - t - i\eta, && t : 0 \to R, \\
C_2 &: \ \theta_2(t) = \eta e^{-it}, && t : \pi/2 \to (3\pi)/2, \\
C_3 &: \ \theta_3(t) = t + i\eta, && t : 0 \to R.
\end{aligned}$$

Then, for $C = C_1 + C_2 + C_3$, we have by the Lebesgue dominated convergence theorem,

$$I^{\pm}_{\varepsilon,R}(\mu,\mu') = \lim_{\eta\downarrow 0} \int_C \frac{1}{D(z)} G^{\varepsilon,\pm}_{\mu,\mu'}(z)dz.$$

We take R such that $R > \max\{\mu^2 - E_0^2, \mu'^2 - E_0^2\}$ and define a curve $C_4 : \theta_4(t) = \sqrt{\eta^2 + R^2}e^{-it}, t : t_s \to t_f$, for $t_s := \arctan(\eta/R)$ and $t_f = 2\pi - t_s$. We consider two cases separately.

(i) The case $\lambda > \lambda_c$. In this case, the function $G^{\varepsilon,\pm}_{\mu,\mu'}(z)/D(z), z \in \mathbb{C}\backslash(0,\infty)$ has two simple poles at $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu'^2 - E_0^2 \pm i\varepsilon$. Then, by the residue theorem, we have

$$\int_C \frac{1}{D(z)} G^{\varepsilon,\pm}_{\mu,\mu'}(z)dz = 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) - \int_{C_4} \frac{1}{D(z)} G^{\varepsilon,\pm}_{\mu,\mu'}(z)dz.$$

Thus, as η tends to 0, we have

$$I^{\pm}_{\varepsilon,R}(\mu,\mu') = 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) - \lim_{\eta\downarrow 0} \int_{C_4} \frac{1}{D(z)} G^{\varepsilon,\pm}_{\mu,\mu'}(z)dz.$$

The definition of line integral implies

$$\int_{C_4} \frac{1}{D(z)} G^{\varepsilon,\pm}_{\mu,\mu'}(z)dz = -i \int_{t_s}^{2\pi - t_s} \frac{G^{\varepsilon,\pm}_{\mu,\mu'}(\sqrt{\eta^2 + R^2}e^{-it})\sqrt{\eta^2 + R^2}e^{-it}}{D(\sqrt{\eta^2 + R^2}e^{-it})} dt.$$

By the triangle inequality, for any $t \in [t_s, t_f]$,

$$|G^{\varepsilon,\pm}_{\mu,\mu'}(\sqrt{\eta^2 + R^2}e^{-it})| \le \frac{|\mu^2 - \mu'^2 \pm 2i\varepsilon|}{(R - |\mu^2 - E_0^2 \pm i\varepsilon|)(R - |\mu'^2 - E_0^2 \mp i\varepsilon|)}.$$

On the other hand, by Lemma 4.5, (4.4) and the Lebesgue dominated convergence theorem, there are constants $\tilde{R} > 0$ and $c_0 > 0$ such that $|D(z)| \ge c_0$ for all $|z| \ge \tilde{R}$. Thus we have

$$I^{\pm}_{\varepsilon,R}(\mu,\mu') = 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) + O(R^{-1}) \ (R \to \infty),$$

where $O(\cdot)$ stands for the well known Landau symbol. Therefore we have

$$J^{\pm}_{\varepsilon}(\mu,\mu') = \frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)}$$

for each $\mu, \mu' \in [E_0, \infty)$. Thus, by (4.11), we have

$$\left\langle R_\pm^{(\varepsilon)*} h, R_\pm^{(\varepsilon)*} f \right\rangle = -\left\langle \left(R_\pm^{(2\varepsilon)}\right)^* h, \left(E_\pm^{(\varepsilon)}\right)^{-1} f \right\rangle - \left\langle \left(E_\pm^{(\varepsilon)}\right)^{-1} h, \left(R_\pm^{(2\varepsilon)}\right)^* f \right\rangle.$$

As in the proof in (1), we obtain $\text{s-}\lim_{\varepsilon\downarrow 0} \left(E_\pm^{(\varepsilon)}\right)^{-1} = I$. Therefore we obtain

$$\lim_{\varepsilon\downarrow 0} \left\langle R_\pm^{(\varepsilon)*} h, R_\pm^{(\varepsilon)*} f \right\rangle = -\left\langle R_\pm^* h, f \right\rangle - \left\langle h, R_\pm^* f \right\rangle.$$

Thus we obtain the desired result.

(ii) The case $\lambda < \lambda_c$. In this case, $G_{\mu,\mu'}^{\varepsilon,\pm}(z)/D(z)$ has a simple pole at $z = x_0$ in addition to $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu'^2 - E_0^2 \pm i\varepsilon$. The residue R_0 of $G_{\mu,\mu'}^{\varepsilon,\pm}(z)/D(z)$ at $z = x_0$ is give by

$$R_0 = \frac{1}{D'(x_0)} \frac{\mu'^2 - \mu^2 \pm 2i\varepsilon}{(x_0 - \mu'^2 + E_0^2 \mp i\varepsilon)(x_0 - \mu^2 + E_0^2 \pm i\varepsilon)}.$$

Thus we have

$$J_\varepsilon^\pm(\mu, \mu') = \frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} + R_0$$

and also

$$\frac{\lambda}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 = -\frac{\lambda}{D'(x_0)} \frac{1}{(\mu'^2 - E_0^2 - x_0 \pm i\varepsilon)(\mu^2 - E_0^2 - x_0 \mp i\varepsilon)}.$$

This implies that

$$\lambda \lim_{\varepsilon\downarrow 0} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 \; d\left\langle h, E(\mu) T^{1/2} g \right\rangle d\left\langle T^{1/2} g, E(\mu') f \right\rangle$$
$$= -\langle h, U_\text{b} \rangle \langle U_\text{b}, f \rangle = -\langle h, Pf \rangle.$$

Thus we obtain the desired result.

□

4.4 Operators U and V

In this subsection, we investigate the operators U and V defined as follows:

$$U := \frac{1}{2}(T^{-1/2}\Omega_+ T^{1/2} + T^{1/2}\Omega_+ T^{-1/2}), V := \frac{1}{2}(T^{-1/2}\Omega_+ T^{1/2} - T^{1/2}\Omega_+ T^{-1/2}),$$

which are used to construct a Bogoliubov transformation. Then, by Lemma 4.8, we can see that $D(U) = D(V) = D(T^{-1/2}) \cap D(T^{1/2})$.

Lemma 4.15. *The operators U and V are bounded.*

Proof. By (4.6) and Lemma 4.8 we have

$$\left\langle h, T^{-1/2}R_{\pm}T^{1/2}f\right\rangle = \frac{\lambda\pi}{2}\int_{[E_0,\infty)} -\left(H\phi^{\pm}_{T^{1/2}g,f}\right)(\mu^2) \mp i\frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2-E_0^2)}\, d\left\langle h, E(\mu)g\right\rangle, \quad (4.12)$$

$$\left\langle h, T^{1/2}R_{\pm}T^{-1/2}f\right\rangle = \frac{\lambda\pi}{2}\int_{[E_0,\infty)} -\left(H\phi^{\pm}_{T^{-1/2}g,f}\right)(\mu^2)\mu \mp i\frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2-E_0^2)}\, d\left\langle h, E(\mu)g\right\rangle. \quad (4.13)$$

By Assumption 3.3 (3), (4.7), (4.8) and a property of Hilbert transform, we can show that

$$\|T^{-1/2}R_{\pm}T^{1/2}f\|, \|T^{1/2}R_{\pm}T^{-1/2}f\| \leq \frac{|\lambda|\pi(C_g+c_g)}{2\delta}\|f\|,$$

where $C_g := (\sup_{E_0<x} x^{-1}\psi_g(x))^{1/2}(\sup_{E_0<x} x\psi_g(x))^{1/2}$. Hence the operators $T^{-1/2}R_{\pm}T^{1/2}$ and $T^{1/2}R_{\pm}T^{-1/2}$ are bounded. □

In the same way as in the proof of Lemma 4.15, we see that $T^{-1/2}R_{\pm}^{*}T^{1/2}$ and $T^{1/2}R_{\pm}^{*}T^{-1/2}$ are bounded on each domain $D(T^{1/2})$ and $D(T^{-1/2})$. In what follows, we write the closed extensions of U and V by the same symbol respectively. Then

$$U^{*} = \frac{1}{2}(\overline{T^{-1/2}\Omega_{+}^{*}T^{1/2}} + \overline{T^{1/2}\Omega_{+}^{*}T^{-1/2}}).$$

Lemma 4.16. *The operators $U^{\sharp}$ and $V^{\sharp}$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.*

Proof. By applying Lemma 4.8 and using the equation

$$U^{\sharp} = I + \frac{1}{2}\left(\overline{T^{-1/2}R_{+}^{\sharp}T^{1/2}} + \overline{T^{1/2}R_{+}^{\sharp}T^{-1/2}}\right),$$

one can easily see that the assertion for $U^{\sharp}$ is true. The proof for $V^{\sharp}$ is similar. □

