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代表者：宮島　静雄・竹尾富貴子・井上　純治

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# 第 8 回関数空間セミナー報告集

## (Seminar on Function Spaces, 1999)

1999年１２月２３日（木）～１２月２５日（土）
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東京理科大学理学部　　：宮島　静雄
お茶の水女子大学理学部　：竹尾富貴子
北海道大学理学研究科　　：井上　純治

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# On powers of $p$-hyponormal and log-hyponormal operators

Masatoshi Ito, Takayuki Furuta, Masahiro Yanagida and Takeaki Yamazaki
Science University of Tokyo

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**Abstract**

This report is based on the following papers:

[FY1] T.Furuta and M.Yanagida, *On powers of p-hyponormal operators*, Scientiae Mathematicae **2** (1999), 279–284.

[FY2] T.Furuta and M.Yanagida, *On powers of p-hyponormal and log-hyponoral operators*, to appear in J. Inequal. Appl.

[I] M.Ito, *Generalizations of the results on powers of p-hyponormal operators*, to appear in J. Inequal. Appl.

[Ym] T.Yamazaki, *Extensions of the results on p-hyponormal and log-hyponormal operators by Aluthge and Wang*, SUT. J. Math. **35** (1999), 139–148.

A bounded linear operator $T$ on a Hilbert space $H$ is said to be $p$-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, and $T$ is said to be log-hyponormal if $T$ is invertible and $\log T^*T \geq \log TT^*$. We shall show several results on powers of $p$-hyponormal and log-hyponormal operators as extensions of the results by Aluthge and Wang [1][2].

## 1 Introduction

以下，作用素とはヒルベルト空間 $H$ 上の有界線形作用素を表すこととする。また，作用素 $T$ が正であるとは正定値，即ち $(Tx, x) \geq 0$ for all $x \in H$ と定義し，$T \geq 0$ と表すこととする。

正作用素の順序を保存する不等式である Löwner-Heinz の不等式 "$A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$" は，$\alpha \in [0,1]$ という強い制限のため応用する上で不便であった。そこで次の結果が確立された。

**Theorem F (Furuta inequality 1987 [8][9]).**

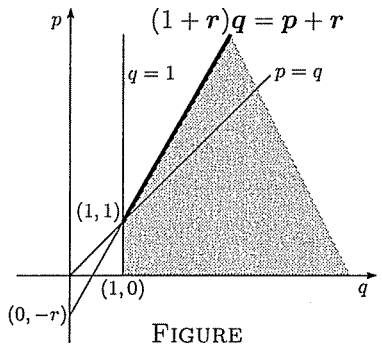
*If* $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

*and*

(ii) $\quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

*hold for* $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p + r$.




FIGURE

Theorem F の (i) または (ii) において $r = 0$ とおくことにより，Löwner-Heinz の不等式が導かれる。また，Theorem F の (i) と (ii) は互いに同値であることが知られている。Theorem F のパラメータ $p, q, r$ の領域を表したのが上図であるが，この領域は best possible であることが示されている [14]。Theorem F の別証明は [4][13] で与えられており，また [9] では 1 ページの証明が示されている。Theorem F の応用は，多くの研究者によって，次の多くの分野で得られている。以下，(A-5) に関連したいくつかの結果を紹介する。

(A) OPERATOR INEQUALITIES

(A-1) Several characterizations of operators satisfying $\log A \geq \log B$ and their applications

(A-2) Applications to the relative operator entropy

(A-3) Applications to Ando-Hiai log-majorization

(A-4) Generalized Aluthge transformation

**(A-5) Several classes associated with log-hyponormal and paranormal operators**

(A-6) Other order preserving operator inequalities

(A-7) Operator functions implying order preserving inequalities

(B) NORM INEQUALITIES

(B-1) Several generalizations of Heinz-Kato theorem

(B-2) Generalizations of some theorems on norms

(B-3) An extension of Kosaki trace inequality and parallel results

(C) OPERATOR EQUATIONS

(C-1) Generalizations of Pedersen-Takesaki theorem and related results

## 2 Results by Aluthge and Wang

正規（normal）作用素（$\overset{\text{def}}{\Longleftrightarrow} T^*T = TT^*$）を含む作用素のクラスとして，次が知られている。

**Definition.**

(i) $T$ : *hyponormal* $\overset{\text{def}}{\Longleftrightarrow} T^*T \geq TT^*$.

(ii) $T$ : *p-hyponormal* for a positive number $p > 0 \overset{\text{def}}{\Longleftrightarrow} (T^*T)^p \geq (TT^*)^p$.

(iii) $T$ : *log-hyponormal* $\overset{\text{def}}{\Longleftrightarrow} T$ is invertible and $\log T^*T \geq \log TT^*$.

(iv) $T$ : *paranormal* $\overset{\text{def}}{\Longleftrightarrow} \|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$.

これらのクラスに関しては今までに非常に多くの研究がされているが，その中で $p$-hyponormal に関しては [17]，log-hyponormal に関しては [15]，paranormal に関しては [3][7][12] を紹介しておく。

これらのクラスの包含関係について，次の結果が知られている。

**Proposition.**

(i) $T$ : *1-hyponormal* $\Longleftrightarrow T$ : *hyponormal.*

(ii) $T$ : *p-hyponormal for* $p > 0 \Longrightarrow T$ : *q-hyponormal for* $p \geq q > 0$.

(iii) $T$ : *invertible p-hyponormal for* $p > 0 \Longrightarrow T$ : *log-hyponormal.*

(iv) $T$ : *p-hyponormal for* $p > 0$ *or log-hyponormal* $\Longrightarrow T$ : *paranormal.*

更に，任意の可逆な正作用素 $X > 0$ について $\lim_{p \to +0} \frac{X^p - I}{p} = \log X$ が成り立つことから，log-hyponormal は $p$-hyponormal の $p \to +0$ の場合であり，その意味で 0-hyponormal と考えることができる。

これらのクラスの作用素のべき乗について考える。任意の自然数 $n$ について，$T$ が normal ならば $T^n$ も normal であることは容易に確かめられる。また，$T$ が paranormal ならば $T^n$ も paranormal であることが示されている [7]。しかし，その中間である hyponormal 作用素 $T$ について，$T^2$ は一般には hyponormal ではないことが知られており，[11, Problem 209] にそのような作用素 $T$ の例が示されている。この場合には Proposition (i)(iv) と前述の結果より $T$ : hyponormal $\Rightarrow T$ : paranormal $\Rightarrow T^2$ : paranormal ということが分かるのみであった。これに関して最近 Aluthge-Wang によって，Theorem F の応用として次の結果が示された。

**Theorem A ([2]).** *Let $T$ be a p-hyponormal operator for $p \in (0, 1]$. Then*

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$$

*hold for all positive integer $n$.*

**Corollary A ([2]).** *Let $T$ be a p-hyponormal operator for $p \in (0, 1]$. Then $T^n$ is $\frac{p}{n}$-hyponormal for all positive integer $n$.*

Corollary A から，任意の hyponormal 作用素 $T$ について $T^2$ は $\frac{1}{2}$-hyponormal であることが分かり，これは $T^2$ は paranormal であるという前述の結果と比べて精密な評価である（Proposition (iv) より）。

## 3 Extensions of the results

Yamazaki [Ym] は，Theorem A の拡張として，次の結果を得た。

**Theorem 1 ([Ym]).** *Let $T$ be a $p$-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n*}T^n)^{\frac{1}{n}} \geq \cdots \geq (T^{2*}T^2)^{\frac{1}{2}} \geq T^*T$$

*and*

$$TT^* \geq (T^2T^{2*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n*})^{\frac{1}{n}}$$

*hold for all positive integer $n$.*

また，log-hyponormal に関しても，同様な次の結果を得た。

**Theorem 2 ([Ym]).** *Let $T$ be a log-hyponormal operator. Then*

$$(T^{n*}T^n)^{\frac{1}{n}} \geq \cdots \geq (T^{2*}T^2)^{\frac{1}{2}} \geq T^*T$$

*and*

$$TT^* \geq (T^2T^{2*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n*})^{\frac{1}{n}}$$

*hold for all positive integer $n$.*

$\log t$ は作用素単調関数であることから，Theorem 2 の系として，次の結果が導かれる。

**Corollary 2 ([Ym]).** *Let $T$ be a log-hyponormal operator. Then*

$$\log(T^{n*}T^n)^{\frac{1}{n}} \geq \log T^*T \geq \log TT^* \geq \log(T^nT^{n*})^{\frac{1}{n}}$$

*hold for all positive integer $n$, i.e., $T^n$ is also log-hyponormal.*

Corollary 2 は Aluthge-Wang による次の結果の一般化になっている。

**Theorem B ([1]).** *Let $T$ be a log-hyponormal operator. Then $T^{2^n}$ is also log-hyponormal for all positive integer $n$.*

Furuta-Yanagida [FY2] は，Theorem 1 を精密化した結果として，次を示した。

**Theorem 3 ([FY2]).** *Let $T$ be a $p$-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n*}T^n)^{\frac{p+1}{n}} \geq \cdots \geq (T^{2*}T^2)^{\frac{p+1}{2}} \geq (T^*T)^{p+1}$$

*and*

$$(TT^*)^{p+1} \geq (T^2T^{2*})^{\frac{p+1}{2}} \geq \cdots \geq (T^nT^{n*})^{\frac{p+1}{n}}$$

*hold for all positive integer $n$.*

この $p$-hyponormal 作用素に関する Theorem 3 の $p=0$ の場合が，ちょうど log-hyponormal 作用素に関する Theorem 2 に対応していることに注意しておく。

ここで以上の結果の証明を紹介する替わりに，簡単のため，Theorem 3 の各不等式の両端を比較したものに当たる次の結果の証明を紹介する。

**Theorem 4 ([FY1]).** *Let $T$ be a $p$-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \text{ and } (TT^*)^{p+1} \geq (T^nT^{n*})^{\frac{p+1}{n}}$$

