Spectral analysis of an abstract pair interaction model

浅原 啓輔

北海道大学 博士 理学 甲第13552号

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Spectral analysis of an abstract pair interaction model
(抽象的な対相互作用モデルのスペクトル解析)

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PRESENTED BY
KEISUKE ASAHARA

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GRADUATE SCHOOL OF SCIENCE
HOKKAIDO UNIVERSITY

ADvised BY
ASAO ARAI

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Abstract

We consider an abstract pair-interaction model in quantum field theory with a coupling constant \( \lambda \in \mathbb{R} \) and analyze the Hamiltonian \( H(\lambda) \) of the model. In the massive case, there exist constants \( \lambda_c < 0 \) and \( \lambda_{c,0} < \lambda_c \) such that, for each \( \lambda \in (\lambda_{c,0}, \lambda_c) \cup (\lambda_c, \infty) \), \( H(\lambda) \) is diagonalized by a proper Bogoliubov transformation, so that the spectrum of \( H(\lambda) \) is explicitly identified, where the spectrum of \( H(\lambda) \) for \( \lambda > \lambda_c \) is different from that for \( \lambda \in (\lambda_{c,0}, \lambda_c) \). As for the case \( \lambda < \lambda_{c,0} \), we show that \( H(\lambda) \) is unbounded from above and below. In the massless case, \( \lambda_c \) coincides with \( \lambda_{c,0} \).

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1 Introduction

This thesis is based on the joint work [6]. We consider an abstract pair-interaction model in quantum field theory. The Hamiltonian of the model is of the form

\[ H(\lambda) := d\Gamma_b(T) + \frac{\lambda}{2}\Phi_s(g)^2 \]

acting in the boson Fock space \( \mathcal{F}_b(\mathcal{H}) \) over a Hilbert space \( \mathcal{H} \) (see Subsection 2.1), where \( T \) is a self-adjoint operator on \( \mathcal{H} \), \( d\Gamma_b(T) \) is the second quantization operator of \( T \), \( \Phi_s(g) \) is the Segal field operator with test vector \( g \) in \( \mathcal{H} \) (see Subsection 2.1) and \( \lambda \in \mathbb{R} \) is a coupling constant. A model of this type is called a \( \phi^2 \)-model.

There have been many studies on massive or massless \( \phi^2 \)-models in concrete forms or abstract forms (see, e.g., [4, 8, 9, 11, 12, 16]). In [11] and [16], the (essential) self-adjointness of the Hamiltonian of a \( \phi^2 \)-model is proved in the case where \( \lambda > 0 \) or \( |\lambda| \) is sufficiently small. In [11], the existence of a ground state of a \( \phi^2 \)-model also is shown in the case where the quantum field under consideration is massive and \( \lambda > 0 \).

It is a well known that Hamiltonians with linear and/or quadratic interactions in quantum fields may be analyzed by the method of Bogoliubov transformations (see, e.g., [1, 2, 3, 4, 7, 8, 10, 12]). A typical Bogoliubov transformation is constructed from bounded linear operators \( U, V \) and a conjugation operator \( J \) on \( \mathcal{H} \) satisfying the following equations:

\[
\begin{align*}
U^*U - V^*V &= I, \\
U^*V - V^*U &= 0, \\
UU^* - V^*_jV_j &= I, \\
UV^* - V^*_jU_j &= 0,
\end{align*}
\]

(1.1)

where \( A_J := JAJ \) and \( A^\ast \) is the adjoint of a densely defined linear operator \( A \). It is well known that there is a unitary operator \( U \) on \( \mathcal{F}_b(\mathcal{H}) \) which implements the Bogoliubov
transformation in question if and only if $V$ is Hilbert-Schmidt [7, 13, 14, 15]. Moreover, it is shown that, under the condition that $V$ is Hilbert-Schmidt and suitable additional conditions, the Hamiltonian under consideration is unitarily equivalent via $U$ to a second quantization operator up to a constant addition. For example, the Pauli-Fierz model with dipole approximation, which can be regarded as a kind of $\phi^2$-model, is analyzed by this method in [10].

Recently, a general quadratic form Hamiltonian with a coupling constant $\lambda \in \mathbb{R}$ has been analyzed in [12] and it is shown that, in the case of a massive quantum field, under suitable conditions, the Hamiltonian is diagonalized by a Bogoliubov transformation. In [8], the sufficient condition formulated in [12] to obtain the result just mentioned has been extended. The spectrum of the standard pair-interaction model in physics, which is a concrete realization of the abstract pair-interaction model, is formally known [9] in the case where $\lambda > \lambda_{c,0}$ and $\lambda \neq \lambda_c$ for the constants $\lambda_c$ and $\lambda_{c,0}$ which satisfy $\lambda_{c,0} < \lambda_c$. The paper [4] gives a rigorous proof for that in the framework of the boson Fock space theory over $\mathcal{H} = L^2(\mathbb{R}^d)$ for any $d \in \mathbb{N}$ and $\lambda > \lambda_c$.

One of the motivations for the present work is to extend the theory developed in [4] with $\mathcal{H} = L^2(\mathbb{R}^d)$ to the theory with $\mathcal{H}$ being an abstract Hilbert space including the case where $\lambda < \lambda_c$. It is a well known fact (see [9]) that the spectral properties of the standard pair-interaction model may depend on whether $\lambda > \lambda_c$ or $\lambda < \lambda_c$. Hence it is important to clarify this aspect mathematically. Therefore we analyze our model also for the region $\lambda < \lambda_c$. We show that, in the massive case with $\lambda \in (\lambda_{c,0}, \lambda_c)$ also, the method of Bogoliubov transformations can be applied to prove that the Hamiltonian $H(\lambda)$ is unitarily equivalent to a second quantization operator up to a constant addition. Then we see that the spectrum of $H(\lambda)$ for $\lambda \in (\lambda_{c,0}, \lambda_c)$ is different from that for $\lambda > \lambda_c$. In the massless case, $\lambda_{c,0}$ coincides with $\lambda_0$.

The main results of the present paper include the following (1)–(3) (see Theorem 2.8 for more details): (1) Identification of the spectra of $H(\lambda)$ for $\lambda > \lambda_c$. (2) Identification of the spectra of $H(\lambda)$ for $\lambda_{c,0} < \lambda < \lambda_c$ it is only in the massive case; in the massless case, $\lambda_{c,0} = \lambda_c$. In this case, bound states different from the ground state appear. (3) Unboundedness of $H(\lambda)$ from above and below for $\lambda < \lambda_{c,0}$.

The outline of this paper is as follows. In Section 2, we define our model and recall a fundamental fact in a general theory of Bogoliubov transformations. We prove the (essential) self-adjointness of $H(\lambda)$ (Theorem 2.3). Then we state the main theorem of this paper (Theorem 3.6). In Section 3, we construct the operators $U$ and $V$ which are used to define the Bogoliubov transformation we need. In Section 4, we show that $U$ and $V$ satisfy (1.1) and $V$ is Hilbert-Schmidt. In Section 5, we prove Theorem 2.8 (1) and calculate the ground
state energy of $H(\lambda)$ in the case $\lambda > \lambda_c$. In Section 6, we prove Theorem 2.8 (2). In Section 7, we prove Theorem 2.8 (3). In Section 8, we consider a slightly generalized Hamiltonian which is of the form $H(\eta, \lambda) := H(\lambda) + \eta \Phi_S(f)$ for $\eta \in \mathbb{R}$ and $f \in \mathcal{H}$. Applying the methods and results in the preceding sections, we analyze $H(\eta, \lambda)$ and identify the spectra of it. In Appendix, we state some basic facts in the theory of boson Fock space.

2 Preliminaries

2.1 The abstract boson Fock Space

Let $\mathcal{H}$ be a Hilbert space over the complex field $\mathbb{C}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The inner product is linear in the second variable and anti-linear in the first one. The symbol $\| \cdot \|_{\mathcal{H}}$ denotes the norm associated with it. We omit $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively if there is no danger of confusion. For each non-negative integer $n = 0, 1, 2, \ldots$, $\hat{\otimes}^n \mathcal{H}$ denotes the $n$-fold symmetric tensor product Hilbert space of $\mathcal{H}$ with convention $\hat{\otimes}^0 \mathcal{H} := \mathbb{C}$. Then $\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \otimes^n \mathcal{H}$ is called the boson Fock space over $\mathcal{H}$. For a dense subspace $\mathcal{D}$ in $\mathcal{H}$, $\hat{\otimes}^n \mathcal{D}$ denotes the algebraic $n$-fold symmetric tensor product of $\mathcal{D}$ with $\hat{\otimes}^0 \mathcal{H} := \mathbb{C}$. Then $\mathcal{F}_b, \text{fin}(\mathcal{D}) := \bigoplus_{n=0}^{\infty} \hat{\otimes}^n \mathcal{D}$ is a dense subspace of $\mathcal{F}_b(\mathcal{H})$, where $\bigoplus_{n=0}^{\infty} \mathcal{D}_n$ stands for the algebraic direct sum of subspace $\mathcal{D}_n \subset \otimes^n \mathcal{H}$, $n = 0, 1, 2, \ldots$. The finite particle vector subspace $\mathcal{F}_b, 0(\mathcal{H}) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H}) \mid \psi^{(n)} \in \otimes^n \mathcal{H}, \ n \geq 0, \text{there is an integer } n_0 \in \mathbb{N} \text{ such that } \psi^{(n)} = 0, \text{ for all } n \geq n_0 \right\}$ satisfies $\mathcal{F}_b, \text{fin}(\mathcal{D}) \subset \mathcal{F}_b, 0(\mathcal{H}) \subset \mathcal{F}_b(\mathcal{H})$, in particular, it is dense in $\mathcal{F}_b(\mathcal{H})$. For a linear operator $T$ on a Hilbert space, the domain of $T$ will be denoted by $D(T)$.

For a densely defined closable operator $T$ on $\mathcal{H}$, let $T^{(n)}_b$ be the densely defined closed operator on $\otimes^n \mathcal{H}$ defined by

$$T^{(n)}_b := \left\{ \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes T \otimes I \otimes \cdots \otimes I | \hat{\otimes}^n D(T), \ n \geq 1, \right\} \bigcup \left\{ 0, \ n = 0, \right\}$$
where $I$ denotes the identity operator on $\mathcal{H}$, $\overline{A}$ denotes the closure of a closable operator $A$ and $A \upharpoonright \mathcal{M}$ denotes the restriction of a linear operator $A$ on a subspace $\mathcal{M}$. The operator

$$d\Gamma_b(T) := \oplus_{n=0}^{\infty} T_b^{(n)}$$

is called the second quantization operator of $T$. If $T$ is self-adjoint or non-negative, then so is $d\Gamma_b(T)$. For each $f \in \mathcal{H}$, there exists a unique densely defined closed operator $A(f)$ on $\mathcal{F}_b(\mathcal{H})$ such that its adjoint $A(f)^*$ is given as follows:

$$D(A(f)^*) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H}) \left| \sum_{n=1}^{\infty} n \| S_n (f \otimes \psi^{(n-1)}) \|^2 < \infty \right. \right\},$$

$$(A(f)^*)^{(n)} = \sqrt{n} S_n (f \otimes \psi^{(n-1)}), \quad n \in \mathbb{N}, \quad (A(f)^*)^{(0)} = 0 \text{ for } \psi \in D(A(f)^*),$$

where $S_n$ is the symmetrization operator on the $n$-fold tensor product $\otimes^n \mathcal{H}$ of $\mathcal{H}$. The operator $A(f)$ (resp. $A(f)^*$) is called the annihilation (resp. creation) operator with test vector $f$. We have

$$\mathcal{F}_{b,0}(\mathcal{H}) \subset D(A(f)) \cap D(A(f)^*)$$

for all $f \in \mathcal{H}$ and $A(f)$ and $A(f)^*$ leave $\mathcal{F}_{b,0}(\mathcal{H})$ invariant. Moreover, they satisfy the following commutation relations:

$$[A(f), A(g)^*] = \langle f, g \rangle, \quad [A(f), A(g)] = 0, \quad [A(f)^*, A(g)^*] = 0, \quad \text{for all } f, g \in \mathcal{H} \quad (2.1)$$

on $\mathcal{F}_{b,0}(\mathcal{H})$, where $[A, B] := AB - BA$ is the commutator of linear operators $A$ and $B$. The relation (2.1) is called the canonical commutation relations (CCR) over $\mathcal{H}$. The symmetric operator

$$\Phi_s(f) := \frac{1}{\sqrt{2}} (A(f) + A(f)^*), \quad f \in \mathcal{H}$$

is called the Segal field operator with test vector $f$. We write its closure by the same symbol.

### 2.2 Bogoliubov Transformation

In this subsection, we define a Bogoliubov transformation and recall an important theorem about it. For a conjugation $J$ on $\mathcal{H}$ (i.e., $J$ is an anti-linear operator on $\mathcal{H}$ satisfying $\|Jf\| = \|f\|$ for all $f \in \mathcal{H}$ and $J^2 = I$) and a linear operator $A$ on $\mathcal{H}$, we define

$$A_J := JAJ.$$

**Definition 2.1.** Let $U$ and $V$ be bounded linear operators on $\mathcal{H}$ and $J$ be a conjugation on $\mathcal{H}$. For each $f \in \mathcal{H}$, let a linear operator $B(f)$ on $\mathcal{F}_b(\mathcal{H})$ be given by

$$B(f) := A(Uf) + A(JVf)^*.$$

Then the correspondence $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is called a Bogoliubov transformation.
By \( \mathcal{F}_{b,0}(\mathcal{H}) \subset D(B(f)) \), the adjoint \( B(f)^* \) exists and the equation \( B(f)^* = A(Uf)^* + A(JVf) \) holds on \( \mathcal{F}_{b,0}(\mathcal{H}) \) for each \( f \in \mathcal{H} \). If the equations

\[
U^*U - V^*V = I, \quad U^*_f V - V^*_f U = 0
\]

hold, then the Bogoliubov transformation preserves CCR, i.e., it holds that

\[
[B(f), B(g)] = \langle f, g \rangle, \quad [B(f), B(g)] = 0, \quad [B(f)^*, B(g)^*] = 0, \quad \text{for all } f, g \in \mathcal{H},
\]
on \( \mathcal{F}_{b,0}(\mathcal{H}) \). The following theorem is well known (see [14, 15]):

**Theorem 2.2.** Let \( \mathcal{H} \) be separable and the operators \( U \) and \( V \) satisfy (1.1). Then there exists a unitary operator \( U \) on \( \mathcal{F}_{b}(\mathcal{H}) \) such that

\[
UB(f)U^{-1} = A(f), \quad f \in \mathcal{H}
\]
if and only if \( V \) is Hilbert-Schmidt.

### 2.3 Hamiltonians

For a self-adjoint operator \( T \) on \( \mathcal{H} \), constants \( \lambda, \eta \in \mathbb{R} \) which are called coupling constants, and vectors \( f, g \in \mathcal{H} \), we define Hamiltonians \( H(\lambda) \) and \( H(\eta, \lambda) \) by

\[
H(\lambda) := d\Gamma_b(T) + \frac{\lambda}{2} \Phi_s(g)^2, \quad H(\eta, \lambda) := H(\lambda) + \eta \Phi_s(f).
\]

If \( g = 0 \), then \( H(\lambda) \) and \( H(\eta, \lambda) \) are well-known operators. Thus, we always assume that \( g \neq 0 \) in the present paper. If \( g \in D(T^{-1/2}) \), let the constant be defined by

\[
\lambda_{c,0} := -\|T^{-1/2}g\|^{-2}.
\]

**Theorem 2.3.** Suppose that \( T \) is an injective, non-negative, self-adjoint operator on \( \mathcal{H} \). Let \( f \in D(T^{-1/2}) \) and \( g \in D(T^{-1/2}) \cap D(T) \). Then the following (1)-(3) hold:

1. Let

\[
\lambda_T(g) := \|T^{-1/2}g\|^{-1}(\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1}
\]

and \( |\lambda| < \lambda_T(g) \). Then \( H(\eta, \lambda) \) is self-adjoint with \( D(H(\eta, \lambda)) = D(d\Gamma_b(T)) \) and essentially self-adjoint on any core of \( d\Gamma_b(T) \) for all \( \eta \in \mathbb{R} \). Moreover, \( H(\eta, \lambda) \) is bounded from below.

2. Let \( |\lambda| \geq \lambda_T(g) \) and \( f \in D(T^{1/2}) \). Then \( H(\eta, \lambda) \) is essentially self-adjoint on any core of \( d\Gamma_b(T) \) for all \( \eta \in \mathbb{R} \). Moreover, if \( \lambda \geq \lambda_T(g) \), then \( H(\eta, \lambda) \) is self-adjoint.
(3) Let \( f \in D(T^{1/2}) \). Then \( H(\lambda_{c,0}) \) is bounded from below. Moreover, if \( \lambda > \lambda_{c,0} \), then \( H(\eta, \lambda) \) is also bounded from below for all \( \eta \in \mathbb{R} \) and \( D(d\Gamma_b(T)) = D((H(\eta, \lambda) + M)^{1/2}) \) for all constant \( M \geq 0 \) satisfying \( H(\eta, \lambda) + M \geq 0 \).