Lemma 4.17. *Let $F(x) = x^{\pm 1/2}, x^{\pm 1}$, a.e. $x \in (0,\infty)$. Then*

$$\Omega_{+}F(T)\Omega_{+}^{*} = (\Omega_{+})_J F(T)(\Omega_{+}^{*})_J \quad \text{on } D(F(T)). \quad (4.14)$$

Proof. By Lemma 4.8, the domain of each side of (4.14) includes $D(F(T))$. By Lemmas 4.11 and 4.12, we have

$$\begin{aligned}(\Omega_{+})_J F(T)(\Omega_{+}^{*})_J &= R_{+}F(T)R_{+}^{*} + R_{+}\{(A_{-})^{*}+I\}F(T)\gamma + F(T)\gamma^{*}(A_{-}+I)R_{+}^{*} \\ &\quad + F(T)\{A_{-}(A_{-})^{*} + A_{-} + (A_{-})^{*} + I\} \\ &= R_{+}F(T)R_{+}^{*} + R_{+}F(T) + F(T)R_{+}^{*} + F(T) \\ &= \Omega_{+}F(T)\Omega_{+}^{*}.\end{aligned}$$

□

5 Commutation relations

In this section, we prove that the pair (U,V) satisfies the condition (1.1), V is Hilbert-Schmidt and

$$B(f) := A(Uf) + A(JVf)^*,\ f \in \mathscr{H}$$

satisfies some commutation relations with $H(\lambda)$. We denote the closure of $B(f)$ by the same symbol. By Lemma 4.16, we have $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) \subset D(B(f)) \cap D(B(f)^*)$ for all $f \in D(T^{-1/2})$.

Theorem 5.1. *The following commutation relations hold:*

(1) *For any $f \in D(T)$ and $\psi \in \mathscr{F}_{\mathrm{b,fin}}(D(T))$,*

$$[H(\lambda), B(f)]\psi = -B(Tf)\psi. \tag{5.1}$$

(2) *For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,*

$$\langle H(\lambda)\phi, B(f)\psi\rangle - \langle B(f)^*\phi, H(\lambda)\psi\rangle = -\langle \phi, B(Tf)\psi\rangle. \tag{5.2}$$

(3) *For any $f \in D(T^{-1/2}) \cap D(T)$, $B(f)$ maps $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{3/2})$ into $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ and for any $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{3/2})$,*

$$[H(\lambda), B(f)]\psi = -B(Tf)\psi. \tag{5.3}$$

The both sides of (5.1),(5.2) and (5.3) have meaning by Lemma 4.16. To prove this theorem, we prove the following lemma:

Lemma 5.2. *For any $f \in D(T)$, the following equations hold:*

$$[U,T]f = (VT+TV)f = \frac{\lambda}{2}\left\langle D_-(T^2-E_0^2)^{-1}g, f\right\rangle g, \tag{5.4}$$

$$(V^*J - U^*)g = -D_-(T^2-E_0^2)^{-1}g. \tag{5.5}$$

Proof. For any $f, h \in D(T^{-1/2}) \cap D(T^{3/2})$, we obtain

$$\langle h, [U,T]f\rangle = \frac{1}{2}\left(\left\langle T^{1/2}R_+^*T^{-1/2}h, Tf\right\rangle - \left\langle Th, T^{1/2}R_+T^{-1/2}f\right\rangle\right).$$

Then, for each $\varepsilon > 0$, we have

$$\begin{aligned}
&\left\langle T^{1/2}R_\pm^{(\varepsilon)*}T^{-1/2}h, Tf\right\rangle - \left\langle Th, T^{1/2}R_\pm^{(\varepsilon)}T^{-1/2}f\right\rangle \\
&= \lambda \int_{[E_0,\infty)}\int_{[E_0,\infty)} \frac{\mu'^2-\mu^2}{(\mu'^2-\mu^2 \pm i\varepsilon)D_\pm(\mu'^2-E_0^2)} d\langle h, E(\mu)g\rangle\, d\langle g, E(\mu')f\rangle \\
&= \lambda \int_{[E_0,\infty)}\int_{[E_0,\infty)} \frac{1}{D_\pm(\mu'^2-E_0^2)} d\langle h, E(\mu)g\rangle\, d\langle E(\mu')g, f\rangle \mp i\varepsilon \left\langle T^{-1/2}h, R_\pm^{(\varepsilon)}T^{-1/2}f\right\rangle.
\end{aligned}$$

Taking the limit $\varepsilon \downarrow 0$, we have

$$\left\langle T^{1/2}R_{\pm}^{*}T^{-1/2}h, Tf\right\rangle - \left\langle Th, T^{1/2}R_{\pm}T^{-1/2}f\right\rangle = \left\langle h, \lambda \left\langle D_{\mp}(T^2 - E_0^2)^{-1}g, f\right\rangle g\right\rangle .$$

Thus we have

$$\langle h, [U,T]f\rangle = \frac{\lambda}{2}\left\langle h, \left\langle D_{-}(T^2 - E_0^2)^{-1}g, f\right\rangle g\right\rangle .$$

Since $D(T^{-1/2}) \cap D(T^{3/2})$ is a core of T, the equation (5.4) holds for $f \in D(T)$. To prove (5.5), we note that

$$\begin{aligned}(V^{*}J - U^{*})g &= \frac{1}{2}(T^{1/2}\Omega_{+}^{*}T^{-1/2}J - T^{-1/2}\Omega_{+}^{*}T^{1/2}J - T^{1/2}\Omega_{+}^{*}T^{-1/2} - T^{-1/2}\Omega_{+}^{*}T^{1/2})g \\ &= -T^{-1/2}\Omega_{+}^{*}T^{1/2}g,\end{aligned}$$

where we have used $Jg = g$. Thus, for any $f \in \mathscr{H}$, we obtain

$$\begin{aligned}&\langle f, (V^{*}J - U^{*})g\rangle \\ &= -\langle f, g\rangle - \lambda \lim_{\varepsilon\downarrow 0}\int_{[E_0,\infty)} \left\langle f, R_{\mu'^2+i\varepsilon}(T^2)D_{-}(T^2 - E_0^2)^{-1}g\right\rangle d\|E(\mu')T^{1/2}g\|^2 \\ &= -\langle f, g\rangle + \lambda \lim_{\varepsilon\downarrow 0}\int_{[E_0,\infty)}\int_{[E_0,\infty)} \frac{\mu'}{\mu'^2 - \mu^2 + i\varepsilon} d\|E(\mu')g\|^2 \frac{1}{D_{-}(\mu^2 - E_0^2)}\langle f, E(\mu)g\rangle \\ &= -\langle f, g\rangle + \int_{[E_0,\infty)} \frac{D_{-}(\mu^2 - E_0^2) - 1}{D_{-}(\mu^2 - E_0^2)} d\langle f, E(\mu)g\rangle \\ &= -\left\langle f, D_{-}(T^2 - E_0^2)^{-1}g\right\rangle .\end{aligned}$$

Hence (5.5) holds. □

Proof of Theorem 5.1.

(1) By Lemma 4.16, for any $f \in D(T)$, $B(f)$ leaves $\mathscr{F}_{\mathrm{b,fin}}(D(T))$ invariant and $H(\lambda)$ maps $\mathscr{F}_{\mathrm{b,fin}}(D(T))$ into $\mathscr{F}_{\mathrm{b,fin}}(\mathscr{H}) \subset D(B(f))$. Thus, by using (2.1) and (10.3), we have for any $\psi \in \mathscr{F}_{\mathrm{b,fin}}(D(T))$,

$$[H(\lambda), B(f)]\psi = \left\{-A(TUf) + A(TJVf)^{*} - \frac{\lambda}{\sqrt{2}}\langle f, (V^{*}J - U^{*})g\rangle \Phi_{\mathrm{s}}(g)\right\}\psi.$$

Hence by Lemma 5.2, (5.1) holds.

(2) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we can see that, for any $f \in D(T^{-1/2})$, $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) \subset D(B(f))$. For any $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$, there are sequences $\psi_n, \phi_n \in \mathscr{F}_{\mathrm{b,fin}}(D(T)), n \in \mathbb{N}$ such that

$\psi_n \to \psi, \phi_n \to \phi, \mathrm{d}\Gamma_{\mathrm{b}}(T)\psi_n \to \mathrm{d}\Gamma_{\mathrm{b}}(T)\psi, \mathrm{d}\Gamma_{\mathrm{b}}(T)\phi_n \to \mathrm{d}\Gamma_{\mathrm{b}}(T)\phi$ as $n \to \infty$, since $\mathscr{F}_{\mathrm{b,fin}}(D(T))$ is a core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. By (1), we have

$$\langle H(\lambda)\phi_n, B(f)\psi_k\rangle - \langle B(f)^*\phi_n, H(\lambda)\psi_k\rangle = -\langle \phi_n, B(Tf)\psi_k\rangle$$

for all $n, k \in \mathbb{N}$ and $f \in D(T^{-1/2}) \cap D(T)$. By the inequalities (10.1) and (10.2) and the $\mathrm{d}\Gamma_{\mathrm{b}}(T)$-boundedness of $\Phi_{\mathrm{s}}(g)^2$, we obtain that $\{B(f)\psi_n\}_{n=1}^{\infty}$, $\{B(f)\phi_n\}_{n=1}^{\infty}$, $\{\Phi_{\mathrm{s}}(g)^2\psi_n\}_{n=1}^{\infty}$, $\{\Phi_{\mathrm{s}}(g)^2\phi_n\}_{n=1}^{\infty}$ and $\{B(Tf)\psi_n\}_{n=1}^{\infty}$ converge. Hence we obtain (5.2).