*hold for all positive integer $n$.*

*Proof of Theorem 4.* 1番目の不等式

$$(T^{n*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \tag{1}$$

を帰納法で証明する。2番目の不等式も同様に証明される。まず $n=1$ の場合は明らかである。次に $n=k$ で成り立つと仮定する。即ち

$$(T^{k*}T^k)^{\frac{p+1}{k}} \geq (T^*T)^{p+1}. \tag{2}$$

ここで各自然数 $n$ について $A_n = (T^{n*}T^n)^{\frac{p}{n}} = |T^n|^{\frac{2p}{n}}$ $B_n = (T^nT^{n*})^{\frac{p}{n}} = |T^{n*}|^{\frac{2p}{n}}$ とおくと

$$A_k = (T^{k*}T^k)^{\frac{p}{k}} \geq (T^*T)^p \geq (TT^*)^p = B_1 \tag{3}$$

が成り立つ。なぜならば，(3) の左の不等号は(2) に Löwner-Heinz の不等式を適用すること，右の不等号は $T$ が $p$-hyponormal であることにより成立するからである。$T$ の極分解を $T=U|T|$ とすると $T^*$ の極分解は $T^*=U^*|T^*|$ である。ここで $p_1=\frac{k}{p}\geq 1$, $r_1=\frac{1}{p}\geq 0$ とおいて，$A_k \geq B_1 \geq 0$ に Theorem F を適用することにより

$$\begin{aligned}
(T^{k+1*}T^{k+1})^{\frac{p+1}{k+1}} &= (U^*|T^*|T^{k*}T^k|T^*|U)^{\frac{p+1}{k+1}} \\
&= U^*(|T^*|T^{k*}T^k|T^*|)^{\frac{p+1}{k+1}}U \\
&= U^*(B_1^{\frac{r_1}{2}}A_k{}^{p_1}B_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}U \\
&\geq U^*B_1^{1+r_1}U \\
&= U^*|T^*|^{2(p+1)}U \\
&= |T|^{2(p+1)} \\
&= (T^*T)^{p+1},
\end{aligned}$$

よって $n=k+1$ で成立することが示された。以上から(1) はすべての自然数 $n$ について成立する。 □

今までに挙げた $p$-hyponormal に関する結果はすべて $p\in(0,1]$ の場合であったが，Itô [I] は，一般の $p>0$ の場合を考えることによって，Theorem 4, Theorem A の拡張である次の結果を示した。

**Theorem 5 ([I]).** *Let $T$ be a $p$-hyponormal operator for $p>0$. Then the following assertions hold:*

(1) $T^{n*}T^n \geq (T^*T)^n$ *and* $(TT^*)^n \geq T^nT^{n*}$ *hold for positive integer $n$ such that $n<p+1$,*

(2) $(T^{n*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1}$ *and* $(TT^*)^{p+1} \geq (T^nT^{n*})^{\frac{p+1}{n}}$ *hold for positive integer $n$ such that $n\geq p+1$.*

**Corollary 5 ([I]).** *Let $T$ be a $p$-hyponormal operator for $p>0$. Then the following assertions hold:*

(1) $T^{n*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n*}$ *hold for positive integer $n$ such that $n<p$,*

(2) $(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$ *hold for positive integer $n$ such that $n\geq p$,*

*i.e., $T^n$ is $\min\{1,\frac{p}{n}\}$-hyponormal for all positive integer $n$.*

更に，Theorem 3 の拡張である次の結果を示した。

**Theorem 6 ([I]).** *For some positive integer $m$, let $T$ be a $p$-hyponormal operator for $m-1<p\leq m$. Then*

$$(T^{n*}T^n)^{\frac{p+1}{n}} \geq \cdots \geq (T^{m+2*}T^{m+2})^{\frac{p+1}{m+2}} \geq (T^{m+1*}T^{m+1})^{\frac{p+1}{m+1}} \geq (T^*T)^{p+1}$$

*and*

$$(TT^*)^{p+1} \geq (T^{m+1}T^{m+1*})^{\frac{p+1}{m+1}} \geq (T^{m+2}T^{m+2*})^{\frac{p+1}{m+2}} \geq \cdots \geq (T^nT^{n*})^{\frac{p+1}{n}}$$

*hold for $n=m+1, m+2, \cdots$.*

## 4 Best possibilities of our results

最後に，前節の結果の best possibility について考察する。Tanahashi [14] は，Theorem F のパラメータの領域が best possible であることを表す次の結果を証明している。

**Theorem C ([14]).** *Let $p > 0$, $q > 0$ and $r > 0$. If $0 < q < 1$ or $(1+r)q < p+r$, there exist positive invertible operators $A$ and $B$ on $\mathbb{R}^2$ such that $A \geq B > 0$ and*

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}.$$

Theorem C を用いて，$p$-hyponormal に関する Theorem 5, Corollary 5 が best possible であることを表す次の Theorem 7, Theorem 8 が示される。

**Theorem 7 ([I]).** *Let $n$ be a positive integer such that $n \geq 2$, $p > 0$ and $\alpha > 1$.*

(1) *In case $n < p+1$, the following assertions hold:*
  (i) *There exists a $p$-hyponormal operator $T$ such that $(T^{n*}T^n)^\alpha \not\geq (T^*T)^{n\alpha}$.*
  (ii) *There exists a $p$-hyponormal operator $T$ such that $(TT^*)^{n\alpha} \not\geq (T^nT^{n*})^\alpha$.*

(2) *In case $n \geq p+1$, the following assertions hold:*
  (i) *There exists a $p$-hyponormal operator $T$ such that $(T^{n*}T^n)^{\frac{(p+1)\alpha}{n}} \not\geq (T^*T)^{(p+1)\alpha}$.*
  (ii) *There exists a $p$-hyponormal operator $T$ such that $(TT^*)^{(p+1)\alpha} \not\geq (T^nT^{n*})^{\frac{(p+1)\alpha}{n}}$.*

**Theorem 8 ([I]).** *Let $n$ be a positive integer such that $n \geq 2$, $p > 0$ and $\alpha > 1$.*

(1) *In case $n < p$, there exists a $p$-hyponormal operator $T$ such that $(T^{n*}T^n)^\alpha \not\geq (T^nT^{n*})^\alpha$.*
(2) *In case $n \geq p$, there exists a $p$-hyponormal operator $T$ such that $(T^{n*}T^n)^{\frac{p\alpha}{n}} \not\geq (T^nT^{n*})^{\frac{p\alpha}{n}}$.*

可逆な正作用素 $A, B > 0$ について $\log A \geq \log B$ で定義される順序を chaotic order と呼ぶ。Chaotic order は，$\log t$ が作用素単調関数であることから，通常の順序 $A \geq B$ よりも弱い順序である。この chaotic order の特徴付けとして，次の結果が知られている。

**Theorem D ([5][6][10][16]).** *For positive invertible operators $A$ and $B$, $\log A \geq \log B$ if and only if*

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

*holds for all $p \geq 0$ and $r \geq 0$.*

Theorem C の応用として，Theorem D が best possible であることを表す次の結果が示されている。

**Theorem E ([18]).** *Let $p > 0$ and $r > 0$. If $\alpha > 1$, there exist positive invertible operators $A$ and $B$ on $\mathbb{R}^2$ such that $\log A \geq \log B$ and*

$$A^{r\alpha} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r\alpha}{p+r}}.$$

Theorem E を用いて，log-hyponormal に関する Theorem 2, Corollary 2 が best possible であることを表す次の Theorem 9, Theorem 10 が示される。

**Theorem 9 ([FY2]).** *Let $n$ be a positive integer such that $n \geq 2$ and $\alpha > 1$. Then the following assertions hold:*

(i) *There exists a log-hyponormal operator $T$ such that $(T^{n*}T^n)^{\frac{\alpha}{n}} \not\geq (T^*T)^\alpha$.*
(ii) *There exists a log-hyponormal operator $T$ such that $(TT^*)^\alpha \not\geq (T^nT^{n*})^{\frac{\alpha}{n}}$.*

**Theorem 10 ([FY2]).** *Let $n$ be a positive integer and $\alpha > 0$. Then there exists a log-hyponormal operator $T$ such that $(T^{n*}T^n)^{\frac{\alpha}{n}} \not\geq (T^nT^{n*})^{\frac{\alpha}{n}}$.*

Theorem 7, Theorem 8, Theorem 9, Theorem 10 は，次の Lemma を用いて証明される。

**Lemma ([FY2]).** *For positive operators $A$ and $B$ on $H$, define the operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{\frac{1}{2}} & 0 & & & & \\ & & B^{\frac{1}{2}} & \boxed{0} & & & \\ & & & A^{\frac{1}{2}} & 0 & & \\ & & & & A^{\frac{1}{2}} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

*where $\boxed{\phantom{0}}$ shows the place of the $(0,0)$ matrix element. Then the following assertions hold:*

(i) *$T$ is $p$-hyponormal for $p > 0$ if and only if $A^p \geq B^p$.*

(ii) *$T$ is log-hyponormal if and only if $A$ and $B$ are invertible and $\log A \geq \log B$.*

*Furthermore, the following assertions hold for $\beta > 0$ and integers $n \geq 2$:*

(iii) $(T^{n*}T^n)^{\frac{\beta}{n}} \geq (T^*T)^\beta$ *if and only if* $(B^{\frac{k}{2}}A^{n-k}B^{\frac{k}{2}})^{\frac{\beta}{n}} \geq B^\beta$ *holds for* $k = 1, 2, \ldots, n-1$.

(iv) $(TT^*)^\beta \geq (T^nT^{n*})^{\frac{\beta}{n}}$ *if and only if* $A^\beta \geq (A^{\frac{k}{2}}B^{n-k}A^{\frac{k}{2}})^{\frac{\beta}{n}}$ *holds for* $k = 1, 2, \ldots, n-1$.

(v) $(T^{n*}T^n)^{\frac{\beta}{n}} \geq (T^nT^{n*})^{\frac{\beta}{n}}$ *if and only if* $A^\beta \geq B^\beta$ *holds and* $(B^{\frac{k}{2}}A^{n-k}B^{\frac{k}{2}})^{\frac{\beta}{n}} \geq B^\beta$ *and* $A^\beta \geq (A^{\frac{k}{2}}B^{n-k}A^{\frac{k}{2}})^{\frac{\beta}{n}}$ *hold for* $k = 1, 2, \ldots, n-1$.