Proof. (1) For any \( \lambda \in \mathbb{R} \), by using (2.1), (10.1), (10.2) and [5, Theorem 5.18.], there are constants \( a, b \geq 0 \) such that for all \( \psi \in D(d\Gamma_b(T)) \),

\[
\left\| \frac{\lambda}{2} \Phi_s(g)^2 \psi \right\| \leq \frac{|\lambda|}{4} (a\|d\Gamma_b(T)\psi\| + b\|\psi\|).
\]

In particular, we can choose \( a \) and \( b \) which satisfy \( a|\lambda|/4 < 1 \) if \( |\lambda| < \lambda_T(g) \). We remark that, to obtain the factor \( \lambda_T(g) \), we need to deform terms \( \|A(g)^2\psi\|, \|A(g)A^*(g)\psi\| \) and \( \|A(g)^2\psi\|^2 \) coming from \( \|\Phi_s(g)^2\psi\|^2 (\psi \in \mathcal{F}_{b,0}(\mathcal{H}) \) to \( \|A(g)A^*(g)\psi\|^2 \) + a marginal term respectively. Thus, for \( |\lambda| < \lambda_T(g) \), by the Kato-Rellich theorem, \( H(\lambda) \) is self-adjoint. It is well known that \( \Phi_s(f) \) is infinitesimally small with respect to \( d\Gamma_b(T) \).

Hence, by the Kato-Rellich theorem, for \( |\lambda| < \lambda_T(g) \), \( H(\lambda, \lambda) \) is self-adjoint.

(2) Firstly, we show that, for any \( f \in D(T^{1/2}) \) and \( \eta, \lambda \in \mathbb{R} \), \( H(\eta, \lambda) \) is essentially self-adjoint on any core of \( d\Gamma_b(T) \). By (10.1), (10.2) and [5, Theorem 5.18.], we can see that there exists \( a > 0 \) such that \( \|H(\eta, \lambda)\psi\| \leq a\|d\Gamma_b(T) + I\psi\| \) for all \( \psi \in D(d\Gamma_b(T)) \). Let \( f \in D(T) \). Then by (2.1) and (10.3), for any \( \psi \in \mathcal{F}_{b,\text{fn}}(D(T)) \), we have

\[
\langle H(\eta, \lambda)\psi, (d\Gamma_b(T) + I)\psi \rangle - \langle (d\Gamma_b(T) + I)\psi, H(\eta, \lambda)\psi \rangle
= \frac{\lambda}{\sqrt{2}} \langle \Phi_s(g)\psi, A(Tg)\psi \rangle - \langle A(Tg)\psi, \Phi_s(g)\psi \rangle + \frac{\eta}{\sqrt{2}} \langle \psi, A(Tf)\psi \rangle - \langle A(Tf)\psi, \psi \rangle.
\]

Thus, by (10.1) and (10.2), we obtain

\[
\left| \langle H(\eta, \lambda)\psi, (d\Gamma_b(T) + I)\psi \rangle - \langle (d\Gamma_b(T) + I)\psi, H(\eta, \lambda)\psi \rangle \right| \leq C\|d\Gamma_b(T) + I\|^{1/2}\|\psi\|^2,
\]

where \( C := \{\|\lambda\|T^{1/2}g\|g\| + 2\|T^{-1/2}g\| + \sqrt{2}\|\eta\|T^{1/2}f\|\} \). By a limiting argument, using the fact that \( \mathcal{F}_{b,\text{fn}}(D(T)) \) is a core of \( d\Gamma_b(T) \) and \( d\Gamma_b(T) \)-boundedness of \( \Phi_s(g)^2 \), we can show that for \( f \in D(T^{1/2}) \) and \( \psi \in D(d\Gamma_b(T)) \), (2.3) holds. Thus, by the Nelson commutator theorem, for all \( \eta, \lambda \in \mathbb{R} \), \( H(\eta, \lambda) \) is essentially self-adjoint and \( H(\eta, \lambda) \) is essentially self-adjoint on any core of \( d\Gamma_b(T) \). The equation \( \overline{H(\eta, \lambda)} \upharpoonright \mathcal{D} = \overline{H(\eta, \lambda)} \upharpoonright \mathcal{D} \) holds for any core \( \mathcal{D} \) of \( d\Gamma_b(T) \). Hence \( H(\eta, \lambda) \) is essentially self-adjoint on any core of \( d\Gamma_b(T) \) for all \( \eta, \lambda \in \mathbb{R} \). Next we show that, if \( \lambda > \|T^{-1/2}g\|^{-1}\|T^{1/2}g\| + \|T^{1/2}g\|^{1/2} \), then \( H(\eta, \lambda) \) is self-adjoint. We can show that, for \( \lambda > 0 \) and any \( 0 < \varepsilon < 1 \), there is a constant \( c_\varepsilon > 0 \) such that

\[
(1 - \varepsilon)\|d\Gamma_b(T)\psi\|^2 + \left\| \frac{\lambda}{2} \Phi_s(g)^2 \psi \right\|^2 \leq \|H(\eta, \lambda)\psi\|^2 + c_\varepsilon\|\psi\|^2, \quad \psi \in D(d\Gamma_b(T)).
\]
Hence $H(\eta, \lambda)$ is closed. In particular, it is self-adjoint.

(3) From the fact that $\Phi_s(f)$ is infinitesimally small with respect to $d\Gamma_b(T)$, for any $\varepsilon > 0$, $\varepsilon d\Gamma_b(T) + \eta \Phi_s(f)$ is bounded from below. By (10.1), for any $\varepsilon > 0$ and $\psi \in D(d\Gamma_b(T)^{1/2})$,

$$| \langle \psi, A(f)\psi \rangle | \leq \|T^{-1/2}f\| \left( \varepsilon \|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 + \frac{1}{4\varepsilon} \|\psi\|^2 \right).$$

Hence if the assertion follows for $\eta = 0$, then so is for all $\eta$. Thus we show that the assertion follows for $\eta = 0$. If $\lambda > 0$, then clearly $H(\lambda) \geq 0$. Let $\lambda < 0$. By (10.1) and (10.2), for any $\psi \in D(d\Gamma_b(T)^{1/2})$, it follows that

$$\|\Phi_s(g)\psi\|^2 \leq 2\|T^{-1/2}g\|^2 \|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 + \|g\|^2 \|\psi\|^2.$$ 

Thus for any $\psi \in D(d\Gamma_b(T))$,

$$\langle \psi, H(\lambda)\psi \rangle = \|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 + \frac{\lambda}{2} \|\Phi_s(g)\psi\|^2$$

$$\geq (1 + |\lambda||T^{-1/2}g|^2) \|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 + \frac{\lambda}{2} \|g\|^2 \|\psi\|^2. \quad (2.4)$$

Hence $H(\lambda)$ is bounded from below if $\lambda \geq \lambda_{c,0}$.

Let $\lambda \geq \lambda_{c,0}$ and $M \geq 0$ be a constant satisfying $H(\lambda) + M \geq 0$. Then for any $\psi \in D(d\Gamma_b(T)) = D(H(\lambda))$,

$$\|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 \leq (1 + |\lambda||T^{-1/2}g|^2) \|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 + \left( \frac{|\lambda|}{2} \|g\|^2 + M \right) \|\psi\|^2.$$ 

(2.5)

By the fact that $D(d\Gamma_b(T))$ is a core of $d\Gamma_b(T)^{1/2}$, we have $D(d\Gamma_b(T)^{1/2}) \subset D((H(\lambda) + M)^{1/2})$ and (2.5) holds on $D(d\Gamma_b(T)^{1/2})$.

In the case $\lambda > 0$, the fact that $\Phi_s(g)^2$ is non-negative implies that $\|H(\lambda)^{1/2}\| \geq \|d\Gamma_b(T)^{1/2}\|$ holds for any $\psi \in D(d\Gamma_b(T))$. In the case $0 > \lambda > \lambda_{c,0}$,

$$\|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 \leq \frac{1}{1 + |\lambda||T^{-1/2}g|^2} \left\{ \|H(\lambda) + M)^{1/2}\| \|\psi\|^2 - \left( \frac{|\lambda|}{2} \|g\|^2 + M \right) \|\psi\|^2 \right\}$$

holds for any $\psi \in D(d\Gamma_b(T))$ by (2.4). Hence for $\lambda > \lambda_{c,0}$ there is a constant $a, b \geq 0$ such that for any $\psi \in D(d\Gamma_b(T))$,

$$\|d\Gamma_b(T)^{1/2}\| \|\psi\|^2 \leq a \|H(\lambda) + M)^{1/2}\| + b \|\psi\|. \quad (2.6)$$

By a functional calculus, $D(d\Gamma_b(T))$ is a core of $(H(\lambda) + M)^{1/2}$. This fact and (2.6) imply that $D((H(\lambda) + M)^{1/2}) \subset D(d\Gamma_b(T)^{1/2})$ and (2.6) holds on $D((H(\lambda) + M)^{1/2})$. \qed
**Remark 2.4.** By [3, Lemma 13-15], if $\mathcal{H}$ is separable, then Theorem 2.3 takes the following forms:

Let $\mathcal{H}$ be separable, $T$ be a non-negative, injective self-adjoint operator, $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$. Then the following (1)-(3) hold:

1. Let $\lambda > \lambda_{c,0}$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.

2. Let $\lambda \leq \lambda_{c,0}$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. In particular, if $\eta = 0$ and $\lambda = \lambda_{c,0}$, then $H(\lambda_{c,0}) = H(0, \lambda_{c,0})$ is bounded from below.

3. Let $\lambda > \lambda_{c,0}$. Then $D(d\Gamma_b(T)^{1/2}) = D((H(\eta, \lambda) + M)^{1/2})$ for all constant $M \geq 0$ satisfying $H(\eta, \lambda) + M \geq 0$.

### 3 The Main Theorem

#### 3.1 Assumptions

To prove our main theorem stated later (Theorem 3.6), we need some assumptions. For a closed operator $A$, $\sigma(A)$ denotes the spectrum of $A$. If $A$ is self-adjoint, then $\sigma_{ac}(A)$ (resp. $\sigma_p(A), \sigma_{sc}(A)$) denotes the absolutely continuous (resp. point, singular continuous) spectrum of $A$. For a self-adjoint operator $A$ which is bounded from below,

$$E_0(A) := \inf \sigma(A)$$

is called the lowest energy of $A$. In particular, it is called the ground state energy of $A$ if $E_0(A) \in \sigma_p(A)$. In this case, an eigenvector of $A$ with eigenvalue $E_0(A)$ is called a ground state of $A$. The ground state is said to be unique if $\dim \text{Ker}(A - E_0(A)) = 1$. For linear operators $A$ and $B$, the symbol $A \subset B$ means that $D(A) \subset D(B)$ and $Af = Bf$ for all $f \in D(A)$, i.e., $B$ is an extension of $A$.

**Definition 3.1.** Let $T$ be a self-adjoint operator on $\mathcal{H}$ and $\{E(B) \mid B \in B^1\}$ be the spectral measure associated with $T$ on the Borel field $B^1$ on $\mathbb{R}$. The operator $T$ is called purely absolutely continuous if, for each $f \in \mathcal{H}$, the measure $\|E(\cdot)f\|^2$ on $B^1$ is absolutely continuous with respect to the one-dimensional Lebesgue measure.
Definition 3.2. For a purely absolutely continuous self-adjoint operator $T$ and vectors $f, g \in \mathcal{H}$, $\psi_{g,f}$ denotes the Radon-Nikodym derivative of the finite complex Borel measure $\langle g, E(\cdot)f \rangle$ on $B^1$. In particular, we set $\psi_g := \psi_{g,g}$.

Assumption 3.3. (1) The operator $T$ is a non-negative, purely absolutely continuous self-adjoint operator.

(2) The fixed vector $g \in \mathcal{H}$ satisfies $g \in D(\hat{T}^{-1/2}) \cap D(T^{1/2})$ and $Jg = g$, where $\hat{T} := T - E_0$, $E_0 := E_0(T)$ and $J$ is a conjugation on $\mathcal{H}$ satisfying $JD(T) \subset D(T)$ and $JT\psi = TJ\psi$ for any $\psi \in D(T)$ (i.e., $JT \subset TJ$).

(3) $\sup_{x \in E_0} x^{\pm 1} \psi_g(x) < \infty$ and $\psi_g(x) > 0$ for all $x \in (E_0, \infty)$,

(4) $\psi_g \in C([E_0, \infty)) \cap C^1((E_0, \infty))$ and $\lim_{x \to E_0} x^{-1} \psi'_g(x) = 0 = \lim_{x \to \infty} x^{-1} \psi'_g(x)$.

Remark 3.4. The operator $T$ is injective since it is a purely absolutely continuous self-adjoint operator. Since $T$ has no eigenvector, the inverse of $\hat{T}$ exists. Assumption 3.3 (2) implies that $T_J = T$. In general, for a self-adjoint operator $A$ and a conjugation $J$, we can choose a vector $f \in D(A)$ satisfying $Jf = f$ if $A_J = A$. Thus the vector $g$ in Assumption 3.3 (2) exists. By Assumption 3.3 (3), one can easily show that $\sup_{x \in \sigma(T)} \psi_g(x) < \infty$ and, for each $f \in \mathcal{H}$, the functions $\psi_{g,f}$ and $\psi_{T^{\pm 1/2}g,f}$ are in $L^2(\mathbb{R})$ and the maps $f \mapsto \psi_{g,f}, \psi_{T^{\pm 1/2}g,f}$ are bounded. Actually, for any $h \in \mathcal{H}$ and $B \in B^1$, the following inequality holds

$$|\langle E(B)h, f \rangle|^2 \leq \|E(B)h\|^2 \|E(B)f\|^2$$

by Schwarz’s inequality. Thus we obtain $|\psi_{h,f}(\mu)|^2 \leq \psi_h(\mu)\psi_f(\mu)$ for almost all $\mu \in \mathbb{R}$ with respect to the Lebesgue measure. Hence, by Assumption 3.3 (3), we have the boundedness of the mappings. Moreover, we see that for any $F \in L^2(\mathbb{R})$, $g \in D(F(T))$, where $F(T)$ denotes the operator defined by $F(T) := \int_{\mathbb{R}} F(\mu)dE(\mu)$. In particular, $g$ is in $D(\psi_{g,f}(T))$ for any $f \in \mathcal{H}$.

Lemma 3.5. Let $T$ be a self-adjoint operator such that $JT \subset TJ$. Then

(1) $E(B)_T = E(B)$, for all $B \in B^1$.

(2) Let $F$ be a Borel measurable function on $\mathbb{R}$. Then $F(T)_T = F^*(T)$, where $F^*$ is complex conjugation of $F$.

Proof. These are proved by using the spectral theorem. \qed
3.2 The Main Theorem

In this subsection, we state the main theorem of the present paper. Let $\lambda_c$ be a constant defined by

$$\lambda_c := - \left( \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} \, d\| E(\mu) g \|^2 \right)^{-1} < 0.$$ 

Then, by a functional calculus, we obtain $\lambda_{c,0} \leq \lambda_c$, and $\lambda_{c,0} = \lambda_c$ if and only if $E_0 = 0$.

**Theorem 3.6.** Let $\mathcal{H}$ be separable. Then the following (1)-(3) hold:

1. Let $T$ and $g$ satisfy Assumption 3.3. If $\lambda > \lambda_c$, then there are a unitary operator $U$ on $\mathcal{F}_b(\mathcal{H})$ and a constant $E_g \in \mathbb{R}$ such that

$$UH(\lambda)U^{-1} = d\Gamma_b(T) + E_g.$$  

   In particular, $U^{-1} \Omega_0$ is the unique ground state of $H(\lambda)$, where $\Omega_0 := (1, 0, 0, \ldots) \in \mathcal{F}_b(\mathcal{H})$ is the Fock vacuum, and

$$\sigma(H(\lambda)) = \{E_g\} \cup [E_0 + E_g, \infty), \quad \sigma_{ac}(H(\lambda)) = [E_0 + E_g, \infty), \quad \sigma_p(H(\lambda)) = \{E_g\}, \quad \sigma_{sc}(H(\lambda)) = \emptyset.$$  

2. Let $T$ and $g$ satisfy Assumption 3.3 and $E_0 > 0$. If $\lambda_{c,0} < \lambda < \lambda_c$, then there exist a unitary operator $V$ on $\mathcal{F}_b(\mathcal{H})$, an injective non-negative self-adjoint operator $\xi$ on $\mathcal{H}$ and a constant $E_b \geq 0$ such that $\xi$ has a ground state and

$$VH(\lambda)V^{-1} = d\Gamma_b(\xi) + E_g - E_b.$$  

   In particular, $V^{-1} \Omega_0$ is the unique ground state of $H(\lambda)$, and

$$\sigma(H(\lambda)) = \bigcup_{n=0}^{\infty} \{n\beta + E_g - E_b\} \cup [E_0 + E_g - E_b, \infty),$$

$$\sigma_{ac}(H(\lambda)) = [E_0 + E_g - E_b, \infty),$$

$$\sigma_p(H(\lambda)) = \bigcup_{n=0}^{\infty} \{n\beta + E_g - E_b\}, \quad \sigma_{sc}(H(\lambda)) = \emptyset,$$

where $\beta > 0$ is the discrete ground state energy of $\xi$.

3. Let $T$ be a non-negative, injective self-adjoint operator. If $g \in D(T^{-1/2})$ and $\lambda < \lambda_{c,0}$, then $H(\lambda)$ is unbounded from above and below.

