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Spectral analysis of an abstract pair interaction model (抽象的な対相互作用モデルのスペクトル解析)

A DISSERTATION SUBMITTED TO HOKKAIDO UNIVERSITY FOR THE DEGREE OF DOCTOR OF SCIENCES

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Abstract

We consider an abstract pair-interaction model in quantum field theory with a coupling constant $\lambda \in \mathbb{R}$ and analyze the Hamiltonian $H(\lambda)$ of the model. In the massive case, there exist constants $\lambda_c < 0$ and $\lambda_{c,0} < \lambda_c$ such that, for each $\lambda \in (\lambda_{c,0}, \lambda_c) \cup (\lambda_c, \infty)$, $H(\lambda)$ is diagonalized by a proper Bogoliubov transformation, so that the spectrum of $H(\lambda)$ is explicitly identified, where the spectrum of $H(\lambda)$ for $\lambda > \lambda_c$ is different from that for $\lambda \in (\lambda_{c,0}, \lambda_c)$. As for the case $\lambda < \lambda_{c,0}$, we show that $H(\lambda)$ is unbounded from above and below. In the massless case, λ_c coincides with $\lambda_{c,0}$.

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1 Introduction

This thesis is based on the joint work [6]. We consider an abstract pair-interaction model in quantum field theory. The Hamiltonian of the model is of the form

$$H(\lambda) := \mathrm{d}\Gamma_{\mathrm{b}}(T) + \frac{\lambda}{2}\Phi_{\mathrm{s}}(g)^{2}$$

acting in the boson Fock space $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ over a Hilbert space \mathscr{H} (see Subsection 2.1), where T is a self-adjoint operator on \mathscr{H} , $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ is the second quantization operator of T, $\Phi_{\mathrm{s}}(g)$ is the Segal field operator with test vector g in \mathscr{H} (see Subsection 2.1) and $\lambda \in \mathbb{R}$ is a coupling constant. A model of this type is called a ϕ^2 -model.

There have been many studies on massive or massless ϕ^2 -models in concrete forms or abstract forms (see, e.g., [4, 8, 9, 11, 12, 16]). In [11] and [16], the (essential) self-adjointness of the Hamiltonian of a ϕ^2 -model is proved in the case where $\lambda > 0$ or $|\lambda|$ is sufficiently small. In [11], the existence of a ground state of a ϕ^2 -model also is shown in the case where the quantum field under consideration is massive and $\lambda > 0$.

It is a well known that Hamiltonians with linear and/or quadratic interactions in quantum fields may be analyzed by the method of Bogoliubov transformations (see, e.g., [1, 2, 3, 4, 7, 8, 10, 12]). A typical Bogoliubov transformation is constructed from bounded linear operators U, V and a conjugation operator J on \mathscr{H} satisfying the following equations:

$$\begin{cases}
U^*U - V^*V = I, \\
U_J^*V - V_J^*U = 0, \\
UU^* - V_J V_J^* = I, \\
UV^* - V_J U_J^* = 0,
\end{cases}$$
(1.1)

where $A_J := JAJ$ and A^* is the adjoint of a densely defined linear operator A. It is well known that there is a unitary operator \mathbb{U} on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ which implements the Bogoliubov transformation in question if and only if V is Hilbert-Schmidt [7, 13, 14, 15]. Moreover, it is shown that, under the condition that V is Hilbert-Schmidt and suitable additional conditions, the Hamiltonian under consideration is unitarily equivalent via U to a second quantization operator up to a constant addition. For example, the Pauli-Fierz model with dipole approximation, which can be regarded as a kind of ϕ^2 -model, is analyzed by this method in [10].

Recently, a general quadratic form Hamiltonian with a coupling constant $\lambda \in \mathbb{R}$ has been analyzed in [12] and it is shown that, in the case of a massive quantum field, under suitable conditions, the Hamiltonian is diagonalized by a Bogoliubov transformation. In [8], the sufficient condition formulated in [12] to obtain the result just mentioned has been extended. The spectrum of the standard pair-interaction model in physics, which is a concrete realization of the abstract pair-interaction model, is formally known [9] in the case where $\lambda > \lambda_{c,0}$ and $\lambda \neq \lambda_c$ for the constants λ_c and $\lambda_{c,0}$ which satisfy $\lambda_{c,0} < \lambda_c$. The paper [4] gives a rigorous proof for that in the framework of the boson Fock space theory over $\mathscr{H} = L^2(\mathbb{R}^d)$ for any $d \in \mathbb{N}$ and $\lambda > \lambda_c$.

One of the motivations for the present work is to extend the theory developed in [4] with $\mathscr{H} = L^2(\mathbb{R}^d)$ to the theory with \mathscr{H} being an abstract Hilbert space including the case where $\lambda < \lambda_c$. It is a well known fact (see [9]) that the spectral properties of the standard pair-interaction model may depend on whether $\lambda > \lambda_c$ or $\lambda < \lambda_c$. Hence it is important to clarify this aspect mathematically. Therefore we analyze our model also for the region $\lambda < \lambda_c$. We show that, in the massive case with $\lambda \in (\lambda_{c,0}, \lambda_c)$ also, the method of Bogoliubov transformations can be applied to prove that the Hamiltonian $H(\lambda)$ is unitarily equivalent to a second quantization operator up to a constant addition. Then we see that the spectrum of $H(\lambda)$ for $\lambda \in (\lambda_{c,0}, \lambda_c)$ is different from that for $\lambda > \lambda_c$. In the massless case, $\lambda_{c,0}$ coincides with λ_0 .

The main results of the present paper include the following (1)–(3) (see Theorem 2.8 for more details): (1) Identification of the spectra of $H(\lambda)$ for $\lambda > \lambda_c$. (2) Identification of the spectra of $H(\lambda)$ for $\lambda_{c,0} < \lambda < \lambda_c$ it is only in the massive case; in the massless case, $\lambda_{c,0} = \lambda_c$). In this case, bound states different from the ground state appear. (3) Unboundedness of $H(\lambda)$ from above and below for $\lambda < \lambda_{c,0}$.

The outline of this paper is as follows. In Section 2, we define our model and recall a fundamental fact in a general theory of Bogoliubov transformations. We prove the (essential) self-adjointness of $H(\lambda)$ (Theorem 2.3). Then we state the main theorem of this paper (Theorem 3.6). In Section 3, we construct the operators U and V which are used to define the Bogoliubov transformation we need. In Section 4, we show that U and V satisfy (1.1) and V is Hilbert-Schmidt. In Section 5, we prove Theorem 2.8 (1) and calculate the ground

state energy of $H(\lambda)$ in the case $\lambda > \lambda_c$. In Section 6, we prove Theorem 2.8 (2). In Section 7, we prove Theorem 2.8 (3). In Section 8, we consider a slightly generalized Hamiltonian which is of the form $H(\eta, \lambda) := H(\lambda) + \eta \Phi_{\rm S}(f)$ for $\eta \in \mathbb{R}$ and $f \in \mathscr{H}$. Applying the methods and results in the preceding sections, we analyze $H(\eta, \lambda)$ and identify the spectra of it. In Appendix, we state some basic facts in the theory of boson Fock space.

2 Preliminaries

2.1 The abstract boson Fock Space

Let \mathscr{H} be a Hilbert space over the complex field \mathbb{C} with the inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$. The inner product is linear in the second variable and anti-linear in the first one. The symbol $\|\cdot\|_{\mathscr{H}}$ denotes the norm associated with it. We omit \mathscr{H} in $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$, respectively if there is no danger of confusion. For each non-negative integer $n = 0, 1, 2, \ldots, \otimes_{s}^{n} \mathscr{H}$ denotes the *n*-fold symmetric tensor product Hilbert space of \mathscr{H} with convention $\otimes_{s}^{0} \mathscr{H} := \mathbb{C}$. Then

$$\mathscr{F}_{\mathrm{b}}(\mathscr{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathscr{H}$$

is called the boson Fock space over \mathscr{H} . For a dense subspace \mathscr{D} in \mathscr{H} , $\hat{\otimes}_{s}^{n} \mathscr{D}$ denotes the algebraic *n*-fold symmetric tensor product of \mathscr{D} with $\hat{\otimes}_{s}^{0} \mathscr{H} := \mathbb{C}$. Then

$$\mathscr{F}_{\mathrm{b,fin}}(\mathscr{D}) := \hat{\oplus}_{n=0}^{\infty} \hat{\otimes}_{\mathrm{s}}^{n} \mathscr{D}$$

is a dense subspace of $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, where $\hat{\oplus}_{n=0}^{\infty} \mathscr{D}_n$ stands for the algebraic direct sum of subspace $\mathscr{D}_n \subset \otimes_{\mathrm{s}}^n \mathscr{H}, n = 0, 1, 2, \ldots$ The finite particle vector subspace

$$\mathscr{F}_{\mathrm{b},0}(\mathscr{H}) := \left\{ \psi^{(n)} \}_{n=0}^{\infty} \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \middle| \begin{array}{c} \psi^{(n)} \in \otimes_{\mathrm{s}}^{n} \mathscr{H}, \ n \ge 0, \text{ there is an integer } n_{0} \in \mathbb{N} \\ \text{ such that } \psi^{(n)} = 0, \text{ for all } n \ge n_{0} \end{array} \right\}$$

satisfies $\mathscr{F}_{\mathrm{b,fin}}(\mathscr{D}) \subset \mathscr{F}_{\mathrm{b},0}(\mathscr{H}) \subset \mathscr{F}_{\mathrm{b}}(\mathscr{H})$, in particular, it is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. For a linear operator T on a Hilbert space, the domain of T will be denoted by D(T).

For a densely defined closable operator T on \mathscr{H} , let $T_{\rm b}^{(n)}$ be the densely defined closed operator on $\otimes_{\rm s}^n \mathscr{H}$ defined by

$$T_{\rm b}^{(n)} := \begin{cases} \overline{\sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \underbrace{T}_{j=1}^{j-{\rm th}} \otimes I \otimes \cdots \otimes I \upharpoonright \hat{\otimes}_{\rm s}^{n} D(T)}, & n \ge 1, \\ 0, & n = 0, \end{cases}$$

where I denotes the identity operator on \mathscr{H} , \overline{A} denotes the closure of a closable operator A and $A \upharpoonright \mathcal{M}$ denotes the restriction of a linear operator A on a subspace \mathcal{M} . The operator

$$\mathrm{d}\Gamma_{\mathrm{b}}(T) := \oplus_{n=0}^{\infty} T_{\mathrm{b}}^{(n)}$$

is called the second quantization operator of T. If T is self-adjoint or non-negative, then so is $d\Gamma_{\rm b}(T)$. For each $f \in \mathscr{H}$, there exists a unique densely defined closed operator A(f) on $\mathscr{F}_{\rm b}(\mathscr{H})$ such that its adjoint $A(f)^*$ is given as follows:

$$D(A(f)^*) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \ \left| \sum_{n=1}^{\infty} n \left\| S_n \left(f \otimes \psi^{(n-1)} \right) \right\|^2 < \infty \right\}, \\ (A(f)^* \psi)^{(n)} = \sqrt{n} S_n (f \otimes \psi^{(n-1)}), \ n \in \mathbb{N}, \quad (A(f)^* \psi)^{(0)} = 0 \text{ for } \psi \in D(A(f)^*),$$

where S_n is the symmetrization operator on the *n*-fold tensor product $\otimes^n \mathscr{H}$ of \mathscr{H} . The operator A(f) (resp. $A(f)^*$) is called the annihilation (resp. creation) operator with test vector f. We have

$$\mathscr{F}_{\mathrm{b},0}(\mathscr{H}) \subset D(A(f)) \cap D(A(f)^*)$$

for all $f \in \mathscr{H}$ and A(f) and $A(f)^*$ leave $\mathscr{F}_{b,0}(\mathscr{H})$ invariant. Moreover, they satisfy the following commutation relations:

 $[A(f), A(g)^*] = \langle f, g \rangle$, [A(f), A(g)] = 0, $[A(f)^*, A(g)^*] = 0$, for all $f, g \in \mathscr{H}$ (2.1) on $\mathscr{F}_{b,0}(\mathscr{H})$, where [A, B] := AB - BA is the commutator of linear operators A and B. The relation (2.1) is called the canonical commutation relations (CCR) over \mathscr{H} . The symmetric

$$\Phi_{\rm s}(f) := \frac{1}{\sqrt{2}} (A(f) + A(f)^*), \ f \in \mathscr{H}$$

is called the Segal field operator with test vector f. We write its closure by the same symbol.

2.2 Bogoliubov Transformation

operator

In this subsection, we define a Bogoliubov transformation and recall an important theorem about it. For a conjugation J on \mathscr{H} (i.e., J is an anti-linear operator on \mathscr{H} satisfying $\|Jf\| = \|f\|$ for all $f \in \mathscr{H}$ and $J^2 = I$) and a linear operator A on \mathscr{H} , we define

$$A_J := JAJ$$

Definition 2.1. Let U and V be bounded linear operators on \mathscr{H} and J be a conjugation on \mathscr{H} . For each $f \in \mathscr{H}$, let a linear operator B(f) on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ be given by

$$B(f) := A(Uf) + A(JVf)^*.$$

Then the correspondence $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is called a Bogoliubov transformation.

By $\mathscr{F}_{b,0}(\mathscr{H}) \subset D(B(f))$, the adjoint $B(f)^*$ exists and the equation $B(f)^* = A(Uf)^* + A(JVf)$ holds on $\mathscr{F}_{b,0}(\mathscr{H})$ for each $f \in \mathscr{H}$. If the equations

$$U^*U - V^*V = I, \quad U^*_IV - V^*_IU = 0$$

hold, then the Bogoliubov transformation preserves CCR, i.e., it holds that

$$[B(f), B(g)^*] = \langle f, g \rangle, [B(f), B(g)] = 0, [B(f)^*, B(g)^*] = 0, \text{ for all } f, g \in \mathscr{H}$$

on $\mathscr{F}_{b,0}(\mathscr{H})$. The following theorem is well known (see [14, 15]):

Theorem 2.2. Let \mathscr{H} be separable and the operators U and V satisfy (1.1). Then there exists a unitary operator \mathbb{U} on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that

$$\mathbb{U}\overline{B(f)}\mathbb{U}^{-1} = A(f), \quad f \in \mathscr{H}$$

if and only if V is Hilbert-Schmidt.

2.3 Hamiltonians

For a self-adjoint operator T on \mathscr{H} , constants $\lambda, \eta \in \mathbb{R}$ which are called coupling constants, and vectors $f, g \in \mathscr{H}$, we define Hamiltonians $H(\lambda)$ and $H(\eta, \lambda)$ by

$$H(\lambda) := \mathrm{d}\Gamma_{\mathrm{b}}(T) + \frac{\lambda}{2} \Phi_{\mathrm{s}}(g)^{2}, \quad H(\eta, \lambda) := H(\lambda) + \eta \Phi_{\mathrm{s}}(f).$$

If g = 0, then $H(\lambda)$ and $H(\eta, \lambda)$ are well-known operators. Thus, we always assume that $g \neq 0$ in the present paper. If $g \in D(T^{-1/2})$, let the constant be defined by

$$\lambda_{\mathbf{c},0} := - \|T^{-1/2}g\|^{-2}.$$

Theorem 2.3. Suppose that T is an injective, non-negative, self-adjoint operator on \mathscr{H} . Let $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T)$. Then the following (1)-(3) hold:

(1) Let

$$\lambda_T(g) := \|T^{-1/2}g\|^{-1} (\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1}$$
(2.2)

and $|\lambda| < \lambda_T(g)$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.

(2) Let $|\lambda| \ge \lambda_T(g)$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_{\rm b}(T)$ for all $\eta \in \mathbb{R}$. Moreover, if $\lambda \ge \lambda_T(g)$, then $H(\eta, \lambda)$ is self-adjoint.

- (3) Let $f \in D(T^{1/2})$. Then $\overline{H(\lambda_{c,0})}$ is bounded from below. Moreover, if $\lambda > \lambda_{c,0}$, then $\overline{H(\eta,\lambda)}$ is also bounded from below for all $\eta \in \mathbb{R}$ and $D(d\Gamma_{\rm b}(T)^{1/2}) = D((\overline{H(\eta,\lambda)} + M)^{1/2})$ for all constant $M \ge 0$ satisfying $\overline{H(\eta,\lambda)} + M \ge 0$.
- *Proof.* (1) For any $\lambda \in \mathbb{R}$, by using (2.1), (10.1), (10.2) and [5, Theorem 5.18.], there are constants $a, b \geq 0$ such that for all $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$\left\|\frac{\lambda}{2}\Phi_{\rm s}(g)^{2}\psi\right\| \leq \frac{|\lambda|}{4}\left(a\|\mathrm{d}\Gamma_{\rm b}(T)\psi\| + b\|\psi\|\right)$$

In particular, we can choose a and b which satisfy $a|\lambda|/4 < 1$ if $|\lambda| < \lambda_T(g)$. We remark that, to obtain the factor $\lambda_T(g)$, we need to deform terms $||A(g)^{*2}\psi||^2$, $||A(g)^*A(g)\psi||^2$ and $||A(g)^2\psi||^2$ coming from $||\Phi_s(g)^2\psi||^2$ ($\psi \in \mathscr{F}_{b,0}(\mathscr{H})$) to $||A(g)A(g)^*\psi||^2$ + a marginal term respectively. Thus, for $|\lambda| < \lambda_T(g)$, by the Kato-Rellich theorem, $H(\lambda)$ is selfadjoint. It is well known that $\Phi_s(f)$ is infinitesimally small with respect to $d\Gamma_b(T)$. Hence, by the Kato-Rellich theorem, for $|\lambda| < \lambda_T(g)$, $H(\eta, \lambda)$ is self-adjoint.