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# Properties on several classes including log-hyponormal operators

Masatoshi Ito, Masahiro Yanagida and Takeaki Yamazaki
Faculty of Science, Science University of Tokyo

**Abstract**

This report is based on the following papers:

[I1] M.Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Math., **35** (1999), 149–165.

[I2] M.Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl., **2** (1999), 569–578.

[Y] T.Yamazaki, *On powers of class $A(k)$ operators including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl., **3** (2000), 97–104.

[YY1] T.Yamazaki and M.Yanagida, *A characterization of log-hyponormal operators via p-paranormality*, to appear in Scientiae Mathematicae.

[YY2] T.Yamazaki and M.Yanagida, *A further generalization of paranormal operators*, to appear in Scientiae Mathematicae.

Recently we introduced several new classes of operators including log-hyponormal operators in [9]. Here we shall discuss some properties and generalizations of these classes.

## 0 Introduction

ここではヒルベルト空間 $H$ 上の有界線形作用素 (以下、作用素と呼ぶ) について考える。作用素 $T$ が positive であるとは positive definite、即ち $(Tx, x) \geq 0$ for all $x \in H$ と定義し、$T \geq 0$ と表す。

Normal ($T^*T = TT^*$) を含む作用素の class として hyponormal ($T^*T \geq TT^*$) や、その拡張である $p$-hyponormal ($(T^*T)^p \geq (TT^*)^p$ for $p > 0$) や log-hyponormal ($T$ is invertible and $\log T^*T \geq \log TT^*$) がよく知られている。最近、この $p$-hyponormal や log-hyponormal と paranormal ($\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$) の関係に対する自然な解釈を与えるために、次の作用素の class が [9] で導入された。

**Definition ([9]).**
$T$ : *class* $A \overset{\text{def}}{\Longleftrightarrow} |T^2| \geq |T|^2$.
$T$ : *class* $A(k)$ *for* $k > 0 \overset{\text{def}}{\Longleftrightarrow} (T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$.
$T$ : *absolute-k-paranormal for* $k > 0 \overset{\text{def}}{\Longleftrightarrow} \||T|^kTx\| \geq \|Tx\|^{k+1}$ *for every unit vector* $x \in H$.

特に、class $A(1) \Longleftrightarrow$ class $A$, absolute-1-paranormal $\Longleftrightarrow$ paranormal である。これらの作用素の class の関係については次の結果が [9] で得られている。

**Theorem 0.A ([9]).**
(1) $T$ : *k-hyponormal for* $k > 0 \Longrightarrow T$ : *class* $A(k)$.
(2) $T$ : *log-hyponormal* $\Longrightarrow T$ : *class* $A(k)$ *for* $k > 0$.
(3) $T$ : *invertible and class* $A(k)$ *for* $k > 0 \Longrightarrow T$ : *class* $A(l)$ *for* $l \geq k$.
(4) $T$ : *absolute-k-paranormal for* $k > 0 \Longrightarrow T$ : *absolute-l-paranormal for* $l \geq k$.
(5) $T$ : *class* $A(k)$ *for* $k > 0 \Longrightarrow T$ : *absolute-k-paranormal.*

Theorem 0.A より、operator inequality で定義された class $A(k)$ と norm inequality で定義された absolute-$k$-paranormal が parallel な関係になっていることがわかる。

また、これらとは別に Aluthge-Wang は [3] で $w$-hyponormal ($|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $T = U|T|$ is the polar decomposition of $T$ and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$) を定義した。$w$-Hyponormal は $T$ の Aluthge 変換 $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ を使って定義されていることに注意しておく。$w$-Hyponormal と他の作用素の class との関係は次のようになっている [1]。

**Theorem 0.B ([1]).**
(1) $T$ : *p-hyponormal for* $p > 0 \Longrightarrow T$ : *w-hyponormal.*
(2) $T$ : *log-hyponormal* $\Longrightarrow T$ : *w-hyponormal.*
(3) $T$ : *w-hyponormal* $\Longrightarrow |T^2| \geq |T|^2$ *(i.e.,* $T$ : *class A) and* $|T^*|^2 \geq |T^{*^2}|$.

第 0 章で紹介した作用素の class の関係は Figure 1 のようになっている。

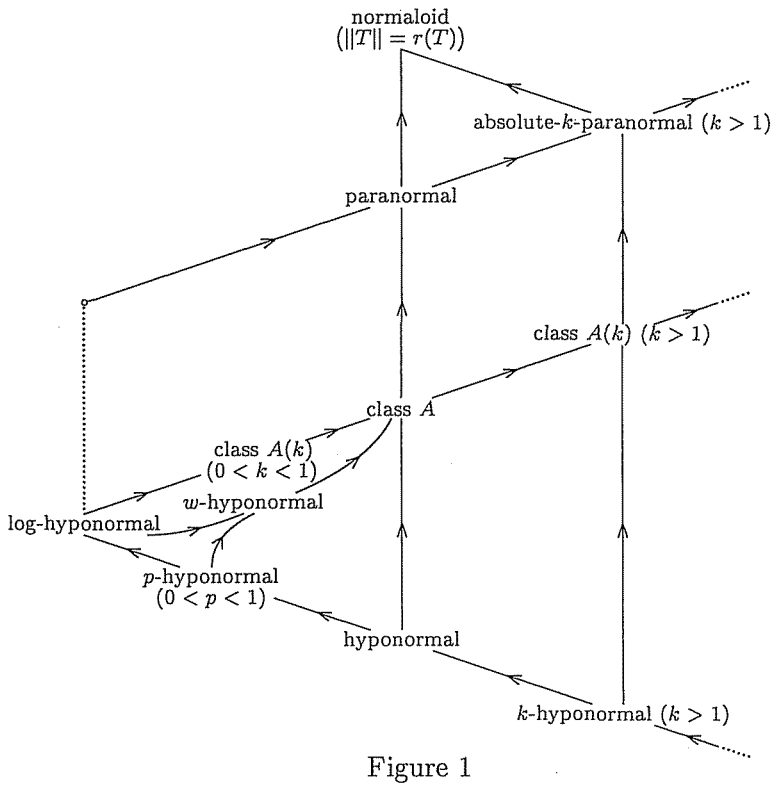


Figure 1

ここでは、これらの作用素の class に関連した 3 つの話題について述べる。

# 1 Some classes of operators associated with generalized Aluthge transformation

Class $A(k)$ と $w$-hyponormal は別々に導入されたものであるが、それらを統一的に理解するために次の作用素の class を定義する。

**Definition 1 ([I1]).** *For $s,t>0$,*
$T$ : *class* $wA(s,t) \overset{\text{def}}{\Longleftrightarrow} (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ *and* $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$.

Class $wA(s,t)$ の定義は $T$ の一般化 Aluthge 変換 $\tilde{T}_{s,t} = |T|^sU|T|^t$ を使って次のように書き換えることができる。

**Proposition 1.1 ([I1]).** *Let $T=U|T|$ be the polar decomposition of $T$. For $s,t>0$,*
$T$ : *class* $wA(s,t) \Longleftrightarrow |\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t}$ *and* $|T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2s}{s+t}}$.

これらのことより、class $wA(k,1) \Longrightarrow$ class $A(k)$, class $wA(\frac{1}{2},\frac{1}{2}) \Longleftrightarrow w$-hyponormal であることがわかる。また、この class $wA(s,t)$ に関して次の結果を得た。

**Theorem 1.2 ([I1]).**
(1) $T$ : *p-hyponormal for* $p>0 \Longrightarrow T$ : *class* $wA(s,t)$ *for* $s,t>0$.
(2) $T$ : *log-hyponormal* $\Longrightarrow T$ : *class* $wA(s,t)$ *for* $s,t>0$.
(3) $T$ : *class* $wA(s,t)$ *for* $s,t>0 \Longrightarrow T$ : *class* $wA(\alpha,\beta)$ *for* $\alpha \geq s$ *and* $\beta \geq t$.

Theorem 1.2 は class $A(k)$ に関する Theorem 0.A の (1)〜(3) と $w$-hyponormal に関する Theorem 0.B を含んだ結果である。また、Theorem 0.A の (3) については invertibility の仮定が必要であるが、Theorem 1.2 の (3) では invertibility は仮定されていないことに注意しておく。

## 2 Several properties on class $A$ and class $A(k)$

Paranormal の性質として、次の結果が知られている。

**Theorem 2.A ([7][8][10][11]).** *Let $T$ be a paranormal operator. Then the following assertions hold for every unit vector $x \in H$;*
(1) $\|T^nx\|^{\frac{1}{n}} \geq \|Tx\|$, *i.e.,* $T$ : *n-paranormal for all positive integer $n$.*
(2) $T^n$ : *paranormal for all positive integer $n$.*
(3) $T^{-1}$ : *paranormal (if $T$ is invertible).*
(4) $\|Tx\| \leq \|T^2x\|^{\frac{1}{2}} \leq \cdots \leq \|T^nx\|^{\frac{1}{n}}$ *hold for all positive integer $n$.*

この結果と対応した class $A$ の性質に関して次の結果を得た。

**Theorem 2.1 ([I2]).** *Let $T$ be an invertible and class $A$ operator. Then the following assertions hold;*
(1) $|T^n|^{\frac{2}{n}} \geq |T|^2$ *for all positive integer $n$.*
(2) $T^n$ : *class $A$ for all positive integer $n$.*
(3) $T^{-1}$ : *class $A$.*
(4) $|T|^2 \leq |T^2| \leq \cdots \leq |T^n|^{\frac{2}{n}}$ *hold for all positive integer $n$.*
(5) $|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^{\frac{2}{n}}$ *hold for all positive integer $n$.*

Class $A$ に対する Theorem 2.1 の (1)〜(4) と paranormal に対する Theorem 2.A の (1)〜(4) はそれぞれ parallel な結果となっていることに注意しておく。

また、[12] では log-hyponormal に対する次の結果が示されている。

**Theorem 2.B ([12]).** *Let $T$ be a log-hyponormal operator. Then the following inequalities hold for all positive integer $n$:*
(1) $T^*T \leq (T^{2^*}T^2)^{\frac{1}{2}} \leq \cdots \leq (T^{n^*}T^n)^{\frac{1}{n}}$.
(2) $TT^* \geq (T^2T^{2^*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n^*})^{\frac{1}{n}}$.

