Study of Holography for Three Dimensional Spin-3 Gravity Coupled to a Scalar Field
(スカラー場と結合した3次元スピン3重力に対するホログラフィーに関する研究)

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Abstract

Since the gravitational theory is not renormalizable, it is necessary to formulate the quantum gravity by using the other framework. The AdS/CFT correspondence is one of the candidates to describe quantum gravity. According to this correspondence, a gravitational theory is equivalent to a gauge theory on the boundary, which does not contain the gravity. Although the complete proof of the AdS/CFT correspondence has not been obtained, a lot of evidence, which guarantees the existence, have been reported. In this paper, we will study the duality between the three-dimensional spin-3 gravity coupled to a scalar field and the W_3 extended conformal field theory. Although the three-dimensional spin-3 gravity is well-studied owing to its equivalence with the Chern-Simons gauge theory, the spin-3 gravity coupled to matter fields is less understood. This is because the action integral of the spin-3 gravity coupled to matters is unknown. The AdS/CFT correspondence gives a new point of view. It is known that the three-dimensional spin-3 gravity is equivalent to the W_3 extended conformal field theory. By considering the bulk scalar field as the operator in the W_3 extended conformal field theory, we find the clue to the difficulty. In this Thesis, we formulate the spin-3 gravity coupled to a scalar field in terms of the action integral. In the large central charge limit, the W_3 algebra reduces to the $SU(1,2)$ algebra. By using this algebra, the state of the bulk scalar is constructed. The commutator $[W_n, W_m]$ gives the Virasoro generator L_{n+m}, and in order to formulate the $SU(1,2)$ invariant theory, it is necessary to introduce an enlarged spacetime by introducing five auxiliary coordinates as in the case of the supersymmetry. When the scalar field exists, half of $SU(1,2)$ invariance is broken. The remaining invariance gives equations, which determine the state of the scalar. The state is an eigenstate of the quadratic Casimir operator of $SU(1,2)$ algebra. The eigenvalue equation is interpreted as the equation of motion for free scalar in the eight-dimensional enlarged spacetime. The action integral is given by the one for the free scalar in the enlarged spacetime. Further, it is shown that the standard AdS/CFT dictionaries hold. The action integral for gravity is also studied. The scalar state is eigenstates of the cubic Casimir operator as well as the quadratic Casimir operator. The metric and the spin-3 gauge field are uniquely determined from the differential equations associated with the Casimir operators. It turns out that these fields are expressed in terms of $SL(3,\mathbb{R})$ gauge connections.

Contents

1 Introduction and Brief Summary

What is gravity? Scientists have been asking this question for a long time. In 2012, the Higgs particle was discovered and the standard model of particle physics was experimentally proved to be correct. In the universe, it is known that there are four fundamental forces: the electromagnetic interaction, the weak interaction, the strong interaction, and the gravitational interaction. The standard model describes three of them except for the gravitational interactions.

In usual quantum field theory, there are divergences at a small distance. These divergences are renormalizable if the dimension of coupling constant is never negative. Since the gravitational constant G has a dimension -1, the gravitational theory is not renormalizable. The authors of [1] obtained proof of non-renormalizability by using loop calculations. Therefore, a new theory, which describes the quantum behavior of gravity, is needed. Fortunately, several theories that avoid these problems appeared. The superstring theory is one of the powerful candidates for quantum gravity. The superstring theory has a typical length scale. This scale plays a role of a cut-off and problematic divergences do not appear.

The superstring theory predicts a non-trivial equivalence between totally different theories. One of the examples is called the AdS/CFT correspondence, which was first discovered by Maldacena [2]. Scientists and mathematicians have been studying this duality and its extension since the birth of the AdS/CFT correspondence. Maldacena showed that Type-IIB superstring theory in ten-dimensional spacetime manifold, which is a product space of five-dimensional AdS spacetime and five-dimensional sphere, is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. The superstring theory contains a rank-2 tensor field, which is called a graviton, while super Yang-Mills theory does not contain the graviton. Roughly speaking, $d+1$ dimensional gravitational theory can be described in terms of a non-gravitational theory living on d dimensional manifold. This map is holographic. This correspondence is one example of the holographic principle, which was proposed in [3] and [4]. According to the holographic principle, the information of quantum gravity is encoded at the boundary of a manifold. For example, the black hole entropy can be quantified by its horizon surface area[5][6]. This implies that the information inside the black hole is encoded at the horizon. Similarly, in the AdS/CFT correspondence, the boundary theory might have complete information to describe the quantum gravity in AdS space. One more important point is the following. the AdS/CFT correspondence is a strong/weak duality. The perturbation theory is a good tool to analyze quantum field theory with small coupling constants. When the non-gravitational theory is strongly-coupled, we can study it using the weakly-coupled string theory and vise versa.

Although the complete proof of the existence of the AdS/CFT correspondence has not been given, a lot of evidence has been reported until now. In [7][8][9], the mathematical methods to calculate correlation functions in the conformal field theory in terms of dual bulk fields were proposed. According to [7][8], a generating functional of correlation functions in the conformal field theory is equal to a partition function in the semi-classical AdS gravity. This prescription is called the differentiating dictionary. Also, it was shown in [9] that the boundary behavior of bulk correlation function coincides with the correlation function in the conformal field theory. This is called the extrapolating dictionary. These two dictionaries are equivalent[10]. Conversely, prescriptions to express a bulk fundamental field in terms of operators in the conformal field theory has been also studied. Local bulk fields can be reconstructed by non-local boundary conformal field theory operators[11][12]. In [13][14][15], it is proposed that a state excited by a fundamental field in AdS spacetime can be expressed as a state in conformal field theory. This prescription is equivalent to that in [11] and [12]

It is worth studying the extension of the applicability of this correspondence. One possible extension will be the three-dimensional higher-spin gravity coupled to a matter field. In particular, we will consider the three-dimensional spin-3 gravity. According to [18], the spin-3 gravity is dual to the W_3 extended conformal field theory. In general, the conformal field theory is characterized by Virasoro generators L_n. The symmetry algebra is $sl(2,\mathbb{R})\otimes sl(2,\mathbb{R})$. The conformal field theory can be extended by extra symmetries. For example, extra gauge symmetries may be realized by adding a current operator with a conformal dimension 1. The W_3 extended conformal field theory is one of the extensions of the Virasoro algebra, which is obtained by adding a current operator with conformal dimension 3[16]. The symmetry algebra is generated by Virasoro generators L_n and W_3 generators W_m, which correspond to Laurent modes of extra current operator, and called the W_3 algebra. In [17][18], it was proposed that three-dimensional higher-spin theories with W_N symmetry is dual to a W_N extended conformal field theory. The higher-spin gravity contains a metric tensor and a spin-N symmetric tensor. The metric and the spin-N field are described by the three dimensional $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ Chern-Simons gauge theory[19]. The authors of [20] [21] studied the three dimensional $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ Chern-Simons gauge theory and showed that the W_3 algebra can be obtained from the infinitesimal gauge transformation of the Chern-Simons gauge connections. Then, it was proposed that the three dimensional $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ Chern-Simons gauge theory is dual to the two dimensional W_3 extended conformal field theory. In [21], two solutions to the equations of motion for gauge fields was studied. Both of them gives the AdS metric. These correspond to two inequivalent embeddings of $SL(2,\mathbb{R})$ in $SL(3,\mathbb{R})$. In one of the two embeddings, $SL(2,\mathbb{R})$ is generated by Virasoro generators (L_1, L_0, L_{-1}), while in the other it is generated by (W_2, L_0, W_{-2}). The vacuum in the former is often called a W_3 vacuum, on the other hand, the one in the latter is called a $W_3^{(2)}$ vacuum. Furthermore, they discovered another solution that interpolates between these two spacetime. At the boundary, this solution gives the W_3 vacuum, while this solution also gives the $W_3^{(2)}$ vacuum far from the boundary. They also obtained black hole solutions. Boundary conditions must be imposed on Chern-Simons gauge connections A and $\bar{A}$ to make the variation problem well-defined. At the boundary, conditions $A_{\bar{x}} = \bar{A}_x = 0$ are often adopted. The authors of [20] [21] adopted other boundary conditions by adding a local term to the action, which deforms the boundary conformal field theory and proposed a black hole solution with both a mass parameter and a spin-3 charge parameter.

There are several remaining problems. The action integral for the gravity sector is well studied. However, the situation is different when a matter field couples to gravity. First, the action integral for a scalar field which has a spin-3 charge has not been found. Then, it is not clear whether standard AdS/CFT dictionaries can be applied or not. Furthermore, in the spin-3 gravity, even the bulk scalar state have not been constructed in terms of boundary W_3 extended conformal field theory. Our motivation is to improve these situations.

The approach is the following. At first, we will construct the bulk scalar field as the operator in the W_3 extended conformal field theory by using the following procedure of [14][15]. The W_3 algebra is different from the Virasoro algebra and the current algebra in usual sense:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{0,m+n} \tag{1.1}$$

$$[L_n, W_m] = (2n-m)W_{n+m} \tag{1.2}$$

$$[W_n, W_m] = \frac{c}{360}n(n^2-1)(n^2-4)\delta_{0,m+n} \tag{1.3}$$

$$+\frac{1}{30}\left(n-m\right)(2n^2-mn+2m^2-8)\,L_{n+m} + \beta(n-m)\Lambda_{n+m} \tag{1.4}$$

plus the similar algebra for anti-chiral counterparts $\bar{L}_n$ and $\bar{W}_m$, where $n, m \in \mathbb{Z}$. Here,

$$\beta = \frac{16}{22+5c} \tag{1.5}$$

$$\Lambda_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq 2} L_k L_{n-k}. \tag{1.6}$$

L_n's denote the Virasoro generators and W_m's denote extra gauge symmetry generators. c is a model dependent constant and often called the central charge. In [22], this is related to the gravitational constant G

$$c = \frac{3}{2G}. \tag{1.7}$$

We restrict the wedge modes subalgebra, which consists of L_n and W_m with $-1 \leq n \leq 1$ and $-2 \leq m \leq 2$. In the large c limit, the third term proportional to β is dropped. By changing the normalization $W_n \to W_n/\sqrt{10}$, the subalgebra can be rewritten as

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} \\ [L_n, W_m] &= (2n-m)W_{n+m} \\ [W_n, W_m] &= \frac{1}{3}(n-m)\left(2n^2 - mn + 2m^2 - 8\right)L_{n+m}. \end{aligned} \tag{1.8}$$

This subalgebra is called $su(1,2)\otimes su(1,2)$ algebra. The commutator of two W_m's gives Virasoro generator L_n. This is similar to the case of supersymmetry. In supersymmetric theory, a theory is invariant under the Pincaré transformations and supertranslations, which is generated by supercharges. The anti-commutator of supercharges gives the translation generator. In order to construct Poincaré invariant and supersymmetric formulation or equivalently Lagrangian, Grassmann odd coordinates, which correspond to supertranslations, are introduced and the dimension of spacetime must be enlarged. For example in $\mathcal{N} = 1$ supersymmetry in four dimensions, there are two supercharges Q and $\bar{Q}$ and corresponding translations are written by

$$G(\theta, \bar{\theta}) = e^{i\theta^\alpha Q_\alpha} e^{i\bar{\theta}^{\bar{\alpha}} \bar{Q}_{\bar{\alpha}}} \tag{1.9}$$

Two Grassmann odd coordinates are introduced and the spacetime dimensions are enlarged to six. In the case of the W_3 extended conformal field theory, we will introduce auxiliary coordinates corresponding to the generators W_{-2}, W_{-1} and their anti-holomorphic counterparts. Since the commutator $[W_n, W_m]$ gives the generator of the conformal transformation L_{n+m}, auxiliary coordinates mix with the ordinary two-dimensional coordinates. Hence, we need to regard these coordinates as the spacetime coordinates instead of the internal gauge variables. In the context of the AdS/CFT correspondence, there are two more coordinates, which correspond to transformations generated by $L_0 + \bar{L}_0$ and $W_0 + \bar{W}_0$. Hence, the dual theory will be formulated in eight dimensions. We consider a massive scalar field at the center of the bulk. Due to the existence of the scalar, half of the isometry is broken. The remaining isometries give conditions, which determine the state of the scalar field at the center of the bulk[14]. In three dimensional AdS spacetime, these conditions are expressed in terms of the Virasoro generator as

$$\left(L_n - (-1)^n \bar{L}_{-n}\right) |\phi(0)\rangle = 0 \tag{1.10}$$

where $n = 0, \pm 1$ and $|\phi(0)\rangle$ denotes the scalar state. In the case of the W_3 extended conformal field theory, there are more isometries, which come from the W_m generators. Although it is not easy to determine conditions for W_m from the isometry, it is possible to show that the state, which satisfies conditions

$$\left(W_n - (-1)^m \bar{W}_{-m}\right) |\phi(0)\rangle = 0, \tag{1.11}$$

where $m = 0, \pm 1, \pm 2$, agrees with the W-primary state, when it is transported to the boundary. Hence, total eight conditions determine the scalar state at the center of the bulk. The state at an arbitrary point can be obtained by multiplying the transformation operator, which contains eight coordinates associated with $(L_{-1},\ \bar{L}_{-1},\ W_{-2},\ \bar{W}_{-2},\ W_{-1},\ \bar{W}_{-1},\ W_0 + \bar{W}_0, L_0 + \bar{L}_0)$.

When the bulk scalar state is obtained, the Killing vector fields, which correspond to the differential representation for generators, can also be obtained. Further, there is two Casimir operator in the W_3 extended conformal field theory, which is called quadratic and cubic Casimir operators. The bulk state is the eigenstate of these Casimir operators. Since the quadratic Casimir operator is written by the sum of the product of two generators, the eigenvalue equation for it gives the second-order differential equation. When we interpret its eigenvalue as the mass of the scalar, the equation of motion can be interpreted as the Klein-Gordon equation for the free massive scalar in eight dimensions. Therefore, the action integral can be written by a sum of the kinetic term and mass term of the scalar in eight dimensions.

We will consider the solution to the eigenvalue equation for the quadratic Casimir operator. Although the differential equation is very complicated, an exact solution will be obtained. The solution has interesting properties. When extra coordinates, which are introduced in order to enlarge the spacetime, vanish, the solution coincides with the bulk to boundary propagator for the scalar in the usual three-dimensional AdS spacetime. Of course, it can be shown that this is invariant under $SU(1,2) \times SU(1,2)$ transformations. Hence, this is interpreted as the bulk to boundary propagator for the scalar field in the enlarged spacetime.

We will also discuss the AdS/CFT dictionary. The differentiating dictionary [7][8] tells us that the partition function in the semi-classical AdS gravity is equal to the generating functional of correlation functions in the conformal field theory. We will investigate whether this statement holds or not in the W_3 extended conformal field theory. The classical solution to the Klein-Gordon equation is constructed in terms of the bulk to boundary propagator with a boundary condition that the scalar field approaches the value ϕ_0 at the boundary:

$$\phi(y, x) = \int d^d x' K(y, x; x') \phi_0(x') \tag{1.12}$$

where K is the bulk to boundary propagator. By substituting the solution into the action and integrating by parts, the boundary action $S_B[\phi_0]$ is obtained. The partition function in the semi-classical AdS gravity is defined by $\exp[S_B[\phi_0]]$. By functionally differentiating this with respect to the boundary value ϕ_0 and setting $\phi_0 = 0$, the boundary two-point function will be obtained. Hence, the differentiating dictionary will be established.

Since we enlarge the spacetime, it is also necessary to construct a new action integral for the gravity sector. In three dimensions, the $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ Chern-Simons action is equivalent to the action for the gravity sector. In even dimensions, however, there is no Chern-Simons action. Hence, we must consider an alternative action for gravity. The vielbein formulation will help us to construct it. From the quadratic Casimir operator, we can read off the metric tensor. Also, there is the cubic Casimir operator and the scalar state is also an eigenstate. This eigenvalue equation gives the third-order differential equation. From this equation, we can read off the completely symmetric rank-3 tensor, which is often called the spin-3 field. When the metric and the spin-3 field are determined, a vielbein field and a spin connection are uniquely determined up to a local frame and two gauge connection as in Chern-Simons gauge theory are defined by taking an appropriate linear combination. These gauge connections satisfy two flatness conditions, which mean that field strengths for two gauge connections vanish. On a hypersurface with constant extra coordinates, these coincide with the equations of motion for the $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ Chern-Simons gauge connections. Then we should adopt the local frame as $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ symmetry. We will consider these flatness conditions as the equations of motion for the vielbein and the spin connection. We will construct the action integral for the gravity sector, which yields the equations of motion.

There exists a new renormalization group flow parameter. In the AdS/CFT correspondence, it is known that the radial coordinate plays the role of a renormalization group flow parameter [24]. For example, in spin-3 gravity [21] studied the holographic renormalization flow and discovered a solution that interpolates between two spacetimes. In the ultraviolet region, this solution gives the $W_3^{(2)}$ vacuum, while in the infrared region this gives the W_3 vacuum. This result can also be observed in our formulation. We will find another renormalization group flow. Two coordinates, which correspond transformations generated by $(W_{-1}, \bar{W}_{-1})$, play the roles of renormalization group flow parameters. In general, the conformal symmetry is broken at arbitrary points because of the extra five coordinates. However, there are points that the conformal symmetry is recovered. One point is the one with vanishing two coordinates, while another one is with infinitely large two coordinates. At the former point, the W_3 vacuum will be realized, while at the latter the $W_3^{(2)}$ vacuum will be realized. Also, we will study the action, which triggers the renormalization group flow. Although the boundary theory at the other hypersurfaces is no longer conformal, the theory is a quantum field theory. It will be shown that fields of the quantum field theory $\mathcal{O}'$ are dressed by $g(\alpha,\beta) = e^{i\alpha W^h_{-2}} e^{\beta W^h_{-1}}$ and given by

$$\mathcal{O}' = g(\alpha,\beta)\mathcal{O}g^{-1}(\alpha,\beta) \tag{1.13}$$

where $\mathcal{O}$ denotes the operator in the conformal field theory.

Also, we will consider other solutions to flatness conditions. The authors of [21] studied a black hole solution. We will obtain new black hole solutions by solving the flatness conditions. In the case of the black hole without the spin-3 charge, we will obtain the exact solution. Taking a hypersurface with vanishing extra coordinates, it coincides with the BTZ black hole [23]. The Hawking temperature can be obtained by solving holonomy condition[25] and it coincides with that in [21]. In the case with the spin-3 charge, however, the situation will be different. The gauge connection corresponding to the black hole with the spin-3 charge is also obtained by solving the flatness conditions perturbatively. It turns out that the perturbative expansion continues as an infinite series. In this Thesis, we carried out up to the third-order perturbation. Interestingly, perturbations more than fourth-order do not affect physical quantities, such as the Hawking temperature or entropy: two holonomy conditions are precisely determined up to the third-order perturbation. Furthermore, it will be shown that holonomy conditions do not contain extra five coordinates. Then, it will be shown that the partition function for the black hole is exactly the same with that in [21]. Although the partition functions are exactly the same, the geometry is totally different. Our gauge connections satisfy the usual boundary conditions $A_{\bar{x}} = \bar{A}_x = 0$. These are gauge inequivalent with gauge connections in [21]. In general, the gauge transformation for the gauge connection A and $\bar{A}$ are determined by using the common matrix U and given by

$$\mathcal{A}' = U^{-1}\mathcal{A}U + U^{-1}dU \tag{1.14}$$

$$\bar{\mathcal{A}}' = U\bar{\mathcal{A}}U^{-1} + UdU^{-1}. \tag{1.15}$$

There is no matrix U from our gauge connections to the ones in [21].

This Thesis is organized as follows. There are three parts. Before discussing our result, we should give several fundamentals in the first part. The first part consists of four sections. In Section.2 and Section.3, we will review fundamentals for the conformal group, operators in conformal field theory and extension to W_3 conformal field theory. Fundamentals of AdS spacetime will be introduced in Section.4 and in Section.5 we will review more general three-dimensional spacetime with a negative cosmological constant. The second part is for the reviews of recent researches by other groups. This part consists of three sections. In Section.6, we will focus on the AdS/CFT correspondence and see how to obtain correlation functions in conformal field theory in terms of bulk language. Conversely, we will introduce the method to express a bulk fundamental field as an operator in conformal field theory in Section.7. It will be shown that a state excited by this filed is expressed in terms of the dual primary operator and its descendants. Also, we will review recent works on spin-3 gravity. The spin-3 gravity can be described by the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons gauge theory. There are several black hole geometries in the spin-3 gravity. We will see these works in Section.8. The third part is the main subject of this Thesis. In this part, the holography between three-dimensional spin-3 gravity and the boundary W_3 conformal field theory will be studied. This part consists of seven sections. In Section.9, we will consider boundary W_3 extended conformal field theory. The bulk scalar field is reconstructed not only by a conformal family generated by the Virasoro generators but also by the one generated by W_m generators. The eigenvalue equation of the quadratic Casimir operator on this field corresponds to the Klein- Gordon equation for the bulk scalar field. In Section.10, the bulk geometry is studied. When we read the metric from the Klein-Gordon equation, the metric field becomes 8×8 symmetric tensor. The role of extra coordinates is studied in Section.11. We will find one of the extra coordinates play the role of a new renormalization group flow parameters. In Section.12, we will introduce the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ gauge connections. These are 8×8 matrix, one of whose indices corresponds to spacetime one and the other to the local $SL(3,\mathbb{R})$ frame. These connections satisfy the flatness conditions. Several gauge connections for black hole metrics will be studied in Section.13. These are parametrized by the mass and the spin-3 charge. On the hypersurface with constant five extra coordinates, the gauge connections are solutions to the three dimensional $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory. Also, we will study the action integral for the gravity sector in Section.14. In even dimension, it is known that there is no Chern-Simons action. Instead of the Chern-Simons action, we will seek the action, which yields the flatness conditions for the gauge connections. Finally, we will summarize our works and discuss future works in Section.15. There are several appendices. Appendices. A and B contain some technical details for obtaining constraints on the boundary scalar state and the local scalar state in the bulk. Appendix.C contains a convention for $SL(3,\mathbb{C})$ symmetry algebra. In Appendix.D, we will express the complete result of the spin-3 field for the black hole without the spin-3 charge. Appendix.E contains some technical details for obtaining the spin-3 field and the black hole solution with the spin-3 charge. This Thesis is based on [26] and [27].

Part I
Fundamentals

2 Fundamentals in Classical Conformal Field Theory

The purpose of this section is to obtain fundamentals for conformal field theory. We refer to instructive textbooks [28] and [29].

2.1 Conformal Group

We start by introducing conformal transformations and determining some conditions for the conformal invariance. Let a metric field in d dimensional spacetime be $g_{\mu\nu}$. In general relativity, the metric transforms under a general coordinate transformation $x \to x'$ as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x). \tag{2.1}$$

The meaning of "conformal" is transformations that locally preserve the angle between arbitrary two curves crossing each other at some point. In more mathematical language, a conformal transformation is defined as follows. Conformal transformations are differentiable maps $x \to x'$ which satisfiy

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{2.2}$$

where the positive function $\Lambda(x)$ is called the scale factor. For simplicity, we assume that the spacetime is flat with a metric $\eta = diag(-1, 1, \cdots, 1)$. Notice that for flat spacetimes the scale factor $\Lambda(x) = 1$ corresponds to the Poincaré group consisting of translations, rotations or boosts.

Next, we will consider some conditions for the conformal invariance. Let us investigate an infinitesimal coordinate transformation $x' = x + \epsilon(x)$. The metric changes as follows up to first order in ϵ:

$$\eta_{\mu\nu} = \eta_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.3}$$

Requiring that the transformation is conformal, the second term in (2.3) must be proportional to the metric

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu}. \tag{2.4}$$

The factor $f(x)$ is determined by taking the trace on both sides

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho. \tag{2.5}$$

By applying an extra derivative ∂^ν on (2.4) and substituting (2.5), we obtain

$$2\partial^2 \epsilon_\mu = (2-d)\partial_\mu f. \tag{2.6}$$

Furthermore, by taking the derivative ∂_μ, (2.6) reduces to

$$(2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f. \tag{2.7}$$

Finally, by contracting with $\eta^{\mu\nu}$, we can obtain

$$(d-1)\partial^2 f = 0. \tag{2.8}$$

Equations (2.4), (2.7) and (2.8) are constraints for conformal transformations. Notice that if $d = 1$, there are no constraints for the factor $f(x)$. Then, in one dimension, any smooth transformations are conformal.

2.2 Conformal Group in $d \geq 3$

In this subsection, we will focus on the conformal group in the case of dimension $d \geq 3$. From the above discussion, there are some constraints for the factor $f(x)$ and in this case there are three constrains

$$\partial^2 f = 0 \tag{2.9}$$

$$\partial_\mu \partial_\nu f = 0 \tag{2.10}$$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu}. \tag{2.11}$$

The above equations imply that the factor $f(x)$ is at most linear in the coordinates

$$f(x) = A + B_\mu x^\mu, \tag{2.12}$$

where A and B_μ are constants. When we substitute this expression into the third constraint, we can find that the infinitesimal transformation ϵ is at most quadratic in the coordinates. So we can write the general expression

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho, \tag{2.13}$$

where a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ are constants and $c_{\mu\nu\rho}$ is symmetric under a permutation in the last two indices. It is clear that the parameter a_μ is free of constraints. This term corresponts to an infinitesimal translation. From the third constraint, $b_{\mu\nu}$ must satisfy the following equation

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\rho_\rho \eta_{\mu\nu}, \tag{2.14}$$

and this implies that $b_{\mu\nu}$ is the sum of a trace and an antisymmetric part

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + M_{\mu\nu} \tag{2.15}$$

where $\alpha = b^\mu_\mu / d$ and $M_{\mu\nu}$ is an antisymmetric constant. The trace part represents an infinitesimal scale transformation or dilatation, while the antisymmetric part represents an infinitesimal rotation. Finally, we will consider the quadratic part. From the third constraint, $c_{\mu\nu\rho}$ must satisfy

$$c_{\mu\nu\rho} + c_{\nu\mu\rho} = \frac{2}{d} c^\sigma_{\sigma\rho}\eta_{\mu\nu}. \tag{2.16}$$

By taking appropriate linear combination, we can find

$$c_{\mu\nu\rho} = \frac{1}{d}(c^\sigma_{\sigma\rho}\eta_{\mu\nu} + c^\sigma_{\sigma\nu}\eta_{\mu\rho} - c^\sigma_{\sigma\mu}\eta_{\rho\nu}) \equiv \eta_{\mu\nu}b_\rho + \eta_{\mu\rho}b_\nu - \eta_{\rho\nu}b_\mu, \tag{2.17}$$

and the corresponding infinitesimal transformation is called a special conformal transformation

$$\epsilon^\mu = 2x^\rho b_\rho x^\mu - b^\mu x^2. \tag{2.18}$$

Notice that conventionally we define a parameter b_μ but there are no relation to $b_{\mu\nu}$.

The finite transformations corresponding to the above are the follows.

1) translation

$$x'^\mu = x^\mu + a^\mu \tag{2.19}$$

2) dilatation

$$x'^\mu = \alpha x^\mu \tag{2.20}$$

3) rotation

$$x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \tag{2.21}$$

4) special conformal transformation

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} \tag{2.22}$$

Also, the generators for the above transformations are listed below.

1) translation

$$P_{\mu} = -i\partial_{\mu} \tag{2.23}$$

2) dilatation

$$D = -ix^{\mu}\partial_{\mu} \tag{2.24}$$

3) rotation

$$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \tag{2.25}$$

4) special conformal transformation

$$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}) \tag{2.26}$$

These generators obey the following commutation relations, which define the conformal algebra:

$$\begin{aligned}
[D, P_{\mu}] &= iP_{\mu} \\
[D, K_{\mu}] &= -iK_{\mu} \\
[K_{\mu}, P_{\nu}] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_{\rho}, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_{\nu} - \eta_{\rho\nu}K_{\mu}) \\
[P_{\rho}, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_{\nu} - \eta_{\rho\nu}P_{\mu}) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}).
\end{aligned} \tag{2.27}$$

The commutator except for the above equals to zero. When we redefine the generators as follows

$$J_{\mu\nu} = L_{\mu\nu},\ J_{-1,0} = D,\ J_{-1\mu} = \frac{1}{2}(P_{\mu} - K_{\mu}),\ J_{0\mu} = \frac{1}{2}(P_{\mu} + K_{\mu}), \tag{2.28}$$

where $J_{ab} = -J_{ba}$ and $a, b \in \{-1, 0, 1, \cdots, d\}$, these new generators obey the $so(2, d)$ algebra

$$[J_{ab}, J_{cd}] = i(\eta_{ad}L_{bc} + \eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac}) \tag{2.29}$$

where the metric η_{ab} is $diag(-1, -1, 1, \cdots, 1)$.

Next, we will seek a matrix representation of the conformal group. Given an infinitesimal conformal transformation parametrized by ω, we are able to seek a matrix representation T such that a field $\Phi(x)$ transforms as

$$\Phi'(x') = (1 - i\omega T)\Phi(x). \tag{2.30}$$

In order to find out the concrete form of these operators, we will start by studying of the Poincaré group. Let us consider the Lorentz transformation. We introduce a matrix representation $S_{\mu\nu}$ to define the action of infinitesimal Lorentz transformation on the field $\Phi(0)$

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \tag{2.31}$$

where $S_{\mu\nu}$ is the spin operators associated with the field Φ. Next, we consider to translate $L_{\mu\nu}$ to a non-zero value of x. This is done by transforming $L_{\mu\nu}$ into $e^{ix\cdot P}L_{\mu\nu}e^{-ix\cdot P}$. By using the commutation relation, we obtain

$$e^{ix\cdot P}L_{\mu\nu}e^{-ix\cdot P} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu. \tag{2.32}$$

Here, we use the Baker-Hausdorff-York formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots \tag{2.33}$$

where A and B are operators. This allows us to write the action of the generators

$$P_\mu \Phi(x) = i\partial_\mu \Phi(x) \tag{2.34}$$
$$L_{\mu\nu}\Phi(x) = S_{\mu\nu}\Phi(x) + i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x). \tag{2.35}$$

Similarly, we can obtain matrix representations for dilatation and special conformal transformation. When we denote the respective values of these generators D and K_μ at the origin by Δ and κ_μ, we obtain representations for these operators at an arbitrary point x by

$$D\Phi(x) = (-ix^\nu\partial_\nu + \Delta)\Phi(x) \tag{2.36}$$
$$K_\mu\Phi(x) = (\kappa_\mu + 2x_\mu\Delta - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu)\Phi(x). \tag{2.37}$$

Also, we can obtain the change in Φ under a finite conformal transformation. For simplicity, we restrict spin-less fields that is $S_{\mu\nu} = 0$. Under a conformal transformation $x \to x'$, a spin-less field $\phi(x)$ transforms as

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi(x) \tag{2.38}$$

where $|\partial x'/\partial x|$ is the Jacobian of the conformal transformation of the coordinates. Fields transforming like the above are called a quasi-primary field.

2.3 Conformal Group in $d = 2$

The conformal group in two dimensions is larger than that in $d \geq 3$. In the preceding subsection, we found that in $d \geq 3$ dimensions conformal transformation consists of translation, rotation, dilatation, and special conformal transformation. The situation is different. In two dimensions, there is only one constraint (2.4). In this subsection, we will consider the conformal group in two dimensions. We will work a Euclidean signature by Wick rotating a time direction as $t = -i\tau$.

The condition (2.4) tells us that a infinitesimal parameter ϵ satisfies

$$\partial_\tau\epsilon_\tau = \partial_x\epsilon_x \ , \ \partial_\tau\epsilon_x = -\partial_x\epsilon_\tau, \tag{2.39}$$

which is just the Cauchy-Riemann equation appearing in complex analysis. It is clear that (2.8) is satisfied trivially by using these condition (2.39). When we introduce complex variables in the following way

$$z = x + i\tau, \ \bar{z} = x - i\tau \tag{2.40}$$
$$\epsilon = \epsilon^x + i\epsilon^\tau, \ \bar{\epsilon} = \epsilon^x - i\epsilon^\tau, \tag{2.41}$$

a holomorphic function $f(z) = z + \epsilon(z)$ and an anti-holomorphic function $\bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$ give rise to an infinitesimal conformal transformation $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$. The metric transforms as

$$ds^2 = d\tau^2 + dx^2 = dzd\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z}, \tag{2.42}$$

which implies that the scale factor is equal to $|\partial f/\partial z|^2$.

All that we have inferred from (2.39) is purely local. It is not obvious that there are any conditions that conformal transformations in two dimensions are defined everywhere or invertible, which must be imposed in order to form a group. Therefore, we should distinguish global conformal transformations, which satisfy the above requirement, from local conformal transformations. The set of global conformal transformations forms the special linear transformation $SL(2,\mathbb{C})$

$$z \to f(z) = \frac{az+b}{cz+d} \tag{2.43}$$

where a, b, c and d are complex constants and satisfy $ad-bc=1$. These transformations contain translations, dilatations, rotations, and special conformal transformations.

Next, we will consider the algebra of conformal group. Any holomorphic infinitesimal transformations are expressed by Laurent expanding ϵ at $z=0$ as

$$z' = z + \epsilon(z) = z + \sum_{n=-\infty}^{\infty} c_n z^{n+1}. \tag{2.44}$$

The effect of such a mapping on a spin-less field $\phi(z,\bar{z})$ is

$$\phi'(z',\bar{z}') = \phi(z,\bar{z}) = \phi(z,\bar{z}) - \epsilon(z')\partial'\phi(z',\bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z',\bar{z}') \tag{2.45}$$

where ∂ and $\bar{\partial}$ are differential operators with respect to z and $\bar{z}$. By introducing the generators

$$l_n = -z^{n+1}\partial \tag{2.46}$$

$$\bar{l}_n = -\bar{z}^{n+1}\bar{\partial}, \tag{2.47}$$

a variation on spin-less field $\delta\phi$ is expressed as

$$\delta\phi = -\epsilon(z)\partial\phi(z,\bar{z}) - \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z,\bar{z}) = \sum_{n=-\infty}^{\infty}(c_n l_n + \bar{c}_n \bar{l}_n)\phi(z,\bar{z}). \tag{2.48}$$

These generators obey the following commutation relation

$$\begin{aligned} &[l_n, l_m] = (n-m)l_{n+m} \\ &[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \\ &[l_n, \bar{l}_m] = 0 \end{aligned} \tag{2.49}$$

where $n, m \in \mathbb{Z}$. The first two algebras are so-called the Witt algebra. Each of these two infinite dimensional algebras contains a finite dimensional algebra generated by $l_{\pm 1}$ and l_0. This is the sub-algebra associated with the global conformal group. Also there is a familiar algebra that is obtained by central extending the Witt algebra by a central charge c namely the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{0,n+m}. \tag{2.50}$$

A central charge c in conforomal field theory is a model dependent quantity. For example, it is equal to 1 for a free boson and 1/2 for a free fermion. We will explain later why it is necessary to central extend the Witt algebra.

2.4 Primary Fields

In this subsection, we will focus on conformal field theory in two dimensions and establish several basic definitions. Let us define two types of fields. First, we will define chiral or anti-chiral fields. Chiral fields are defined as fields only depending on z, while anti-chiral fields are defined as fields depending on an anti-holomnorphic variable $\bar{z}$. An another type that we will define is called a primary field. Primary fields are defined as fields that transform under conformal transformation $z \to f(z)$ according to

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(z,\bar{z}) \tag{2.51}$$

where $(h,\bar{h})$ are the holomorphic conformal dimension and its anti-holomorphic counterpart. In two dimensions, given a field with scaling dimension Δ and planar spin s, we can define the holomorphic conformal dimension and its anti-holomorphic counterpart as

$$h = \frac{1}{2}(\Delta + s),\ \bar{h} = \frac{1}{2}(\Delta - s). \tag{2.52}$$

In a special case that a field transforms as (2.51) for global conformal transformation, the field is called a quasi-primary field. By definition, primary fields is always quasi-primary fields but the reverse is not true.