**Example 3.7.** A concrete realization of the abstract model is given as follows (see [9, Chapter 12]):

$$\mathcal{H} \leftrightarrow L^2(\mathbb{R}^d), \quad T \leftrightarrow \omega, \quad g \leftrightarrow \frac{\hat{\rho}}{\sqrt{\omega}},$$
where \( \omega \) is the multiplication operator associated with the function \( \omega(k) := \sqrt{|k|^2 + m^2}, k \in \mathbb{R}^d \) for a fixed \( m \geq 0 \) and \( \hat{\rho} \) is the Fourier transform of a function \( \rho \in L^2(\mathbb{R}^d) \) satisfying \( \hat{\rho}/\sqrt{\omega} \in L^2(\mathbb{R}^d) \). Assume that \( \hat{\rho} \) is rotation invariant, i.e., there exists a function \( v \) on \([0, \infty)\) such that \( \hat{\rho}(k) = v(|k|) \) for all \( k \in \mathbb{R}^d \). Then we have \( \psi_g(s) = |S^{d-1}| \omega_1^{-1}(s)^{d-2} |v(\omega_1^{-1}(s))|^2 \) for \( s > m \), where \( |S^{d-1}| \) is the surface area of the \((d - 1)\)-dimensional unite sphere with convention \( |S^0| = 2\pi \) and \( \omega_1(r) = \sqrt{r^2 + m^2}, r \geq 0 \). Set \( \psi_g(m) := 0 \). Hence, with \( J \) being the complex conjugation, the following conditions (2)'-(4)' imply that the present model satisfies Assumption 3.3:

(2)' \( \hat{\rho}(k)^* = \hat{\rho}(k) \) and 
\[
\hat{\rho} \in L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} \frac{|\hat{\rho}(k)|^2}{|k|^2} dk < \infty.
\]

(3)' \( \hat{\rho} \) is rotation invariant. \( \sup_{k \in \mathbb{R}^d} \omega(k)^{1/2} |k|^{(d-2)/2} |\hat{\rho}(k)| < \infty. \hat{\rho}(k) > 0, \) for all \( k \neq 0 \).

(4)' \( v \in C^1([0, \infty)) \) and
\[
\lim_{|k| \to 0} |k|^{d-4} \hat{\rho}(k) \{(d - 2)\hat{\rho}(k) + 2|k|v'(|k|)\} = 0,
\]
\[
\lim_{|k| \to \infty} |k|^{d-4} \hat{\rho}(k) \{(d - 2)\hat{\rho}(k) + 2|k|v'(|k|)\} = 0.
\]

We can show that \( \psi_g \) is right continuous at \( m \) by \( \int_{\mathbb{R}^d} |\hat{\rho}(k)|^2 |k|^{-2} dk < \infty \) and \( v \in C^1([0, \infty)) \). For example, one can easily check that the function
\[
\hat{\rho}(k) := \exp \left( -\frac{1}{|k|^2} - |k|^2 \right), k \in \mathbb{R}^d \setminus \{0\}, \hat{\rho}(0) := 0
\]
satisfies the above conditions (2)'-(4)'.

4 Definitions and properties of some functions and operators

In this section, we introduce some functions and operators. We assume that \( \mathcal{H} \) is separable and Assumption 3.3 from this section to Section 6.

4.1 Functions \( D \) and \( D_\pm \)

Lemma 4.1. Let \( D : \mathbb{C}\setminus(0, \infty) \to \mathbb{C} \) be the function
\[
D(z) := 1 + \lambda \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2 - z} d\|E(\mu)g\|^2, \quad z \in \mathbb{C}\setminus(0, \infty).
\]
Then \(D\) is well-defined and analytic in \(\mathbb{C}\backslash[0, \infty)\). Moreover, the following hold:

1. For all \(\lambda > \lambda_c\), \(D(z)\) has no zeros in \(\mathbb{C}\backslash[0, \infty)\).
2. For all \(\lambda < \lambda_c\), \(D(z)\) has a unique simple zero in the negative real axis \((-\infty, 0)\).

**Proof.** If \(\text{Im} z \neq 0\) (resp. \(\text{Re} z < 0\)), then for any \(n \in \mathbb{N}\),

\[
\int_{[E_0, \infty)} \left| \frac{\mu}{(\mu^2 - E_0^2 - z)^n} \right| d\| E(\mu)g \| < c^{-n}\| T^{1/2} \| < \infty,
\]

where \(c = |\text{Im} z|\) (resp. \(|\text{Re} z|\)). If \(z = 0\), then

\[
\int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} d\| E(\mu)g \| < \| T^{-1/2} \| < \infty.
\]

Thus, by using the Lebesgue dominated convergence theorem, \(D\) is well-defined and analytic in \(\mathbb{C}\backslash[0, \infty)\).

1. If \(\lambda = 0\), then \(D(z) = 1\) for all \(z \in \mathbb{C}\backslash(0, \infty)\), so it has no zeros. Let \(\lambda \neq 0\) and \(z = x + iy \in \mathbb{C}\backslash(0, \infty)\). Then we see that

\[
\text{Im} \ D(z) = y\lambda \int_{[E_0, \infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2 + y^2} d\| E(\mu)g \|^2.
\]

Thus \(\text{Im} \ D(z) = 0\) is equivalent to \(y = 0\). Therefore \(D(z) = 0\) if and only if \(D(x) = 0\).

Let \(y = 0\). In the case \(\lambda > 0\), one has \(D(x) > 0\) for all \(x \in (-\infty, 0]\). Thus \(D\) has no zeros. Next, we consider the case \(\lambda < 0\). We have for \(x < 0\),

\[
D'(x) = \lambda \int_{[E_0, \infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2} d\| E(\mu)g \|^2 < 0.
\]

Thus \(D\) is monotone decreasing in \((-\infty, 0)\). If \(\lambda > \lambda_c\), then \(D(0) > 0\). Hence \(D\) has no zeros.

(2) Let \(\lambda < \lambda_c\). We can see that

\[
D(0) = 1 + \lambda \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} d\| E(\mu)g \|^2 = 1 - \frac{\lambda}{\lambda_c} < 0.
\]

By the Lebesgue dominated convergence theorem, \(D(x) \to 1\) as \(x \to -\infty\). Since \(D\) is monotone decreasing in \((-\infty, 0)\), \(D\) has a unique simple zero in \((-\infty, 0)\).

\[\square\]
Let
\[ \phi_{g}(x) := \psi_{g}(\sqrt{x})\chi_{[E_{0}^{2}, \infty)}(x), \quad x \in \mathbb{R}, \]
where \( \chi_{B} \) is the characteristic function of \( B \in \mathbf{B}^{1} \).

**Lemma 4.2.** The following hold:

1. The function \( \phi_{g} \) satisfies \( \phi_{g} \in C^{1}(\mathbb{R}) \cap L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R}) \) and \( \sup_{x \in \mathbb{R}} |\phi'_{g}(x)| < \infty \).

2. Let
\[ A_{\varepsilon}^{(1)}(x) := \frac{x}{\pi(x^{2} + \varepsilon^{2})}, \quad A_{\varepsilon}^{(2)}(x) := \frac{\varepsilon}{\pi(x^{2} + \varepsilon^{2})}, \quad x \in \mathbb{R}, \ \varepsilon > 0 \]
be the conjugate poisson kernel and the poisson kernel respectively and \( f \ast h \) denote the convolution of functions \( f \) and \( h \). Let
\[ (H_{\varepsilon}f)(s) := \frac{1}{\pi} \int_{|x - s| \geq \varepsilon} \frac{f(x)}{s - x} \, dx, \quad (Hf)(s) := \lim_{\varepsilon \downarrow 0}(H_{\varepsilon}f)(s), \quad s \in \mathbb{R}, \ \varepsilon > 0, \]
where \( Hf \) is called the Hilbert transform of \( f \). Then for all \( x \in \mathbb{R} \),
\[ \lim_{\varepsilon \downarrow 0} \left( A_{\varepsilon}^{(1)} \ast \phi_{g} \right)(x) = (H\phi_{g})(x), \quad \lim_{\varepsilon \downarrow 0} \left( A_{\varepsilon}^{(2)} \ast \phi_{g} \right)(x) = \phi_{g}(x), \]
hold uniformly in \( x \).

**Proof.** For any \( h > 0 \), by Assumption 3.3 (1), (4) and the mean value theorem, there exists \( \theta \in (E_{0} + h/2, E_{0} + h) \) such that
\[ \int_{E_{0} + h/2}^{E_{0} + h} \frac{\psi_{g}(\mu)}{\mu - E_{0}} \, d\mu = \frac{h}{\theta - E_{0}} \psi_{g}(\theta). \]
This fact and \( \theta < E_{0} + h \) imply that
\[ \|E([E_{0}, E_{0} + h])\hat{T}^{-1/2}g\|^{2} = \int_{[E_{0}, E_{0} + h]} \frac{\psi_{g}(\mu)}{\mu - E_{0}} \, d\mu > \frac{\psi_{g}(\theta)}{2}. \tag{4.1} \]
By taking the limit \( h \downarrow 0 \) and Assumption 3.3 (1), the left hand side of (4.1) tends to zero. Thus we obtain \( \lim_{h \to E_{0} + 0} \psi_{g}(h) = 0 \). This fact and \( \psi_{g} \in C([E_{0}, \infty)) \) imply that \( \psi_{g}(E_{0}) = 0 \). Since \( \psi_{g} \) is the Radon-Nikodym derivative of \( \|E(\cdot)g\|^{2} \) and \( E_{0} \leq T \), we have \( \psi_{g}(x) = 0 \) for \( x < E_{0} \). Thus \( \phi_{g} \in C(\mathbb{R}) \). By the differentiability of \( \psi_{g} \), we obtain \( \phi'_{g}(x) = \psi'_{g}(\sqrt{x})/(2\sqrt{x}) \) for \( x > E_{0}^{2} \) and \( \phi'_{g}(x) = 0 \) for \( x < E_{0}^{2} \). Thus, \( \phi'_{g} \) is continuous on \( (-\infty, E_{0}^{2}) \cup (E_{0}^{2}, \infty) \). Since \( \phi'_{g}(x) = 0 \) for \( x < E_{0}^{2} \) and \( \lim_{h \to 0+}(E_{0} + h)^{-1}\psi_{g}(E_{0} + h) = 0 \), we have \( \lim_{h \to 0+}\phi'_{g}(E_{0}^{2} + h) = 0 \). By this fact and the l'Hôpital theorem, we obtain \( \lim_{h \to 0+}(\phi_{g}(E_{0}^{2} + h) - \phi_{g}(E_{0}^{2})) / h = 0 \). We
have $\lim_{h \to 0^-}\phi_g(E_0^2 + h) - \phi_g(E_0^2)) = 0$ since $\phi_g(x) = 0$ for $x < E_0^2$. Thus $\phi_g$ is continuous at $E_0^2$. Hence $\phi_g \in C^1(\mathbb{R})$. By the fact that $\psi'_g(x) = 0$ for $x < E_0$ and Assumption 3.3 (4) imply that $\phi_g \in C^1(\mathbb{R})$ and $\phi'_g(E_0^2) = 0$. By Assumption 3.3 (2) and a change of variable, we have $\phi_g \in L^1(\mathbb{R})$. We obtain $\phi_g \in L^2(\mathbb{R})$ by Assumption 3.3 (3) and a change of variable. The inequality $\sup_{x \in \mathbb{R}} |\phi'_g(x)| < \infty$ is given by Assumption 3.3 (4). The assertion (1) holds. Next we consider the assertion (2). By (1), in particular, $\phi_g$ is bounded and uniformly continuous. Thus it is easy to see that $A_\varepsilon^{(2)} \ast \phi_g$ converges uniformly to $\phi_g$. Moreover, by (1), Hölder’s inequality, the mean value theorem and a similar estimate to the proof of [17, Theorem 92.], we can show that $(A_\varepsilon^{(1)} \ast \phi_g)(x) - (H_\varepsilon \phi_g)(x)$ tends to 0 uniformly in $x$ as $\varepsilon \downarrow 0$. Hence the assertion (2) holds. □

Detailed studies of the Hilbert transform are given in [17].

**Lemma 4.3.** For all $s \geq 0$, $D_\pm(s) := \lim_{\varepsilon \downarrow 0} D(s \pm i\varepsilon)$ are uniformly convergent and continuous in $s \geq 0$ with

$$D_\pm(s) = 1 - \frac{\lambda \pi}{2} (H \phi_g)(E_0^2 + s) \pm i \frac{\lambda \pi}{2} \psi_g(\sqrt{E_0^2 + s}), \quad s \geq 0. \quad (4.2)$$

**Proof.** For any $s \geq 0$ and $\varepsilon > 0$, we have by a change of variable

$$D(s \pm i\varepsilon) = 1 - \frac{\lambda \pi}{2} (A_\varepsilon^{(1)} \ast \phi_g)(E_0^2 + s) \pm i \frac{\lambda \pi}{2} (A_\varepsilon^{(2)} \ast \phi_g)(E_0^2 + s).$$

Thus, by Lemma 4.2, $D_\pm$ converge uniformly in $s \geq 0$ and (4.2) holds. The continuity of $D_\pm$ is due to the uniform convergence. □

**Remark 4.4.** For all $\mu \in \left[E_0^0, \infty\right)$, we have

$$i\pi \lambda \psi_g(\mu) = D_+(\mu^2 - E_0^2) - D_-(\mu^2 - E_0^2). \quad (4.3)$$

**Lemma 4.5.** Let $\lambda \neq \lambda_c$, then $\delta := \inf_{s \geq 0} |D_\pm(s)| > 0$.

**Proof.** If $\lambda = 0$, then clearly $D_\pm(s) = 1 > 0$ for all $s \in [0, \infty)$. Let $\lambda \neq 0, \lambda_c$. Then $D_\pm(0) = D(0) \neq 0$. Hence, by the continuity of $D_\pm$, $D_\pm$ has no zeros near $s = 0$. For any $\varepsilon > 0$ and $s > E_0^2 + 1$, we have

$$(H_\varepsilon \phi_g)(s) = I_1^{(e)}(s) + \sum_{j=2}^{4} I_j(s),$$

$$I_1^{(e)}(s) = \int_{\varepsilon}^{1} \frac{\phi_g(s - x) - \phi_g(s + x)}{x} dx, \quad I_2(s) = \int_{E_0^2}^{s-1} \frac{\phi_g(x)}{s - x} dx,$$

$$I_3(s) = \int_{s+1}^{2s} \frac{\phi_g(x)}{s - x} dx, \quad I_4(s) = \int_{2s}^{\infty} \frac{\phi_g(x)}{s - x} dx.$$
Then, by the Lebesgue dominated convergence theorem, each \( I_j(s), \ j = 2, 3, 4 \) tends to zero as \( s \to \infty \). By the mean value theorem and the property that \( \phi'_y(x) \to 0 \) as \( x \to \infty \), we have \( \lim_{s \to \infty} \lim_{\varepsilon \downarrow 0} I_1^{(\varepsilon)}(s) = 0 \). Hence we can see that \( (H\phi_y)(s) \to 0 \) as \( s \to \infty \). This fact implies that \( \inf_{s_0 \leq s} \Re D_\pm(s) > 0 \) for a sufficiently large number \( s_0 > 0 \). In addition, \( \Im D_\pm(s) \) are positive for any closed interval included in \((0, \infty)\) by Assumption 3.3 (3) and the continuity of \( \psi_g \). Hence we can see that \( \inf_{s \geq 0} |D_\pm(s)| > 0 \). \( \Box \)

**Remark 4.6.** By Lemmas 4.3 and 4.5, we can see that there are constants \( c, d, \varepsilon_0 > 0 \) with \( 0 < c < d \) such that

\[
c \leq \frac{|D(s \pm i\varepsilon)|}{D_\pm(s)} \leq d
\]

for all \( s \geq 0, 0 < \varepsilon < \varepsilon_0 \).

### 4.2 Operators \( R_\pm \)

Through this subsection, we assume \( \lambda \neq \lambda_c \).

**Lemma 4.7.** One can define bounded operators \( R_\pm \) on \( \mathcal{H} \) as follows:

\[
R_\pm f := -\lambda \lim_{\varepsilon \downarrow 0} \int_{(E_0, \infty)} \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle , \quad f \in \mathcal{H},
\]

where \( R_\pm(A) := (A - z)^{-1} \) is the resolvent of a linear operator \( A \) at \( z \in \rho(A) \) (the resolvent set of a linear operator \( A \)).