(2) Firstly, we show that, for any $f \in D(T^{1/2})$ and $\eta, \lambda \in \mathbb{R}$, $H(\eta, \lambda)$ is essentially selfadjoint on any core of $d\Gamma_{\rm b}(T)$. By (10.1), (10.2) and [5, Theorem 5.18.], we can see that there exists a > 0 such that $||H(\eta, \lambda)\psi|| \le a||(d\Gamma_{\rm b}(T) + I)\psi||$ for all $\psi \in D(d\Gamma_{\rm b}(T))$. Let $f \in D(T)$. Then by (2.1) and (10.3), for any $\psi \in \mathscr{F}_{\rm b,fin}(D(T))$, we have

Thus, by (10.1) and (10.2), we obtain

$$|\langle H(\eta,\lambda)\psi, (\mathrm{d}\Gamma_{\mathrm{b}}(T)+I)\psi\rangle - \langle (\mathrm{d}\Gamma_{\mathrm{b}}(T)+I)\psi, H(\eta,\lambda)\psi\rangle| \le C ||(\mathrm{d}\Gamma_{\mathrm{b}}(T)+I)^{1/2}\psi||^{2},$$
(2.3)

where $C := \{ |\lambda| || T^{1/2}g || (||g|| + 2||T^{-1/2}g||) + \sqrt{2}|\eta| ||T^{1/2}f|| \}$. By a limiting argument, using the fact that $\mathscr{F}_{\mathrm{b,fin}}(D(T))$ is a core of $d\Gamma_{\mathrm{b}}(T)$ and $d\Gamma_{\mathrm{b}}(T)$ -boundedness of $\Phi_{\mathrm{s}}(g)^2$, we can show that for $f \in D(T^{1/2})$ and $\psi \in D(d\Gamma_{\mathrm{b}}(T))$, (2.3) holds. Thus, by the Nelson commutator theorem, for all $\eta, \lambda \in \mathbb{R}$, $H(\eta, \lambda)$ is essentially self-adjoint and $\overline{H(\eta, \lambda)}$ is essentially self-adjoint on any core of $d\Gamma_{\mathrm{b}}(T)$. The equation $\overline{H(\eta, \lambda)} \upharpoonright \mathscr{D} = \overline{H(\eta, \lambda)} \upharpoonright \mathscr{D}$ holds for any core \mathscr{D} of $d\Gamma_{\mathrm{b}}(T)$. Hence $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_{\mathrm{b}}(T)$ for all $\eta, \lambda \in \mathbb{R}$. Next we show that, if $\lambda > ||T^{-1/2}g||^{-1}(||T^{-1/2}g|| + ||T^{1/2}g||)^{-1}$, then $H(\eta, \lambda)$ is self-adjoint. We can show that, for $\lambda > 0$ and any $0 < \varepsilon < 1$, there is a constant $c_{\varepsilon} > 0$ such that

$$(1-\varepsilon)\|\mathrm{d}\Gamma_{\mathrm{b}}(T)\psi\|^{2} + \left\|\frac{\lambda}{2}\Phi_{\mathrm{s}}(g)^{2}\psi\right\|^{2} \leq \|H(\eta,\lambda)\psi\|^{2} + c_{\varepsilon}\|\psi\|^{2}, \quad \psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)).$$

Hence $H(\eta, \lambda)$ is closed. In particular, it is self-adjoint.

(3) From the fact that $\Phi_{\rm s}(f)$ is infinitesimally small with respect to $\mathrm{d}\Gamma_{\rm b}(T)$, for any $\varepsilon > 0$, $\varepsilon \mathrm{d}\Gamma_{\rm b}(T) + \eta \Phi_{\rm s}(f)$ is bounded from below. By (10.1), for any $\varepsilon > 0$ and $\psi \in D(\mathrm{d}\Gamma_{\rm b}(T)^{1/2})$,

$$|\langle \psi, A(f)\psi\rangle| \leq ||T^{-1/2}f|| \left(\varepsilon ||\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi||^{2} + \frac{1}{4\varepsilon}||\psi||^{2}\right).$$

Hence if the assertion follows for $\eta = 0$, then so is for all η . Thus we show that the assertion follows for $\eta = 0$. If $\lambda > 0$, then clearly $H(\lambda) \ge 0$. Let $\lambda < 0$. By (10.1) and (10.2), for any $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$, it follows that

$$\|\Phi_{\rm s}(g)\psi\|^2 \le 2\|T^{-1/2}g\|^2 \|\mathrm{d}\Gamma_{\rm b}(T)^{1/2}\psi\|^2 + \|g\|^2 \|\psi\|^2.$$

Thus for any $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$\langle \psi, H(\lambda)\psi \rangle = \| \mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi \|^{2} + \frac{\lambda}{2} \| \Phi_{\mathrm{s}}(g)\psi \|^{2}$$

$$\geq (1+\lambda \|T^{-1/2}g\|^{2}) \| \mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi \|^{2} + \frac{\lambda}{2} \|g\|^{2} \|\psi\|^{2}.$$
 (2.4)

Hence $H(\lambda)$ is bounded from below if $\lambda \geq \lambda_{c,0}$.

Let $\lambda \geq \lambda_{c,0}$ and $M \geq 0$ be a constant satisfying $H(\lambda) + M \geq 0$. Then for any $\psi \in D(d\Gamma_{b}(T)) = D(H(\lambda))$,

$$\|(\overline{H(\lambda)} + M)^{1/2}\psi\|^{2} \le (1 + |\lambda| \|T^{-1/2}g\|^{2}) \|d\Gamma_{\rm b}(T)^{1/2}\psi\|^{2} + \left(\frac{|\lambda|}{2} \|g\|^{2} + M\right) \|\psi\|^{2}.$$
(2.5)

By the fact that $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ is a core of $\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}$, we have $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) \subset D((\overline{H(\lambda)} + M)^{1/2})$ and (2.5) holds on $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$.

In the case $\lambda > 0$, the fact that $\Phi_{\rm s}(g)^2$ is non-negative implies that $||H(\lambda)^{1/2}\psi|| \ge$ $||\mathrm{d}\Gamma_{\rm b}(T)^{1/2}\psi||$ holds for any $\psi \in D(\mathrm{d}\Gamma_{\rm b}(T))$. In the case $0 > \lambda > \lambda_{\rm c,0}$,

$$\|\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi\|^{2} \leq \frac{1}{1+\lambda}\|T^{-1/2}g\|^{2}}\left\{\|(\overline{H(\lambda)}+M)^{1/2}\psi\|^{2} - \left(\frac{\lambda}{2}\|g\|^{2}+M\right)\|\psi\|^{2}\right\}$$

holds for any $\psi \in D(d\Gamma_{\rm b}(T))$ by (2.4). Hence for $\lambda > \lambda_{\rm c,0}$ there is a constant $a, b \ge 0$ such that for any $\psi \in D(d\Gamma_{\rm b}(T))$,

$$\|\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi\| \le a\|(\overline{H(\lambda)} + M)^{1/2}\psi\| + b\|\psi\|.$$
(2.6)

By a functional calculus, $D(d\Gamma_{\rm b}(T))$ is a core of $(\overline{H(\lambda)} + M)^{1/2}$. This fact and (2.6) imply that $D((\overline{H(\lambda)} + M)^{1/2}) \subset D(d\Gamma_{\rm b}(T)^{1/2})$ and (2.6) holds on $D((\overline{H(\lambda)} + M)^{1/2})$.

Remark 2.4. By [3, Lemma 13-15], if \mathscr{H} is separable, then Theorem 2.3 takes the following forms:

Let \mathscr{H} be separable, T be a non-negative, injective self-adjoint operator, $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$. Then the following (1)-(3) hold:

- (1) Let $\lambda > \lambda_{c,0}$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.
- (2) Let $\lambda \leq \lambda_{c,0}$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_{\rm b}(T)$ for all $\eta \in \mathbb{R}$. In particular, if $\eta = 0$ and $\lambda = \lambda_{c,0}$, then $H(\lambda_{c,0}) = H(0, \lambda_{c,0})$ is bounded from below.
- (3) Let $\lambda > \lambda_{c,0}$. Then $D(d\Gamma_b(T)^{1/2}) = D((H(\eta, \lambda) + M)^{1/2})$ for all constant $M \ge 0$ satisfying $H(\eta, \lambda) + M \ge 0$.

3 The Main Theorem

3.1 Assumptions

To prove our main theorem stated later (Theorem 3.6), we need some assumptions. For a closed operator A, $\sigma(A)$ denotes the spectrum of A. If A is self-adjoint, then $\sigma_{\rm ac}(A)$ (resp. $\sigma_{\rm p}(A)$, $\sigma_{\rm sc}(A)$) denotes the absolutely continuous (resp. point, singular continuous) spectrum of A. For a self-adjoint operator A which is bounded from below,

$$E_0(A) := \inf \sigma(A)$$

is called the lowest energy of A. In particular, it is called the ground state energy of A if $E_0(A) \in \sigma_p(A)$. In this case, an eigenvector of A with eigenvalue $E_0(A)$ is called a ground state of A. The ground state is said to be unique if dim $\text{Ker}(A - E_0(A)) = 1$. For linear operators A and B, the symbol $A \subset B$ means that $D(A) \subset D(B)$ and Af = Bf for all $f \in D(A)$, i.e., B is an extension of A.

Definition 3.1. Let T be a self-adjoint operator on \mathscr{H} and $\{E(B) \mid B \in \mathbf{B}^1\}$ be the spectral measure associated with T on the Borel field \mathbf{B}^1 on \mathbb{R} . The operator T is called purely absolutely continuous if, for each $f \in \mathscr{H}$, the measure $||E(\cdot)f||^2$ on \mathbf{B}^1 is absolutely continuous with respect to the one-dimensional Lebesgue measure.

Definition 3.2. For a purely absolutely continuous self-adjoint operator T and vectors $f, g \in \mathcal{H}, \psi_{g,f}$ denotes the Radon-Nikodym derivative of the finite complex Borel measure $\langle g, E(\cdot)f \rangle$ on \mathbf{B}^1 . In particular, we set $\psi_g := \psi_{g,g}$.

- **Assumption 3.3.** (1) The operator T is a non-negative, purely absolutely continuous selfadjoint operator.
 - (2) The fixed vector $g \in \mathscr{H}$ satisfies $g \in D(\hat{T}^{-1/2}) \cap D(T^{1/2})$ and Jg = g, where $\hat{T} := T E_0$, $E_0 := E_0(T)$ and J is a conjugation on \mathscr{H} satisfying $JD(T) \subset D(T)$ and $JT\psi = TJ\psi$ for any $\psi \in D(T)$ (i.e., $JT \subset TJ$),
 - (3) $\sup_{E_0 < x} x^{\pm 1} \psi_g(x) < \infty$ and $\psi_g(x) > 0$ for all $x \in (E_0, \infty)$,
 - (4) $\psi_g \in C([E_0,\infty)) \cap C^1((E_0,\infty))$ and $\lim_{x \downarrow E_0} x^{-1} \psi'_g(x) = 0 = \lim_{x \to \infty} x^{-1} \psi'_g(x).$

Remark 3.4. The operator T is injective since it is a purely absolutely continuous selfadjoint operator. Since T has no eigenvector, the inverse of \hat{T} exists. Assumption 3.3 (2) implies that $T_J = T$. In general, for a self-adjoint operator A and a conjugation J, we can choose a vector $f \in D(A)$ satisfying Jf = f if $A_J = A$. Thus the vector g in Assumption 3.3 (2) exists. By Assumption 3.3 (3), one can easily show that $\sup_{x \in \sigma(T)} \psi_g(x) < \infty$ and, for each $f \in \mathscr{H}$, the functions $\psi_{g,f}$ and $\psi_{T^{\pm 1/2}g,f}$ are in $L^2(\mathbb{R})$ and the maps : $f \mapsto \psi_{g,f}, \psi_{T^{\pm 1/2}g,f}$ are bounded. Actually, for any $h \in \mathscr{H}$ and $B \in \mathbf{B}^1$, the following inequality holds

$$|\langle E(B)h, f \rangle|^2 \le ||E(B)h||^2 ||E(B)f||^2$$

by Schwarz's inequality. Thus we obtain $|\psi_{h,f}(\mu)|^2 \leq \psi_h(\mu)\psi_f(\mu)$ for almost all $\mu \in \mathbb{R}$ with respect to the Lebesgue measure. Hence, by Assumption 3.3 (3), we have the boundedness of the mappings. Moreover, we see that for any $F \in L^2(\mathbb{R})$, $g \in D(F(T))$, where F(T) denotes the operator defined by $F(T) := \int_{\mathbb{R}} F(\mu)dE(\mu)$. In particular, g is in $D(\psi_{g,f}(T))$ for any $f \in \mathscr{H}$.

Lemma 3.5. Let T be a self-adjoint operator such that $JT \subset TJ$. Then

- (1) $E(B)_J = E(B)$, for all $B \in \mathbf{B}^1$.
- (2) Let F be a Borel measurable function on \mathbb{R} . Then $F(T)_J = F^*(T)$, where F^* is complex conjugation of F.

Proof. These are proved by using the spectral theorem.

3.2 The Main Theorem

In this subsection, we state the main theorem of the present paper. Let λ_c be a constant defined by

$$\lambda_{\rm c} := -\left(\int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2} \ d\|E(\mu)g\|^2\right)^{-1} < 0.$$

Then, by a functional calculus, we obtain $\lambda_{c,0} \leq \lambda_c$, and $\lambda_{c,0} = \lambda_c$ if and only if $E_0 = 0$.

Theorem 3.6. Let \mathscr{H} be separable. Then the following (1)-(3) hold:

(1) Let T and g satisfy Assumption 3.3. If $\lambda > \lambda_c$, then there are a unitary operator \mathbb{U} on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ and a constant $E_{\mathrm{g}} \in \mathbb{R}$ such that

$$\mathbb{U}H(\lambda)\mathbb{U}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(T) + E_{\mathrm{g}}.$$
(3.1)

In particular, $\mathbb{U}^{-1}\Omega_0$ is the unique ground state of $H(\lambda)$, where $\Omega_0 := (1, 0, 0, \ldots) \in \mathscr{F}_{\mathrm{b}}(\mathscr{H})$ is the Fock vacuum, and

$$\sigma(H(\lambda)) = \{E_{g}\} \cup [E_{0} + E_{g}, \infty), \qquad (3.2)$$

$$\sigma_{\rm ac}(H(\lambda)) = [E_0 + E_{\rm g}, \infty), \ \sigma_{\rm p}(H(\lambda)) = \{E_{\rm g}\}, \ \sigma_{\rm sc}(H(\lambda)) = \emptyset.$$
(3.3)

(2) Let T and g satisfy Assumption 3.3 and $E_0 > 0$. If $\lambda_{c,0} < \lambda < \lambda_c$, then there exist a unitary operator \mathbb{V} on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, an injective non-negative self-adjoint operator ξ on \mathscr{H} and a constant $E_{\mathrm{b}} \geq 0$ such that ξ has a ground state and

$$\mathbb{V}H(\lambda)\mathbb{V}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(\xi) + E_{\mathrm{g}} - E_{\mathrm{b}}.$$

In particular, $\mathbb{V}^{-1}\Omega_0$ is the unique ground state of $H(\lambda)$, and

$$\begin{aligned} \sigma(H(\lambda)) &= \bigcup_{n=0}^{\infty} \{ n\beta + E_{\rm g} - E_{\rm b} \} \cup [E_0 + E_{\rm g} - E_{\rm b}, \infty), \\ \sigma_{\rm ac}(H(\lambda)) &= [E_0 + E_{\rm g} - E_{\rm b}, \infty), \\ \sigma_{\rm p}(H(\lambda)) &= \bigcup_{n=0}^{\infty} \{ n\beta + E_{\rm g} - E_{\rm b} \}, \ \sigma_{\rm sc}(H(\lambda)) = \emptyset, \end{aligned}$$

where $\beta > 0$ is the discrete ground state energy of ξ .

(3) Let T be a non-negative, injective self-adjoint operator. If $g \in D(T^{-1/2})$ and $\lambda < \lambda_{c,0}$, then $H(\lambda)$ is unbounded from above and below.

Example 3.7. A concrete realization of the abstract model is given as follows (see [9, Chapter 12]):

$$\mathscr{H} \leftrightarrow L^2(\mathbb{R}^d), \ T \leftrightarrow \omega, \ g \leftrightarrow \frac{\hat{\rho}}{\sqrt{\omega}},$$

where ω is the multiplication operator associated with the function $\omega(k) := \sqrt{|k|^2 + m^2}, k \in \mathbb{R}^d$ for a fixed $m \ge 0$ and $\hat{\rho}$ is the Fourier transform of a function $\rho \in L^2(\mathbb{R}^d)$ satisfying $\hat{\rho}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Assume that $\hat{\rho}$ is rotation invariant, i.e., there exists a function v on $[0, \infty)$ such that $\hat{\rho}(k) = v(|k|)$ for all $k \in \mathbb{R}^d$. Then we have $\psi_g(s) = |S^{d-1}| \omega_1^{-1}(s)^{d-2} |v(\omega_1^{-1}(s))|^2$ for s > m, where $|S^{d-1}|$ is the surface area of the (d-1)-dimensional unite sphere with convention $|S^0| = 2\pi$ and $\omega_1(r) = \sqrt{r^2 + m^2}, r \ge 0$. Set $\psi_g(m) := 0$. Hence, with J being the complex conjugation, the following conditions (2)'-(4)' imply that the present model satisfies Assumption 3.3:

(2)' $\hat{\rho}(k)^*=\hat{\rho}(k)$ and

$$\hat{\rho} \in L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} \frac{|\hat{\rho}(k)|^2}{|k|^2} dk < \infty.$$

(3)' $\hat{\rho}$ is rotation invariant. $\sup_{k \in \mathbb{R}^d} \omega(k)^{\pm 1/2} |k|^{(d-2)/2} |\hat{\rho}(k)| < \infty$. $\hat{\rho}(k) > 0$, for all $k \neq 0$. (4)' $v \in C^1([0,\infty))$ and

$$\lim_{|k|\to 0} |k|^{d-4} \hat{\rho}(k) \{ (d-2)\hat{\rho}(k) + 2|k|v'(|k|) \} = 0,$$
$$\lim_{|k|\to\infty} |k|^{d-4} \hat{\rho}(k) \{ (d-2)\hat{\rho}(k) + 2|k|v'(|k|) \} = 0.$$

We can show that ψ_g is right continuous at m by $\int_{\mathbb{R}^d} |\hat{\rho}(k)|^2 |k|^{-2} dk < \infty$ and $v \in C^1([0,\infty))$. Thus, $\psi_g \in C([m,\infty))$. For example, one can easily check that the function

$$\hat{\rho}(k) := \exp\left(-\frac{1}{|k|^2} - |k|^2\right), \ k \in \mathbb{R}^d \setminus \{0\}, \ \hat{\rho}(0) := 0$$

satisfies the above conditions (2)'-(4)'.