We will consider infinitesimal conformal transformations and the variation of primary fields. If we consider an infinitesimal map $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{f}(\bar{z})$, the variation of primary fields up to first order in ϵ is given by

$$\phi'(z,\bar{z}) = \phi(z,\bar{z}) + \epsilon(z)\partial\phi(z,\bar{z}) + h\partial\epsilon(z)\phi(z,\bar{z}) + \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z,\bar{z}) + \bar{h}\bar{\partial}\bar{\epsilon}(z)\phi(z,\bar{z}) \equiv \phi(z,\bar{z}) + \delta\phi(z,\bar{z}). \tag{2.53}$$

2.5 Stress-Energy Tensor

In usual field theory, there is a Lagrangian that is a functional of fundamental fields. This quantity is determined by the symmetries of the theory. The Noether's theorem tells us that there is a conserved current j^{μ} for all continuous symmetry transformation. For example, a stress-energy tensor is a Noether current corresponding to infinitesimal coordinate transformation. We denote the Lagrangian as $\mathcal{L}$. When the fundamental field ϕ changes

$$\phi(x) \to \phi'(x') = \phi(x) + \epsilon\delta\phi(x) \tag{2.54}$$

under the infinitesimal coordinate transformation, the Lagrangian changes

$$\mathcal{L} \to \mathcal{L} + \epsilon\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi\right) + \epsilon\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu})\phi}\right)\right]\delta\phi. \tag{2.55}$$

The third term is just the Euler-Lagrange equation. If this transformation is a symmetry transformation, we can allow the Lagrangian to change by a total derivative

$$\mathcal{L} \to \mathcal{L} + \epsilon\partial_{\mu}\mathcal{J}^{\mu}. \tag{2.56}$$

Therefore, we find a conserved current

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi - \mathcal{J}^{\mu} \tag{2.57}$$

In this section, we will focus on a current corresponding to the conformal transformation.

In order to obtain the current for conformal field theory in d dimension, let us recall Noether's theorem. We are interested in current for a conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$ and we have a current

$$j_\mu = T_{\mu\nu}\epsilon^\nu \tag{2.58}$$

where the tensor $T_{\mu\nu}$ is a symmetric tensor. We often call this a stress-energy tensor.

As this current conserves, for constant ϵ, we have

$$\partial_\mu T^\mu_\nu = 0. \tag{2.59}$$

For more general transformation $\epsilon(x)$, a conservation law of this current implies

$$T_{\mu\nu}\partial^\mu\epsilon^\nu = \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) = \frac{1}{2}T^\mu_\mu f = 0. \tag{2.60}$$

In the last equality, we use a constraint for conformal transformations (2.4). Above two equation imply that in the conformal field theory, the stress-energy tensor is conserved and traceless.

In two dimensional conformal field theory, it is shown that the stress-energy tensor can be decomposed into chiral and anti-chiral part. To do so, we again perform the change of coordinates from real to the complex ones and we find

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{tt} - T_{xx} - 2iT_{tx}) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{tt} - T_{xx} + 2iT_{tx}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}(T_{tt} + T_{xx}) \end{aligned} \tag{2.61}$$

We adopt a Euclidean flat metric $\eta_{\mu\nu} = diag(1,1)$. Since a stress tensor is traceless, we find $T_{tt} = -T_{xx}$. By substituting this relation, non-diagonal component in (2.61) vanishes. Finally by using a conservation law (2.59), we find

$$\partial_t T_{tt} + \partial_x T_{xt} = 0,\ \partial_t T_{tx} + \partial_x T_{xx} = 0, \tag{2.62}$$

from which it follows that

$$\partial_z T_{\bar{z}\bar{z}} = \partial_{\bar{z}} T_{zz} = 0. \tag{2.63}$$

This implies that two non-vanishing component of a stress-energy tensor are chiral and anti-chiral field and we define

$$T(z) = T_{zz}(z),\ \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z}). \tag{2.64}$$

3 Operator Formalism of Conformal Field Theory

In the preceding section, we introduced the fundamental definition of the conformal transformation and the classical field theory. When we treat the field theory quantum mechanically, we must quantize fundamental fields by an appropriate procedure like the canonical quantization. In the conformal field theory, one of the appropriate methods is called the radial quantization. In this section, we will review a procedure of the radial quantization and treat the field or current as operators.

3.1 Radial Quantization

The operator formalism distinguishes a time direction from a space direction. We choose the space direction along a circle centered at the origin and the time direction to be orthogonal to the space direction. Then, the whole spacetime forms the cylinder. This choice of spacetime leads to the radial quantization of two-dimensional conformal field theory.

In order to make this choice, we will define conformal field theory on an infinite cylinder with a time t going from $-\infty$ to ∞ along an orthogonal direction of a circle and space being compactified with a coordinate x going from 0 to L. Here, the points $(0, t)$ and (L, t) are identified. If we analytically continue to Euclidean signature, the cylinder is described by a single complex coordinate $\xi = t + ix$. We perform a change of variable by mapping the cylinder to the complex plane via

$$z = e^{\frac{2\pi}{L}\xi}. \tag{3.1}$$

Then, former time translations $t \to t + t_0$, with a positive t_0, are mapped to complex dilatation $z \to e^{\frac{2\pi}{L}t_0}z$ and space translations $x \to x + x_0$ are mapped to rotation $z \to e^{\frac{2\pi}{L}ix_0}z$. Then, according to the change of variable (3.1), a time direction corresponds to a radial direction of circles on the complex plane and a space direction corresponds to a angle direction, which is depicted in Figure 1.

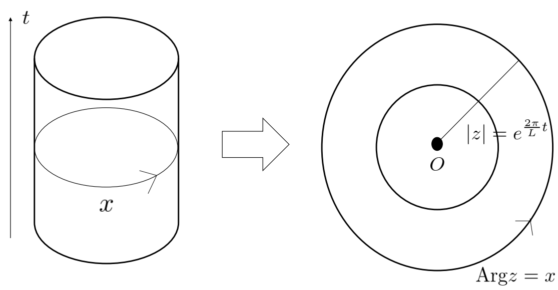

Figure 1: Conformal Transformation from cylinder to circle

We assume the existence of a vacuum state $|0\rangle$. From now, we will consider a primary field $\phi(z, \bar{z})$ with conformal weights $(h, \bar{h})$, where $\bar{z} = e^{\frac{2\pi}{L}\bar{\xi}}$ and $\bar{\xi} = t - ix$. We assume that interactions are weakened as $t \to -\infty$, and the asymptotic field is given by

$$\phi_{in} \propto \lim_{t\to-\infty} \phi(x, t) = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z}). \tag{3.2}$$

This field reduces to a single operator, which create a single asymptotic in-state acting on $|0\rangle$

$$|\phi\rangle_{in} = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z}) |0\rangle . \tag{3.3}$$

Next, we will consider the Hermitian conjugation of this asymptotic state $|\phi\rangle_{in}$. In Euclidean signature $\tau = it$, the Hermitian conjugation of the field $\phi(\tau, x)$ is proportional to $\phi(-\tau, x)$. So, in radial quantization, this justifies the following definition of Hermitian conjugation

$$\phi^\dagger(z, \bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \tag{3.4}$$

where a coefficient on the right hand side is necessary to obtain definite inner product of ϕ. We define am asymptotic out-state as

$$_{out}\langle\phi| = |\phi\rangle_{in}^\dagger . \tag{3.5}$$

Finally, we comment on some constraints for asymptotic state. A primary field ϕ is expanded as

$$\phi(z,\bar{z}) = \sum_{n,m} z^{-m-h}\bar{z}^{-m-\bar{h}}\phi_{m,n} \tag{3.6}$$

where $\phi_{m,n}$'s are Laurent coefficients around $z = \bar{z} = 0$ given by

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}). \tag{3.7}$$

A straightforward Hermitian conjugation tells us

$$\phi^{\dagger}(z,\bar{z}) = \sum_{n,m} \bar{z}^{-m-h}z^{-m-\bar{h}}\phi^{\dagger}_{m,n}, \tag{3.8}$$

while from (3.4) we obtain

$$\phi^{\dagger}(z,\bar{z}) = \sum_{n,m} \bar{z}^{-m-h}z^{-m-\bar{h}}\phi_{-m,-n}. \tag{3.9}$$

Comparing these two expressions, we can obtain the Hermitian conjugation of modes $\phi^{\dagger}_{m,n} = \phi_{-m,-n}$. In order to obtain a well-defined in-state, a constraint that an in-state is not singular at any point is imposed. It tells us that for $m > -h$ and $n > -\bar{h}$,

$$\phi_{m,n}\left|0\right\rangle = 0, \tag{3.10}$$

and the in-state simplifies

$$\left|\phi\right\rangle_{in} = \phi_{-h,-\bar{h}}\left|0\right\rangle. \tag{3.11}$$

In later discussion, we will omit subscripts *in* or *out*.

3.2 Operator Product Expansion

In the preceding subsection, it was shown that a time direction corresponds to a radial direction of circle on the complex plane. This fact tells us that the time ordering, which appears in correlation functions, becomes a radial ordering, which is defined explicitly as

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & for\ |z| > |w| \\ B(w)A(z) & for\ |z| < |w| \end{cases} \tag{3.12}$$

for arbitrary operators A and B.

With this definition, it is clear that equal time commutators or operator product expansions for some operators are given by a contour integral of radial ordering. Let $a(z)$ and $b(z)$ be two holomorphic operators and consider the integral

$$\oint_{C_w} dz a(z)b(w) \tag{3.13}$$

where the contour C_w is a circle counterclockwise aroud w. When we split the contour into two parts depicted in Figure2, the integral reduces a commutator

$$\oint_{C_w} dz R(a(z)b(w)) = \oint_{C_1} a(z)b(w) - \oint_{C_2} a(z)b(w) = \oint dz[a(z),b(w)]. \tag{3.14}$$

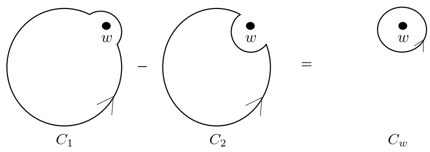

Figure 2: Contours

An equal time commutator $[a, b]$ can be obtained by integrating with respect to w

$$[a,b] = \oint_{C_0} dw \oint_{C_w} dz R(a(z)b(w)) \tag{3.15}$$

where C_0 is circle counterclockwise around the origin.

In quantum field theories, the Noether charge associated to symmetry transformations Q relates to a variation of fields $\delta\phi$ and it is given by

$$\delta\phi = [Q, \phi]. \tag{3.16}$$

Recall that the conserved current associated to the conformal transformation is the stress tensor $j_\mu = T_{\mu\nu}\epsilon^\nu$. There exists a conserved charge

$$Q = \int dx j_0 \tag{3.17}$$

where the integral is performed at a hypersurface with constant t. In radial quantization, we refer that the hypersurface corresponds to one with constant $|z|$. So, the integral $\int dx$ reduce to a contour integral $\oint dz$. we define a conserved charge by dividing by $2\pi i$ in our convention

$$Q = \frac{1}{2\pi i}\oint dz T(z)\epsilon(z) + \frac{1}{2\pi i}\oint d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}). \tag{3.18}$$

By substituting this into (3.16), we obtain the relation between a variation of a field and the radial ordering

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint dz[T(z)\epsilon(z), \phi(w,\bar{w})] + \frac{1}{2\pi i}\oint d\bar{z}[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w})]. \tag{3.19}$$

In the case of a primary field, since $\delta\phi$ is given by (2.53), we find that the left hand side of (3.19) is given in terms of contour integrals by

$$\delta\phi = \frac{1}{2\pi i}\oint dz \left(\frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial\phi(w,\bar{w})\right)$$

where anti-holomorphic parts are omitted for simplicity. Therefore, we obtain operator product expansions for the primary field ϕ as

$$R(T(z)\phi(w)) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial\phi(w,\bar{w}) + \cdots \tag{3.20}$$

$$R(\bar{T}(\bar{z})\phi(w)) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\phi(w,\bar{w}) + \cdots \tag{3.21}$$

where we omit regular terms at $z = w$ and the dots symbol denotes these terms. The operator product expansion with the stress tensor is often regarded as an alternative definition of a primary field. In later discussion, we will omit the radial ordering symbol R in our convention.

Finally, we will comment on the operator product expansion $T(z)T(w)$. If the stress tensor was a primary field, one could obtain

$$T(z)T(w) = \frac{\Delta_T}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \cdots$$

where Δ_T denotes a conformal dimension of the stress tensor. Unfortunately, it is not correct. For example, we will consider a theory containing a free boson φ. This theory is described by the action

$$S = \frac{1}{2}\int d^2x \partial_\mu \varphi \partial^\mu \varphi. \tag{3.22}$$

The propagator is given by

$$\langle \varphi(z,\bar{z})\varphi(w,\bar{w})\rangle = -\log{(z-w)^2} - \log{(\bar{z}-\bar{w})^2}. \tag{3.23}$$

the holomorphic part and the ant-holomorphic part are separated by taking the derivatives and we read the operator product expansion for $\partial\varphi$ or $\bar{\partial}\varphi$ as

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z-w)^2} + \cdots \tag{3.24}$$

$$\bar{\partial}\varphi(\bar{z})\bar{\partial}\varphi(\bar{w}) = -\frac{1}{(\bar{z}-\bar{w})^2} + \cdots . \tag{3.25}$$

In this theory, a stress tensor is defined by

$$T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi : (z) \equiv -\frac{1}{2}\lim_{w\to z}[\partial\varphi(z)\partial\varphi(w) - \langle\partial\varphi(z)\partial\varphi(w)\rangle]. \tag{3.26}$$

By using Wick's theorem, we can obtain

$$T(z)T(w) = \frac{\frac{1}{2}}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \cdots . \tag{3.27}$$

In the free boson theory, an anomalous term appears. In the free fermion theory, an anomalous term also appears, however, the coefficient is one quarter. In ghosts system, an anomalous term with coefficient 1 appears. Due to this term, the stress tensor is not strictly the primary field. In general, the operator product expansion $T(z)T(w)$ is written by

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \cdots$$

where c is a model dependent quantity, for example $c = 1$ for a free scalar, and called the central charge. It is well-known that an extra term is added into the conformal transformation $z \to w$ of the stress tensor

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z,w\} \tag{3.28}$$

where the second term is the Shwarzian derivative given by

$$\{f,z\} = \frac{f'''(z)}{f(z)} - \frac{3}{2}\left(\frac{f''(z)}{f(z)}\right)^2. \tag{3.29}$$

It is clear that the Shwarzian derivative exactly vanishes when conformal transformations are global transformations. This means that the stress tensor is a quasi-primary operator with conformal weight 2.

3.3 Virasoro Algebra

In the last subsection, we stated that the operator product expansion of a stress tensors reads

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \cdots \tag{3.30}$$

with the central charge c. In this section, we will focus on the equal time commutator or the algebra of stress tensors. Since a stress tensor is quasi-primary, we perform a Laurent expansion of $T(z)$ as

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n \tag{3.31}$$

where L_n is a Laurent coefficient defined by

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z). \tag{3.32}$$

The equal time commutator is given by (3.15) and after performing the contour integral, we can obtain

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{0,n+m}(n^3 - n). \tag{3.33}$$

Taking the anti-chiral stress tensor into account, one can obtain the Virasoro algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}\delta_{0,n+m}(n^3 - n) \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}\delta_{0,n+m}(n^3 - n) \\
[L_n, \bar{L}_m] &= 0.
\end{aligned} \tag{3.34}$$

This is a central extension of the Witt algebra. We can interpret $L_0, L_{\pm 1}$ as global conformal transformation generators.

Finally we will comment on the definition of a primary field in terms of Laurent modes. A operator, which can be expanded in terms of Laurent modes

$$\phi(z) = \sum_n z^{-n-h}\phi_n, \tag{3.35}$$

is called a primary operator with a conformal dimension h, when it satisfies the equal time commutator

$$[L_n, \phi_m] = ((h-1)n - m)\phi_{n+m}. \tag{3.36}$$

If the above condition is satisfies only for $L_0, L_{\pm 1}$, a operator is called a quasi-primary operator. When we restrict global generators, the second terms in (3.34) are eliminated and the stress tensor is interpreted as a quasi-primary operator with a conformal dimension 2 from the definition (3.36).

3.4 Correlation functions of Chiral Primary Operators

The objects of interest in quantum field theory are correlation functions, which is calculated by a perturbative method. In usual quantum field theory, it can be obtained using either canonical quantization or path integral. To do so, we must consider an action integral. In contrast, conformal field theory is very powerful and some correlation functions, such as two-point or three-point functions, are determined by the conformal invariance. In this subsection, we will see how to obtain the correlation functions by the conformal invariance. These expressions will allow us to derive a general formula for operator product expansion of quasi-primary operators.

First, let us focus on the two-point function of two chiral quasi-primary operators. Let two chiral quasi-primary operators with conformal dimension h_1 and h_2 be $\phi_1(z)$ and $\phi_2(z)$. We will write two-point function of these as

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = g(z_1, z_2). \tag{3.37}$$

The translation invariance generated by L_{-1} tells us $g(z_1, z_2) = g(z_1 - z_2)$. g is invariant under an identical translation $z_i \to z_i + a$. Next, we consider scale transformation $z \to \lambda z$ generated by L_0. The invariance under rescaling implies that

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle \to \lambda^{h_1+h_2} \langle \phi_1(\lambda z_1)\phi_2(\lambda z_2)\rangle = g(z_1 - z_2)$$

from which we conclude

$$g(z_1 - z_2) = \frac{d_{12}}{(z_1 - z_2)^{h_1+h_2}}$$

where a constant d_{12} is called a structure constant. Finally, the invariance under L_1 implies the invariance under inversion transformation $z \to -1/z$ for which we find

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle \to z_1^{-2h_1} z_2^{-2h_2} \frac{d_{12}}{(-\frac{1}{z_1} + \frac{1}{z_2})^{h_1+h_2}} = g(z_1 - z_2).$$

This is satisfied only if the case $h_1 = h_2$. We conclude that the two-point function of two quasi-primary operators is determined by global conformal transformations and the form is given by

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{d_{12}\delta_{h_1,h_2}}{(z_1 - z_2)^{2h_1}}. \tag{3.38}$$

For example, in the case of stress tensors, we find

$$\langle T(z_1)T(z_2)\rangle = \frac{\frac{c}{2}}{(z_1 - z_2)^4}. \tag{3.39}$$

Next, we go to the three-point function. Let an extra quasi-primary operator with conformal dimension h_3 be $\phi_3(z)$. The translation invariance tells us that the three-point function is a function with respect to the difference $z_{ij} = z_i - z_j$

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(z_{12}, z_{23}, z_{13}). \tag{3.40}$$

Similarly in two-point function, scale and special conformal invariance are required and we actually obtain the three point function by solving these constraints

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \tag{3.41}$$

with some structure constant C_{123}.

3.5 Normal Ordered Product

Consider the product operator of free bosons at the same point $\partial\varphi\partial\varphi(z)$. This is singular, since the operator product expansion of free bosons (3.24) has a singularity at $z = w$, and we should regularize to define the product operator. This is simply done by subtracting the corresponding vacuum expectation value. We define the regularized product by

$$:\partial\varphi\partial\varphi:(z)=\lim_{w\to z}[\partial\varphi(z)\partial\varphi(w)-\langle\partial\varphi(z)\partial\varphi(w)\rangle]. \tag{3.42}$$

This is just the stress tensor for a free scalar theory. In terms of modes, this is equivalent to the normal ordering in which the operators annihilating the vacuum are put at the rightmost position.

However, this prescription does not work for some operators. For example, we consider a product of stress tensors at the same point $TT(z)$. Even if we subtract the vacuum expectation value of $T(z)T(w)$, only most singular term is eliminated and two singular terms remain.

It is clear how this prescription should be generalized. Instead of subtracting only the vacuum expectation value, we should subtract all singular terms of the operator product expansion. This is just a definition of the normal ordered product operator. We will write the normal-ordered version of $A(z)B(w)$ by $(AB)(z)$.

More mathematically, if the operator product expansion of A and B is written as

$$A(z)B(w)=\sum_{n=-N}^{\infty}(z-w)^n\{AB\}_n(w), \tag{3.43}$$

where N is some positive integer, then

$$(AB)(w)=\{AB\}_0(w)=\frac{1}{2\pi i}\oint dz\frac{A(z)B(w)}{z-w}. \tag{3.44}$$

It is convenient to define the singular operator product expansion or contraction as the singular parts of operator product expansion by

$$\overbrace{A(z)B}(w)=\sum_{n=-N}^{-1}(z-w)^n\{AB\}_n(w). \tag{3.45}$$

Hence, the normal ordered product can be rewritten as

$$(AB)(w)=\lim_{z\to w}\left[A(z)B(w)-\overbrace{A(z)B}(w)\right]. \tag{3.46}$$

In the case for a stress tensor, the operator product expansion containing the normal order product is given by

$$T(z)T(w)=\frac{\frac{c}{2}}{(z-w)^4}+\frac{2}{(z-w)^2}T(w)+\frac{1}{z-w}\partial T(w)+(TT)(w)+(z-w)(\partial TT)(w)+\cdots. \tag{3.47}$$

Notice that the operator (TT) is not primary. This is because the operator product expansion of (TT) with T is given by

$$\overbrace{T(z)(TT})(w)=\frac{10c}{(z-w)^6}+\frac{12}{(z-w)^4}T(w)+\frac{10}{(z-w)^3}\partial T(w)+\frac{4}{(z-w)^2}(TT)(w)+\frac{1}{z-w}\partial(TT)(w).$$

However, by subtracting a term proportional to $\partial^2 T$, one can construct a new quasi-primary operator

$$\Lambda(z)=(TT)(z)-\frac{3}{10}\partial^2T(z) \tag{3.48}$$

Assume Λ can be expanded as

$$\Lambda(z) = \sum_{n=-\infty}^{\infty} z^{-n-4}\Lambda_n \tag{3.49}$$

with

$$\Lambda_n = \frac{1}{2\pi i}\oint dz z^{n+3}\Lambda(z). \tag{3.50}$$

Then, the equal time commutator $[L_n, \Lambda_m]$ is given by

$$[L_n, \Lambda_m] = \frac{8}{15\beta}n(n^2-1)L_{n+m} + (3n-m)\Lambda_{n+m} \tag{3.51}$$

where β is constant related to the central charge

$$\beta = \frac{16}{22+5c}. \tag{3.52}$$

If we restrict to global conformal transformations (L_{-1}, L_0, L_1), the first term in right hand side vanishes. Then, it turns out that the operator $\Lambda(z)$ is a quasi-primary operator with a conformal dimension 4.

Finally, we will summerise several properties of the Hilbert space of conformal field theory. First, we consider the stress tensor $T(z)$. For Laurent expansion of $T(z)$, we find an asymptotic in-state, which corresponds to an excited state by the stress tensor

$$T(z) \leftrightarrow \lim_{z\to 0} T(z)\left|0\right\rangle = L_{-2}\left|0\right\rangle \tag{3.53}$$

where we employ similar procedure for the primary in-state in the previous subsection. Similarly for the derivative of that and normal ordered product, we find

$$\begin{aligned}
&\partial T(z) \leftrightarrow L_{-3}\left|0\right\rangle \\
&(TT)(z) \leftrightarrow L_{-2}L_{-2}\left|0\right\rangle .
\end{aligned}$$

These exapmle motivates the following statement: For each state $\left|\Phi\right\rangle$ in the Verma module

$$\left\{\left(\prod_{n=1}^{N} L_{k_n}\right)\left|0\right\rangle ; k_n \leq -2\right\}, \tag{3.54}$$

we can find a field $F \in \{T, \partial T, \cdots, (TT), \cdots\}$ with the property that $\lim_{z\to 0} F(z)\left|0\right\rangle = \left|\Phi\right\rangle$.

Now let us consider a primary operator with conformal dimension h. This gives rise to the in-state $\left|\phi\right\rangle = \phi_{-h}\left|0\right\rangle = \left|h\right\rangle$. For $n > 0$, this state satisfies $L_n\left|\phi\right\rangle = 0$ by definition of a primary field. While for $n < 0$, the states that L_n's are multiplied give rise to descendant states. For example, $L_{-1}\left|h\right\rangle \leftrightarrow \partial\phi$, $L_{-1}L_{-1}\left|h\right\rangle \leftrightarrow \partial^2\phi$, $L_{-2}\left|h\right\rangle \leftrightarrow (T\phi)$ and so on. The L_n for $n < 0$ work as the raising operators and the lowest state $\left|h\right\rangle$ is called a highest weight state. Each primary operator ϕ gives rise to an infinite set of descendant operators by taking derivatives and taking normal ordered product with T. The set of these operators

$$[\phi] = \{\phi, \partial\phi, \cdots, (T\phi), \cdots\} = \left\{\left(\prod_{n=1}^{N} L_{k_n}\right)\phi ; k_n \leq -1\right\} \tag{3.55}$$

is called a conformal family.

3.6 Current Operator and Its Algebra

The conformal symmetry is generated by the stress tensor and characterized by the Virasoro algebra (3.34). One can ask the following question. How about the conformal field theory with Lie-algebraic symmetry? The Noether's theorem tells us that an additional symmetry, for example, gauge symmetry, is generated by its corresponding current. Hence, when we consider the conformal field theory with an additional Lie-algebraic symmetry, we should introduce the chiral current $J(z)$ and the anti-chiral current $\bar{J}(\bar{z})$ associated with the symmetry transformation. It is well-known that in the case of theory of a free boson with $su(2)$ symmetry the current is expressed in terms of a free boson φ as $J^3(z) = i\partial\varphi(z),\ J^{\pm} = e^{\pm i\sqrt{2}\varphi(z)}$.

Let us investigate the algebra of the additional symmetry more precisely. We consider the following action

$$S_0 = \frac{1}{4a^2}\int d^2x \mathrm{Tr}[\partial^{\mu}g^{-1}\partial_{\mu}g] \tag{3.56}$$

where a^2 is a positive constant and g is a matrix bosonic field living on the goup manifold G. The trace is taken over a representation of the group t_a

$$\mathrm{Tr}t_a t_b = \eta_{ab}. \tag{3.57}$$

where η is a Killing metric, which represents the group manifold. From the equations of motion $\delta S_0/\delta g = 0$, one can find a conserved current

$$J_{\mu} = g^{-1}\partial_{\mu}g. \tag{3.58}$$

However, the components J_z and $J_{\bar{z}}$ are not separately conserved. In order to conserve the current components separately, we should add a more complicated action to the above action

$$\Gamma = -\frac{i}{24\pi}\int_B d^3x \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(\bar{g}^{-1}\partial^{\alpha}\bar{g}\bar{g}^{-1}\partial^{\beta}\bar{g}\bar{g}^{-1}\partial^{\gamma}\bar{g}). \tag{3.59}$$

This is defined on a three dimensional manifold B, whose boundary is original two-dimensional space. the expansion of the field g is denoted as $\bar{g}$.

Then we consider the action

$$S = S_0 + k\Gamma \tag{3.60}$$

where k is an integer. By taking the variation, we find

$$\left(1 + \frac{a^2 k}{4\pi}\right)\partial_z(g^{-1}\partial_{\bar{z}}g) + \left(1 - \frac{a^2 k}{4\pi}\right)\partial_{\bar{z}}(g^{-1}\partial_z g) = 0. \tag{3.61}$$

Thus, for $a^2 = 4\pi/k$, we find the desired conservation law. The conservation of the dual current is obtained for $a^2 = -4\pi/k$ with a negative k. The separate conservation implies that the action S is invariant under the following transformation

$$g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\bar{\Omega}^{-1}(\bar{z}) \tag{3.62}$$

where Ω and $\bar{\Omega}$ are arbitrary matrices valued in G. For infinitesimal transformation

$$\Omega(z) = 1 + \omega(z),\ \bar{\Omega}(z) = 1 + \bar{\omega}(\bar{z}),$$

the action is transformed as

$$\delta S = \frac{k}{2\pi}\int d^2x \mathrm{Tr}[\omega(z)\partial_{\bar{z}}(\partial_z g g^{-1}) - \bar{\omega}(\bar{z})\partial_z(g^{-1}\partial_{\bar{z}}g)] \tag{3.63}$$

with $a^2 = 4\pi/k$. The first term vanishes by integrating by parts. Thus $G \times G$ invariance is extended a local $G(z) \times G(\bar{z})$ invariance.

We redefine the conserved currents by

$$J = -k\partial_z g g^{-1} \tag{3.64}$$

$$\bar{J} = k g^{-1}\partial_{\bar{z}} g. \tag{3.65}$$

The conserved charge is calculated by a contour integral using an analogous discussion in Section.3

$$Q_G = -\frac{1}{2\pi i}\oint dz \omega^a J_a + \frac{1}{2\pi i}\oint d\bar{z}\bar{\omega}^a \bar{J}_a \tag{3.66}$$

here we perform the trace by using (3.57). A variation of field $\delta\phi$ is also obtained by $\delta\phi = [Q_G, \phi]$. When we consider $\phi = J$, we can find the operator product expansion between current operators

$$J^a(z)J^b(w) = \frac{k\eta^{ab}}{(z-w)^2} + if_{abc}\frac{J^c(w)}{z-w} + \cdots . \tag{3.67}$$

Here we use the Lie algebra

$$[t_a, t_c] = if_{ab}^c t_c. \tag{3.68}$$

Introducing the Laurent modes

$$J^a(z) = \sum_{n=-\infty}^{\infty} z^{-n-1} J_n^a \tag{3.69}$$

the operator product expansion briefly reduced to an affine Kac-Moody algebra

$$[J_n^a, J_m^b] = if_{abc}J_{n+m}^c + kn\eta_{ab}\delta_{n+m,0}. \tag{3.70}$$

It turns out that an integer k denotes central extension and we call this c-number the level of the Kac-Moody algebra. Also it is easily shown that the current operator is a quasi-primary operator with conformal dimension 1.

3.7 $\mathcal{W}$-extended Conformal Field Theory

The Hilbert space is spanned by the stress tensor and the current with conformal dimension 1. Here, one natural question occurs. Is it possible to extend the conformal symmetry by a current with conformal dimension $h > 2$? The answer is yes. Zamolodchikov [16] discovered an extension of the Virasoro algebra by including a current with conformal dimension 3. This extended conformal field theory and its algebra are called $\mathcal{W}_3$-extended conformal field theory and $\mathcal{W}_3$ algebra.

The $\mathcal{W}_3$ algebra [30] has the stress energy tensor $T(z)$ and a primary field $W(z)$ whose conformal weight is 3. Their singular operator product expansion are given by

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{3.71}$$

$$T(z)W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} \tag{3.72}$$

$$W(z)W(w) = \frac{\frac{c}{3}}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \tag{3.73}$$

$$\frac{1}{(z-w)^2}\left[2\beta\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right] + \frac{1}{z-w}\left[\beta\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right], \tag{3.74}$$

where

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z) \tag{3.75}$$

and β is given by

$$\beta = \frac{16}{22+5c}. \tag{3.76}$$

c is the central charge. These operators have following Laurent expansion:

$$W(z) = \sum_n W_n z^{-n-3}. \tag{3.77}$$

The commutators are given by

$$[L_n, L_m] = \quad (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{0,m+n} \tag{3.78}$$

$$[L_n, W_m] = \quad (2n-m)W_{n+m} \tag{3.79}$$

$$[W_n, W_m] = \quad \frac{c}{360}n(n^2-1)(n^2-4)\delta_{0,m+n} \tag{3.80}$$

$$+\frac{1}{30}\left(n-m\right)(2n^2-mn+2m^2-8\big) L_{n+m} + \beta(n-m)\Lambda_{n+m}. \tag{3.81}$$

Here, Λ_n is a Laurent mode for $\Lambda(z)$ and defined by (3.50). Taking the limit $c \to \infty$ and disregarding the normal-ordered operator, we can obtain the subalgebra of the wedge modes,L_n ($n = 0, \pm 1$) and W_m ($m = 0, \pm 1, \pm 2$). Choosing one normalization $W_n \to W_n/\sqrt{10}$, the subalgebra is rewritten as

$$[L_n, L_m] = (n-m)L_{n+m} \tag{3.82}$$

$$[L_n, W_m] = (2n-m)W_{n+m} \tag{3.83}$$

$$[W_n, W_m] = \frac{1}{3}\left(2n^2 - mn + 2m^2 - 8\right) L_{n+m}. \tag{3.84}$$

We call this subalgebra $su(1,2)$ algebra. While choosing other normalization $W_n \to iW_n/\sqrt{10}$, a different subalgebra appears:

$$[L_n, L_m] = (n-m)L_{n+m} \tag{3.85}$$

$$[L_n, W_m] = (2n-m)W_{n+m} \tag{3.86}$$

$$[W_n, W_m] = -\frac{1}{3}\left(2n^2 - mn + 2m^2 - 8\right) L_{n+m}. \tag{3.87}$$

We call this subalgebra $sl(3,\mathbb{R})$ algebra.

4 Anti-de Sitter space

In this section, we will review a spacetime that has a negative curvature, named anti-de Sitter space or AdS space. An AdS space is a maximally symmetric spacetime. Due to this property, this spacetime is defined as an embedded space in one higher dimensional Minkowski space. We will also discuss a symmetry of AdS space. The isometry of d dimensional AdS space is $SO(2,d)$.

4.1 Definition of AdS space

AdS space is one of solutions to the Einstein equation with a negative cosmological constant and it is a maximally symmetric spacetime.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{4.1}$$

where R and $R_{\mu\nu}$ are the Ricci scalar and the Ricci tensor and Λ is a cosmological constant, which is negative. We define the Ricci tensor, the Ricci scalar and the Christoffel symbol by the below expressions[31]:

- the Ricci tensor

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\rho\alpha}\Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\rho} \tag{4.2}$$

- the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{4.3}$$

- the Christoffel symbol

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho} \right) \tag{4.4}$$

Here, $g_{\mu\nu}$ denotes the metric tensor and $g^{\mu\nu}$ denotes the inversion of the metric tensor. We often call $d+1$ dimensional AdS space AdS$_{d+1}$.

Due to maximally symmetric, AdS$_{d+1}$ has the maximal number of spacetime symmetries $(d+1)(d+2)/2$. This is same with one of the hyperboloids embedded in $d+2$ dimensional Minkowski space $M^{(2,d)}$.

For a while, we will discuss the hyperboloid embedded in $M^{(2,d)}$. This space has a flat metric

$$\eta_{AB} = diag(-1, -1, 1, \cdots, 1). \tag{4.5}$$

The hyperboloid is defied as

$$X_A X^A = \eta_{AB} X^A X^B = -(X^0)^2 - (X^{d+1})^2 + \sum_{i=2}^{d} (X^i)^2 = -l_{AdS}^2, \tag{4.6}$$

where l_{AdS} is a constant. Subscripts are raised by the inversion of (4.5) and superscripts are lowered by (4.5). Generators of symmetric transformation are $2d$ boosts and $(d-2)(d-1)/2$ rotations namely $SO(2,d)$ generators. In general , AdS$_{d+1}$ is defined as a hyperboloid (4.6) and l_{AdS} is called the AdS radius. Roughly sketch of two dimensional AdS space is Figure3.

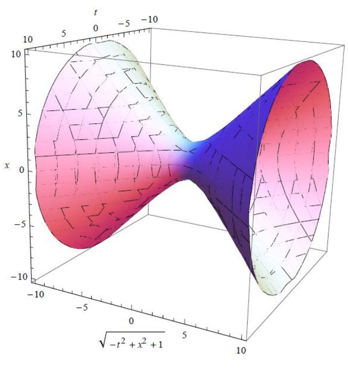

Figure 3: AdS space in two dimension

We can solve (4.6) with respect to one variable X_{d+1}

$$X_{d+1} = \sqrt{l_{AdS}^2 + \sum_{i=0}^{d} (X^i)^2} = \sqrt{l_{AdS}^2 + \eta_{\mu\nu} X^\mu X^\nu}, \tag{4.7}$$

where $\eta_{\mu\nu} = diag(-1, 1, \cdots, 1)$. Substituting it into the line element $ds^2 = \eta_{AB} dX^A dX^B$, the induced metric can be obtained

$$ds^2 = \left(\eta_{\mu\nu} - \frac{X_\mu X_\nu}{l_{AdS}^2 + \eta_{\mu\nu} X^\mu X^\nu} \right) dX^\mu dX^\nu. \tag{4.8}$$

From this expression, we can calculate the Ricci scalar and it is given by

$$R = -\frac{d(d+1)}{l_{AdS}^2} < 0. \tag{4.9}$$

Furthermore, from the Einstein equation (4.1) and the Ricci scalar (4.9), the cosmological constant is given by

$$\Lambda = \frac{d-1}{2(d+1)} R = -\frac{d(d-1)}{2 l_{AdS}^2} < 0. \tag{4.10}$$

The AdS space has a negative cosmological constant and a negative curvature. Also AdS$_{d+1}$ space has the isometry $SO(2, d)$. All of these transformation are generated by the following Killing vectors

- Boost

  $$J_{AB} = X^A \partial_B + X^B \partial_A \tag{4.11}$$

  in $A \in \{0, d+1\}$ and $B \notin \{0, d+1\}$ planes, and

- Rotation

  $$L_{AB} = X^A \partial_B - X^B \partial_A \tag{4.12}$$

  in $A, B \notin \{0, d+1\}$ planes.