**Proof.** For a fixed \( \varepsilon > 0 \) and any \( f \in \mathcal{H} \),

\[
\int_{(E_0, \infty)} \left\| \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu^2 - E_0^2)} \right\| d\|E(\mu')f\|^2 \leq \frac{\|f\|^2\|T^{1/2}g\|}{\delta\varepsilon} < \infty
\]

by Lemma 4.5 and a property of a resolvent. Thus we can define linear operators \( R_\pm^{(\varepsilon)} \) on \( \mathcal{H} \) by

\[
R_\pm^{(\varepsilon)} f := -\lambda \int_{(E_0, \infty)} \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle
\]

in the sense of Bochner integral with the polarization identity. For any \( h, f \in \mathcal{H} \),

\[
\left\langle h, R_\pm^{(\varepsilon)} f \right\rangle = -\lambda \int_{(E_0, \infty)} \frac{\left\langle h, R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g \right\rangle}{D_\pm(\mu^2 - E_0^2)} d\left\langle T^{1/2}g, E(\mu')f \right\rangle
\]

\[
= -\lambda \int_{(E_0, \infty)} \int_{(E_0, \infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \pm i\varepsilon)D_\pm(\mu^2 - E_0^2)} d\left\langle h, E(\mu)g \right\rangle d\left\langle T^{1/2}g, E(\mu')f \right\rangle,
\]

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where we have used a functional calculus. By change of variables in the Lebesgue-Stieltjes integration, a functional calculus and Fubini's theorem, we have

\[
\left\langle h, R_{\pm}^{(\epsilon)} f \right\rangle = \frac{\lambda \pi}{2} \int_{[E_0, \infty)} - (A_{\pm}^{(1)} * \phi_{g, f}^\pm) (\mu^2) \mu^{1/2} T \mp i \left( A_{\pm}^{(2)} * \phi_{g, f}^\pm \right) (\mu^2) \mu^{1/2} d\langle h, E(\mu)g \rangle,
\]

where \( \phi_{g, f}^\pm(x) = \psi_{g, f}(\sqrt{\epsilon}) x^{-1/4} D_{\pm}(x - E_0^2)^{-1} \chi_{[E_0^2, \infty)}(x), x \in \mathbb{R} \). We have \( \phi_{g, f}^\pm \in L^2(\mathbb{R}) \) by Remark 3.4, and the function \( \left( A_{\pm}^{(j)} * \phi_{g, f}^\pm \right) (\mu^2) \mu^{1/2} (\mu \in \mathbb{R}) \) is in \( L^2(\mathbb{R}) \) for each \( j = 1, 2 \). Thus, by a change of variable, we have

\[
\left\| \left( R_{\pm}^{(\epsilon)} f - \left( -\frac{\pi \lambda}{2} (H \phi_{g, f}^\pm)(T^2) T^{1/2} g \mp \frac{1}{2} A_{\pm} f \right) \right) \right\|^2 \leq \left( \frac{\lambda \pi}{2} \right)^2 c_g \int_{[E_0^2, \infty)} |(A_{\pm}^{(1)} * \phi_{g, f}^\pm)(x) - (H \phi_{g, f}^\pm)(x)|^2 dx
\]

\[
+ \left( \frac{\lambda \pi}{2} \right)^2 c_g \int_{[E_0^2, \infty)} |(A_{\pm}^{(2)} * \phi_{g, f}^\pm)(x) - \phi_{g, f}^\pm(x)|^2 dx,
\]

where \( c_g := \sup_{x \in [E_0, \infty)} \psi_{g, f}(x) \) and the linear operators

\[A_{\pm} f := i\pi \lambda \psi_{g, f}(T) D_{\pm}(T^2 - E_0^2)^{-1} g, \quad f \in \mathcal{H}\]

are well-defined (see Remark 3.4 and Lemma 4.5). Hence, by \( \phi_{g, f}^\pm \in L^2(\mathbb{R}) \), we have

\[R_{\pm}^{(\epsilon)} f \to -(\pi \lambda/2)(H \phi_{g, f}^\pm)(T^2) T^{1/2} g \mp (1/2) A_{\pm} f \text{ as } \epsilon \downarrow 0.\]

Moreover, by change of variables, the isometricity of Hilbert transform and Remark 3.4, we can show that the inequalities

\[
\| (H \phi_{g, f}^\pm)(T^2) T^{1/2} g \| \leq \frac{c_g}{\delta} \| f \|, \quad \| A_{\pm} f \| \leq \frac{\pi |\lambda| c_g}{\delta} \| f \|
\]

hold for all \( f \in \mathcal{H} \). Hence \( R_{\pm} \) are bounded. \( \square \)

By the definition of the adjoint operator, \( R_{\pm}^\ast := (R_{\pm})^* \) are given as follows: for \( f \in \mathcal{H} \),

\[
R_{\pm}^{(\epsilon)\ast} f = \lambda \int_{[E_0, \infty)} R_{\mu^2 \pm i \epsilon}(T^2) D_{\pm}(T^2 - E_0^2)^{-1} T^{1/2} g \ d\langle T^{1/2} g, E(\mu') f \rangle, \quad (4.5)
\]

\[
R_{\pm}^\ast f = \lim_{\epsilon \downarrow 0} R_{\pm}^{(\epsilon)\ast} f.
\]

For a densely defined linear operator \( A \) on a Hilbert space, \( A^\sharp \) denotes \( A \) or \( A^* \).

**Lemma 4.8.** The ranges of \( R_{\pm}^\sharp \) are included in \( D(T^{-1}) \cap D(T) \) and \( R_{\pm}^\sharp \) map \( D(T) \) into \( D(T^2) \).
Proof. For any $f,h \in \mathcal{H}$, we have

$$\langle h, R_{\pm} f \rangle = \frac{\lambda \pi}{2} \int_{[E_0, \infty)} - (H_{\phi_{g,f}^\pm}) (\mu)^2 \mu^{1/2} \mp i \frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} d \langle h, E(\mu)g \rangle. \quad (4.6)$$

By a change of variable, we have

$$(H_{\phi_{g,f}^\pm})(\mu^2) = \left( H_{\psi_{T-1/2g,f}^\pm} \right) (\mu) + \left( H_{\psi_{T-1/2g,f}^\pm} \right) (-\mu), \quad \mu \in \mathbb{R}, \quad (4.7)$$

where $\psi_{h,f}^\pm(x) := \psi_{h,f}(x)D_{\pm}(x^2 - E_0)(x^2 - E_0)^{-1}\chi_{[E_0, \infty)}(x), x \in \mathbb{R}$ for $h,f \in \mathcal{H}$. Thus we see by Assumption 3.3 (3) and a functional calculus that $\text{Ran}(R_{\pm}) \subset D(T^{-1})$. The equation

$$\mu \left( H_{\phi_{g,f}^\pm} \right)(\mu^2) = \left( H_{\psi_{T1/2g,f}^\pm} \right)(\mu) - \left( H_{\psi_{T1/2g,f}^\pm} \right)(-\mu), \quad \mu \in \mathbb{R}, \quad (4.8)$$

(4.6), Assumption 3.3 (3) and operational calculus imply that $\text{Ran}(R_{\pm}) \subset D(T)$. For any $f \in D(T)$ and $\mu \in \mathbb{R}$,

$$\mu^2 \left( H_{\phi_{g,f}^\pm} \right)(\mu^2) = \left( H_{\psi_{T1/2g,Tf}^\pm} \right)(\mu) + \left( H_{\psi_{T1/2g,Tf}^\pm} \right)(-\mu) + \frac{2}{\pi} \int_{[E_0, \infty)} \psi_{T1/2g,f}(x) dx. \quad (4.10)$$

Hence $R_{\pm} f \in D(T^2)$ and the following equation holds for any $h \in \mathcal{H}$,

$$\langle h, T^2 R_{\pm} f \rangle = \frac{\lambda \pi}{2} \int_{[E_0, \infty)} - \left\{ \left( H_{\psi_{T1/2g,Tf}^\pm} \right)(\mu) + \left( H_{\psi_{T1/2g,Tf}^\pm} \right)(-\mu) + \frac{2c}{\pi} \right\} \mu^{1/2} d \langle h, E(\mu)g \rangle$$

$$\mp i \frac{\lambda \pi}{2} \int_{[E_0, \infty)} \psi_{T1/2g,Tf}(\mu) \mu^{1/2} d \langle h, E(\mu)g \rangle,$$

where $c := \int_{\mathbb{R}} \psi_{T1/2g,f}(x) dx$. In quite the same manner as in the case of $R_{\pm}$, we can prove the statement for $R_{\pm}^*$.

\[ \square \]

**Lemma 4.9.** The operator equations $(R_{\pm})_f = R_{\mp}$ hold.

*Proof.* This follows from Assumption 3.3 (1) and Lemma 3.5. \[ \square \]

**Lemma 4.10.** The operator equation $R_- = R_+ \gamma + A_-$ holds, where

$$\gamma := D_+(T^2 - E_0^2)D_-(T^2 - E_0^2)^{-1}$$

is a bounded operator.

*Proof.* The first resolvent formula gives that, for any $\mu', \mu'' \in \mathbb{R}, \varepsilon > 0$,

$$R_{\mu'^2-i\varepsilon}(T^2) - R_{\mu'^2+i\varepsilon}(T^2) = -2i\varepsilon R_{\mu'^2-i\varepsilon}(T^2) R_{\mu'^2+i\varepsilon}(T^2).$$
Then, for any $f \in \mathcal{H}$,

$$R_-(^\varepsilon f) = -\lambda \int_{[E_0, \infty)} \frac{R_{\mu^2+i\varepsilon}(T^2)^{1/2}g}{D_-(\mu^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle$$

$$+ 2i\lambda \int_{[E_0, \infty)} \frac{R_{\mu^2+i\varepsilon}(T^2)R_{\mu^2-i\varepsilon}(T^2)^{1/2}g}{D_-(\mu^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle.$$

Thus, by a change of variable, we have for any $h \in \mathcal{H}$

$$\langle h, R_-^\varepsilon f \rangle = \langle h, R_+^\varepsilon \gamma f \rangle + 2i\lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu)g \rangle d \langle T^{1/2}g, E(\mu')f \rangle$$

$$\times \frac{\mu^{1/2}}{((\mu^2 - \mu'^2)^2 + \varepsilon^2)D_-(\mu^2 - E_0^2)}$$

$$= \langle h, R_+^\varepsilon \gamma f \rangle + i\pi \lambda \int_{[E_0, \infty)} (A^{(2)}_\varepsilon \ast \phi^\varepsilon_{g,f}) (\mu^2)^{1/2} d \langle h, E(\mu)g \rangle.$$

By a property of the Poisson kernel, the function $(A^{(2)}_\varepsilon \ast \psi^\varepsilon_{g,f}) (\mu^2)^{1/2}$ converges to $\psi_{g,f}(\mu)/D_-(\mu^2 - E_0^2)$ as $\varepsilon \to +0$ in the sense of $L^2(\mathbb{R})$. Hence the continuity of the inner product with $L^2(\mathbb{R})$ implies that

$$\langle h, R_- f \rangle = \langle h, R_+ \gamma f \rangle + i\pi \lambda \int_{[E_0, \infty)} \frac{\psi_{g,f}(\mu)}{D_-(\mu^2 - E_0^2)} d \langle h, E(\mu)g \rangle$$

$$= \langle h, R_+ \gamma f \rangle + \langle h, A_- f \rangle.$$

Since $f$ and $h$ are arbitrary, one obtains the conclusion. \hfill \Box

By the definitions of $A_\pm$, we have

$$(A_-)^* = -A_+.$$

**Lemma 4.11.** For any Borel measurable function $F$ on $\mathbb{R}$, $A_\pm F(T) \subseteq F(T)A_\pm$.

**Proof.** For any $f \in D(F(T))$, an operational calculus implies that $\psi_{g,F(T)} = F\psi_{g,f} \in L^2(\mathbb{R})$. This fact imply that $\psi_{g,f}(T)g \in D(F(T))$ and $F(T)\psi_{g,f}(T)g = \psi_{g,F(T)}f(T)g$. Hence $A_\pm f \in D(F(T))$ and $F(T)A_\pm f = A_\pm F(T)f$ by Lemma 4.5. \hfill \Box

**Lemma 4.12.** The following operator equations hold:

$$A_- R^*_\pm = (\gamma - I)R^*_\pm, \quad A_-(A_-)^* = -A_ - (A_-)^*.$$
Proof. By applying Lemma 4.11 to the case $F = \chi_B$, one can easily see that $A\pm E(B) = E(B)A\pm$ hold for any $B \in \mathbb{B}^1$. For any $f, h \in \mathcal{H}$, we have

$$\langle (A_-)^* h, R^{(c)}_\pm f \rangle$$

Then, by a limiting argument, we obtain $A_\pm R^{(c)}_\pm = (\gamma - 1)R^{(c)}_\pm$. Moreover, (4.3) and the equation $(A_-)^* = -A_+$ imply that

$$\langle h, A_- (A_-)^* f \rangle = -(i\pi\lambda)^2 \int_{(E_0,\infty)} \frac{\psi_{g,f}(\mu)\psi_{g}(\mu)}{D_+(\mu^2 - E_0^2)D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle$$

Thus, by a limiting argument, we obtain $A_\pm R^{(c)}_\pm = (\gamma - 1)R^{(c)}_\pm$. Moreover, (4.3) and the equation $(A_-)^* = -A_+$ imply that

$$\langle h, A_- (A_-)^* f \rangle = -\langle h, (A_-)^* f + A_- f \rangle .$$

Hence the equation $A_- (A_-)^* = -A_+ - (A_-)^*$ holds. \qed

### 4.3 Operators $\Omega_\pm$

In this subsection we consider the bounded operators

$$\Omega_\pm := I + R_\pm.$$  

Let $x_0 < 0$ be the zero of $D(z)$ given in Lemma 4.1 (2) and

$$U_b := \sqrt{\frac{\lambda}{D'(x_0)}} R_{E_0^2 + x_0} (T^2)^{1/2} g, \ P := \langle U_b, \cdot \rangle U_b.$$  

Then, by functional calculus, we see that $\|U_b\| = 1, U_b \in D(T^{-1}) \cap D(T^2)$ and

$$T U_b = \sqrt{\lambda/D'(x_0)} T^{-1/2} g + (E_0^2 + x_0) T^{-1} U_b.$$  

Hence $P$ is a projection operator.

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Lemma 4.13. Let $\lambda \neq \lambda_c$. Then the following equations hold:

\begin{align}
\Omega^*_{\pm} \Omega_{\pm} &= I, \\
\Omega_{\pm} \Omega^*_{\pm} &= I - \theta(\lambda_c - \lambda)P,
\end{align}

where $\theta$ is the Heaviside function:

\[ \theta(t) = \begin{cases} 
1 & \text{if } t > 0, \\
0 & \text{if } t < 0.
\end{cases} \]

Remark 4.14. Lemma 4.13 implies that $\Omega_{\pm}$ are unitary operators if $\lambda > \lambda_c$ and partial isometries with their final subspace $\text{Ran}(I - P)$ if $\lambda < \lambda_c$.

Proof. (1) We first prove (4.9).

It is sufficient to prove that $R^*_\pm R_\pm = -(R_\pm + R^*_\pm)$ hold. For any $f, h \in \mathcal{H}$ and $\varepsilon > 0$,

\begin{align*}
\langle R^{(e)}_\pm h, R^{(e)}_\pm f \rangle &= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\langle h, E(\mu')T^{1/2} g \rangle d\langle T^{1/2} g, E(\mu'') f \rangle \\
&\quad \times \left\langle \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu^2 - E^2_0)} , \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu''^2 - E^2_0)} \right\rangle.
\end{align*}

By the definition of the function $D$, we have

\[ \lambda \langle T^{1/2} g, R_\varepsilon(T^2)T^{1/2}g \rangle = D(z - E^2_0) - 1, \quad z \in \mathbb{C}\backslash(E^2_0, \infty). \]

By this formula and the resolvent identity, we obtain

\begin{align*}
\langle R^{(e)}_\pm h, R^{(e)}_\pm f \rangle &= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\langle h, E(\mu')T^{1/2} g \rangle d\langle T^{1/2} g, E(\mu'') f \rangle \\
&\quad \times \frac{D(\mu^2 - E^2_0 \pm i\varepsilon) - D(\mu''^2 - E^2_0 \pm i\varepsilon)}{(\mu^2 - \mu''^2 \mp 2i\varepsilon)D_\pm(\mu^2 - E^2_0)D_\pm(\mu''^2 - E^2_0)} \\
&= -\left\langle E^{(e)}_\pm h, R^{(2e)}_\pm f \right\rangle - \left\langle R^{(2e)}_\pm h, E^{(e)}_\pm f \right\rangle,
\end{align*}

where the operators $E^{(e)}_\pm$ on $\mathcal{H}$ are given as follows:

\[ E^{(e)}_\pm := D(T^2 - E^2_0 \pm i\varepsilon)D_\pm(T^2 - E^2_0)^{-1}. \]

The inequality (4.4) implies that $E^{(e)}_\pm$ are bounded for all $0 < \varepsilon < \varepsilon_0$. Thus, by the Lebesgue dominated convergence theorem, we have $\text{s-lim}_{\varepsilon \to 0} E^{(e)}_\pm = I.$ Hence we obtain $R^*_\pm R_\pm = -(R_\pm + R^*_\pm)$.
(2) We next prove (4.10) for \( \lambda \neq \lambda_c \).

It is sufficient to prove that
\[
R_+^* R_+^* = -(R_+ + R_+^*) - \theta(\lambda_c - \lambda) P \quad \text{for any } f, h \in \mathcal{A}^c
\]
and a fixed \( \varepsilon > 0 \), (4.5) implies
\[
\left\langle R_+^{(\varepsilon)*} h, R_+^{(\varepsilon)*} f \right\rangle = \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\langle h, E(\mu)T^{1/2}g \rangle d\langle T^{1/2}g, E(\mu')f \rangle \times \langle R_{\mu^2 \pm i\varepsilon}(T^2)D_+(T^2 - E_0^2)^{-1}T^{1/2}g, R_{\mu^2 \pm i\varepsilon}(T^2)D_+(T^2 - E_0^2)^{-1}T^{1/2}g \rangle.
\]

Then, by operational calculus, we see that
\[
\left\langle R_+^{(\varepsilon)*} h, R_+^{(\varepsilon)*} f \right\rangle = \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu''} \frac{1}{\mu''} \left( \frac{1}{\mu'' - \mu^2 \pm i\varepsilon} - \frac{1}{\mu'' - \mu^2 \mp i\varepsilon} \right) d\|E(\mu'')g\|^2.
\]

where, for any \( \mu, \mu' \in [E_0, \infty) \),
\[
J_{\varepsilon}^\pm(\mu, \mu') \quad \text{is the curve given as follows:}
\]
\[
J_{\varepsilon}^\pm(\mu, \mu') = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{0}^{R} \left( \frac{1}{D_+(s)} - \frac{1}{D_-(s)} \right) C_{\mu, \mu'}(s)ds.
\]

Then, by a change of variable and (4.3), one can show that
\[
J_{\varepsilon}^\pm(\mu, \mu') = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{0}^{R} \left( \frac{1}{D_+(s)} - \frac{1}{D_-(s)} \right) C_{\mu, \mu'}(s)ds.
\]

where, for \( R > 0 \),
\[
I_{\varepsilon, R}^\pm(\mu, \mu') := \int_{0}^{R} \left( \frac{1}{D_+(s)} - \frac{1}{D_-(s)} \right) G_{\mu, \mu'}(s)ds
\]
and
\[
G_{\mu, \mu'}(z) := \frac{1}{z - \mu^2 + E_0^2 \pm i\varepsilon} - \frac{1}{z - \mu^2 + E_0^2 \mp i\varepsilon}, \quad z \in \mathbb{C}.
\]

For \( 0 < \eta < \varepsilon \) and \( R > 0 \), let \( C_i \) \( (i = 1, 2, 3) \) be the curve given as follows:
\[
C_1 : \quad \theta_1(t) = R - t - i\eta, \quad t : 0 \to R,
C_2 : \quad \theta_2(t) = \eta e^{-it}, \quad t : \pi/2 \to (3\pi)/2,
C_3 : \quad \theta_3(t) = t + i\eta, \quad t : 0 \to R.
\]
Then, for $C = C_1 + C_2 + C_3$, we have by the Lebesgue dominated convergence theorem,

$$I_{\varepsilon,R}^\pm(\mu, \mu') = \lim_{\eta \to 0} \int_C \frac{1}{D(z)} G^\pm_{\mu,\mu'}(z) dz.$$  

We take $R$ such that $R > \max\{\mu^2 - E_0^2, \mu'^2 - E_0^2\}$ and define a curve $C_4 : \theta_4(t) = \sqrt{\eta^2 + R^2 e^{-it}}, t : t_s \to t_f$, for $t_s := \arctan(\eta/R)$ and $t_f = 2\pi - t_s$. We consider two cases separately.