4 Definitions and properties of some functions and operators

In this section, we introduce some functions and operators. We assume that \mathscr{H} is separable and Assumption 3.3 from this section to Section 6.

4.1 Functions D and D_{\pm}

Lemma 4.1. Let $D : \mathbb{C} \setminus (0, \infty) \to \mathbb{C}$ be the function

$$D(z) := 1 + \lambda \int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2 - z} d\|E(\mu)g\|^2, \quad z \in \mathbb{C} \setminus (0,\infty).$$

Then D is well-defined and analytic in $\mathbb{C}\setminus[0,\infty)$. Moreover, the following hold:

- (1) For all $\lambda > \lambda_c$, D(z) has no zeros in $\mathbb{C} \setminus [0, \infty)$.
- (2) For all $\lambda < \lambda_c$, D(z) has a unique simple zero in the negative real axis $(-\infty, 0)$.

Proof. If $\text{Im} z \neq 0$ (resp. Re z < 0), then for any $n \in \mathbb{N}$,

$$\int_{[E_0,\infty)} \left| \frac{\mu}{(\mu^2 - E_0^2 - z)^n} \right| d\|E(\mu)g\|^2 \le c^{-n} \|T^{1/2}g\|^2 < \infty,$$

where c is |Imz| (resp. |Rez|). If z = 0, then

$$\int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 \le \|\hat{T}^{-1/2}g\|^2 < \infty.$$

Thus, by using the Lebesgue dominated convergence theorem, D is well-defined and analytic in $\mathbb{C}\setminus[0,\infty)$.

(1) If $\lambda = 0$, then D(z) = 1 for all $z \in \mathbb{C} \setminus (0, \infty)$, so it has no zeros. Let $\lambda \neq 0$ and $z = x + iy \in \mathbb{C} \setminus (0, \infty)$. Then we see that

Im
$$D(z) = y\lambda \int_{[E_0,\infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2 + y^2} d\|E(\mu)g\|^2$$
.

Thus Im D(z) = 0 is equivalent to y = 0. Therefore D(z) = 0 if and only if D(x) = 0. Let y = 0. In the case $\lambda > 0$, one has D(x) > 0 for all $x \in (-\infty, 0]$. Thus D has no zeros. Next, we consider the case $\lambda < 0$. We have for x < 0,

$$D'(x) = \lambda \int_{[E_0,\infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2} d\|E(\mu)g\|^2 < 0.$$

Thus D is monotone decreasing in $(-\infty, 0)$. If $\lambda > \lambda_c$, then D(0) > 0. Hence D has no zeros.

(2) Let $\lambda < \lambda_c$. We can see that

$$D(0) = 1 + \lambda \int_{[E_0,\infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 = 1 - \frac{\lambda}{\lambda_c} < 0.$$

By the Lebesgue dominated convergence theorem, $D(x) \to 1$ as $x \to -\infty$. Since D is monotone decreasing in $(-\infty, 0)$, D has a unique simple zero in $(-\infty, 0)$.

Let

$$\phi_g(x) := \psi_g(\sqrt{x})\chi_{[E_0^2,\infty)}(x), \ x \in \mathbb{R},$$

where χ_B is the characteristic function of $B \in \mathbf{B}^1$.

Lemma 4.2. The following hold :

- (1) The function ϕ_g satisfies $\phi_g \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sup_{x \in \mathbb{R}} |\phi'_g(x)| < \infty$.
- (2) Let

$$A_{\varepsilon}^{(1)}(x) := \frac{x}{\pi(x^2 + \varepsilon^2)}, \ A_{\varepsilon}^{(2)}(x) := \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}, \ x \in \mathbb{R}, \ \varepsilon > 0$$

be the conjugate poisson kernel and the poisson kernel respectively and f * h denote the convolution of functions f and h. Let

$$(H_{\varepsilon}f)(s) := \frac{1}{\pi} \int_{|x-s| \ge \varepsilon} \frac{f(x)}{s-x} dx, \ (Hf)(s) := \lim_{\varepsilon \downarrow 0} (H_{\varepsilon}f)(s), \ s \in \mathbb{R}, \ \varepsilon > 0,$$

where Hf is called the Hilbert transform of f. Then for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} \left(A_{\varepsilon}^{(1)} \ast \phi_g \right)(x) = (H\phi_g)(x), \quad \lim_{\varepsilon \downarrow 0} \left(A_{\varepsilon}^{(2)} \ast \phi_g \right)(x) = \phi_g(x),$$

hold uniformly in x.

Proof. For any h > 0, by Assumption 3.3 (1), (4) and the mean value theorem, there exists $\theta \in (E_0 + h/2, E_0 + h)$ such that

$$\int_{E_0+h/2}^{E_0+h} \frac{\psi_g(\mu)}{\mu - E_0} d\mu = \frac{h}{2} \frac{\psi_g(\theta)}{\theta - E_0}$$

This fact and $\theta < E_0 + h$ imply that

$$\|E([E_0, E_0 + h])\hat{T}^{-1/2}g\|^2 = \int_{[E_0, E_0 + h]} \frac{\psi_g(\mu)}{\mu - E_0} d\mu > \frac{\psi_g(\theta)}{2}.$$
(4.1)

By taking the limit $h \downarrow 0$ and Assumption 3.3 (1), the left hand side of (4.1) tends to zero. Thus we obtain $\lim_{h\to E_0+0} \psi_g(h) = 0$. This fact and $\psi_g \in C([E_0,\infty))$ imply that $\psi_g(E_0) = 0$. Since ψ_g is the Radon-Nikodym derivative of $||E(\cdot)g||^2$ and $E_0 \leq T$, we have $\psi_g(x) = 0$ for $x < E_0$. Thus $\phi_g \in C(\mathbb{R})$. By the differentiability of ψ_g , we obtain $\phi'_g(x) = \psi'_g(\sqrt{x})/(2\sqrt{x})$ for $x > E_0^2$ and $\phi'_g(x) = 0$ for $x < E_0^2$. Thus, ϕ'_g is continuous on $(-\infty, E_0^2) \cup (E_0^2, \infty)$. Since $\phi'_g(x) = 0$ for $x < E_0^2$ and $\lim_{h\to 0+0} (E_0+h)^{-1}\psi_g(E_0+h) = 0$, we have $\lim_{h\to 0} \phi'_g(E_0^2+h) = 0$. By this fact and the l'Hôpital theorem, we obtain $\lim_{h\to 0+0} (\phi_g(E_0^2+h) - \phi_g(E_0^2))/h = 0$. have $\lim_{h\to 0-0} (\phi_g(E_0^2+h)-\phi_g(E_0^2))/h = 0$ since $\phi_g(x) = 0$ for $x < E_0^2$. Thus ϕ_g is continuous at E_0^2 . Hence $\phi_g \in C^1(\mathbb{R})$. By the fact that $\psi'_g(x) = 0$ for $x < E_0$ and Assumption 3.3 (4) imply that $\phi_g \in C^1(\mathbb{R})$ and $\phi'_g(E_0^2) = 0$. By Assumption 3.3 (2) and a change of variable, we have $\phi_g \in L^1(\mathbb{R})$. We obtain $\phi_g \in L^2(\mathbb{R})$ by Assumption 3.3 (3) and a change of variable. The inequality $\sup_{x \in \mathbb{R}} |\phi'_g(x)| < \infty$ is given by Assumption 3.3 (4). The assertion (1) holds. Next we consider the assertion (2). By (1), in particular, ϕ_g is bounded and uniformly continuous. Thus it is easy to see that $A_{\varepsilon}^{(2)} * \phi_g$ converges uniformly to ϕ_g . Moreover, by (1), Hölder's inequality, the mean value theorem and a similar estimate to the proof of [17, Theorem 92.], we can show that $(A_{\varepsilon}^{(1)} * \phi_g)(x) - (H_{\varepsilon}\phi_g)(x)$ tends to 0 uniformly in x as $\varepsilon \downarrow 0$. Hence the assertion (2) holds.

Detailed studies of the Hilbert transform are given in [17].

Lemma 4.3. For all $s \ge 0$, $D_{\pm}(s) := \lim_{\epsilon \downarrow 0} D(s \pm i\epsilon)$ are uniformly convergent and continuous in $s \ge 0$ with

$$D_{\pm}(s) = 1 - \frac{\lambda \pi}{2} (H\phi_g) (E_0^2 + s) \pm i \frac{\lambda \pi}{2} \psi_g \left(\sqrt{E_0^2 + s}\right), \quad s \ge 0.$$
(4.2)

Proof. For any $s \ge 0$ and $\varepsilon > 0$, we have by a change of variable

$$D(s\pm i\varepsilon) = 1 - \frac{\lambda\pi}{2} \left(A_{\varepsilon}^{(1)} * \phi_g \right) \left(E_0^2 + s \right) \pm i \frac{\lambda\pi}{2} \left(A_{\varepsilon}^{(2)} * \phi_g \right) \left(E_0^2 + s \right)$$

Thus, by Lemma 4.2, D_{\pm} converge uniformly in $s \ge 0$ and (4.2) holds. The continuity of D_{\pm} is due to the uniform convergence.

Remark 4.4. For all $\mu \in [E_0, \infty)$, we have

$$i\pi\lambda\psi_g(\mu) = D_+(\mu^2 - E_0^2) - D_-(\mu^2 - E_0^2).$$
(4.3)

Lemma 4.5. Let $\lambda \neq \lambda_c$, then $\delta := \inf_{s \ge 0} |D_{\pm}(s)| > 0$.

Proof. If $\lambda = 0$, then clearly $D_{\pm}(s) = 1 > 0$ for all $s \in [0, \infty)$. Let $\lambda \neq 0, \lambda_c$. Then $D_{\pm}(0) = D(0) \neq 0$. Hence, by the continuity of D_{\pm} , D_{\pm} has no zeros near s = 0. For any $\varepsilon > 0$ and $s > E_0^2 + 1$, we have

$$(H_{\varepsilon}\phi_g)(s) = I_1^{(\varepsilon)}(s) + \sum_{j=2}^4 I_j(s),$$

$$I_1^{(\varepsilon)}(s) = \int_{\varepsilon}^1 \frac{\phi_g(s-x) - \phi_g(s+x)}{x} dx, \ I_2(s) = \int_{E_0^2}^{s-1} \frac{\phi_g(x)}{s-x} dx$$

$$I_3(s) = \int_{s+1}^{2s} \frac{\phi_g(x)}{s-x} dx, \ I_4(s) = \int_{2s}^{\infty} \frac{\phi_g(x)}{s-x} dx.$$

Then, by the Lebesgue dominated convergence theorem, each $I_j(s)$, j = 2, 3, 4 tends to zero as $s \to \infty$. By the mean value theorem and the property that $\phi'_g(x) \to 0$ as $x \to \infty$, we have $\lim_{s\to\infty} \lim_{\varepsilon \downarrow 0} I_1^{(\varepsilon)}(s) = 0$. Hence we can see that $(H\phi_g)(s) \to 0$ as $s \to \infty$. This fact implies that $\inf_{s_0 \leq s} \operatorname{Re}D_{\pm}(s) > 0$ for a sufficiently large number $s_0 > 0$. In addition, $\operatorname{Im}D_{\pm}(s)$ are positive for any closed interval included in $(0, \infty)$ by Assumption 3.3 (3) and the continuity of ψ_g . Hence we can see that $\inf_{s\geq 0} |D_{\pm}(s)| > 0$.

Remark 4.6. By Lemmas 4.3 and 4.5, we can see that there are constants $c, d, \varepsilon_0 > 0$ with 0 < c < d such that

$$c \le \left| \frac{D(s \pm i\varepsilon)}{D_{\pm}(s)} \right| \le d \tag{4.4}$$

for all $s \ge 0, 0 < \varepsilon < \varepsilon_0$.

4.2 Operators R_{\pm}

Through this subsection, we assume $\lambda \neq \lambda_c$.

Lemma 4.7. One can define bounded operators R_{\pm} on \mathscr{H} as follows:

$$R_{\pm}f := -\lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0,\infty)} \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle, \quad f \in \mathscr{H},$$

where $R_z(A) := (A - z)^{-1}$ is the resolvent of a linear operator A at $z \in \rho(A)$ (the resolvent set of a linear operator A).

Proof. For a fixed $\varepsilon > 0$ and any $f \in \mathscr{H}$,

$$\int_{[E_0,\infty)} \left\| \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)} \right\| d\|E(\mu')f\|^2 \le \frac{\|f\|^2 \|T^{1/2}g\|}{\delta\varepsilon} < \infty$$

by Lemma 4.5 and a property of a resolvent. Thus we can define linear operators $R_{\pm}^{(\varepsilon)}$ on \mathscr{H} by

$$R_{\pm}^{(\varepsilon)}f := -\lambda \int_{[E_0,\infty)} \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle$$

in the sense of Bochner integral with the polarization identity. For any $h, f \in \mathscr{H}$,

$$\begin{split} \left\langle h, R_{\pm}^{(\varepsilon)} f \right\rangle \\ &= -\lambda \int_{[E_0,\infty)} \frac{\left\langle h, R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g \right\rangle}{D_{\pm}(\mu'^2 - E_0^2)} d\left\langle T^{1/2} g, E(\mu') f \right\rangle \\ &= -\lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \mp i\varepsilon) D_{\pm}(\mu'^2 - E_0^2)} d\left\langle h, E(\mu) g \right\rangle d\left\langle T^{1/2} g, E(\mu') f \right\rangle , \end{split}$$

where we have used a functional calculus. By change of variables in the Lebesgue-Stieltjes integration, a functional calculus and Fubini's theorem, we have

$$\left\langle h, R_{\pm}^{(\varepsilon)}f \right\rangle = \frac{\lambda\pi}{2} \int_{[E_0,\infty)} - \left(A_{\varepsilon}^{(1)} * \phi_{g,f}^{\pm}\right) (\mu^2) \mu^{1/2} \mp i \left(A_{\varepsilon}^{(2)} * \phi_{g,f}^{\pm}\right) (\mu^2) \mu^{1/2} d \left\langle h, E(\mu)g \right\rangle,$$

where $\phi_{g,f}^{\pm}(x) = \psi_{g,f}(\sqrt{x})x^{-1/4}D_{\pm}(x-E_0^2)^{-1}\chi_{[E_0^2,\infty)}(x), x \in \mathbb{R}$. We have $\phi_{g,f}^{\pm} \in L^2(\mathbb{R})$ by Remark 3.4, and the function $\left(A_{\varepsilon}^{(j)} * \phi_{g,f}^{\pm}\right)(\mu^2)\mu^{1/2}$ $(\mu \in \mathbb{R})$ is in $L^2(\mathbb{R})$ for each j = 1, 2. Thus, by a change of variable, we have

$$\begin{aligned} \left\| R_{\pm}^{(\varepsilon)} f - \left(-\frac{\pi\lambda}{2} (H\phi_{g,f}^{\pm})(T^2) T^{1/2} g \mp \frac{1}{2} A_{\pm} f \right) \right\|^2 \\ &\leq \left(\frac{\lambda\pi}{2} \right)^2 c_g \int_{[E_0^2,\infty)} \left| (A_{\varepsilon}^{(1)} * \phi_{g,f}^{\pm})(x) - (H\phi_{g,f}^{\pm})(x) \right|^2 dx \\ &+ \left(\frac{\lambda\pi}{2} \right)^2 c_g \int_{[E_0^2,\infty)} \left| (A_{\varepsilon}^{(2)} * \phi_{g,f}^{\pm})(x) - \phi_{g,f}^{\pm}(x) \right|^2 dx, \end{aligned}$$

where $c_g := \sup_{x \in [E_0,\infty)} \psi_g(x)$ and the linear operators

$$A_{\pm}f := i\pi\lambda\psi_{g,f}(T)D_{\pm}(T^2 - E_0^2)^{-1}g, \ f \in \mathscr{H}$$

are well-defined (see Remark 3.4 and Lemma 4.5). Hence, by $\phi_{g,f}^{\pm} \in L^2(\mathbb{R})$, we have

$$R_{\pm}^{(\varepsilon)}f \to -(\pi\lambda/2)(H\phi_{g,f}^{\pm})(T^2)T^{1/2}g \mp (1/2)A_{\pm}f \text{ as } \varepsilon \downarrow 0.$$

Moreover, by change of variables, the isometricity of Hilbert transform and Remark 3.4, we can show that the inequalities

$$\left\| (H\phi_{g,f}^{\pm})(T^2)T^{1/2}g \right\| \le \frac{c_g}{\delta} \|f\|, \quad \|A_{\pm}f\| \le \frac{\pi|\lambda|c_g}{\delta} \|f\|$$

hold for all $f \in \mathscr{H}$. Hence R_{\pm} are bounded.

By the definition of the adjoint operator, $R_{\pm}^* := (R_{\pm})^*$ are given as follows: for $f \in \mathscr{H}$,

$$R_{\pm}^{(\varepsilon)*}f = \lambda \int_{[E_0,\infty)} R_{\mu'^2 \pm i\varepsilon}(T^2) D_{\mp}(T^2 - E_0^2)^{-1} T^{1/2} g \ d\left\langle T^{1/2} g, E(\mu') f \right\rangle, \tag{4.5}$$
$$R_{\pm}^* f = \lim_{\varepsilon \downarrow 0} R_{\pm}^{(\varepsilon)*} f.$$

For a densely defined linear operator A on a Hilbert space, A^{\sharp} denotes A or A^* .