Also there is a characteristic $SO(2, d)$ invariant quantity. We denote two point as X_1^A and X_2^A. The invariant distance or AdS distance can be defined by

$$P(X_1|X_2) = \eta_{AB}(X_1^A - X_2^A)(X_1^B - X_2^B). \tag{4.13}$$

4.2 Coordinates System of AdS Space

From the definition of AdS space (4.6), it is shown that AdS$_{d+1}$ can be parametrized by $d+1$ coodinates. Although we can parametrize AdS space in arbitrary dimensions, we restrict three dimensional AdS space for simplicity. There are two typical coordinates that represent AdS$_3$ namely global and Poincaré. Note that for simplicity we set $l_{AdS} = 1$ in the following discussion.

- the global coordinates (ρ, τ, ϕ)
  The global coordinates of AdS$_3$ can be embedded as follows:

  $$\begin{aligned} X^0 &= \cosh\rho \cos\tau \\ X^1 &= \sinh\rho \cos\phi \\ X^2 &= \sinh\rho \sin\phi \\ X^3 &= \cosh\rho \sin\tau \end{aligned} \tag{4.14}$$

  The metric in this coordinates is given by

  $$ds^2 = -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\phi^2 \tag{4.15}$$

  where $0 \le \rho < \infty, -\infty < \tau < \infty, 0 \le \varphi < 2\pi$. The boundary is located at $\rho \to \infty$.

- the Poincaré coordinates $(y, x, \bar{x})$
  The Poincaré coordinates can be defined as

$$\begin{aligned} X^0 &= \frac{1 + x'^2 - t'^2 + y^2}{2y} \\ X^1 &= \frac{1 - x'^2 + t'^2 - y^2}{2y} \\ X^2 &= \frac{x}{y} \\ X^3 &= \frac{t}{y}. \end{aligned} \tag{4.16}$$

  Furthermore, we define a light cone coordinates $x = t' - x'$ and $\bar{x} = t' + x'$ and the metric can be given by

$$ds^2 = \frac{1}{y^2}(dy^2 - dx d\bar{x}) \tag{4.17}$$

  where $0 < y < \infty, -\infty < x, \bar{x} < \infty$. From the definition of embedding, this coordinate system covers half of the hyperboloid. The boundary direction is y direction and the boundary of AdS_3 is located at a hypersurface $y = 0$.

4.3 Killing Vectors and $sl(2, \mathbb{R})$ Algebra

As we discussed in the above, d dimensional AdS spacetime is defined as a hyperboloid in $d+1$ dimensional Minkowski spacetime (4.6). This hyperboloid is invariant under the $SO(2, d)$ transformation. So, AdS spacetime has two kind of Killing vectors, which corresponds to boost and rotation given by (4.11) and (4.12).

In three dimensions, this symmetry is holomorphic to $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ symmetry and those generators can be given by the linear combination of Killing vectors for AdS spacetime. The $SL(2, \mathbb{R})_L$ generators are given by

$$J_0 = \frac{L_{03} - L_{21}}{2}, \ J_1 = \frac{J_{02} + J_{31}}{2}, \ J_2 = \frac{J_{32} - J_{01}}{2}, \tag{4.18}$$

and the $SL(2, \mathbb{R})_R$ generators are given by

$$\bar{J}_0 = \frac{L_{03} + L_{21}}{2}, \ \bar{J}_1 = -\frac{J_{02} - J_{31}}{2}, \ \bar{J}_2 = -\frac{J_{32} + J_{01}}{2}. \tag{4.19}$$

They satisfy the commutation relations

$$[J_a, J_b] = -\epsilon_{ab}{}^{c} J_c, \tag{4.20}$$

where $\epsilon_{ab}{}^{c}$ is the Levi-Civita symbol that equals to 1 for even permutations of (0,1,2). Similar relations are holds for the $\bar{J}$ generators. Of course, Killing vectors are uniquely determined in arbitrary dimensions. Those in general dimensions are given in the review of AdS/CFT correspondence [32]. For simplicity, we omit the symbols L, R in the following.

We will review some representations for $SL(2, \mathbb{R})$ generators. Here, we will focus on holomorphic parts J_n. Anti-holomorphic parts are discussed similarly.

First, we will consider the global coordinates in three dimension given by (4.15). Substituting embedding coordinates into $SO(2, 2)$ generators L_{AB} and J_{AB} and taking the linear combination in order to form J_n, we obtain the Killing vectors

$$\begin{aligned} J_0 &= \frac{1}{2}(\partial_\tau + \partial_\phi) \\ J_1 &= \sin(\tau+\phi)\partial_\rho + \frac{1}{2}\coth\rho\cos(\tau+\phi)\partial_\phi + \frac{1}{2}\tanh\rho\cos(\tau+\phi)\partial_\tau \\ J_2 &= -\cos(\tau-\phi)\partial_\rho - \frac{1}{2}\coth\rho\sin(\tau-\phi)\partial_\phi + \frac{1}{2}\tanh\rho\sin(\tau-\phi)\partial_\tau. \end{aligned} \tag{4.21}$$

We consider to diagonalize J_0 and obtain the representation in the eliptic basis

$$L_0 = iJ_0,\ L_{\pm 1} = i(J_1 \pm J_2). \tag{4.22}$$

They satisfy the global Virasoro algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1},\ [L_1, L_{-1}] = 2L_0. \tag{4.23}$$

The Killing vectors can be rewritten as follows

$$\begin{aligned} L_0 &= i\partial_{x^+} \\ L_{\pm 1} &= ie^{\pm ix^+}\left[\coth 2\rho\partial_{x^+} - \sinh^{-2} 2\rho\partial_{x^-} \mp \frac{i}{2}\partial_\rho\right] \end{aligned} \tag{4.24}$$

where we define $x^\pm = \tau \pm \phi$. Notice that L_0 is a Hermitian operator while $L_{\pm 1}$ are adjoint operators. It is appropriate for the global coordinates since it diagonalizes the Hamiltonian in the global coordinate. Anti-holomorphic parts are obtained by replacing L_n and (x^+, x^-) by $\bar{L}_n$ and (x^-, x^+).

Second, we consider to diagonalize J_2 as

$$L^h_0 = -J_2,\ L^h_{\pm 1} = i(J_1 \mp J_0). \tag{4.25}$$

These also satisfy the Virasoro algebra, however, L^h_0 is anti-Hermitian and $L^h_{\pm 1}$ are Hermitian. This representation is appropriate for Poincaré coordinates, which have basis diagonalizing the dilatation operator. The Killing vectors can be rewritten as follows

$$\begin{aligned} L^h_0 &= -\frac{y}{2}\partial_y - x\partial_x \\ L^h_{-1} &= i\partial_x \\ L^h_1 &= -ixy\partial_y - ix^2\partial_x - iy^2\partial_{\bar{x}}. \end{aligned} \tag{4.26}$$

It turns out that the explicit relation between the basis for the global coordinates and ones for the Poincaré coordinates is given by

$$L_0 = -\frac{1}{2}(L^h_1 - L^h_{-1}),\ L_{\pm 1} = \pm L^h_0 + \frac{1}{2}(L^h_1 + L^h_{-1}). \tag{4.27}$$

Anti-holomorphic parts are also obtained by replacing L^h_n and $(x, \bar{x})$ by $\bar{L}^h_n$ and $(\bar{x}, x)$.

5 Gravity in Three Dimension

5.1 Einstein-Hilbert Action

In three dimensions, the action integral for the gravity sector is written by three dimensional Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R - 2\Lambda) + (\text{boundary terms}) \tag{5.1}$$

where G is gravitational constant, g is a determinant of the metric $g = \det g_{\mu\nu}$ and Λ is a cosmological constant. The spacetime is distinguished by a value of Λ, which can be positive, negative or null.:

1) $\Lambda > 0$: de Sitter space or dS space

2) $\Lambda = 0$: flat space

3) $\Lambda < 0$: Anti-de Sitter space or AdS space

Here, we will focus on AdS space. Boundary terms must be added in order to make the action principle well-defined. This action (5.1) gives an equation of motion for the metric by taking a variation with respect to $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{5.2}$$

Pure AdS spacetime (4.6) is also one of the solutions.

An important point is that any solutions of this vacuum equation with $\Lambda < 0$ is locally AdS. In three dimensions, the full curvature is determined by the Ricci tensor, since the Weyl tensor identically vanishes,

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{5.3}$$

By substituting (5.2) into this expression, it turns out that any solutions of Einstein equation have a constant curvature

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{5.4}$$

This means that there are no local degrees of freedom on three dimensional Einstein manifold. So, gravity in three dimensions is not dynamical but topological.

Although there are no degrees of freedom, gravity in three dimensions has interesting properties. Every spacetime is equivalent locally, however, causal structures may be different. For example, it is known that there exists an AdS black hole solution.

5.2 Black Hole Solution for the Einstein Equation

A three-dimensional AdS black hole was first discovered in 1992 by Bañados, Teitelboim and Zanelli [23]. We call this black hole solution the BTZ black hole.

First of all, we will give the metric for the BTZ black hole. The BTZ black hole is a solution for the Einstein equation with a negative cosmological constant. The Einstein equation gives a solution

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2(N^\phi(r)dt + d\phi)^2, \tag{5.5}$$

where the squared lapse N^2 and the angular shift N^ϕ are given by

$$N^2(r) = -8GM + r^2 + \frac{16G^2J^2}{r^2} \tag{5.6}$$

$$N^\phi(r) = -\frac{4GJ^2}{r^2} \tag{5.7}$$

with $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \phi \leq 2\pi$. The two constant parameter M and J are the conserved charge associated with asymptotic invariance under a time displacement and a rotation namely a mass and an angular momentum.

The BTZ metric (5.5) is stationary and axially symmetric. There are two horizons at the point $N(r_\pm) = 0$, which is given by

$$r_\pm = \left[4GM\left(1 \pm \sqrt{1 - \frac{J^2}{M^2}}\right)\right]^{\frac{1}{2}}. \tag{5.8}$$

When $|J| \leq M$, the BTZ has an event horizon at $r = r_+$ and a inner horizon at $r = r_-$. If $|J| = M$, both horizons coincide. This solutions is the extremal BTZ black hole.

When $M < 0$ or $J > M$, the horizon disappears and a naked singularity appears at $r = 0$. Therefore, the condition $|J| \leq M$ plays the role of a cosmic censorship condition. However, there is a special case $M = -1/8G$ and $J = 0$. In this case, the metric exactly coincides with global AdS space

$$ds^2_{global} = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2. \tag{5.9}$$

AdS spacetime is separated from the continuous spectrum of the BTZ black hole since the solution with $-1/8G < M < 0$ corresponds to naked singularities. While in the case of massless $M = J = 0$, the black hole looks like the Poincaré AdS

$$ds^2_{massless} = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2 \tag{5.10}$$

with an identification $\phi \sim \phi + 2\pi$. It seems a little odd that global AdS differs in mass from the Poincaré AdS. The difference arises since the time directions of these patches do not agree. This is because it gives rise to different definition of energy. The spectrum of the BTZ black hole is summarized in Figure 4[33].

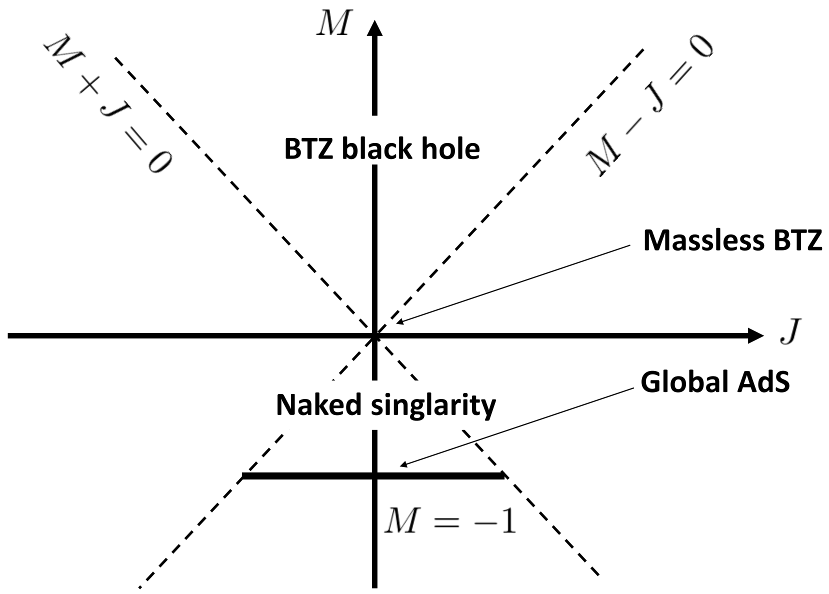

Figure 4: Spectrum of the BTZ black hole

Since the horizon size is $2\pi r_+$, the Hawking-Bekenstein entropy [5][6] is given by

$$S = \frac{2\pi r_+}{4G}. \tag{5.11}$$

From (5.11), one can determine the Hawking temperature of the BTZ black hole

$$T = \left(\frac{\partial S}{\partial M}\right)^{-1} = \frac{r_+^2 - r_-^2}{2\pi r_+}. \tag{5.12}$$

It was shown in [35], the most general asymptotically AdS solution to the Einstein equation is given by

$$ds^2 = \frac{dr^2}{r^2} - \left(rdx - \frac{1}{r}L(\bar{x})d\bar{x}\right)\left(rd\bar{x} - \frac{1}{r}\bar{L}(x)dx\right) \tag{5.13}$$

where L and $\bar{L}$ are two arbitrary functions and we introduce light-cone coordinates $x = t - \phi$ and $\bar{x} = t + \phi$.

5.3 Boundary Action Term

Before continuing the discussion of three-dimensional gravity, we will comment on the boundary action terms in (5.1). The boundary action contains the extrinsic curvature term, which makes the variation problem well-defined, and the counterterm action, which regularize the boundary stress tensor. The former is called the Gibbons-Hawking-York term and given by

$$S_{GHY} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma}K \tag{5.14}$$

where K is the trace of the extrinsic curvature and $\partial\mathcal{M}$ expresses the boundary region. In [34], Brown and York defined the quasilocal stress tensor associated with a spacetime region. This is given by variation of the gravitational action with respect to the boundary metric $\gamma_{\mu\nu}$

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{\mu\nu}}. \tag{5.15}$$

Let us investigate a concrete expression for the stress energy tensor. The three-dimensional gravitational action S consists of the Einstein-Hilbert action, the Gibbons-Hawking-York term, and a counterterm action

$$S = S_{EH} + S_{GHY} + \frac{1}{8\pi G}S_{ct}[\gamma], \tag{5.16}$$

The Gibbons-Hawking-York term is required to make a variation principle well-defined. On the other hand, the counterterm action S_{ct} is introduced in order to make the stress tensor finite.

To compute the concrete form of the stress tensor, we have to know the variation of the action with respect to $\gamma_{\mu\nu}$. The Einstein equation comes from the variation of the first action. When we consider the solution for the Einstein equation, the first term vanishes. So, only boundary terms contribute to the stress tensor

$$\delta S = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^dx\sqrt{-\gamma}\delta\gamma_{\mu\nu}\left(K^{\mu\nu} - K\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right). \tag{5.17}$$

Thus the quasilocal stress tensor is given by

$$T^{\mu\nu} = \frac{1}{8\pi G}\left(K^{\mu\nu} - K\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right).. \tag{5.18}$$

Here, the extrinsic curvature tensor is given by

$$K^{\mu\nu} = -\frac{1}{2}(\nabla^\mu n^\nu + \nabla^\nu n^\mu), \tag{5.19}$$

where n^μ is an outward pointing unit normal vector to the boundary $\partial\mathcal{M}$.

For example, in the case of Poincaré AdS space or massless BTZ

$$ds^2 = \frac{1}{y^2}dy^2 + \frac{1}{dy^2}dxd\bar{x}, \tag{5.20}$$

the normal vector on constant y is

$$n^\mu = y\delta^{\mu,y}. \tag{5.21}$$

Substituting this into (5.18), it is find

$$T_{x\bar{x}} = \frac{1}{8\pi G}\left(-\frac{1}{2y^2} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{x\bar{x}}}\right). \tag{5.22}$$

It is clear that up to a contribution from the counterterm action, the stress tensor is diverge at the boundary $y \to 0$. Thus, we have to choose the counterterm action to cancel such a divergence

$$S_{ct} = -\int d^2x\sqrt{-\gamma}. \tag{5.23}$$

As we choose such an action, we can obtain $T_{\mu\nu} = 0$, which is obviously free of divergence. In higher dimensions, it is necessary to add additional counterterms proportional to R and R^2.

Part II

Reviews of Recent Researches by other groups

6 AdS/CFT Correspondence and Dictionaries

The AdS/CFT correspondence, which was originally conjectured by Maldacena [2], says that there is an equivalence between two totally different theories.

The key points of AdS/CFT correspondence are the following. First, there is a mapping between a quantum gravity namely string theory and non-gravitational quantum theory. According to this feature, we may describe a quantum gravity in terms of non-gravitational language. Secondly, the AdS/CFT correspondence is a strong/weak duality. When the non-gravitational theory is strongly-coupled, we can study it using the weakly-coupled string theory and vise versa. Thirdly, This map is holographic. A $d+1$ dimensional quantum gravity can be described by using a d dimensional non-gravitational theory. Indeed, Maldacena showed the duality between five-dimensional gravity and four-dimensional non-gravitational theory.

Scientists and mathematicians have been studying this duality and its extension since the birth of the AdS/CFT correspondence. Two convenient dictionaries, which describe the equivalence at the correlation function level, are discovered. One of them tells us the relation between the generating functional of conformal field theory and that of the AdS gravity [7][8]. This dictionary namely GKP/W or differentiating dictionary says that the generating functional of correlation functions in the conformal field theory is equivalent to the partition function of the semi-classical AdS gravity

$$Z_{CFT} = Z_{AdS}. \tag{6.1}$$

where Z_{CFT} is the generating functional of correlation functions in conformal field theory and Z_{AdS} is the partition function of gravitational theory. We will review the details in the next subsection. Another one, which is called BDHM or extrapolating dictionary[9], says that a state in conformal field theory is proportional to that in an AdS space on the boundary limit.

$$|\psi\rangle_{CFT} \sim |\psi\rangle_{AdS}\,|_{boundary}. \tag{6.2}$$

Then, the correlation functions in conformal field theory are obtained from the bulk ones by taking the boundary limit.

6.1 GKP/W Dictionary

Let us consider $d+1$ dimensional Euclidean AdS space, which is obtained by Wick rotating a time direction. Then, the metric is given by

$$ds^2 = \frac{1}{y^2}(dy^2 + dt^2 + dx_1^2 + dx_2^2 + \cdots + dx_{d-1}^2) \tag{6.3}$$

where y direction is boundary direction and $y = 0$ corresponds to the boundary of AdS space. Witten conjectured that the semi-classical approximation to the path-integral with the boundary condition that at $y \to 0$ fundamental fields ϕ approaches a boundary value ϕ_0

$$Z_{AdS}[\phi_0] = \exp\left[-S[\phi_0]\right], \tag{6.4}$$

where S is a classical gravitational action, is equal to the generating functional of correlation functions in the conformal field theory

$$Z_{CFT} = \left\langle \exp \int \phi_0 \mathcal{O} \right\rangle \tag{6.5}$$

where $\mathcal{O}$ denotes operator in the conformal field theory. The boundary values of fundamental fields behave as source functions. Then, we can calculate n-point functions by functionally differentiating this generating functional with respect to ϕ_0

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_{CFT} = \frac{1}{Z_{AdS}} \frac{\delta}{\delta\phi_0(x_1)} \frac{\delta}{\delta\phi_0(x_2)} \cdots \frac{\delta}{\delta\phi_0(x_n)} Z_{AdS}[\phi_0]. \tag{6.6}$$

For simplicity, let us consider a free boson ϕ with a mass m in $d+1$ dimensional Eucliden AdS space. The action is given by

$$S(\phi) = \int d^{d+1}x\sqrt{g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2\right). \tag{6.7}$$

and the equation of motion is given by

$$\left(\partial_y^2 + \partial_t^2 + \partial_{x_1}^2 + \cdots + \partial_{x_{d-1}}^2 + \frac{d-1}{y}\partial_y - \frac{m^2}{y^2}\right)\phi = 0 \tag{6.8}$$

We set the boundary value of the free boson as ϕ_0 and assume the solution as

$$\phi(y,x) = \int d^dx' G(y, x-x')\phi_0(x') \tag{6.9}$$

where a function G is a Green function that is required to be regular at $y \to 0,\ x \neq x'$. Thus the solution is given by

$$G(y, x-x') = \frac{y^\Delta}{(y^2 + (x-x')^2)^\Delta} \tag{6.10}$$

where $\Delta = (d + \sqrt{d^2 + 4m^2})/2$.

By substituting (6.9) into the action (6.7) and integrating by parts, we can obtain a surface integral term

$$S(\phi) = \frac{1}{2}\int d^dx y^{1-d}\phi_0(x)\frac{\partial}{\partial y}\phi(y,x)|_{z=0} = \frac{d}{2}\int d^dx d^dx' \frac{\phi_0(x)\phi_0(x')}{(x-x')^{2\Delta}}. \tag{6.11}$$

So, the classical generating functional for the free boson in AdS space is given by

$$Z_{AdS} = \exp\left(-\frac{d}{2}\int d^dx d^dx' \frac{\phi_0(x)\phi_0(x')}{(x-x')^{2\Delta}}\right) \tag{6.12}$$

and by differentiating this with respect to the boundary value ϕ_0 we can obtain the two-point function of quasi-primary operators with conformal weight Δ

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{CFT} \sim \frac{1}{(x_1 - x_2)^{2\Delta}}. \tag{6.13}$$

6.2 BDHM Dictionary

There is another method to calculate the correlation function in conformal field theory. In [9], the bulk n-point functions are considered. After calculating the bulk correlators and taking the boundary limit, it turns out that the leading behavior can be extracted to give correlators of the operators dual to the bulk fields

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_{CFT} = \lim_{y\to 0} y^{-n\Delta} \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle_{bulk}\,. \tag{6.14}$$

In this prescription, it is necessary to quantize the bulk field. So, let us seek the vacuum state in the bulk.

As in the preceding subsection, let us consider a free boson ϕ with mass m in $d+1$ dimensional AdS space. The action is given by (6.7) and the equation of motion is given by

$$\left(\partial_y^2 + \partial_t^2 + \partial_{x_1}^2 + \cdots + \partial_{x_{d-1}}^2 + \frac{d-1}{y}\partial_y - \frac{m^2}{y^2}\right)\phi = 0. \tag{6.15}$$

Assume that the solution to this equation of motion is given by

$$\phi(y,x) = y^{\frac{d}{2}} f(ky) e^{ik\cdot x} \tag{6.16}$$

and a function f satisfies the following equation

$$\frac{d^2}{dz^2}f(z) + \frac{1}{z}\frac{d}{dz}f(z) - \left(1+\frac{\nu^2}{z^2}\right)f(z) = 0 \tag{6.17}$$

where $z = ky$ and $\nu = \sqrt{d^2+4m^2}$. This equation is just the modified Bessel equation and the solution that is regular at $z \to \infty$ is given by a modified Bessel function of the second kind $K_\nu(ky)$. Then, the solution for (6.15) is written as

$$\phi(y,x) = \int \frac{d^dk}{(2\pi)^d}\left(y^{\frac{d}{2}} a(\boldsymbol{k}) K_\nu(ky) e^{ik\cdot x} + (h.c.)\right) \tag{6.18}$$

where $h.c.$ denotes the Hermitian conjugation of the first term.

Here, we consider the canonical quantization by requiring the following commutation relation

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = \delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{6.19}$$

and we define the vacuum state $|0\rangle$ such that $a(\boldsymbol{k})\,|0\rangle = 0$. Notice that (6.19) is hols only the case of bosons, while in the case of fermions we should replace anti-commutation relations. Then, the bulk two-point function between free bosons is defined by

$$G_{bulk}(y,x|y',x') = \langle 0|\,\phi(y,x)\phi(y',x')\,|0\rangle \tag{6.20}$$

and the correct form may be given by integrating double k integral. However, any correlation functions should be $SO(2,d)$ invariant and $SO(2,d)$ symmetry tells us that the bulk correlator can be rewritten in terms of $SO(2,d)$ invariant quantity such as the invariant distance

$$P(y,x|y',x') = \frac{y^2 + y'^2 - (x-x')^2}{2yy'}. \tag{6.21}$$

The equation of motion (6.15) is rewritten in terms of P as

$$(P^2-1)\partial_P^2 G_{bulk} + (d+1)P\partial_P G_{bulk} - m^2 G_{bulk} = 0. \tag{6.22}$$

Here, we will comment on the boundary condition for the solution. There is a conservation current

$$j_\mu(y,x) = \phi^*(y,x)\partial_\mu\phi(y,x) - \phi(y,x)\partial_\mu\phi^*(y,x) \tag{6.23}$$

and from the conservation law, a Dirichlet condition that on the boundary the z component of this probability density j_z has zero value is imposed. Thus, the solution, which satisfies the Dirichret condition, is obtained by

$$G_{bulk}(P) = \frac{1}{(1+P)^\Delta} F\left(\Delta, \Delta - \frac{d}{2} + 1, 2\Delta + 1 - d; \frac{2}{1+P}\right) \tag{6.24}$$

where F denotes the hypergeometric function. By taking the boundary limit $y, y' \to 0$, this reduces to

$$G_{bulk}(P) \to \frac{1}{P^\Delta} \sim \frac{(yy')^\Delta}{(x-x')^2\Delta} \tag{6.25}$$

and we conclude that the boundary behavior of the bulk two-point function between free bosons is proportional to thetwo-point function of quasi-primary operators dual to the bulk bosons

$$G_{bulk}(y,x|y',x')|_{y,y'\to 0} \sim y^\Delta y'^\Delta \langle \mathcal{O}(x)\mathcal{O}(x')\rangle \,. \tag{6.26}$$

7 Bulk Reconstruction

The AdS/CFT dictionaries, which were reviewed in the previous section, are the prescription to calculate the correlation functions in conformal field theory in terms of semi-classical bulk fundamental fields. Conversely, there are several prescriptions to represent fields in AdS spacetime by using some operators on the boundary. This prescription is so-called bulk reconstruction.

In 2005, Hamilton *et al* [11] proposed that a free scalar field in two-dimensional AdS space can be expressed in terms of operators living on a finite boundary region. Generalization to arbitrary dimension or BTZ geometry are proposed in [12],[38],[39].

In this subsection, we will focus on bulk reconstruction and review these two methods.

7.1 HKLL Prescription

The extrapolating dictionary implies that the boundary behavior of a fundamental field corresponds to an operator in conformal field theory. For example, we consider a d dimensional bulk scalar field ϕ with mass m. By taking to have normalizable fall-off near the boundary, the field behaves as

$$\phi(y,x) \sim y^\Delta \phi_0(x) \tag{7.1}$$

where $\Delta = (d + \sqrt{d^2+4m^2})/2$ and ϕ_0 is a boundary condition of scalar field. Then, from BDHM dictionary, the leading behavior of the bulk two-point function in gravity side coincides with the two-point function in conformal field theory

$$\langle \phi(y,x)\phi(y',x')\rangle \sim y^\Delta y'^\Delta \langle \mathcal{O}(x)\mathcal{O}(x')\rangle \tag{7.2}$$

where $\mathcal{O}$ is a operator dual to ϕ with conformal dimension Δ. This implies that the boundary behavior of the fundamental bulk field corresponds to the boundary operator dual to bulk field

$$\phi_0(x) \leftrightarrow \mathcal{O}(x). \tag{7.3}$$

Then, this implies a correspondence between local fields in the bulk and non-local operators in conformal field theory

$$\phi(y,x) \leftrightarrow \int dx' K(y,x|x')\mathcal{O}(x'). \tag{7.4}$$

Here, we will refer to the kernel function K as a smearing function. Correlators in the bulk are equal to that of the non-local operators dual to field

$$\langle\phi(y,x)\phi(y',x')\rangle = \int dx_1 dx_2 K(y,x|x_1)K(y',x'|x_2)\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\,. \tag{7.5}$$

We will review the construction of smearing function. For simplicity we will consider a free scalar in three dimensional Lorentzian Poincaré AdS. The metric is given by

$$ds^2 = \frac{1}{y^2}(dy^2 - dt^2 + dx^2). \tag{7.6}$$

The scalar should satisfy the Klein-Gordon equation

$$\left(\partial_y^2 - \partial_t^2 + \partial_x^2 + \frac{1}{y}\partial_y - \frac{m^2}{y^2}\right)\phi = 0. \tag{7.7}$$

In this coordinate, the mode expansion of a scalar is obtained by solving the equation of motion

$$\phi(y,t,x) = \int_{\omega>|k|} d\omega dk\; a_{\omega k} e^{-i\omega t} e^{ikx} y J_\nu(\sqrt{\omega^2-k^2}y) \tag{7.8}$$

where J_ν is a Bessel function of order $\nu = \Delta - 1$. The boundary field is

$$\begin{aligned}\phi_0(t,x) &= \lim_{y\to 0}\frac{1}{y^\Delta}\phi(y,t,x)\\ &= \frac{1}{2^\nu\Gamma(\nu+1)}\int_{|\omega|>|k|} d\omega dk\; a_{\omega k}(\omega^2-k^2)^{\frac{\nu}{2}} e^{-i\omega t}e^{ikx}\end{aligned} \tag{7.9}$$

Therefore, we can express the bulk field in terms of the boundary field

$$\phi(y,x) = \int dt'dx' K(y,t,x|t',x')\phi_0(t',x') \tag{7.10}$$

where the kernel function, which is so-called a smearing function, is written by

$$K(y,t,x|t',x') = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^2}\int_{|\omega|>|k|} d\omega dk \frac{y^{\Delta-\nu}}{(\omega^2-k^2)^{\frac{\nu}{2}}} J_\nu(\sqrt{\omega^2-k^2}y)e^{-i\omega(t-t')}e^{ik(x-x')} \tag{7.11}$$

Evaluating the integral (7.11) is highly mathematical. Thanks to Poincaré invariance, we can set $t = x = x' = 0$ without the loss of generality. Setting

$$\omega \pm k = re^{\pm\xi}, \tag{7.12}$$

we have

$$\begin{aligned}K(y,0,0|t',0) &= \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^2}\int_0^\infty rdr\int_{-\infty}^\infty d\xi \frac{y}{(2r)^\nu}J_\nu(2ry)e^{2irt'\cosh\xi}\\ &= -\frac{y^\Delta}{2\pi^2 t'^2}F\left(1,1,\Delta;\frac{y^2}{t'^2}\right)\end{aligned} \tag{7.13}$$

where F is a hypergeometric function. So, we can conclude that the smearing function is given by

$$K(y,t,x|t',x') = -\frac{1}{2\pi^2}\frac{y^\Delta}{(t-t')^2-(x-x')^2}F\left(1,1,\Delta;\frac{y^2}{(t-t')^2-(x-x')^2}\right). \tag{7.14}$$

At a first glance it is correct, however, there is an ambiguity. The smearing function (7.14) has support on the entire boundary of Poincaré AdS. Then, even the boundary field should be constructed by dual operators living on the entire boundary. There is no correspondence between the local boundary field and local primary operator in conformal field theory (7.3). To manifest this requirement, we should make the support compact. It is realized to complexify the boundary spacial coordinate x.

Again, we go back to (7.11), we now use the following integral

$$
\begin{aligned}
2\pi J_0(r\sqrt{\omega^2-k^2}) &= \int_0^{2\pi} d\theta \exp[-ir\omega\sin\theta - kr\cos\theta] \\
2^{\nu-1}\Gamma(\nu)b^{-\nu}J_\nu(b) &= \int_0^1 drr(1-r^2)^{\nu-1}J_0(br)
\end{aligned}
$$

and we have

$$
\frac{J_\nu(\sqrt{\omega^2-k^2}y)}{(\omega^2-k^2)^{\nu/2}} = \frac{y^{-\nu}}{2^\nu\Gamma(\nu)}\int_{T^2+X^2<y^2} dTdX(y^2-T^2-X^2)^{\nu-1}e^{-i\omega T}e^{-kX} \tag{7.15}
$$

where we define $T = r\sin\theta, X = r\cos\theta$. By substituting this into (7.11), we can obtain an alternative expression for the bulk scalar by

$$
\phi(y,t,x) = \frac{\nu}{\pi}\int_{T^2+X^2<y^2} dTdX\left(\frac{y^2-T^2-X^2}{y}\right)^{\nu-1}\phi_0(t+T,x+iX). \tag{7.16}
$$

where $\nu = \Delta - 1$. In this case, it is obvious that the smearing function has the support on the finite boundary region and the boundary field can be constructed by single dual operator. Finally we replace the boundary field by a dual primary operator in conformal field theory, we can reconstruct the bulk field in terms of the boundary operators

$$
\phi(y,t,x) = \frac{\nu}{\pi}\int_{T^2+X^2<y^2} dTdX\left(\frac{y^2-T^2-X^2}{y}\right)^{\nu-1}\mathcal{O}(t+T,x+iX). \tag{7.17}
$$

Since (7.17) gives the correct two-point function in the bulk, this replacement can be guaranteed. Notice that the above ambiguity is occurred in arbitrary odd dimensions, but in even dimensions there are no ambiguities and the smearing function can be always written as (7.17).

7.2 Bulk Local State Reconstruction

The HKLL prescription for reconstruction of local bulk fields has several weaknesses. The map assumes the existence of a gravity dual. It is assumed that the boundary value of the bulk field is equivalent to the dual quasi-primary operator in conformal field theory. Next, the bulk field is a state-dependent operator, since the smearing kernel is a solution to the equation of motion for the bulk field and depends on the bulk geometry. Finally, it is difficult to prove the existence of a well-defined smearing function for black hole geometries.

It is interesting to seek an alternative definition of local bulk fields, which may overcome the above difficulties. It is required that bulk local fields are

1) determined by information of the conformal field theory

2) geometry-independent

3) applicable to black holes

Miyaji et al [13] and Verlinde [40] proposed that a bulk local state and corresponding operators can be constructed by using Ishibashi state $|\phi\rangle$, which is defined by

$$(L_n - (-1)^n \bar{L}_{-n}) |\phi\rangle = 0. \tag{7.18}$$

Nakayama and Ooguri [14] showed that the construction of bulk local fields is equivalent to HKLL bulk reconstruction. Furthermore, Goto and Takayanagi [15] proposed that bulk state on the BTZ black hole can be written by using Ishibashi state [41].

For simplicity, let us consider three dimensional global AdS space. Then, the metric is given by

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \tag{7.19}$$

where $0 < \rho < \infty, 0 < \phi < 2\pi, -\infty < t < \infty$. We consider a bulk local scalar operator $\phi_\Delta(\boldsymbol{x})$. The only requirement is that the actions of the conformal symmetry and the bulk isometry on ϕ_Δ are compatible

$$[J, \phi_\Delta(\boldsymbol{x})] = i\mathcal{L}_J \phi_\Delta(\boldsymbol{x}) \tag{7.20}$$

where J is a generator of the conformal symmetry and $\mathcal{L}_{\mathcal{J}}$ is the Lie derivative in the AdS coordinates with respect to the Killing vector field $\mathcal{J}$ corresponding J. The $SO(2,2)$ generators can be organized by Hamiltonian H, anti-symmetric rotatons M_{ab}, translations P_a and special conformal transformations K_a where $a, b = 1, 2$.

Let us investigate the implication of the requirement (7.20) when the bulk operator is located at the origin of AdS space $(\rho, t, \phi) = (0, 0, 0)$. The isometry of this point is generated by M_{ab} and $P_a + K_a$. In the case of scalar operator, the right hand side in (7.20) is equal to zero and we have

$$[M_{ab}, \phi_\Delta(0)] = 0 \tag{7.21}$$

$$[P_a + K_a, \phi_\Delta(0)] = 0 \tag{7.22}$$

Acting these on the conformally invariant vacuum $|0\rangle$, these condition are written by

$$M_{ab} |\phi_\Delta(0)\rangle = 0 \tag{7.23}$$

$$(P_a + K_a) |\phi_\Delta(0)\rangle = 0 \tag{7.24}$$

where we define $|\phi_\Delta(0)\rangle = \phi_\Delta |0\rangle$. When we rewrite these conditions in terms of Virasoro generators, the requirement (7.20) are given by

$$(L_0 - \bar{L}_0) |\phi_\Delta(0)\rangle = (L_{\pm 1} + \bar{L}_{\mp 1}) |\phi_\Delta(0)\rangle = 0. \tag{7.25}$$

These are just the conditions for the Ishibashi state. These can be solved by introducing a formal primary state $|\mathcal{O}\rangle$, which satisfies

$$L_1 |\mathcal{O}\rangle = \bar{L}_1 |\mathcal{O}\rangle = 0 \tag{7.26}$$

$$L_0 |\mathcal{O}\rangle = \bar{L}_0 |\mathcal{O}\rangle = \frac{\Delta}{2} |\mathcal{O}\rangle \tag{7.27}$$

and the solution is given by

$$|\phi_\Delta(0)\rangle = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\Delta)}{n!\Gamma(\Delta + n)} L_{-1}^n \bar{L}_{-1}^n |\mathcal{O}\rangle . \tag{7.28}$$

Remarkably, the solution not only contains a primary state but also its descendant states. So, we can reconstruct the bulk local state at the origin by using Ishibashi state, which contains a primary state and its descendant states. This prescription satisfies the requirement 1) bulk fields is inherent to the conformal field theory since we use only conformal field theoretical properties.