(3) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we see that, for any $f \in D(T^{-1/2}) \cap D(T)$, $B(f)$ maps $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{3/2})$ into $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$. Therefore, by (5.2) and the density of $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$, we have (5.3). ☐

5.1 Relations between U and V

Lemma 5.3. *Let $\lambda \neq \lambda_{\mathrm{c}}$. Then the following equations hold:*

$$\begin{cases} U^*U - V^*V &= I, \\ U_J^*V - V_J^*U &= 0, \\ UU^* - V_JV_J^* &= I - \theta(\lambda_c - \lambda)Q_+, \\ UV^* - V_JU_J^* &= \theta(\lambda_c - \lambda)Q_-, \end{cases} \tag{5.6}$$

where

$$Q_\pm := \frac{1}{2}\left(\left\langle T^{1/2}U_{\mathrm{b}}, \cdot\right\rangle T^{-1/2}U_{\mathrm{b}} \pm \left\langle T^{-1/2}U_{\mathrm{b}}, \cdot\right\rangle T^{1/2}U_{\mathrm{b}}\right)$$

are bounded operators on $\mathscr{H}$.

Proof. It is sufficient to prove (5.6) on $D(T^{-1/2}) \cap D(T^{1/2})$. Using (4.9), one can show that the first equation in (4.9) holds. We have

$$U_J^*V - V_J^*U = \frac{1}{2}(-T^{1/2}(\Omega_+^*)_J\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_J\Omega_+T^{1/2}).$$

Multiplying the equation by $(\Omega_+)_J$ from the left, and using Lemma 4.17, we obtain

$$\begin{aligned}(\Omega_+)_J(U_J^*V - V_J^*U) &= (\Omega_+)_J(-T^{1/2}(\Omega_+^*)_J\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_J\Omega_+T^{1/2}) \\ &= \Omega_+(-T^{1/2}\Omega_+^*\Omega_+T^{-1/2} + T^{-1/2}\Omega_+^*\Omega_+T^{1/2}) = 0.\end{aligned}$$

By (4.9), this implies that $U_J^* V - V_J^* U = 0$. By Lemma 3.5 and Lemma 4.17, we have

$$\begin{aligned} V_J V_J^* &= \frac{1}{4}\{T^{-1/2}(\Omega_+ T \Omega_+^*)_J T^{-1/2} - T^{-1/2}(\Omega_+\Omega_+^*)_J T^{1/2} \\ &\quad - T^{1/2}(\Omega_+\Omega_+^*)_J T^{-1/2} + T^{1/2}(\Omega_+ T^{-1}\Omega_+^*)_J T^{1/2}\} \\ &= \frac{1}{4}(T^{-1/2}\Omega_+ T\Omega_+^* T^{-1/2} - T^{-1/2}\Omega_+\Omega_+^* T^{1/2} \\ &\quad - T^{1/2}\Omega_+\Omega_+^* T^{-1/2} + T^{1/2}\Omega_+ T^{-1}\Omega_+^* T^{1/2}) \\ &= VV^*. \end{aligned}$$

Hence, by direct calculations and (4.10), one obtains $UU^* - V_J V_J^* = I - \theta(\lambda_c - \lambda)Q_+$. Similarly one can prove the last equation in (5.6) (note that $P_J = P$). □

5.2 Hilbert-Schmidtness of V

In this subsection, we show that V is Hilbert-Schmidt. Then we can use Theorem 2.2 in the case of $\lambda > \lambda_c$.

Lemma 5.4. *The operator V is Hilbert-Schmidt.*

Proof. On $D(T^{-1/2}) \cap D(T^{1/2})$, V^*V is calculated as follows:

$$\begin{aligned} V^*V &= \frac{1}{4}(T^{-1/2}R_+T^{1/2} + T^{1/2}R_+^*T^{-1/2} + T^{1/2}[R_+^*, T^{-1}]R_+T^{1/2} \\ &\quad + T^{1/2}R_+T^{-1/2} + T^{-1/2}R_+^*T^{1/2} + T^{-1/2}[R_+^*, T]R_+T^{-1/2} \\ &\quad + T^{1/2}R_+^*R_+T^{-1/2} + T^{-1/2}R_+^*R_+T^{1/2}) \\ &= \frac{1}{4}(T^{1/2}[R_+^*, T^{-1}]R_+T^{1/2} + T^{-1/2}[R_+^*, T]R_+T^{-1/2}), \end{aligned}$$

where we have used the formula $R_+^*R_+ = -(R_+ + R_+^*)$ in the proof of Lemma 4.13 and Lemma 4.8. Thus, for any $f \in D(T^{-1/2}) \cap D(T^{1/2})$ and $\varepsilon > 0$, we have

$$\begin{aligned} &\left\langle f, (T^{1/2}[R_+^{(\varepsilon)*}, T^{-1}]R_+^{(\varepsilon)}T^{1/2} + T^{-1/2}[R_+^{(\varepsilon)*}, T]R_+^{(\varepsilon)}T^{-1/2})f \right\rangle \\ &= \int_{[E_0,\infty)}\int_{[E_0,\infty)} \frac{\lambda\mu'}{(\mu'^2 - \mu^2 + i\varepsilon)D_+(\mu'^2 - E_0^2)} d\left\langle [T^{-1}, R_+^{(\varepsilon)}]T^{1/2}f, E(\mu)T^{1/2}g \right\rangle d\left\langle E(\mu')g, f\right\rangle \\ &\quad + \int_{[E_0,\infty)}\int_{[E_0,\infty)} \frac{\lambda\mu}{(\mu'^2 - \mu^2 + i\varepsilon)D_+(\mu'^2 - E_0^2)} d\left\langle [T, R_+^{(\varepsilon)}]T^{-1/2}f, E(\mu)T^{-1/2}g \right\rangle d\left\langle E(\mu')g, f\right\rangle. \end{aligned}$$

Then, for any $B \in \mathbf{B}^1$, we can see

$$\begin{aligned} &\left\langle [T^{-1}, R_+^{(\varepsilon)}]T^{1/2}f, E(B)T^{1/2}g \right\rangle \\ &= \lambda \int_B \int_{[E_0,\infty)} \frac{\mu'' - \mu}{(\mu''^2 - \mu^2 - i\varepsilon)D_-(\mu''^2 - E_0^2)} d\left\langle f, E(\mu'')g\right\rangle d\|E(\mu)g\|^2. \end{aligned} \tag{5.7}$$

Similarly, we obtain

$$\begin{aligned}
&\left\langle [T, R_+^{(\varepsilon)}]T^{-1/2}f, E(B)T^{-1/2}g \right\rangle \\
&= \lambda \int_B \int_{[E_0,\infty)} \frac{\mu - \mu''}{(\mu''^2 - \mu^2 - i\varepsilon)D_-(\mu''^2 - E_0^2)} d\left\langle f, E(\mu'')g \right\rangle d\|E(\mu)g\|^2.
\end{aligned}$$

Thus, by the formula of a change of variable in Lebesgue-Stieltjes integration and Fubini's theorem, we have

$$\begin{aligned}
&\left\langle f, (T^{1/2}[R_+^{(\varepsilon)*}, T^{-1}]R_+^{(\varepsilon)}T^{1/2} + T^{-1/2}[R_+^{(\varepsilon)*}, T]R_+^{(\varepsilon)}T^{-1/2})f \right\rangle \\
&= \lambda^2 \int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\|E(\mu)g\|^2 d\left\langle f, E(\mu'')g \right\rangle d\left\langle E(\mu')g, f \right\rangle \\
&\quad \times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)}.
\end{aligned}$$

Then it is easy to see that for any $\mu, \mu', \mu'' \in [E_0, \infty)$,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)} \\
&= \frac{1}{(\mu' + \mu)(\mu'' + \mu)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)}.
\end{aligned}$$

For any $\varepsilon > 0$ and $\mu, \mu', \mu'' \in [E_0, \infty)$, we have, by Lemma 4.5 and the arithmetic-geometric mean inequality,

$$\left| \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)} \right| \leq \frac{1}{4\delta^2 \mu \sqrt{\mu'\mu''}}.$$

On the other side, for any $\alpha, \beta \in \mathbb{C}$, we see

$$\begin{aligned}
&\int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu\sqrt{\mu'\mu''}} d\|E(\mu)g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\
&= \|T^{-1/2}g\|^2 \|T^{-1/4}(f + \alpha g)\|^2 \|T^{-1/4}(f + \beta g)\|^2 < \infty.
\end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\|E(\mu)g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\
&\quad \times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)} \\
&= \int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\|E(\mu)g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\
&\quad \times \frac{1}{(\mu' + \mu)(\mu'' + \mu)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)}.
\end{aligned}$$

In particular, for each $\alpha, \beta = \pm 1, \pm i$, the polarization identity and Fubini's theorem give

$$\langle f, V^* V f \rangle = \frac{\lambda^2}{4} \int_{[E_0,\infty)} \left| \left\langle f, R_{-\mu}(T) D_-(T^2 - E_0^2)^{-1} g \right\rangle \right|^2 d\|E(\mu)g\|^2.$$