Lemma 4.8. The ranges of R^{\sharp}_{\pm} are included in $D(T^{-1}) \cap D(T)$ and R^{\sharp}_{\pm} map D(T) into $D(T^2)$.

Proof. For any $f, h \in \mathcal{H}$, we have

$$\langle h, R_{\pm}f \rangle = \frac{\lambda \pi}{2} \int_{[E_0,\infty)} -\left(H\phi_{g,f}^{\pm}\right) (\mu^2) \mu^{1/2} \mp i \frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle.$$
(4.6)

By a change of variable, we have

$$(H\phi_{g,f}^{\pm})(\mu^2) = \left(H\psi_{T^{-1/2}g,f}^{\pm}\right)(\mu) + \left(H\psi_{T^{-1/2}g,f}^{\pm}\right)(-\mu), \ \mu \in \mathbb{R},\tag{4.7}$$

where $\psi_{h,f}^{\pm}(x) := \psi_{h,f}(x)D_{\pm}(x^2 - E_0)^{-1}\chi_{[E_0,\infty)}(x), x \in \mathbb{R}$ for $h, f \in \mathscr{H}$. Thus we see by Assumption 3.3 (3) and a functional calculus that $\operatorname{Ran}(R_{\pm}) \subset D(T^{-1})$. The equation

$$\mu \left(H\phi_{g,f}^{\pm} \right) (\mu^2) = \left(H\psi_{T^{1/2}g,f}^{\pm} \right) (\mu) - \left(H\psi_{T^{1/2}g,f}^{\pm} \right) (-\mu), \ \mu \in \mathbb{R},$$
(4.8)

(4.6), Assumption 3.3 (3) and operational calculus imply that $\operatorname{Ran}(R_{\pm}) \subset D(T)$. For any $f \in D(T)$ and $\mu \in \mathbb{R}$,

$$\mu^2 \left(H\phi_{g,f}^{\pm} \right) (\mu^2) = \left(H\psi_{T^{1/2}g,Tf}^{\pm} \right) (\mu) + \left(H\psi_{T^{1/2}g,Tf}^{\pm} \right) (-\mu) + \frac{2}{\pi} \int_{[E_0,\infty)} \psi_{T^{1/2}g,f}^{\pm} (x) \ dx.$$

Hence $R_{\pm}f \in D(T^2)$ and the following equation holds for any $h \in \mathscr{H}$,

$$\begin{split} \left\langle h, T^2 R_{\pm} f \right\rangle = & \frac{\lambda \pi}{2} \int_{[E_0,\infty)} - \left\{ \left(H \psi_{T^{1/2}g,Tf}^{\pm} \right) (\mu) + \left(H \psi_{T^{1/2}g,Tf}^{\pm} \right) (-\mu) + \frac{2c}{\pi} \right\} \mu^{1/2} d \left\langle h, E(\mu)g \right\rangle \\ & \mp i \frac{\lambda \pi}{2} \int_{[E_0,\infty)} \psi_{T^{1/2}g,Tf}^{\pm}(\mu) \mu^{1/2} d \left\langle h, E(\mu)g \right\rangle, \end{split}$$

where $c := \int_{\mathbb{R}} \psi_{T^{1/2}g,f}^{\pm}(x) dx$. In quite the same manner as in the case of R_{\pm} , we can prove the statement for R_{\pm}^* .

Lemma 4.9. The operator equations $(R_{\pm})_J = R_{\mp}$ hold.

Proof. This follows from Assumption 3.3(1) and Lemma 3.5.

Lemma 4.10. The operator equation $R_{-} = R_{+}\gamma + A_{-}$ holds, where

$$\gamma := D_+ (T^2 - E_0^2) D_- (T^2 - E_0^2)^{-1}$$

is a bounded operator.

Proof. The first resolvent formula gives that, for any $\mu', \mu'' \in \mathbb{R}, \varepsilon > 0$,

$$R_{\mu'^2 - i\varepsilon}(T^2) - R_{\mu'^2 + i\varepsilon}(T^2) = -2i\varepsilon R_{\mu'^2 - i\varepsilon}(T^2)R_{\mu'^2 + i\varepsilon}(T^2).$$

Then, for any $f \in \mathscr{H}$,

$$\begin{split} R_{-}^{(\varepsilon)}f &= -\lambda \int_{[E_{0},\infty)} \frac{R_{\mu'^{2}+i\varepsilon}(T^{2})T^{1/2}g}{D_{-}(\mu'^{2}-E_{0}^{2})} d\left\langle T^{1/2}g, E(\mu')f\right\rangle \\ &+ 2i\lambda\varepsilon \int_{[E_{0},\infty)} \frac{R_{\mu'^{2}+i\varepsilon}(T^{2})R_{\mu'^{2}-i\varepsilon}(T^{2})T^{1/2}g}{D_{-}(\mu'^{2}-E_{0}^{2})} d\left\langle T^{1/2}g, E(\mu')f\right\rangle \end{split}$$

Thus, by a change of variable, we have for any $h \in \mathscr{H}$

$$\begin{split} \left\langle h, R_{-}^{(\varepsilon)} f \right\rangle &= \left\langle h, R_{+}^{(\varepsilon)} \gamma f \right\rangle + 2i\lambda \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} d\left\langle h, E(\mu)g \right\rangle d\left\langle T^{1/2}g, E(\mu')f \right\rangle \\ &\times \frac{\mu^{1/2}\varepsilon}{\{(\mu^{2} - \mu'^{2})^{2} + \varepsilon^{2}\}D_{-}(\mu'^{2} - E_{0}^{2})} \\ &= \left\langle h, R_{+}^{(\varepsilon)} \gamma f \right\rangle + i\pi\lambda \int_{[E_{0},\infty)} \left(A_{\varepsilon}^{(2)} * \phi_{g,f}^{-}\right)(\mu^{2})\mu^{1/2}d\left\langle h, E(\mu)g \right\rangle. \end{split}$$

By a property of the Poisson kernel, the function $\left(A_{\varepsilon}^{(2)} * \phi_{g,f}^{-}\right)(\mu^{2})\mu^{1/2}$ $(\mu \in \mathbb{R})$ converges to $\psi_{g,f}(\mu)/D_{-}(\mu^{2}-E_{0}^{2})$ as $\varepsilon \to +0$ in the sense of $L^{2}(\mathbb{R})$. Hence the continuity of the inner product with $L^{2}(\mathbb{R})$ implies that

$$\langle h, R_{-}f \rangle = \langle h, R_{+}\gamma f \rangle + i\pi\lambda \int_{[E_{0},\infty)} \frac{\psi_{g,f}(\mu)}{D_{-}(\mu^{2} - E_{0}^{2})} d\langle h, E(\mu)g \rangle$$
$$= \langle h, R_{+}\gamma f \rangle + \langle h, A_{-}f \rangle .$$

Since f and h are arbitrary, one obtains the conclusion.

By the definitions of A_{\pm} , we have

$$(A_{-})^{*} = -A_{+}$$

Lemma 4.11. For any Borel measurable function F on \mathbb{R} , $A_{\pm}F(T) \subset F(T)A_{\pm}$.

Proof. For any $f \in D(F(T))$, an operational calculus implies that $\psi_{g,F(T)f} = F\psi_{g,f} \in L^2(\mathbb{R})$. This fact imply that $\psi_{g,f}(T)g \in D(F(T))$ and $F(T)\psi_{g,f}(T)g = \psi_{g,F(T)f}(T)g$. Hence $A_{\pm}f \in D(F(T))$ and $F(T)A_{\pm}f = A_{\pm}F(T)f$ by Lemma 4.5.

Lemma 4.12. The following operator equations hold:

$$A_{-}R_{\pm}^{*} = (\gamma - I)R_{\pm}^{*}, \quad A_{-}(A_{-})^{*} = -A_{-} - (A_{-})^{*}.$$

Proof. By applying Lemma 4.11 to the case $F = \chi_B$, one can easily see that $A_{\pm}E(B) = E(B)A_{\pm}$ hold for any $B \in \mathbf{B}^1$. For any $f, h \in \mathcal{H}$, we have

$$\left\langle (A_{-})^{*}h, R_{\pm}^{(\varepsilon)*}f \right\rangle$$

= $\int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \frac{i\pi\lambda^{2}\mu^{1/2}\psi_{g}(\mu)}{(\mu^{2} - \mu'^{2} \mp i\varepsilon)D_{\mp}(\mu^{2} - E_{0}^{2})D_{-}(\mu^{2} - E_{0}^{2})} d\langle h, E(\mu)g \rangle d\langle T^{1/2}g, E(\mu')f \rangle.$

Then, since γ and E(B) commute on \mathscr{H} for any $B \in \mathbf{B}^1$, (4.3) gives

$$\begin{split} \left\langle (A_{-})^{*}h, R_{\pm}^{(\varepsilon)*}f \right\rangle \\ &= \lambda \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \frac{\mu^{1/2}}{(\mu^{2} - \mu^{\prime 2} \mp i\varepsilon)D_{\mp}(\mu^{2} - E_{0}^{2})} d\left\langle h, E(\mu)(\gamma - 1)g \right\rangle d\left\langle T^{1/2}g, E(\mu^{\prime})f \right\rangle \\ &= \left\langle h, (\gamma - 1)R_{\pm}^{(\varepsilon)*}f \right\rangle. \end{split}$$

Thus, by a limiting argument, we obtain $A_-R_{\pm}^* = (\gamma - 1)R_{\pm}^*$. Moreover, (4.3) and the equation $(A_-)^* = -A_+$ imply that

$$\begin{split} \langle h, A_{-}(A_{-})^{*}f \rangle &= -(i\pi\lambda)^{2} \int_{[E_{0},\infty)} \frac{\psi_{g,f}(\mu)\psi_{g}(\mu)}{D_{+}(\mu^{2}-E_{0}^{2})D_{-}(\mu^{2}-E_{0}^{2})} d\left\langle h, E(\mu)g \right\rangle \\ &= -i\pi\lambda \int_{[E_{0},\infty)} \frac{(D_{+}(\mu^{2}-E_{0}^{2})-D_{-}(\mu^{2}-E_{0}^{2}))\psi_{g,f}(\mu)}{D_{+}(\mu^{2}-E_{0}^{2})D_{-}(\mu^{2}-E_{0}^{2})} d\left\langle h, E(\mu)g \right\rangle \\ &= -\left\langle h, (A_{-})^{*}f + A_{-}f \right\rangle. \end{split}$$

Hence the equation $A_{-}(A_{-})^{*} = -A_{-} - (A_{-})^{*}$ holds.

4.3 Operators Ω_{\pm}

In this subsection we consider the bounded operators

$$\Omega_{\pm} := I + R_{\pm}.$$

Let $x_0 < 0$ be the zero of D(z) given in Lemma 4.1 (2) and

$$U_{\rm b} := \sqrt{\frac{\lambda}{D'(x_0)}} R_{E_0^2 + x_0}(T^2) T^{1/2} g, \ P := \langle U_{\rm b}, \cdot \rangle \, U_{\rm b}.$$

Then, by functional calculus, we see that $||U_{\rm b}|| = 1, U_{\rm b} \in D(T^{-1}) \cap D(T^2)$ and

$$TU_{\rm b} = \sqrt{\lambda/D'(x_0)}T^{-1/2}g + (E_0^2 + x_0)T^{-1}U_{\rm b}.$$

Hence P is a projection operator.

Lemma 4.13. Let $\lambda \neq \lambda_c$. Then the following equations hold:

$$\Omega_{\pm}^*\Omega_{\pm} = I, \tag{4.9}$$

$$\Omega_{\pm}\Omega_{\pm}^* = I - \theta(\lambda_c - \lambda)P, \qquad (4.10)$$

where θ is the Heaviside function:

$$\theta(t) = \begin{cases} 1 & if \quad t > 0, \\ 0 & if \quad t < 0. \end{cases}$$

Remark 4.14. Lemma 4.13 implies that Ω_{\pm} are unitary operators if $\lambda > \lambda_{c}$ and partial isometries with their final subspace $\operatorname{Ran}(I - P)$ if $\lambda < \lambda_{c}$.

Proof. (1) We first prove (4.9).

It is sufficient to prove that $R_{\pm}^* R_{\pm} = -(R_{\pm} + R_{\pm}^*)$ hold. For any $f, h \in \mathscr{H}$ and $\varepsilon > 0$,

$$\begin{split} \left\langle R_{\pm}^{(\varepsilon)}h, R_{\pm}^{(\varepsilon)}f \right\rangle &= \lambda^2 \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu')T^{1/2}g \right\rangle d\left\langle T^{1/2}g, E(\mu'')f \right\rangle \\ & \times \left\langle \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)}, \frac{R_{\mu''^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu''^2 - E_0^2)} \right\rangle. \end{split}$$

By the definition of the function D, we have

$$\lambda \langle T^{1/2}g, R_z(T^2)T^{1/2}g \rangle = D(z - E_0^2) - 1, \ z \in \mathbb{C} \setminus (E_0^2, \infty).$$

By this formula and the resolvent identity, we obtain

$$\begin{split} \left\langle R_{\pm}^{(\varepsilon)}h, R_{\pm}^{(\varepsilon)}f\right\rangle &= \lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\left\langle h, E(\mu')T^{1/2}g\right\rangle d\left\langle T^{1/2}g, E(\mu'')f\right\rangle \\ &\times \frac{D(\mu'^2 - E_0^2 \mp i\varepsilon) - D(\mu''^2 - E_0^2 \pm i\varepsilon)}{(\mu'^2 - \mu''^2 \mp 2i\varepsilon)D_{\mp}(\mu'^2 - E_0^2)D_{\pm}(\mu''^2 - E_0^2)} \\ &= -\left\langle E_{\pm}^{(\varepsilon)}h, R_{\pm}^{(2\varepsilon)}f\right\rangle - \left\langle R_{\pm}^{(2\varepsilon)}h, E_{\pm}^{(\varepsilon)}f\right\rangle, \end{split}$$

where the operators $E_{\pm}^{(\varepsilon)}$ on $\mathscr H$ are given as follows:

$$E_{\pm}^{(\varepsilon)} := D(T^2 - E_0^2 \pm i\varepsilon)D_{\pm}(T^2 - E_0^2)^{-1}.$$

The inequality (4.4) implies that $E_{\pm}^{(\varepsilon)}$ are bounded for all $0 < \varepsilon < \varepsilon_0$. Thus, by the Lebesgue dominated convergence theorem, we have s- $\lim_{\varepsilon \downarrow 0} E_{\pm}^{(\varepsilon)} = I$. Hence we obtain $R_{\pm}^* R_{\pm} = -(R_{\pm} + R_{\pm}^*)$.

(2) We next prove (4.10) for $\lambda \neq \lambda_c$.

It is sufficient to prove that $R_{\pm}R_{\pm}^* = -(R_{\pm} + R_{\pm}^*) - \theta(\lambda_c - \lambda)P$ holds. For any $f, h \in \mathscr{H}$ and a fixed $\varepsilon > 0$, (4.5) implies

$$\left\langle R_{\pm}^{(\varepsilon)*}h, R_{\pm}^{(\varepsilon)*}f \right\rangle$$

= $\lambda^2 \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\langle h, E(\mu)T^{1/2}g \rangle d\langle T^{1/2}g, E(\mu')f \rangle$
 $\times \left\langle R_{\mu^2 \pm i\varepsilon}(T^2)D_{\mp}(T^2 - E_0^2)^{-1}T^{1/2}g, R_{\mu'^2 \pm i\varepsilon}(T^2)D_{\mp}(T^2 - E_0^2)^{-1}T^{1/2}g \rangle.$

Then, by operational calculus, we see that

$$\left\langle R_{\pm}^{(\varepsilon)*}h, R_{\pm}^{(\varepsilon)*}f \right\rangle$$

$$= \lambda^{2} \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} d\left\langle h, E(\mu)T^{1/2}g \right\rangle d\left\langle T^{1/2}g, E(\mu')f \right\rangle d\|E(\mu'')g\|^{2}$$

$$\times \frac{\mu''}{(\mu''^{2} - \mu^{2} \pm i\varepsilon)(\mu''^{2} - \mu'^{2} \mp i\varepsilon)D_{\pm}(\mu''^{2} - E_{0}^{2})D_{\mp}(\mu''^{2} - E_{0}^{2})}$$

$$= \lambda \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \frac{1}{\mu^{2} - \mu'^{2} \mp 2i\varepsilon} J_{\varepsilon}^{\pm}(\mu,\mu')d\left\langle h, E(\mu)T^{1/2}g \right\rangle d\left\langle T^{1/2}g, E(\mu')f \right\rangle, \quad (4.11)$$

where, for any $\mu, \mu' \in [E_0, \infty)$,

$$J_{\varepsilon}^{\pm}(\mu,\mu') = \int_{[E_0,\infty)} \frac{\lambda\mu''}{D_{\pm}(\mu''^2 - E_0^2)D_{\mp}(\mu''^2 - E_0^2)} \left(\frac{1}{\mu''^2 - \mu^2 \pm i\varepsilon} - \frac{1}{\mu''^2 - \mu'^2 \mp i\varepsilon}\right) d\|E(\mu'')g\|^2.$$

Then, by a change of variable and (4.3), one can show that

$$J_{\varepsilon}^{\pm}(\mu,\mu') = \lim_{R \to \infty} \frac{1}{2\pi i} I_{\varepsilon,R}^{\pm}(\mu,\mu'),$$

where, for R > 0,

$$I_{\varepsilon,R}^{\pm}(\mu,\mu') := \int_0^R \left(\frac{1}{D_+(s)} - \frac{1}{D_-(s)}\right) G_{\mu,\mu'}^{\varepsilon,\pm}(s) ds$$

and

$$G_{\mu,\mu'}^{\varepsilon,\pm}(z) := \frac{1}{z - \mu'^2 + E_0^2 \mp i\varepsilon} - \frac{1}{z - \mu^2 + E_0^2 \pm i\varepsilon}, \ z \in \mathbb{C}.$$

For $0 < \eta < \varepsilon$ and R > 0, let C_i (i = 1, 2, 3) be the curve given as follows:

$$C_{1}: \ \theta_{1}(t) = R - t - i\eta, \quad t: 0 \to R, \\ C_{2}: \ \theta_{2}(t) = \eta e^{-it}, \qquad t: \pi/2 \to (3\pi)/2, \\ C_{3}: \ \theta_{3}(t) = t + i\eta, \qquad t: 0 \to R. \end{cases}$$

Then, for $C = C_1 + C_2 + C_3$, we have by the Lebesgue dominated convergence theorem,

$$I_{\varepsilon,R}^{\pm}(\mu,\mu') = \lim_{\eta \downarrow 0} \int_C \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz.$$

We take R such that $R > \max\{\mu^2 - E_0^2, \mu'^2 - E_0^2\}$ and define a curve $C_4 : \theta_4(t) = \sqrt{\eta^2 + R^2}e^{-it}, t : t_s \to t_f$, for $t_s := \arctan(\eta/R)$ and $t_f = 2\pi - t_s$. We consider two cases separately.