The local bulk field at an arbitrary point is easily constructed. It is done by acting the translation generator

$$g = e^{i(L_0+\bar{L}_0)t} e^{i(L_0-\bar{L}_0)\phi} e^{-\frac{\rho}{2}(L_1-L_{-1}+\bar{L}_1-\bar{L}_{-1})} \tag{7.29}$$

on (7.28). The state is an eigenstate of the conformal Casimir operator $C_2 = L_0^2 - (L_1L_{-1} + L_{-1}L_1)/2 + \bar{L}_0^2 - (\bar{L}_1\bar{L}_{-1} + \bar{L}_{-1}\bar{L}_1)/2$ with the eigenvalue $\Delta(\Delta-2)$. The Killing vectors can be defined in Section.4 and given by

$$\begin{aligned} L_0 &= i\partial_{x^+} \\ L_{\pm 1} &= ie^{\pm ix^+}\left[\coth 2\rho\partial_{x^+} - \sinh^{-2} 2\rho\partial_{x^-} \mp \frac{i}{2}\partial_\rho\right] \end{aligned}$$

When we substitute these into the quadratic Casimir operator, the quadratic Casimir operator coincides with the d'Alembertian in the global AdS coordinates. So, we can interpret the eigenvalue equation of C_2 as the Klein-Gordon equation with mass $m^2 = \Delta(\Delta-2)$. From this viewpoint, we can interpret the primary operator as the one dual to the bulk operator with conformal dimension $\Delta = (1+\sqrt{1+4m^2})/2$. Then, we conclude that the equation of motion for bulk fields naturally appears as the eigenvalue equation for Casimir operator and requirement 2) bulk fields are state-independent is satisfied.

From now, we review the equivalence with the HKLL prescription[11][12] [15][38]. First, we move from the global AdS to the Poincaré AdS. The coordinate transformations and the Killing vectors are given in Section.4. The bulk state at arbitrarily points $(y, x, \bar{x})$ in Poincaré AdS space are obtained by performing $SL(2,\mathbb{R})$ translations

$$|\phi_\Delta(y,x,\bar{x})\rangle = g(y,x,\bar{x})\,|\phi_\Delta(0)\rangle\,, \tag{7.30}$$

where $g(y,x,\bar{x})$ is given by

$$g(y,x,\bar{x}) = e^{\frac{i}{2}(x+\bar{x})(L^h_{-1}+\bar{L}^h_{-1})} e^{\frac{i}{2}(x-\bar{x})(L^h_{-1}-\bar{L}^h_{-1})} y^{L^h_0+\bar{L}^h_0}. \tag{7.31}$$

After a bit calculation, we can obtain a concrete form for $|\phi(y,x,\bar{x})\rangle$

$$|\phi(y,x,\bar{x})\rangle = \frac{\pi}{1-\Delta}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\Delta)_n} y^{\Delta+2n}(L^h_{-1}\bar{L}^h_{-1})^n \mathcal{O}(x,\bar{x})\,|0\rangle\,. \tag{7.32}$$

The scalar fields are expressed in Lorentzian signature as

$$\phi_{HKLL}(y,x,\bar{x}) = \int_{x'\bar{x}'\le y^2} dx'd\bar{x}'\left(\frac{y^2+x'\bar{x}'}{y}\right)^{\Delta-2}\mathcal{O}(x+x',\bar{x}+\bar{x'}). \tag{7.33}$$

Expanding in Taylor series of a primary operator, we can obtain

$$\begin{aligned} \phi_{HKLL}(y,x,\bar{x}) &= \int_{x'\bar{x}'\le y^2} dx'd\bar{x}'\left(\frac{y^2+x'\bar{x}'}{y}\right)^{\Delta-2}\sum_{m,n}\frac{1}{m!n!}x'^m\bar{x}'^n\partial_x^m\partial_{\bar{x}}^n\mathcal{O}_\Delta(x,\bar{x}) \\ &= \frac{\pi}{1-\Delta}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\Delta)_n}y^{\Delta+2n}(L^h_{-1}\bar{L}^h_{-1})^n\mathcal{O}_\Delta(x,\bar{x}). \end{aligned} \tag{7.34}$$

Acting this operator into the vacuum state in CFT, it is found that this state is exactly same with (7.32).

8 3D Spin-3 Gravity as the Chern-Simons Theory

The Einstein-Hilbert action in three dimension (5.1) is equivalent to an action which describes the dynamics of an ordinary gauge field namely the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons action [42]. This equivalence can be extended to higher-spin gravity theories. It was shown in [19] that the higher-spin gravity can be formulated as a Chern-Simons gauge theory when matter fields are not coupled to gravity and higher-spin fields. In the case of spin-N gravity, the gauge group is $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$.

8.1 Einstein Gravity and $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons Theory

First, we will focus on the Einstein gravity. There is an alternative formulation to describe Einstein gravity [43]. In this formalism, a fundamental field is an auxiliary quantity ${e^a}_\mu$ with a local frame index $a = 0, 1, 2$. This is often called a vielbein. The vielbein can be thought as the square root of the metric

$$g_{\mu\nu}(x) = \eta_{ab}{e^a}_\mu(x){e^b}_\nu(x) \tag{8.1}$$

where η_{ab} is the local flat frame metric $\eta_{ab} = diag(-1, 1, 1)$.

From the definition (8.1), the vielbein transforms as a spacetime vector field. We assume ${e^a}_\mu$ is not singular in order to make the determinant of the metric $\det g$ non-zero. Henceforth, there is an inverse ${e^\mu}_a$ such that ${e^a}_\mu {e_a}^\nu = {\delta_\mu}^\nu$ and ${e^a}_\mu e_b{}^\mu = {\delta^a}_b$. Notice that the vielbein is not unique since all frame fields related with a local Lorentz transformation $e'^a_\mu = {\Lambda^a}_b {e^b}_\mu$ are equivalent.

In the general relativity, the covariant derivative of a tensor is given by its partial derivative plus connections. Similarly in tetrad formalism, the covariant derivative of a tensor ${X^a}_b$ is given by its partial derivative and connections, however, it is necessary to replace the Christoffel symbol Γ by the antisymmetric spin connection, which is denoted by $\omega^{ab} = {\omega^{ab}}_\mu dx^\mu$

$$\nabla_\mu {X^a}_b = \partial_\mu {X^a}_b + {(\omega_\mu)^a}_c {X^c}_b - {(\omega_\mu)^c}_b {X^a}_c. \tag{8.2}$$

The relation between the Christoffel symbol and the spin connection is given by

$$\omega^{ab}{}_\mu = {e^a}_\nu e^{b\lambda}\Gamma^\nu{}_{\mu\lambda} - e^{b\lambda}\partial_\mu {e^b}_\lambda. \tag{8.3}$$

We define the one-form $e^a = {e^a}_\mu dx^\mu$. There are two Cartan structure equations, called a torsion and a curvature

$$\begin{aligned} T^a &= de^a + {\omega^a}_b \wedge e^b & (8.4)\\ R^{ab} &= d\omega^{ab} + {\omega^a}_c \wedge \omega^{cb}. & (8.5) \end{aligned}$$

We define the Levi-Civita tensor by

$$\epsilon_{\mu\nu\rho} = e^{-1}\epsilon_{abc}{e^a}_\mu {e^b}_\nu {e^c}_\rho \tag{8.6}$$

and it turns out that the invariant volume element is rewritten in terms of vielbein

$$\sqrt{-g}d^3x = \frac{1}{3!}\epsilon_{abc}e^a \wedge e^b \wedge e^c$$

and we have

$$\sqrt{-g}d^3xR = \epsilon_{abc}e^a \wedge R^{bc}.$$

By substituting these into the Einstein-Hilbert action (5.1), the action is rewritten as

$$S_{EH} = \frac{1}{16\pi G}\int \left(2e^a \wedge R_a - \frac{\Lambda}{3}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right) \tag{8.7}$$

where we adopt $R^{ab} = -\epsilon^{abc}R_c$.

In [36] and [37], it was shown that three dimensional Einstein gravity with $\Lambda < 0$ is equivalent to the Chern-Simons theory for $SO(2,2)$ gauge group. The Chern-Simons action for a compact gauge group is given by

$$S_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right] \tag{8.8}$$

where k is a constant and A is a gauge field one-form $A = A_\mu dx^\mu$. Integrated by parts, the variation of this action (8.8) takes the form

$$\delta S_{CS} = \frac{k}{2\pi}\int_{\mathcal{M}} \mathrm{Tr}[\delta A \wedge (dA + A \wedge A)] - \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}[A \wedge \delta A]. \tag{8.9}$$

The value of δA is chosen to eliminate the second boundary action. The first one implies that there is an equation of motion

$$F = dA + A \wedge A = 0. \tag{8.10}$$

This equation solves by introducing an arbitrary function matrix G and the solution is given by

$$A = G^{-1}dG, \tag{8.11}$$

which means that A is a gauge transformation of the trivial field and there is no physical degree of freedom.

As we discussed in Section.4, the generators of $SO(2,2)$ gauge group satisfy the following commutation relation

$$[J^a, J^b] = -\epsilon^{ab}{}_cJ^c,\ [\bar{J}^a, \bar{J}^b] = -\epsilon^{ab}{}_c\bar{J}^c,\ [J^a, \bar{J}^b] = 0. \tag{8.12}$$

where ϵ_{abc} is Levi-Civita symbol and it is isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ gauge group. $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators can be defined by taking the following linear combination

$$J_0 = \frac{1}{2}(t_1 + t_3),\ J_1 = \frac{1}{2}(t_1 - t_3),\ J_2 = t_2. \tag{8.13}$$

The superscripts are lowered by the metric $h_{ab} = \mathrm{Tr}(t_at_b)/2$ and the subscripts are raised by its inversion.

Thanks to this splitting, we can rewrite the Chern-Simons action for $SO(2,2)$ as the sum of two Chern-Simons action $S_{CS}[A]$ and $S_{CS}[\bar{A}]$. Assuming that the Chern-Simons gauge field A and $\bar{A}$ can be decomposed into vielbein and spin connection as

$$A_\mu = (\omega^a_\mu + e^a_\mu)t_a,\ \bar{A}_\mu = (\omega^a_\mu - e^a_\mu)t_a, \tag{8.14}$$

where we define $\omega^a = \epsilon^a_{bc}\omega^{bc}_\mu dx^\mu$, The Chern-Simons action can be rewritten as

$$S_{CS}[A,\bar{A}] = S_{CS}[A] - S_{CS}[\bar{A}] = \frac{k}{4\pi}\int \left(2e^a \wedge R_a + \frac{1}{3}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right). \tag{8.15}$$

Hence, it turns out that the Einstein gravity with a negative cosmological constant is equivalent to a Chern-Simons theory with $SO(2,2)$ gauge group, provided that the level acquires the value

$$k = \frac{1}{4G}. \tag{8.16}$$

8.2 Boundary Conditions for the Gauge Fields A and $\bar{A}$

To solve the equations (8.10) and the one for $\bar{A}$, it is necessary to consider boundary conditions for A and $\bar{A}$. Boundary conditions are adopted in order to make the action principle well-defined. The first variation for the action is given by

$$\delta S_{CS}[A] = -\frac{1}{8\pi G}\int_M \mathrm{Tr}[\delta A \wedge (dA + A \wedge A)] + \frac{1}{16\pi G}\int_{\partial M} \mathrm{Tr}[A \wedge \delta A]. \tag{8.17}$$

The first term in the above expression gives the equation of motion for A. Writing down the second term in terms of components of A, we can obtain a radial boundary term

$$S_B[A] = \frac{1}{16\pi G}\int_{\partial M} \mathrm{Tr}[A_x \delta A_{\bar{x}} - A_{\bar{x}} \delta A_x]. \tag{8.18}$$

To make the variation problem well-defined, the Chern-Simons gauge field should satisfy boundary conditions $A_x = 0$ or $A_{\bar{x}} = 0$. Similarly in the case for $\bar{A}$, the gauge field $\bar{A}$ must satisfy boundary conditions $\bar{A}_x = 0$ or $\bar{A}_{\bar{x}} = 0$.

8.3 Spin-3 Gravity and $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons Theory

Campoleoni et al generaliszd this situation to the case of the higher-spin gravity. As we discussed in the above, a spin-2 field like the metric field can be described by a vielbein with one local Lorentz frame index. Then, it is straightforward to consider that a spin-s fully symmetric tensor field $\varphi_{\mu_1,\mu_2\cdots\mu_s}$ should be described by a one-form with $s-1$ local Lorentz indeces $e_\mu^{a_1 a_2 \cdots a_{s-1}}$. However, due to the spacetime index, it carries a (1,s-1) component and it does not makes $\varphi_{\mu_1,\mu_2\cdots\mu_s}$ symmetric. To eliminate it, one consider the local Lorentz gauge transformation

$$\delta e_\mu^{a_1,a2\cdots a_{s-1}} = D_\mu \xi^{a_1 a_2 \cdots a_{s-1}} + \bar{e}_{\mu b}\Lambda^{b,a_1 a_2 \cdots a_{s-1}} \tag{8.19}$$

where D_μ is the Lorentz-covariant derivative and $\bar{e}$ is the background vielbein. This leads to introduce a traceless connection $\omega_\mu^{b,a_1 a_2 \cdots a_{s-1}}$. It is the higer-spin analogue of the spin connection of gravity. It should be expressed in terms of $e_\mu^{a_1 a_2 \cdots a_{s-1}}$ and its derivatives in order to guarantee a appropriate torsion-less condition. In arbitrary dimension, the local Lorentz transformation for $\omega_\mu^{b,a_1 a_2 \cdots a_{s-1}}$ leads to introduce an extra auxiliary field

$$\delta\omega_\mu^{b,a_1 a_2 \cdots a_{s-1}} = \partial_\mu \Lambda^{b,a_1 a_2 \cdots a_{s-1}} + \bar{e}_{\mu c}\Theta^{bc,a_1 a_2 \cdots a_{s-1}}, \tag{8.20}$$

however, in three dimension the extra field Θ vanishes. Hence, a spin-s field can be described by the pair of one-forms $e_\mu^{a_1 a_2 \cdots a_{s-1}}$ and $\omega_\mu^{b,a_1 a_2 \cdots a_{s-1}}$ and the fully symmetric tensor can be described as

$$\varphi_{\mu_1 \mu_2 \cdots \mu_s} = \frac{1}{s!}\bar{e}^{a_1}_{(\mu_1} \cdots \bar{e}^{a_{s-1}}_{\mu_{s-1}} e_{\mu_s) a_1 \cdots a_{s-1}}. \tag{8.21}$$

It is convenient to introduce a spin connection $\omega_\mu^{a_1 \cdots a_{s-1}}$ by contracting local Lorentz indeces by the Levi-Civita symbol ϵ_{abc}. It is analogue of usual $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chen-Simons theory $\omega_\mu^a = \epsilon^{abc}\omega_{\mu bc}/2$.

From now we will focus on the three dimensional spin-3 gravity. This theory has a symmetric spin-3 field $\varphi_{\mu\nu\rho}$ and it is described by the pair of one-forms e_μ^{ab} and ω_μ^{ab}. We can define gauge connections A and $\bar{A}$ by

$$A = (j_\mu^a J_a + t_\mu^{ab} T_{ab}) dx^\mu \tag{8.22}$$

$$\bar{A} = (\bar{j}_\mu^a J_a + \bar{t}_\mu^{ab} T_{ab}) dx^\mu \tag{8.23}$$

where J_a are $SO(2,2)$ generators and T_{ab} are higher-spin generators. Here, we define

$$j^a_\mu = \omega^a_\mu + e^a_\mu,\ \bar{j}^a_\mu = \omega^a_\mu - e^a_\mu \tag{8.24}$$

$$t^{ab}_\mu = \omega^{ab}_\mu + e^{ab}_\mu,\ \bar{t}^{ab}_\mu = \omega^{ab}_\mu - e^{ab}_\mu. \tag{8.25}$$

Since $t_{ab}T^{ab}$ must transform as $SO(2,2)$ tensor, T^{ab} must be symmetric and traceless. As J_a satisfy the commutator

$$[J_a, J_b] = -\epsilon_{ab}{}^c J_c, \tag{8.26}$$

T_{ab} must satisfy

$$[J_a, T_{bc}] = -\epsilon_{ab}{}^m T_{mc}. \tag{8.27}$$

Finally, the Jaccobi identity fixes the commutator between the higher-spin generators T_{ab} as

$$[T_{ab}, T_{bc}] = \sigma(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am})J^m \tag{8.28}$$

where σ is a normalization factor. By taking an appropriate linear combination

$$T_{00} = \frac{1}{4}(t_4 + t_8 + 2t_6),\ T_{01} = \frac{1}{4}(t_4 - t_8),\ T_{11} = \frac{1}{4}(t_4 + t_8 - 2t_6) \tag{8.29}$$

$$T_{02} = \frac{1}{2}(t_5 + t_6),\ T_{22} = t_6,\ T_{12} = \frac{1}{2}(t_5 - t_6), \tag{8.30}$$

the Lie algebra for t_a yields W_3 wedge algebra

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} \\
&[L_n, W_m] = (2n-m)W_{n+m} \\
&[W_n, W_m] = \frac{\sigma}{3}(n-m)(2n^2 + 2m^2 - nm - 8)L_{n+m}
\end{aligned}$$

where we adopt $(L_1, L_0, L_{-1}, W_2, W_1, W_0, W_{-1}, W_{-2}) = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)$. For $\sigma = -1$, $SL(3,\mathbb{R})$ gauge group is realized. Therefore, the spin-3 gravity is described by the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory with the action (8.8). When we redefine the vielbein field and spin connection by using generators t_a as

$$e = e^a{}_\mu t_a dx^\mu,\ \omega = \omega^a{}_\mu t_a dx^\mu, \tag{8.31}$$

(8.22) and (8.23) are rewritten as

$$A = \omega + e,\ \bar{A} = \omega - e. \tag{8.32}$$

Of course, in the case of the three dimensional $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory, the flatness conditions

$$dA + A \wedge A = d\bar{A} + \bar{A} \wedge \bar{A} = 0 \tag{8.33}$$

are the equations of motion. Then, The relations between vielbein and metric or spin-3 field are given by

$$g_{\mu\nu} = \frac{1}{2!}\mathrm{Tr}(e_\mu e_\nu) \tag{8.34}$$

$$\varphi_{\mu\nu\rho} = \frac{1}{3!}\mathrm{Tr}(e_{(\mu} e_\nu e_{\rho)}). \tag{8.35}$$

8.4 Solutions to Flatness Conditions

In the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory, the gauge connections satisfy the equations of motion (8.33). The AdS spacetime is realized by the following solutions

$$\begin{aligned} A_{AdS} &= \frac{t_1}{y}dx - \frac{t_2}{y}dy \\ \bar{A}_{AdS} &= -\frac{t_3}{y}d\bar{x} + \frac{t_2}{y}dy. \end{aligned} \tag{8.36}$$

More generally, we can consider solutions that approach this vacuum asymptotically. In [20], the authors consider the following connections

$$\begin{aligned} A &= (\frac{t_1}{y} - \frac{2\pi}{k}\mathcal{L}(x)yt_3 - \frac{\pi}{2k}\mathcal{W}(x)y^2t_8)dx - \frac{t_2}{y}dy \\ \bar{A} &= -(\frac{t_3}{y} - \frac{2\pi}{k}\bar{\mathcal{L}}(\bar{x})yt_1 - \frac{\pi}{2k}\bar{\mathcal{W}}(\bar{x})y^2t_8)d\bar{x} + \frac{t_2}{y}dy, \end{aligned} \tag{8.37}$$

where $\mathcal{L}$ and $\mathcal{W}$ are arbitrary functions.

The most general asymptotically solutions are obtained as follows. First, we must consider boundary conditions. As we discussed in Section.8.2, the gauge connection must satisfy $A_{\bar{x}} = 0$. There is an additional condition. Asymptotically AdS solutions imply that its difference to the AdS solution is finite at the boundary

$$(A - A_{AdS})|_{y=0} = \mathcal{O}(1). \tag{8.38}$$

We consider the gauge connection at $y = 1$, since y dependence are realized by multiplying translation operator $b = y^{t_2}$ such as $A(y,x) = b^{-1}a\mathcal{A}(x)b + b^{-1}db$ and $\bar{A}(y,\bar{x}) = b\bar{\mathcal{A}}(\bar{x})b^{-1} + bdb^{-1}$. We expand $\mathcal{A}$ in the $SL(3,\mathbb{R})$ basis

$$\mathcal{A}(x) = \sum_{n=1}^{3} l^n(x)t_n + \sum_{m=4}^{8} w^m(x)t_m. \tag{8.39}$$

Recall that we define $(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) = (L_1,L_0,L_{-1},W_2,W_1,W_0,W_{-1},W_{-2})$. The additional boundary condition imposes the following condition on the components l^n, w^m

$$l^1 = 1, w^4 = w^5 = 0. \tag{8.40}$$

We set

$$l^0 = 0,\ w^6 = w^7 = 0 \tag{8.41}$$

to fix the gauge degrees of freedom[19]. There are two physical degrees of freedom corresponding to l^3 and w^8. If we take $l^3 = -2\pi\mathcal{L}(x)/k$ and $w^8 = -\pi\mathcal{W}(x)/2k$, the connection coinsides with (8.37).

Let us consider the symmetry algebra of this asymptotically AdS spacetime. It is done by considering an infinitesimal gauge transformation. We consider the global gauge transformation

$$U = e^{\lambda^a(x)t_a} \tag{8.42}$$

with

$$\lambda^a(x) = \sum_{n=1}^{3} \epsilon_n t_n + \sum_{m=4}^{8} \chi_m t_m, \tag{8.43}$$

which leaves the structure of (8.37) invariant. When we set $\epsilon = \epsilon^1$ and $\chi = \chi^8$, the other parameters are given by

$$\epsilon^2 = -\epsilon', \ \epsilon^3 = \frac{1}{2}\epsilon'' + \frac{2\pi}{k}\epsilon\mathcal{L} + \frac{4\pi}{k}\chi\mathcal{W} \tag{8.44}$$

and

$$\begin{aligned}
\chi^7 &= -\chi', \ \chi^6 = \frac{1}{2}\chi'' + \frac{4\pi}{k}\chi\mathcal{L} \\
\chi^5 &= -\frac{1}{6}\chi''' - \frac{10\pi}{3k}\chi'\mathcal{L} - \frac{4\pi}{3k}\chi\mathcal{L}' \\
\chi^4 &= \frac{1}{24}\chi'''' + \frac{4\pi}{3k}\chi''\mathcal{L} + \frac{7\pi}{6k}\chi\mathcal{L}'' + \frac{4\pi^2}{k^2}\chi\mathcal{L}^2 - \frac{\pi}{2k}\epsilon\mathcal{W}.
\end{aligned} \tag{8.45}$$

Here, the prime symbol denotes the derivative with respect to x. Under this transformations the functions $\mathcal{L}$ and $\mathcal{W}$ vary as

$$\delta\mathcal{L} = \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} + \frac{k}{4\pi}\epsilon''' + 2\chi\mathcal{W}' + 3\chi'\mathcal{W} \tag{8.46}$$

$$\delta\mathcal{W} = \epsilon\mathcal{W}' + 3\epsilon'\mathcal{W} - \frac{1}{3}\left(2\chi\mathcal{L}''' + 9\chi'\mathcal{L}'' + 15\chi''\mathcal{L}' + 10\chi'''\mathcal{L} + \frac{k}{4\pi}\chi''''' + \frac{64\pi}{k}(\chi\mathcal{L}\mathcal{L}' + \chi'\mathcal{L}^2)\right)$$

These are identical to the variations

$$\delta\mathcal{O} = \frac{1}{2\pi i}\oint dw J(z)\mathcal{O}(w) \tag{8.47}$$

with the current

$$J = \epsilon\mathcal{L} + \chi\mathcal{W}. \tag{8.48}$$

Here we use complex coordinate $z = x$. Then, we can read an operator product expansion from (8.46). The result is just the W_3 algebra with a central charge $c = 6k$[22]. There is an embedding of $SL(2,\mathbb{R})$ with $\mathcal{W} = 0$, which is generated by generators (t_1, t_2, t_3). This is often called a principal embedding. AdS spacetime is realized by the following flat solution

$$\begin{aligned}
A_{AdS} &= \frac{t_1}{y}dx - \frac{t_2}{y}dy \\
\bar{A}_{AdS} &= -\frac{t_3}{y}d\bar{x} + \frac{t_2}{y}dy.
\end{aligned}$$

When we calculate the metric by using (8.34), it coincides with the Poincaré AdS metric given by (4.17). The spin-3 field is identically zero. Notice that we set AdS radius unit $l_{AdS} = 1$.

8.5 $W_3^{(2)}$ algebra

The AdS spacetime is also realized by another configurations

$$\begin{aligned}
A'_{AdS} &= \frac{t_4}{4y}dx - \frac{t_2}{2y}dy \\
\bar{A}'_{AdS} &= \frac{t_8}{4y}d\bar{x} + \frac{t_2}{2y}dy.
\end{aligned} \tag{8.49}$$

This yields the Poincaré AdS metric with half an unit AdS radius

$$ds^2 = \frac{1}{4y^2}(dy^2 - dxd\bar{x}) \tag{8.50}$$

and vanishing spin-3 field. This fact implies that there are two inequivalent embedding of $SL(2,\mathbb{R})$. As was discussed in [21], in the case of $SL(3,\mathbb{R})$, there are two inequivalent embeddings of $SL(2,\mathbb{R})$. One is called a principal embedding in $SL(3,\mathbb{R})$. In another type, $SL(2,\mathbb{R})$ is generated by (W_{-2}, L_0, W_2). We define new $SL(2,\mathbb{R})$ generators $\hat{L}_n$ by rescaling as

$$\hat{L}_1 = \frac{1}{4}W_2,\ \hat{L}_0 = \frac{1}{2}L_0,\ \hat{L}_{-1} = -\frac{1}{4}W_{-2} \tag{8.51}$$

and this embedding gives rise to a new algebra known as $W_3^{(2)}$.

Let us consider the unbarred gauge connection

$$\mathcal{A} = (\hat{t}_4 - \mathcal{T}\hat{t}_8 + jt_6 + g_1t_3 + g_2t_5)dx \tag{8.52}$$

and the gauge transformation

$$\lambda = \epsilon_1\hat{t}_4 + \epsilon_0\hat{t}_2 + \epsilon_{-1}\hat{t}_8 + \gamma t_6 + \delta_1 t_1 + \delta_{-1}t_5 + \rho_1 t_7 + \rho_{-1}t_3. \tag{8.53}$$

Here, we define $(\hat{t}_4, \hat{t}_2, \hat{t}_8) = (W_2/4, L_0/2, W_{-2}/4)$. Defining new fields

$$\begin{aligned}
j &= \frac{9}{2\hat{c}}U \\
\mathcal{T} &= -\frac{6}{\hat{c}}T - \frac{27}{\hat{c}^2}U^2 \\
g_1 &= \frac{3}{\sqrt{2}\hat{c}}(G_+ + G_-) \\
g_2 &= \frac{3}{\sqrt{2}\hat{c}}(G_+ - G_-)
\end{aligned} \tag{8.54}$$

and gauge parameters

$$\begin{aligned}
\epsilon_1 &= \epsilon \\
\gamma &= -\frac{1}{2}\eta + \frac{9}{2\hat{c}}U\epsilon \\
\delta_1 &= \frac{1}{2\sqrt{2}}(\alpha_+ + \alpha_-) \\
\delta_2 &= \frac{1}{2\sqrt{2}}(\alpha_+ - \alpha_-),
\end{aligned} \tag{8.55}$$

the variations of (U, T, G_+, G_-) take the form

$$\begin{aligned}
\delta U &= \epsilon' U + \epsilon U' - \alpha_+ G_+ \alpha_- G_- - \frac{\hat{c}}{9}\eta' \\
\delta T &= \frac{\hat{c}}{12}\epsilon''' + 2\epsilon' T + \epsilon T' + \frac{3}{2}\alpha'_+ G_+ + \frac{1}{2}\alpha_+ G'_+ + \frac{3}{2}\alpha'_- G_- + \frac{1}{2}\alpha_- G'_- + \eta' U \\
\delta G_+ &= \frac{\hat{c}}{6}\alpha''_- + \frac{3}{2}\epsilon' G_+ + \epsilon G'_+ + \alpha_-(T + \frac{18}{\hat{c}}U^2 + \frac{3}{2}U') + 3\alpha'_- U + \eta G_+ \\
\delta G_- &= -\frac{\hat{c}}{6}\alpha''_+ + \frac{3}{2}\epsilon' G_- + \epsilon G'_- - \alpha_+(T + \frac{18}{\hat{c}}U^2 + \frac{3}{2}U') + 3\alpha'_+ U - \eta G_-
\end{aligned} \tag{8.56}$$

As we discussed in the preceding subsection, these can be translated into the following operator product expansion

$$
\begin{aligned}
U(z)U(0) &= -\frac{\hat{c}}{9z^2} \\
U(z)G_{\pm}(0) &= \pm\frac{G_{\pm}(0)}{z} \\
T(z)U(0) &= \frac{U(0)}{z^2} + \frac{\partial U(0)}{z} \\
T(z)G_{\pm}(0) &= \frac{3}{2z^2}G_{\pm}(0) + \frac{\partial G_{\pm}(0)}{z} \\
T(z)T(0) &= \frac{\hat{c}}{2z^4} + \frac{2}{z^2}T(0) + \frac{\partial T(0)}{z} \\
G_{+}(z)G_{-}(0) &= -\frac{\hat{c}}{3z^3} + \frac{3}{z^2}U(0) - \frac{1}{z}(T(0) + \frac{18}{\hat{c}}U^2(0) - \frac{3}{2}\partial U(0)).
\end{aligned}
\tag{8.57}
$$

This is the $W_3^{(2)}$ algebra. This contains a stress tensor T, current operator U with conformal dimension 1 and two quasi-primary operators with conformal dimension 3/2 denoted by $G_{\pm}$. $\hat{c}$ is the central charge. The embedding of $SL(2,\mathbb{R})$ with $j = g_1 = g_2 = 0$ is generated by generators $(\hat{t}_4, \hat{t}_2, \hat{t}_8)$. This gives the same $SL(2,\mathbb{R})$ commutation relation. The only difference arises from the rescaled trace relation. Generators $(\hat{t}_4, \hat{t}_2, \hat{t}_8)$ have traces

$$
\mathrm{Tr}(\hat{t}_2\hat{t}_2) = \frac{1}{4}\mathrm{Tr}(t_2t_2) \tag{8.58}
$$

$$
\mathrm{Tr}(\hat{t}_4\hat{t}_8) = \frac{1}{4}\mathrm{Tr}(t_1t_3). \tag{8.59}
$$

This implies that the Chern-Simons level is divided by four. According to [22], the level is related to the central charge. Therefore, the central charge of the $W_3^{(2)}$ is modified and given by

$$
\hat{c} = \frac{1}{4}c = \frac{3k}{2}. \tag{8.60}
$$

In [21], the interpolating solution between $W_3^{(2)}$ vacuum in the ultraviolet region $y = 0$ and W_3 vacuum in the infrared region $y \to \infty$ is discovered. The connections are given by

$$
A_{INT} = \frac{\lambda}{y}t_1dx + \frac{t_4}{4y^2}d\bar{x} - \frac{t_2}{y}dy \tag{8.61}
$$

$$
\bar{A}_{INT} = -\frac{\lambda}{y}t_3d\bar{x} + \frac{t_8}{4y^2}dx + \frac{t_2}{y}dy \tag{8.62}
$$

where the parameter λ plays the role of renormalization group flow. When $\lambda = 0$, $W_3^{(2)}$ vacuum is realized. While for $\lambda \to \infty$, we can find W_3 vacuum is realized by rescaling $y \to \lambda y$.

The corresponding metric and spin-3 field are

$$
ds^2 = \frac{dy^2}{y^2} - \left(\frac{1}{4y^4} + \frac{\lambda^2}{y^2}\right)dxd\bar{x} \tag{8.63}
$$

$$
\varphi = -\frac{\lambda^2}{8y^4}dx^3 + \frac{\lambda^2}{8y^4}d\bar{x}^3. \tag{8.64}
$$

At large y region equivalently large λ region, the metric yields the ome for the Poincaré AdS. At small y region region, the metric yields the one with half an unit radius. The spin-3 field is asymptotes to zero at both region. Notice that x and $\bar{x}$ interchange when y goes from zero to infinity.

The central charge decreases after the flow occurs. It seems odd since it is inconsistent with the c-theorem [44]. However, the proof of c-theorem applies to Lorentz invariant renormalization flow. Although the fixed points of this flow are Lorentz invariant theories, the full flow is not. Then, there is no contradiction.

8.6 Black Hole Solutions in Spin-3 Gravity

The BTZ black hole is realized as the Chern-Simons flat connection and it is given by

$$A_{BTZ} = \left(yt_1 - \frac{a}{y}t_3\right)dx + \frac{t_2}{y}dy \tag{8.65}$$

$$\bar{A}_{BTZ} = -\left(yt_3 - \frac{\bar{a}}{y}t_1\right)d\bar{x} - \frac{t_2}{y}dy \tag{8.66}$$

where a and $\bar{a}$ are parameters related to mass M and angular momentum J

$$a = 2G(M-J),\ \bar{a} = 2G(M+J). \tag{8.67}$$

Using these connection, the metric can be written as

$$ds^2 = \frac{dy^2}{y^2} - \frac{dxd\bar{x}}{y^2} + adx^2 + \bar{a}d\bar{x}^2 - a\bar{a}y^2 dxd\bar{x} \tag{8.68}$$

Notice that x and $\bar{x}$ has two periodicities: $x \sim x + 2\pi \sim x + 2\pi i$. The Hawking temperature can be determined by imposing a condition that the conical singularity at the horizon does not appear in the metric (8.68). In Euclidean signature $(x, \bar{x}) = (z, -\bar{z})$, this constraint implies the periodicity condition

$$(z, \bar{z}) \sim (z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) \tag{8.69}$$

and we can obtain

$$\tau = \frac{i}{2\sqrt{a}},\ \bar{\tau} = \frac{i}{\sqrt{\bar{a}}}. \tag{8.70}$$

This result is obtained by a different method. We use holonomy[25][45]. On a hypersurface with $y = 1$, the flat connections are given by

$$\mathcal{A}_{BTZ} = (t_1 - at_3)\,dx \tag{8.71}$$

$$\bar{\mathcal{A}}_{BTZ} = -(t_3 - \bar{a}t_1)\,d\bar{x}. \tag{8.72}$$

Translations to a point with an arbitrary y are realized by acting $g = y^{-t_2}$ on these connection

$$A_{BTZ} = g\mathcal{A}_{BTZ}g^{-1} \tag{8.73}$$

$$\bar{A}_{BTZ} = g^{-1}\bar{\mathcal{A}}_{BTZ}g. \tag{8.74}$$

We can define the Wilson line as

$$W(x_i; x_f) = \mathcal{P}\mathrm{exp}\left(-\int_{x_i}^{x_f} \mathcal{A}_{BTZ}\right),\ \bar{W}(\bar{x}_i; \bar{x}_f) = \mathcal{P}\mathrm{exp}\left(-\int_{\bar{x}_i}^{\bar{x}_f} \bar{\mathcal{A}}_{BTZ}\right) \tag{8.75}$$

where the symbol $\mathcal{P}$ denotes the path-ordered product. Since $\mathcal{A}_{BTZ}$ and $\bar{\mathcal{A}}_{BTZ}$ do not depend on x and $\bar{x}$, the Wilson lines take the form

$$W(x_i; x_f) = \exp(A_{BTZ}(x_f) - A_{BTZ}(x_i)),\ \bar{W}(x_i; x_f) = \exp\left(\bar{\mathcal{A}}_{BTZ}(\bar{x}_f) - \bar{\mathcal{A}}_{BTZ}(\bar{x}_i)\right). \tag{8.76}$$

We set $(x_f, x_i) = (z + 2\pi\tau, z)$. Then, the holonomy matrix is defined by $\omega = \log W$. The requirement that the flat connections are non-singular is equivalent with the statement that the matrix w and $\bar{w}$ should be required to have eigenvalues 0, $2\pi i$ or $-2\pi i$. So, we need that they satisfy the following two conditions

$$\mathrm{det} w = 0 \tag{8.77}$$

$$\mathrm{Tr} w^2 = -8\pi^2 \tag{8.78}$$

and similar equations for $\bar{w}$. Solutions to these four equations coincide with (8.70).