(i) The case $\lambda > \lambda_c$. In this case, the function $G^\pm_{\mu,\mu'}(z)/D(z), z \in \mathbb{C}\setminus(0, \infty)$ has two simple poles at $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu'^2 - E_0^2 \pm i\varepsilon$. Then, by the residue theorem, we have

$$\int_C \frac{1}{D(z)} G^\pm_{\mu,\mu'}(z) dz = 2\pi i \left( \frac{1}{D(\mu^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} \right) - \int_{C_4} \frac{1}{D(z)} G^\pm_{\mu,\mu'}(z) dz.$$  

Thus, as $\eta$ tends to 0, we have

$$I_{\varepsilon,R}^\pm(\mu, \mu') = 2\pi i \left( \frac{1}{D(\mu^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} \right) - \lim_{\eta \to 0} \int_{C_4} \frac{1}{D(z)} G^\pm_{\mu,\mu'}(z) dz.$$  

The definition of line integral implies

$$\int_{C_4} \frac{1}{D(z)} G^\pm_{\mu,\mu'}(z) dz = -i \int_{t_s}^{2\pi - t_s} \frac{G^\pm_{\mu,\mu'}(\sqrt{\eta^2 + R^2 e^{-it}}) \sqrt{\eta^2 + R^2 e^{-it}}}{D(\sqrt{\eta^2 + R^2 e^{-it}})} dt.$$  

By the triangle inequality, for any $t \in [t_s, t_f]$,

$$|G^\pm_{\mu,\mu'}(\sqrt{\eta^2 + R^2 e^{-it}})| \leq \frac{|\mu^2 - \mu'^2 \pm 2i\varepsilon|}{(R - |\mu^2 - E_0^2 \mp i\varepsilon|)(R - |\mu'^2 - E_0^2 \mp i\varepsilon|)}.$$  

On the other hand, by Lemma 4.5, (4.4) and the Lebesgue dominated convergence theorem, there are constants $\tilde{R} > 0$ and $c_0 > 0$ such that $|D(z)| \geq c_0$ for all $|z| \geq \tilde{R}$. Thus we have

$$I_{\varepsilon,R}^\pm(\mu, \mu') = 2\pi i \left( \frac{1}{D(\mu^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} \right) + O(R^{-1}) \ (R \to \infty),$$  

where $O(\cdot)$ stands for the well known Landau symbol. Therefore we have

$$J_{\varepsilon}^\pm(\mu, \mu') = \frac{1}{D(\mu^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)}$$
for each $\mu, \mu' \in [E_0, \infty)$. Thus, by (4.11), we have

$$\langle R^{(\varepsilon)}_\pm h, R^{(\varepsilon)}_\pm f \rangle = -\left( R^{(2\varepsilon)}_\pm h, (E^{(\varepsilon)}_\pm)^{-1} f \right) - \left( (E^{(\varepsilon)}_\pm)^{-1} h, (R^{(2\varepsilon)}_\pm)^* f \right).$$

As in the proof in (1), we obtain $\operatorname{s-lim}_{\varepsilon \downarrow 0} (E^{(\varepsilon)}_\pm)^{-1} = I$. Therefore we obtain

$$\lim_{\varepsilon \downarrow 0} \langle R^{(\varepsilon)}_\pm h, R^{(\varepsilon)}_\pm f \rangle = -\langle R^*_\pm h, f \rangle - \langle h, R^*_\pm f \rangle.$$

Thus we obtain the desired result.

(ii) The case $\lambda < \lambda_c$. In this case, $G^{\varepsilon, \pm}_{\mu, \mu'}(z)/D(z)$ has a simple pole at $z = x_0$ in addition to $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu^2 - E_0^2 \pm i\varepsilon$. The residue $R_0$ of $G^{\varepsilon, \pm}_{\mu, \mu'}(z)/D(z)$ at $z = x_0$ is given by

$$R_0 = \frac{1}{D'(x_0)} \frac{\mu^2 - \mu^2 \pm 2i\varepsilon}{(x_0 - \mu^2 + E_0^2 \mp i\varepsilon)(x_0 - \mu^2 + E_0^2 \pm i\varepsilon)}.$$

Thus we have

$$J^\pm_\varepsilon(\mu, \mu') = \frac{1}{D(\mu^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} + R_0$$

and also

$$\frac{\lambda}{\mu^2 - \mu^2 \mp 2i\varepsilon} R_0 = -\frac{\lambda}{D'(x_0)} \frac{\mu^2 - \mu^2 \mp 2i\varepsilon}{(\mu^2 - E_0^2 \mp x_0 \mp i\varepsilon)(\mu^2 - E_0^2 \pm x_0 \pm i\varepsilon)}.$$

This implies that

$$\lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu^2 - \mu^2 \mp 2i\varepsilon} R_0 \, d \langle h, E(\mu) T^{1/2} g \rangle \, d \langle T^{1/2} g, E(\mu') f \rangle$$

$$= -\langle h, U_b \rangle \langle U_b, f \rangle = -\langle h, P f \rangle.$$

Thus we obtain the desired result.

\[\Box\]

### 4.4 Operators $U$ and $V$

In this subsection, we investigate the operators $U$ and $V$ defined as follows:

$$U := \frac{1}{2}(T^{-1/2} \Omega_+ T^{1/2} + T^{1/2} \Omega_+ T^{-1/2}), \quad V := \frac{1}{2}(T^{-1/2} \Omega_+ T^{1/2} - T^{1/2} \Omega_+ T^{-1/2}),$$

which are used to construct a Bogoliubov transformation. Then, by Lemma 4.8, we can see that $D(U) = D(V) = D(T^{-1/2}) \cap D(T^{1/2})$. 

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Proof. By (4.6) and Lemma 4.8 we have
\[
\langle h, T^{-1/2} R_\pm T^{1/2} f \rangle = \frac{\lambda \pi}{2} \int_{(E_0, \infty)} - \left( \phi_{T^{1/2} g}^\pm \right) (\mu^2) \mp i \frac{\psi_{g,f}(\mu)}{D_\pm (\mu^2 - E_0^2)} \, d \langle h, E(\mu)g \rangle ,
\]
(4.12)
\[
\langle h, T^{1/2} R_\pm T^{-1/2} f \rangle = \frac{\lambda \pi}{2} \int_{(E_0, \infty)} - \left( \phi_{T^{-1/2} g}^\pm \right) (\mu^2) \mu \mp i \frac{\psi_{g,f}(\mu)}{D_\pm (\mu^2 - E_0^2)} \, d \langle h, E(\mu)g \rangle .
\]
(4.13)

By Assumption 3.3 (3), (4.7), (4.8) and a property of Hilbert transform, we can show that
\[
\|T^{-1/2} R_\pm T^{1/2} f\|, \|T^{1/2} R_\pm T^{-1/2} f\| \leq \frac{|\lambda| \pi (C_g + c_g)}{2 \delta} \|f\|,
\]
where
\[
C_g := (\sup_{E_0 < x} x^{-1} \psi_g(x))^{1/2} (\sup_{E_0 < x} x^2 \psi_g(x))^{1/2}.
\]
Hence the operators $T^{-1/2} R_\pm T^{1/2}$ and $T^{1/2} R_\pm T^{-1/2}$ are bounded.

In the same way as in the proof of Lemma 4.15, we see that $T^{-1/2} R_\pm^* T^{1/2}$ and $T^{1/2} R_\pm^* T^{-1/2}$ are bounded on each domain $D(T^{1/2})$ and $D(T^{-1/2})$. In what follows, we write the closed extensions of $U$ and $V$ by the same symbol respectively. Then
\[
U^* = \frac{1}{2} \left( T^{-1/2} \Omega_+^* T^{1/2} + T^{1/2} \Omega_+^* T^{-1/2} \right).
\]

Lemma 4.16. The operators $U^*$ and $V^*$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2})$, $D(T)$) invariant.

Proof. By applying Lemma 4.8 and using the equation
\[
U^* = I + \frac{1}{2} \left( T^{-1/2} R_\pm^* T^{1/2} + T^{1/2} R_\pm^* T^{-1/2} \right),
\]
one can easily see that the assertion for $U^*$ is true. The proof for $V^*$ is similar.

Lemma 4.17. Let $F(x) = x^{\pm 1/2}, x^{\pm 1}, a.e. \ x \in (0, \infty)$. Then
\[
\Omega_+ F(T) \Omega_+^* = (\Omega_+)^{\pm} F(T) (\Omega_+^*)^{\pm} \text{ on } D(F(T)).
\]
(4.14)

Proof. By Lemma 4.8, the domain of each side of (4.14) includes $D(F(T))$. By Lemmas 4.11 and 4.12, we have
\[
(\Omega_+)^{\pm} F(T) (\Omega_+^*)^{\pm} = R_+ F(T) R_+^* + R_+ \{(A_-)^* + I\} F(T) \gamma + F(T) \gamma^* (A_- + I) R_+^*
+ F(T) \{A_- (A_-)^* + A_- + (A_-)^* + I\}
= R_+ F(T) R_+^* + R_+ F(T) + F(T) R_+^* + F(T)
= \Omega_+ F(T) \Omega_+^*.
\]
5 Commutation relations

In this section, we prove that the pair \((U, V)\) satisfies the condition (1.1), \(V\) is Hilbert-Schmidt and

\[
B(f) := A(U f) + A(JV f)^*, \quad f \in \mathcal{H}
\]
satisfies some commutation relations with \(H(\lambda)\). We denote the closure of \(B(f)\) by the same symbol. By Lemma 4.16, we have \(D(d\Gamma_b(T)^{1/2}) \subset D(B(f)) \cap D(B(f)^*)\) for all \(f \in D(T^{-1/2})\).

**Theorem 5.1.** The following commutation relations hold:

1. For any \(f \in D(T)\) and \(\psi \in \mathcal{F}_{b,\text{fin}}(D(T))\),

\[
[H(\lambda), B(f)]\psi = -B(T f)\psi. \tag{5.1}
\]

2. For any \(f \in D(T^{-1/2}) \cap D(T)\) and \(\psi, \phi \in D(d\Gamma_b(T))\),

\[
\langle H(\lambda)\phi, B(f)\psi \rangle - \langle B(f)^*\phi, H(\lambda)\psi \rangle = -\langle \phi, B(T f)\psi \rangle. \tag{5.2}
\]

3. For any \(f \in D(T^{-1/2}) \cap D(T)\), \(B(f)\) maps \(D(d\Gamma_b(T)^{3/2})\) into \(D(d\Gamma_b(T))\) and for any \(\psi \in D(d\Gamma_b(T)^{3/2})\),

\[
[H(\lambda), B(f)]\psi = -B(T f)\psi. \tag{5.3}
\]

The both sides of (5.1),(5.2) and (5.3) have meaning by Lemma 4.16. To prove this theorem, we prove the following lemma:

**Lemma 5.2.** For any \(f \in D(T)\), the following equations hold:

\[
[U, T]f = (VT + TV) f = \frac{\lambda}{2} \langle D_-(T^2 - E_0^2)^{-1}g, f \rangle g, \tag{5.4}
\]

\[
(V^* J - U^*)g = -D_-(T^2 - E_0^2)^{-1}g. \tag{5.5}
\]

**Proof.** For any \(f, h \in D(T^{-1/2}) \cap D(T^{3/2})\), we obtain

\[
\langle h, [U, T]f \rangle = \frac{1}{2} \left( \langle T^{1/2} R^\pm_+ T^{-1/2} h, T f \rangle - \langle T h, T^{1/2} R^\pm_+ T^{-1/2} f \rangle \right).
\]

Then, for each \(\varepsilon > 0\), we have

\[
\begin{align*}
\langle T^{1/2} R^\pm_+ T^{-1/2} h, T f \rangle - \langle T h, T^{1/2} R^\pm_+ T^{-1/2} f \rangle \\
= \lambda \int_{E_0} \int_{E_0} \frac{\mu^2 - \mu^2}{(\mu^2 - \mu^2 \pm i\varepsilon) D_\pm(\mu^2 - E_0^2)} d \langle h, E(\mu) g \rangle d \langle g, E(\mu') f \rangle \\
= \lambda \int_{E_0} \int_{E_0} \frac{1}{D_\pm(\mu^2 - E_0^2)} d \langle h, E(\mu) g \rangle d \langle E(\mu') g, f \rangle + i\varepsilon \langle T^{-1/2} h, R^\pm_+ T^{-1/2} f \rangle.
\end{align*}
\]
Taking the limit $\varepsilon \downarrow 0$, we have
\[
\langle T^{1/2} R^*_\pm T^{-1/2} h, T f \rangle - \langle Th, T^{1/2} R_\pm T^{-1/2} f \rangle = \langle h, \lambda \langle D_\pm (T^2 - E_0^2)^{-1} g, f \rangle \rangle.
\]
Thus we have
\[
\langle h, [U, T] f \rangle = \frac{\lambda}{2} \langle h, \langle D_-(T^2 - E_0^2)^{-1} g, f \rangle \rangle.
\]
Since $D(T^{-1/2}) \cap D(T^{3/2})$ is a core of $T$, the equation (5.4) holds for $f \in D(T)$. To prove (5.5), we note that
\[
(V^* J - U^*)g = \frac{1}{2}(T^{1/2} \Omega_+ T^{-1/2} J - T^{-1/2} \Omega_+^* T^{1/2} J - T^{1/2} \Omega_+^* T^{-1/2} - T^{-1/2} \Omega_+^* T^{1/2})g
\]
where we have used $Jg = g$. Thus, for any $f \in \mathcal{H}$, we obtain
\[
\langle f, (V^* J - U^*)g \rangle
\]
\[
= - \langle f, g \rangle - \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \langle f, R_{\mu^2+i\varepsilon}(T^2)D_-(T^2 - E_0^2)^{-1} g \rangle d\|E(\mu')T^{1/2}g\|^2
\]
\[
= - \langle f, g \rangle + \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu'}{\mu^2 - \mu^2 + i\varepsilon} d\|E(\mu')g\|^2 \frac{1}{D_-(\mu^2 - E_0^2)} \langle f, E(\mu)g \rangle
\]
\[
= - \langle f, g \rangle + \int_{[E_0, \infty)} \frac{D_-(\mu^2 - E_0^2) - 1}{D_-(\mu^2 - E_0^2)} d\langle f, E(\mu)g \rangle
\]
\[
= - \langle f, D_-(T^2 - E_0^2)^{-1} g \rangle.
\]
Hence (5.5) holds.

Proof of Theorem 5.1.

(1) By Lemma 4.16, for any $f \in D(T)$, $B(f)$ leaves $\mathcal{F}_{b,\text{fin}}(D(T))$ invariant and $H(\lambda)$ maps $\mathcal{F}_{b,\text{fin}}(D(T))$ into $\mathcal{F}_{b,\text{fin}}(\mathcal{H}) \subset D(B(f))$. Thus, by using (2.1) and (10.3), we have for any $\psi \in \mathcal{F}_{b,\text{fin}}(D(T))$,
\[
[H(\lambda), B(f)]\psi = \left\{-A(TUf) + A(TJVf)^* - \frac{\lambda}{\sqrt{2}} \langle f, (V^* J - U^*)g \rangle \Phi_s(g) \right\} \psi.
\]
Hence by Lemma 5.2, (5.1) holds.

(2) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we can see that, for any $f \in D(T^{-1/2})$, $D(d\Gamma_b(T)^{1/2}) \subset D(B(f))$. For any $\psi, \phi \in D(d\Gamma_b(T))$, there are sequences $\psi_n, \phi_n \in \mathcal{F}_{b,\text{fin}}(D(T))$, $n \in \mathbb{N}$ such that
\[ \psi_n \to \psi, \phi_n \to \phi, d\Gamma_b(T)\psi_n \to d\Gamma_b(T)\psi, d\Gamma_b(T)\phi_n \to d\Gamma_b(T)\phi \text{ as } n \to \infty, \text{ since } \mathcal{F}_{b,\text{fin}}(D(T)) \text{ is a core of } d\Gamma_b(T). \] By (1), we have

\[ \langle H(\lambda)\phi_n, B(f)\psi_k \rangle - \langle B(f)^*\phi_n, H(\lambda)\psi_k \rangle = -\langle \phi_n, B(Tf)\psi_k \rangle \]

for all \( n, k \in \mathbb{N} \) and \( f \in D(T^{-1/2}) \cap D(T) \). By the inequalities (10.1) and (10.2) and the \( d\Gamma_b(T) \)-boundedness of \( \Phi_s(g)^2 \), we obtain that \( \{ B(f)\psi_n \}_{n=1}^\infty \), \( \{ B(f)^*\phi_n \}_{n=1}^\infty \), \( \{ \Phi_s(g)^2\psi_n \}_{n=1}^\infty \), \( \{ \Phi_s(g)^2\phi_n \}_{n=1}^\infty \) and \( \{ B(Tf)\psi_n \}_{n=1}^\infty \) converge. Hence we obtain (5.2).