(i) The case $\lambda > \lambda_c$. In this case, the function $G_{\mu,\mu'}^{\varepsilon,\pm}(z)/D(z), z \in \mathbb{C} \setminus (0,\infty)$ has two simple poles at $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu'^2 - E_0^2 \pm i\varepsilon$. Then, by the residue theorem, we have

$$\int_{C} \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz = 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) - \int_{C_4} \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz.$$

Thus, as η tends to 0, we have

$$\begin{split} I^{\pm}_{\varepsilon,R}(\mu,\mu') &= 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) \\ &- \lim_{\eta \downarrow 0} \int_{C_4} \frac{1}{D(z)} G^{\varepsilon,\pm}_{\mu,\mu'}(z) dz. \end{split}$$

The definition of line integral implies

$$\int_{C_4} \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz = -i \int_{t_s}^{2\pi - t_s} \frac{G_{\mu,\mu'}^{\varepsilon,\pm}(\sqrt{\eta^2 + R^2} e^{-it})\sqrt{\eta^2 + R^2} e^{-it}}{D(\sqrt{\eta^2 + R^2} e^{-it})} dt.$$

By the triangle inequality, for any $t \in [t_s, t_f]$,

$$|G^{\varepsilon,\pm}_{\mu,\mu'}(\sqrt{\eta^2 + R^2}e^{-it})| \le \frac{|\mu^2 - {\mu'}^2 \pm 2i\varepsilon|}{(R - |\mu^2 - E_0^2 \pm i\varepsilon|)(R - |{\mu'}^2 - E_0^2 \mp i\varepsilon|)}$$

On the other hand, by Lemma 4.5, (4.4) and the Lebesgue dominated convergence theorem, there are constants $\tilde{R} > 0$ and $c_0 > 0$ such that $|D(z)| \ge c_0$ for all $|z| \ge \tilde{R}$. Thus we have

$$I_{\varepsilon,R}^{\pm}(\mu,\mu') = 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) + O(R^{-1}) \ (R \to \infty),$$

where $O(\cdot)$ stands for the well known Landau symbol. Therefore we have

$$J_{\varepsilon}^{\pm}(\mu,\mu') = \frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)}$$

for each $\mu, \mu' \in [E_0, \infty)$. Thus, by (4.11), we have

$$\left\langle R_{\pm}^{(\varepsilon)*}h, R_{\pm}^{(\varepsilon)*}f \right\rangle = -\left\langle \left(R_{\pm}^{(2\varepsilon)}\right)^*h, \left(E_{\pm}^{(\varepsilon)}\right)^{-1}f \right\rangle - \left\langle \left(E_{\pm}^{(\varepsilon)}\right)^{-1}h, \left(R_{\pm}^{(2\varepsilon)}\right)^*f \right\rangle.$$

As in the proof in (1), we obtain s- $\lim_{\varepsilon \downarrow 0} \left(E_{\pm}^{(\varepsilon)} \right)^{-1} = I$. Therefore we obtain

$$\lim_{\varepsilon \downarrow 0} \left\langle R_{\pm}^{(\varepsilon)*}h, R_{\pm}^{(\varepsilon)*}f \right\rangle = -\left\langle R_{\pm}^*h, f \right\rangle - \left\langle h, R_{\pm}^*f \right\rangle$$

Thus we obtain the desired result.

(ii) The case $\lambda < \lambda_c$. In this case, $G_{\mu,\mu'}^{\varepsilon,\pm}(z)/D(z)$ has a simple pole at $z = x_0$ in addition to $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = {\mu'}^2 - E_0^2 \pm i\varepsilon$. The residue R_0 of $G_{\mu,\mu'}^{\varepsilon,\pm}(z)/D(z)$ at $z = x_0$ is give by

$$R_0 = \frac{1}{D'(x_0)} \frac{\mu'^2 - \mu^2 \pm 2i\varepsilon}{(x_0 - \mu'^2 + E_0^2 \mp i\varepsilon)(x_0 - \mu^2 + E_0^2 \pm i\varepsilon)}$$

Thus we have

$$J_{\varepsilon}^{\pm}(\mu,\mu') = \frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} + R_0$$

and also

$$\frac{\lambda}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 = -\frac{\lambda}{D'(x_0)} \frac{1}{(\mu'^2 - E_0^2 - x_0 \pm i\varepsilon)(\mu^2 - E_0^2 - x_0 \mp i\varepsilon)}.$$

This implies that

$$\lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 \ d\left\langle h, E(\mu)T^{1/2}g \right\rangle d\left\langle T^{1/2}g, E(\mu')f \right\rangle$$
$$= -\left\langle h, U_{\rm b} \right\rangle \left\langle U_{\rm b}, f \right\rangle = -\left\langle h, Pf \right\rangle.$$

Thus we obtain the desired result.

4.4 Operators U and V

In this subsection, we investigate the operators U and V defined as follows:

$$U := \frac{1}{2} (T^{-1/2} \Omega_{+} T^{1/2} + T^{1/2} \Omega_{+} T^{-1/2}), V := \frac{1}{2} (T^{-1/2} \Omega_{+} T^{1/2} - T^{1/2} \Omega_{+} T^{-1/2}),$$

which are used to construct a Bogoliubov transformation. Then, by Lemma 4.8, we can see that $D(U) = D(V) = D(T^{-1/2}) \cap D(T^{1/2})$.

Lemma 4.15. The operators U and V are bounded.

Proof. By (4.6) and Lemma 4.8 we have

$$\left\langle h, T^{-1/2} R_{\pm} T^{1/2} f \right\rangle = \frac{\lambda \pi}{2} \int_{[E_0,\infty)} - \left(H \phi_{T^{1/2}g,f}^{\pm} \right) (\mu^2) \mp i \frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} \, d \left\langle h, E(\mu)g \right\rangle, \quad (4.12)$$

$$\left\langle h, T^{1/2} R_{\pm} T^{-1/2} f \right\rangle = \frac{\lambda \pi}{2} \int_{[E_0,\infty)} - \left(H \phi_{T^{-1/2}g,f}^{\pm} \right) (\mu^2) \mu \mp i \frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} \, d \left\langle h, E(\mu)g \right\rangle.$$

$$(4.13)$$

By Assumption 3.3 (3), (4.7), (4.8) and a property of Hilbert transform, we can show that

$$||T^{-1/2}R_{\pm}T^{1/2}f||, ||T^{1/2}R_{\pm}T^{-1/2}f|| \le \frac{|\lambda|\pi(C_g + c_g)}{2\delta}||f||$$

where $C_g := (\sup_{E_0 < x} x^{-1} \psi_g(x))^{1/2} (\sup_{E_0 < x} x \psi_g(x))^{1/2}$. Hence the operators $T^{-1/2} R_{\pm} T^{1/2}$ and $T^{1/2} R_{\pm} T^{-1/2}$ are bounded.

In the same way as in the proof of Lemma 4.15, we see that $T^{-1/2}R_{\pm}^*T^{1/2}$ and $T^{1/2}R_{\pm}^*T^{-1/2}$ are bounded on each domain $D(T^{1/2})$ and $D(T^{-1/2})$. In what follows, we write the closed extensions of U and V by the same symbol respectively. Then

$$U^* = \frac{1}{2} \left(\overline{T^{-1/2} \Omega_+^* T^{1/2}} + \overline{T^{1/2} \Omega_+^* T^{-1/2}} \right).$$

Lemma 4.16. The operators U^{\sharp} and V^{\sharp} leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.

Proof. By applying Lemma 4.8 and using the equation

$$U^{\sharp} = I + \frac{1}{2} \left(\overline{T^{-1/2} R_{+}^{\sharp} T^{1/2}} + \overline{T^{1/2} R_{+}^{\sharp} T^{-1/2}} \right),$$

one can easily see that the assertion for U^{\sharp} is true. The proof for V^{\sharp} is similar.

Lemma 4.17. Let $F(x) = x^{\pm 1/2}, x^{\pm 1}, a.e. \ x \in (0, \infty)$. Then

$$\Omega_{+}F(T)\Omega_{+}^{*} = (\Omega_{+})_{J}F(T)(\Omega_{+}^{*})_{J} \quad on \ D(F(T)).$$
(4.14)

Proof. By Lemma 4.8, the domain of each side of (4.14) includes D(F(T)). By Lemmas 4.11 and 4.12, we have

$$\begin{aligned} (\Omega_{+})_{J}F(T)(\Omega_{+}^{*})_{J} &= R_{+}F(T)R_{+}^{*} + R_{+}\{(A_{-})^{*} + I\}F(T)\gamma + F(T)\gamma^{*}(A_{-} + I)R_{+}^{*} \\ &+ F(T)\{A_{-}(A_{-})^{*} + A_{-} + (A_{-})^{*} + I\} \\ &= R_{+}F(T)R_{+}^{*} + R_{+}F(T) + F(T)R_{+}^{*} + F(T) \\ &= \Omega_{+}F(T)\Omega_{+}^{*}. \end{aligned}$$

5 Commutation relations

In this section, we prove that the pair (U, V) satisfies the condition (1.1), V is Hilbert-Schmidt and

$$B(f) := A(Uf) + A(JVf)^*, \ f \in \mathscr{H}$$

satisfies some commutation relations with $H(\lambda)$. We denote the closure of B(f) by the same symbol. By Lemma 4.16, we have $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) \subset D(B(f)) \cap D(B(f)^*)$ for all $f \in D(T^{-1/2})$.

Theorem 5.1. The following commutation relations hold:

(1) For any $f \in D(T)$ and $\psi \in \mathscr{F}_{b,fin}(D(T))$,

$$[H(\lambda), B(f)]\psi = -B(Tf)\psi.$$
(5.1)

(2) For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$\langle H(\lambda)\phi, B(f)\psi\rangle - \langle B(f)^*\phi, H(\lambda)\psi\rangle = -\langle \phi, B(Tf)\psi\rangle.$$
(5.2)

(3) For any $f \in D(T^{-1/2}) \cap D(T)$, B(f) maps $D(d\Gamma_{\rm b}(T)^{3/2})$ into $D(d\Gamma_{\rm b}(T))$ and for any $\psi \in D(d\Gamma_{\rm b}(T)^{3/2})$,

$$[H(\lambda), B(f)]\psi = -B(Tf)\psi.$$
(5.3)

The both sides of (5.1),(5.2) and (5.3) have meaning by Lemma 4.16. To prove this theorem, we prove the following lemma:

Lemma 5.2. For any $f \in D(T)$, the following equations hold:

$$[U,T]f = (VT + TV)f = \frac{\lambda}{2} \left\langle D_{-}(T^{2} - E_{0}^{2})^{-1}g, f \right\rangle g, \qquad (5.4)$$

$$(V^*J - U^*)g = -D_{-}(T^2 - E_0^2)^{-1}g.$$
(5.5)

Proof. For any $f, h \in D(T^{-1/2}) \cap D(T^{3/2})$, we obtain

$$\langle h, [U,T]f \rangle = \frac{1}{2} \left(\langle T^{1/2}R_{+}^{*}T^{-1/2}h, Tf \rangle - \langle Th, T^{1/2}R_{+}T^{-1/2}f \rangle \right)$$

Then, for each $\varepsilon > 0$, we have

$$\left\langle T^{1/2} R_{\pm}^{(\varepsilon)*} T^{-1/2} h, Tf \right\rangle - \left\langle Th, T^{1/2} R_{\pm}^{(\varepsilon)} T^{-1/2} f \right\rangle$$

= $\lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{\mu'^2 - \mu^2}{(\mu'^2 - \mu^2 \pm i\varepsilon) D_{\pm}(\mu'^2 - E_0^2)} d\langle h, E(\mu)g \rangle d\langle g, E(\mu')f \rangle$
= $\lambda \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{D_{\pm}(\mu'^2 - E_0^2)} d\langle h, E(\mu)g \rangle d\langle E(\mu')g, f \rangle \mp i\varepsilon \left\langle T^{-1/2}h, R_{\pm}^{(\varepsilon)} T^{-1/2}f \right\rangle.$

Taking the limit $\varepsilon \downarrow 0$, we have

$$\langle T^{1/2}R_{\pm}^*T^{-1/2}h, Tf \rangle - \langle Th, T^{1/2}R_{\pm}T^{-1/2}f \rangle = \langle h, \lambda \langle D_{\mp}(T^2 - E_0^2)^{-1}g, f \rangle g \rangle.$$

Thus we have

$$\langle h, [U,T]f \rangle = \frac{\lambda}{2} \langle h, \langle D_-(T^2 - E_0^2)^{-1}g, f \rangle g \rangle.$$

Since $D(T^{-1/2}) \cap D(T^{3/2})$ is a core of T, the equation (5.4) holds for $f \in D(T)$. To prove (5.5), we note that

$$\begin{split} (V^*J - U^*)g &= \frac{1}{2} (T^{1/2} \Omega_+^* T^{-1/2} J - T^{-1/2} \Omega_+^* T^{1/2} J - T^{1/2} \Omega_+^* T^{-1/2} - T^{-1/2} \Omega_+^* T^{1/2})g \\ &= -T^{-1/2} \Omega_+^* T^{1/2} g, \end{split}$$

where we have used Jg = g. Thus, for any $f \in \mathcal{H}$, we obtain

$$\begin{split} &\langle f, (V^*J - U^*)g \rangle \\ = &- \langle f, g \rangle - \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \langle f, R_{\mu'^2 + i\varepsilon}(T^2) D_-(T^2 - E_0^2)^{-1}g \rangle \, d \|E(\mu')T^{1/2}g\|^2 \\ = &- \langle f, g \rangle + \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu'}{\mu'^2 - \mu^2 + i\varepsilon} d \|E(\mu')g\|^2 \frac{1}{D_-(\mu^2 - E_0^2)} \, \langle f, E(\mu)g \rangle \\ = &- \langle f, g \rangle + \int_{[E_0, \infty)} \frac{D_-(\mu^2 - E_0^2) - 1}{D_-(\mu^2 - E_0^2)} d \, \langle f, E(\mu)g \rangle \\ = &- \langle f, D_-(T^2 - E_0^2)^{-1}g \rangle \,. \end{split}$$

Hence (5.5) holds.

Proof of Theorem 5.1.

(1) By Lemma 4.16, for any $f \in D(T)$, B(f) leaves $\mathscr{F}_{b,fin}(D(T))$ invariant and $H(\lambda)$ maps $\mathscr{F}_{b,fin}(D(T))$ into $\mathscr{F}_{b,fin}(\mathscr{H}) \subset D(B(f))$. Thus, by using (2.1) and (10.3), we have for any $\psi \in \mathscr{F}_{b,fin}(D(T))$,

$$[H(\lambda), B(f)]\psi = \left\{-A(TUf) + A(TJVf)^* - \frac{\lambda}{\sqrt{2}}\left\langle f, (V^*J - U^*)g\right\rangle \Phi_{\rm s}(g)\right\}\psi.$$

Hence by Lemma 5.2, (5.1) holds.

(2) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we can see that, for any $f \in D(T^{-1/2})$, $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) \subset D(B(f))$. For any $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$, there are sequences $\psi_n, \phi_n \in \mathscr{F}_{\mathrm{b,fin}}(D(T)), n \in \mathbb{N}$ such that $\psi_n \to \psi, \phi_n \to \phi, d\Gamma_{\rm b}(T)\psi_n \to d\Gamma_{\rm b}(T)\psi, d\Gamma_{\rm b}(T)\phi_n \to d\Gamma_{\rm b}(T)\phi$ as $n \to \infty$, since $\mathscr{F}_{\rm b,fin}(D(T))$ is a core of $d\Gamma_{\rm b}(T)$. By (1), we have

$$\langle H(\lambda)\phi_n, B(f)\psi_k \rangle - \langle B(f)^*\phi_n, H(\lambda)\psi_k \rangle = -\langle \phi_n, B(Tf)\psi_k \rangle$$

for all $n, k \in \mathbb{N}$ and $f \in D(T^{-1/2}) \cap D(T)$. By the inequalities (10.1) and (10.2) and the $d\Gamma_{\rm b}(T)$ -boundedness of $\Phi_{\rm s}(g)^2$, we obtain that $\{B(f)\psi_n\}_{n=1}^{\infty}$, $\{B(f)\phi_n\}_{n=1}^{\infty}$, $\{\Phi_{\rm s}(g)^2\psi_n\}_{n=1}^{\infty}$, $\{\Phi_{\rm s}(g)^2\phi_n\}_{n=1}^{\infty}$ and $\{B(Tf)\psi_n\}_{n=1}^{\infty}$ converge. Hence we obtain (5.2).

(3) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we see that, for any $f \in D(T^{-1/2}) \cap D(T)$, B(f) maps $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{3/2})$ into $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$. Therefore, by (5.2) and the density of $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$, we have (5.3).