In [20][21], the authors discovered a black hole solution with spin-3 charge. Two parameters b and μ related to spin-3 charge and its chemical potential are introduced. In order to obtain the connections, which corresponds to a black hole with a spin-3 charge, they required the following constraints:

1) The Euclidean geometry is smooth at the horizon.

2) When we take the limit $\mu \to 0$, the geometry asymptotes to the BTZ black hole.

3) The thermal partition function should be written like

$$Z = \mathrm{Tr} e^{a\tau + b\nu},$$

where ν is a function of (μ, τ), (a, b) should obey the integrability condition

$$\left(\frac{\partial a}{\partial \nu}\right)_\tau = \left(\frac{\partial b}{\partial \tau}\right)_\nu \tag{8.79}$$

They proposed that the flat conditions that satisfy the above three requirement were given by

$$A = \left(\frac{t_1}{y} - ayt_3 - by^2 t_8\right) dx + \mu \left(8byt_3 + a^2 y^2 t_8 - 2at_6 + \frac{t_8}{y^2}\right) d\bar{x} - \frac{t_2}{y} dy \tag{8.80}$$

$$\bar{A} = -\left(\frac{t_3}{y} - \bar{a}yt_1 - \bar{b}y^2 t_4\right) d\bar{x} - \bar{\mu} \left(8\bar{b}yt_1 + \bar{a}^2 y^2 t_4 - 2\bar{a}t_6 + \frac{t_8}{y^2}\right) dx + \frac{t_2}{y} dy \tag{8.81}$$

where we define mass and spin-3 charge parameters as a and b. These are related to $\mathcal{L}$ and $\mathcal{W}$ in [21] by

$$a = \frac{2\pi}{k}\mathcal{L},\ b = \frac{\pi}{2k}\mathcal{W}. \tag{8.82}$$

The integrability condition is guaranteed by the four holonomy conditions

$$1728b^2\mu^3\tau^3 + 9b\tau^3(3 + 48a) + a^2\mu\tau^3(27 - 16a\zeta_1^3\mu^2 - 9(-2 + 8a\mu^2)) = 0 \tag{8.83}$$

$$3 + 12a\tau^2 - 144b\mu\tau^2 + 64a^2\mu^2\tau^2 = 0. \tag{8.84}$$

plus counterpart conditions. First solving the second equation for b and differentiating with respect to τ, we can obtain an expression $\partial b/\partial\tau$. Next, by substituting b into the first equation and differentiating with respect to $\nu = 4\mu\tau$, it turns out that the integrability condition is satisfied and the thermal partition function can be given by

$$Z = \mathrm{Tr} e^{\frac{i\pi}{2G}(a\tau + b\nu)} e^{\frac{i\pi}{2G}(\bar{a}\bar{\tau} + \bar{b}\bar{\nu})} \tag{8.85}$$

where an overall factor $i\pi/2G$ is introduced in order to obtain an appropriate energy of black hole.

Part III
Original Works

9 Bulk Local State in 3D Spin-3 Gravity

In [17][18], it was proposed that three-dimensional higher-spin theories with W_N symmetry are dual to a W_N extended conformal field theory. In particular, for $N = 3$, the spin-3 gravity is equivalent to the $SL(3,\mathrm{R}) \times SL(3,\mathrm{R})$ Chern-Simons theory.

In this section, we will construct the bulk scalar state with spin-3 charge by extending the prescription proposed in [14] to W_3 symmetry. Before we go to the concrete construction, we must take care about the dimension of manifold. As we discussed in Section.3, conformal field theory is characterized by the Virasoro algebra and current algebras. In general, current algebras come from additional gauge symmetries and the algebras are closed only among its generators. One example is the Kac-Moody algebra discussed in Section.3. Zamolodchikov [16] proposed an algebra by introducing a current with conformal dimension 3. The symmetry algebra is called $su(1,2) \oplus su(1,2)$ algebra

$$[L_n, L_m] = \quad (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{0,m+n} \tag{9.1}$$

$$[L_n, W_m] = \quad (2n-m)W_{n+m} \tag{9.2}$$

$$[W_n, W_m] = \quad \frac{c}{360}n(n^2-1)(n^2-4)\delta_{0,m+n} \tag{9.3}$$

$$+\frac{1}{30}\left(n-m)(2n^2-mn+2m^2-8\right)L_{n+m} + \beta(n-m)\Lambda_{n+m} \tag{9.4}$$

plus the similar algebra for anti-chiral counterpart $\bar{L}_n$ and $\bar{W}_m$, where

$$\beta = \frac{16}{22+5c} \tag{9.5}$$

$$\Lambda_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq 2} L_k L_{n-k}. \tag{9.6}$$

Here, L_n's denote the Virasoro generator corresponding to the conformal symmetry transformation and W_m's denote extra gauge symmetry generators.

In the large c limit, the third term proportional to β is dropped. Restricting the wedge mode L_n with $n = 0, \pm 1$ and W_m with $m = 0, \pm 1, \pm 2$ and changing normalization $W_n \to W_n/\sqrt{10}$, the subalgebra can be rewritten as

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} \\
&[L_n, W_m] = (2n-m)W_{n+m} \\
&[W_n, W_m] = \frac{1}{3}(n-m)\left(2n^2-mn+2m^2-8\right)L_{n+m}.
\end{aligned} \tag{9.7}$$

9.1 Enlarged spacetime

The algebra is different from the Kac-Moody algebra. The algebra (9.7) is totally closed, however, the commutator $[W_n, W_m]$ is not closed. Twice action of W_3 generators gives the conformal transformation. This is similar to the supersymmetry algebra. The generators of supersymmetry contain usual Poincaré generators, which consist of translations and rotations, and anti-commuting spinors Q and its conjugates[46][47]:

$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0. \tag{9.8}$$

The anti-commutator of Q and $Q^\dagger$ gives a translation generator

$$\{Q_\alpha, Q^\dagger_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \tag{9.9}$$

where σ is the Pauili matrices. The commutators of Q and P vanish. We have to enlarge the spacetime to include commmuting and anti-commuting variable in order to describe supersymmetric field theory. The enlarged spacetime is called superspace and contains ordinary spacetime coordinates and a set of anti-commuting Grassmann coordinate $(\theta, \bar{\theta})$. The finite supersymmetric transformation

$$G(x, \theta, \bar{\theta}) = e^{-ix^\mu P_\mu} e^{i\theta Q} e^{i\bar{\theta} Q^\dagger} \tag{9.10}$$

can be defined on the superspace $(x^\mu, \theta, \bar{\theta})$.

If there exists a current operator $W(z)$ with spin-s, the primary conditions associated with W_k's $(k = -s+1, -s+2, \cdots, s-1)$ are given by

$$W_m |\psi\rangle = 0 \tag{9.11}$$

$$W_0 |\psi\rangle = w |\psi\rangle \tag{9.12}$$

where $m > 0$ and w is a constant. In the conformal field theory, the primary state is annihilated by L_1 and the descendant state is created by L_{-1}. All descendant states are created by multiplying $|\psi\rangle$ by an exponent $e^{ixL_{-1}}$. Similarly, in the case of W_3 extended conformal field theory, W_{-2} and W_{-1} create new descendant states. In order to create all descendant state for W_m, it is necessary to introduce coordinates α and β, which correspond to W_{-2} and W_{-1}. Because of the commutator $[W_n, W_m] \propto L_{n+m}$, for example $[W_0, W_{-1}]$ gives a translation, these coordinates α and β mix with ordinary coordinates under $SU(1,2)$ transformations. Hence, we need to regard new coordinates as spacetime coordinates instead of simple gauge variables, and we must introduce an enlarged spacetime, which contain ordinary two coordinate $(x, \bar{x})$ and a set of coordinates $(\alpha, \beta, \bar{\alpha}, \bar{\beta})$ related to $SU(1,2)$ transformation, in order to formulate the $SU(1,2)$ invariant theory. As we will discuss later, L_{-1} moves an operator to an arbitrary point of x. Similarly, W_{-2} and W_{-1} move an operator to an arbitrary point of α, β. In the context of the AdS/CFT correspondence, we must also introduce extra coordinates, which corresponds to translation generated by diagonalized generator L_0, in order to move a boundary operator to a bulk point. In W_3 extended conformal field theory, there is another diagonalized generator W_0. This generates extra transformation. Therefore, the total dimension of the bulk spacetime becomes eight.

9.2 Boundary State

In this subsection, we will consider a local state of a scalar field with a spin-3 charge. In the case of three dimensional AdS spacetime, the condition for the local state of a scalar field $|\psi\rangle$ located at the center of the bulk is given in terms of the genelators of the wedge algebra $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ and given by [14][15]

$$(L_n - (-1)^n \bar{L}_{-n}) |\psi\rangle = 0 \quad (n = -1, 0, 1). \tag{9.13}$$

While in the case of the spin-3 gravity, the symmetry algebra $su(1,2) \oplus su(1,2)$ is generated by L_n and W_m. Conditions for the local state of the scalar will be expressed in terms of generators of the $su(1,2) \oplus su(1,2)$ symmetry which contains the above $sl(2,\mathrm{R}) \oplus sl(2,\mathrm{R})$. Then, conditions for local scalar will be (9.13) and

$$(W_m - (-1)^m \bar{W}_{-m}) |\psi\rangle = 0. \tag{9.14}$$

The commutator $[W_n, W_m]$ gives the Virasoro generator and it must satisfy the condition (9.13). (9.14) is a sufficient and necessary condition such that the commutator $[W_n, W_m]$ satisfies (9.13).

First, we will consider the boundary state. Since the conformal field theory is living on the boundary in the text of AdS/CFT correspondence, it is convenient to rewrite down the state at the boundary rather than the one at the center of the bulk. And for later convenience, we will move the representation from the elliptic basis into the hyperbolic basis, which is discussed in Section.4.

Let $|\psi\rangle_B$ be a scalar state extrapolated to the boundary. This state and $|\psi\rangle$ are related by the following transformation:

$$|\psi\rangle_B = \lim_{\rho\to\infty} g(\rho)\,|\psi\rangle\,, \tag{9.15}$$

where

$$g(\rho) = \exp\left[-\frac{1}{2}\rho(L_1 - L_{-1} + \bar{L}_1 - \bar{L}_{-1})\right]. \tag{9.16}$$

The state on the boundary satisfies the following conditions instead of (9.13) and (9.14):

$$\lim_{\rho\to\infty} g(\rho)(L_n - (-1)^n\bar{L}_{-n})g^{-1}(\rho)\,|\psi\rangle_B = 0 \tag{9.17}$$

$$\lim_{\rho\to\infty} g(\rho)(W_n - (-1)^n\bar{W}_{-n})g^{-1}(\rho)\,|\psi\rangle_B = 0 \tag{9.18}$$

First, let us consider (9.17). Before taking the limit, (9.17) are given by

$$\begin{aligned}
&\left[(L_0 - \bar{L}_0)\cosh\rho - \frac{1}{2}(L_1 + L_{-1} - \bar{L}_1 - \bar{L}_{-1})\sinh\rho\right]|\psi\rangle_B \\
=\ &\left[\frac{1}{2}(L_1 - L_{-1} + \bar{L}_1 - \bar{L}_{-1}) + \frac{1}{2}(L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1})\cosh\rho - (L_0 + \bar{L}_0)\sinh\rho\right]|\psi\rangle_B \\
=\ &\left[\frac{1}{2}(L_{-1} - L_1 + \bar{L}_{-1} - \bar{L}_1) + \frac{1}{2}(L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1})\cosh\rho - (L_0 + \bar{L}_0)\sinh\rho\right]|\psi\rangle_B = 0
\end{aligned} \tag{9.19}$$

In the limit $\rho \to \infty$, after taking an appropriate linear combination, the above conditions degenerate into holomorphic and anti-holomorphic parts:

$$\left[L_0 - \frac{1}{2}(L_1 + L_{-1})\right]|\psi\rangle_B = 0 \tag{9.20}$$

$$\left[\bar{L}_0 - \frac{1}{2}(\bar{L}_1 + \bar{L}_{-1})\right]|\psi\rangle_B = 0 \tag{9.21}$$

A solution to these conditions is constructed by the primary state

$$|\mathcal{O}_\Delta, q\rangle = \lim_{t\to i\infty} \mathcal{O}_{\Delta,q}(\phi = 0, t)\,|0\rangle\,, \tag{9.22}$$

where $\mathcal{O}_{\Delta,q}$ is a scalar primary operator with a conformal weight $(h, \bar{h}) = (\Delta/2, \Delta/2)$. This state should satisfy

$$L_1\,|\mathcal{O}_{\Delta,q}\rangle = \bar{L}_1\,|\mathcal{O}_{\Delta,q}\rangle = 0 \tag{9.23}$$

$$L_0\,|\mathcal{O}_{\Delta,q}\rangle = \bar{L}_0\,|\mathcal{O}_{\Delta,q}\rangle = \frac{\Delta}{2}\,|\mathcal{O}_{\Delta,q}\rangle \tag{9.24}$$

and extra conditions for W_3 generators

$$W_2\,|\mathcal{O}_{\Delta,q}\rangle = \bar{W}_2\,|\mathcal{O}_{\Delta,q}\rangle = 0 \tag{9.25}$$

$$W_1\,|\mathcal{O}_{\Delta,q}\rangle = \bar{W}_1\,|\mathcal{O}_{\Delta,q}\rangle = 0 \tag{9.26}$$

$$W_0\,|\mathcal{O}_{\Delta,q}\rangle = \bar{W}_0\,|\mathcal{O}_{\Delta,q}\rangle = iq\,|\mathcal{O}_{\Delta,q}\rangle\,. \tag{9.27}$$

From (9.23) and (9.24), it turns out the boundary state $|\psi\rangle_B$ is solved as

$$|\psi\rangle_B = e^{L_{-1}+\bar{L}_{-1}} |\mathcal{O}_\Delta, q\rangle \,. \tag{9.28}$$

It implies that the boundary state of the scalar field can be represented by using corresponding quasi-primary operator and its descendant. It can also be shown that the operator which annihilates $|\psi\rangle_B$ coincides with the Virasoro generator L_1^h in the hyperbolic representation in Section.4.

$$L_1^h = -L_0 + \frac{1}{2}(L_1 + L_{-1}) \tag{9.29}$$
$$L_0^h = \frac{1}{2}(L_1 - L_{-1}) \tag{9.30}$$
$$L_{-1}^h = L_0 + \frac{1}{2}(L_1 + L_{-1}). \tag{9.31}$$

Notice that L_0^h is anti-Hermitian and others are Hermitian. Similarly, there are relations for $\bar{L}_n^h$. The state (9.28) satisfies the quasi-primary conditions for Virasoro generators in the hyperbolic representation

$$\begin{aligned} &L_1^h |\psi\rangle_B = \bar{L}_1^h |\psi\rangle_B = 0 \\ &L_0^h |\psi\rangle_B = \bar{L}_0^h |\psi\rangle_B = \frac{\Delta}{2} |\psi\rangle_B \end{aligned} \tag{9.32}$$

It can be shown that the exponential operator in (9.28) carries a quasi-primary state $|\mathcal{O}_\Delta\rangle$ at the origin of Euclidean plane in the global coordinates to the boundary in the Poincaré coordinates. In order to describe at an arbitrary point, we should act a translation operator g. (9.32) lead to introduce parameters $(x, \bar{x})$, which correspond to a translation generated by $(L_{-1}^h, \bar{L}_{-1}^h)$ and the translation to y direction is generated by $L_0^h + \bar{L}_0^h$. Then, the translation operator can be written as

$$g(y, x, \bar{x}) = e^{ixL_{-1}^h} e^{i\bar{x}\bar{L}_{-1}^h} y^{L_0^h + \bar{L}_0^h}. \tag{9.33}$$

Notice that L_{-1}^h is Hermitian.

Next, we will consider the quasi-primary conditions for W generators (9.18). Transformations of W_m under $g(\rho)$ are given in Appendix.A. The boundary limits of the quasi-primary conditions for W (9.18) are reduced to the following conditions:

$$(W_2 - 4W_1 + 6W_0 - 4W_{-1} + W_{-2}) |\psi\rangle_B = 0 \tag{9.34}$$
$$(\bar{W}_2 - 4\bar{W}_1 + 6\bar{W}_0 - \bar{W}_{-1} + \bar{W}_{-2}) |\psi\rangle_B = 0 \tag{9.35}$$

As in the case of Virasoro generators, these conditions define generators W_2^h and $\bar{W}_2^h$, which annihilate the boundary state. Also the $su(1,2)$ algebra defines remaining W_m generators

$$W_2^h = \frac{i}{4}(W_2 - 4W_1 + 6W_0 - 4W_{-1} + W_{-2}) \tag{9.36}$$
$$W_1^h = \frac{i}{4}(W_2 - 2W_1 + 2W_{-1} - W_{-2}) \tag{9.37}$$
$$W_0^h = \frac{i}{4}(W_2 - 2W_0 + W_{-2}) \tag{9.38}$$
$$W_{-1}^h = \frac{i}{4}(W_2 + 2W_1 - 2W_{-1} - W_{-2}) \tag{9.39}$$
$$W_{-2}^h = \frac{i}{4}(W_2 + 4W_1 + 6W_0 + 4W_{-1} + W_{-2}) \tag{9.40}$$

We adopt the rule of the Hermitian conjugation $(W_m)^\dagger = -W_{-m}$. Note that W^h_m with odd m are Hermitian and the others are anti-Hermitian. The boundary state satisfies

$$\begin{aligned} W^h_2 \left|\psi\right\rangle_B &= W^h_1 \left|\psi\right\rangle_B = 0 \\ W^h_0 \left|\psi\right\rangle_B &= q \left|\psi\right\rangle_B . \end{aligned} \tag{9.41}$$

and similar conditions for $\bar{W}^h_m$. Since the boundary state $|\psi\rangle$ satisfies the primary conditions (9.32) and (9.41), the state is a primary state in $SU(1,2)$ algebra. The state at the center of the bulk will be constructed in terms of this primary state on the hyperbolic basis.

From similar discussion in the above, we should introduce transformation from the boundary to an arbitrary point. According to (9.41) and its counterparts, it is carried out to introduce four parameter $(\alpha, \beta, \bar{\alpha}, \bar{\beta})$, which are related with the transformations generated by $(W^h_{-2}, W^h_{-1}, \bar{W}^h_{-2}, \bar{W}^h_{-1})$ plus one coordinate γ, which are related with the transformations generated by $W_0 + \bar{W}_0$. Then, the transformation takes the form

$$g(y, x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma) = e^{ixL^h_{-1} + i\bar{x}\bar{L}^h_{-1}} e^{i\alpha W^h_{-2} + i\bar{\alpha}\bar{W}^h_{-2}} e^{\beta W^h_{-1} + \bar{\beta}\bar{W}^h_{-1}} e^{\frac{i}{2}\gamma(W^h_0 + \bar{W}^h_0)} y^{L^h_0 + \bar{L}^h_0}. \tag{9.42}$$

Since W^h_{-1} is anti-Hermitian, corresponding finite transformations can be defined without the imaginary unit i.

9.3 CFT Description of Local State in the Bulk

We will solve the quasi-primary conditions for the local state at the center of the bulk $|\psi\rangle$. This state will be built from the boundary state $|\psi\rangle_B$.

For a while, we will see the case without W extension. In [13], this was solved by a series expansion. Here, we will obtain the state in terms of an integral form. Assume that the local state is represented in a form

$$|\psi\rangle = \int dx d\bar{x} f(x, \bar{x}) e^{xL^h_{-1}} e^{\bar{x}\bar{L}^h_{-1}} \left|\psi\right\rangle_B . \tag{9.43}$$

Let us first consider the condition (9.13) with $n = 0$. When L^h_0 is applied on $|\psi\rangle_B$, we obtain

$$L^h_0 \left|\psi\right\rangle = \int dx d\bar{x} f(x, \bar{x}) e^{xL^h_{-1}} (e^{-xL^h_{-1}} L^h_0 e^{xL^h_{-1}}) e^{\bar{x}\bar{L}^h_{-1}} \left|\psi\right\rangle_B \tag{9.44}$$

Calculating the commutator

$$e^{-xL^h_{-1}} L^h_0 e^{xL^h_{-1}} = L^h_0 + xL^h_{-1} \tag{9.45}$$

and replacing L^h_0 and L^h_{-1} by $\Delta/2$ and ∂_x, we can obtain

$$L^h_0 \left|\psi\right\rangle = \int dx d\bar{x} \left(\frac{\Delta}{2} - x\partial_x\right) f(x, \bar{x}) e^{xL^h_{-1}} e^{\bar{x}\bar{L}^h_{-1}} \left|\psi\right\rangle_B . \tag{9.46}$$

After a similar calculation for $\bar{L}^h_0$ is done, the condition $(L^h_0 - \bar{L}^h_0)\left|\psi\right\rangle = 0$ yields an equation

$$(x\partial_x - \bar{x}\partial_{\bar{x}}) f(x, \bar{x}) = 0 \tag{9.47}$$

Similarly, the other two conditions yield the following equations:

$$\left(x^2\partial_x + \Delta x\right) f(x, \bar{x}) = 0 \tag{9.48}$$

$$\left(\bar{x}^2\partial_{\bar{x}} + \Delta \bar{x}\right) f(x, \bar{x}) = 0 \tag{9.49}$$

From the above three equations, we can obtain a solution up to an overall constant

$$f(x,\bar{x}) = (1+x\bar{x})^{\Delta-2}, \tag{9.50}$$

and the local state at the center of the bulk is obtained by

$$|\psi\rangle = \int dx d\bar{x} (1+x\bar{x})^{\Delta-2} e^{xL^h_{-1}} e^{\bar{x}\bar{L}^h_{-1}} |\psi\rangle_B = \frac{\pi}{1-\Delta} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\Delta)_n} (L^h_{-1}\bar{L}^h_{-1})^n |\psi\rangle_B . \tag{9.51}$$

Here, $(\Delta)_n = \Delta(\Delta+1)\cdots(\Delta+n-1)$. This expression coincide with the one, which is obtained by a different way in [15].

In the case of spin-3 gravity, it is necessary to introduce four extra variables, which correspond to W generators. The state at the center of the bulk can be obtained in a similar way.

$$|\psi\rangle = \int dxdydzd\bar{x}d\bar{y}d\bar{z} F(x,y,z,\bar{x},\bar{y},\bar{z}) e^{xW^h_{-2}+\bar{x}\bar{W}^h_{-2}} e^{yW^h_{-1}+\bar{y}\bar{W}^h_{-1}} e^{zL^h_{-1}+\bar{z}\bar{L}^h_{-1}} |\psi\rangle_B . \tag{9.52}$$

Here, $\bar{x}, \bar{y}$ and $\bar{z}$ are complex conjugates of x, y and z. The function F is obtained by solving eight equations (9.13) and (9.14) and given by the following expression

$$F(x,y,z,\bar{x},\bar{y},\bar{z}) = \left(\frac{R_+}{R_-}\right)^{\frac{3}{4}iq} T^{\frac{\Delta-8}{4}}, \tag{9.53}$$

where

$$\begin{aligned}
R_+ = \quad & 1 + 2\zeta|1+\xi_2|^2 + \zeta^2|1-4\xi_1-2\xi_2-\xi_2^2|^2 & (9.54)\\
R_- = \quad & 1 + 2\zeta|1-\xi_2|^2 + \zeta^2|1+4\xi_1+2\xi_2-\xi_2^2|^2 & (9.55)\\
T = \quad & 1 + 4\zeta(1+|\xi_2|^2) + 2\zeta^2(|4\xi_1+2\bar{\xi}_2|^2 + 3|\xi_3^2-1|^2) & \\
& +4\zeta^3(1+|\xi_2|^2)^{-1}(|1+|\xi_2|^4-\xi_2^2-\bar{\xi}_2^2|^2 + |4(1+|\xi_2|^2)\xi_1+3\xi_2-\xi_2^3+\bar{\xi}_2+\xi_2^2\bar{\xi}_2|^2) & \\
& +\zeta^4|1-16\xi_1^2-16\xi_1\xi_2-6\xi_2^2+\xi_2^4|^2 & (9.56)
\end{aligned}$$

Here, ζ and ξ_i are variables defined by

$$\zeta = z\bar{z},\ \xi_1 = \frac{x}{z^2},\ \xi_2 = \frac{y}{z} \tag{9.57}$$

and the bar symbol denotes complex the conjugate. Substituting this solution F into (9.52), we can obtain the formal integral expression for a state at the center of the bulk. As in the case of $SL(2,\mathbb{R})$, a local state in an arbitrarily bulk point can be given by multiplying the transformation g in (9.42):

$$|\Phi(y,x,\bar{x},\alpha,\bar{\alpha},\beta,\bar{\beta},\gamma)\rangle = g(y,x,\bar{x},\alpha,\bar{\alpha},\beta,\bar{\beta},\gamma)\,|\psi\rangle\,, \tag{9.58}$$

where

$$g(y,x,\bar{x},\alpha,\bar{\alpha},\beta,\bar{\beta},\gamma) = e^{ixL^h_{-1}+i\bar{x}\bar{L}^h_{-1}} e^{i\alpha W^h_{-2}+i\bar{\alpha}\bar{W}^h_{-2}} e^{\beta W^h_{-1}+\bar{\beta}\bar{W}^h_{-1}} e^{\frac{i}{2}\gamma(W^h_0+\bar{W}^h_0)} y^{L^h_0+\bar{L}^h_0}.$$

There are six integrals and it is technically difficult to perform. From the equivalence with the HKLL reconstruction, however, we should reconstruct the local state at an arbitrary point by using a bulk to boundary propagator for the bulk scalar field with a conformal weight $(\Delta/2, \Delta/2)$. In the next section, we will consider the bulk scalar field, which satisfies the Klein-Gordon equation and solve this equation. Actually, it will turn out that a solution is invariant under $SU(1,2)$ transformation for the bulk.

9.4 Differential Representation for $SU(1,2)$ Generators

In order to obtain the second-order differential equation, we should consider a differential representation or an infinite dimensional representation for $SU(1,2)$ generators. As we discussed in Section.2, given an infinitesimal transformation parameter ϵ, we can seek a differential representation T such that a field Φ transforms as

$$\Phi'(x') = (1 - \epsilon T)\Phi(x)$$

From now, we will seek a differential representation of W generators.

The local state at an arbitrary point is obtained by performing a transformation $g(y, x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)$ for the center state $|\psi\rangle$. By studying $SU(1,2) \times SU(1,2)$ transformation of this state, we will be able to obtain an infinite dimensional representation of L^h_n and W^h_m in the bulk. First, let us apply $e^{-\epsilon L^h_{-1}}$ on $|\Phi(y, x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)\rangle$, where a parameter ϵ is an infinitesimal constant. We have

$$\begin{aligned} e^{-\epsilon L^h_{-1}} |\Phi(y, x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)\rangle &= |\Phi(y, x + i\epsilon, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)\rangle \\ &= (1 - i\epsilon\partial_x) |\Phi(y, x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)\rangle \end{aligned} \tag{9.59}$$

and this defines a representation $\hat{L}^h_{-1} = i\partial_x$. In the below, we will omit the coordinates $(y, x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)$ for simplicity. For the other generators, the exponent of the generator is moved to the right by using the $SU(1,2)$ algebra until it reaches the center state $|\psi\rangle$, and the condition (9.13) and (9.14) are used and then the exponent is moved to the left. For example, we consider an exponent $e^{-\epsilon L^h_0}$.

$$\begin{aligned} e^{-\epsilon L^h_0} |\Phi\rangle &= e^{-\epsilon L^h_0} e^{ixL^h_{-1}} e^{\epsilon L^h_0} e^{-\epsilon L^h_0} e^{\alpha W^h_{-2}} e^{\epsilon L^h_0} \cdots e^{-\epsilon L^h_0} y^{L^h_0 + \bar{L}^h_0} |\psi\rangle \\ &= |\Phi\rangle - \epsilon \left(x\partial_x + 2\alpha\partial_\alpha + \beta\partial_\beta + \frac{1}{2} y\partial_y \right) |\Phi\rangle + \mathcal{O}(\epsilon^2). \end{aligned} \tag{9.60}$$

In the second line, we use the Baker-Campbell-Hausdorff formula

$$e^{sX} Y e^{-sX} = Y + s[X, Y] + \frac{1}{2} s^2 [X, [X, Y]] + \cdots .$$

By this procedure, we identify the following representations

$$\begin{aligned} \hat{L}^h_{-1} = & \quad i\partial_x \\ \hat{L}^h_0 = & \quad x\partial_x + 2\alpha\partial_\alpha + \beta\partial_\beta + \frac{1}{2} y\partial_y \\ \hat{L}^h_1 = & \quad -ixy\partial_y - i(x^2 + 3\beta^2)\partial_x - 3i\beta\partial_\gamma - i(2\beta^3 + 4x\alpha)\partial_\alpha - i(2x\beta + 4\alpha)\partial_\beta \\ & \quad -iy^2 \cosh 2\gamma \partial_{\bar{x}} - iy^2 \bar{\beta} \cosh 2\gamma \partial_{\bar{\alpha}} - iy^2 \sinh 2\gamma \partial_{\bar{\beta}} \\ \hat{W}^h_{-2} = & \quad i\partial_\alpha \\ \hat{W}^h_{-1} = & \quad -x\partial_\alpha - \partial_\beta \\ \hat{W}^h_0 = & \quad -2i\beta\partial_x - i(x^2 + \beta^2)\partial_\alpha - 2ix\partial_\beta - i\partial_\gamma \\ \hat{W}^h_1 = & \quad 3x\partial_\gamma + (6x\beta - 4\alpha)\partial_x + (x^3 + 3x\beta^2)\partial_\alpha + (\beta^2 + 3x^2)\partial_\beta + \beta y\partial_y \\ & \quad +y^2 \sinh 2\gamma \partial_{\bar{x}} + y^2 \bar{\beta} \sinh 2\gamma \partial_{\bar{\alpha}} + y^2 \cosh 2\gamma \partial_{\bar{\beta}} \\ \hat{W}^h_2 = & \quad -i(8\alpha - 4x\beta) y\partial_y + 6i(x^2 - \beta^2)\partial_\gamma + i(12\beta x^2 - 4\beta^3 - 16x\alpha)\partial_x \\ & \quad +i(4x^3 + 4x\beta^2 - 16\alpha\beta)\partial_\beta - i(3\beta^4 - x^4 + 16\alpha^2 - 6x^2\beta^2)\partial_\alpha \\ & \quad +i[y^4 - 4y^2\bar{\beta}(\beta \cosh 2\gamma - x \sinh 2\gamma)]\partial_{\bar{\alpha}} - 4i(\beta \sinh 2\gamma - x \cosh 2\gamma) y^2 \partial_{\bar{\beta}} \\ & \quad -4i(\beta \cosh 2\gamma - x \sinh 2\gamma) y^2 \partial_{\bar{x}}. \end{aligned} \tag{9.61}$$

The expressions for $\bar{L}^h_n$ and $\bar{W}^h_m$ are obtained by replacing $(x, \bar{x}, \alpha, \bar{\alpha}, \beta, \bar{\beta})$ by $(\bar{x}, x, \bar{\alpha}, \alpha, \bar{\beta}, \beta)$. These generators satisfy the wedge-mode algebra (9.7). In the following discussion, we will omit the hat symbol for simplicity.

10 Correlation functions and Bulk Geometry

In Section.7, we reviewed the construction of bulk scalar state as a state in conformal field theory. $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetric theory has a quadratic Casimir operator. According to [15], an eigenvalue equation of the quadratic Casimir operator for a bulk scalar state coincides with the Klein-Gordon equation for the scalar. This implies that the Klein-Gordon equation is inherent to the conformal field theory and the differential representation of the quadratic Casimir operator corresponds to the Klein-Gordon operator.

There is a quadratic Casimir operator in the W_3 extended conformal field theory. In Section.9, the infinite dimensional representation of $SU(1,2)$ generators was also obtained. Then, the eigenvalue equation becomes a second-order differential equation and we should interpret this equation as the equation of motion for the bulk scalar field. Since this theory has two diagonalized generator L_0 and W_0, eigenvalues contain both the conformal dimension Δ and W_3 charge q.

Using this representation, we will study correlation functions of quasi-primary operators in the boundary. the bulk to boundary propagators is constructed by solving an eigenvalue problem of a $SU(1,2)$ quadratic Casimir operator.

10.1 Correlation Functions on the Boundary

First, we need a representation of the holomorphic $SU(1,2)$ generators on the boundary. At the boundary $y \to 0$, the boundary state $|\phi\rangle_B$ is the eigenstate of L_0 and W_0 with the eigenvalue $\Delta/2$ and iq. Then, in order to obtain the representation, we replace

$$y\partial_y \;\rightarrow\; \Delta \tag{10.1}$$
$$\partial_\gamma \;\rightarrow\; q \tag{10.2}$$

and take the limit $y \to 0$. Generators on the boundary are expressed as follows:

$$\begin{aligned}
L_{-1} &= i\partial_x \\
L_0 &= -x\partial_x - 2\alpha\partial_\alpha - \beta\partial_\beta - \frac{\Delta}{2} \\
L_1 &= -i(x^2 + 3\beta^2)\partial_x - ix\Delta + 3iq\beta - i(2\beta^3 + 4a\alpha)\partial_\alpha - i(2x\beta + 4\alpha)\partial_\beta \\
W_{-2} &= i\partial_\alpha \\
W_{-1} &= -x\partial_\alpha - \partial_\beta \\
W_0 &= -2i\beta\partial_x - i(x^2 + \beta^2)\partial_\alpha - 2xi\partial_\beta - iq \\
W_1 &= (-4\alpha + 6x\beta)\partial_x + (x^3 + 3x\beta^2)\partial_\alpha + (3x^2 + \beta^2)\partial_\beta + \beta\Delta + 3ixq \\
W_2 &= -i(4\beta^3 - 12\beta x^2 + 16x\alpha)\partial_x - i(3\beta^4 - x^4 + 16\alpha^2 - 6x^2\beta^2)\partial_\beta \\
&\quad + i(-16\alpha\beta + 4x\beta^2 + 4x^3)\partial_\beta - 6i(x^2 - \beta^2)q + i(4x\beta - 8\alpha)\Delta.
\end{aligned} \tag{10.3}$$

Similarly, generators $\bar{L}_n, \bar{W}_m$ are obtained from the above by interchanging $x \to \bar{x}, \alpha \to \bar{\alpha}, \beta \to \bar{\beta}$.