**Corollary 2.C ([12]).** *Let $T$ be a log-hyponormal operator. Then $T^n$ is also a log-hyponormal operator for all positive integer $n$.*

$|T| = (T^*T)^{\frac{1}{2}}$ なので、Theorem 2.1 の (4),(5) は Theorem 2.B の仮定を log-hyponormal から class $A$ にまで弱められることを示している。

また、class $A(k)$ のべき乗に関して次の結果を得た。

**Theorem 2.2 ([Y]).** *Let $T$ be an invertible and class $A(k)$ operator for $k \in (0,1]$. Then $T^n$ is a class $A(\frac{k}{n})$ operator for all positive integer $n$.*

**Corollary 2.3 ([Y]).** *Let $T$ be an invertible and class $A$ operator. Then $T^n$ is a class $A(\frac{1}{n})$ operator for all positive integer $n$.*

Invertible class $A(\frac{1}{n}) \Longrightarrow$ class $A$ なので、Corollary 2.3 は Theorem 2.1 の (2) より精密な結果となっている。

また、Aluthge-Wang [2] は "$T$ : $p$-hyponormal for $p \in (0,1] \Longrightarrow T^n$ : $\frac{p}{n}$-hyponormal for all positive integer $n$" であることを示した。この結果と、Corollary 2.C, Theorem 2.2 をまとめてみると次のようになる。

- $T$ : $p$-hyponormal for $p \in (0,1] \Longrightarrow T^n$ : $\frac{p}{n}$-hyponormal for all positive integer $n$.
- $T$ : log-hyponormal $\Longrightarrow T^n$ : log-hyponormal for all positive integer $n$.
- $T$ : invertible class $A(k)$ for $k \in (0,1] \Longrightarrow T^n$ : class $A(\frac{k}{n})$ for all positive integer $n$.

$T$ が $p$-hyponormal のとき $T^n$ は $T$ より広い class に含まれ、$T$ が class $A(k)$ のとき $T^n$ は $T$ より狭い class に含まれる。また、それぞれにおいて $T^n$ が元の class より log-hyponormal に近い class に含まれることがわかる。

ところで、最近 [6] で class $A(p,r)$ $((|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ for $p,r>0)$ が class $A(k)$ の拡張として定義された。また、class $AI(p,r) \overset{\text{def}}{\Longleftrightarrow}$ invertible class $A(p,r)$ である [6]。このとき class $A(k,1) \Longleftrightarrow$ class $A(k)$、class $AI(p,r) \Longleftrightarrow$ invertible class $wA(p,r)$ であることに注意しておく。この class $AI(p,r)$ のべき乗についても、次の結果を得た。

**Theorem 2.4 ([Y]).** *Let $T$ be a class $AI(s,t)$ operator for $s,t \in (0,1]$. Then $T^n$ is a class $AI(\frac{s}{n},\frac{t}{n})$ operator for all positive integer $n$.*

さらに、invertible $w$-hyponormal $\Longleftrightarrow$ class $AI(\frac{1}{2},\frac{1}{2})$ より、$w$-hyponormal のべき乗は次のようになることがわかる。

**Corollary 2.5 ([Y]).** *Let $T$ be an invertible and $w$-hyponormal operator. Then $T^n$ is a class $AI(\frac{1}{2n},\frac{1}{2n})$ operator for all positive integer $n$.*

Class $AI(p,r) \Longleftrightarrow$ invertible class $wA(p,r)$ と Theorem 1.2 の (3) より、Corollary 2.5 は Aluthge-Wang [4] によって得られた "$T$ : invertible $w$-hyponormal $\Longrightarrow T^2$ : $w$-hyponormal" を含んだ結果であることがわかる。

## 3 A further generalization of paranormal operators

第 0 章で class $A(k)$ と absolute-$k$-paranormal が parallel な関係になっていることを述べたが、この章では absolute-$k$-paranormal の拡張であり、また第 2 章で紹介した class $A(p,r)$ と parallel な関係にある作用素の class として absolute-$(p,r)$-paranormal を次のように定義し、それについての議論を行う。

**Definition 2 ([YY2]).** *For $p,r>0$,*
*$T$ : absolute-$(p,r)$-paranormal* $\overset{\text{def}}{\Longleftrightarrow} \left\| |T|^p|T^*|^r x \right\|^r \geq \left\| |T^*|^r x \right\|^{p+r}$ *for every unit vector $x \in H$.*

特に、absolute-$(k,1)$-paranormal $\Longleftrightarrow$ absolute-$k$-paranormal であり、absolute-$(p,p)$-paranormal は [5] で定義された $p$-paranormal ($\left\| |T|^p U|T|^p x \right\| \geq \left\| |T|^p x \right\|^2$ for every unit vector $x \in H$ where $T = U|T|$ is the polar decomposition of $T$) と一致することがわかる。また、absolute-$(p,r)$-paranormal に関する次の結果を得た。

**Theorem 3.1 ([YY2]).** *For each $p,r>0$,*
*$T$ : class $A(p,r) \Longrightarrow T$ : absolute-$(p,r)$-paranormal.*

**Theorem 3.2 ([YY2]).** *For each $p,r>0$,*
*$T$ : absolute-$(p,r)$-paranormal $\Longrightarrow T$ : absolute-$(s,t)$-paranormal for $s \geq p$ and $t \geq r$.*

また、[6] では次のような log-hyponormal の class $AI(p,r)$ を用いた characterization が示されている。

**Theorem 3.A ([6]).** *The following assertions are mutually equivalent:*
(1) *$T$ : log-hyponormal.*
(2) *$T$ : class $AI(p,p)$ for all $p>0$.*
(3) *$T$ : class $AI(p,r)$ for all $p,r>0$.*

この Theorem 3.A と parallel な結果として次の定理を得た。

**Theorem 3.3 ([YY1][YY2]).** *The following assertions are mutually equivalent:*
(1) *$T$ : log-hyponormal.*
(2) *$T$ : invertible and $p$-paranormal for all $p>0$.*
(3) *$T$ : invertible and absolute-$(p,r)$-paranormal for all $p,r>0$.*

Theorem 3.3 では、class $AI(p,r)$ より広い作用素の class である (invertible) absolute-$(p,r)$-paranormal を用いても、Theorem 3.A と同様な log-hyponormal の characterization が得られることを示している。

これらの結果より、class $A(p,r)$, absolute-$(p,r)$-paranormal は Figure 2 のような位置付けにあることがわかる。

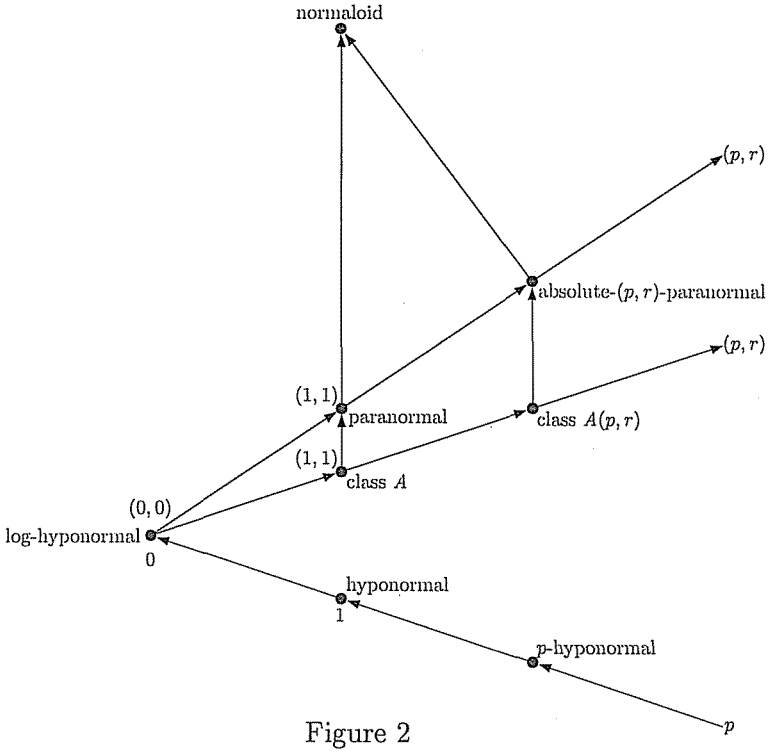


Figure 2

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# FURTHER EXTENSIONS OF CHARACTERIZATIONS OF CHAOTIC ORDER ASSOCIATED WITH KANTOROVICH TYPE INEQUALITIES

MASASHI HASHIMOTO, MASAHIRO YANAGIDA AND TAKEAKI YAMAZAKI
SCIENCE UNIVERSITY OF TOKYO

ABSTRACT. We showed characterizations of chaotic order via Kantorovich inequality in [23]. Recently as a nice application of generalized Furuta inequality, Furuta and Seo showed an extension of one of our results and a related result on operator equations in [15]. In this report, we shall show further extensions of the results by Furuta and Seo by using essentially the same idea as theirs.

## 1. INTRODUCTION

This report is based on the following papers:

M.Hashimoto and T.Yamazaki, *Further extensions of characterizations of chaotic order associated with Kantorovich type inequalities*, to appear in Scientiae Mathematicae.

M.Hashimoto and M.Yanagida, *Further characterizations of chaotic order associated with Kantorovich type inequalities via Furuta inequality*, preprint.