Let $\{e_n\}_{n=1}^\infty \subset D(T^{-1/2}) \cap D(T^{1/2})$ be a CONS of $\mathscr{H}$. The termwise integration implies that

$$\begin{aligned}
\sum_{n=1}^\infty \langle e_n, V^* V e_n \rangle &= \frac{\lambda^2}{4} \int_{[E_0,\infty)} \|R_{-\mu}(T) D_-(T^2 - E_0^2)^{-1} g\|^2 d\|E(\mu)g\|^2 \\
&= \frac{\lambda^2}{4} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{(\mu' + \mu)^2 |D_-(\mu'^2 - E_0^2)|^2} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 \qquad (5.8) \\
&\le \frac{\lambda^2}{16\delta^2} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu'\mu} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 < \infty,
\end{aligned}$$

where we have used the arithmetic-geometric mean inequality and Lemma 4.5. Hence V is Hilbert-Schmidt. □

Lemma 5.5. *If $\lambda > \lambda_{\mathrm{c}}$, then there is a unitary operator $\mathbb{U}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that for all $f \in \mathscr{H}$,*

$$\mathbb{U} B(f) \mathbb{U}^{-1} = A(f).$$

Proof. By Lemma 5.3 and Lemma 5.4, we can apply Theorem 2.2. □

6 Analysis in the case $\lambda > \lambda_{\mathrm{c}}$

In this section we prove Theorem 3.6 (1). Before starting the proof, we need to know a property of the Hamiltonian $H(\lambda)$.

6.1 Time evolution

Theorem 6.1 (Time evolution)**.** *If $\lambda > \lambda_{\mathrm{c},0}$, then for all $f \in D(T^{-1/2})$, $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$ and $t \in \mathbb{R}$,*

$$e^{itH(\lambda)} B(f) e^{-itH(\lambda)} \psi = B(e^{itT} f) \psi, \qquad (6.1)$$

$$e^{itH(\lambda)} B(f)^* e^{-itH(\lambda)} \psi = B(e^{itT} f)^* \psi. \qquad (6.2)$$

Proof. It is sufficient to prove (6.1), because (6.2) follows from taking the adjoint of (6.1). We define a function $v : \mathbb{R} \to \mathbb{C}$ by $v(t) := \left\langle \phi, e^{itH(\lambda)} B(e^{-itT} f) e^{-itH(\lambda)} \psi \right\rangle, t \in \mathbb{R}$ for any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$. Then v is well-defined by an operational calculus and Theorem 2.3. The function v is differentiable and, by Theorem 5.1 (2), we have for any $t \in \mathbb{R}$,

$$\begin{aligned}\frac{d}{dt}v(t) =& i\left\langle H(\lambda)e^{-itH(\lambda)}\phi, B(e^{-itT}f)e^{-itH(\lambda)}\psi\right\rangle - i\left\langle B(e^{-itT}f)^* e^{-itH(\lambda)}\phi, H(\lambda)e^{-itH(\lambda)}\psi\right\rangle \\ &+ i\left\langle e^{-itH(\lambda)}\phi, B(Te^{-itT}f)e^{-itH(\lambda)}\psi\right\rangle \\ =&0.\end{aligned}$$

Hence $v(t) = v(0)$ for all $t \in \mathbb{R}$. Hence the equation

$$\left\langle \phi, e^{itH(\lambda)}B(e^{-itT}f)e^{-itH(\lambda)}\psi\right\rangle = \langle \phi, B(f)\psi\rangle$$

holds for all $t \in \mathbb{R}$. By replacing f with $e^{itT}f$, one has for all $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$e^{itH(\lambda)}B(f)e^{-itH(\lambda)}\psi = B(e^{itT}f)\psi.$$

Since $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ is a core of $(H(\lambda)+M)^{1/2}$ and $D(H(\lambda)+M)^{1/2} = D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$ by Theorem 2.3 (3), we obtain (6.1) for $f \in D(T^{-1/2}) \cap D(T)$ and $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. Finally we extend (6.1) for all $f \in D(T^{-1/2})$. Let $f \in D(T^{-1/2})$ and $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. Then we set $f_n := E((-\infty, n])f$ for each $n \in \mathbb{N}$. Then $f_n \in D(T^{-1/2}) \cap D(T)$ for all $n \in \mathbb{N}$ and one can easily show that $f_n \to f$, $T^{-1/2}f_n \to T^{-1/2}f$ as $n \to \infty$ by using a functional calculus and the Lebesgue dominated convergence theorem. Thus we have $Uf_n \to Uf$, $JVf_n \to JVf$ as $n \to \infty$ by the boundedness of U and V. By using the linearity of the Hilbert transform and that of the map $f \mapsto \psi_{g,f}$, (4.12), (4.13) and (4.7), we can show that $T^{-1/2}Uf_n \to T^{-1/2}Uf$, $T^{-1/2}JVf_n \to T^{-1/2}JVf$ as $n \to \infty$. Therefore we obtain $B(f_n)\phi \to B(f)\phi$ and $B(e^{itT}f_n)\phi \to B(e^{itT}f)\phi$ as $n \to \infty$ for any $\phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$ by [3, Lemma 4-28]. By the preceding result, we have for any $n \in \mathbb{N}$,

$$B(f_n)e^{-itH(\lambda)}\psi = e^{-itH(\lambda)}B(e^{itT}f_n)\psi.$$

The equation $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) = D((H(\lambda) + M)^{1/2})$ in Theorem 2.3 (3) implies that

$$e^{-itH(\lambda)}D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) = D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}).$$

Hence, by taking the limit $n \to \infty$, we obtain (6.1) for $f \in D(T^{-1/2}), \psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. □

6.2 Proof of Theorem 3.6 (1)

In this subsection, we assume that $\lambda > \lambda_{\mathrm{c}}$.

Lemma 6.2. *Let $\Omega := \mathbb{U}^{-1}\Omega_0$, where $\mathbb{U}$ is the unitary operator in Lemma 5.5. Then there is an eigenvalue E_{g} of $H(\lambda)$ and Ω is the corresponding eigenvector: $H(\lambda)\Omega = E_{\mathrm{g}}\Omega$.*

Proof. In general, by [3, Proposition 4-4] for a dense subspace $\mathscr{D} \subset \mathscr{H}$, if $\psi \in \cap_{f\in\mathscr{D}} D(A(f))$ satisfies $A(f)\psi = 0$ for all $f \in \mathscr{D}$, then there is a constant $\alpha \in \mathbb{C}$ such that $\psi = \alpha\Omega_0$. Thus, by Lemma 5.5, if $B(f)\phi = 0$ for all $f \in D(T^{-1/2})$, there is a constant $\alpha \in \mathbb{C}$ such that $\phi = \alpha\Omega$. For any $f \in D(T^{-1/2})$ and $t \in \mathbb{R}$,

$$B(f)e^{-itH(\lambda)}\Omega = e^{-itH(\lambda)}B(e^{itT}f)\Omega = 0$$

by Lemma 5.5 and Theorem 6.1. Thus, for each $t \in \mathbb{R}$, there is a constant $\alpha(t) \in \mathbb{C}$ such that $e^{-itH(\lambda)}\Omega = \alpha(t)\Omega$. Then we have $|\alpha(t)| = 1, \alpha(t+s) = \alpha(t)\alpha(s)$ for all $t, s \in \mathbb{R}$, since $\{e^{-itH(\lambda)}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group. Thus there exists a constant $E_{\mathrm{g}} \in \mathbb{R}$ such that $\alpha(t) = e^{-itE_{\mathrm{g}}}, t \in \mathbb{R}$. The differentiation of the equation $e^{-itH(\lambda)}\Omega = e^{-itE_{\mathrm{g}}}\Omega$ in t implies that $\Omega \in D(H(\lambda))$ and $\Omega \in \mathrm{Ker}(H(\lambda) - E_{\mathrm{g}})$. □

Proof of Theorem 3.6 (1).
The subspace $\mathcal{U} := \mathscr{L}(\{B(f_1)^* \cdots B(f_n)^*\Omega, \Omega \mid f_j \in D(T^{-1/2}), j = 1, \dots, n,\ n \in \mathbb{N}\})$ is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ by the fact that $\mathcal{U} = \mathbb{U}^{-1}\mathscr{F}_{\mathrm{b,fin}}(D(T^{-1/2}))$, where $\mathscr{L}(\mathscr{D})$ denotes the subspace algebraically spanned by the vectors in a subset $\mathscr{D}$ of a Hilbert space. By Lemma 6.1 and Lemma 10.3, for any $t \in \mathbb{R}$ and $f_j \in D(T^{-1/2}), j = 1, \dots, n$, we have

$$\begin{aligned} e^{itH(\lambda)}B(f_1)^* \cdots B(f_n)^*\Omega =& B(e^{itT}f_1)^* \cdots B(e^{itT}f_n)^* e^{itH(\lambda)}\Omega \\ =& B(e^{itT}f_1)^* \cdots B(e^{itT}f_n)^* e^{itE_{\mathrm{g}}}\Omega \\ =& e^{itE_{\mathrm{g}}}\mathbb{U}^{-1}e^{it\mathrm{d}\Gamma_{\mathrm{b}}(T)}A(f_1)^* \cdots A(f_n)^*\Omega_0 \\ =& \mathbb{U}^{-1}e^{it(\mathrm{d}\Gamma_{\mathrm{b}}(T)+E_{\mathrm{g}})}\mathbb{U}B(f_1)^* \cdots B(f_n)^*\Omega. \end{aligned}$$