(3) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we see that, for any \( f \in D(T^{-1/2}) \cap D(T), B(f) \) maps \( D(d\Gamma_b(T)^{3/2}) \) into \( D(d\Gamma_b(T)) \). Therefore, by (5.2) and the density of \( D(d\Gamma_b(T)) \), we have (5.3). \( \square \)

### 5.1 Relations between \( U \) and \( V \)

**Lemma 5.3.** Let \( \lambda \neq \lambda_c \). Then the following equations hold:

\[
\begin{align*}
U^*U - V^*V &= I, \\
U_j^*V - V_j^*U &= 0, \\
UU^* - V_jV_j^* &= I - \theta(\lambda_c - \lambda)Q_+, \\
UV^* - V_jU_j^* &= \theta(\lambda_c - \lambda)Q_-,
\end{align*}
\]

where

\[ Q_\pm := \frac{1}{2} \left( \langle T^{1/2}U_b, \cdot \rangle T^{-1/2}U_b \pm \langle T^{-1/2}U_b, \cdot \rangle T^{1/2}U_b \right) \]

are bounded operators on \( \mathcal{H} \).

**Proof.** It is sufficient to prove (5.6) on \( D(T^{-1/2}) \cap D(T^{1/2}) \). Using (4.9), one can show that the first equation in (4.9) holds. We have

\[ U_j^*V - V_j^*U = \frac{1}{2}(-T^{1/2}(\Omega_+^*)_j\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_j\Omega_+T^{1/2}). \]

Multiplying the equation by \((\Omega_+)_j\) from the left, and using Lemma 4.17, we obtain

\[ (\Omega_+)_j(U_j^*V - V_j^*U) = (\Omega_+)_j(-T^{1/2}(\Omega_+^*)_j\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_j\Omega_+T^{1/2}) \]
\[ = \Omega_+(-T^{1/2}\Omega_+^*\Omega_+T^{-1/2} + T^{-1/2}\Omega_+^*\Omega_+T^{1/2}) = 0. \]

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By (4.9), this implies that \( U^*_j V - V^*_j U = 0 \). By Lemma 3.5 and Lemma 4.17, we have

\[
V_j V_j^* = \frac{1}{4} \{ T^{-1/2} (\Omega_+ T \Omega_+^*) T^{-1/2} - T^{-1/2} (\Omega_+^* T \Omega_+^* \mathcal{J}) T^{1/2} \} - T^{1/2} \Omega_+ T \Omega_+^* T^{-1/2} + T^{1/2} \Omega_+^* T \Omega_+^* T^{1/2} - T^{1/2} \Omega_+ \Omega_+^* T^{-1/2} + T^{1/2} \Omega_+^* \Omega_+^* T^{1/2} \}
\]

\[
= \frac{1}{4} (T^{-1/2} \Omega_+ T \Omega_+^* T^{-1/2} - T^{-1/2} \Omega_+^* T \Omega_+^* T^{1/2})
\]

\[
= VV^*.
\]

Hence, by direct calculations and (4.10), one obtains \( UU^* - V_j V_j^* = I - \theta (\lambda_c - \lambda) Q_+ \). Similarly, one can prove the last equation in (5.6) (note that \( P_j = P \)).

\[\square\]

### 5.2 Hilbert-Schmidtness of \( V \)

In this subsection, we show that \( V \) is Hilbert-Schmidt. Then we can use Theorem 2.2 in the case of \( \lambda > \lambda_c \).

**Lemma 5.4.** The operator \( V \) is Hilbert-Schmidt.

**Proof.** On \( D(T^{-1/2}) \cap D(T^{1/2}) \), \( V^* V \) is calculated as follows:

\[
V^* V = \frac{1}{4} \{ T^{-1/2} R_+ T^{1/2} + T^{1/2} R_+^* T^{-1/2} + T^{1/2} [R_+^*, T] R_+ T^{1/2} \}
\]

\[
+ T^{1/2} R_+ T^{-1/2} + T^{-1/2} R_+^* T^{1/2} + T^{-1/2} [R_+^*, T] R_+ T^{-1/2} \}
\]

\[
+ T^{1/2} R_+^* R_+ T^{-1/2} + T^{-1/2} R_+^* R_+ T^{1/2} \}
\]

\[
= \frac{1}{4} (T^{1/2} [R_+^*, T] R_+ T^{1/2} + T^{-1/2} [R_+^*, T] R_+ T^{-1/2} \}
\]

where we have used the formula \( R_+^* R_+ = -(R_+ + R_+^*) \) in the proof of Lemma 4.13 and Lemma 4.8. Thus, for any \( f \in D(T^{-1/2}) \cap D(T^{1/2}) \) and \( \varepsilon > 0 \), we have

\[
\left\langle f, (T^{1/2} [R_+^* , T^{-1} ] R_+^* T^{1/2} + T^{-1/2} [R_+^* , T ] R_+^* T^{-1/2} ) f \right\rangle
\]

\[
= \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\lambda \mu'}{\mu^2 - \mu^2 + i \varepsilon} d \left( [T^{-1}, R_+^* ] T^{1/2} f, E(\mu) T^{1/2} g \right) d \left( E(\mu') g, f \right)
\]

\[
+ \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\lambda \mu}{\mu^2 - \mu^2 + i \varepsilon} d \left( [T, R_+^* ] T^{-1/2} f, E(\mu) T^{-1/2} g \right) d \left( E(\mu') g, f \right).
\]

Then, for any \( B \in \mathcal{B}^1 \), we can see

\[
\left\langle [T^{-1}, R_+^* ] T^{1/2} f, E(B) T^{1/2} g \right\rangle
\]

\[
= \lambda \int_B \int_{[E_0, \infty)} \frac{\mu'' - \mu}{\mu''^2 - \mu^2 - i \varepsilon} d \left( f, E(\mu'') g \right) d \left( E(\mu) g \right) d \| E(\mu) g \|^2.
\]
Similarly, we obtain
\[
\left\langle [T, R_+^{(e)}]T_{-1/2}f, E(B)T_{-1/2}g \right\rangle
= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu - \mu''}{(\mu'' - \mu^2 - i\varepsilon)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)} d\left\langle f, E(\mu'')g \right\rangle d\|E(\mu)g\|^2.
\]
Thus, by the formula of a change of variable in Lebesgue-Stieltjes integration and Fubini’s theorem, we have
\[
\left\langle f, (T^{1/2}[R_+^{(e)}, T^{-1}]R_+^{(e)}T_{1/2} + T^{-1/2}[R_+^{(e)} T]R_+^{(e)}T_{-1/2})f \right\rangle
= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu)g\|^2 d\left\langle f, E(\mu'')g \right\rangle d\left\langle E(\mu'), f \right\rangle
\times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'' - \mu^2 + i\varepsilon)(\mu'' - \mu^2 - i\varepsilon)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)}.
\]
Then it is easy to see that for any \( \mu, \mu', \mu'' \in [E_0, \infty) \),
\[
\lim_{\varepsilon \downarrow 0} \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'' - \mu^2 \pm i\varepsilon)(\mu'' - \mu^2 - i\varepsilon)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)} = \frac{1}{(\mu' + \mu)(\mu'' + \mu)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)}.
\]
For any \( \varepsilon > 0 \) and \( \mu, \mu', \mu'' \in [E_0, \infty) \), we have, by Lemma 4.5 and the arithmetic-geometric mean inequality,
\[
\left| \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'' - \mu^2 \pm i\varepsilon)(\mu'' - \mu^2 - i\varepsilon)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)} \right| \leq \frac{1}{4\delta^2 \mu \sqrt{\mu' \mu''}}.
\]
On the other side, for any \( \alpha, \beta \in \mathbb{C} \), we see
\[
\int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu \sqrt{\mu' \mu''}} d\|E(\mu)g\|^2 d\|E(\mu')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2
= \|T_{-1/2}g\|^2 \|T_{-1/4}(f + \alpha g)\|^2 \|T_{-1/4}(f + \beta g)\|^2 < \infty.
\]
Thus, by the Lebesgue dominated convergence theorem, we have
\[
\lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu)g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2
\times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'' - \mu^2 \pm i\varepsilon)(\mu'' - \mu^2 - i\varepsilon)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)}
= \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu)g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2
\times \frac{1}{(\mu' + \mu)(\mu'' + \mu)D_+(\mu'' - E_0^2)D_- (\mu'' - E_0^2)}.
\]
In particular, for each \( \alpha, \beta = \pm 1, \pm i \), the polarization identity and Fubini’s theorem give
\[
\langle f, V^* V f \rangle = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \left| \langle f, R_{-\mu}(T) D_-(T^2 - E_0^2) g \rangle \right|^2 d\|E(\mu)g\|^2.
\]

Let \( \{e_n\}_{n=1}^{\infty} \subset D(T^{-1/2}) \cap D(T^{1/2}) \) be a CONS of \( \mathcal{H} \). The termwise integration implies that
\[
\sum_{n=1}^{\infty} \langle e_n, V^* V e_n \rangle = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \|R_{-\mu}(T) D_-(T^2 - E_0^2)^{-1} g \| d\|E(\mu)g\|^2
\]
\[
= \frac{\lambda^2}{4} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{(\mu' + \mu)^2 |D_-(\mu^2 - E_0^2)|^2} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 \quad (5.8)
\]
\[
\leq \frac{\lambda^2}{16\delta^2} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu'\mu} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 < \infty,
\]
where we have used the arithmetic-geometric mean inequality and Lemma 4.5. Hence \( V \) is Hilbert-Schmidt.

\[ \square \]

**Lemma 5.5.** If \( \lambda > \lambda_c \), then there is a unitary operator \( U \) on \( \mathcal{F}_b(\mathcal{H}) \) such that for all \( f \in \mathcal{H} \),
\[
UB(f)U^{-1} = A(f).
\]

**Proof.** By Lemma 5.3 and Lemma 5.4, we can apply Theorem 2.2. \[ \square \]

### 6 Analysis in the case \( \lambda > \lambda_c \)

In this section we prove Theorem 3.6 (1). Before starting the proof, we need to know a property of the Hamiltonian \( H(\lambda) \).

#### 6.1 Time evolution

**Theorem 6.1** (Time evolution). If \( \lambda > \lambda_{c,0} \), then for all \( f \in D(T^{-1/2}) \), \( \psi \in D(d\Gamma_b(T)^{1/2}) \) and \( t \in \mathbb{R} \),
\[
e^{itH(\lambda)}B(f)e^{-itH(\lambda)}\psi = B(e^{itT}f)\psi, \quad (6.1)
\]
\[
e^{itH(\lambda)}B(f)^*e^{-itH(\lambda)}\psi = B(e^{itT}f)^*\psi. \quad (6.2)
\]

**Proof.** It is sufficient to prove (6.1), because (6.2) follows from taking the adjoint of (6.1). We define a function \( v : \mathbb{R} \to \mathbb{C} \) by \( v(t) := \langle \phi, e^{itH(\lambda)}B(e^{-itT}f)e^{-itH(\lambda)}\psi \rangle, t \in \mathbb{R} \) for any \( f \in D(T^{-1/2}) \cap D(T) \) and \( \psi, \phi \in D(d\Gamma_b(T)) \). Then \( v \) is well-defined by an operational
calculus and Theorem 2.3. The function \( v \) is differentiable and, by Theorem 5.1 (2), we have for any \( t \in \mathbb{R} \),
\[
\frac{d}{dt} v(t) = i \langle H(\lambda) e^{-iH(\lambda)T} \phi, B(e^{-iH(\lambda)T} f) e^{-iH(\lambda)\psi} \rangle - i \langle B(e^{-iH(\lambda)T} f)^* e^{-iH(\lambda)\phi}, H(\lambda) e^{-iH(\lambda)\psi} \rangle \\
+ i \langle e^{-iH(\lambda)\phi}, B(T e^{-iH(\lambda)T} f) e^{-iH(\lambda)\psi} \rangle \\
= 0.
\]
Hence \( v(t) = v(0) \) for all \( t \in \mathbb{R} \). Hence the equation
\[
\langle \phi, e^{iT(\lambda)} B(e^{-iT(\lambda)} f) e^{-iT(\lambda)} \psi \rangle = \langle \phi, B(\psi) \rangle
\]
holds for all \( t \in \mathbb{R} \). By replacing \( f \) with \( e^{i\lambda T} f \), one has for all \( \psi \in D(d\Gamma_b(T)) \),
\[
e^{iT(\lambda)} B(\psi) e^{-iT(\lambda)} B(e^{i\lambda T} f) = B(e^{i\lambda T} f) \psi.
\]
Since \( D(d\Gamma_b(T)) \) is a core of \( (H(\lambda) + M)^{1/2} \) and \( D(H(\lambda) + M)^{1/2} = D(d\Gamma_b(T)^{1/2}) \) by Theorem 2.3 (3), we obtain (6.1) for \( f \in D(T^{-1/2}) \cap D(T) \) and \( \psi \in D(d\Gamma_b(T)^{1/2}) \). Finally we extend (6.1) for all \( f \in D(T^{-1/2}) \). Let \( f \in D(T^{-1/2}) \) and \( \psi \in D(d\Gamma_b(T)^{1/2}) \). Then we set \( f_n := E((-\infty, n]) f \) for each \( n \in \mathbb{N} \). Then \( f_n \in D(T^{-1/2}) \cap D(T) \) for all \( n \in \mathbb{N} \) and one can easily show that \( f_n \to f, T^{-1/2} f_n \to T^{-1/2} f \) as \( n \to \infty \) by using a functional calculus and the Lebesgue dominated convergence theorem. Thus we have \( U f_n \to U f, JV f_n \to JV f \) as \( n \to \infty \) by the boundedness of \( U \) and \( V \). By using the linearity of the Hilbert transform and that of the map \( f \mapsto \psi_{g,f} \), (4.12), (4.13) and (4.7), we can show that \( T^{-1/2} U f_n \to T^{-1/2} U f, T^{-1/2} JV f_n \to T^{-1/2} JV f \) as \( n \to \infty \). Therefore we obtain \( B(f_n) \phi \to B(f) \phi \) and \( B(e^{iT} f_n) \phi \to B(e^{iT} f) \phi \) as \( n \to \infty \) for any \( \phi \in D(d\Gamma_b(T)^{1/2}) \) by [3, Lemma 4-28]. By the preceding result, we have for any \( n \in \mathbb{N} \),
\[
B(f_n) e^{-iH(\lambda)\psi} = e^{-iH(\lambda)\psi} B(e^{iT} f_n) \psi.
\]
The equation \( D(d\Gamma_b(T)^{1/2}) = D((H(\lambda) + M)^{1/2}) \) in Theorem 2.3 (3) implies that
\[
e^{-iH(\lambda)\lambda} D(d\Gamma_b(T)^{1/2}) = D(d\Gamma_b(T)^{1/2}).
\]
Hence, by taking the limit \( n \to \infty \), we obtain (6.1) for \( f \in D(T^{-1/2}), \psi \in D(d\Gamma_b(T)^{1/2}) \).  

### 6.2 Proof of Theorem 3.6 (1)

In this subsection, we assume that \( \lambda > \lambda_c \).