ヒルベルト空間 $H$ 上の有界線形作用素について考える。作用素 $T$ が positive であるとは $(Tx,x)\geq 0$ for all $x\in H$ と定義し、$T\geq 0$ と表す。また、$T$ が positive かつ invertible であるとき、$T$ は strictly positive であるといい、$T>0$ と表す。Positive operator の順序を保存する不等式として有名な Löwner-Heinz の定理: $A\geq B\geq 0$ *ensures* $A^\alpha\geq B^\alpha$ *for any* $\alpha\in[0,1]$ があるが、$\alpha>1$ の時は必ずしも成立しないので応用上不便であった。そこで応用上便利なように次の定理が確立された。

**Theorem F** ([8]).

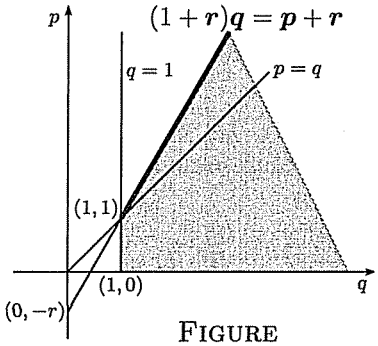
*If* $A\geq B\geq 0$, *then for each* $r\geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}}\geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

*and*

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}}\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

*hold for* $p\geq 0$ *and* $q\geq 1$ *with* $(1+r)q\geq p+r$.




FIGURE

Theorem F の (i) または (ii) において $r=0$ とおくことにより Löwner-Heinz の定理が得られる。Theorem F の別証明は [4][16] で与えられており、また [9] では one-page proof が示されている。Theorem F のパラメータ $p,q,r$ の範囲を示したのが上図であるが、この領域は best possible であることが [19] で示された。

Theorem F の拡張として [12] で次の Theorem G が確立された。

**Theorem G** ([12]). *If* $A\geq B\geq 0$ *with* $A>0$, *then for each* $t\in[0,1]$ *and* $p\geq 1$,

$$A^{1-t+r}=\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}A^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}\geq\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

*holds for any* $s\geq 1$ *and* $r\geq t$.

Ando-Hiai [2] では log majorization に関する主定理と同値なものとして、次の作用素不等式が示されている: *If* $A\geq B\geq 0$ *with* $A>0$, *then*

$$A^r\geq\{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

*holds for any* $p\geq 1$ *and* $r\geq 1$.

Theorem G は Ando-Hiai による上の不等式と Theorem F 自身を interpolate するものである。Theorem G の別証明は [7] で示され、[14] では (1.1) の one-page proof が示されている。作用素不等式(1.1) における

外側の指数 $\frac{1-t+r}{(p-t)s+r}$ が best possible であることが [20] で示され、[6],[22] で簡潔な別証明が与えられた。

また、前述したとおり Löwner-Heinz の定理に関連して次の命題も有名である: $A \geq B \geq 0$ *does not always assure* $A^\alpha \geq B^\alpha$ *for any* $\alpha > 1$. この応用上の不便さを解消する一つの方法として Theorem F とは別に [13] で次の定理が示された。

**Theorem A.1** ([13]). *If* $A \geq B \geq 0$ *and* $MI \geq A \geq mI > 0$, *then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m,M,p)A^p \geq B^p \text{ for } p \geq 1,$$

*where*

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - m^pM)^{p-1}}. \tag{1.2}$$

Theorem A.1 は Hölder-McCarthy inequality [17] と Kantorovich inequality: *If* $MI \geq A \geq mI > 0$, *then* $(A^2x,x) \leq \frac{(m+M)^2}{4mM}(Ax,x)^2$ *holds for every unit vector* $x$ *in* $H$ の両方に関連した結果であるということを注意しておきたい。定数 $\frac{(m+M)^2}{4mM}$ は Kantorovich 定数と呼ばれ、簡単な計算により $\frac{(m+M)^2}{4mM} = \left(\frac{m+M}{2}\right)^2/(\sqrt{mM})^2$ と変形できる。また、$K_+(m,M,2) = \frac{(m+M)^2}{4mM}$ という関係を満たしていることより $K_+(m,M,p)$ は Kantorovich 定数の一般化であると言える。

さて positive invertible operator $A$ と $B$ に対して $\log A \geq \log B$ で定義された順序を chaotic order と呼ぶ。これは $\log t$ が operator monotone function であることから、通常の順序 $A \geq B$ よりも弱い順序になっていることがわかる。Theorem F の応用として次のような chaotic order の特徴付けがよく知られている。

**Theorem A.2** ([5][11]). *Let* $A$ *and* $B$ *be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$

(ii) $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ *for all* $p \geq 0$.

(iii) $A^u \geq (A^{\frac{u}{2}}B^pA^{\frac{u}{2}})^{\frac{u}{p+u}}$ *for all* $p \geq 0$ *and* $u \geq 0$.

(i)⇔(ii) は [1] で示されている。最近、[21] で (i)⇒(iii) の Theorem F だけを用いた非常に簡潔な証明が示され、(ii)⇒(i) の簡単な証明が [15] で示された。

以前私たちは、Theorem A.1 と Theorem A.2 の応用として、以下のような chaotic order の特徴付けを示した。

**Theorem B.1** ([23]). *Let* $A$ *and* $B$ *be positive invertible operators satisfying* $MI \geq A \geq mI > 0$. *Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $\dfrac{(m^p + M^p)^2}{4m^pM^p}A^p \geq B^p$ *for all* $p \geq 0$.

**Theorem B.2** ([23]). *Let* $A$ *and* $B$ *be positive invertible operators satisfying* $MI \geq A \geq mI > 0$. *Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $M_h(p)A^p \geq B^p$ *for all* $p > 0$, *where* $h = \frac{M}{m} > 1$ *and*

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}}. \tag{1.3}$$

以下の 2 つの Lemman により、全ての $p > 0$ に対して $\frac{(m^p+M^p)^2}{4m^pM^p} \geq M_h(p)$ が成り立つので、chaotic order に対する十分条件として Theorem B.2 の方が Theorem B.1 よりも精密な結果になっていることがわかる。

**Lemma B.3** ([23]). *Let* $K_+(m,M,p)$ *be defined in* (1.2). *Then*

$$F(p,r,m,M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

*is an increasing function of* $p$, $r$, *and* $M$, *and also a decreasing function of* $m$ *for* $p > 0$, $r > 0$ *and* $M > m > 0$. *And the following inequality holds:*

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq 1. \tag{1.4}$$

**Lemma B.4** ([23]). *Let $M > m > 0$, $p > 0$ and $K_+(m, M, p)$ be defined in* (1.2). *Then*

$$\lim_{r \to +0} K_+\left(m^r, M^r, \frac{p+r}{r}\right) = M_h(p),$$

*where $h = \frac{M}{m} > 1$ and $M_h(p)$ is defined in* (1.3).

なお、$M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ は Specht's ratio [3][18] と呼ばれている。この定数は $M \geq x_i \geq m > 0$ ( $i = 1, 2, \cdots, n$ ) を満たすような実数 $x_i$ に対し、相加平均、相乗平均に関する次の不等式を満たす。

$$\frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

また、定数 $M_h(1)$ は上の不等式において最良であることも示されている。

Theorem G の応用として、Furuta-Seo により [15] で次の定理が確立された。

**Theorem C.1** ([15]). *Let $A$ and $B$ be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \in [0, 1]$, $p \geq 0$, $u \geq 0$ and $s \geq 1$ such that $(p + \alpha u)s \geq (1 - \alpha)u$, there exists the unique invertible positive contraction $T$ satisfying*

$$TA^{(p+\alpha u)s}T = (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s.$$

(iii) *For each $\alpha \in [0, 1]$, $p \geq u \geq 0$ and $s \geq 1$, there exists the unique invertible positive contraction $T$ satisfying*

$$TA^{(p+\alpha u)s}T = (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s.$$

(iv) *For each $p \geq 0$, there exists the unique invertible positive contraction $T$ satisfying*

$$TA^pT = B^p.$$

さらに [15] で Furuta-Seo は 上の Theorem C.1 の応用として次の結果も示している。これは Theorem B.1 を含んだ結果になっている。

**Theorem C.2** ([15]). *Let $A$ and $B$ be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \in [0, 1]$, $p \geq 0$ and $u \geq 0$,*

$$\frac{\left(M^{(p+\alpha u)s} + m^{(p+\alpha u)s}\right)^2}{4M^{(p+\alpha u)s} m^{(p+\alpha u)s}} A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

*holds for any $s \geq 1$ and $(p + \alpha u)s \geq (1 - \alpha)u$.*

(iii) *For each $\alpha \in [0, 1]$ and $p \geq u \geq 0$,*

$$\frac{\left(M^{(p+\alpha u)s} + m^{(p+\alpha u)s}\right)^2}{4M^{(p+\alpha u)s} m^{(p+\alpha u)s}} A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

*holds for any $s \geq 1$.*

(iv) $\frac{(M^p + m^p)^2}{4M^p m^p} A^p \geq B^p$ *holds for all $p \geq 0$.*

ここでは、Theorem C.1 のさらなる拡張を示す。そして Theorem C.2 の拡張として Theorem B.1 と Theorem B.2 を同時に interpolate する定理を示す。

## 2. Extensions of the results by Furuta and Seo

最初に、Theorem C.1 の拡張として、作用素方程式を用いた以下の chaotic order の特徴付けを紹介する。

**Theorem 1.** *Let $A$ and $B$ be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number $n$, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,r,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}.$$

(iii) *For each natural number $n$, $\alpha \in [0,1]$, $p \geq nu \geq 0$, $s \geq 1$ and real number $r$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,r,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}.$$

(iv) *For each natural number $n$ and $p \geq 0$, there exists the unique invertible positive contraction $T = T(n,p)$ satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

次の Corollary 2 は Theorem 1 の (ii),(iii) において、$r = \frac{(p+\alpha u)s}{nu}$ と置くことによって得られる。

**Corollary 2.** *Let $A$ and $B$ be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number $n$, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$ and $s \geq 1$ such that $(p+\alpha u)s \geq n(1-\alpha)u$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iii) *For each natural number $n$, $\alpha \in [0,1]$, $p \geq nu \geq 0$ and $s \geq 1$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,u,s)$ satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iv) *For each natural number $n$ and $p \geq 0$, there exists the unique invertible positive contraction $T = T(n,p)$ satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