By this equation and a limiting argument, we obtain $\mathbb{U}e^{itH(\lambda)}\mathbb{U}^{-1} = e^{it(\mathrm{d}\Gamma_{\mathrm{b}}(T)+E_{\mathrm{g}})}$. By the unitary covariance of functional calculus, we have

$$\mathbb{U}e^{itH(\lambda)}\mathbb{U}^{-1} = e^{it\mathbb{U}H(\lambda)\mathbb{U}^{-1}}, \quad t \in \mathbb{R}.$$

Hence (3.1) holds. The equation (3.1) and the well-known spectral properties of $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ imply that E_{g} is the ground state energy of $H(\lambda)$ and Ω is the unique ground state of $H(\lambda)$. □

Lemma 6.3. *The ground state energy E_{g} is given as follows:*

$$E_{\mathrm{g}} = \frac{\lambda}{4}\|g\|^2 - \mathrm{Tr}(T^{1/2}V^*VT^{1/2}), \tag{6.3}$$

$$\mathrm{Tr}(T^{1/2}V^*VT^{1/2}) = \frac{\lambda^2}{4}\int_{[E_0,\infty)}\int_{[E_0,\infty)} \frac{\mu}{(\mu+\mu')^2|D_-(\mu^2-E_0^2)|^2} d\|E(\mu)g\|^2 d\|E(\mu')g\|^2. \tag{6.4}$$

Proof. The operator $\mathbb{U}$ leaves $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ invariant by Theorem 3.6 (1). In particular, $\mathbb{U}\Omega_0 \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. Thus, by Lemma 10.4, the isometricity of $\mathbb{U}$ and the definition of $B(\cdot)$, we have $\langle \Omega_0, (H(\lambda) - E_{\mathrm{g}})\Omega_0 \rangle = \mathrm{Tr}(T^{1/2}V^*VT^{1/2})$. By the definition of $H(\lambda)$ and (2.1), we have $\langle \Omega_0, H(\lambda)\Omega_0 \rangle = \lambda \|g\|^2/4$. Hence (6.3) holds. The formula (6.4) can be proved in the same way as (5.8). □

7 Analysis in the case $\lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}$

In Section 5, we proved Theorem 3.6 (1). But the proof is valid only for the case $\lambda > \lambda_{\mathrm{c}}$. Therefore it is necessary to find another pair of operators U and V if one wants to use a Bogoliubov transformation for the spectral analysis of $H(\lambda)$ in the case $\lambda \leq \lambda_{\mathrm{c}}$. In this section we assume that T and g satisfy Assumption 3.3, $E_0 > 0$ and $\lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}$. Under these conditions, we can define the operators ξ, X, Y and $T_\pm$ as follows:

$$
\begin{aligned}
&\xi := \Omega_+ T \Omega_+^* + \beta P, \\
&X := U\Omega_+^* + T_+ P, \ Y := V\Omega_+^* + T_- P, \\
&T_\pm := \frac{1}{2}(\beta^{1/2}T^{-1/2} \pm \beta^{-1/2}T^{1/2}),
\end{aligned}
$$

where $\beta := (E_0^2 + x_0)^{1/2}$.

Remark 7.1. *The definition of x_0 implies that*

$$
E_0^2 + x_0 \begin{cases} > 0, & \text{if} \quad \lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}, \\ = 0, & \text{if} \quad \lambda = \lambda_{\mathrm{c},0}, \\ < 0, & \text{if} \quad \lambda < \lambda_{\mathrm{c},0}. \end{cases}
$$

Thus, in the case $\lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}$, we see that the inequality $0 < \beta < E_0$ holds. Let

$$
C(f) := A(Xf) + A(JYf)^*, f \in \mathscr{H}.
$$

Then $C(f)$ is a densely defined closable operator. We denotes its closure by the same symbol.

7.1 Properties of X, Y and ξ

In this subsection, we study the operators X, Y and ξ. Firstly, we consider ξ. Let

$$
\tilde{T} := \Omega_+ T \Omega_+^*.
$$

Lemma 7.2. *The operator $\tilde{T}$ is a self-adjoint operator with $D(\tilde{T}) = D(T)$.*

Proof. By Lemma 4.8 we see that $D(\tilde{T}) = D(T)$. Hence $\tilde{T}$ is symmetric. For any $\phi \in D((\tilde{T})^*)$ and $\psi \in D(T) = D(\tilde{T})$, we have $\left\langle \Omega_+^*(\tilde{T})^*\phi, \psi \right\rangle = \left\langle \Omega_+^*\phi, T\psi \right\rangle$. This implies that $\Omega_+^*\phi \in D(T)$. Hence $\tilde{T}$ is self-adjoint. □

Lemma 7.3. *The spectra of $\tilde{T}$ are as follows:*

$$\sigma(\tilde{T}) = \{0\} \cup \sigma(T), \sigma_{\rm ac}(\tilde{T}) = \sigma(T), \sigma_{\rm p}(\tilde{T}) = \{0\}, \sigma_{\rm sc}(\tilde{T}) = \emptyset.$$

Proof. We define a family of projection operators $\{E_P(B) \mid B \in \mathbf{B}^1\}$ on $\mathscr{H}$ as follows: $E_P(B) = 0$ if $0 \notin B$ and $E_P(B) = P$ if $0 \in B$ for each $B \in \mathbf{B}^1$. By the definition of the spectral measure, we can see that $\{E_{\tilde{T}}(B) := \Omega_+ E(B)\Omega_+^* + E_P(B) | \ B \in \mathbf{B}^1\}$ is a spectral measure. Using a functional calculus, we see that $E_{\tilde{T}}(\cdot)$ is the spectral measure of $\tilde{T}$. The absolutely continuous part (resp. singular part) of $\tilde{T}$ is $\tilde{T} \upharpoonright \mathrm{Ran}(I-P)$ (resp. $\tilde{T} \upharpoonright \mathrm{Ran}(P)$) since T is absolutely continuous and $\Omega_\pm$ are partial isometries. Thus we see $\sigma(\tilde{T}) = \{0\} \cup \sigma_{\rm ac}(\tilde{T}), \sigma_{\rm p}(\tilde{T}) = \{0\}, \sigma_{\rm sc}(\tilde{T}) = \emptyset$.

We next show that $\sigma_{\rm ac}(\tilde{T}) = \sigma(T)$. For any $\mu \in \sigma(T)$, there is a sequence $\psi_n \in D(T), n \in \mathbb{N}$ such that $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|(T-\mu)\psi_n\| = 0$. For each $n \in \mathbb{N}$, there is a $\phi_n \in \mathrm{Ran}(I-P)$ such that $\psi_n = \Omega_+^*\phi_n$. Then $\|\phi_n\| = \|\Omega_+\psi_n\| = \|\psi_n\| = 1$ and $\|(\tilde{T}-\mu)\phi_n\| = \|(T-\mu)\psi_n\| \to 0$ as $n \to \infty$. Thus we have $\mu \in \sigma(\tilde{T} \upharpoonright \mathrm{Ran}(I-P)) = \sigma_{\rm ac}(\tilde{T})$. For any $\mu \in \sigma_{\rm ac}(\tilde{T})$, there is a sequence $\eta_n \in D(\tilde{T}) \cap \mathrm{Ran}(I-P)$ such that $\|\eta_n\| = 1$ and $\lim_{n\to\infty} \|(\tilde{T}-\mu)\eta_n\| = 0$. Then we easily see that $\Omega_+^*\eta_n \in D(T)$ for all $n \in \mathbb{N}$. The equation $\Omega_+\Omega_+^*\eta_n = \eta_n$ implies that $\|\Omega_+^*\eta_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|(T-\mu)\Omega_+^*\eta_n\| = \|(\tilde{T}-\mu)\eta_n\| \to 0, \quad n \to \infty.$$

Thus $\mu \in \sigma(T)$. Hence $\sigma_{\rm ac}(\tilde{T}) = \sigma(T)$. □

Lemma 7.4. *The operator ξ is an injective, non-negative self-adjoint operator with $D(\xi) = D(T)$ and we have the following equations:*

$$\sigma(\xi) = \{\beta\} \cup \sigma(T), \sigma_{\rm ac}(\xi) = \sigma(T), \sigma_{\rm p}(\xi) = \{\beta\}, \sigma_{\rm sc}(\xi) = \emptyset. \tag{7.1}$$

In particular, β is the ground state energy of ξ, which is an isolated eigenvalue of ξ, and $U_{\rm b}$ is the unique ground state of ξ.