5.1 Relations between U and V

Lemma 5.3. Let $\lambda \neq \lambda_c$. Then the following equations hold:

$$\begin{cases}
U^{*}U - V^{*}V = I, \\
U_{J}^{*}V - V_{J}^{*}U = 0, \\
UU^{*} - V_{J}V_{J}^{*} = I - \theta(\lambda_{c} - \lambda)Q_{+}, \\
UV^{*} - V_{J}U_{J}^{*} = \theta(\lambda_{c} - \lambda)Q_{-},
\end{cases}$$
(5.6)

where

$$Q_{\pm} := \frac{1}{2} \left(\left\langle T^{1/2} U_{\rm b}, \cdot \right\rangle T^{-1/2} U_{\rm b} \pm \left\langle T^{-1/2} U_{\rm b}, \cdot \right\rangle T^{1/2} U_{\rm b} \right)$$

are bounded operators on \mathcal{H} .

Proof. It is sufficient to prove (5.6) on $D(T^{-1/2}) \cap D(T^{1/2})$. Using (4.9), one can show that the first equation in (4.9) holds. We have

$$U_J^*V - V_J^*U = \frac{1}{2}(-T^{1/2}(\Omega_+^*)_J\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_J\Omega_+T^{1/2}).$$

Multiplying the equation by $(\Omega_+)_J$ from the left, and using Lemma 4.17, we obtain

$$(\Omega_{+})_{J}(U_{J}^{*}V - V_{J}^{*}U) = (\Omega_{+})_{J}(-T^{1/2}(\Omega_{+}^{*})_{J}\Omega_{+}T^{-1/2} + T^{-1/2}(\Omega_{+}^{*})_{J}\Omega_{+}T^{1/2})$$

= $\Omega_{+}(-T^{1/2}\Omega_{+}^{*}\Omega_{+}T^{-1/2} + T^{-1/2}\Omega_{+}^{*}\Omega_{+}T^{1/2}) = 0.$

By (4.9), this implies that $U_J^*V - V_J^*U = 0$. By Lemma 3.5 and Lemma 4.17, we have

$$V_{J}V_{J}^{*} = \frac{1}{4} \{ T^{-1/2} (\Omega_{+}T\Omega_{+}^{*})_{J}T^{-1/2} - T^{-1/2} (\Omega_{+}\Omega_{+}^{*})_{J}T^{1/2} - T^{1/2} (\Omega_{+}\Omega_{+}^{*})_{J}T^{-1/2} + T^{1/2} (\Omega_{+}T^{-1}\Omega_{+}^{*})_{J}T^{1/2} \} = \frac{1}{4} (T^{-1/2}\Omega_{+}T\Omega_{+}^{*}T^{-1/2} - T^{-1/2}\Omega_{+}\Omega_{+}^{*}T^{1/2} - T^{1/2}\Omega_{+}\Omega_{+}^{*}T^{-1/2} + T^{1/2}\Omega_{+}T^{-1}\Omega_{+}^{*}T^{1/2}) = VV^{*}.$$

Hence, by direct calculations and (4.10), one obtains $UU^* - V_J V_J^* = I - \theta(\lambda_c - \lambda)Q_+$. Similarly one can prove the last equation in (5.6) (note that $P_J = P$).

5.2 Hilbert-Schmidtness of V

In this subsection, we show that V is Hilbert-Schmidt. Then we can use Theorem 2.2 in the case of $\lambda > \lambda_c$.

Lemma 5.4. The operator V is Hilbert-Schmidt.

Proof. On $D(T^{-1/2}) \cap D(T^{1/2}), V^*V$ is calculated as follows:

$$V^*V = \frac{1}{4} (T^{-1/2}R_+T^{1/2} + T^{1/2}R_+^*T^{-1/2} + T^{1/2}[R_+^*, T^{-1}]R_+T^{1/2} + T^{1/2}R_+T^{-1/2} + T^{-1/2}R_+^*T^{1/2} + T^{-1/2}[R_+^*, T]R_+T^{-1/2} + T^{1/2}R_+^*R_+T^{-1/2} + T^{-1/2}R_+^*R_+T^{1/2}) = \frac{1}{4} (T^{1/2}[R_+^*, T^{-1}]R_+T^{1/2} + T^{-1/2}[R_+^*, T]R_+T^{-1/2}),$$

where we have used the formula $R_+^*R_+ = -(R_+ + R_+^*)$ in the proof of Lemma 4.13 and Lemma 4.8. Thus, for any $f \in D(T^{-1/2}) \cap D(T^{1/2})$ and $\varepsilon > 0$, we have

$$\begin{split} &\left\langle f, (T^{1/2}[R_{+}^{(\varepsilon)*}, T^{-1}]R_{+}^{(\varepsilon)}T^{1/2} + T^{-1/2}[R_{+}^{(\varepsilon)*}, T]R_{+}^{(\varepsilon)}T^{-1/2})f\right\rangle \\ = &\int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \frac{\lambda \mu'}{(\mu'^{2} - \mu^{2} + i\varepsilon)D_{+}(\mu'^{2} - E_{0}^{2})} d\left\langle [T^{-1}, R_{+}^{(\varepsilon)}]T^{1/2}f, E(\mu)T^{1/2}g\right\rangle d\left\langle E(\mu')g, f\right\rangle \\ &+ \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \frac{\lambda \mu}{(\mu'^{2} - \mu^{2} + i\varepsilon)D_{+}(\mu'^{2} - E_{0}^{2})} d\left\langle [T, R_{+}^{(\varepsilon)}]T^{-1/2}f, E(\mu)T^{-1/2}g\right\rangle d\left\langle E(\mu')g, f\right\rangle . \end{split}$$

Then, for any $B \in \mathbf{B}^1$, we can see

$$\left\langle [T^{-1}, R_{+}^{(\varepsilon)}] T^{1/2} f, E(B) T^{1/2} g \right\rangle$$

= $\lambda \int_{B} \int_{[E_{0},\infty)} \frac{\mu'' - \mu}{(\mu''^{2} - \mu^{2} - i\varepsilon) D_{-}(\mu''^{2} - E_{0}^{2})} d\langle f, E(\mu'')g \rangle d\|E(\mu)g\|^{2}.$ (5.7)

Similarly, we obtain

$$\left\langle [T, R_{+}^{(\varepsilon)}] T^{-1/2} f, E(B) T^{-1/2} g \right\rangle$$

= $\lambda \int_{B} \int_{[E_{0},\infty)} \frac{\mu - \mu''}{(\mu''^{2} - \mu^{2} - i\varepsilon) D_{-}(\mu''^{2} - E_{0}^{2})} d\langle f, E(\mu'')g \rangle d\|E(\mu)g\|^{2}.$

Thus, by the formula of a change of variable in Lebesgue-Stieltjes integration and Fubini's theorem, we have

$$\begin{split} \left\langle f, (T^{1/2}[R_{+}^{(\varepsilon)*}, T^{-1}]R_{+}^{(\varepsilon)}T^{1/2} + T^{-1/2}[R_{+}^{(\varepsilon)*}, T]R_{+}^{(\varepsilon)}T^{-1/2})f \right\rangle \\ &= \lambda^{2} \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} \int_{[E_{0},\infty)} d\|E(\mu)g\|^{2} d\left\langle f, E(\mu'')g\right\rangle d\left\langle E(\mu')g, f\right\rangle \\ &\times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^{2} - \mu^{2} + i\varepsilon)(\mu''^{2} - \mu^{2} - i\varepsilon)D_{+}(\mu'^{2} - E_{0}^{2})D_{-}(\mu''^{2} - E_{0}^{2})}. \end{split}$$

Then it is easy to see that for any $\mu, \mu', \mu'' \in [E_0, \infty)$,

$$\lim_{\varepsilon \downarrow 0} \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)} = \frac{1}{(\mu' + \mu)(\mu'' + \mu)D_+(\mu'^2 - E_0^2)D_-(\mu''^2 - E_0^2)}.$$

For any $\varepsilon > 0$ and $\mu, \mu', \mu'' \in [E_0, \infty)$, we have, by Lemma 4.5 and the arithmetic-geometric mean inequality,

$$\left|\frac{(\mu-\mu')(\mu-\mu'')}{(\mu'^2-\mu^2+i\varepsilon)(\mu''^2-\mu^2-i\varepsilon)D_+(\mu'^2-E_0^2)D_-(\mu''^2-E_0^2)}\right| \le \frac{1}{4\delta^2\mu\sqrt{\mu'\mu''}}.$$

On the other side, for any $\alpha, \beta \in \mathbb{C}$, we see

$$\int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu \sqrt{\mu' \mu''}} d\|E(\mu)g\|^2 d\|E(\mu'')(f+\alpha g)\|^2 d\|E(\mu')(f+\beta g)\|^2$$
$$= \|T^{-1/2}g\|^2 \|T^{-1/4}(f+\alpha g)\|^2 \|T^{-1/4}(f+\beta g)\|^2 < \infty.$$

Thus, by the Lebesgue dominated convergence theorem, we have

$$\begin{split} \lim_{\varepsilon \downarrow 0} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\|E(\mu)g\|^2 d\|E(\mu'')(f+\alpha g)\|^2 d\|E(\mu')(f+\beta g)\|^2 \\ & \times \frac{(\mu-\mu')(\mu-\mu'')}{(\mu'^2-\mu^2+i\varepsilon)(\mu''^2-\mu^2-i\varepsilon)D_+(\mu'^2-E_0^2)D_-(\mu''^2-E_0^2)} \\ = \int_{[E_0,\infty)} \int_{[E_0,\infty)} \int_{[E_0,\infty)} d\|E(\mu)g\|^2 d\|E(\mu'')(f+\alpha g)\|^2 d\|E(\mu')(f+\beta g)\|^2 \\ & \times \frac{1}{(\mu'+\mu)(\mu''+\mu)D_+(\mu'^2-E_0^2)D_-(\mu''^2-E_0^2)}. \end{split}$$

In particular, for each $\alpha, \beta = \pm 1, \pm i$, the polarization identity and Fubini's theorem give

$$\langle f, V^* V f \rangle = \frac{\lambda^2}{4} \int_{[E_0,\infty)} \left| \langle f, R_{-\mu}(T) D_{-}(T^2 - E_0^2)^{-1} g \rangle \right|^2 d \| E(\mu) g \|^2.$$

Let $\{e_n\}_{n=1}^{\infty} \subset D(T^{-1/2}) \cap D(T^{1/2})$ be a CONS of \mathscr{H} . The termwise integration implies that

$$\sum_{n=1}^{\infty} \langle e_n, V^* V e_n \rangle = \frac{\lambda^2}{4} \int_{[E_0,\infty)} \|R_{-\mu}(T) D_- (T^2 - E_0^2)^{-1} g\|^2 d\|E(\mu)g\|^2$$

$$= \frac{\lambda^2}{4} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{(\mu' + \mu)^2 |D_- (\mu'^2 - E_0^2)|^2} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 \quad (5.8)$$

$$\leq \frac{\lambda^2}{16\delta^2} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{1}{\mu'\mu} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 < \infty,$$

where we have used the arithmetic-geometric mean inequality and Lemma 4.5. Hence V is Hilbert-Schmidt. $\hfill \Box$

Lemma 5.5. If $\lambda > \lambda_c$, then there is a unitary operator \mathbb{U} on $\mathscr{F}_{b}(\mathscr{H})$ such that for all $f \in \mathscr{H}$,

$$\mathbb{U}B(f)\mathbb{U}^{-1} = A(f).$$

Proof. By Lemma 5.3 and Lemma 5.4, we can apply Theorem 2.2.

6 Analysis in the case $\lambda > \lambda_c$

In this section we prove Theorem 3.6 (1). Before starting the proof, we need to know a property of the Hamiltonian $H(\lambda)$.

6.1 Time evolution

Theorem 6.1 (Time evolution). If $\lambda > \lambda_{c,0}$, then for all $f \in D(T^{-1/2})$, $\psi \in D(d\Gamma_{\rm b}(T)^{1/2})$ and $t \in \mathbb{R}$,

$$e^{itH(\lambda)}B(f)e^{-itH(\lambda)}\psi = B(e^{itT}f)\psi,$$
(6.1)

$$e^{itH(\lambda)}B(f)^*e^{-itH(\lambda)}\psi = B(e^{itT}f)^*\psi.$$
(6.2)

Proof. It is sufficient to prove (6.1), because (6.2) follows from taking the adjoint of (6.1). We define a function $v : \mathbb{R} \to \mathbb{C}$ by $v(t) := \langle \phi, e^{itH(\lambda)}B(e^{-itT}f)e^{-itH(\lambda)}\psi \rangle, t \in \mathbb{R}$ for any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$. Then v is well-defined by an operational calculus and Theorem 2.3. The function v is differentiable and, by Theorem 5.1 (2), we have for any $t \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}v(t) &= i \left\langle H(\lambda)e^{-itH(\lambda)}\phi, B(e^{-itT}f)e^{-itH(\lambda)}\psi \right\rangle - i \left\langle B(e^{-itT}f)^*e^{-itH(\lambda)}\phi, H(\lambda)e^{-itH(\lambda)}\psi \right\rangle \\ &+ i \left\langle e^{-itH(\lambda)}\phi, B(Te^{-itT}f)e^{-itH(\lambda)}\psi \right\rangle \\ &= 0. \end{aligned}$$

Hence v(t) = v(0) for all $t \in \mathbb{R}$. Hence the equation

$$\left\langle \phi, e^{itH(\lambda)}B(e^{-itT}f)e^{-itH(\lambda)}\psi \right\rangle = \left\langle \phi, B(f)\psi \right\rangle$$

holds for all $t \in \mathbb{R}$. By replacing f with $e^{itT}f$, one has for all $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$e^{itH(\lambda)}B(f)e^{-itH(\lambda)}\psi = B(e^{itT}f)\psi$$

Since $D(d\Gamma_{\rm b}(T))$ is a core of $(H(\lambda)+M)^{1/2}$ and $D(H(\lambda)+M)^{1/2} = D(d\Gamma_{\rm b}(T)^{1/2})$ by Theorem 2.3 (3), we obtain (6.1) for $f \in D(T^{-1/2}) \cap D(T)$ and $\psi \in D(d\Gamma_{\rm b}(T)^{1/2})$. Finally we extend (6.1) for all $f \in D(T^{-1/2})$. Let $f \in D(T^{-1/2})$ and $\psi \in D(d\Gamma_{\rm b}(T)^{1/2})$. Then we set $f_n := E((-\infty, n])f$ for each $n \in \mathbb{N}$. Then $f_n \in D(T^{-1/2}) \cap D(T)$ for all $n \in \mathbb{N}$ and one can easily show that $f_n \to f$, $T^{-1/2}f_n \to T^{-1/2}f$ as $n \to \infty$ by using a functional calculus and the Lebesgue dominated convergence theorem. Thus we have $Uf_n \to Uf$, $JVf_n \to JVf$ as $n \to \infty$ by the boundedness of U and V. By using the linearity of the Hilbert transform and that of the map $f \mapsto \psi_{g,f}$, (4.12), (4.13) and (4.7), we can show that $T^{-1/2}Uf_n \to T^{-1/2}Uf_n \to T^{-1/2}JVf_n \to T^{-1/2}JVf$ as $n \to \infty$. Therefore we obtain $B(f_n)\phi \to B(f)\phi$ and $B(e^{itT}f_n)\phi \to B(e^{itT}f)\phi$ as $n \to \infty$ for any $\phi \in D(d\Gamma_{\rm b}(T)^{1/2})$ by [3, Lemma 4-28]. By the preceding result, we have for any $n \in \mathbb{N}$,

$$B(f_n)e^{-itH(\lambda)}\psi = e^{-itH(\lambda)}B(e^{itT}f_n)\psi.$$

The equation $D(d\Gamma_b(T)^{1/2}) = D((H(\lambda) + M)^{1/2})$ in Theorem 2.3 (3) implies that

$$e^{-itH(\lambda)}D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) = D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}).$$

Hence, by taking the limit $n \to \infty$, we obtain (6.1) for $f \in D(T^{-1/2}), \psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. \Box

6.2 Proof of Theorem 3.6 (1)

In this subsection, we assume that $\lambda > \lambda_c$.

Lemma 6.2. Let $\Omega := \mathbb{U}^{-1}\Omega_0$, where \mathbb{U} is the unitary operator in Lemma 5.5. Then there is an eigenvalue E_g of $H(\lambda)$ and Ω is the corresponding eigenvector: $H(\lambda)\Omega = E_g\Omega$.

Proof. In general, by [3, Proposition 4-4] for a dense subspace $\mathscr{D} \subset \mathscr{H}$, if $\psi \in \bigcap_{f \in \mathscr{D}} D(A(f))$ satisfies $A(f)\psi = 0$ for all $f \in \mathscr{D}$, then there is a constant $\alpha \in \mathbb{C}$ such that $\psi = \alpha \Omega_0$. Thus, by Lemma 5.5, if $B(f)\phi = 0$ for all $f \in D(T^{-1/2})$, there is a constant $\alpha \in \mathbb{C}$ such that $\phi = \alpha \Omega$. For any $f \in D(T^{-1/2})$ and $t \in \mathbb{R}$,

$$B(f)e^{-itH(\lambda)}\Omega = e^{-itH(\lambda)}B(e^{itT}f)\Omega = 0$$

by Lemma 5.5 and Theorem 6.1. Thus, for each $t \in \mathbb{R}$, there is a constant $\alpha(t) \in \mathbb{C}$ such that $e^{-itH(\lambda)}\Omega = \alpha(t)\Omega$. Then we have $|\alpha(t)| = 1, \alpha(t+s) = \alpha(t)\alpha(s)$ for all $t, s \in \mathbb{R}$, since $\{e^{-itH(\lambda)}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group. Thus there exists a constant $E_{g} \in \mathbb{R}$ such that $\alpha(t) = e^{-itE_{g}}, t \in \mathbb{R}$. The differentiation of the equation $e^{-itH(\lambda)}\Omega = e^{-itE_{g}}\Omega$ in t implies that $\Omega \in D(H(\lambda))$ and $\Omega \in \text{Ker}(H(\lambda) - E_{g})$.

Proof of Theorem 3.6 (1).