**Remark 1.** Corollary 2 の (ii) [(iii),(iv) も同様] において、$n = 1$ と置く事により Theorem C.1 の (ii) [(iii),(iv) も同様] を導く事ができる。つまり Theorem 1 は Theorem C.1 を特別な場合として含んでいる事がわかる。

次に、Theorem C.2 の拡張として、Kantorovich 型の作用素不等式を用いた、以下の chaotic order の特徴付けを紹介する。

**Theorem 3.** *Let $A$ and $B$ be positive invertible operators satisfying $MI \geq A \geq mI > 0$ and $K_+(m,M,p)$ be defined in (1.2). Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number $n$, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right)A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$

*holds for all $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$.*

(iii) *For each natural number $n$, $\alpha \in [0,1]$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right)A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$

*holds for all $s \geq 1$ and real numbers $r$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$.*

(iv) *For each natural number $n$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{p+ru}{n+1}}, M^{\frac{p+ru}{n+1}}, n+1\right)A^p \geq B^p$$

*holds for real numbers $r$ such that $nru \geq p$.*

**Remark 2.** Theorem 3 の (ii) [(iii) も同様] において $n = 1$, $r = \frac{(p+\alpha u)s}{u}$ とおくと Theorem C.2 の (ii) [(iii) も同様] を得る。また Theorem 3 の (iv) において $n = 1$, $r = \frac{p}{u}$ とおくと Theorem C.2 の (iv) を得る。

ここまで見てきたように、Theorem 3 は Theorem C.2 を含み、Theorem B.1 の拡張にもなっている。さらに、Theorem 3 の (ii),(iii) において、$r = \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha$ と置くことによって次の Theorem 4 を導く事ができる。これは Theorem B.1 よりも精密な結果である Theorem B.2 の拡張になっている。つまり、Theorem 3 が Theorem B.1 と Theorem B.2 を interpolate する結果であるといえる。

**Theorem 4.** *Let $A$ and $B$ be positive invertible operators satisfying $MI \geq A \geq mI > 0$, and $K_+(m, M, p)$ and $M_h(p)$ be defined in (1.2) and (1.3), respectivery. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number $n$, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

*holds for all $s \geq 1$ such that $(p+\alpha u)s \geq (n+\alpha)u$.*

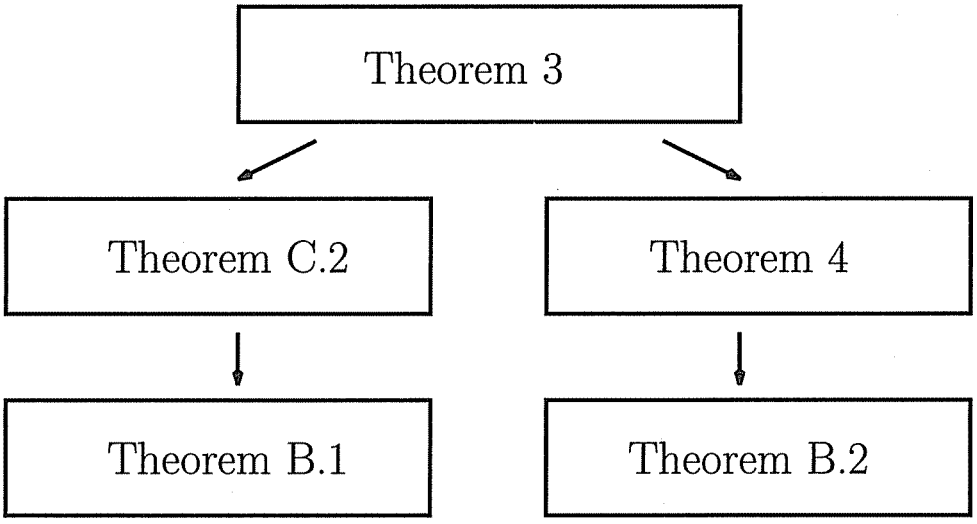
(iii) *For each natural number $n$, $\alpha \in [0,1]$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

*holds for all $s \geq 1$.*

(iv) *$M_h(p)\, A^p \geq B^p$ holds for all $p \geq 0$, where $h = \frac{M}{m} > 1$.*

ここまでの定理の関係を図にまとめると次のようになる。




さて、ここまでは Theorem G を応用し Theorem 1 を示し、それを基に Theorem 3 を示してきた。実は、Theorem 1 に相当する定理を、Theorem F を応用してつくり、それを基に Theorem 3 に相当するものを示すと、定理の形が簡潔になり、パラメータの領域も広げられる事がわかった。それらが以下の Theorem 5, Theorem 6 である。

**Theorem 5.** *Let $A$ and $B$ be positive and invertible operators, and $n$ be a natural number. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \geq 0$, $p \geq 0$, $s \geq 0$ and $r \geq \max\left\{0, \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha\right\}$, there exists the unique invertible positive contraction $T = T(n, \alpha, p, s, r)$ satisfying*

$$T(A^{\frac{(p+\alpha)s+r}{n+1}} T)^n = A^{\frac{-(p+\alpha)s+nr}{2(n+1)}} \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s A^{\frac{-(p+\alpha)s+nr}{2(n+1)}}.$$

(iii) *For each $\alpha \geq 0$, $p \geq 0$ and $s \geq 0$, there exists the unique invertible positive contraction $T = T(n, \alpha, p, s)$ satisfying*

$$T(A^{\frac{(p+\alpha)s}{n}} T)^n = \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s.$$

(iv) *For each $p \geq 0$, there exists the unique invertible positive contraction $T = T(n, p)$ satisfying*

$$T(A^{\frac{p}{n}} T)^n = B^p.$$

**Theorem 6.** *Let $A$ and $B$ be positive and invertible operators satisfying $MI \geq A \geq mI > 0$, and let $K_+(m, M, p)$ and $M_h(p)$ be defined in* (1.2) *and* (1.3), *respectively. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number $n$, $\alpha \geq 0$ and $p \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha)s+r}{n+1}}, M^{\frac{(p+\alpha)s+r}{n+1}}, n+1\right) A^{(p+\alpha)s} \geq \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s$$

*holds for $s \geq 0$ and $r \geq \max\left\{0, \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha\right\}$.*

(iii) *For each natural number $n$, $\alpha \geq 0$ and $p \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha)s-\alpha}{n}}, M^{\frac{(p+\alpha)s-\alpha}{n}}, n+1\right) A^{(p+\alpha)s} \geq \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s$$

*holds for $s \geq 0$ such that $(p+\alpha)s \geq (n+1)\alpha$.*

(iv) *For each natural number $n$ and $p \geq 0$,*

$$K_+\left(m^{\frac{p}{n}}, M^{\frac{p}{n}}, n+1\right) A^p \geq B^p$$

*holds.*

(v) $\dfrac{(m^p + M^p)^2}{4m^p M^p} A^p \geq B^p$ *holds for all $p \geq 0$.*

(vi) $M_h(p) A^p \geq B^p$ *holds for all $p \geq 0$, where $h = \frac{M}{m} > 1$.*

実際、Theorem 1 と Theorem 3 の (ii),(iii) では、$s \geq 1$ であったが、Theorem 5 と Theorem 6 の (ii),(iii) では $s \geq 0$ と広がっていることがわかる。

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# THE BEST POSSIBILITY FOR THE GRAND FURUTA INEQUALITY

MASATOSHI FUJII

ABSTRACT. In this note, we give a short proof to the best possibility for the grand Furuta inequality: For given $p,\ s \geq 1,\ t \in [0,1],\ r \geq t$ and $\alpha > 1$, there exist positive invertible operators $S$ and $T$ such that $S \geq T$ and

$$S^{(1-t+r)\alpha} \not\geq [S^{\frac{r}{2}}(S^{-\frac{t}{2}}T^pS^{-\frac{t}{2}})^sS^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}\alpha}.$$

**1. Introduction.** Throughout this note, an operator $T$ means a bounded linear operator acting on a Hilbert space $H$. An operator $A$ is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$, and we denote $A > 0$ if $A \geq 0$ is invertible.

One of the most important inequalities is the Löwner-Heinz inequality:

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha \quad \text{for} \quad \alpha \in [0,1]. \tag{1}$$

Furthermore it is known that $[0,1]$ is the best possible for (1). That is, for $\alpha > 1$ there exist $A,\ B > 0$ such that

$$A \geq B \geq 0 \quad \text{and} \quad A^\alpha \not\geq B^\alpha. \tag{2}$$

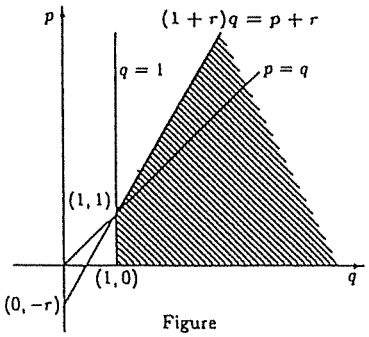
In 1987, Furuta established the following historical extension of (1), which is called the Furuta inequality now:

**Furuta inequality.** [11] *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{3}$$

*holds for all $p \geq 0$ and $q \geq 1$ such that*

$$(1+r)q \geq p+r. \tag{*}$$



Figure

The condition (*) is expressed as in the right.
See [12] for a one-page proof and also [4, 21]. Recently the best possibility of the Furuta inequality was discussed by Tanahashi [22]. He proved that the condition (*) is complete. More precisely,

**Theorem A.** *Let $p > 0$ and $r \geq 0$ be given. If either $0 < q < 1$ or $(1+r)q < p+r$, then there exist $A$ and $B$ such that $A \geq B > 0$ and*

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}.$$

In the case of $p \geq 1$, Theorem A is rephrased as follows:

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**Theorem A'.** *Let $p \geq 1$ and $r \geq 0$ be given. For $\alpha > 1$, there exist $A$ and $B$ such that $A \geq B > 0$ and*

$$A^{(1+r)\alpha} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}\alpha}.$$