Proof. By Lemma 7.3 and the spectral property of direct sum of self-adjoint operators, we have the equation (7.1). Thus β is an isolated ground state energy by Remark 7.1. By $\Omega_+^* U_{\rm b} = 0$, $U_{\rm b}$ is a ground state of ξ. Assume that $f \in \mathrm{Ker}(\xi - \beta)$ satisfies $(I-P)f \neq 0$. Then $\Omega_+^* f \neq 0$ by Lemma 4.13. This implies that $T\Omega_+^* f = \beta\Omega_+^* f$, but this contradicts Assumption 3.3 (1). Hence $(I-P)f = 0$ and this implies that the ground state of ξ is unique. □

Lemma 7.5. *The operators $\xi^{\pm 1/2}$ are given by*

$$\xi^{1/2} = \Omega_+ T^{1/2}\Omega_+^* + \beta^{1/2}P, \tag{7.2}$$
$$\xi^{-1/2} = \Omega_+ T^{-1/2}\Omega_+^* + \beta^{-1/2}P \tag{7.3}$$

with $D(\xi^{\pm 1/2}) = D(T^{\pm 1/2})$.

Proof. We can show in the same way as in the proof of Lemma 7.4 that the right hand side of (7.2) is non-negative, self-adjoint operator with its domain $D(T^{1/2})$. We have $\xi \subset (\Omega_+ T^{1/2}\Omega_+^* + \beta^{1/2}P)^2$. Since a self-adjoint operator has no non-trivial symmetric extension, (7.2) holds. In the same way as in the proof of (7.2), we can show that the right hand side of (7.3) is a self-adjoint operator. We have $D(\Omega_+ T^{-1/2}\Omega_+^* + \beta^{-1/2}P) \subset \mathrm{Ran}(\xi^{1/2})$ and $\xi^{1/2}(\Omega_+ T^{-1/2}\Omega_+^* + \beta^{-1/2}P) = I$ on $D(\Omega_+ T^{-1/2}\Omega_+^*)$. Hence $\Omega_+ T^{-1/2}\Omega_+^* + \beta^{-1/2}P \subset \xi^{-1/2}$. Thus the equation (7.3) holds. □

Next, we study X and Y.

Lemma 7.6. *The operators $X^\sharp$ and $Y^\sharp$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.*

Proof. The assertion follows from Lemma 4.8, Lemma 4.16, Lemma 7.5 and the definitions of X and Y. □

Lemma 7.7. *The following equations hold:*

$$\begin{cases} X^*X - Y^*Y &= I, \\ X_J^*Y - Y_J^*X &= 0, \\ XX^* - Y_JY_J^* &= I, \\ XY^* - Y_JX_J^* &= 0. \end{cases} \tag{7.4}$$

Proof. The operator P (resp. $T_\pm$) satisfies $P_J = P$ (resp. $(T_\pm)_J = T_\pm$). By (4.10), we have $\Omega_+^* U_{\mathrm{b}} = 0$. Hence we obtain $(U^* \pm V^*)T^{\pm 1/2}U_{\mathrm{b}} = 0$ and $(U^*T_\pm - V^*T_\mp)U_{\mathrm{b}} = 0$. The equations $T_+T_+ - T_-T_- = I$ and $T_+T_- - T_-T_+ = 0$ hold on $D(T^{-1}) \cap D(T)$. By (5.6) and direct calculations, we have $X^*X - Y^*Y = I$ and $X_J^*Y - Y_J^*X = 0$. By similar calculations, we have $XX^* - Y_JY_J^* = I$ and $XY^* - Y_JX_J^* = 0$ on $D(T^{-1/2}) \cap D(T^{1/2})$. Then, by a limiting argument, we obtain (7.4). □

Lemma 7.8. *The operator Y is Hilbert-Schmidt.*

Proof. We can easily show that the assertion follows from Lemma 5.4, Lemma 7.6 and the choice a CONS $\{e_n\}_{n=0}^\infty \subset D(T^{-1/2}) \cap D(T^{1/2})$ with $e_0 = U_{\mathrm{b}}$. □

Lemma 7.9. *There is a unitary operator $\mathbb{V}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that for all $f \in \mathscr{H}$,*

$$\mathbb{V}C(f)\mathbb{V}^{-1} = A(f).$$

Proof. By Theorem 2.2, (7.4) and Lemma 7.8, we can prove this assertion. □

7.2 Commutation relations

Theorem 7.10. *The following commutation relations hold:*

(1) *For any $f \in D(T)$ and $\psi \in \mathscr{F}_{\mathrm{b,fin}}(D(T))$,*

$$[H(\lambda), C(f)]\psi = -C(\xi f)\psi.$$

(2) *For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,*

$$\langle H(\lambda)\phi, C(f)\psi\rangle - \langle C(f)^*\phi, H(\lambda)\psi\rangle = -\langle \phi, C(\xi f)\psi\rangle .$$

(3) *For any $f \in D(T^{-1/2}) \cap D(T)$, $C(f)$ maps $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{3/2})$ into $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ and for any $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{3/2})$,*

$$[H(\lambda), C(f)]\psi = -C(\xi f)\psi.$$

Theorem 7.10 follows, in the same manner as in the proof of Theorem 5.1, from Lemma 4.16, Lemma 7.5 and the next lemma:

Lemma 7.11. *For any $f \in D(T)$ the following equations hold:*

$$-TXf + \frac{\lambda}{2}\langle (Y^*J - X^*)g, f\rangle g = -X\xi f, \tag{7.5}$$

$$TJYf + \frac{\lambda}{2}\langle f, (Y^*J - X^*)g\rangle g = -JY\xi f. \tag{7.6}$$

Remark 7.12. *By Lemma 4.16 and the definition of ξ, the both sides of* (7.5) *and* (7.6) *have meaning.*

Proof. Let $a := \sqrt{\lambda/D'(x_0)}$. Then we can see by the definition of x_0 and (5.5),

$$(Y^*J - X^*)g = -\Omega_+ D_-(T^2 - E_0^2)^{-1}g + \frac{\beta^{-1/2}a}{\lambda}U_{\mathrm{b}}.$$

We have

$$\begin{aligned} TT_\pm U_{\mathrm{b}} =& \frac{1}{2}(\beta^{1/2}T^{1/2}U_{\mathrm{b}} \pm \beta^{-1/2}T^{3/2}U_{\mathrm{b}}) \\ =& \frac{1}{2}(\beta^{1/2}T^{1/2}U_{\mathrm{b}} \pm \beta^{3/2}T^{-1/2}U_{\mathrm{b}} \pm \beta^{-1/2}ag). \end{aligned} \tag{7.7}$$

Thus, for any $f \in D(T)$, we have

$$\begin{aligned} & - TXf + \frac{\lambda}{2}\langle (Y^*J - X^*)g, f\rangle g \\ =& - TU\Omega_+^* f - \frac{\lambda}{2}\left\langle D_-(T^2 - E_0^2)^{-1}g, \Omega_+^* f\right\rangle g - TT_+Pf + \frac{\beta^{-1/2}a}{2}\langle U_{\mathrm{b}}, f\rangle g. \end{aligned}$$

Then, by (5.4) and (7.7), we have

$$-TXf + \frac{\lambda}{2}\langle (Y^*J - X^*)g, f\rangle g = - UT\Omega_+^* f - \beta \langle U_{\rm b}, f\rangle T_+ U_{\rm b}$$
$$= - X(\Omega_+ T\Omega_+^* + \beta P)f.$$

Thus we obtain (7.5). Similarly one can prove (7.6). □

7.3 Proof of Theorem 3.6 (2)

Theorem 7.13. *For all $f \in D(T^{-1/2}), \psi \in D(\mathrm{d}\Gamma_{\rm b}(T)^{1/2})$ and $t \in \mathbb{R}$,*

$$e^{itH(\lambda)} C(f) e^{-itH(\lambda)}\psi = C(e^{it\xi} f)\psi,$$
$$e^{itH(\lambda)} C(f)^* e^{-itH(\lambda)}\psi = C(e^{it\xi} f)^*\psi.$$

Proof. These are proved in the same way as in the proof of Theorem 6.1 by Theorem 7.10. □

Lemma 7.14. *Let $\Omega := \mathbb{V}^{-1}\Omega_0$ where $\mathbb{V}$ is the unitary operator in Lemma 7.9. Then:*

(1) *There is an eigenvalue $\tilde{E}_{\rm g}$ of $H(\lambda)$ and Ω is an eigenvector of $H(\lambda)$ with the eigenvalue $\tilde{E}_{\rm g}$.*

(2) *The following equation holds:*

$$\mathbb{V}H(\lambda)\mathbb{V}^{-1} = \mathrm{d}\Gamma_{\rm b}(\xi) + \tilde{E}_{\rm g}.$$

(3) *The constant $\tilde{E}_{\rm g}$ is given as follows:*

$$\tilde{E}_{\rm g} = E_{\rm g} - \beta \|T_- U_{\rm b}\|^2. \tag{7.8}$$

Proof. The assertions (1) and (2) can be proved in the same way as in the proof of Theorem 3.6 (1).
(3) We have

$$\tilde{E}_{\rm g} = \frac{\lambda}{4}\|g\|^2 - \mathrm{Tr}(\xi^{1/2} Y^* Y \xi^{1/2})$$

in the same way as in the proof of Lemma 6.2. Then, by Lemma 7.5, we have

$$\xi^{1/2} Y^* Y \xi^{1/2} = \Omega_+ T^{1/2} V^* V T^{1/2} \Omega_+^* + \Omega_+ T^{1/2} V^* \beta^{1/2} T_- P + \beta^{1/2} P T_- V T^{1/2} \Omega_+^* + \beta P T_- T_- P.$$