**Lemma 6.2.** Let \( \Omega := U^{-1} \Omega_0 \), where \( U \) is the unitary operator in Lemma 5.5. Then there is an eigenvalue \( E_g \) of \( H(\lambda) \) and \( \Omega \) is the corresponding eigenvector: \( H(\lambda) \Omega = E_g \Omega \).

\[32\]
Proof. In general, by [3, Proposition 4-4] for a dense subspace \( \mathcal{D} \subset \mathcal{H} \), if \( \psi \in \cap_{f \in \mathcal{D}} \mathcal{D}(A(f)) \) satisfies \( A(f)\psi = 0 \) for all \( f \in \mathcal{D} \), then there is a constant \( \alpha \in \mathbb{C} \) such that \( \psi = \alpha \Omega_0 \). Thus, by Lemma 5.5, if \( B(f)\phi = 0 \) for all \( f \in D(T^{-1/2}) \), there is a constant \( \alpha \in \mathbb{C} \) such that \( \phi = \alpha \Omega \). For any \( f \in D(T^{-1/2}) \) and \( t \in \mathbb{R} \),
\[
B(f)e^{-itH(\lambda)}\Omega = e^{-itH(\lambda)}B(e^{it}f)\Omega = 0
\]
by Lemma 5.5 and Theorem 6.1. Thus, for each \( t \in \mathbb{R} \), there is a constant \( \alpha(t) \in \mathbb{C} \) such that \( e^{-itH(\lambda)}\Omega = \alpha(t)\Omega \). Then we have \( |\alpha(t)| = 1, \alpha(t + s) = \alpha(t)\alpha(s) \) for all \( t, s \in \mathbb{R} \), since \( \{e^{-itH(\lambda)}\}_{t \in \mathbb{R}} \) is a strongly continuous one-parameter unitary group. Thus there exists a constant \( E_g \in \mathbb{R} \) such that \( \alpha(t) = e^{-itE_g}, t \in \mathbb{R} \). The differentiation of the equation \( e^{-itH(\lambda)}\Omega = e^{-itE_g}\Omega \) in \( t \) implies that \( \Omega \in \mathcal{D}(H(\lambda)) \) and \( \Omega \in \text{Ker}(H(\lambda) - E_g) \). \( \square \)

Proof of Theorem 3.6 (1). The subspace \( \mathcal{U} := \mathcal{L}(\{B(f_1)^* \cdots B(f_n)^*\Omega, \Omega \mid f_j \in D(T^{-1/2}), j = 1, \ldots, n, n \in \mathbb{N}\}) \) is dense in \( \mathcal{F}_b(\mathcal{H}) \) by the fact that \( \mathcal{U} = \bigcup_{t \in \mathbb{R}} \mathcal{F}_{b, \text{fin}}(D(T^{-1/2})) \), where \( \mathcal{L}(\mathcal{D}) \) denotes the subspace algebraically spanned by the vectors in a subset \( \mathcal{D} \) of a Hilbert space. By Lemma 6.1 and Lemma 10.3, for any \( t \in \mathbb{R} \) and \( f_j \in D(T^{-1/2}), j = 1, \ldots, n \), we have
\[
e^{itH(\lambda)}B(f_1)^* \cdots B(f_n)^*\Omega = B(e^{it}f_1)^* \cdots B(e^{it}f_n)^*e^{itH(\lambda)}\Omega
\]
\[
= B(e^{it}f_1)^* \cdots B(e^{it}f_n)^*e^{itE_g}\Omega
\]
\[
= e^{itE_g} \mathcal{U}^{-1} e^{itd\Gamma_b(T)} A(f_1)^* \cdots A(f_n)^*\Omega_0
\]
\[
= \mathcal{U}^{-1} e^{it(d\Gamma_b(T) + E_g)} \mathcal{U} B(f_1)^* \cdots B(f_n)^*\Omega.
\]
By this equation and a limiting argument, we obtain \( \mathcal{U} e^{itH(\lambda)} \mathcal{U}^{-1} = e^{it(d\Gamma_b(T) + E_g)} \). By the unitary covariance of functional calculus, we have
\[
\mathcal{U} e^{itH(\lambda)} \mathcal{U}^{-1} = e^{itH(\lambda)}\mathcal{U}^{-1}, \quad t \in \mathbb{R}.
\]
Hence (3.1) holds. The equation (3.1) and the well-known spectral properties of \( d\Gamma_b(T) \) imply that \( E_g \) is the ground state energy of \( H(\lambda) \) and \( \Omega \) is the unique ground state of \( H(\lambda) \). \( \square \)

Lemma 6.3. The ground state energy \( E_g \) is given as follows:
\[
E_g = \frac{\lambda}{4} \|g\|^2 - \text{Tr}(T^{1/2}V^*VT^{1/2}), \quad (6.3)
\]
\[
\text{Tr}(T^{1/2}V^*VT^{1/2}) = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu}{(\mu + \mu')^2[D_-(\mu^2 - E_0^2)]^2} d\|E(\mu)g\|^2 d\|E(\mu')g\|^2. \quad (6.4)
\]
Proof. The operator $U$ leaves $D(d\Gamma_b(T))$ invariant by Theorem 3.6 (1). In particular, $U\Omega_0 \in D(d\Gamma_b(T)^{1/2})$. Thus, by Lemma 10.4, the isometricity of $U$ and the definition of $B(\cdot)$, we have $\langle \Omega_0, (H(\lambda) - E_\delta)\Omega_0 \rangle = \text{Tr}(T^{1/2}V^*VT^{1/2})$. By the definition of $H(\lambda)$ and (2.1), we have $\langle \Omega_0, H(\lambda)\Omega_0 \rangle = \lambda\|g\|^2/4$. Hence (6.3) holds. The formula (6.4) can be proved in the same way as (5.8). \hfill $\Box$

7 Analysis in the case $\lambda_{c,0} < \lambda < \lambda_c$

In Section 5, we proved Theorem 3.6 (1). But the proof is valid only for the case $\lambda > \lambda_c$. Therefore it is necessary to find another pair of operators $U$ and $V$ if one wants to use a Bogoliubov transformation for the spectral analysis of $H(\lambda)$ in the case $\lambda \leq \lambda_c$. In this section we assume that $T$ and $g$ satisfy Assumption 3.3, $E_0 > 0$ and $\lambda_{c,0} < \lambda < \lambda_c$. Under these conditions, we can define the operators $\xi, X, Y$ and $T_\pm$ as follows:

\[
\xi := \Omega_+ T\Omega_+^* + \beta P, \\
X := U\Omega_+^* + T_+ P, \quad Y := V\Omega_+^* + T_- P, \\
T_\pm := \frac{1}{2}(\beta^{1/2}T^{-1/2} \pm \beta^{-1/2}T^{1/2}),
\]

where $\beta := (E_0^2 + x_0)^{1/2}$.

Remark 7.1. The definition of $x_0$ implies that

\[
E_0^2 + x_0 \begin{cases} 
> 0, & \text{if } \lambda_{c,0} < \lambda < \lambda_c, \\
= 0, & \text{if } \lambda = \lambda_{c,0}, \\
< 0, & \text{if } \lambda < \lambda_{c,0}.
\end{cases}
\]

Thus, in the case $\lambda_{c,0} < \lambda < \lambda_c$, we see that the inequality $0 < \beta < E_0$ holds. Let

\[
C(f) := A(Xf) + A(JYf)^*, \quad f \in \mathcal{H}.
\]

Then $C(f)$ is a densely defined closable operator. We denotes its closure by the same symbol.

7.1 Properties of $X, Y$ and $\xi$

In this subsection, we study the operators $X, Y$ and $\xi$. Firstly, we consider $\xi$. Let

\[
\tilde{T} := \Omega_+ T\Omega_+^*.
\]

Lemma 7.2. The operator $\tilde{T}$ is a self-adjoint operator with $D(\tilde{T}) = D(T)$. 

Proof. By Lemma 4.8 we see that \( D(\bar{T}) = D(T) \). Hence \( \bar{T} \) is symmetric. For any \( \phi \in D((\bar{T})^*) \) and \( \psi \in D(T) = D(\bar{T}) \), we have \( \langle \Omega_+^*(\bar{T})^*\phi, \psi \rangle = \langle \Omega_+^*\phi, T\psi \rangle \). This implies that \( \Omega_+^*\phi \in D(T) \). Hence \( \bar{T} \) is self-adjoint. \( \Box \)

**Lemma 7.3.** The spectra of \( \bar{T} \) are as follows:

\[
\sigma(\bar{T}) = \{0\} \cup \sigma(T), \sigma_{ac}(\bar{T}) = \sigma(T), \sigma_p(\bar{T}) = \{0\}, \sigma_{sc}(\bar{T}) = \emptyset.
\]

**Proof.** We define a family of projection operators \( \{E_p(B) \mid B \in B^1\} \) on \( \mathcal{H} \) as follows: \( E_p(B) = 0 \) if \( 0 \notin B \) and \( E_p(B) = P \) if \( 0 \in B \) for each \( B \in B^1 \). By the definition of the spectral measure, we can see that \( \{E_\bar{T}(B) := \Omega_+E(B)\Omega_+^* + E_p(B) \mid B \in B^1\} \) is a spectral measure. Using a functional calculus, we see that \( E_\bar{T}(-) \) is the spectral measure of \( \bar{T} \). The absolutely continuous part (resp. singular part) of \( \Omega_+ \) is \( \Omega_+ \) is a ground state of \( \bar{T} \). By Lemma 4.8 we see that \( \Omega_+ \) is the unique ground state of \( \bar{T} \). Thus \( \Omega_+ \) is a ground state of \( \bar{T} \). Hence \( \Omega_+ \) is self-adjoint.

We next show that \( \sigma_{ac}(\bar{T}) = \sigma(T) \). For any \( \mu \in \sigma(T) \), there is a sequence \( \psi_n \in D(T), n \in \mathbb{N} \) such that \( \|\psi_n\| = 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \|(T - \mu)\psi_n\| = 0 \). For each \( n \in \mathbb{N} \), there is a \( \phi_n \in \text{Ran}(I - P) \) such that \( \psi_n = \Omega_+^*\phi_n \). Then \( \|\phi_n\| = \|\Omega_+\psi_n\| = \|\psi_n\| = 1 \) and \( \|(\bar{T} - \mu)\phi_n\| = \|(T - \mu)\psi_n\| \to 0 \) as \( n \to \infty \). Thus we have \( \mu \in \sigma(\bar{T} \upharpoonright \text{Ran}(I - P)) = \sigma_{ac}(\bar{T}) \).

For any \( \mu \in \sigma_{ac}(\bar{T}) \), there is a sequence \( \eta_n \in D(\bar{T}) \cap \text{Ran}(I - P) \) such that \( \|\eta_n\| = 1 \) and \( \lim_{n \to \infty} \|(\bar{T} - \mu)\eta_n\| = 0 \). Then we easily see that \( \Omega_+\eta_n \in D(T) \) for all \( n \in \mathbb{N} \). The equation \( \Omega_+\Omega_+^*\eta_n = \eta_n \) implies that \( \|\Omega_+^*\eta_n\| = 1 \) for all \( n \in \mathbb{N} \) and

\[
\|(T - \mu)\Omega_+^*\eta_n\| = \|(\bar{T} - \mu)\eta_n\| \to 0, \quad n \to \infty.
\]

Thus \( \mu \in \sigma(T) \). Hence \( \sigma_{ac}(\bar{T}) = \sigma(T) \). \( \Box \)

**Lemma 7.4.** The operator \( \xi \) is an injective, non-negative self-adjoint operator with \( D(\xi) = D(T) \) and we have the following equations:

\[
\sigma(\xi) = \{\beta\} \cup \sigma(T), \sigma_{ac}(\xi) = \sigma(T), \sigma_p(\xi) = \{\beta\}, \sigma_{sc}(\xi) = \emptyset.
\]

In particular, \( \beta \) is the ground state energy of \( \xi \), which is an isolated eigenvalue of \( \xi \), and \( U_b \) is the unique ground state of \( \xi \).

**Proof.** By Lemma 7.3 and the spectral property of direct sum of self-adjoint operators, we have the equation (7.1). Thus \( \beta \) is an isolated ground state energy by Remark 7.1. By \( \Omega_+U_b = 0 \), \( U_b \) is a ground state of \( \xi \). Assume that \( f \in \text{Ker}(\xi - \beta) \) satisfies \( (I - P)f \neq 0 \). Then \( \Omega_+f \neq 0 \) by Lemma 4.13. This implies that \( T\Omega_+f = \beta\Omega_+f \), but this contradicts Assumption 3.3 (1). Hence \( (I - P)f = 0 \) and this implies that the ground state of \( \xi \) is unique. \( \Box \)
Lemma 7.5. The operators $\xi^{\pm 1/2}$ are given by
\[
\begin{align*}
\xi^{1/2} &= \Omega_+ T^{1/2} \Omega_+^* + \beta^{1/2} P, \\
\xi^{-1/2} &= \Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P
\end{align*}
\] (7.2) (7.3)
with $D(\xi^{\pm 1/2}) = D(T^{\pm 1/2})$.

Proof. We can show in the same way as in the proof of Lemma 7.4 that the right hand side of (7.2) is non-negative, self-adjoint operator with its domain $D(T^{1/2})$. We have $\xi \subset (\Omega_+ T^{1/2} \Omega_+^* + \beta^{1/2} P)^2$. Since a self-adjoint operator has no non-trivial symmetric extension, (7.2) holds. In the same way as in the proof of (7.2), we can show that the right hand side of (7.3) is a self-adjoint operator. We have $D(\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P) \subset \text{Ran}(\xi^{1/2})$ and $\xi^{1/2}(\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P) = I$ on $D(\Omega_+ T^{-1/2} \Omega_+^*)$. Hence $\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P \subset \xi^{-1/2}$. Thus the equation (7.3) holds.

Next, we study $X$ and $Y$.

Lemma 7.6. The operators $X^2$ and $Y^2$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2})$, $D(T)$) invariant.

Proof. The assertion follows from Lemma 4.8, Lemma 4.16, Lemma 7.5 and the definitions of $X$ and $Y$.

Lemma 7.7. The following equations hold:
\[
\begin{align*}
X^* X - Y^* Y &= I, \\
X^*_j Y - Y^*_j X &= 0, \\
XX^* - Y_j Y^*_j &= I, \\
XY^* - Y_j X^*_j &= 0.
\end{align*}
\] (7.4)

Proof. The operator $P$ (resp. $T_\pm$) satisfies $P = P$ (resp. $(T_\pm)_J = T_\pm$). By (4.10), we have $\Omega_+^* U_b = 0$. Hence we obtain $(U^* \pm V^*) T^{\pm 1/2} U_b = 0$ and $(U^* T_\pm - V^* T_\mp) U_b = 0$. The equations $T_+ T_+ - T_- T_- = I$ and $T_+ T_- - T_- T_+ = 0$ hold on $D(T^{-1}) \cap D(T)$. By (5.6) and direct calculations, we have $X^* X - Y^* Y = I$ and $X^*_j Y - Y^*_j X = 0$. By similar calculations, we have $XX^* - Y_j Y^*_j = I$ and $XY^* - Y_j X^*_j = 0$ on $D(T^{-1/2}) \cap D(T^{1/2})$. Then, by a limiting argument, we obtain (7.4).

Lemma 7.8. The operator $Y$ is Hilbert-Schmidt.

Proof. We can easily show that the assertion follows from Lemma 5.4, Lemma 7.6 and the choice a CONS $\{e_n\}_{n=0}^\infty \subset D(T^{-1/2}) \cap D(T^{1/2})$ with $e_0 = U_b$.

Lemma 7.9. There is a unitary operator $\mathcal{V}$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $f \in \mathcal{H}$,
\[
\mathcal{V} C(f) \mathcal{V}^{-1} = A(f).
\]

Proof. By Theorem 2.2, (7.4) and Lemma 7.8, we can prove this assertion.
7.2 Commutation relations

Theorem 7.10. The following commutation relations hold:

1. For any \( f \in D(T) \) and \( \psi \in \mathcal{F}_{b,\text{fin}}(D(T)) \),
\[
[H(\lambda), C(f)]\psi = -C(\xi f)\psi.
\]

2. For any \( f \in D(T^{-1/2}) \cap D(T) \) and \( \psi, \phi \in D(d\Gamma_b(T)) \),
\[
\langle H(\lambda)\phi, C(f)\psi \rangle - \langle C(f)^*\phi, H(\lambda)\psi \rangle = -\langle \phi, C(\xi f)\psi \rangle.
\]

3. For any \( f \in D(T^{-1/2}) \cap D(T) \), \( C(f) \) maps \( D(d\Gamma_b(T)^{3/2}) \) into \( D(d\Gamma_b(T)) \) and for any \( \psi \in D(d\Gamma_b(T)^{3/2}) \),
\[
[H(\lambda), C(f)]\psi = -C(\xi f)\psi.
\]

Theorem 7.10 follows, in the same manner as in the proof of Theorem 5.1, from Lemma 4.16, Lemma 7.5 and the next lemma:

Lemma 7.11. For any \( f \in D(T) \) the following equations hold:

\[
-TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g = -X\xi f, \tag{7.5}
\]

\[
TJYf + \frac{\lambda}{2} \langle f, (Y^*J - X^*)g \rangle g = -JY\xi f. \tag{7.6}
\]

Remark 7.12. By Lemma 4.16 and the definition of \( \xi \), the both sides of (7.5) and (7.6) have meaning.

Proof. Let \( a := \sqrt{\lambda/D'(x_0)} \). Then we can see by the definition of \( x_0 \) and (5.5),
\[
(Y^*J - X^*)g = -\Omega_+D_-(T^2 - E_0^2)^{-1}g + \frac{\beta^{-1/2}a}{\lambda}U_b.
\]

We have
\[
TT_\pm U_b = \frac{1}{2}(\beta^{1/2}T^{1/2}U_b \pm \beta^{-1/2}T^{3/2}U_b)
\]
\[
= \frac{1}{2}(\beta^{1/2}T^{1/2}U_b \pm \beta^{3/2}T^{-1/2}U_b \pm \beta^{-1/2}ag). \tag{7.7}
\]

Thus, for any \( f \in D(T) \), we have
\[
-TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g
\]
\[
= -TU\Omega_+^*f - \frac{\lambda}{2} \langle D_-(T^2 - E_0^2)^{-1}g, \Omega_+^*f \rangle g - TT_+ Pf + \frac{\beta^{-1/2}a}{2} \langle U_b, f \rangle g.
\]
Then, by (5.4) and (7.7), we have
\[-T X f + \frac{\lambda}{2} \langle (Y^* J - X^*) g, f \rangle g = -U T \Omega_f^* f - \beta \langle U_f, f \rangle T_+ U_+ \]
\[= -X(\Omega_+ T \Omega_f^* + \beta P) f.\]
Thus we obtain (7.5). Similarly one can prove (7.6).

\[\square\]

7.3 Proof of Theorem 3.6 (2)

Theorem 7.13. For all \( f \in D(T^{-1/2}), \psi \in D(d \Gamma_!(T^{1/2})) \) and \( t \in \mathbb{R} \),
\[e^{i t H(\lambda)} C(f) e^{-i t H(\lambda)} \psi = C(e^{it} f) \psi,
\[e^{i t H(\lambda)} C(f)^* e^{-i t H(\lambda)} \psi = C(e^{it} f)^* \psi.\]

Proof. These are proved in the same way as in the proof of Theorem 6.1 by Theorem 7.10.

Lemma 7.14. Let \( \Omega := \mathbb{V}^{-1} \Omega_0 \) where \( \mathbb{V} \) is the unitary operator in Lemma 7.9. Then:

1. There is an eigenvalue \( \tilde{E}_g \) of \( H(\lambda) \) and \( \Omega \) is an eigenvector of \( H(\lambda) \) with the eigenvalue \( \tilde{E}_g \).