**2. Grand Furuta inequality.** In 1995, Furuta [14] extended his inequality to an interpolational form combining with the Ando-Hiai inequality [2], which is called the grand Furuta inequality in [10]:

**Grand Furuta inequality.** *If $S \geq T \geq 0$ and $S > 0$, then for each $t \in [0,1]$*

$$S^{1-t+r} \geq [S^{\frac{r}{2}}(S^{-\frac{t}{2}}T^pS^{-\frac{t}{2}})^sS^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \tag{4}$$

*holds for all $p$, $s \geq 1$ and $r \geq t$.*

It was given a mean theoretic proof in [10] and very recently an elementary one-page proof in [15]. See also [16, 17, 18, 19, 20]. Now Tanahashi [23] considered the best possibility for the grand Furuta inequality:

**Theorem B.** *Let $p$, $s \geq 1$, $t \in [0,1]$, $r \geq t$. Then for each $\alpha > 1$ there exist $S$, $T > 0$ such that $S \geq T$ and*

$$S^{(1-t+r)\alpha} \not\geq [S^{\frac{r}{2}}(S^{-\frac{t}{2}}T^pS^{-\frac{t}{2}})^sS^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}\alpha}. \tag{5}$$

His discussion is analogous to Theorem A by himself and so more complicated. Very recently Yamazaki [24] presents a simplified proof to Theorem B, which is based on the Furuta inequality, Theorem A and Yanagida's recent result [26] on the best possibility for a Furuta's type operator inequality equivalent to the chaotic order $\log A \geq \log B$ for $A$, $B > 0$, cited below:

**Theorem C.** *Let $p > 0$ and $r > 0$ be given. For $\alpha > 1$, there exist $A$, $B > 0$ such that $\log A \geq \log B$ and*

$$A^{r\alpha} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}\alpha}.$$

We note that Theorem C says the best possibility for the following characterization of the chaotic order, see [1, 3, 5, 6, 7, 8, 9, 13, 25]: For $A$, $B > 0$, $\log A \geq \log B$ if and only if

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

holds for all $p$, $r \geq 0$.

Yamazaki's simplified proof in [24] was surprising to us because both Theorem A' and Theorem C were used very well. To prove Theorem B, he divides into two cases; $0 \leq t < 1$ and $t = 1$. The former needs Theorem A' and the latter does Theorem C. This striking contrast is the motivation of this note. We present a short proof to Theorem B with no use of Theorem C, in this note. Though our basic idea is essentially similar to Yamazaki's one, we use Theorem A' only, where (2) is regarded as the special case $p = 1$ and $r = 0$ in Theorem A'.

**3. The best possibility of grand Furuta inequality.** In this section, we give a straightforward proof to Theorem B.

*Proof of Theorem B.* Assume that $p \geq 1$, $s \geq 1$, $r \geq t$, $t \in (0,1]$ and $\alpha > 1$ are given. Incidentally, the case $t = 0$ is just Theorem A' and so it can be omitted.

First of all, under the assumption $p > t$, we take $\beta = \frac{1}{1-t}$ if $0 < t < 1$ and $\beta$ is sufficiently large if $t = 1$. Next we put

$$r_1 = r\beta,\ \delta = \frac{t\beta}{2},\ p_1 = (p-t)s\beta \text{ and } \alpha_1 = \frac{1-t+r}{1+r_1}\alpha\beta. \tag{6}$$

Then we have $r_1,\ \delta \geq 0,\ p_1 \geq 1$ and $\alpha_1 > 1$. Hence it follows from Theorem A' that there exist $A,\ B > 0$ such that $A \geq B > 0$ and

$$A^{(1+r_1)\alpha_1} \not\geq (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}\alpha_1}. \tag{7}$$

We here put

$$S = A^{\beta} \text{ and } T = (A^{\delta} B^{\frac{p_1}{s}} A^{\delta})^{\frac{1}{p}};$$

we have an example for Theorem B. As a matter of fact, $S \geq T$ is ensured by the Furuta inequality (3) because $p \geq 1,\ \frac{p_1}{s} \geq 0,\ \delta > 0$ and $(1+2\delta)p \geq \frac{p_1}{s} + 2\delta$. On the other hand, it is easily checked that (5) is just the same as (7) by the set of (6).

Finally we give a counterexample for the case $p = t = 1$ (and $r \geq t = 1,\ s \geq 1,\ \alpha > 1$). For this, we apply (2), that is, there exist $A, B > 0$ such that $A \geq B$ and $A^{\alpha} \not\geq B^{\alpha}$. And we put

$$S = A^{\frac{1}{r}} \text{ and } T = S^{\frac{1}{2}}(S^{-\frac{r}{2}} B S^{-\frac{r}{2}})^{\frac{1}{s}} S^{\frac{1}{2}} = A^{\frac{1}{2r}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{s}} A^{\frac{1}{2r}};$$

in other words,

$$A = S^{r} \text{ and } B = S^{\frac{r}{2}}(S^{-\frac{1}{2}} T S^{-\frac{1}{2}})^{s} S^{\frac{r}{2}},$$

Then $S$ and $T$ are as desired. Actually $S \geq T$ is shown as follows:

$$S \geq T \Longleftrightarrow 1 \geq (S^{-\frac{1}{2}} T S^{-\frac{1}{2}})^{s} \Longleftrightarrow S^{r} \geq S^{\frac{r}{2}}(S^{-\frac{1}{2}} T S^{-\frac{1}{2}})^{s} S^{\frac{r}{2}} \Longleftrightarrow A \geq B.$$

Furthermore $A^{\alpha} \not\geq B^{\alpha}$ is an equivalent expression of (5) in this case $p = t = 1$.

So the proof is complete.

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DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, KASHIWARA, OSAKA 582-8582, JAPAN

# Bloomfield–Watson Type Inequalities

**T. Ando (Hokusei Gakuen Univ.)**

**1.** Let $\Phi$ be a **unital positive linear map** $\Phi$ between $C^*$-algebras. Here **unital** and **positive** mean that $\Phi$ is unit-preserving and positivity-preserving, respectively.

For such $\Phi$ and $A > 0$ the following inequalities are well known

$$\Phi(A)^2 \leq \Phi(A^2) \quad \text{or} \quad 0 \leq \Phi(A^2) - \Phi(A)^2, \tag{1}$$

$$\Phi(A^{-1})^{-1} \leq \Phi(A) \quad \text{or} \quad 0 \leq \Phi(A) - \Phi(A^{-1})^{-1}. \tag{2}$$

Our aim is to obtain **inverse estimates** in an extended sens. Let

$$\lambda_{\max}(A) \overset{def}{=} \text{maximum spectre of } A = ||A||,$$

$$\lambda_{\min}(A) \overset{def}{=} \text{minimum spectre of } A = ||A^{-1}||^{-1}.$$

In a general $C^*$-algebra $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the only natural values associated to a general positive element $A > 0$.

The following are known extensions of **Kantorovic** type inequalities.

## Theorem 1.

$$\Phi(A^2) - \Phi(A)^2 \leq \frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4}, \tag{3}$$

$$\Phi(A) - \Phi(A^{-1})^{-1} \leq \left(\sqrt{\lambda_{\max}(A)} - \sqrt{\lambda_{\min}(A)}\right)^2, \tag{4}$$

*and*

$$\Phi(A^2) \leq \frac{\{\lambda_{\max}(A) + \lambda_{\min}(A)\}^2}{4\lambda_{\max}(A) \cdot \lambda_{\min}(A)} \Phi(A)^2, \tag{5}$$

$$\Phi(A) \leq \frac{\{\lambda_{\max}(A) + \lambda_{\min}(A)\}^2}{4\lambda_{\max}(A) \cdot \lambda_{\min}(A)} \Phi(A^{-1})^{-1}. \tag{6}$$

**2.** These inequalities are equivalent to the following numearical inequalities:

$$\lambda_{\max}\left(\Phi(A^2)-\Phi(A)^2\right) \leq \frac{\{\lambda_{\max}(A)-\lambda_{\min}(A)\}^2}{4}, \tag{7}$$

$$\lambda_{\max}\left(\Phi(A)-\Phi(A^{-1})^{-1}\right) \leq \left(\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right)^2, \tag{8}$$

and

$$\lambda_{\max}\left(\Phi(A)^{-1}\Phi(A^2)\Phi(A)^{-1}\right) \leq \frac{\{\lambda_{\max}(A)+\lambda_{\min}(A)\}^2}{4\lambda_{\max}(A)\cdot\lambda_{\min}(A)}, \tag{9}$$

$$\lambda_{\max}\left(\Phi(A^{-1})^{\frac{1}{2}}\Phi(A)\Phi(A^{-1})^{\frac{1}{2}}\right) \leq \frac{\{\lambda_{\max}(A)+\lambda_{\min}(A)\}^2}{4\lambda_{\max}(A)\cdot\lambda_{\min}(A)}. \tag{10}$$

When the $C^*$-algebra is the **matrix algebra** $M_n$ of $n\times n$ matrices, for a positive elemement (matrix) $A$ we can consider its $n$ **eigenvalues**, arranged in decreasing order

$$\lambda_1(A)\geq\lambda_2(A)\geq\cdots\geq\lambda_n(A).$$

Since

$$\lambda_1(A)=\lambda_{\max}(A),\quad \lambda_n(A)=\lambda_{\min}(A),$$

the inequalities (7) to (10) suggest some estimates for the remaining eigenvalues of the matrices on the left hand side.

In the following we assume that $\Phi$ is a **unital positive** linear map from $M_n$ to $M_m$ with $m\leq\frac{n}{2}$.

We need the notion of **majorization**. Given two sequences of $m$ positive numbers

$$\{\alpha_j\}_{j=1}^m \quad \text{with } \alpha_1\geq\alpha_2\geq\cdots\geq\alpha_m, \quad \text{and} \quad \{\beta_j\}_{j=1}^m \quad \text{with } \beta_1\geq\beta_2\geq\cdots\geq\beta_m$$

$\{\alpha_j\}_{j=1}^m$ is said to be **weakly majorized** by $\{\beta_j\}_{j=!}^m$ if

$$\sum_{j=1}^k\alpha_j \leq \sum_{j=1}^k\beta_j \qquad (k=1,2,\cdots,m)$$

If equality occurs for $k=m$, we say **majorization**.