Let $\mathcal{O}_{\Delta_i,q_i}(x_i, \alpha_i, \beta_i)$ be holomorphic quasi-primary operators at points (x_i, α_i, β_i). Due to the $su(1,2)$ invariance, the two-point function of these operators

$$G_{12}(x_1, \alpha_1, \beta_1; x_2, \alpha_2, \beta_2) = \langle 0| T\mathcal{O}_{\Delta_1,q_1}(x_1, \alpha_1, \beta_1)\mathcal{O}_{\Delta_2,q_2}(x_2, \alpha_2, \beta_2) |0\rangle\,, \tag{10.4}$$

where the symbol T denotes the time-ordering product, must satisfy the following conditions:

$$(L_n^{(1)} + L_n^{(2)})G_{12} = 0 \tag{10.5}$$
$$(W_m^{(1)} + W_m^{(2)})G_{12} = 0. \tag{10.6}$$

Here, superscripts (1) and (2) refer to each operator $\mathcal{O}_{\Delta_i,q_i}$. By $n = -1$ in (10.5) and $m = -2$ in (10.6), G_{12} must be a function of $x_1 - x_2$ and $\alpha_1 - \alpha_2$. Taking other constraints into account, only if $\Delta_1 = \Delta_2$ and $q_1 = q_2$, this function is non-zero and G_{12} is determined up to overall constant. For a time order $t_1 > t_2$, the two-point function is given by

$$G_{12} = (D_{12})^{-\frac{\Delta+3iq}{4}}(D_{12}^*)^{-\frac{\Delta-3iq}{4}}, \tag{10.7}$$

where D_{12} and D_{12}^* are defined by

$$D_{12} = x_{12}^2 - \beta_{12}^2 - 2x_{12}(\beta_1 + \beta_2) + 4\alpha_{12} \tag{10.8}$$
$$D_{12}^* = x_{12}^2 - \beta_{12}^2 + 2x_{12}(\beta_1 + \beta_2) - 4\alpha_{12}. \tag{10.9}$$

Notice that in order to avoid the light-cone singularity we adopt the Feynmann prescription and replace x_{12} by $x_{12} - i\epsilon$. We omit $i\epsilon$ term for simplicity. The two-point function of anti-holomorphic quasi-primary operator is calculable in the similar way but we need to take the replacement $D_{12} \to \bar{D}_{12}$ and $D_{12}^* \to \bar{D}_{12}^*$. Here $\bar{D}_{12}$ and $\bar{D}_{12}^*$ are defined by

$$\bar{D}_{12} = \bar{x}_{12}^2 - \bar{\beta}_{12}^2 - 2\bar{x}_{12}(\bar{\beta}_1 + \bar{\beta}_2) + 4\bar{\alpha}_{12} \tag{10.10}$$
$$\bar{D}_{12}^* = \bar{x}_{12}^2 - \bar{\beta}_{12}^2 + 2\bar{x}_{12}(\bar{\beta}_1 + \bar{\beta}_2) + 4\bar{\alpha}_{12}. \tag{10.11}$$

Next, let us consider three point function of quasi-primary operators:

$$G_{123}(x_1, \alpha_1, \beta_1; x_2, \alpha_2, \beta_2; x_3, \alpha_3, \beta_3) = \tag{10.12}$$
$$\langle 0| T\mathcal{O}_{\Delta_1,q_1}(x_1, \alpha_1, \beta_1)\mathcal{O}_{\Delta_2,q_2}(x_2, \alpha_2, \beta_2)\mathcal{O}_{\Delta_3,q_3}(x_3, \alpha_3, \beta_3) |0\rangle . \tag{10.13}$$

In a similar fashion, G_{123} is a solution to the following equations:

$$\sum_{i=1}^{3} L_n^{(i)} G_{123} = 0 \tag{10.14}$$
$$\sum_{i=1}^{3} W_m^{(i)} G_{123} = 0. \tag{10.15}$$

G_{123} is non-zero only for $q_1 + q_2 + q_3 = 0$. Unlike the case of the conformal field theory, there are three linearly independent solutions to these equations:

$$G_{123} = (D_{12}D_{12}^*)^{-\frac{1}{8}(\Delta_1+\Delta_2-\Delta_3)}(D_{23}D_{23}^*)^{-\frac{1}{8}(\Delta_2+\Delta_3-\Delta_1)}(D_{13}D_{13}^*)^{-\frac{1}{8}(\Delta_1+\Delta_3-\Delta_2)}\sum_{i=1}^{3} c_i K_i \tag{10.16}$$

where c_i is constant and

$$\begin{aligned} K_1 &= \left(\frac{D_{13}}{D_{13}^*}\right)^{-\frac{3}{4}iq_1}\left(\frac{D_{23}}{D_{23}^*}\right)^{-\frac{3}{4}iq_2} \\ K_2 &= \left(\frac{D_{21}}{D_{21}^*}\right)^{-\frac{3}{4}iq_2}\left(\frac{D_{31}}{D_{31}^*}\right)^{-\frac{3}{4}iq_3} \\ K_3 &= \left(\frac{D_{32}}{D_{32}^*}\right)^{-\frac{3}{4}iq_3}\left(\frac{D_{12}}{D_{12}^*}\right)^{-\frac{3}{4}iq_1} \end{aligned} \tag{10.17}$$

However, there are several restrictions. When one of the operator is an identity, this reduces to a two point function. For example, let $\mathcal{O}_{\Delta_3,q_3}$ be an identity. In this case, K_1 cannnot appear. So, the coefficient c_1 should be a polynomial with respect to Δ_3 and q_3. Similarly, c_2 and c_3 should be a polynomial with respect to (Δ_1, q_1) and (Δ_2, q_2).

For $\alpha, \beta, \bar{\alpha}, \bar{\beta}, \gamma \neq 0$, the conformal invariance is broken in these correlation functions, since extra five coordinates are introduced. If $\alpha = \beta = \bar{\alpha} = \bar{\beta} = \gamma = 0$, the invariance is recovered and the two-point function G_{12} and the three-point function G_{123} coincide with the ones discussed in Section.3. It is natural, because the scalar state (9.58) at $(\alpha, \beta, \bar{\alpha}, \bar{\beta}, \gamma) = \mathbf{0}$ coincides with the usual conformal covariant state in Section.7.

At $\beta = 0$, this leads to usual conformal invariant two-point function with conformal dimension Δ. There is another point where conformal invariance is recovered. That is the point where $\beta \to \infty$. After multiplying a normalization factor, the two-point function behaves as $x_{12}^{-\Delta}$. This implies that the conformal invariance is recovered and the dual operator with the bulk scalar is a quasi-primary operator with conformal dimension $\Delta/2$. We will discuss in detail in the later section.

10.2 Bulk to Boundary Propagator

From the structure of the local state in the bulk, the scalar local state in the bulk satisfies a differential equation which is associated with the quadratic Casimir operator of $su(1, 2)$

$$\begin{aligned} C_2(L, W) = \quad & (L_0)^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) - \frac{1}{8}(W_2 W_{-2} + W_{-2} W_2) \\ & + \frac{1}{2}(W_1 W_{-1} + W_{-1} W_1) - \frac{3}{4}(W_0)^2. \end{aligned} \tag{10.18}$$

It can be shown that the eigenvalue of C_2 on $|\mathcal{O}_{\Delta,q}\rangle$ is given by $(\Delta^2 - 8\Delta - 3q^2)/4$. Substituting the infinite dimensional representation (9.61), a second-order differential equation for a scalar field is obtained:

$$\begin{aligned} & \big[y^2\partial_y^2 - 7y\partial_y + 3\partial_\gamma^2 - 4y^2\cosh 2\gamma\partial_x\partial_{\bar{x}} - 4y^2\cosh 2\gamma(\bar{\beta}\partial_x\partial_{\bar{\alpha}} + \beta\partial_{\bar{x}}\partial_\alpha) \\ & -4y^2\sinh 2\gamma(\partial_x\partial_{\bar{\beta}} + \partial_{\bar{x}}\partial_\beta) - 4y^2\sinh 2\gamma(\beta\partial_\alpha\partial_{\bar{\beta}} + \bar{\beta}\partial_{\bar{\alpha}}\partial_\beta) \\ & +(y^4 - 4\beta\bar{\beta}y^2\cosh 2\gamma)\partial_\alpha\partial_{\bar{\alpha}} - 4y^2\cosh 2\gamma\partial_\beta\partial_{\bar{\beta}} - m^2\big]\,|\Phi\rangle = 0, \end{aligned} \tag{10.19}$$

where m is a mass of the scalar field and defined by

$$m^2 = \Delta^2 - 8\Delta - 3q^2 \tag{10.20}$$

This equation is interpreted as the Klein-Gordon equation for a scalar field in eight dimensional enlarged spacetime. The spacetime is isometric under $SU(1, 2) \times SU(1, 2)$ transformation and the scalar field transforms non-trivially under this symmetry transformation (9.58).

The construction of the bulk to boundary propagator is carried out by solving (10.19). Let $K_{\Delta,q}$ be the bulk to boundary propagator. Near the boundary, $K_{\Delta,q}$ can be expressed as the power series expansion in y:

$$K_{\Delta,q}(y, \gamma, X_1; X_2) = \sum_{n=0}^{\infty} y^{\Delta+2n} e^{-iq\gamma} f_n(\gamma, X_1; X_2), \tag{10.21}$$

where X_i denotes 6 cordinates $(x_i, \bar{x}_i, \alpha_i, \bar{\alpha}_i, \beta_i, \bar{\beta}_i)$. The leading order term f_0 is just the product of a holomorphic and an anti-holomorphic two-point functions in the boundary conformal field theory:

$$f_0 = (D_{12}\bar{D}_{12})^{-\frac{\Delta+3iq}{4}} (D_{12}^*\bar{D}_{12}^*)^{-\frac{\Delta-3iq}{4}} \tag{10.22}$$

Substituting (10.22) into (10.19), an equation for f_1 can be obtained and it is easily solved. By repeating this process sequentially, we can obtain other functions f_n. Up to order $y^{\Delta+4}$, the results are expressed in Appendix.B.

Up to order $y^{\Delta+4}$, the series (10.21) can be summed up and formed into

$$
\begin{aligned}
K_{\Delta,q} = y^{\Delta} e^{-iq\gamma} \quad & \left[D_{12}\bar{D}_{12} - 2y^2(x_{12}+\beta_{12})(\bar{x}_{12}+\bar{\beta}_{12})e^{-2\gamma} + y^4\right]^{-\frac{\Delta+3iq}{4}} \\
& \left[D^*_{12}\bar{D}^*_{12} - 2y^2(x_{12}-\beta_{12})(\bar{x}_{12}-\bar{\beta}_{12})e^{2\gamma} + y^4\right]^{-\frac{\Delta-3iq}{4}}. \qquad (10.23)
\end{aligned}
$$

It can be proved that this expression solves the Klein-Gorodon equation (10.19) exactly. Also, it is checked that this satisfies the bulk $su(1,2) \times su(1,2)$ invariance conditions:

$$
\begin{aligned}
(L^{(1)}_n + L^{(2)}_n) K_{\Delta,q} &= 0 \qquad (10.24)\\
(W^{(1)}_m + W^{(2)}_m) K_{\Delta,q} &= 0. \qquad (10.25)
\end{aligned}
$$

Notice that in the above expressions, the superscript (1) denotes the generators in the bulk (9.61) and (2) denotes the generators in the boundary (10.3). Hence, the solution (10.23) is $su(1,2) \times su(1,2)$ invariant and the exact solution for (10.19).

By using the bulk to boundary propagator, the scalar field inside the bulk should be reconstructed in terms of boundary operators. This provides a more explicit expression for the local state at an arbitrary bulk point than that in Section.9:

$$
|\Phi(y,x,\bar{x},\alpha,\bar{\alpha},\beta\bar{\beta},\gamma)\rangle = \int d^6X' K_{\Delta,q}(y,\gamma,X;X')\, |\phi\rangle_B \,. \qquad (10.26)
$$

10.3 Geometry of 8D Manifold

In the quantum field theory in the curved spacetime, the Klein-Gorodon equation for a scalar field can be expressed as

$$
g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{-g}} g^{\mu\nu}\partial_\mu\left(\sqrt{-g}\partial_\nu\phi\right) = m^2\phi, \qquad (10.27)
$$

where $g_{\mu\nu}$ is a metric of the manifold and m is a mass of a scalar field. It is possible to read off the metric from (10.27). Since the spacetime must be enlarged by five extra coordinates that corresponds to W_m transformation, the metric contains their components.

The equation (10.19) coincides with the Klein-Gordon equation for a scalar field (10.27). Then, it is possible to read off a metric from this equation:

$$
\begin{aligned}
ds_0^2 &= g_{\mu\nu}dx^\mu dx^\nu \\
&= \frac{dy^2}{y^2} - \frac{1}{y^4}(y^2\cosh 2\gamma - 4\beta\bar{\beta})dxd\bar{x} + \frac{4}{y^4}d\alpha d\bar{\alpha} - \frac{1}{y^2}\cosh 2\gamma d\beta d\bar{\beta} \\
&\quad - \frac{4}{y^4}(\beta dx d\alpha + \bar{\beta}d\bar{x}d\alpha) + \frac{1}{y^2}\sinh 2\gamma(dxd\bar{\beta} + d\bar{x}d\beta) + \frac{1}{3}d\gamma^2 \qquad (10.28)
\end{aligned}
$$

The determinant of this metric is $g = \det g_{\mu\nu} = -y^{-18}/12$. This spacetime is no longer a 8 dimensional pure AdS space. One of the evidences is the number of minus signs. A pure AdS space has a signature $(-,+,+,+,+,+,+,+)$, but this metric has a signature $(-,-,-,+,+,+,+,+)$. Secondarily, it can be shown that this metric satisfies the Einstein equation

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_8 g_{\mu\nu} = 0 \qquad (10.29)
$$

with a negative cosmological constant $\Lambda_8 = -36$. However, the fact that a pure AdS space is maximally symmetric spacetime tells us a cosmological constant should be -21. Furthermore, the symmetry does not coincide with the one of the pure AdS space. In fact, (10.28) is invariant under $SU(1,2) \times SU(1,2)$ rather than $SO(2,7)$.

The ordinary AdS space can be realized by taking a three dimensional hypersurface Σ_0 embedded at $\alpha = \bar{\alpha} = \beta = \bar{\beta} = \gamma = 0$. Other hypersurface with constant $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ and γ can be also considered. An induced metric on this hypersurface Σ_C is given by

$$ds_0^2|_{\Sigma_C} = \frac{dy^2}{y^2} - \frac{1}{y^4}(y^2 \cosh 2\gamma - 4\beta\bar{\beta})dxd\bar{x}. \tag{10.30}$$

This geometry is interesting. For $\beta = \bar{\beta} = 0$, the hypersurface is of course an AdS space with an unit radius. When we set $\beta = -\bar{\beta} = \lambda$ and take the limit $\lambda \to \infty$, the term proportional to y^{-4} dominates in the second term. So, hypersurface asymptotes to an AdS space with half an unit radius. For $\beta\bar{\beta} < 0$, the spacetime is not an AdS, but asymptotically AdS. This is a solution interpolating the W_3 and $W_3^{(2)}$ vacua, which are discussed in Section.8. $W_3^{(2)}$ vacuum is realized in the ultraviolet region $y = 0$, while in the infrared region $y \to \infty$ W_3 vacuum is realized.

10.4 AdS/CFT Dictionaries in Spin-3 Gravity

In Section.6, we reviewed the AdS/CFT correspondence. There is an equivalence between correlation functions in the AdS gravity and ones in conformal field theory. This equivalence is described by two equivalent methods, which are called the differentiating and extrapolating dictionaries, respectively.

Again we briefly recall the two dictionaries. The differentiating dictionary tells us that the generating functional in both side are equivalent[7][8]

$$Z_{AdS} = Z_{CFT}$$

The partition function in AdS gravity is expressed in terms of a classical action S of fundamental fields ϕ in Euclidean signature with a boundary condition ϕ_0

$$Z_{AdS} = \exp\left[-S[\phi_0]\right].$$

While the generating functional of correlation functions in the conformal field theory is expressed in terms of the product between a quasi-primary operator $\mathcal{O}$ and its source ϕ_0

$$Z_{CFT} = \left\langle \exp\left[\int d^d x \phi_0 \mathcal{O}\right]\right\rangle.$$

where the source function ϕ_0 is the boundary value at the boundary of AdS spacetime on the gravity side. Two-point function is given by functionaly differentiating the source function.

In another dictionary[9], the leading behavior of the bulk correlator is equal to the correlation function in conformal field theory. More mathematically, we consider a bulk fundamental field ϕ and their correlation function. Then, the leading boundary behavior is equal to the correlation function in conformal field theory with normalization

$$y^{2\Delta} \langle \phi(x_1)\phi(x_2)\rangle \sim \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$$

where $\mathcal{O}$ is a quasi-primary operator dual to the bulk field ϕ. This dictionary holds in our construction since the leading behavior of the bulk to boundary propagator (10.23) is equal to the boundary to boundary propagator (10.22) up to normalization factor.

Here, there is one question whether the differentiating dictionary is valid in the spin-3 gravity or not. More explicitly, can we write down the classical action integral for scalar field coupled to the spin-3 gravity? In this subsection, we will show that the differentiating dictionary also holds in the spin-3 gravity.

We should switch to an Euclidean signature by a Wick rotation

$$x = z,\ \beta = i\xi,\ \alpha = i\zeta \tag{10.31}$$

$$\bar{x} = -\bar{z},\ \bar{\beta} = i\bar{\xi},\ \bar{\alpha} = -i\bar{\zeta}. \tag{10.32}$$

Here $\bar{z}, \bar{\xi}$ and $\bar{\zeta}$ are the complex conjugations of z, ξ and ζ. It can be shown that in the region $z_{12}, \xi_{12}, \zeta_{12} \sim 0$ and in the $y \to 0$ limit, (10.23) behaves as

$$\begin{aligned} K_{\Delta,q} = \quad & \mathcal{N}(\gamma_1) y^{8-\Delta} e^{-iq\gamma_1} \delta^2(z_{12}) \delta^2(\xi_{12}) \delta^2(\zeta_{12}) + \cdots \\ & + y^{\Delta} e^{-iq\gamma_1} (D^E_{12} \bar{D}^E_{12})^{-\frac{\Delta+3iq}{4}} (D^{*E}_{12} \bar{D}^{*E}_{12})^{-\frac{\Delta-3iq}{4}} + \cdots \end{aligned} \tag{10.33}$$

where the dots stand for terms with higher order powers of y and E means Wick rotated version of D_{12}'s, according to the rule (10.31) and (10.32). $\mathcal{N}(\gamma_1)$ is given by

$$\begin{aligned} \mathcal{N}(\gamma_1) = \int d^2z d^2\xi d^2\zeta \quad & (1 + 2e^{-2\gamma_1} |z + i\xi|^2 + |z^2 + \xi^2 - 2iz\xi + 4i\zeta|^2)^{-\frac{\Delta+3iq}{4}} \\ & (1 + 2e^{2\gamma_1} |z - i\xi|^2 + |z^2 + \xi^2 + 2iz\xi - 4i\zeta|^2)^{-\frac{\Delta-3iq}{4}}. \end{aligned} \tag{10.34}$$

We define an action integral for a bulk scalar field coupled to spin-3 gravity by

$$S_{matter} = \int dy d^2z d^2\xi d^2\zeta d\gamma \sqrt{g} \left(g^{\mu\nu} \nabla_\mu \Phi^* \nabla_\nu \Phi + m^2 \Phi^* \Phi \right). \tag{10.35}$$

By substituting a solution for the Klein-Gordon equation

$$\Phi(y,\gamma,z,\xi,\zeta,\bar{z},\bar{\xi},\bar{\zeta}) = \int d^2z' d^2\xi' d^2\zeta' K_{\Delta,q}(y,\gamma,z,\xi,\zeta,\bar{z},\bar{\xi},\bar{\zeta}; z',\xi',\zeta',\bar{z}',\bar{\xi}',\bar{\zeta}') \Phi_0(z',\xi',\zeta',\bar{z}',\bar{\xi}',\bar{\zeta}') \tag{10.36}$$

, where Φ_0 is a boundary condition, into the action integral (10.35), a generating functional for two-point function is obtained as a surface integral on the boundary

$$S_{matter} = \left(\int_{-\infty}^{\infty} d\gamma \mathcal{N}(\gamma) e^{-iq\gamma} \right) \int d^2z_1 d^2\xi_1 d^2\zeta_1 d^2z_2 d^2\xi_2 d^2\zeta_2 f_0 \Phi_0^*(z_1,\xi_1,\zeta_1,\bar{z}_1,\bar{\xi}_1,\bar{\zeta}_1) \Phi_0(z_2,\xi_2,\zeta_2,\bar{z}_2,\bar{\xi}_2,\bar{\zeta}_2) \tag{10.37}$$

where f_0 is the boundary two-point function given by (10.22). Therefore, the differentiating dictionary is established in the case of spin-3 gravity coupled to a scalar field.

11 Vielbein Formulation in 8D

In addition to a quadratic Casimir operator, there is one more Casimir operator in the W_3 extended conformal field theory. This is a cubic Casimir operator. Substituting (9.61) into this operator, a third-order differential equation are obtained. Although this is an independent equation in the context of the boundary theory, it will turn out that a solution (10.23) also satisfies this eigenvalue equation. This equation allows us to read off a completely symmetric rank-3 tensor field, which is called a spin-3 gauge field.

In this section, we will study the geometry of the enlarged spacetime. At a hypersurface with vanishing extra coordinates, this coincides with the three-dimensional AdS spacetime. We will discuss the vielbein formulation of this geometry analogous to Section.8. The vielbein is uniquely determined by the metric and the spin-3 field up to a local frame transformation. We will introduce 8×8 gauge connection matrices A and $\bar{A}$ with an analogy of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ case. Surprisingly, these field strengths are flat.

11.1 Spin-3 Gauge Field and Vielbein Formalism

We will consider an another Casimir operator for $SU(1,2)$ symmetry, which is called a cubic Casimir operator. This is defined by

$$\begin{aligned}
C_3 = & -\frac{i}{4}(L_1L_1W_{-2}+L_1W_{-2}L_1+W_{-2}L_1L_1)-i(L_0L_0W_0+L_0W_0L_0+W_0L_0L_0)\\
&+\frac{i}{2}(L_1L_0W_{-1}+L_0L_1W_{-1}+L_1W_{-1}L_0+L_0W_{-1}L_1+W_{-1}L_0L_1+W_{-1}L_1L_0)\\
&-\frac{i}{4}(L_1L_{-1}W_0+L_{-1}L_1W_0+L_1W_0L_{-1}+L_{-1}W_0L_1+W_0L_1L_{-1}+W_0L_{-1}L_1)\\
&+\frac{i}{2}(L_0L_{-1}W_1+L_{-1}L_0W_1+L_0W_1L_{-1}+L_{-1}W_1L_0+W_1L_0L_{-1}+W_1L_{-1}L_0) \qquad (11.1)\\
&-\frac{i}{4}(L_{-1}L_{-1}W_2+L_{-1}W_2L_{-1}+W_2L_{-1}L_{-1})-\frac{i}{4}(W_2W_{-1}W_{-1}+W_{-1}W_2W_{-1}+W_{-1}W_{-1}W_2)\\
&+\frac{i}{8}(W_2W_0W_{-2}+W_0W_2W_{-2}+W_2W_{-2}W_0+W_0W_{-2}W_0+W_{-2}W_0W_2+W_{-2}W_2W_0)\\
&-\frac{i}{4}(W_2W_{-1}W_{-1}+W_{-1}W_2W_{-1}+W_{-1}W_{-1}W_2)-\frac{i}{4}(W_1W_1W_{-2}+W_1W_{-2}W_1+W_{-2}W_1W_1)\\
&+\frac{i}{4}(W_1W_0W_{-1}+W_0W_1W_{-1}+W_1W_{-1}W_0+W_0W_{-1}W_1+W_{-1}W_0W_1+W_{-1}W_1W_0)-\frac{3}{4}iW_0^3.
\end{aligned}$$

The local state for the scalar field in the bulk also satisfies an eigenstate equation for the cubic Casimir operator:

$$\left[C_3(L,W)+C_3(\bar{L},\bar{W})\right]|\Phi\rangle = -\frac{3}{2}iq(\Delta^2-8\Delta+16+q^2)\,|\Phi\rangle \qquad (11.2)$$

After Substituting the infinite dimensional representation for generators into the above equation, a differential equation for the scalar field Φ is obtained. This can be written in the following form

$$\phi^{\mu\nu\rho}\nabla_\mu\nabla_\nu\nabla_\rho\Phi = -\frac{3}{2}iq(\Delta^2-8\Delta+16+q^2)\Phi, \qquad (11.3)$$

where ∇_μ is a covariant derivative for the metric (10.28). The tensor $\phi_{\mu\nu\rho}$ is completely symmetric and given by

$$\begin{aligned}
\phi &= \phi_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho\\
&= \frac{1}{36y^4}\left[27\bar{\beta}d\bar{x}dx^2+27\beta dxd\bar{x}^2-27y\sinh 2\gamma dydxd\bar{x}-27dx^2d\bar{\alpha}-27d\bar{x}^2d\alpha\right.\\
&\quad +27d\alpha d\bar{\beta}^2+27d\bar{\alpha}d\beta^2-72d\alpha d\bar{\alpha}d\gamma-18y^2dy^2d\gamma+2y^4d\gamma^3-27\bar{\beta}d\bar{x}d\beta^2\\
&\quad -27\beta dxd\bar{\beta}^2+72\bar{\beta}d\bar{x}d\alpha d\gamma+72\beta dxd\bar{\alpha}d\gamma-72\beta\bar{\beta}dxd\bar{x}d\gamma\\
&\quad +(27ydxd\bar{\beta}dy+27yd\bar{x}d\beta dy-9y^2dxd\bar{x}d\gamma-9y^2d\beta d\bar{\beta}d\gamma)\cosh 2\gamma\\
&\quad \left.+(-27yd\beta d\bar{\beta}dy+9y^2dxd\bar{\beta}d\gamma+9y^2d\bar{x}d\beta d\gamma)\sinh 2\gamma\right]. \qquad (11.4)
\end{aligned}$$

On the hypersurface Σ_C, (11.4) simplifies to

$$\phi|_{\Sigma_C} = \frac{3}{4y^4}\left[\bar{\beta}d\bar{x}dx^2+\beta dxd\bar{x}^2-y\sinh 2\gamma dydxd\bar{x}\right]. \qquad (11.5)$$

It is clear that ϕ is equal to zero at the hypersurface Σ_0. It can be shown that 8 dimensional covariant derivative of $\phi_{\mu\nu\rho}$ vanishes.

From now, let us show that the field ϕ coincides with a spin-3 gauge field. To do so, we introduce a vielbein e^a_μ with $SL(3,\mathbb{R})$ local symmetry. Here, the superscript a denotes local indices and the subscript μ denotes spacetime indices. Notice that this is different from the 8×3 rectangular vielbein introduced in [19] but this is an 8×8 square vielbein.

We consider $SL(3,\mathbb{R})$ gauge connections

$$A = \omega + e = t_a({\omega^a}_\mu + {e^a}_\mu)dx^\mu \tag{11.6}$$
$$\bar{A} = \omega - e = t_a({\omega^a}_\mu - {e^a}_\mu)dx^\mu \tag{11.7}$$

and require flatness conditions on them

$$F = dA + A\wedge A = 0 \tag{11.8}$$
$$\bar{F} = d\bar{A} + \bar{A}\wedge\bar{A} = 0. \tag{11.9}$$

Here, ω denotes a spin connection and t_a are generator matrices for $SL(3,\mathbb{R})$ described in Appendix.C. Note that in even dimensions there is no Chern-Simons action which yields the flatness conditions. Taking a hypersurce with constant $\alpha,\bar{\alpha},\beta,\bar{\beta}$ and γ, there are the equations of motion for the gauge connection, which are written by (11.8) and (11.9). On this surface that frame fields (11.6) and (11.7) reduce to 8×3 rectangular connection, equations (11.8) and (11.9) will coincide with the equations of motion for connections in $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ Chern-Simons gauge theory.

11.2 Solutions to Flatness Conditions

In the preceding subsections, we obtained the metric (10.28). In Cartan formulation, the metric can be obtained by

$$g_{\mu\nu} = \frac{1}{2}\mathrm{tr}[e^2] = h_{ab}e^a_\mu e^b_\nu, \tag{11.10}$$

where h_{ab} is the Killing mertic for $SL(3,\mathbb{R})$ and defined as

$$h_{ab} = \frac{1}{2}\mathrm{tr}[t_a t_b]. \tag{11.11}$$

It turns out there are two solutions to (11.8) and (11.9), which produce (10.28).

- Solution (I)

$$\begin{aligned} A = \quad & \frac{1}{y}dyt_2 + \frac{1}{y}(\cosh\gamma dx - \sinh\gamma d\beta)t_3 + \frac{1}{2}d\gamma t_6 \\ & + \frac{1}{y}(\sinh\gamma dx - \cosh\gamma d\beta)t_7 - \frac{1}{y^2}(\bar{\beta}d\bar{x} - d\bar{\alpha})t_8 \end{aligned} \tag{11.12}$$

$$\begin{aligned} \bar{A} = \quad & -\frac{1}{y}dyt_2 - \frac{1}{y}(\cosh\gamma d\bar{x} - \sinh\gamma d\bar{\beta})t_1 - \frac{1}{2}d\gamma t_6 \\ & - \frac{1}{y}(\sinh\gamma d\bar{x} - \cosh\gamma d\bar{\beta})t_5 + \frac{1}{y^2}(\beta dx - d\alpha)t_4 \end{aligned} \tag{11.13}$$

- Solution (II)

$$\begin{aligned} A = \quad & \frac{1}{y}dyt_2 + \frac{1}{y}(\cosh\gamma dx - \sinh\gamma d\beta)t_3 - \frac{1}{2}d\gamma t_6 \\ & + \frac{1}{y}(\sinh\gamma dx - \cosh\gamma d\beta)t_7 + \frac{1}{y^2}(\beta dx - d\alpha)t_8 \end{aligned} \tag{11.14}$$

$$\begin{aligned} \bar{A} = \quad & -\frac{1}{y}dyt_2 - \frac{1}{y}(\cosh\gamma d\bar{x} - \sinh\gamma d\bar{\beta})t_1 + \frac{1}{2}d\gamma t_6 \\ & - \frac{1}{y}(\sinh\gamma d\bar{x} - \cosh\gamma d\bar{\beta})t_5 - \frac{1}{y^2}(\bar{\beta}d\bar{x} - d\bar{\alpha})t_4 \end{aligned} \tag{11.15}$$

As β and $\bar{\beta}$ increase from 0 to ∞, the leading terms in the solution (I) interchange between A_x and $A_{\bar{x}}$, while ones in the solution (II) does not interchange. It is found that the solution (I) has both A_x and $A_{\bar{x}}$ components. This solution does not satisfy boundary condition $A_{\bar{x}} = \bar{A}_x = 0$ on the hypersurface Σ_C. In the other hand, the solution (II) satisfies these boundary conditions.

For the solutions (I) and (II), the metrics are exactly the same and coincide with (10.28). The spin-3 fields, which is defined in terms of a vielbein field

$$\varphi = \frac{1}{3!}\mathrm{tr}(e^3), \tag{11.16}$$

are not the same. One for the solution (II) exactly coincides with ϕ in (11.4) up to an overall constant. By performing an $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ gauge transformation U for A and $\bar{A}$

$$A \to A' = U^{-1}AU + U^{-1}dU \tag{11.17}$$

$$\bar{A} \to \bar{A}' = UAU^{-1} + UdU^{-1}, \tag{11.18}$$

$$\tag{11.19}$$

it is possible to change the spin-3 field. Although it may be possible to transform the solution (I) into (II) by such a transformation, this will change the metric field. This is because the solutions (I) and (II) are not equivalent.

Although there are two independent solutions for flat conditions (11.8) and (11.9), only for the solution (II) both the metric and the spin-3 field, which is defined in terms of a vielbein field coincide with (10.28) and (11.4). Hence, from the viewpoint of bulk reconstruction, the solution (II) is appropriate.

Both solutions satisfy the flatness condition (11.8) and (11.9). In the following section, we will construct black hole solutions as extensions on this solution.

12 New Scale for the Renormalization Flow

InSection.9 and Section.10, we obtained the differential representation for $SU(1,2)$ generators. We also obtained the two-point function of the scalar fields on the boundary which is invariant under the $SU(1,2) \times SU(1,2)$ transformations. Furthermore, from the eigenvalue equation for the quadratic Casimir operator, we read off the metric field in Section.11.

In this section, we will analyze a boundary scalar two-point function and discuss a new scale for the renormalization flow.

12.1 Conformal Dimension for Scalar Field

The scalar two-point function on the boundary is given by (10.22). This is determined by the boundary $SU(1,2)$ symmetry. For simplicity, we will focus on the holomorphic part of the two-point function. However, a similar discussion holds for the anti-holomorphic part. So, we define the holomorphic two-point function as

$$G_{12} = (D_{12})^{-\frac{\Delta+3iq}{4}}(D^*_{12})^{-\frac{\Delta-3iq}{4}}, \tag{12.1}$$

where D_{12} and D^*_{12} are defined by

$$D_{12} = x_{12}^2 - \beta_{12}^2 - 2x_{12}(\beta_1+\beta_2) + 4\alpha_{12} \tag{12.2}$$

$$D^*_{12} = x_{12}^2 - \beta_{12}^2 + 2x_{12}(\beta_1+\beta_2) - 4\alpha_{12}. \tag{12.3}$$

If two operators are put on a hypersurface Σ_C with constant α, β and γ, then two-point function reduces to

$$G_{12}|_{\Sigma_C} = (x_{12}^2 - 4x_{12}\beta)^{-\frac{\Delta+3iq}{4}}(x_{12}^2 + 4x_{12}\beta)^{-\frac{\Delta-3iq}{4}}. \tag{12.4}$$

This breaks conformal invariance in general. For $\beta = 0$, however, this corresponds to a two-point function with conformal dimension $(\Delta/2, \Delta/2)$. This result comes from ordinary conformal filed theory, and on the hypersurface Σ_0 conformal symmetry are recovered. Furthermore, conformal symmetry is recovered by taking β to infinity after changing the normalization of the quasi-primary field by multiplying a factor $\beta^{\Delta/4}$. Then, the conformal dimension is reduced by half and spin-3 charge q vanishes at the end of the limit. The same conclusion can be reached for the three-point function. Therefore, the conformal field theory flows to the another one.

To summerize, an operator

$$\mathcal{O}(x,\alpha,\beta) = \beta^{\frac{\Delta}{4}} e^{ixL^h_{-1}} e^{i\alpha W^h_{-2}} e^{\beta W^h_{-1}} e^{L_{-1}} \mathcal{O}(0) e^{-L_{-1}} e^{-\beta W^h_{-1}} e^{-i\alpha W^h_{-2}} e^{-ixL^h_{-1}} \tag{12.5}$$

where $\mathcal{O}(0)$ is the holomorphic primary operator with $h = \Delta/2$ at the origin of Euclidean plane, have conformal correlation function of conformal weight $h/2$ in the limit $\beta \to \infty$. This renormalization group flow proceeds along β direction at $y = 0$ in the higher dimensional space with the metric (10.28). This is similar to the renormalization group flow in the radial direction y. Because β appears in the metric (10.28) in the combination $\beta\bar{\beta}/y^4$ with y, the limit $\beta \to \infty$ corresponds to $y \to 0$ that is ultraviolet limit.

12.2 Renormalization Group Flow between Two Vacua

As we discussed in Section.8, there are two vacua correponding to two types of the embeddings of $sl(2,\mathbb{R})$ in $sl(3,\mathbb{R})$. One is called principal embedding in $SL(3,\mathbb{R})$, where $SL(2,\mathbb{R})$ is generated by (L_{-1}, L_0, L_1). In another type, $SL(2,\mathbb{R})$ is generated by (W_{-2}, L_0, W_2). We define new $SL(2,\mathbb{R})$ generators $\hat{L}_n$ by rescaling as

$$\hat{L}_1 = \frac{1}{4}W_2,\ \hat{L}_0 = \frac{1}{2}L_0,\ \hat{L}_{-1} = \frac{1}{4}W_{-2} \tag{12.6}$$

and this embedding gives rise to a new algebra known as $W_3^{(2)}$.

It is clear that in this embedding a scaling dimension $\Delta/2$ changes to $\Delta/4$, then the conformal weight of primary field change one half.

Again, in a hypersurface $\beta = 0$, two-point function of scalar fields tales the form

$$G_{12}|_{\beta=0} \sim \frac{1}{x_{12}^{\Delta}}$$

,while in a hypersurface $\beta \to \infty$ it takes the form

$$G_{12}|_{\beta\to\infty} \sim \frac{1}{x_{12}^{\frac{\Delta}{2}}}.$$

So, this observation implies that vacuum change from principal vacuum to $W_3^{(2)}$ vacuum when β goes from zero to infinity.

Our renormalization group flow will be triggered by some additional terms in the action. We will argue that these terms are given by

$$\Delta S = -i\beta W^h_{-1} - i\bar{\beta}\bar{W}^h_{-1}. \tag{12.7}$$

Recall that under the flow, an operator $\mathcal{O}(x)$ transforms as

$$\begin{aligned}\mathcal{O}(x) &\to e^{ixL^h_{-1}} e^{\beta W^h_{-1}} \mathcal{O}(0) e^{-\beta W^h_{-1}} e^{-ixL^h_{-1}} \\ &= e^{\beta W^h_{-1}} \mathcal{O}'(x) e^{-\beta W^h_{-1}}\end{aligned} \tag{12.8}$$

where

$$\mathcal{O}'(x) = e^{ix(L^h_{-1}-\beta W^h_{-2})}\mathcal{O}(0)e^{-ix(L^h_{-1}-\beta W^h_{-2})}. \tag{12.9}$$

In the above discussion, we omitted α dependence because $[W_{-1}, W_{-2}] = 0$. The fact that L^h_{-1} change to $L_{-1} - i\beta W^h_{-2}$ is consistent with the observation that for $\beta = 0$ the translation generator is L^h_{-1} and for $\beta \to \infty$ it is proportional to W^h_{-2}.

Then, arbitrary states of form $|\psi\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)|0\rangle$ are transformed according to

$$|\psi\rangle \to e^{\beta W^h_{-1}}\mathcal{O}'_1(x_1)\mathcal{O}'_2(x_2)\cdots\mathcal{O}'_n(x_n)|0\rangle \tag{12.10}$$

In path integral formalism, this translation is done by adding a term $\Delta S = i\beta W^h_{-1}$ to the action. We regard this additional term as the origin of the change of the translation generator L^h_{-1}.