2. The following equation holds:
\[\mathbb{V} H(\lambda) \mathbb{V}^{-1} = d \Gamma_b(\xi) + \tilde{E}_g.\]

3. The constant \( \tilde{E}_g \) is given as follows:
\[\tilde{E}_g = E_g - \beta \| T_- U_b \|^2. \tag{7.8}\]

Proof. The assertions (1) and (2) can be proved in the same way as in the proof of Theorem 3.6 (1).

(3) We have
\[\tilde{E}_g = \frac{\lambda}{4} \| g \|^2 - \text{Tr}(\xi^{1/2} Y^* Y \xi^{1/2})\]
in the same way as in the proof of Lemma 6.2. Then, by Lemma 7.5, we have
\[\xi^{1/2} Y^* Y \xi^{1/2} = \Omega_+ T^{1/2} V^* V T^{1/2} \Omega_+ + \Omega_+ T^{1/2} V^* \beta^{1/2} T_- P + \beta^{1/2} P T_- V T^{1/2} \Omega_+ + \beta P T_- T_- P.\]

We choose a CONS \( \{ e_n \}_{n=0}^\infty \subset D(T) \) satisfying \( e_0 = U_b \). Then it is easy to see that \( \{ \Omega_+^* e_n \}_{n=1}^\infty \) is a CONS of \( \mathcal{H} \) by Lemma 4.13. Hence we have
\[\text{Tr}(\xi^{1/2} Y^* Y \xi^{1/2}) = \sum_{n=1}^\infty \langle e_n, \Omega_+ T^{1/2} V^* V T^{1/2} \Omega_+^* e_n \rangle + \beta \| T_- U_b \|^2 \]
\[= \text{Tr}(T^{1/2} V^* V T^{1/2}) + \beta \| T_- U_b \|^2.\]

Thus we obtain (7.8).
In particular, \( H(\lambda) \) have eigenvectors as follows:

\[
\phi_n := \mathcal{V}^{-1}A(U_b)^n\Omega_0, \quad H(\lambda)\phi_n = (n\beta + \tilde{E}_\phi)\phi_n, \quad n \in \mathbb{N} \cup \{0\}.
\]

Hence the spectral properties of \( H(\lambda) \) as stated in Theorem 3.6 (2) follow.

### 8 Analysis in the case \( \lambda < \lambda_{c,0} \)

In this section, we show that \( H(\lambda) \) is unbounded from above and below.

**Theorem 8.1.** Let \( g \in D(T^{-1/2}) \). Then \( H(\lambda) \) is unbounded above for any \( \lambda \in \mathbb{R} \). If \( \lambda < \lambda_{c,0} \), then \( H(\lambda) \) is unbounded below.

**Proof.** For any \( f \in D(T) \setminus \{0\} \), we set \( \psi_n := a_nA(f)^n\Omega_0, \quad a_n \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N} \cup \{0\} \). Then we have the following equations:

\[
\begin{align*}
\text{d}\Gamma_b(T)\psi_n &= n\frac{a_n}{a_{n-1}}A(Tf)^n\psi_{n-1}, \quad A(g)\psi_n = n\langle g, f \rangle \frac{a_n}{a_{n-1}}\psi_{n-1}, \\
\|\psi_n\|^2 &= |a_n|^2n!\|f\|^{2n}, \quad \|A(g)^n\psi_n\|^2 = \|g\|^2\|\psi_n\|^2 + \|A(g)\psi_n\|^2,
\end{align*}
\]

where \( \psi_{-1} := 0 \). Then we have

\[
\langle \psi_n, H(\lambda)\psi_n \rangle = \|\psi_n\|^2 \left( \frac{\lambda}{4}\|g\|^2 + n^2\frac{\|T^{1/2}f\|^2 + |\langle g, f \rangle|^2}{2\|f\|^2} \right).
\]

We take \( f \) such that \( \langle g, f \rangle = 0 \). Then we have \( \langle \psi_n, H(\lambda)\psi_n \rangle /\|\psi_n\|^2 \to \infty \) as \( n \to \infty \) for any \( \lambda \in \mathbb{R} \). Thus \( H(\lambda) \) is unbounded above for any \( \lambda \in \mathbb{R} \).

Let \( \phi_N := \sum_{n=0}^{N} \psi_n, N = 0, 1, 2, \ldots \) Then we have \( \|\phi_N\|^2 = \sum_{n=0}^{N} \|\psi_n\|^2 \) and

\[
\langle \phi_N, H(\lambda)\phi_N \rangle = \sum_{n=2}^{N} \|\psi_n\|^2 \left( \frac{\lambda\|g\|^2}{4} + n^2\frac{\|T^{1/2}f\|^2 + \lambda|\langle g, f \rangle|^2}{2\|f\|^2} \right) + \|\psi_1\|^2 \left( \frac{\lambda\|g\|^2}{4} + \frac{\|T^{1/2}f\|^2}{\|f\|^2} + \frac{\lambda|\langle g, f \rangle|^2}{2\|f\|^2} \right) + \lambda\|\psi_0\|^2\|g\|^2.
\]

Let \( a_0 := 1, \quad a_n := n^{-3/4}n^{-1/2}, n \in \mathbb{N} \) and, for any \( 0 < \delta, \ 0 < \epsilon < 1 \),

\[
\begin{align*}
f &= f_\delta := \frac{T^{-1}E((\delta, \infty))g}{\|T^{-1}E((\delta, \infty))g\|}, \\
c_{\lambda}(\epsilon, \delta) &:= \|T^{1/2}f_\delta\|^2 \left\{ 1 + \frac{\lambda}{2}(2 - \epsilon)\|T^{-1/2}E((\delta, \infty))g\|^2 \right\}.
\end{align*}
\]
Then $\sum_{n=0}^{\infty} \|\psi_n\|^2$ converges and, for any $N \in \mathbb{N},$

$$\langle \phi_N, H(\lambda)\phi_N \rangle = \sum_{n=2}^{N} \|\psi_n\|^2 n c_{\lambda} (\varepsilon, \delta) + \frac{\lambda}{2} \sum_{n=2}^{N} \|\psi_n\|^2 \left( \frac{a_{n-2}}{a_n} - n(1 - \varepsilon) \right) \langle g, f_{\delta} \rangle^2 + C_N, \quad (8.1)$$

where

$$C_N := \frac{\lambda \|g\|^2}{4} \sum_{n=0}^{N} \|\psi_n\|^2 + \|\psi_1\|^2 \left( \|T_{1/2} f_{\delta}\|^2 + \frac{\lambda}{2} \langle g, f_{\delta} \rangle^2 \right).$$

For all $0 < \delta, 0 < \varepsilon < 1$, we have

$$- \frac{2}{\|T^{-1/2} E((\delta, \infty))g\|^2 (2 - \varepsilon)} < \lambda_{c,0}. \quad (8.2)$$

The left hand side of (8.2) tends to $\lambda_{c,0}$ as $\varepsilon, \delta \downarrow 0$. Since $\lambda < \lambda_{c,0}$, we can take a pair $(\varepsilon, \delta)$ satisfying $c_{\lambda}(\varepsilon, \delta) < 0$. We fix such a pair. There is an $n_0 \in \mathbb{N}$ such that $a_{n-2}/a_n - n(1 - \varepsilon) > 0$ for all $n \geq n_0$. Hence we can see that $\langle \phi_N, H(\lambda)\phi_N \rangle / \|\phi_N\|^2$ tends to $-\infty$ as $N \to \infty$, because the first term of the right hand side of (8.1) tends to $-\infty$ as $N \to \infty$. \qed

9 Generalization of the $\phi^2$-model

In this section we consider $H(\eta, \lambda)$ defined in Subsection 2.3.

**Assumption 9.1.** We need the following assumptions:

1. $f \in D(T^{1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2}),$
2. $f \in D(T^{-1})$ and $\Re \langle T^{-1} f, g \rangle = 0,$
3. $f, g \in D(T^{-1})$ and $\Re \langle T^{-1} f, g \rangle \neq 0.$

We can prove a slight generalization of Theorem 3.6.

**Theorem 9.2.** Let $\mathcal{H}$ be separable. Then the following (1)-(5) hold:

1. Suppose that Assumption 3.3 and, Assumption 9.1 (2) or (3) hold. Let $\lambda > \lambda_c$. Then there is a unitary operator $U$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta \in \mathbb{R},$

$$U H(\eta, \lambda) U^{-1} = d\Gamma_b(T) + E_g + E_{f,g},$$

where the constant $E_{f,g} \in \mathbb{R}$ is defined by

$$E_{f,g} = -\frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{(\Re \langle T^{-1} f, g \rangle)^2 \eta^2 \lambda}{2(1 + \lambda \|T^{-1/2} g\|^2)}. $$
Suppose that Assumption 3.3 and Assumption 9.1 (2) or (3) hold. Let $E_0 > 0$ and $\lambda_{c,0} < \lambda < \lambda_c$. Then there is a unitary operator $\mathcal{U}$ on $\mathcal{F}_b(\mathcal{H})$ and a non-negative, injective self-adjoint operator $\xi$ on $\mathcal{H}$ such that, for all $\eta \in \mathbb{R}$,

$$\mathcal{U}H(\eta, \lambda)^{-1} = d\Gamma_b(\xi) + E_g - E_b + E_{f,g}.$$  

Let $T$ be a non-negative, injective self-adjoint operator and suppose that $f$ and $g$ satisfy Assumption 9.1 (1) and (2). Then there is a unitary operator $\mathcal{W}$ on $\mathcal{F}_b(\mathcal{H})$ such that, for all $\eta \in \mathbb{R}$,

$$\mathcal{W}H(\eta, \lambda_{c,0})\mathcal{W}^{-1} = \overline{H(\lambda_{c,0})} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2.$$  

Let $T$ be a non-negative, injective self-adjoint operator and suppose that $f$ and $g$ satisfy Assumption 9.1 (1) and (3). Then, for all $\eta \in \mathbb{R}\backslash\{0\}$,

$$\sigma(H(\eta, \lambda_{c,0})) = \mathbb{R}, \quad \sigma_p(H(\eta, \lambda_{c,0})) = \emptyset.$$  

Let $T$ be a non-negative, injective self-adjoint operator and suppose that $f$ and $g$ satisfy Assumption 9.1 (1). Moreover, suppose that Assumption 9.1 (2) or (3) holds. Let $\lambda < \lambda_{c,0}$. Then, for all $\eta \in \mathbb{R}$, $\overline{H(\eta, \lambda)}$ is unbounded from above and below.

Theorem 9.2 is immediately proved by the following lemma and Theorem 3.6.

**Lemma 9.3.** Let $T$ be a non-negative, injective self-adjoint operator, $f \in D(T^{-1})$ and $g \in D(T^{-1/2}) \cap D(T)$.

1. Let $\text{Re} \langle T^{-1}f, g \rangle = 0$. Then there is a unitary operator $\mathcal{U}_1$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta, \lambda \in \mathbb{R}$,

$$\mathcal{U}_1H(\eta, \lambda)\mathcal{U}_1^{-1} = \overline{H(\lambda)} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2. \quad (9.1)$$

2. Let $\text{Re} \langle T^{-1}f, g \rangle \neq 0$ and $g \in D(T^{-1}).$

   (i) If $\lambda \neq \lambda_{c,0}$, then there is a unitary operator $\mathcal{U}_2$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta \in \mathbb{R}$,

   $$\mathcal{U}_2H(\eta, \lambda)\mathcal{U}_2^{-1} = \overline{H(\lambda)} + E_{f,g}.$$  

   (ii) If $\lambda = \lambda_{c,0}$, then for all $\eta \in \mathbb{R}\backslash\{0\}$,

   $$\sigma(H(\eta, \lambda_{c,0})) = \mathbb{R}, \quad \sigma_p(H(\eta, \lambda_{c,0})) = \emptyset. \quad (9.2)$$
Proof. Let \( \mathcal{U}_1 := e^{-i\Phi_s(\eta \tau^{-1} f)} \) for any \( \eta \in \mathbb{R} \). Then, by direct calculations, we obtain
\[
\mathcal{U}_1 H(\eta, \lambda) \mathcal{U}_1^{-1} = H(\lambda) - \eta^2 \|T^{-1/2} f\|^2 - \lambda \eta \kappa \Phi_s(g) + \frac{\lambda}{2} \eta^2 \kappa^2 \tag{9.3}
\]
on \( \mathcal{S}_{b, \text{fin}}(D(T)) \) for all \( \eta, \lambda \in \mathbb{R} \), where \( \kappa := \text{Re}(\tau^{-1} f, g) \). In the case of (1), we have (9.1) by \( \kappa = 0 \) and a limiting argument. Next, we prove (2). We assume that \( g \in D(T^{-1}) \) and \( \text{Re}(\tau^{-1} f, g) \neq 0 \). Let \( \mathcal{V}_1 := e^{i\Phi_s(\eta \tau^{-1} g)} \) for any \( \alpha \in \mathbb{R} \) and define a unitary operator \( \mathcal{U}_2 := \mathcal{V}_1 \mathcal{U}_1 \). Then it follows that
\[
\mathcal{U}_2 H(\eta, \lambda) \mathcal{U}_2^{-1} = H(\lambda) + \left(\alpha + \lambda \alpha \|T^{-1/2} g\|^2 - \lambda \eta \kappa\right) \Phi_s(g)
- \frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{\lambda}{2} \eta^2 \kappa^2 + \frac{\alpha}{2} \|T^{-1/2} g\|^2 \left(\alpha + \lambda \alpha \|T^{-1/2} g\|^2 - 2 \lambda \eta \kappa\right)
\]
on \( \mathcal{S}_{b, \text{fin}}(D(T)) \) in the same way as (9.3). For \( \lambda \neq \lambda_{c,0} \), let \( \alpha = \lambda \eta \kappa (1 + \lambda \|T^{-1/2} g\|^2)^{-1} \). Then we obtain
\[
\mathcal{U}_2 H(\eta, \lambda) \mathcal{U}_2^{-1} = H(\lambda) - \frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{\lambda \eta \kappa^2}{2(1 + \lambda \|T^{-1/2} g\|^2)} \tag{9.4}
\]
by a limiting argument. If \( \lambda = \lambda_{c,0} \), then, for all \( \eta, \alpha \in \mathbb{R} \), we have
\[
\mathcal{U}_2 H(\eta, \lambda_{c,0}) \mathcal{U}_2^{-1} = H(\lambda_{c,0}) - \frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{\lambda_{c,0} \eta^2 \kappa^2}{2} + \kappa \alpha
\]
in the same way as (9.4), where \( H(\nu, \lambda_{c,0}) := H(\lambda_{c,0}) + \nu \Phi_s(g) \) for all \( \nu \in \mathbb{R} \). We can see that \( \sigma(H(\nu, \lambda_{c,0})) = \mathbb{R} \) and \( \sigma_p(H(\nu, \lambda_{c,0})) = \emptyset \) for all \( \nu \in \mathbb{R} \setminus \{0\} \), because \( \mathcal{V}_1 H(\nu, \lambda_{c,0}) \mathcal{V}_1^{-1} = H(\nu, \lambda_{c,0}) + \nu \alpha \|T^{-1/2} g\|^2 \) and \( \alpha \in \mathbb{R} \) is arbitrary. Hence we have (9.2).

\[ \blacksquare \]

Remark 9.4. If \( \mathcal{H} \) is separable, then the condition \( g \in D(T^{-1/2}) \cap D(T) \) in the above lemma is weakened to the condition \( g \in D(T^{-1/2}) \cap D(T^{1/2}) \).

10 Appendix

In this section, we recall some known facts in the Fock space theory. Let \( T \) be a non-negative, injective self-adjoint operator on \( \mathcal{H} \).

Lemma 10.1. [5, Theorem 5.16.]
Let \( f \in D(T^{-1/2}) \) and \( \psi \in D(d\Gamma_b(T)^{1/2}) \). Then \( \psi \in D(A(f)) \cap D(A(f)^*) \) and the following inequalities hold:
\[
\|A(f)\psi\| \leq \|T^{-1/2} f\| \|d\Gamma_b(T)^{1/2} \psi\|, \tag{10.1}
\]
\[
\|A(f)^*\psi\|^2 \leq \|T^{-1/2} f\|^2 \|d\Gamma_b(T)^{1/2} \psi\|^2 + \|f\|^2 \|\psi\|^2. \tag{10.2}
\]
**Lemma 10.2.** [5, Proposition 5.10.] For any $f \in D(T)$, the following commutation relations hold on $\mathcal{F}_{b,fin}(D(T))$:

$$\left[ d\Gamma_b(T) , A(f) \right] = -A(Tf) , \quad \left[ d\Gamma_b(T) , A(f)^* \right] = A(Tf)^* .$$

(10.3)

**Lemma 10.3.** [5, Lemma 5.21.] For any $t \in \mathbb{R}$ and $f \in \mathcal{H}$, the following equations hold:

$$e^{itd\Gamma_b(T)}A(f)^2e^{-itd\Gamma_b(T)} = A(e^{itT}f)^2 .$$

**Lemma 10.4.** [5, Theorem 5.21.] Assume that $\mathcal{H}$ be separable. Let $\{e_n\}_{n=1}^\infty \subset D(T^{1/2})$ be a CONS of $\mathcal{H}$. Then, for any $\psi \in D(d\Gamma_b(T)^{1/2})$, $\sum_{n=1}^\infty \|A(T^{1/2}e_n)\psi\|^2$ converges and the following equation holds:

$$\sum_{n=1}^\infty \|A(T^{1/2}e_n)\psi\|^2 = \|d\Gamma_b(T)^{1/2}\psi\|^2 .$$

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**References**