We choose a CONS $\{e_n\}_{n=0}^\infty \subset D(T)$ satisfying $e_0 = U_{\rm b}$. Then it is easy to see that $\{\Omega_+^* e_n\}_{n=1}^\infty$ is a CONS of $\mathscr{H}$ by Lemma 4.13. Hence we have

$$\mathrm{Tr}(\xi^{1/2} Y^* Y \xi^{1/2}) = \sum_{n=1}^\infty \left\langle e_n, \Omega_+ T^{1/2} V^* V T^{1/2} \Omega_+^* e_n \right\rangle + \beta \|T_- U_b\|^2$$
$$= \mathrm{Tr}(T^{1/2} V^* V T^{1/2}) + \beta \|T_- U_b\|^2.$$

Thus we obtain (7.8). □

In particular, $H(\lambda)$ have eigenvectors as follows:

$$\phi_n := \mathbb{V}^{-1} A(U_{\mathrm{b}})^{*n}\Omega_0,\ \ H(\lambda)\phi_n = (n\beta + \tilde{E}_{\mathrm{g}})\phi_n\ ,n \in \mathbb{N}\cup\{0\}.$$

Hence the spectral properties of $H(\lambda)$ as stated in Theorem 3.6 (2) follow.

8 Analysis in the case $\lambda < \lambda_{\mathrm{c},0}$

In this section, we show that $H(\lambda)$ is unbounded from above and below.

Theorem 8.1. *Let $g \in D(T^{-1/2})$. Then $H(\lambda)$ is unbounded above for any $\lambda \in \mathbb{R}$. If $\lambda < \lambda_{\mathrm{c},0}$, then $H(\lambda)$ is unbounded below.*

Proof. For any $f \in D(T)\backslash\{0\}$, we set $\psi_n := a_n A(f)^{*n}\Omega_0,\ a_n \in \mathbb{C}\backslash\{0\},\ n \in \mathbb{N}\cup\{0\}$. Then we have the following equations:

$$\begin{aligned}
&\mathrm{d}\Gamma_{\mathrm{b}}(T)\psi_n = n\frac{a_n}{a_{n-1}}A(Tf)^*\psi_{n-1}, && A(g)\psi_n = n\left\langle g,f\right\rangle \frac{a_n}{a_{n-1}}\psi_{n-1},\\
&\|\psi_n\|^2 = |a_n|^2 n!\|f\|^{2n}, && \|A(g)^*\psi_n\|^2 = \|g\|^2\|\psi_n\|^2 + \|A(g)\psi_n\|^2,
\end{aligned}$$

where $\psi_{-1} := 0$. Then we have

$$\left\langle \psi_n, H(\lambda)\psi_n\right\rangle = \|\psi_n\|^2\left(\frac{\lambda}{4}\|g\|^2 + n\frac{2\|T^{1/2}f\|^2 + \lambda|\left\langle g,f\right\rangle|^2}{2\|f\|^2}\right).$$

We take f such that $\langle g,f\rangle = 0$. Then we have $\left\langle \psi_n, H(\lambda)\psi_n\right\rangle/\|\psi_n\|^2 \to \infty$ as $n\to\infty$ for any $\lambda\in\mathbb{R}$. Thus $H(\lambda)$ is unbounded above for any $\lambda\in\mathbb{R}$.

Let $\phi_N := \sum_{n=0}^N \psi_n, N = 0,1,2,\ldots$. Then we have $\|\phi_N\|^2 = \sum_{n=0}^N \|\psi_n\|^2$ and

$$\begin{aligned}
\left\langle \phi_N, H(\lambda)\phi_N\right\rangle =& \sum_{n=2}^N \|\psi_n\|^2\left(\frac{\lambda\|g\|^2}{4} + n\frac{2\|T^{1/2}f\|^2 + \lambda|\left\langle g,f\right\rangle|^2}{2\|f\|^2} + \frac{\lambda}{2}\mathrm{Re}\frac{a_{n-2}{}^*}{a_n{}^*}\frac{\left\langle g,f\right\rangle^2}{\|f\|^4}\right)\\
&+\|\psi_1\|^2\left(\frac{\lambda\|g\|^2}{4} + \frac{\|T^{1/2}f\|^2}{\|f\|^2} + \frac{\lambda|\left\langle g,f\right\rangle|^2}{2\|f\|^2}\right) + \frac{\lambda\|\psi_0\|^2\|g\|^2}{4}.
\end{aligned}$$

Let $a_0 := 1,\ a_n := n^{-3/4}n!^{-1/2}, n\in\mathbb{N}$ and, for any $0<\delta,\ 0<\varepsilon<1$,

$$\begin{aligned}
&f = f_\delta := \frac{T^{-1}E((\delta,\infty))g}{\|T^{-1}E((\delta,\infty))g\|},\\
&c_\lambda(\varepsilon,\delta) := \|T^{1/2}f_\delta\|^2\left\{1 + \frac{\lambda}{2}(2-\varepsilon)\|T^{-1/2}E((\delta,\infty))g\|^2\right\}.
\end{aligned}$$

Then $\sum_{n=0}^{\infty} \|\psi_n\|^2$ converges and, for any $N \in \mathbb{N}$,

$$\langle \phi_N, H(\lambda)\phi_N \rangle = \sum_{n=2}^{N} \|\psi_n\|^2 n c_\lambda(\varepsilon, \delta) + \frac{\lambda}{2} \sum_{n=2}^{N} \|\psi_n\|^2 \left(\frac{a_{n-2}}{a_n} - n(1-\varepsilon) \right) \langle g, f_\delta \rangle^2 + C_N, \quad (8.1)$$

where

$$C_N := \frac{\lambda \|g\|^2}{4} \sum_{n=0}^{N} \|\psi_n\|^2 + \|\psi_1\|^2 \left(\|T^{1/2} f_\delta\|^2 + \frac{\lambda}{2} \langle g, f_\delta \rangle^2 \right).$$

For all $0 < \delta, 0 < \varepsilon < 1$, we have

$$-\frac{2}{\|T^{-1/2} E((\delta, \infty)) g\|^2 (2 - \varepsilon)} < \lambda_{\mathrm{c},0}. \quad (8.2)$$

The left hand side of (8.2) tends to $\lambda_{\mathrm{c},0}$ as $\varepsilon, \delta \downarrow 0$. Since $\lambda < \lambda_{\mathrm{c},0}$, we can take a pair (ε, δ) satisfying $c_\lambda(\varepsilon, \delta) < 0$. We fix such a pair. There is a $n_0 \in \mathbb{N}$ such that $a_{n-2}/a_n - n(1-\varepsilon) > 0$ for all $n \geq n_0$. Hence we can see that $\langle \phi_N, H(\lambda)\phi_N \rangle / \|\phi_N\|^2$ tends to $-\infty$ as $N \to \infty$, because the first term of the right hand side of (8.1) tends to $-\infty$ as $N \to \infty$. □

9 Generalization of the ϕ^2-model

In this section we consider $H(\eta, \lambda)$ defined in Subsection 2.3.

Assumption 9.1. *We need the following assumptions:*

(1) $f \in D(T^{1/2})$ *and* $g \in D(T^{-1/2}) \cap D(T^{1/2})$,

(2) $f \in D(T^{-1})$ *and* $\mathrm{Re}\, \langle T^{-1} f, g \rangle = 0$,

(3) $f, g \in D(T^{-1})$ *and* $\mathrm{Re}\, \langle T^{-1} f, g \rangle \neq 0$.

We can prove a slight generalization of Theorem 3.6.

Theorem 9.2. *Let $\mathscr{H}$ be separable. Then the following* (1)-(5) *hold:*

(1) *Suppose that Assumption 3.3 and, Assumption 9.1* (2) *or* (3) *hold. Let $\lambda > \lambda_{\mathrm{c}}$. Then there is a unitary operator $\mathbb{U}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that for all $\eta \in \mathbb{R}$,*

$$\mathbb{U} H(\eta, \lambda) \mathbb{U}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(T) + E_{\mathrm{g}} + E_{f,g},$$

where the constant $E_{f,g} \in \mathbb{R}$ is defined by

$$E_{f,g} = -\frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{(\mathrm{Re}\, \langle T^{-1} f, g \rangle)^2 \eta^2 \lambda}{2(1 + \lambda \|T^{-1/2} g\|^2)}.$$

(2) *Suppose that Assumption 3.3 and, Assumption 9.1* (2) *or* (3) *hold. Let* $E_0 > 0$ *and* $\lambda_{c,0} < \lambda < \lambda_c$. *Then there are a unitary operator* $\mathbb{V}$ *on* $\mathscr{F}_b(\mathscr{H})$ *and a non-negative, injective self-adjoint operator* ξ *on* $\mathscr{H}$ *such that, for all* $\eta \in \mathbb{R}$,

$$\mathbb{V}H(\eta,\lambda)\mathbb{V}^{-1} = d\Gamma_b(\xi) + E_g - E_b + E_{f,g}.$$

(3) *Let* T *be a non-negative, injective self-adjoint operator and suppose that* f *and* g *satisfy Assumption 9.1* (1) *and* (2). *Then there is a unitary operator* $\mathbb{W}$ *on* $\mathscr{F}_b(\mathscr{H})$ *such that, for all* $\eta \in \mathbb{R}$,

$$\mathbb{W}\overline{H(\eta,\lambda_{c,0})}\mathbb{W}^{-1} = \overline{H(\lambda_{c,0})} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2.$$

(4) *Let* T *be a non-negative, injective self-adjoint operator and suppose that* f *and* g *satisfy Assumption 9.1* (1) *and* (3). *Then, for all* $\eta \in \mathbb{R}\backslash\{0\}$,

$$\sigma(\overline{H(\eta,\lambda_{c,0})}) = \mathbb{R}, \quad \sigma_p(\overline{H(\eta,\lambda_{c,0})}) = \emptyset.$$

(5) *Let* T *be a non-negative, injective self-adjoint operator and suppose that* f *and* g *satisfy Assumption 9.1* (1). *Moreover, suppose that Assumption 9.1* (2) *or* (3) *holds. Let* $\lambda < \lambda_{c,0}$. *Then, for all* $\eta \in \mathbb{R}$, $\overline{H(\eta,\lambda)}$ *is unbounded from above and below.*

Theorem 9.2 is immediately proved by the following lemma and Theorem 3.6.