The subspace $\mathcal{U} := \mathscr{L}(\{B(f_1)^* \cdots B(f_n)^*\Omega, \Omega \mid f_j \in D(T^{-1/2}), j = 1, \ldots, n, n \in \mathbb{N}\})$ is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ by the fact that $\mathcal{U} = \mathbb{U}^{-1}\mathscr{F}_{\mathrm{b,fin}}(D(T^{-1/2}))$, where $\mathscr{L}(\mathscr{D})$ denotes the subspace algebraically spanned by the vectors in a subset \mathscr{D} of a Hilbert space. By Lemma 6.1 and Lemma 10.3, for any $t \in \mathbb{R}$ and $f_j \in D(T^{-1/2}), j = 1, \ldots, n$, we have

$$e^{itH(\lambda)}B(f_1)^*\cdots B(f_n)^*\Omega = B(e^{itT}f_1)^*\cdots B(e^{itT}f_n)^*e^{itH(\lambda)}\Omega$$
$$= B(e^{itT}f_1)^*\cdots B(e^{itT}f_n)^*e^{itE_g}\Omega$$
$$= e^{itE_g}\mathbb{U}^{-1}e^{itd\Gamma_b(T)}A(f_1)^*\cdots A(f_n)^*\Omega_0$$
$$= \mathbb{U}^{-1}e^{it(d\Gamma_b(T)+E_g)}\mathbb{U}B(f_1)^*\cdots B(f_n)^*\Omega.$$

By this equation and a limiting argument, we obtain $\mathbb{U}e^{itH(\lambda)}\mathbb{U}^{-1} = e^{it(\mathrm{d}\Gamma_{\mathrm{b}}(T)+E_{\mathrm{g}})}$. By the unitary covariance of functional calculus, we have

$$\mathbb{U}e^{itH(\lambda)}\mathbb{U}^{-1} = e^{it\mathbb{U}H(\lambda)\mathbb{U}^{-1}}, \quad t \in \mathbb{R}.$$

Hence (3.1) holds. The equation (3.1) and the well-known spectral properties of $d\Gamma_{\rm b}(T)$ imply that $E_{\rm g}$ is the ground state energy of $H(\lambda)$ and Ω is the unique ground state of $H(\lambda)$.

Lemma 6.3. The ground state energy E_g is given as follows:

$$E_{\rm g} = \frac{\lambda}{4} \|g\|^2 - \operatorname{Tr}(T^{1/2}V^*VT^{1/2}), \tag{6.3}$$

$$\operatorname{Tr}(T^{1/2}V^*VT^{1/2}) = \frac{\lambda^2}{4} \int_{[E_0,\infty)} \int_{[E_0,\infty)} \frac{\mu}{(\mu+\mu')^2 |D_-(\mu^2 - E_0^2)|^2} d\|E(\mu)g\|^2 d\|E(\mu')g\|^2.$$
(6.4)

Proof. The operator \mathbb{U} leaves $D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$ invariant by Theorem 3.6 (1). In particular, $\mathbb{U}\Omega_{0} \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. Thus, by Lemma 10.4, the isometricity of \mathbb{U} and the definition of $B(\cdot)$, we have $\langle \Omega_{0}, (H(\lambda) - E_{\mathrm{g}})\Omega_{0} \rangle = \mathrm{Tr}(T^{1/2}V^{*}VT^{1/2})$. By the definition of $H(\lambda)$ and (2.1), we have $\langle \Omega_{0}, H(\lambda)\Omega_{0} \rangle = \lambda ||g||^{2}/4$. Hence (6.3) holds. The formula (6.4) can be proved in the same way as (5.8).

7 Analysis in the case $\lambda_{\mathrm{c},0} < \lambda < \lambda_{\mathrm{c}}$

In Section 5, we proved Theorem 3.6 (1). But the proof is valid only for the case $\lambda > \lambda_c$. Therefore it is necessary to find another pair of operators U and V if one wants to use a Bogoliubov transformation for the spectral analysis of $H(\lambda)$ in the case $\lambda \leq \lambda_c$. In this section we assume that T and g satisfy Assumption 3.3, $E_0 > 0$ and $\lambda_{c,0} < \lambda < \lambda_c$. Under these conditions, we can define the operators ξ, X, Y and T_{\pm} as follows:

$$\begin{split} \xi &:= \Omega_+ T \Omega_+^* + \beta P, \\ X &:= U \Omega_+^* + T_+ P, \ Y &:= V \Omega_+^* + T_- P, \\ T_\pm &:= \frac{1}{2} (\beta^{1/2} T^{-1/2} \pm \beta^{-1/2} T^{1/2}), \end{split}$$

where $\beta := (E_0^2 + x_0)^{1/2}$.

Remark 7.1. The definition of x_0 implies that

$$E_0^2 + x_0 \begin{cases} > 0, & \text{if} \quad \lambda_{c,0} < \lambda < \lambda_c, \\ = 0, & \text{if} \quad \lambda = \lambda_{c,0}, \\ < 0, & \text{if} \quad \lambda < \lambda_{c,0}. \end{cases}$$

Thus, in the case $\lambda_{c,0} < \lambda < \lambda_c$, we see that the inequality $0 < \beta < E_0$ holds. Let

$$C(f) := A(Xf) + A(JYf)^*, f \in \mathscr{H}.$$

Then C(f) is a densely defined closable operator. We denotes its closure by the same symbol.

7.1 Properties of X, Y and ξ

In this subsection, we study the operators X, Y and ξ . Firstly, we consider ξ . Let

$$\tilde{T} := \Omega_+ T \Omega_+^*.$$

Lemma 7.2. The operator \tilde{T} is a self-adjoint operator with $D(\tilde{T}) = D(T)$.

Proof. By Lemma 4.8 we see that $D(\tilde{T}) = D(T)$. Hence \tilde{T} is symmetric. For any $\phi \in D((\tilde{T})^*)$ and $\psi \in D(T) = D(\tilde{T})$, we have $\left\langle \Omega^*_+(\tilde{T})^*\phi, \psi \right\rangle = \left\langle \Omega^*_+\phi, T\psi \right\rangle$. This implies that $\Omega^*_+\phi \in D(T)$. Hence \tilde{T} is self-adjoint.

Lemma 7.3. The spectra of \tilde{T} are as follows:

$$\sigma(\tilde{T}) = \{0\} \cup \sigma(T), \sigma_{\rm ac}(\tilde{T}) = \sigma(T), \sigma_{\rm p}(\tilde{T}) = \{0\}, \sigma_{\rm sc}(\tilde{T}) = \emptyset$$

Proof. We define a family of projection operators $\{E_P(B) \mid B \in \mathbf{B}^1\}$ on \mathscr{H} as follows: $E_P(B) = 0$ if $0 \notin B$ and $E_P(B) = P$ if $0 \in B$ for each $B \in \mathbf{B}^1$. By the definition of the spectral measure, we can see that $\{E_{\tilde{T}}(B) := \Omega_+ E(B)\Omega_+^* + E_P(B) \mid B \in \mathbf{B}^1\}$ is a spectral measure. Using a functional calculus, we see that $E_{\tilde{T}}(\cdot)$ is the spectral measure of \tilde{T} . The absolutely continuous part (resp. singular part) of \tilde{T} is $\tilde{T} \upharpoonright \operatorname{Ran}(I - P)$ (resp. $\tilde{T} \upharpoonright \operatorname{Ran}(P)$) since T is absolutely continuous and Ω_{\pm} are partial isometries. Thus we see $\sigma(\tilde{T}) = \{0\} \cup \sigma_{\operatorname{ac}}(\tilde{T}), \sigma_{\operatorname{p}}(\tilde{T}) = \{0\}, \sigma_{\operatorname{sc}}(\tilde{T}) = \emptyset$.

We next show that $\sigma_{\rm ac}(\tilde{T}) = \sigma(T)$. For any $\mu \in \sigma(T)$, there is a sequence $\psi_n \in D(T), n \in \mathbb{N}$ such that $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|(T-\mu)\psi_n\| = 0$. For each $n \in \mathbb{N}$, there is a $\phi_n \in \operatorname{Ran}(I-P)$ such that $\psi_n = \Omega_+^*\phi_n$. Then $\|\phi_n\| = \|\Omega_+\psi_n\| = \|\psi_n\| = 1$ and $\|(\tilde{T}-\mu)\phi_n\| = \|(T-\mu)\psi_n\| \to 0$ as $n \to \infty$. Thus we have $\mu \in \sigma(\tilde{T} \upharpoonright \operatorname{Ran}(I-P)) = \sigma_{\rm ac}(\tilde{T})$. For any $\mu \in \sigma_{\rm ac}(\tilde{T})$, there is a sequence $\eta_n \in D(\tilde{T}) \cap \operatorname{Ran}(I-P)$ such that $\|\eta_n\| = 1$ and $\lim_{n\to\infty} \|(\tilde{T}-\mu)\eta_n\| = 0$. Then we easily see that $\Omega_+^*\eta_n \in D(T)$ for all $n \in \mathbb{N}$. The equation $\Omega_+\Omega_+^*\eta_n = \eta_n$ implies that $\|\Omega_+^*\eta_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|(T-\mu)\Omega_+^*\eta_n\| = \|(\tilde{T}-\mu)\eta_n\| \to 0, \quad n \to \infty.$$

Thus $\mu \in \sigma(T)$. Hence $\sigma_{\rm ac}(\tilde{T}) = \sigma(T)$.

Lemma 7.4. The operator ξ is an injective, non-negative self-adjoint operator with $D(\xi) = D(T)$ and we have the following equations:

$$\sigma(\xi) = \{\beta\} \cup \sigma(T), \sigma_{\rm ac}(\xi) = \sigma(T), \sigma_{\rm p}(\xi) = \{\beta\}, \sigma_{\rm sc}(\xi) = \emptyset.$$
(7.1)

In particular, β is the ground state energy of ξ , which is an isolated eigenvalue of ξ , and $U_{\rm b}$ is the unique ground state of ξ .

Proof. By Lemma 7.3 and the spectral property of direct sum of self-adjoint operators, we have the equation (7.1). Thus β is an isolated ground state energy by Remark 7.1. By $\Omega^*_+U_b = 0$, U_b is a ground state of ξ . Assume that $f \in \text{Ker}(\xi - \beta)$ satisfies $(I - P)f \neq 0$. Then $\Omega^*_+f \neq 0$ by Lemma 4.13. This implies that $T\Omega^*_+f = \beta\Omega^*_+f$, but this contradicts Assumption 3.3 (1). Hence (I - P)f = 0 and this implies that the ground state of ξ is unique.

Lemma 7.5. The operators $\xi^{\pm 1/2}$ are given by

$$\xi^{1/2} = \Omega_+ T^{1/2} \Omega_+^* + \beta^{1/2} P, \tag{7.2}$$

$$\xi^{-1/2} = \Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P \tag{7.3}$$

with $D(\xi^{\pm 1/2}) = D(T^{\pm 1/2}).$

Proof. We can show in the same way as in the proof of Lemma 7.4 that the right hand side of (7.2) is non-negative, self-adjoint operator with its domain $D(T^{1/2})$. We have $\xi \subset (\Omega_+ T^{1/2} \Omega_+^* + \beta^{1/2} P)^2$. Since a self-adjoint operator has no non-trivial symmetric extension, (7.2) holds. In the same way as in the proof of (7.2), we can show that the right hand side of (7.3) is a self-adjoint operator. We have $D(\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P) \subset \operatorname{Ran}(\xi^{1/2})$ and $\xi^{1/2}(\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P) = I$ on $D(\Omega_+ T^{-1/2} \Omega_+^*)$. Hence $\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P \subset \xi^{-1/2}$. Thus the equation (7.3) holds.

Next, we study X and Y.

Lemma 7.6. The operators X^{\sharp} and Y^{\sharp} leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.

Proof. The assertion follows from Lemma 4.8, Lemma 4.16, Lemma 7.5 and the definitions of X and Y. \Box

Lemma 7.7. The following equations hold:

$$\begin{cases}
X^*X - Y^*Y = I, \\
X_J^*Y - Y_J^*X = 0, \\
XX^* - Y_JY_J^* = I, \\
XY^* - Y_JX_J^* = 0.
\end{cases}$$
(7.4)

Proof. The operator P (resp. T_{\pm}) satisfies $P_J = P$ (resp. $(T_{\pm})_J = T_{\pm}$). By (4.10), we have $\Omega^*_{\pm}U_b = 0$. Hence we obtain $(U^* \pm V^*)T^{\pm 1/2}U_b = 0$ and $(U^*T_{\pm} - V^*T_{\mp})U_b = 0$. The equations $T_+T_+ - T_-T_- = I$ and $T_+T_- - T_-T_+ = 0$ hold on $D(T^{-1}) \cap D(T)$. By (5.6) and direct calculations, we have $X^*X - Y^*Y = I$ and $X_J^*Y - Y_J^*X = 0$. By similar calculations, we have $XX^* - Y_JY_J^* = I$ and $XY^* - Y_JX_J^* = 0$ on $D(T^{-1/2}) \cap D(T^{1/2})$. Then, by a limiting argument, we obtain (7.4).

Lemma 7.8. The operator Y is Hilbert-Schmidt.

Proof. We can easily show that the assertion follows from Lemma 5.4, Lemma 7.6 and the choice a CONS $\{e_n\}_{n=0}^{\infty} \subset D(T^{-1/2}) \cap D(T^{1/2})$ with $e_0 = U_{\rm b}$.

Lemma 7.9. There is a unitary operator \mathbb{V} on $\mathscr{F}_{b}(\mathscr{H})$ such that for all $f \in \mathscr{H}$,

$$\mathbb{V}C(f)\mathbb{V}^{-1} = A(f).$$

Proof. By Theorem 2.2, (7.4) and Lemma 7.8, we can prove this assertion.

7.2 Commutation relations

Theorem 7.10. The following commutation relations hold:

(1) For any $f \in D(T)$ and $\psi \in \mathscr{F}_{b,fin}(D(T))$,

$$[H(\lambda), C(f)]\psi = -C(\xi f)\psi.$$

(2) For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$,

$$\langle H(\lambda)\phi, C(f)\psi\rangle - \langle C(f)^*\phi, H(\lambda)\psi\rangle = -\langle \phi, C(\xi f)\psi\rangle$$

(3) For any $f \in D(T^{-1/2}) \cap D(T)$, C(f) maps $D(d\Gamma_{\rm b}(T)^{3/2})$ into $D(d\Gamma_{\rm b}(T))$ and for any $\psi \in D(d\Gamma_{\rm b}(T)^{3/2})$,

$$[H(\lambda), C(f)]\psi = -C(\xi f)\psi.$$

Theorem 7.10 follows, in the same manner as in the proof of Theorem 5.1, from Lemma 4.16, Lemma 7.5 and the next lemma:

Lemma 7.11. For any $f \in D(T)$ the following equations hold:

$$-TXf + \frac{\lambda}{2} \left\langle (Y^*J - X^*)g, f \right\rangle g = -X\xi f, \tag{7.5}$$

$$TJYf + \frac{\lambda}{2} \langle f, (Y^*J - X^*)g \rangle g = -JY\xi f.$$
(7.6)

Remark 7.12. By Lemma 4.16 and the definition of ξ , the both sides of (7.5) and (7.6) have meaning.

Proof. Let $a := \sqrt{\lambda/D'(x_0)}$. Then we can see by the definition of x_0 and (5.5),

$$(Y^*J - X^*)g = -\Omega_+ D_- (T^2 - E_0^2)^{-1}g + \frac{\beta^{-1/2}a}{\lambda} U_{\rm b}.$$

We have

$$TT_{\pm}U_{\rm b} = \frac{1}{2} (\beta^{1/2} T^{1/2} U_{\rm b} \pm \beta^{-1/2} T^{3/2} U_{\rm b})$$

$$= \frac{1}{2} (\beta^{1/2} T^{1/2} U_{\rm b} \pm \beta^{3/2} T^{-1/2} U_{\rm b} \pm \beta^{-1/2} ag).$$
(7.7)

Thus, for any $f \in D(T)$, we have

$$- TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g$$

= $-TU\Omega_+^*f - \frac{\lambda}{2} \langle D_-(T^2 - E_0^2)^{-1}g, \Omega_+^*f \rangle g - TT_+Pf + \frac{\beta^{-1/2}a}{2} \langle U_{\rm b}, f \rangle g.$

Then, by (5.4) and (7.7), we have

$$-TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g = -UT\Omega^*_+ f - \beta \langle U_{\mathbf{b}}, f \rangle T_+ U_{\mathbf{b}}$$
$$= -X(\Omega_+ T\Omega^*_+ + \beta P)f.$$

Thus we obtain (7.5). Similarly one can prove (7.6).

7.3 Proof of Theorem 3.6 (2)

Theorem 7.13. For all
$$f \in D(T^{-1/2}), \psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$$
 and $t \in \mathbb{R}$,
 $e^{itH(\lambda)}C(f)e^{-itH(\lambda)}\psi = C(e^{it\xi}f)\psi,$
 $e^{itH(\lambda)}C(f)^*e^{-itH(\lambda)}\psi = C(e^{it\xi}f)^*\psi.$

Proof. These are proved in the same way as in the proof of Theorem 6.1 by Theorem 7.10. \Box Lemma 7.14. Let $\Omega := \mathbb{V}^{-1}\Omega_0$ where \mathbb{V} is the unitary operator in Lemma 7.9. Then:

- (1) There is an eigenvalue \tilde{E}_{g} of $H(\lambda)$ and Ω is an eigenvector of $H(\lambda)$ with the eigenvalue \tilde{E}_{g} .
- (2) The following equation holds:

$$\mathbb{V}H(\lambda)\mathbb{V}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(\xi) + \tilde{E}_{\mathrm{g}}.$$

(3) The constant \tilde{E}_{g} is given as follows:

$$\tilde{E}_{\rm g} = E_{\rm g} - \beta \|T_{-}U_{\rm b}\|^2.$$
(7.8)

Proof. The assertions (1) and (2) can be proved in the same way as in the proof of Theorem 3.6 (1).