Next, we will consider the geometry. If the renormalization flow from W_3 vacuum to $W_3^{(2)}$ vacuum is realized, the geometry will change. More strictly, for $\beta = 0$ W_3 vacuum is realized, while for $\beta \to \infty$ another vacuum is realized. As we discussed in the previous section, the metric of eight dimensional spacetime is given by (10.28)

$$\begin{aligned} ds_0^2 \;=\; & \frac{dy^2}{y^2} - \frac{1}{y^4}(y^2\cosh 2\gamma - 4\beta\bar{\beta})dxd\bar{x} + \frac{4}{y^4}d\alpha d\bar{\alpha} - \frac{1}{y^2}\cosh 2\gamma d\beta d\bar{\beta} \\ & - \frac{4}{y^4}(\beta dx d\alpha + \bar{\beta}d\bar{x}d\alpha) + \frac{1}{y^2}\sinh 2\gamma(dxd\bar{\beta} + d\bar{x}d\beta) + \frac{1}{3}d\gamma^2. \end{aligned}$$

when we restrict a hypersurface Σ_C with constant $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ and γ, it reduces to

$$ds_0^2|_{\Sigma_C} = \frac{dy^2}{y^2} - \frac{1}{y^4}(y^2\cosh 2\gamma - 4\beta\bar{\beta})dxd\bar{x}.$$

For $\beta = \bar{\beta} = 0$, this yields the Poincaré AdS spacetime. While for $\beta\bar{\beta} \to \infty$, this asymptotes to

$$ds_0^2|_{\Sigma_C} \sim \frac{dy^2}{y^2} - \frac{4}{y^4}\lambda^2 dxd\bar{x}$$

where we set $\beta = -\bar{\beta} = \lambda$. After we rescale a variable y to $4\lambda z = y^2$, we can find

$$ds_0^2|_{\Sigma_C} \sim \frac{dz^2}{4z^2} - \frac{1}{4z^2}dxd\bar{x}.$$

It is just the Poincaré AdS with half an AdS radius. The roughly sketch of geometry is given by Figure 5.

For $0 < \beta < \infty$, there is an interpolating geometry. When the ultraviolet region $y = 0$, $W_3^{(2)}$ vacuum is realize. While when the infrared region $y \to \infty$, W_3 vacuum is realize.

This fact is almost consistent with the result in [21]. In our result, variables x and $\bar{x}$ do not interchange after and before renormalization flow occurs, however in [21], authors claim that these variables interchange.

What field theories do live on the hypersurface? Of course, At the hypersurface Σ_0 and the one where $\beta \to \infty$, the theories on the boundary are the conformal field theory. At the other hypersurface, although there is no conformal field theory, there are operators dressed by W_m generators. We denote the operators as $\mathcal{O}'$. The relation between $\mathcal{O}'$ and the primary operator $\mathcal{O}$ in the conformal field theory is given by

$$\mathcal{O}'(0) = e^{i\alpha W^h_{-2}}e^{\beta W^h_{-1}}\mathcal{O}(0)e^{-\beta W^h_{-1}}e^{-i\alpha W^h_{-2}} \tag{12.11}$$

This implies that at the hypersurface the boundary theory is the quantum field theory with dressed operators $\mathcal{O}'$. When the extra coordinates go from zero to infinity, the theory living

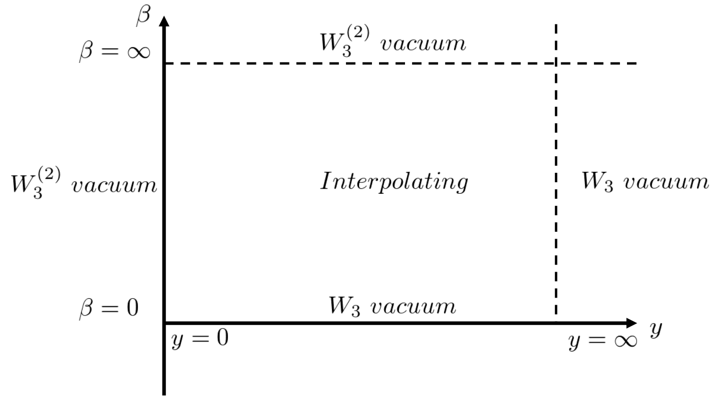

Figure 5: The flow of geometry

on the boundary flows from the conformal field theory to the conformal field theory. In the intermediate region, however, the theory is not conformal. As a result of the flow, the conformal dimension of the field operator changes from $(\Delta/2, \Delta/2)$ to $(\Delta/4, \Delta/4)$.

Finally, we will consider the role of β. In the ordinary AdS/CFT correspondence, the radial direction y corresponds to the energy scale of the boundary theory. The boundary conformal field theory with a metric

$$ds^2_{CFT} = \eta_{\mu\nu} dx^\mu dx^\nu \tag{12.12}$$

is invariant under a scale transformation $x^\mu \to \alpha' x^\nu$, which rescales the energy $E \to E/\alpha'$. In the AdS spacetime, the metric is given by

$$ds^2 = \frac{1}{y^2}(dy^2 + \eta_{\mu\nu} dx^\mu dx^\nu). \tag{12.13}$$

When we take a hypersurface with constant y, the induced metric is given by

$$ds^2 = \frac{1}{y^2}\eta_{\mu\nu} dx^\mu dx^\nu \tag{12.14}$$

and this implies that rescaled energy is given by $E' \propto 1/y$. In the pure AdS spacetime, the symmetry is $SO(2,2)$. When we take the holographic screen at constant y, the symmetry on the screen coincides with conformal symmetry. In general, asymptotically AdS spacetime, this statement is not true because the bulk has less symmetry than pure AdS spacetime. Near the boundary, however, the conformal symmetry is recovered. This means that the theory on the holographic screen at a constant y change. In the large y or the infrared region, the theory is not conformal. When y approaches the boundary, the theory flows to conformal field theory.

In the case of our model, the metric is given by (10.28). When we consider the hypersurface Σ_C with constant α, β, γ, the induced metric is given by (10.30). By performing the general coordinate transformation $4\beta z = y^2$ with $\beta = -\bar{\beta} \neq 0$, the radial coordinate changes to z.

$$ds^2 = \frac{dz^2}{z^2} - \frac{1}{4\beta z}(\cosh 2\gamma + \frac{\beta}{z})dx d\bar{x} \tag{12.15}$$

We fix $y = \epsilon \ll 1$ where the two-point function of two scalar behaves as (12.1). Then, the radial coordinate z determines the value of β uniquely. When z moves from the infrared region $z \sim \infty$ to the ultraviolet region $z \sim 0$, β moves from 0 to ∞. This is shown in Figure 6.

At $\beta = 0$ the theory is a conformal field theory where the scaling dimension of the scalar primary is $\Delta/2$, on the other hand, at $\beta \to \infty$ the theory is another conformal field theory with the scaling dimension of the scalar $\Delta/4$. At intermediate regions, the conformal symmetry is broken because of a non-zero β. The boundary theory is a quantum field theory where field operators are given by dressed operators (12.11). Therefore, the theory flows from one conformal field theory to another conformal field theory and β triggers off this flow.

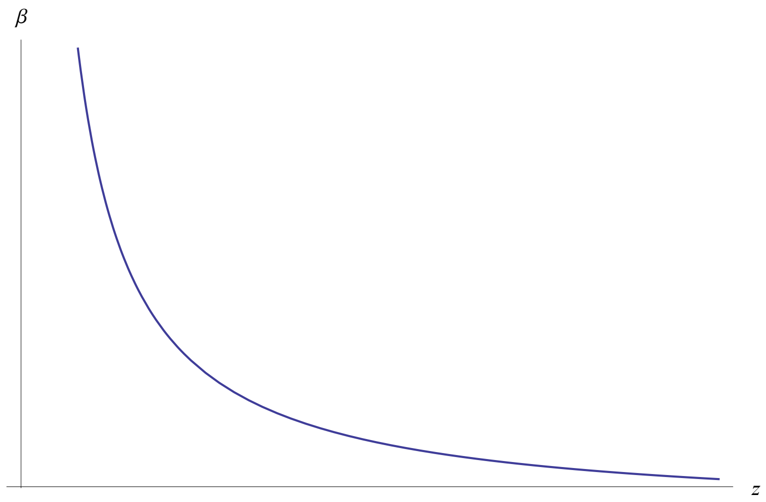

Figure 6: Relation between z and β with constant y

13 Black Hole Solution for the Chern-Simons flatness condition

In this section, we will consider new solutions for the Chern-Simons gauge connections A and $\bar{A}$. In particular, we will focus on connections for black holes with or without a spin-3 charge. For this purpose, we will consider a perturbation around the background (11.6) and (11.7). First, we will consider flat connections at $y = 0$, which are denoted as $\mathcal{A}_0$ and $\bar{\mathcal{A}}_0$

$$\begin{aligned} \mathcal{A}_0 = & \quad (\cosh\gamma dx - \sinh\gamma d\beta)t_3 - \frac{1}{2}d\gamma t_6 \\ & + \quad (\sinh\gamma dx - \cosh\gamma d\beta)t_7 + (\beta dx - d\alpha)t_8 \end{aligned} \tag{13.1}$$

$$\begin{aligned} \bar{\mathcal{A}}_0 = & \quad (-\cosh\gamma d\bar{x} + \sinh\gamma d\bar{\beta})t_1 + \frac{1}{2}d\gamma t_6 \\ & - \quad (\sinh\gamma d\bar{x} - \cosh\gamma d\bar{\beta})t_5 - (\bar{\beta}d\bar{x} - d\bar{\alpha})t_4. \end{aligned} \tag{13.2}$$

Gauge connections at an arbitrary y are obtained by carrying out the following gauge transformations

$$A_0 = b^{-1}(y)\mathcal{A}_0 b(y) + b^{-1}(y)db(y) \tag{13.3}$$
$$\bar{A}_0 = b(y)\bar{\mathcal{A}}_0 b^{-1}(y) + b(y)db^{-1}(y), \tag{13.4}$$

where $b(y) = y^{t_2}$. These connection (13.1) and (13.2) satisfy the flatness conditions

$$\mathcal{F}_0 = d\mathcal{A}_0 + \mathcal{A}_0 \wedge \mathcal{A}_0 = 0 \tag{13.5}$$
$$\bar{\mathcal{F}}_0 = d\bar{\mathcal{A}}_0 + \bar{\mathcal{A}}_0 \wedge \bar{\mathcal{A}}_0 = 0. \tag{13.6}$$

From now, we add small perturbations ψ and $\bar{\psi}$ to (13.1) and (13.2):

$$\mathcal{A} = \mathcal{A}_0 + \psi \tag{13.7}$$
$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_0 + \bar{\psi} \tag{13.8}$$

We require that these satisfy the flatness conditions (11.8) and (11.9). The transformation $b(y)$ is used to obtain connections A and $\bar{A}$.

When ψ and $\bar{\psi}$ are expanded as $\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \cdots$, we can obtain the flatness conditions to each orders of perturbation systematically. The i-th order perturbation $\psi^{(i)}$ should satisfy

$$d\psi^{(i)} + \mathcal{A}_0 \wedge \psi^{(i)} + \psi^{(i)} \wedge \mathcal{A}_0 + \sum_{k=1}^{i-1} \psi^{(k)} \wedge \psi^{(i-k)} = 0. \tag{13.9}$$

Let us solve the above equation for $i = 1$

$$d\psi^{(1)} + \mathcal{A}_0 \wedge \psi^{(1)} + \psi^{(1)} \wedge \mathcal{A}_0 = 0. \tag{13.10}$$

By expanding $\psi^{(1)}$ and $\mathcal{A}_0$ into a basis of $SL(3,\mathbb{R})$ generators as

$$\psi^{(1)} = \psi^{(1)a} t_a \tag{13.11}$$
$$\mathcal{A}_0 = \mathcal{A}_0^a t_a, \tag{13.12}$$

(13.10) is transformed into

$$d\psi^{(1)a} + f^a{}_{bc} \mathcal{A}_0^b \wedge \psi^{(1)c} = 0, \tag{13.13}$$

where $f^a{}_{bc}$ is the structure constant for $sl(3,\mathbb{R})$ algebra given in Appendix.C. Explicit forms of these equation are given by

$$\begin{aligned}
&d\psi^{(1)1} + 4(\sinh\gamma dx - \cosh\gamma d\beta) \wedge \psi^{(1)4} + d\gamma \wedge \psi^{(1)5} = 0\\
&d\psi^{(1)2} - 2(\cosh\gamma dx - \sinh\gamma d\beta) \wedge \psi^{(1)1} + 16(\beta dx - d\alpha) \wedge \psi^{(1)4} - 2(\sinh\gamma dx - \cosh\gamma d\beta) \wedge \psi^{(1)5} = 0\\
&d\psi^{(1)3} - (\cosh\gamma dx - \sinh\gamma d\beta) \wedge \psi^{(1)2} + 4(\beta dx - d\alpha) \wedge \psi^{(1)5} - 2(\sinh\gamma dx - \cosh\gamma d\beta) \wedge \psi^{(1)6} - d\gamma \wedge \psi^{(1)7} = 0\\
&d\psi^{(1)4} = 0\\
&d\psi^{(1)5} - 4(\cosh\gamma dx - \sinh\gamma d\beta) \wedge \psi^{(1)4} + d\gamma \wedge \psi^{(1)1} = 0\\
&d\psi^{(1)6} - 3(\sinh\gamma dx - \cosh\gamma d\beta) \wedge \psi^{(1)1} - 3(\cosh\gamma dx - \sinh\gamma d\beta) \wedge \psi^{(1)5} = 0\\
&d\psi^{(1)7} - (\sinh\gamma dx - \cosh\gamma d\beta) \wedge \psi^{(1)2} - 4(\beta dx - d\alpha) \wedge \psi^{(1)1} - 2(\cosh\gamma dx - \sinh\gamma d\beta) \wedge \psi^{(1)6} - d\gamma \wedge \psi^{(1)3} = 0\\
&d\psi^{(1)8} - 2(\beta dx - d\alpha) \wedge \psi^{(1)2} + (\sinh\gamma dx - \cosh\gamma d\beta) \wedge \psi^{(1)3} - (\cosh\gamma dx - \sinh\gamma d\beta) \wedge \psi^{(1)7} = 0
\end{aligned} \tag{13.14}$$

For $a = 4$, the equation is simple and solved by introducing an arbitrary function Q_1: $\psi^{(1)4} = -dQ_1$. Other equations are also solved by taking appropriate linear combinations of some equations and introducing seven arbitrary functions Q_i where $i = 2, 3, \cdots, 7$. Then, the most general solutions are given by

$$\begin{aligned}
\psi^{(1)1} &= (dQ_3 - 4Q_1 dx)\sinh\gamma + (dQ_2 + 4Q_1 d\beta)\cosh\gamma\\
\psi^{(1)2} &= -(16\beta Q_1 + 2Q_2)dx + 16Q_1 d\alpha - 2Q_3 d\beta + dQ_4\\
\psi^{(1)3} &= [dQ_6 + 4Q_3 d\alpha - 2Q_5 d\beta - (Q_4 + 4\beta Q_3)dx]\cosh\gamma - [dQ_7 - 4Q_2 d\alpha - Q_4 d\beta - (2Q_5 - 4\beta Q_2)]\sinh\gamma\\
\psi^{(1)4} &= -dQ_1\\
\psi^{(1)5} &= -(dQ_3 - 4Q_1 dx)\cosh\gamma - (dQ_2 + 4Q_1 d\beta)\sinh\gamma\\
\psi^{(1)6} &= -dQ_5 + 3Q_2 d\beta + 3Q_3 dx\\
\psi^{(1)3} &= [dQ_6 + 4Q_3 d\alpha - 2Q_5 d\beta - (Q_4 + 4\beta Q_3)dx]\sinh\gamma - [dQ_7 - 4Q_2 d\alpha - Q_4 d\beta - (2Q_5 - 4\beta Q_2)]\cosh\gamma\\
\psi^{(1)8} &= -dQ_8 - Q_6 d\beta + 2Q_4 d\alpha + (Q_7 - 2\beta Q_4)dx.
\end{aligned} \tag{13.15}$$

Similar calculation holds for the perturbation $\bar{\psi}^{(1)}$. Only we need is to replace variables, arbitrary functions and perturbations in (13.15) $(x, \alpha, \beta, Q_i, \psi^{(1)1}, \psi^{(1)2}, \psi^{(1)3}, \psi^{(1)4}, \psi^{(1)5}, \psi^{(1)6}, \psi^{(1)7}, \psi^{(1)8})$

by $(\bar{x}, \bar{\alpha}, \bar{\beta}, \bar{Q}_i, \bar{\psi}^{(1)3}, \bar{\psi}^{(1)2}, \bar{\psi}^{(1)1}, \bar{\psi}^{(1)8}, \bar{\psi}^{(1)7}, \bar{\psi}^{(1)6}, \bar{\psi}^{(1)5}, \bar{\psi}^{(1)4})$. Hence there are sixteen perturbative gauge modes Q_i and $\bar{Q}_i$. For static or stationary black hole solutions, these functions must be chosen such that $\psi^{(1)}$ and $\bar{\psi}^{(1)}$ are periodic in the direction of x and $\bar{x}$. Using results of first order perturbation, we can solve the equations for second order perturbation. By continuing these procedure, we can solve i-th order perturbation systematically.

13.1 Asymptotically AdS Black Hole Solution without Spin-3 Charge

The flat connections for a black hole without spin-3 charge are obtained by choosing suitable Q_i and $\bar{Q}_i$, which are determined to make $\psi^{(1)}$ and $\bar{\psi}^{(1)}$ are periodic in x and $\bar{x}$

$$\begin{aligned} Q_1 &= Q_3 = 0 \\ Q_2 &= -ax \\ Q_4 &= -ax^2 \\ Q_5 &= -3az\beta \\ Q_6 &= -\frac{1}{3}ax^3 - 3ax\beta^2 \\ Q_7 &= -ax^2\beta - 4ax\alpha \\ Q_8 &= \frac{1}{3}ax^3\beta - 2ax^2\alpha + ax\beta^3. \end{aligned} \tag{13.16}$$

$\bar{Q}_i$ is obtained by replacing $(Q_i, x, \alpha, \beta, a)$ into $(\bar{Q}_i, \bar{x}, \bar{\alpha}, \bar{\beta}, -\bar{a})$. Then $\psi^{(1)}$ and $\bar{\psi}^{(1)}$ are given by

$$\begin{aligned} \psi^{(1)} = \quad & [-a\cosh\gamma t_1 - a(3\beta^2\cosh\gamma - 4\alpha\sinh\gamma)t_3 \\ & +a\sinh\gamma t_5 + 3a\beta t_6 + a(4\alpha\cosh\gamma - 3\beta^2\sinh\gamma)t_7 - a\beta^3 t_8]dx \end{aligned} \tag{13.17}$$

$$\begin{aligned} \bar{\psi}^{(1)} = \quad & [\bar{a}\cosh\gamma t_3 + \bar{a}(3\bar{\beta}^2\cosh\gamma - 4\bar{\alpha}\sinh\gamma)t_1 \\ & -\bar{a}\sinh\gamma t_7 - 3\bar{a}\bar{\beta}t_6 - \bar{a}(4\bar{\alpha}\cosh\gamma - 3\bar{\beta}^2\sinh\gamma)t_5 + \bar{a}\bar{\beta}^3 t_4]d\bar{x} \end{aligned} \tag{13.18}$$

In the above equations, parameters a and $\bar{a}$ are the following constants

$$a = 2G(M+J), \ \bar{a} = 2G(M-J), \tag{13.19}$$

where M and J are mass and angular momentum, G is gravitational constant.

So far, we consider a first-order perturbation corresponding a stationary black hole. Generally, we have to consider the next order solution in order to obtain an exact solution. It is found that a solution $\mathcal{A}_0 + \psi^{(1)}$ satisfies the full order flat condition (13.5). This is because the first order perturbation has only x component. The second order equation contains a wedge product $\psi^{(1)} \wedge \psi^{(1)}$ but this term vanishes and we can take a solution $\psi^{(2)} = 0$.

The vielbein $e = (A - \bar{A})/2$ yields the metric

$$\begin{aligned} ds^2 = \quad & ds_0^2 + adx^2 + \bar{a}d\bar{x}^2 - 2a\beta d\gamma dx - 2\bar{a}\bar{\beta}d\gamma d\bar{x} - \frac{4}{y^4}(a\bar{\beta}\beta^3 + \bar{a}\beta\bar{\beta}^3)dxd\bar{x} \\ + \quad & \frac{a}{y^2}(3\beta^2\cosh 2\gamma - 4\alpha\sinh 2\gamma)dxd\bar{x} + \frac{\bar{a}}{y^2}(3\bar{\beta}^2\cosh 2\gamma - 4\bar{\alpha}\sinh 2\gamma)dxd\bar{x} \\ - \quad & a\bar{a}\left(y^2\cosh 2\gamma - 6\beta\bar{\beta} - \frac{4}{y^4}\beta^3\bar{\beta}^3 + \frac{9\beta^2\bar{\beta}^2}{y^2}\cosh 2\gamma + \frac{16\alpha\bar{\alpha}}{y^2}\cosh 2\gamma - \frac{12\beta^2\bar{\alpha}}{y^2}\sinh 2\gamma - \frac{12\bar{\beta}^2\alpha}{y^2}\sinh 2\gamma\right)dxd\bar{x} \\ - \quad & \frac{a}{y^2}(3\beta^2\sinh 2\gamma - 4\alpha\cosh 2\gamma)dxd\bar{\beta} - \frac{\bar{a}}{y^2}(3\bar{\beta}^2\sinh 2\gamma - 4\bar{\alpha}\cosh 2\gamma)d\bar{x}d\beta \\ + \quad & \frac{4a}{y^4}\beta^3 dxd\bar{\alpha} + \frac{4\bar{a}}{y^4}\bar{\beta}^3 d\bar{x}d\alpha, \end{aligned} \tag{13.20}$$

where ds_0^2 is the metric (10.28). It also turns out that this metric satisfies eight dimensional vacuum Einstein equation (10.29).

On the hypersurface Σ_C, the metric reduces to

$$\begin{aligned} ds^2 = & \frac{dy^2}{y^2} - \frac{1}{y^4}(y^2\cosh 2\gamma - 4\beta\bar{\beta})dxd\bar{x} + adx^2 + \bar{a}d\bar{x}^2 - \frac{4}{y^4}(a\bar{\beta}\beta^3 + \bar{a}\beta\bar{\beta}^3)dxd\bar{x} \\ + & \frac{a}{y^2}(3\beta^2\cosh 2\gamma - 4\alpha\sinh 2\gamma)dxd\bar{x} + \frac{\bar{a}}{y^2}(3\bar{\beta}^2\cosh 2\gamma - 4\bar{\alpha}\sinh 2\gamma)dxd\bar{x} \\ - & a\bar{a}\left(y^2\cosh 2\gamma - 6\beta\bar{\beta} - \frac{4}{y^4}\beta^3\bar{\beta}^3 + \frac{9\beta^2\bar{\beta}^2}{y^2}\cosh 2\gamma + \frac{16\alpha\bar{\alpha}}{y^2}\cosh 2\gamma - \frac{12\beta^2\bar{\alpha}}{y^2}\sinh 2\gamma - \frac{12\bar{\beta}^2\alpha}{y^2}\sinh 2\gamma\right)dxd\bar{x}. \end{aligned} \tag{13.21}$$

The induced metric (13.21) is also a black hole solution to the equation of motion of three dimensional $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ Chern-Simons theory. This black hole does not have a spin-3 charge. In particular, on a hypersurface Σ_0 with $\alpha=\bar{\alpha}=\beta=\bar{\beta}=\gamma=0$, the metric coincides with the one of BTZ black hole [23].

It also turns out that a spin-3 field $\varphi=\mathrm{tr}(e^3)/6$ for these flat connections $\mathcal{A}$ and $\bar{\mathcal{A}}$ satisfies the eight dimensional equation

$$\nabla_\mu\varphi_{\nu\rho\sigma}=0. \tag{13.22}$$

The complete result of the spin-3 field is very complex and it is expressed in Appendix.D. On the hypersurface Σ_0, it vanishes

$$\varphi|_{\Sigma_0}=0. \tag{13.23}$$

13.2 Hawking Temperature for Black Hole Solution without Spin-3 Charge

The Hawking temperature of this black hole (13.21) can be obtained by holonomy conditions [25]. We recall the procedure in Section.8. Let us consider the case of finite and non-vanishing a and $\bar{a}$. The flatness condition (11.8) solves for the more general form

$$\mathcal{A}=U^{-1}dU \tag{13.24}$$

where U is a matrix function and U^{-1} is its inversion. On the hypersurface Σ_C, this reduces to $\mathcal{A}_x=U^{-1}\partial_x U$ and U is given by $U=\exp(x\mathcal{A}_x)$. On the three dimensional asymptotically Euclidean AdS space, which is obtained by Wick rotating a time direction, the coordinate $z\equiv x$ and $\bar{z}\equiv\bar{x}$ are identified as $(z,\bar{z})\sim(z+2\pi\tau,\bar{z}+2\pi\bar{\tau})$, where τ and $\bar{\tau}$ are modular parameters of the boundary tori. A holonomy matrix w are defined as a Willson line

$$e^w=U(z,\bar{z})^{-1}U(z+2\pi\tau,\bar{z}+2\pi\bar{\tau}). \tag{13.25}$$

So, w is given by

$$w=2\pi\tau\mathcal{A}_x. \tag{13.26}$$

Similar discussion holds for anti-holomorphic part and a holonomy matrix $\bar{w}$ is defined by

$$\bar{w}=2\pi\bar{\tau}\bar{\mathcal{A}}_{\bar{x}}. \tag{13.27}$$

By requiring that the flat connections are non-singular, the matrix w and $\bar{w}$ should be required to have eigenvalues 0, $2\pi i$ or $-2\pi i$. So, we require that the holonomy matrices satisfy the following two conditions

$$\mathrm{det}w=0 \tag{13.28}$$

$$\mathrm{tr}w^2=-8\pi^2 \tag{13.29}$$

and similar equations for $\bar{w}$. The connections for the black hole without a spin-3 charge satisfy the first condition trivially. Substituting this solution into the second condition, we find

$$\tau = \frac{i}{2\sqrt{a}},\ \bar{\tau} = \frac{i}{2\sqrt{\bar{a}}}. \tag{13.30}$$

A periodicity of a time direction relates to the inverse of temperature T and we define the inverse right and left temperature as

$$\beta_R = -2\pi i\tau,\ \beta_L = -2\pi i\bar{\tau}, \tag{13.31}$$

where $\beta_R = 1/T_R$ and $\beta_L = 1/T_L$. Then we obtain

$$\beta_R = \frac{\pi}{\sqrt{a}} = \frac{\pi}{\sqrt{2G(M+J)}},\ \beta_L = \frac{\pi}{\sqrt{\bar{a}}} = \frac{\pi}{\sqrt{2G(M-J)}}. \tag{13.32}$$

13.3 Asymptotically Black Hole Solution with Spin-3 Charge

In this section, we will consider a black hole solution for (13.5) with the spin-3 charge. Although procedures are similar in the preceding subsection, Solving the equations is more difficult. In the preceding section, there are two parameters related to the mass and the angular momentum. But, in this case, we have to add some extra parameters related to spin-3 charges and its chemical potentials, which denotes $b, \bar{b}$ and $\mu, \bar{\mu}$. Furthermore, the integrability condition for the partition function is imposed as in Section.8.

First, let us solve the equation (13.5) for a first order perturbation $\psi^{(1)}$. When contribution from spin-3 charge is taken into account, it turns out that arbitrary functions Q_i takes form

$$\begin{aligned}
Q_1 &= bx \\
Q_2 &= Q_2^{BTZ} - 4bx\beta - 8b\alpha \\
Q_3 &= 2bx^2 \\
Q_4 &= Q_4^{BTZ} + 4bx^2\beta - 16bx\alpha \\
Q_5 &= Q_5^{BTZ} + 2bx^3 - 5bx\beta^2 - 24b\alpha\beta \\
Q_6 &= Q_6^{BTZ} - 8bx^2\alpha + 4bx^3\beta - 4bx\beta^3 - 24b\alpha\beta^2 \\
Q_7 &= Q_7^{BTZ} - 16bx\alpha\beta - 16b\alpha^2 + 2bx^2\beta^2 + bx^4 \\
Q_8 &= Q_8^{BTZ} + \mu x + 8bx^2\alpha\beta - 2bx^3\beta^2 - 16bx\alpha^2 + \frac{b}{5}x^5 + bx\beta^4 + 8b\alpha\beta^3
\end{aligned} \tag{13.33}$$

where Q_i^{BTZ} are given by (13.16). Substituting these into $\psi^{(1)}$ (13.14), we can find that α components appear in the connection $\mathcal{A}$ in addition to x components. In preceding subsection, flat connections up to first order perturbation is exact solutions for (13.5). In this case, however, these are not exact. the first order perturbation has α components and then $\psi^{(1)} \wedge \psi^{(1)}$ term in the second order equation does not vanish.

Then let us consider the flatness conditions for the second order perturbation. The differential equation is given by

$$d\psi^{(2)} + \mathcal{A}_0 \wedge \psi^{(2)} + \psi^{(2)} \wedge \mathcal{A}_0 + \psi^{(1)} \wedge \psi^{(1)} = 0. \tag{13.34}$$

Inhomogeneous solutions for this equation are given by

$$\begin{aligned}
\psi_I^{(2)3} &= -96ab\alpha^2\beta\cosh\gamma dx - 32b\mu\alpha\sinh\gamma dx \\
\psi_I^{(2)6} &= 48ab\alpha^2 dx \\
\psi_I^{(2)7} &= -96ab\alpha^2\beta\sinh\gamma - 32b\mu\alpha\cosh\gamma dx \\
\psi_I^{(2)8} &= -48ab\alpha^2\beta^2 dx.
\end{aligned}$$

There is an ambiguity associated with the solutions to the homogeneous parts for $\psi^{(2)}$. Making $\psi^{(2)}$ periodic in x and keeping the Fefferman-Graham gauge for the metric $\psi^{(2)2} = 0$, this ambiguity is fixed. We take into account these contributions and obtain the second order solution

$$\begin{aligned}
\psi^{(2)1} =& \quad \zeta_2 b\mu \sinh\gamma d\beta + \zeta_2 b\mu \cosh\gamma dx \\
\psi^{(2)2} =& \quad 0 \\
\psi^{(2)3} =& \quad \psi_I^{(2)3} + 2\zeta_1 a\mu\beta \cosh\gamma dx + (4\zeta_2 b\mu\beta d\alpha - 2\zeta_2 b\mu\beta^2 dx)\cosh\gamma - 4\zeta_2 b\mu\alpha \sinh\gamma dx \\
\psi^{(2)4} =& \quad 0 \\
\psi^{(2)5} =& \quad -\zeta_2 b\mu \sinh\gamma dx - \zeta_2 b\mu \cosh\gamma d\beta \\
\psi^{(2)6} =& \quad \psi_I^{(2)6} - \zeta_1 a\mu dx \\
\psi^{(2)7} =& \quad \psi_I^{(2)7} + 2\zeta_1 a\mu\beta \sinh\gamma dx - 4\zeta_2 b\mu\alpha \cosh\gamma dx + (4\zeta_2 b\mu\beta d\alpha - 2\zeta_2 b\mu\beta^2 dx)\cosh\gamma \\
\psi^{(2)8} =& \quad \psi_I^{(2)8} + \zeta_1 a\mu\beta^2 dx + 2\zeta_2 b\mu\beta^2 d\alpha - \frac{2}{3}\zeta_2 b\mu\beta^3 dx
\end{aligned} \tag{13.35}$$

where ζ_1 and ζ_2 are constants and are determined by the integrability condition that we will discuss in the later. By repeating this procedure order by order, we will obtain the solution for i-th order perturbation $\psi^{(i)}$ in principle. The solutions for the third perturbation are written in Appendix.E, which contain the extra constant parameter ζ_3. Of course we can add higher-order perturvations, which do not depend on α, β, such as $\zeta_4 a^2 b^2$ and $\zeta_5 a^2\mu^2$ and so on. These terms are restricted from the holnomy conditions, which will be discussed later, and we find $\zeta_i = 0$ except for ζ_1, ζ_2 and ζ_3. Hence, we will consider only non-zero constant and omit other terms in the following discussion.

Let us consider the holonomy up to this order. As in preceding subsection, a holonomy matrix w can be defined and imposed two constraints (13.28) and (13.29). Those constraints are given by

$$\begin{aligned}
&27b^2\zeta_2^2\mu^3\tau^3 + 9b\tau^3[3 + a(-6\zeta_2 + \zeta_1(8+\zeta_2)\mu^2)] + a^2\mu\tau^3(27 - 2a\zeta_1^3\mu^2 - 9[\zeta_1 + 8a\zeta_3\mu^2)] = (\text{4th order tems}) \\
&3 + 12a\tau^2 - 12b(-4+\zeta_2)\mu\tau^2 + 4a^2(\zeta_1^2 + 12\zeta_3)\mu^2\tau^2 = (\text{4th order tems}).
\end{aligned}$$

On the right hand side of the above equations, there are terms which are higher order in the expansion parameter and these depend on α and β. Surprisingly if we consider the next order solution, fourth-order terms are exactly canceled out by a new contribution from the next order and their terms do not change any terms in the left hand side. Hence, if we consider all order perturbation, the holonomy conditions may reduce to the following form ultimately

$$27b^2\zeta_2^2\mu^3\tau^3 + 9b\tau^3[3 + a(-6\zeta_2 + \zeta_1(8+\zeta_2)\mu^2)] + a^2\mu\tau^3(27 - 2a\zeta_1^3\mu^2 - 9[\zeta_1 + 8a\zeta_3\mu^2)] = 0 \tag{13.36}$$

$$3 + 12a\tau^2 - 12b(-4+\zeta_2)\mu\tau^2 + 4a^2(\zeta_1^2 + 12\zeta_3)\mu^2\tau^2 = 0. \tag{13.37}$$

In the following discussions, we assume that this observation is valid.

Then, we will solve holonomy conditions (13.36) and (13.37) with respect to a and b. (13.37) is solved as

$$b = \frac{1 + 4a\tau^2 + \frac{1}{12}(\zeta_1^2 + 12\zeta_3)\nu^2 a^2}{(\zeta_2 - 4)\nu\tau} \tag{13.38}$$

where we define $\nu = 4\mu\tau$. By substituting this into (13.36) and differentiating the result with respect to τ and ν. Then , we solve the differential equation for $(\partial a/\partial\tau)_\nu$ and $(\partial b/\partial\nu)_\tau$ and require that these two differential coefficients should satisfy

$$\left(\frac{\partial a}{\partial \nu}\right)_\tau = \left(\frac{\partial b}{\partial \tau}\right)_\nu. \tag{13.39}$$

This equation is the integrability condition for the partition function of the black hole [20]

$$Z = \mathrm{Tr} e^{\frac{i\pi}{2G}(a\tau + \frac{1}{4}b\nu)} e^{\frac{i\pi}{2G}(\bar{a}\bar{\tau} + \frac{1}{4}\bar{b}\bar{\nu})}. \tag{13.40}$$

The integrability condition tells us that non-zero parameter ζ_i are determined uniquely and given by

$$(\zeta_1, \zeta_2, \zeta_3) = (-2, -8, 1). \tag{13.41}$$

As we discussed, we can add some higher order terms which only depend on x, such as $\zeta_4 a^2 b^2$ and $\zeta_5 a^2 \mu^2$. The constraint (13.39) eliminates ζ_i for $i \geq 4$. Then, the right-mover temperature is given by $T_R = i/2\pi\tau$.

A similar analysis for the left mover can also be carried out. The holonomy conditions for bar parts require that a solution which guarantee the integrability condition should satisfy

$$(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3) = (-2, 8, 1). \tag{13.42}$$

The entropy of this black hole $S = S_L + S_R$ can be obtained by solving differential equation in [20]

$$\tau = \frac{i}{4\pi^2}\frac{\partial S_R}{\partial a}, \ \nu = \frac{i}{4\pi^2}\frac{\partial S_R}{\partial b}. \tag{13.43}$$

The right-moving pert S_R is given by

$$S_R = \frac{\pi}{2G}\sqrt{a} f\left(\frac{27b^2}{2a^3}\right) \tag{13.44}$$

where

$$f(x) = \cos\left[\arctan\left(\frac{\sqrt{x(2-x)}}{6(1-y)}\right)\right].$$

Surprisingly, the entropy and partition function do not depend on α and β.