Lemma 9.3. *Let* T *be a non-negative, injective self-adjoint operator,* $f \in D(T^{-1})$ *and* $g \in D(T^{-1/2}) \cap D(T)$.

(1) *Let* $\mathrm{Re}\,\langle T^{-1}f, g\rangle = 0$. *Then there is a unitary operator* $\mathbb{U}_1$ *on* $\mathscr{F}_b(\mathscr{H})$ *such that for all* $\eta, \lambda \in \mathbb{R}$,

$$\mathbb{U}_1\overline{H(\eta,\lambda)}\mathbb{U}_1^{-1} = \overline{H(\lambda)} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2. \tag{9.1}$$

(2) *Let* $\mathrm{Re}\,\langle T^{-1}f, g\rangle \neq 0$ *and* $g \in D(T^{-1})$.

(i) If $\lambda \neq \lambda_{c,0}$, *then there is a unitary operator* $\mathbb{U}_2$ *on* $\mathscr{F}_b(\mathscr{H})$ *such that for all* $\eta \in \mathbb{R}$,

$$\mathbb{U}_2\overline{H(\eta,\lambda)}\mathbb{U}_2^{-1} = \overline{H(\lambda)} + E_{f,g}.$$

(ii) If $\lambda = \lambda_{c,0}$, *then for all* $\eta \in \mathbb{R}\backslash\{0\}$,

$$\sigma(\overline{H(\eta,\lambda_{c,0})}) = \mathbb{R}, \quad \sigma_p(\overline{H(\eta,\lambda_{c,0})}) = \emptyset. \tag{9.2}$$

Proof. Let $\mathbb{U}_1 := e^{-i\Phi_{\rm s}(i\eta T^{-1}f)}$ for any $\eta \in \mathbb{R}$. Then, by direct calculations, we obtain

$$\mathbb{U}_1 H(\eta,\lambda)\mathbb{U}_1^{-1} = H(\lambda) - \frac{\eta^2}{2}\|T^{-1/2}f\|^2 - \lambda\eta\kappa\Phi_{\rm s}(g) + \frac{\lambda}{2}\eta^2\kappa^2 \tag{9.3}$$

on $\mathscr{F}_{\rm b,fin}(D(T))$ for all $\eta, \lambda \in \mathbb{R}$, where $\kappa := {\rm Re}\langle T^{-1}f, g\rangle$. In the case of (1), we have (9.1) by $\kappa = 0$ and a limiting argument. Next, we prove (2). We assume that $g \in D(T^{-1})$ and ${\rm Re}\langle T^{-1}f, g\rangle \neq 0$. Let $\mathbb{V}_1 := e^{i\Phi_{\rm s}(i\alpha T^{-1}g)}$ for any $\alpha \in \mathbb{R}$ and define a unitary operator $\mathbb{U}_2 := \mathbb{V}_1\mathbb{U}_1$. Then it follows that

$$\begin{aligned}\mathbb{U}_2 H(\eta,\lambda)\mathbb{U}_2^{-1} =& H(\lambda) + \Big(\alpha + \lambda\alpha\|T^{-1/2}g\|^2 - \lambda\eta\kappa\Big)\Phi_{\rm s}(g)\\ &- \frac{\eta^2}{2}\|T^{-1/2}f\|^2 + \frac{\lambda}{2}\eta^2\kappa^2 + \frac{\alpha}{2}\|T^{-1/2}g\|^2\Big(\alpha + \lambda\alpha\|T^{-1/2}g\|^2 - 2\lambda\eta\kappa\Big)\end{aligned}$$

on $\mathscr{F}_{\rm b,fin}(D(T))$ in the same way as (9.3). For $\lambda \neq \lambda_{\rm c,0}$, let $\alpha = \lambda\eta\kappa(1 + \lambda\|T^{-1/2}g\|^2)^{-1}$. Then we obtain

$$\mathbb{U}_2\overline{H(\eta,\lambda)}\mathbb{U}_2^{-1} = \overline{H(\lambda)} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2 + \frac{\lambda\eta^2\kappa^2}{2(1+\lambda\|T^{-1/2}g\|^2)} \tag{9.4}$$

by a limiting argument. If $\lambda = \lambda_{\rm c,0}$, then, for all $\eta, \alpha \in \mathbb{R}$, we have

$$\mathbb{U}_2\overline{H(\eta,\lambda_{\rm c,0})}\mathbb{U}_2^{-1} = \overline{H_g(-\kappa\eta\lambda_{\rm c,0},\lambda_{\rm c,0})} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2 + \frac{\lambda_{\rm c,0}\eta^2\kappa^2}{2} + \kappa\eta\alpha$$

in the same way as (9.4), where $H_g(\nu,\lambda_{\rm c,0}) := H(\lambda_{\rm c,0}) + \nu\Phi_{\rm s}(g)$ for all $\nu \in \mathbb{R}$. We can see that $\sigma(\overline{H_g(\nu,\lambda_{\rm c,0})}) = \mathbb{R}$ and $\sigma_{\rm p}(\overline{H_g(\nu,\lambda_{\rm c,0})}) = \emptyset$ for all $\nu \in \mathbb{R}\backslash\{0\}$, because $\mathbb{V}_1\overline{H_g(\nu,\lambda_{\rm c,0})}\mathbb{V}_1^{-1} = \overline{H_g(\nu,\lambda_{\rm c,0})} + \nu\alpha\|T^{-1/2}g\|^2$ and $\alpha \in \mathbb{R}$ is arbitrary. Hence we have (9.2). □

Remark 9.4. *If $\mathscr{H}$ is separable, then the condition $g \in D(T^{-1/2}) \cap D(T)$ in the above lemma is weakened to the condition $g \in D(T^{-1/2}) \cap D(T^{1/2})$.*

10 Appendix

In this section, we recall some known facts in the Fock space theory. Let T be a non-negative, injective self-adjoint operator on $\mathscr{H}$.

Lemma 10.1. [5, Theorem 5.16.]
Let $f \in D(T^{-1/2})$ and $\psi \in D({\rm d}\Gamma_{\rm b}(T)^{1/2})$. Then $\psi \in D(A(f)) \cap D(A(f)^*)$ and the following inequalities hold:

$$\|A(f)\psi\| \leq \|T^{-1/2}f\|\|{\rm d}\Gamma_{\rm b}(T)^{1/2}\psi\|, \tag{10.1}$$

$$\|A(f)^*\psi\|^2 \leq \|T^{-1/2}f\|^2\|{\rm d}\Gamma_{\rm b}(T)^{1/2}\psi\|^2 + \|f\|^2\|\psi\|^2. \tag{10.2}$$

Lemma 10.2. [5, Proposition 5.10.] *For any $f \in D(T)$, the following commutation relations hold on $\mathscr{F}_{\mathrm{b,fin}}(D(T))$:*

$$[\mathrm{d}\Gamma_{\mathrm{b}}(T), A(f)] = -A(Tf), \quad [\mathrm{d}\Gamma_{\mathrm{b}}(T), A(f)^*] = A(Tf)^*. \tag{10.3}$$

Lemma 10.3. [5, Lemma 5.21.] *For any $t \in \mathbb{R}$ and $f \in \mathscr{H}$, the following equations hold:*

$$e^{it\mathrm{d}\Gamma_{\mathrm{b}}(T)} A(f)^\sharp e^{-it\mathrm{d}\Gamma_{\mathrm{b}}(T)} = A(e^{itT} f)^\sharp.$$

Lemma 10.4. [5, Theorem 5.21.] *Assume that $\mathscr{H}$ be separable. Let $\{e_n\}_{n=1}^\infty \subset D(T^{1/2})$ be a CONS of $\mathscr{H}$. Then, for any $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$, $\sum_{n=1}^\infty \|A(T^{1/2}e_n)\psi\|^2$ converges and the following equation holds*:

$$\sum_{n=1}^\infty \|A(T^{1/2}e_n)\psi\|^2 = \|\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi\|^2.$$

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