(3) We have

$$\tilde{E}_{\rm g} = \frac{\lambda}{4} \|g\|^2 - \text{Tr}(\xi^{1/2} Y^* Y \xi^{1/2})$$

in the same way as in the proof of Lemma 6.2. Then, by Lemma 7.5, we have

$$\xi^{1/2}Y^*Y\xi^{1/2} = \Omega_+T^{1/2}V^*VT^{1/2}\Omega_+^* + \Omega_+T^{1/2}V^*\beta^{1/2}T_-P + \beta^{1/2}PT_-VT^{1/2}\Omega_+^* + \beta PT_-T_-P.$$

We choose a CONS $\{e_n\}_{n=0}^{\infty} \subset D(T)$ satisfying $e_0 = U_b$. Then it is easy to see that $\{\Omega_+^* e_n\}_{n=1}^{\infty}$ is a CONS of \mathscr{H} by Lemma 4.13. Hence we have

$$\operatorname{Tr}(\xi^{1/2}Y^*Y\xi^{1/2}) = \sum_{n=1}^{\infty} \left\langle e_n, \Omega_+ T^{1/2}V^*VT^{1/2}\Omega_+^*e_n \right\rangle + \beta \|T_-U_b\|^2$$
$$= \operatorname{Tr}(T^{1/2}V^*VT^{1/2}) + \beta \|T_-U_b\|^2.$$

Thus we obtain (7.8).

In particular, $H(\lambda)$ have eigenvectors as follows:

$$\phi_n := \mathbb{V}^{-1} A(U_{\mathbf{b}})^{*n} \Omega_0, \ H(\lambda) \phi_n = (n\beta + \tilde{E}_{\mathbf{g}}) \phi_n \ , n \in \mathbb{N} \cup \{0\}.$$

Hence the spectral properties of $H(\lambda)$ as stated in Theorem 3.6 (2) follow.

8 Analysis in the case $\lambda < \lambda_{\mathrm{c},0}$

In this section, we show that $H(\lambda)$ is unbounded from above and below.

Theorem 8.1. Let $g \in D(T^{-1/2})$. Then $H(\lambda)$ is unbounded above for any $\lambda \in \mathbb{R}$. If $\lambda < \lambda_{c,0}$, then $H(\lambda)$ is unbounded below.

Proof. For any $f \in D(T) \setminus \{0\}$, we set $\psi_n := a_n A(f)^{*n} \Omega_0$, $a_n \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N} \cup \{0\}$. Then we have the following equations:

$$\begin{split} \mathrm{d}\Gamma_{\mathrm{b}}(T)\psi_{n} &= n\frac{a_{n}}{a_{n-1}}A(Tf)^{*}\psi_{n-1}, \quad A(g)\psi_{n} = n\left\langle g, f\right\rangle \frac{a_{n}}{a_{n-1}}\psi_{n-1}, \\ \|\psi_{n}\|^{2} &= |a_{n}|^{2}n!\|f\|^{2n}, \qquad \|A(g)^{*}\psi_{n}\|^{2} = \|g\|^{2}\|\psi_{n}\|^{2} + \|A(g)\psi_{n}\|^{2}, \end{split}$$

where $\psi_{-1} := 0$. Then we have

$$\langle \psi_n, H(\lambda)\psi_n \rangle = \|\psi_n\|^2 \left(\frac{\lambda}{4}\|g\|^2 + n\frac{2\|T^{1/2}f\|^2 + \lambda|\langle g, f \rangle|^2}{2\|f\|^2}\right).$$

We take f such that $\langle g, f \rangle = 0$. Then we have $\langle \psi_n, H(\lambda)\psi_n \rangle / ||\psi_n||^2 \to \infty$ as $n \to \infty$ for any $\lambda \in \mathbb{R}$. Thus $H(\lambda)$ is unbounded above for any $\lambda \in \mathbb{R}$.

Let $\phi_N := \sum_{n=0}^{N} \psi_n, N = 0, 1, 2, \dots$ Then we have $\|\phi_N\|^2 = \sum_{n=0}^{N} \|\psi_n\|^2$ and

$$\begin{aligned} \langle \phi_N, H(\lambda)\phi_N \rangle &= \sum_{n=2}^N \|\psi_n\|^2 \left(\frac{\lambda \|g\|^2}{4} + n \frac{2\|T^{1/2}f\|^2 + \lambda |\langle g, f \rangle|^2}{2\|f\|^2} + \frac{\lambda}{2} \operatorname{Re} \frac{a_{n-2}^*}{a_n^*} \frac{\langle g, f \rangle^2}{\|f\|^4} \right) \\ &+ \|\psi_1\|^2 \left(\frac{\lambda \|g\|^2}{4} + \frac{\|T^{1/2}f\|^2}{\|f\|^2} + \frac{\lambda |\langle g, f \rangle|^2}{2\|f\|^2} \right) + \frac{\lambda \|\psi_0\|^2 \|g\|^2}{4}. \end{aligned}$$

Let $a_0 := 1, \ a_n := n^{-3/4} n!^{-1/2}, n \in \mathbb{N}$ and, for any $0 < \delta, \ 0 < \varepsilon < 1$,

$$f = f_{\delta} := \frac{T^{-1}E((\delta,\infty))g}{\|T^{-1}E((\delta,\infty))g\|},$$
$$c_{\lambda}(\varepsilon,\delta) := \|T^{1/2}f_{\delta}\|^{2} \left\{ 1 + \frac{\lambda}{2}(2-\varepsilon)\|T^{-1/2}E((\delta,\infty))g\|^{2} \right\}$$

Then $\sum_{n=0}^{\infty} \|\psi_n\|^2$ converges and, for any $N \in \mathbb{N}$,

$$\langle \phi_N, H(\lambda)\phi_N \rangle = \sum_{n=2}^N \|\psi_n\|^2 nc_\lambda(\varepsilon, \delta) + \frac{\lambda}{2} \sum_{n=2}^N \|\psi_n\|^2 \left(\frac{a_{n-2}}{a_n} - n(1-\varepsilon)\right) \langle g, f_\delta \rangle^2 + C_N, \quad (8.1)$$

where

$$C_N := \frac{\lambda \|g\|^2}{4} \sum_{n=0}^N \|\psi_n\|^2 + \|\psi_1\|^2 \left(\|T^{1/2} f_\delta\|^2 + \frac{\lambda}{2} \langle g, f_\delta \rangle^2 \right).$$

For all $0 < \delta, 0 < \varepsilon < 1$, we have

$$\frac{2}{\|T^{-1/2}E((\delta,\infty))g\|^2(2-\varepsilon)} < \lambda_{\rm c,0}.$$
(8.2)

The left hand side of (8.2) tends to $\lambda_{c,0}$ as $\varepsilon, \delta \downarrow 0$. Since $\lambda < \lambda_{c,0}$, we can take a pair (ε, δ) satisfying $c_{\lambda}(\varepsilon, \delta) < 0$. We fix such a pair. There is a $n_0 \in \mathbb{N}$ such that $a_{n-2}/a_n - n(1-\varepsilon) > 0$ for all $n \ge n_0$. Hence we can see that $\langle \phi_N, H(\lambda)\phi_N \rangle / \|\phi_N\|^2$ tends to $-\infty$ as $N \to \infty$, because the first term of the right of (8.1) tends to $-\infty$ as $N \to \infty$.

9 Generalization of the ϕ^2 -model

In this section we consider $H(\eta, \lambda)$ defined in Subsection 2.3.

Assumption 9.1. We need the following assumptions:

- (1) $f \in D(T^{1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$,
- (2) $f \in D(T^{-1})$ and $\operatorname{Re} \langle T^{-1}f, g \rangle = 0$,
- (3) $f, g \in D(T^{-1})$ and $\operatorname{Re} \langle T^{-1}f, g \rangle \neq 0$.

We can prove a slight generalization of Theorem 3.6.

Theorem 9.2. Let \mathscr{H} be separable. Then the following (1)-(5) hold:

(1) Suppose that Assumption 3.3 and, Assumption 9.1 (2) or (3) hold. Let $\lambda > \lambda_{c}$. Then there is a unitary operator \mathbb{U} on $\mathscr{F}_{b}(\mathscr{H})$ such that for all $\eta \in \mathbb{R}$,

$$\mathbb{U}H(\eta,\lambda)\mathbb{U}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(T) + E_{\mathrm{g}} + E_{f,g},$$

where the constant $E_{f,g} \in \mathbb{R}$ is defined by

$$E_{f,g} = -\frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{(\operatorname{Re}\langle T^{-1}f,g\rangle)^2 \eta^2 \lambda}{2(1+\lambda \|T^{-1/2}g\|^2)}.$$

(2) Suppose that Assumption 3.3 and, Assumption 9.1 (2) or (3) hold. Let E₀ > 0 and λ_{c,0} < λ < λ_c. Then there are a unitary operator V on 𝓕_b(ℋ) and a non-negative, injective self-adjoint operator ξ on ℋ such that, for all η ∈ ℝ,

$$\mathbb{V}H(\eta,\lambda)\mathbb{V}^{-1} = \mathrm{d}\Gamma_{\mathrm{b}}(\xi) + E_{\mathrm{g}} - E_{\mathrm{b}} + E_{f,g}.$$

(3) Let T be a non-negative, injective self-adjoint operator and suppose that f and g satisfy Assumption 9.1 (1) and (2). Then there is a unitary operator W on 𝔅_b(𝔅) such that, for all η ∈ ℝ,

$$\mathbb{W}\overline{H(\eta,\lambda_{\mathrm{c},0})}\mathbb{W}^{-1} = \overline{H(\lambda_{\mathrm{c},0})} - \frac{\eta^2}{2} \|T^{-1/2}f\|^2.$$

(4) Let T be a non-negative, injective self-adjoint operator and suppose that f and g satisfy Assumption 9.1 (1) and (3). Then, for all $\eta \in \mathbb{R} \setminus \{0\}$,

$$\sigma(\overline{H(\eta,\lambda_{\mathrm{c},0})})=\mathbb{R},\quad \sigma_\mathrm{p}(\overline{H(\eta,\lambda_{\mathrm{c},0})})=\emptyset.$$

(5) Let T be a non-negative, injective self-adjoint operator and suppose that f and g satisfy Assumption 9.1 (1). Moreover, suppose that Assumption 9.1 (2) or (3) holds. Let $\lambda < \lambda_{c,0}$. Then, for all $\eta \in \mathbb{R}$, $\overline{H(\eta, \lambda)}$ is unbounded from above and below.

Theorem 9.2 is immediately proved by the following lemma and Theorem 3.6.

Lemma 9.3. Let T be a non-negative, injective self-adjoint operator, $f \in D(T^{-1})$ and $g \in D(T^{-1/2}) \cap D(T)$.

(1) Let $\operatorname{Re} \langle T^{-1}f, g \rangle = 0$. Then there is a unitary operator \mathbb{U}_1 on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that for all $\eta, \lambda \in \mathbb{R}$,

$$\mathbb{U}_1 \overline{H(\eta,\lambda)} \mathbb{U}_1^{-1} = \overline{H(\lambda)} - \frac{\eta^2}{2} \|T^{-1/2}f\|^2.$$
(9.1)

(2) Let $\operatorname{Re} \langle T^{-1}f, g \rangle \neq 0$ and $g \in D(T^{-1})$.

(i) If $\lambda \neq \lambda_{c,0}$, then there is a unitary operator \mathbb{U}_2 on $\mathscr{F}_{b}(\mathscr{H})$ such that for all $\eta \in \mathbb{R}$,

$$\mathbb{U}_2\overline{H(\eta,\lambda)}\mathbb{U}_2^{-1} = \overline{H(\lambda)} + E_{f,g}.$$

(ii) If $\lambda = \lambda_{c,0}$, then for all $\eta \in \mathbb{R} \setminus \{0\}$,

$$\sigma(\overline{H(\eta, \lambda_{c,0})}) = \mathbb{R}, \quad \sigma_{p}(\overline{H(\eta, \lambda_{c,0})}) = \emptyset.$$
(9.2)

Proof. Let $\mathbb{U}_1 := e^{-i\Phi_s(i\eta T^{-1}f)}$ for any $\eta \in \mathbb{R}$. Then, by direct calculations, we obtain

$$\mathbb{U}_{1}H(\eta,\lambda)\mathbb{U}_{1}^{-1} = H(\lambda) - \frac{\eta^{2}}{2} \|T^{-1/2}f\|^{2} - \lambda\eta\kappa\Phi_{s}(g) + \frac{\lambda}{2}\eta^{2}\kappa^{2}$$
(9.3)

on $\mathscr{F}_{\mathrm{b,fin}}(D(T))$ for all $\eta, \lambda \in \mathbb{R}$, where $\kappa := \mathrm{Re}\langle T^{-1}f, g \rangle$. In the case of (1), we have (9.1) by $\kappa = 0$ and a limiting argument. Next, we prove (2). We assume that $g \in D(T^{-1})$ and $\mathrm{Re}\langle T^{-1}f, g \rangle \neq 0$. Let $\mathbb{V}_1 := e^{i\Phi_s(i\alpha T^{-1}g)}$ for any $\alpha \in \mathbb{R}$ and define a unitary operator $\mathbb{U}_2 := \mathbb{V}_1\mathbb{U}_1$. Then it follows that

$$\mathbb{U}_{2}H(\eta,\lambda)\mathbb{U}_{2}^{-1} = H(\lambda) + \left(\alpha + \lambda\alpha \|T^{-1/2}g\|^{2} - \lambda\eta\kappa\right)\Phi_{s}(g) \\ - \frac{\eta^{2}}{2}\|T^{-1/2}f\|^{2} + \frac{\lambda}{2}\eta^{2}\kappa^{2} + \frac{\alpha}{2}\|T^{-1/2}g\|^{2}\left(\alpha + \lambda\alpha\|T^{-1/2}g\|^{2} - 2\lambda\eta\kappa\right)$$

on $\mathscr{F}_{b,\text{fin}}(D(T))$ in the same way as (9.3). For $\lambda \neq \lambda_{c,0}$, let $\alpha = \lambda \eta \kappa (1 + \lambda ||T^{-1/2}g||^2)^{-1}$. Then we obtain

$$\mathbb{U}_{2}\overline{H(\eta,\lambda)}\mathbb{U}_{2}^{-1} = \overline{H(\lambda)} - \frac{\eta^{2}}{2}\|T^{-1/2}f\|^{2} + \frac{\lambda\eta^{2}\kappa^{2}}{2(1+\lambda\|T^{-1/2}g\|^{2})}$$
(9.4)

by a limiting argument. If $\lambda = \lambda_{c,0}$, then, for all $\eta, \alpha \in \mathbb{R}$, we have

$$\mathbb{U}_2 \overline{H(\eta, \lambda_{\mathrm{c},0})} \mathbb{U}_2^{-1} = \overline{H_g(-\kappa\eta\lambda_{\mathrm{c},0}, \lambda_{\mathrm{c},0})} - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{\lambda_{\mathrm{c},0}\eta^2\kappa^2}{2} + \kappa\eta\alpha$$

in the same way as (9.4), where $H_g(\nu, \lambda_{c,0}) := H(\lambda_{c,0}) + \nu \Phi_s(g)$ for all $\nu \in \mathbb{R}$. We can see that $\sigma(\overline{H_g(\nu, \lambda_{c,0})}) = \mathbb{R}$ and $\sigma_p(\overline{H_g(\nu, \lambda_{c,0})}) = \emptyset$ for all $\nu \in \mathbb{R} \setminus \{0\}$, because $\mathbb{V}_1 \overline{H_g(\nu, \lambda_{c,0})} \mathbb{V}_1^{-1} = \overline{H_g(\nu, \lambda_{c,0})} + \nu \alpha \|T^{-1/2}g\|^2$ and $\alpha \in \mathbb{R}$ is arbitrary. Hence we have (9.2).

Remark 9.4. If \mathscr{H} is separable, then the condition $g \in D(T^{-1/2}) \cap D(T)$ in the above lemma is weakened to the condition $g \in D(T^{-1/2}) \cap D(T^{1/2})$.

10 Appendix

In this section, we recall some known facts in the Fock space theory. Let T be a non-negative, injective self-adjoint operator on \mathscr{H} .

Lemma 10.1. [5, Theorem 5.16.]

Let $f \in D(T^{-1/2})$ and $\psi \in D(d\Gamma_{\rm b}(T)^{1/2})$. Then $\psi \in D(A(f)) \cap D(A(f)^*)$ and the following inequalities hold:

$$\|A(f)\psi\| \le \|T^{-1/2}f\| \|d\Gamma_{\rm b}(T)^{1/2}\psi\|,\tag{10.1}$$

$$||A(f)^*\psi||^2 \le ||T^{-1/2}f||^2 ||d\Gamma_{\rm b}(T)^{1/2}\psi||^2 + ||f||^2 ||\psi||^2.$$
(10.2)

Lemma 10.2. [5, Proposition 5.10.] For any $f \in D(T)$, the following commutation relations hold on $\mathscr{F}_{b,fin}(D(T))$:

$$[d\Gamma_{\rm b}(T), A(f)] = -A(Tf), \quad [d\Gamma_{\rm b}(T), A(f)^*] = A(Tf)^*.$$
(10.3)

Lemma 10.3. [5, Lemma 5.21.] For any $t \in \mathbb{R}$ and $f \in \mathcal{H}$, the following equations hold:

$$e^{itd\Gamma_{\rm b}(T)}A(f)^{\sharp}e^{-itd\Gamma_{\rm b}(T)} = A(e^{itT}f)^{\sharp}.$$

Lemma 10.4. [5, Theorem 5.21.] Assume that \mathscr{H} be separable. Let $\{e_n\}_{n=1}^{\infty} \subset D(T^{1/2})$ be a CONS of \mathscr{H} . Then, for any $\psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$, $\sum_{n=1}^{\infty} ||A(T^{1/2}e_n)\psi||^2$ converges and the following equation holds:

$$\sum_{n=1}^{\infty} \|A(T^{1/2}e_n)\psi\|^2 = \|\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\psi\|^2.$$

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