Thanks to the miraculous cancellation, holonomy conditions (13.36) and (13.37) do not depend on the variables $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ and γ. This situation is similar to that in the black hole without spin-3 charge. So, let us study the holonomy conditions for the flat connections on the hypersurface Σ_0 with $\alpha = \beta = \bar{\alpha} = \bar{\beta} = \gamma = 0$. The flat connections are given by

$$\mathcal{A}|_{\Sigma_0} = \left[(-a + \zeta_2 b\mu)t_1 + t_3 - (b + \zeta_3 a^2 \mu)t_4 - \zeta_1 a\mu t_6 - \mu t_8\right] dx \tag{13.45}$$

$$\bar{\mathcal{A}}|_{\Sigma_0} = \left[(\bar{a} + \bar{\zeta}_2 \bar{b}\bar{\mu})t_3 + t_1 - (\bar{b} + \bar{\zeta}_3 \bar{a}^2 \bar{\mu})t_8 - \bar{\zeta}_1 \bar{a}\bar{\mu} t_6 - \bar{\mu} t_4\right] d\bar{x}. \tag{13.46}$$

These are solutions to the equations of motion in three dimensional Chern-Simons theory with the boundary condition $\mathcal{A}_{\bar{x}} = \bar{\mathcal{A}}_x = 0$. Since x component and $\bar{x}$ component are decomposed, the flatness conditions are trivially satisfied and there are no constraints for constants ζ_i. These are determined by requiring the integrability condition (13.39) and coincides with (13.41) and (13.42). The metric is obtained by

$$\begin{aligned} ds^2 = \ & \frac{dy^2}{y^2} + \left(a + 12b\mu + \frac{16}{3}a^2\mu^2\right) dx^2 + \left(\bar{a} + 12\bar{b}\bar{\mu} + \frac{16}{3}\bar{a}^2\bar{\mu}^2\right) d\bar{x}^2 \\ & - \frac{1}{3y^4}\left[3y^2 + 12\mu\bar{\mu} + 8a\bar{a}\mu\bar{\mu}y^4 + 12y^8(b + a^2\mu)(\bar{b} + \bar{a}^2\bar{\mu})\right] dx d\bar{x} \\ & - y^2(a + 8b\mu)(\bar{a} + 8\bar{b}\bar{\mu}) dx d\bar{x}. \end{aligned} \tag{13.47}$$

This coincides with the BTZ black hole when we take $b = \bar{b} = \mu = \bar{\mu} = 0$.

The most general black hole solution in the spin-3 gravity will be obtained by replacing all constant parameter a, b, μ into functions that depend on x. Similarly, constants with bar symbol are replaced by functions of $\bar{x}$. The flat connections are given by

$$\begin{aligned} \mathcal{A}|_{\Sigma_0} = \ & [(-a(x) + \zeta_2 b(x)\mu(x))t_1 + t_3 - (b(x) + \zeta_3 a(x)^2 \mu(x))t_4 \\ & -\zeta_1 a(x)\mu(x)t_6 - \mu(x)t_8]dx \end{aligned} \tag{13.48}$$

$$
\begin{aligned}
\bar{\mathcal{A}}|_{\Sigma_0} = \quad & [(\bar{a}(\bar{x}) + \bar{\zeta}_2\bar{b}(\bar{x})\bar{\mu}(\bar{x}))t_3 + t_1 - (\bar{b}(\bar{x}) + \bar{\zeta}_3\bar{a}(\bar{x})^2\bar{\mu}(\bar{x}))t_8 \\
& -\bar{\zeta}_1\bar{a}(\bar{x})\bar{\mu}(\bar{x})t_6 - \bar{\mu}(\bar{x})t_4]d\bar{x}. \qquad (13.49)
\end{aligned}
$$

It turns out that (13.48) and (13.49) still satisfy the flatness conditions. In fact, when we take $b = \bar{b} = \mu = \bar{\mu} = 0$, the metric coincides with the most general BTZ black hole in the Fefferman-Graham gauge [35].

Next we consider connections which do not satisfy the boundary condition $\mathcal{A}_{\bar{x}} = \bar{\mathcal{A}}_x = 0$, which are introduced in [21].

$$
\mathcal{A}' = (t_3 - at_1 - bt_4)dx + \mu(8bt_1 + a^2t_4 - 2at_6 + t_8)d\bar{x} \qquad (13.50)
$$

$$
\bar{\mathcal{A}}' = -(t_1 - \bar{a}t_3 - \bar{b}t_8)d\bar{x} + \bar{\mu}(8\bar{b}t_3 + \bar{a}^2t_8 - 2\bar{a}t_6 + t_4)dx. \qquad (13.51)
$$

To make the variation problem well-defined, the extra local terms

$$
S_{Extra} = \frac{k}{4\pi}\int_{y=\epsilon} d^2x\left[\frac{4}{y}A'^1_{\bar{x}} + \frac{16\mu}{y^2}A'^4_x\right] \qquad (13.52)
$$

must be added. The metric is given by

$$
\begin{aligned}
ds'^2 = \quad & \frac{dy^2}{y^2} + \frac{1}{3}\left[3a(1 + 8\bar{b}\bar{\mu}y^2) + 4\bar{a}^2\bar{\mu}(3by^4 + 4\bar{\mu})\right]dx^2 + \frac{1}{3}\left[3\bar{a}(1 + 8b\mu y^2) + 4a^2\mu(3\bar{b}y^4 + 4\mu)\right]d\bar{x}^2 \\
& -\left[\frac{1}{y^2} + \frac{4\mu\bar{\mu}}{y^4} + \frac{4}{3}(9b\mu + 9\bar{b}\bar{\mu} + 2a\bar{a}\mu\bar{\mu}) + 4y^4(b\bar{b} + a^2\bar{a}^2\mu\bar{\mu}) + y^2(a\bar{a} + 64b\bar{b}\mu\bar{\mu})\right]dxd\bar{x} \quad (13.53)
\end{aligned}
$$

and this obviously does not coincide with (13.47). It is checked that the spin-3 field is also different from that obtained from (13.45) and (13.46). Surprisingly the holonomy conditions are exactly coincide with (13.36) and (13.37). The gauge connections are not equivalent to ours. There are the following two reasons. First, the asymptotics of this metric is totally different from the one of ours. Second, if the gauge connections are equivalent, there is common matrix U to connect these

$$
\mathcal{A}' = U^{-1}\mathcal{A}U + U^{-1}dU \qquad (13.54)
$$

$$
\bar{\mathcal{A}}' = U\bar{\mathcal{A}}U^{-1} + UdU^{-1}. \qquad (13.55)
$$

It turns out that the matrix does not exist. Hence, although the partition function is exactly the same, two sets of flat connections define two distinct bulk geometries.

14 Action Integral for 3D Spin-3 Gravity Coupled to a Scalar Field

In this work, the formulation of the three-dimensional spin-3 gravity coupled to a scalar field in eight-dimensional bulk space is studied. There is a natural action for the scalar field written by (10.35). To make this formulation complete, we should consider an action for the gravity sector.

We introduced the eight dimensional vielbein e^a_μ and spin connection ω^a_μ in this work. The gauge connections could be defined by $A = \omega + e$ and $\bar{A} = \omega - e$. By solving the flatness conditions (11.8) and (11.9) for the connections, we found a new black hole solutions. If we restrict to hypersurface Σ_C with constant $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ and γ, the connections become solutions to equations of motion of $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory.

However, Chen-Simons formulations for even dimension does not exist and it is not possible to formulate the gravity sector by means of the Chern-Simons action. So, we should consider an action that reproduces the equations of motion for e and ω at the semiclassical level.

Let us consider flatness conditions (11.8) and (11.9) again. By taking an appropriate linear combination, the conditions yields the equations of motion for e and ω

$$de^a + f^a{}_{bc}\omega^b \wedge e^c = 0 \tag{14.1}$$

$$d\omega^a + \frac{1}{2}f^a{}_{bc}\omega^b \wedge \omega^c + \frac{1}{2}f^a{}_{bc}e^b \wedge e^c = 0. \tag{14.2}$$

Here, the structure constant f and its contraction is written in Appendix.C. The first condition is rewritten as a torsionless condition in terms of components

$$\nabla_\mu e^a_\nu + f^a{}_{bc}\omega^b_\mu e^c_\nu - (\mu \leftrightarrow \nu) = 0. \tag{14.3}$$

Recall the condition of metricity $\nabla_\mu g_{\nu\rho} = 0$. The full covariant derivative of the vielbein should vanish

$$D_\mu e^a_\nu = \nabla_\mu e^a_\nu + f^a{}_{bc}\omega^b_\mu e^c_\nu = 0. \tag{14.4}$$

Then, we adopt this condition as the equation of motion for the vielbein. We solve the equation with respect to ω^a_μ and the solution is given by

$$\tilde{\omega}^a_\mu = \frac{1}{12}f^a{}_{bc}e^b_\nu \nabla_\mu e^{c\nu}. \tag{14.5}$$

We define the curvature for the local $sl(3,\mathbb{R})$ transformation

$$R^a_{\mu\nu}(\omega) = \partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + f^a{}_{bc}\omega^b_\mu \omega^c_\nu \tag{14.6}$$

then the second equation (14.2) is rewritten as

$$R^a_{\mu\nu}(\tilde{\omega}) + f^a{}_{bc}e^b_\mu e^c_\nu = 0. \tag{14.7}$$

It can be shown that the following action yields this equation

$$S_{gravity} = \frac{1}{16\pi G}\int d^8x|e|\left(f_a{}^{bc}e^\mu_b e^\nu_c R^a_{\mu\nu}(\tilde{\omega}) + \frac{10}{3}\Lambda_8\right) \tag{14.8}$$

where we define $|e| = \det e^a_\mu$ and Λ_8 is a cosmological constant in eight dimension, which is equal to -36.

In the preceding section, an action for a scalar field has been obtained and it is written by (10.35). Then, We propose that the total action

$$S = S_{gravity} + S_{matter}. \tag{14.9}$$

describes the three-dimensional spin-3 gravity coupled to a scalar field. In the large central charge limit, the action for the spin-3 gravity determines the geometry of the eight-dimensional spacetime semi-classically. The action for the scalar S_{matter} describes the scalar field in this background.

15 Summary and Future Works

In this Thesis, the three-dimensional spin-3 gravity was studied from the point of view of the AdS/CFT correspondence. According to preceding works by other groups, the spin-3 gravity is dual to the W_3 extended conformal field theory. Also, this theory can be described as the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory. So far, this duality is guaranteed only at the level of the operator product expansion. In [21][20], the gauge transformations for the gauge connection A and $\bar{A}$ give the W_3 algebra.

It was unknown how the bulk field can be expressed as operators in W_3 extended conformal field theory and how the matter action can be written. We found answers to these questions. Our results are the following:

1) A bulk scalar state is reconstructed from a conformal family generated by the generator in the W_3 extended conformal field theory. Since the commutators $[W_n, W_m]$ give conformal transformations, we should enlarge the three-dimensional AdS spacetime to an eight-dimensional spacetime in order to construct W_3 invariant theory by analogy with superspace. The enlarged spacetime is defined by introducing five extra coordinate $(\alpha, \beta, \bar{\alpha}, \bar{\beta}, \gamma)$. On a hypersurface with vanishing extra coordinate, the usual three-dimensional AdS spacetime is realized. We performed the bulk reconstruction by imposing conditions (9.13) and (9.14), which is a natural extension of ones in [15]. The solution (9.58) is expressed in terms of the primary operator and its descendants in terms of L_{-1}, W_{-1} and W_{-2}.

2) There is a quadratic Casimir operator in the W_3 extended conformal field theory. The reconstructed state satisfies the eigenvalue equation for the Casimir operator with eigenvalue $(\Delta^2 - 8\Delta - 3q^2)/4$. We interpreted this eigenvalue as the mass of the bulk scalar. Also, we obtained the differential representations for generators. Substituting these representations into the quadratic Casimir operator, then the quadratic Casimir operator yields a second-order differential operator. This equation is interpreted as the Klein-Gordon equation for a free scalar with the mass $m^2 = (\Delta^2 - 8\Delta - 3q^2)/4$ in the enlarged spacetime. The action integral, which yields the Klein-Gordon equation, is just the action for the massive free scalar.

3) There is also a cubic Casimir operator. The eigenvalue equation for the Casimir operator yields the third-order differential equation. Thanks to the two differential equations, we read off the metric and the spin-3 field. These allowed us to determine the vielbein and the spin connection uniquely. We defined the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ gauge connections. These satisfies the flatness conditions in the enlarged spacetime (11.8) and (11.9). We adopted these conditions as the equations of motion for the vielbein and the spin connection. We also constructed the action for gravity sector (14.8), which yields the equations of motion.

4) Coordinates β and $\bar{\beta}$ play the role of a new renormalization flow parameter. The two-point function for the massive scalar fields is obtained by solving $SU(1,2) \times SU(1,2)$ invariant conditions (10.5) and (10.6). The solution violates conformal invariance since the extra coordinates are introduced. There are two points where the conformal invariance is recovered. One is the point where $\beta = \bar{\beta} = 0$. Another is the one where $(\beta, \bar{\beta})$ goes to infinity. At the former point, the metric coincides with the Poincaré AdS metric with a unit AdS radius. At the latter one, the metric coincides with that of the AdS with half a unit radius. This implies that a conformal vacuum changes when β and $\bar{\beta}$ go from zero to infinity. There are two inequivalent $SL(2,\mathbb{R})$ embeddings of $SU(1,2)$ namely W_3 vacuum and $W_3^{(2)}$ vacuum. The gauge connections of the former is expressed in terms of matrices (t_1, t_2, t_3) and corresponding metric is the Poincaré AdS metric with a unit radius, while the ones of the latter is expressed in terms of matrices (t_4, t_2, t_8) and the corresponding metric is the one with half a unit radius. Therefore, coordinate β plays the role of a new flow parameter. And when β goes from zero to infinity, the corresponding vacuum goes from W_3 to $W_3^{(2)}$. At the other points for other values of $(\beta, \bar{\beta})$, interpolating solution is realized. The conformal invariance is broken on the hypersurface. So, the dual theory living on the boundary is no longer the conformal field theory but a quantum field theory. The field operators is dressed by $g(\alpha, \beta) = e^{i\alpha W^h_{-2} + i\bar{\alpha}\bar{W}^h_{-2}} e^{\beta W^h_{-1} + \bar{\beta}\bar{W}^h_{-1}}$.

5) We obtained new black hole solutions by solving the flatness conditions. In the case of the black hole without the spin-3 charge, taking a hypersurface Σ_0, it coincides with the BTZ black hole. The Hawking temperature is obtained by solving holonomy condition and it coincides with that in [21]. However, in the case of the black hole with the spin-3 charge,

the situation is different. The gauge connection corresponding to the black hole with the spin-3 charge was also obtained by solving the flatness conditions perturbatively. In this Thesis, we carried out up to the third-order perturbation. The perturbative expansion continued as an infinite series. It was checked that higher-order terms did not affect physical quantities, such as the Hawking temperature and the entropy: two holonomy conditions whose solution gives the Hawking temperature were determined by perturbations up to the third-order perturbation. Then, the partition function for the black hole coincided with that in [21]. Although the partition functions are exactly the same, the geometry is totally different. Furthermore, our gauge connections satisfies the usual boundary condition $A_{\bar{x}} = \bar{A}_x = 0$.

There remain several problems to resolve.

First, we obtained the action integral on the enlarged spacetime. This will provide a starting point to investigate higher-spin gravity. In the case of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory, the corresponding action coincides with the Einstein-Hilbert action as we reviewed in Section8. In the spin-3 gravity case, we obtained the action integral (14.8) in vielbein formalism. It is still unknown whether the action can be rewritten in terms of the metric-like fields, such as the metric $g_{\mu\nu}$ and the spin-3 gauge field $\varphi_{\mu\nu\rho}$, because the number of degrees of freedom of the metric and the spin-3 field is much larger than that of the vielbein: the number of degrees of freedom of the vielbein is 56, while the one of the metric is 36 and the one of the spin-3 field is 120. Because $\varphi_{\mu\nu\rho}$ must satisfy the condition $\nabla_\mu \varphi_{\nu\rho\lambda} = 0$, however, components of $g_{\mu\nu}$ and $\varphi_{\mu\nu\rho}$ are not independent. It may be necessary to leave the vielbein in the action. This is not a drawback of the theory, because the vielbeins will be necessary when fermions are introduced.

Secondly, we obtained a new black hole solution with the spin-3 charge. Although the geometry is totally different from that in [21], the thermodynamical partition function is the same. So are the conformal field theories living on the boundaries the same?

Thirdly, the black hole geometry has less symmetry than AdS space intuitively in general. In three dimensions, however, the AdS spacetime and the BTZ black hole are locally equivalent. The BTZ black hole can be obtained by an appropriate coordinate transformation from the AdS spacetime. It is worth investigating the existence of the coordinate transformation from enlarged AdS to enlarged BTZ or black hole with the spin-3 charge in eight dimensions. If the transformation exists, it will be possible to construct the scalar state in the black hole spacetime and the bulk-boundary propagator for the black hole geometry.

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A Transformation of W_m under $g(\rho)$

Transformations of W_m under

$$g(\rho) = \exp\left[-\frac{1}{2}\rho(L_1 - L_{-1} + \bar{L}_1 - \bar{L}_{-1})\right] \tag{A.1}$$

are given by

$$\begin{aligned} g(\rho)W_2 g^{-1}(\rho) = \quad & \frac{1}{4}(\cosh^2 2rho + 2\cosh\rho + 1)W_2 - \sinh\rho(\cosh\rho + 1)W_1 \\ & + \frac{3}{2}\sinh^2 2\rho W_0 + \sinh\rho(1 - \cosh\rho)W_{-1} \\ & + \frac{1}{4}(\cosh^2 2\rho - 2\cosh\rho + 1)W_{-2} \end{aligned} \tag{A.2}$$

$$\begin{aligned} g(\rho)W_1 g^{-1}(\rho) = \quad & -\frac{1}{8}(\sinh 2\rho + 2\sinh\rho)W_2 + \frac{1}{2}(\cosh 2\rho + \cosh\rho)W_1 \\ & - \frac{3}{4}\sinh 2\rho W_0 + \frac{1}{2}(\cosh 2\rho - \cosh\rho)W_{-1} \\ & - \frac{1}{8}(\sinh 2\rho + 2\sinh\rho)W_{-2} \end{aligned} \tag{A.3}$$

$$\begin{aligned} g(\rho)W_0 g^{-1}(\rho) = \quad & \frac{1}{8}(\cosh 2\rho - 1)W_2 - \frac{1}{2}\sinh 2\rho W_1 + \frac{1}{4}(3\cosh 2\rho + 1)W_0 \\ & - \frac{1}{2}\sinh 2\rho W_{-1} + \frac{1}{8}(\cosh 2\rho - 1)W_{-2} \end{aligned} \tag{A.4}$$

$$\begin{aligned} g(\rho)W_{-1} g^{-1}(\rho) = \quad & -\frac{1}{8}(\sinh 2\rho + 2\sinh\rho)W_2 + \frac{1}{2}(\cosh 2\rho - \cosh\rho)W_1 \\ & - \frac{3}{4}\sinh 2\rho W_0 + \frac{1}{2}(\cosh 2\rho + \cosh\rho)W_{-1} \\ & - \frac{1}{8}(\sinh 2\rho + 2\sinh\rho)W_{-2} \end{aligned} \tag{A.5}$$

$$\begin{aligned} g(\rho)W_{-2} g^{-1}(\rho) = \quad & \frac{1}{4}(\cosh^2 2rho - 2\cosh\rho + 1)W_2 + \sinh\rho(-\cosh\rho + 1)W_1 \\ & + \frac{3}{2}\sinh^2 2\rho W_0 - \sinh\rho(1 + \cosh\rho)W_{-1} \\ & + \frac{1}{4}(\cosh^2 2\rho + 2\cosh\rho + 1)W_{-2}. \end{aligned} \tag{A.6}$$

Similar relations are obtained for anti-holomorphic part. Then, at the boundary $\rho \to \infty$ $W_m - (-1)^m \bar{W}_{-m}$ gives

$$\begin{aligned} g(\rho)(W_2 - \bar{W}_{-2})g^{-1}(\rho) \to \quad & \frac{1}{6}e^{2\rho}(W_2 - 4W_1 + 6W_0 - 4W_{-1} + W_{-2}) \\ & - \frac{1}{6}e^{2\rho}(\bar{W}_2 - 4\bar{W}_1 + 6\bar{W}_0 - 4\bar{W}_{-1} + \bar{W}_{-2}) \\ g(\rho)(W_1 + \bar{W}_{-1})g^{-1}(\rho) \to \quad & \frac{1}{16}e^{2\rho}(-W_2 + 4W_1 - 6W_0 + 4W_{-1} - W_{-2}) \\ & - \frac{1}{16}e^{2\rho}(\bar{W}_2 - 4\bar{W}_1 + 6\bar{W}_0 - 4\bar{W}_{-1} + \bar{W}_{-2}) \\ g(\rho)(W_0 - \bar{W}_{-0})g^{-1}(\rho) \to \quad & \frac{1}{16}e^{2\rho}(W_2 - 4W_1 + 6W_0 - 4W_{-1} + W_{-2}) \\ & - \frac{1}{16}e^{2\rho}(\bar{W}_2 - 4\bar{W}_1 + 6\bar{W}_0 - 4\bar{W}_{-1} + \bar{W}_{-2}) \end{aligned}$$

It is clear that the following two independent conditions on the boundary state are obtained

$$(W_2 - 4W_1 + 6W_0 - 4W_{-1} + W_{-2})\,|\psi\rangle_B = 0 \tag{A.7}$$
$$(\bar{W}_2 - 4\bar{W}_1 + 6\bar{W}_0 - 4\bar{W}_{-1} + \bar{W}_{-2})\,|\psi\rangle_B = 0. \tag{A.8}$$

B Solution to the Klein-Gordon Equation

We consider the bulk- boundary propagator, which is a solution to the Klein-Gordon equation (10.19). We assume that the solution $K_{\Delta,q}$ can be expressed as

$$K_{\Delta,q}(y,\gamma,X_1;X_2) = \sum_{n=0}^{\infty} y^{\Delta+2n} e^{-iq\gamma} f_n(\gamma,X_1;X_2) \tag{B.1}$$

near the boundary. Here, X_i denotes $(x_i,\bar{x}_i,\alpha_i,\bar{\alpha}_i,\beta,\bar{\beta}_i)$. f_0 is equal to the boundary two-point function of quasi-primary operators, which is given by (10.22). We can solve the equation perturbatively. First, we consider the sub-leading equation at the order of $y^{\Delta+2}$. By substituting the boundary two-point function, an equation for f_1 is obtained and it is readily solved

$$\begin{aligned} f_1 = \quad & (\Delta+3iq)\frac{(x_{12}+\beta_{12})(\bar{x}_{12}+\bar{\beta}_{12})}{2D_{12\bar{D}_{12}}} e^{-(iq+2)\gamma} f_0 \\ & +(\Delta-3iq)\frac{(x_{12}-\beta_{12})(\bar{x}_{12}-\bar{\beta}_{12})}{2D^*_{12\bar{D}^*_{12}}} e^{-(iq-2)\gamma} f_0. \end{aligned} \tag{B.2}$$

By repeating this procedure, we can obtain the function f_2 as

$$\begin{aligned} f_2 = \quad & \frac{(\Delta+3iq)(\Delta+3iq+4)}{8}\frac{(x_{12}+\beta_{12})^2(\bar{x}_{12}+\bar{\beta}_{12})^2}{(D_{12\bar{D}_{12}})^2} e^{-(iq+4)\gamma} f_0 \\ & +\frac{(\Delta-3iq)(\Delta-3iq+4)}{8}\frac{(x_{12}-\beta_{12})^2(\bar{x}_{12}-\bar{\beta}_{12})^2}{(D^*_{12\bar{D}^*_{12}})^2} e^{-(iq-4)\gamma} f_0 \\ & +\frac{\Delta^2-4\Delta+12iq+9q^2}{16D^*_{12}\bar{D}^*_{12}} e^{-iq\gamma} f_0 + \frac{\Delta^2-4\Delta-12iq+9q^2}{16D_{12}\bar{D}_{12}} e^{-iq\gamma} f_0 \\ & +\frac{\Delta^2+9q^2}{16}\left(\frac{1}{\bar{D}_{12}D^*_{12}} + \frac{1}{\bar{D}^*_{12}D_{12}}\right) e^{-iq\gamma} f_0. \end{aligned} \tag{B.3}$$

C $sl(3,\mathbb{C})$ Algebra

Generators of $sl(3,\mathbb{C})$ in the fundamental representation are given by [19]

$$\begin{aligned} & t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; t_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; t_3 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \\ & t_4 = \sqrt{-\sigma}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \; t_5 = \sqrt{-\sigma}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; t_6 = \sqrt{-\sigma}\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & t_7 = \sqrt{-\sigma}\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \; t_8 = \sqrt{-\sigma}\begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{C.1}$$

For $\sigma < 0$, the algebra, $t_4, t_5, \cdots, t_8$ are real and the algebra coincides with $sl(3, \mathbb{R})$. While for $\sigma > 0$, these generators are pure imaginary and the algebra coincides with $su(1, 2)$. The Killing metric can be defined by

$$h_{ab} = \frac{1}{2}\mathrm{tr}[t_a t_b]. \tag{C.2}$$

Its non-zero components are the follows

$$h_{22} = 1,\ h_{13} = -2,\ h_{48} = -8\sigma,\ h_{57} = 2\sigma,\ h_{66} = -\frac{4}{3}\sigma. \tag{C.3}$$

Indices of the local frame are lowered by the Killing metric h_{ab} and raised by its inversion h^{ab}. The structure constant f_{abc} are defined by

$$[t_a, t_b] = f_{ab}{}^{c} t_c. \tag{C.4}$$

It is completely symmetric and non-zero components are given by

$$f_{123} = -2,\ f_{158} = -8\sigma,\ f_{167} = 4\sigma,\ f_{248} = 16\sigma,\ f_{257} = -2\sigma,\ f_{347} = -8\sigma,\ f_{356} = 4\sigma. \tag{C.5}$$

The completely symmetric tensor d_{abc} is defined by

$$d_{abc} = \frac{1}{2}\mathrm{tr}[\{t_a, t_b\} t_c]. \tag{C.6}$$

Casimir operators are given by

$$C_2 = h^{ab} T_a T_b \tag{C.7}$$

$$C_3 = d^{abc} T_a T_b T_c \tag{C.8}$$

where T_a is some irreducible representation of $sl(3, \mathbb{C})$.

From the adjoint representation $(T^a)^b{}_c = \sigma f_b{}^a{}_c$, the following relations are obtained

$$f_a{}^{bc} f_{bcd} = \frac{12}{\sigma^2} h_{ad} \tag{C.9}$$

$$d^{cde} f_{caf} f_d{}^{fg} f_{egb} = 0. \tag{C.10}$$

D Spin-3 Field without Spin-3 Charge

Given the vielbein field e, the spin-3 field can be given by

$$\varphi = \frac{1}{3!}\mathrm{tr}(e^3). \tag{D.1}$$

The spin-3 field without mass parameters ϕ coincides with (11.4) up to a multiplicative constant. In the case of a black hole, owing to the existence of perturbations ψ and $\bar{\psi}$, the spin-3 field becomes a polynomial of the second degree for a and $\bar{a}$.

Complete result is

$$\varphi = 4\phi + \phi_P \tag{D.2}$$

where

$$
\begin{aligned}
\phi_P = & \quad \frac{6a\beta}{y^2}dy^2dx + \frac{12a^2\alpha}{y}dydx^2 - 3a^2dx^2d\alpha + \frac{3a}{y^4}(16a\alpha^2 - 2\beta^2 - a\beta^4)dx^2d\bar{\alpha} \\
+ & \quad \frac{3}{y^4}\left[-\bar{a}\bar{\beta}^2 + a^2\left\{-3\bar{a}y^4\beta^2 + \beta^4 - \bar{a}\beta^4\bar{\beta}^2 + 16\alpha^2(-1+\bar{a}\bar{\beta}^2)\right\} - a(-2\beta^2 + \bar{a}(y^4 + 2\beta^2\bar{\beta}^2))\right. \\
& \quad +ay^2\beta\left\{-1 + 3\bar{a}\bar{\beta}^2 + a(-\beta^2 + \bar{a}(y^4 + 16\alpha\bar{\alpha} + 3\beta^3\bar{\beta}^2))\right\}\cosh 2\gamma \\
& \quad \left.-4ay^2\beta(\bar{a}\bar{\alpha} + a(-\alpha + \bar{a}\bar{\alpha}\beta^2 + 3\bar{a}\alpha\bar{\beta}^2))\sinh 2\gamma\right] dx^2d\bar{x} \\
+ & \quad \frac{3a\beta}{y^2}(-4a\alpha\cosh 2\gamma + (1 + a\beta^2)\sinh 2\gamma)dx^2d\bar{\beta} + a(1 + 3a\beta^2)dx^2d\gamma + \frac{3\bar{a}\beta\beta^3}{y^4}d\beta^2d\bar{x} \\
- & \quad 2a\beta d\gamma^2dx - \frac{3a}{y}dydxd\beta - \frac{3a}{y^3}(3\beta^2\cosh 2\gamma - 4\alpha\sinh 2\gamma)dydxd\bar{\beta} \\
- & \quad \frac{3}{2y^3}\left[4(-\bar{a}\alpha + a(-\alpha + 3\bar{a}\bar{\alpha}\beta^2 + 3\bar{a}\alpha\bar{\beta}^2))\cosh 2\gamma\right. \\
& \quad \left.+(3\bar{a}\bar{\beta}^2 + a(3\beta^2 + \bar{a}(y^4 - 16\alpha\bar{\alpha} - 9\beta^2\bar{\beta}^2)))\sinh 2\gamma\right] dydxd\bar{x} \\
+ & \quad \frac{6a}{y^4}\left[4\beta\bar{\beta}(-1 + \bar{a}\bar{\beta}^2) - y^2(-1 + 3\bar{a}\bar{\beta}^2)\cosh 2\gamma + 4\bar{a}y^2\bar{\alpha}\sinh 2\gamma\right] dxd\bar{x}d\alpha \\
- & \quad \frac{3a}{y^4}\left[8\alpha\bar{\beta}(-1 + \bar{a}\bar{\beta}^2 + 4\bar{a}y^2\bar{\alpha}\beta)\cosh 2\gamma - y^2\beta(-1 + 3\bar{a}\bar{\beta}^2)\sinh 2\gamma\right] dxd\bar{x}d\beta \\
+ & \quad \frac{1}{2y^4}\left[-y^2\left\{-3\bar{a}\bar{\beta}^2 + a(-3\beta^2 + \bar{a}(y^4 + 16\alpha\bar{\alpha} + 9\beta^2\bar{\beta}^2))\right\}\cosh 2\gamma\right. \\
& \quad +4\beta\bar{\beta}(3a\bar{a}y^4 + 2a\beta^2 + 2\bar{a}\bar{\beta}^2 - 2a\bar{a}\beta^2\bar{\beta}^2) \\
& \quad \left.+4y^2(-a\alpha - \bar{a}\bar{\alpha} + 3a\bar{a}\bar{\alpha}\beta^2 + 3a\bar{a}\alpha\bar{\beta}^2)\sinh 2\gamma\right] dxd\bar{x}d\gamma \\
+ & \quad \frac{24a\beta}{y^4}dxd\alpha d\bar{\alpha} - \frac{6a^2}{y}\sinh 2\gamma dxd\alpha d\bar{\beta} - \frac{24}{y^4}a\alpha dxd\bar{\alpha}d\beta - \frac{8a\beta^3}{y^4}dxd\bar{\alpha}d\gamma \\
+ & \quad \frac{3a\beta}{y^2}\cosh 2\gamma dxd\beta d\bar{\beta} + \frac{a}{y^2}(4\alpha\cosh 2\gamma - 3\beta^2\sinh 2\gamma)dxd\bar{\beta}d\gamma \\
+ & \quad (x \leftrightarrow \bar{x}, \alpha \leftrightarrow \bar{\alpha}, \beta \leftrightarrow \bar{\beta}, a \leftrightarrow \bar{a}).
\end{aligned}
\tag{D.3}
$$

E Solution for Third Perturbation

Third-order perturbations for flatness conditions are given as follows

$$
\begin{aligned}
\psi^{(3)1} = \quad & \cosh\gamma(H_1 + dQ_2^{(3)}) - \sinh\gamma H_2 - 4\zeta_3 a^2\mu\beta\cosh\gamma dx - 8\zeta_3 a^2\mu\cosh\gamma d\alpha \\
\psi^{(3)2} = \quad & 0 \\
\psi^{(3)3} = \quad & \cosh\gamma H_3 + \sinh\gamma(-H_4 - 4\beta Q_2^{(3)}dx + 4Q_2^{(3)}d\alpha) \\
& -4\zeta_3 a^2\mu\beta^3\cosh dx - 24\zeta_3 a^2\mu\beta^2\cosh\gamma d\alpha \\
\psi^{(3)4} = \quad & \zeta_2 ab\mu\beta dx + 2\zeta_2 b^2\mu\beta^2 dx + 8\zeta_2 b^2\mu\beta d\alpha - \zeta_3\mu a^2 dx \\
\psi^{(3)5} = \quad & \cosh\gamma H_2 - \sinh\gamma(H_1 + dQ_2^{(3)}) + 4\zeta_3 a^2\mu\beta\sinh\gamma dx + 8\zeta_3 a^2\mu\sinh\gamma d\alpha \\
\psi^{(3)6} = \quad & -48b^2\mu(8+\zeta_2)\alpha^2 dx - 8ab\mu\zeta_2\beta^3 dx - 8b^2\mu\zeta_2\beta^4 dx \\
& -64b^2\mu\zeta_2\beta^3 d\alpha + 3Q_2^{(3)}d\beta + 6\zeta_3 a^2\mu\beta^2 dx + 24\zeta_3 a^2\mu\beta d\alpha \\
\psi^{(3)7} = \quad & \sinh\gamma H_3 + \cosh\gamma(-H_4 - 4\beta Q_2^{(3)}dx + 4Q_2^{(3)}d\alpha) \\
& -4\zeta_3 a^2\mu\beta^3\sinh\gamma dx - 24\zeta_3 a^2\mu\beta^2\sinh\gamma d\alpha \\
\psi^{(3)8} = \quad & 48b^2\mu(\zeta_2+8)\alpha^2\beta^2 dx + 256ab^2\alpha^4\beta dx - 32ab\mu\zeta_1\alpha^2\beta dx \\
& +\frac{2}{3}b^2\mu\zeta_2\beta^6 dx + 2b\mu\zeta_2\beta^2 dx + ab\mu\zeta_2\beta^5 dx + 8b\mu\zeta_2\beta^5 d\alpha - \zeta_3 a^2\mu\beta^4 dx - 8\zeta_3 a^2\mu\beta^3 d\alpha.
\end{aligned}
\tag{E.1}
$$

Here, one-form H_i and the function $Q_2^{(3)}$ are given by

$$
\begin{aligned}
H_1 = \quad & 5ab\mu\zeta_2\beta^2 dx + \frac{20}{3}b^2\mu\zeta_2\beta^3 dx + 40b^2\mu\zeta_2\beta^2 d\alpha \\
H_2 = \quad & 256ab^2\alpha^3 dx - 16ab\mu\zeta_1\alpha dx \\
H_3 = \quad & 96b^2\mu(\zeta_2+8)\alpha^2\beta dx + 256ab^2\alpha^4 dx - 32ab\mu\zeta_1\alpha^2 dx \\
& +4b^2\mu\zeta_2\beta^5 dx + 5ab\mu\zeta_2\beta^4 dx + 4b\mu^2\zeta_2\beta dx + 40b^2\mu\zeta_2\beta^4 d\alpha \\
H_4 = \quad & -256ab^2\alpha^3\beta^2 dx + 4ab\mu(4\zeta_1+\zeta_2)\alpha\beta^2 dx
\end{aligned}
\tag{E.2}
$$

$$
Q_2^{(3)} = \quad -256ab^2\alpha^3\beta + 4ab\mu(4\zeta_1+\zeta_2)\alpha\beta. \tag{E.3}
$$

$\bar{\phi}^{(3)a}$ is obtained by the following replacement $(x, \alpha, \beta, Q_i, \psi^{(1)1}, \psi^{(1)2}, \psi^{(1)3}, \psi^{(1)4}, \psi^{(1)5}, \psi^{(1)6}, \psi^{(1)7}, \psi^{(1)8})$ by $(\bar{x}, \bar{\alpha}, \bar{\beta}, \bar{Q}_i, \bar{\psi}^{(1)3}, \bar{\psi}^{(1)2}, \bar{\psi}^{(1)1}, \bar{\psi}^{(1)8}, \bar{\psi}^{(1)7}, \bar{\psi}^{(1)6}, \bar{\psi}^{(1)5}, \bar{\psi}^{(1)4})$ and $(a, b, \mu, \zeta_1, \zeta_2, \zeta_3)$ by $(-\bar{a}, \bar{b}, \bar{\mu}, \bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3)$.

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