If $\{\alpha_j\}_{j=1}^m$ is the sequence of the eigenvalues of $0<B\in M_m$, that is ,

$$\alpha_j = \lambda_j(B) \quad (j=1,2,\cdots,m)$$

we write $B\prec_w\{\beta_j\}_{j=1}^m$.

We aim to establish the following majorizations:

$$\Phi(A^2) - \Phi(A)^2 \prec_w \left\{ \frac{\{\lambda_j(A) - \lambda_{n-j+1}(A)\}^2}{4} \right\}_{j=1}^m,$$

$$\Phi(A) - \Phi(A^{-1})^{-1} \prec_w \left\{ \left( \sqrt{\lambda_j(A)} - \sqrt{\lambda_{n-j+1}(A)} \right)^2 \right\}_{j=1}^m.$$

and

$$\Phi(A)^{-1}\Phi(A^2)\Phi(A)^{-1} \prec_w \left\{ \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \right\}_{j=1}^m,$$

$$\Phi(A^{-1})^{\frac{1}{2}}\Phi(A)\Phi(A^{-1})^{\frac{1}{2}} \prec_w \left\{ \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \right\}_{j=1}^m.$$

**3. These majorizations do not hold for general unital positive linear map** from $M_n$ to $M_m$. We have to restrict our consideration to the case of **compression**.

For an **orthoprojection** $P$ of rank $m \leq \frac{n}{2}$, according to the decomposition $I = P + P^{\perp}$, each $A \in M_n$ is written in block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

The **compression** $\Phi_P(A)$ is

$$\Phi_P(A) \stackrel{def}{=} PAP = A_{11}.$$

$\Phi_P(\cdot)$ is considered as a unital positive linear map from $M_n$ to $M_m$.

Fix $A > 0$ and write $\lambda_j = \lambda_j(A)$ $j = 1, 2, \ldots, n$. Then the following identities:

$$\Phi_P(A^2) - \Phi_P(A)^2 = |A_{21}|^2, \tag{11}$$

$$\Phi_P(A) - \Phi_P(A^{-1})^{-1} = |A_{22}^{-1/2}A_{21}|^2 \tag{12}$$

and

$$\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1} = 1 + |A_{21}A_{11}^{-1}|^2. \tag{13}$$

$$\Phi_P(A^{-1})^{\frac{1}{2}}\Phi_P(A)\Phi_P(A^{-1})^{\frac{1}{2}} = (1 - |A_{22}^{-\frac{1}{2}}A_{21}A_{11}^{-\frac{1}{2}}|^2)^{-1} \tag{14}$$

## Theorem 2.

$$\Phi(A^2) - \Phi(A)^2 \prec_w \left\{ \frac{(\lambda_j - \lambda_{n-j+1})^2}{4} \right\}_{j=1}^m,$$

*that is,*

$$|A_{21}|^2 \prec_w \left\{ \frac{(\lambda_j - \lambda_{n-j+1})^2}{4} \right\}_{j=1}^m$$

*More precisely*

$$|A_{21}| \prec_w \left\{ \frac{\lambda_j - \lambda_{n-j+1}}{2} \right\}_{j=1}^{m}. \tag{15}$$

In this connection Bloomfield–Watson [1] obtained the trace inequality :

$$\mathrm{Tr}\left(\Phi_P(A^2) - \Phi_P(A)^2\right) \leq \sum_{j=1}^{m} \frac{(\lambda_j - \lambda_{n-j+1})^2}{4}.$$

## Theorem 3.

$$\Phi_P(A) - \Phi_P(A^{-1})^{-1} \prec_w \left\{ \left(\sqrt{\lambda_j} - \sqrt{\lambda_{n-j+1}}\right)^2 \right\}_{j=1}^{m}$$

*that is,*

$$|A_{22}^{-1/2} A_{21}|^2 \prec_w \left\{ \left(\sqrt{\lambda_j} - \sqrt{\lambda_{n-j+1}}\right)^2 \right\}_{j=1}^{m}. \tag{16}$$

In this connection Rao [6] obtained the following trace inequality :

$$\mathrm{Tr}\left(\Phi_P(A) - \Phi_P(A^{-1})^{-1}\right) \leq \sum_{j=1}^{m} \left(\sqrt{\lambda_j} - \sqrt{\lambda_{n-j+1}}\right)^2. \tag{17}$$

## Theorem 4.

$$\log\left(\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1}\right) \prec_w \left\{ \log\left( \frac{(\lambda_j + \lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}} \right) \right\}_{j=1}^{m}.$$

*This implies*

$$\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1} \prec_w \left\{ \frac{(\lambda_j + \lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}} \right\}_{j=1}^{m}.$$

*that is,*

$$|A_{21}A_{11}^{-1}|^2 \prec_w \left\{ \frac{(\lambda_j - \lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}} \right\}_{j=1}^{m}. \tag{18}$$

In this connection, Khatri– Rao [3] obtained the following determinantal inequaltiy :

$$\det\left(\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1}\right) \leq \prod_{j=1}^{m} \frac{(\lambda_j + \lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}}. \tag{19}$$

**Theorem 5.**

$$\log\left(\Phi_P(A^{-1})^{1/2}\Phi_P(A)\Phi_P(A^{-1})^{1/2}\right) \prec_w \left\{\log\left(\frac{(\lambda_j+\lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}}\right)\right\}_{j=1}^m .$$

*This impies*

$$\Phi_P(A^{-1})^{1/2}\Phi_P(A)\Phi_P(A^{-1})^{1/2} \prec_w \left\{\frac{(\lambda_j+\lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}}\right\}_{j=1}^m ,$$

*that is,*

$$(1-|A_{22}^{-1/2}A_{21}A_{11}^{-1/2}|^2)^{-1} \prec_w \left\{\frac{(\lambda_j+\lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}}\right\}_{j=1}^m . \tag{20}$$

In this connection, Bloomfield – Watson [1] as well as Knott [4] obtained the following determinantal inequality :

$$\det\left(\Phi(A^{-1})^{1/2}\Phi(A)\Phi(A^{-1})^{1/2}\right) \le \prod_{j=1}^m \frac{(\lambda_j+\lambda_{n-j+1})^2}{4\lambda_j\lambda_{n-j+1}}. \tag{21}$$

**4.** For the proofs of Theorem 3 – 5 we need **Ky Fan's principle**: for $0 < B \in M_m$ and $1 \le k \le m$

$$\sum_{j=1}^k \lambda_j(B) = \max\{\mathrm{Tr}(QB); Q \text{ orghoprojection } \mathrm{rank}(Q)=k\}$$

and

$$\prod_{j=1}^k \lambda_j(B) = \max\{\det(QBQ); Q \text{ orghoprojection } \mathrm{rank}(Q)=k\}.$$

Here $\det(QBQ)$ is understood as the product of positive eigenvalues of $QBQ$.

**(1)** To apply the principle to the proof of Theorem 3, we show that for an orthoprojection $Q \le P$ with $\mathrm{rank}(Q)=k$

$$\mathrm{Tr}\left(Q\cdot(\Phi_P(A)-\Phi_P(A^{-1})^{-1})\right) \le \mathrm{Tr}\left(\Phi_Q(A)-\Phi_Q(A^{-1})^{-1}\right) \tag{22}$$

and, then appeal to the inequalitiy (17) of Rao.

**(2)** To apply the principle to the proof of Theorem 4, we show that for an orthoprojection $Q \le P$ with $\mathrm{rank}(Q)=k$

$$\det\left(Q\cdot\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1}\cdot Q\right) \le \det\left(\Phi_Q(A)^{-1}\Phi_Q(A^2)\Phi_Q(A)^{-1}\right) \tag{23}$$

and then appeal to the inequality (19) of Khatri-Rao.

**(3)** To apply the principle to the proof of Theorem 5, we show that for an orthoprojection $Q \leq P$ with $\mathrm{rank}(Q) = k$

$$\det\left(Q \cdot \Phi_P(A^{-1})^{\frac{1}{2}}\Phi_P(A)\Phi_P(A^{-1})^{\frac{1}{2}}\right) \;\leq\; \det\left(\Phi_Q(A^{-1})^{\frac{1}{2}}\Phi_Q(A)\Phi_Q(A^{-1})^{\frac{1}{2}}\right) \tag{24}$$

and then appeal to the inequality (21) of Bloomfield–Watson , Knott .

Therefore, for our majorization results the most essential are the classical inequatities due to Rao, Khatri–Rao, Bloomfield–Watson and Knott.

**5.** What is the reason for non-validity of the theorems for a general unital positive linear map $\Phi$ between $M_n$ and $M_m$ ?

By **Stinespring's Theorem** there is a *-representation $\pi(\cdot)$ of $M_n$ to the $C^*$–algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and a projection $P$ (of $\mathcal{H}$) with rank $k$ such that

$$\Phi(X) \;=\; \Phi_P(\pi(X)) \qquad (X \in M_n).$$

Then

$$\lambda_{\max}(A) = \lambda_{\max}(\pi(A)), \quad \lambda_{\min}(A) \;=\; \lambda_{\min}(\pi(A))$$

But usually $\mathrm{Tr}(\pi(A)) = \infty$, and the above approach can not be applied.

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# The real part of an outer function and a Helson-Szegö weight

Takahiko Nakazi, Hokkaido University
Takanori Yamamoto, Hokkai-Gakuen University

When $F$ is a strongly outer function in $H^1$, we study the set

$$\{f\ ;\ f \text{ is outer and } |F| \le Re\ f \text{ a.e.on } \partial D\},$$

where $\partial D$ is a unit circle. When $\gamma$ is a positive constant, we describe the set

$$\left\{f\ ;\ f \text{ is outer},\quad |F| \le \gamma\ Re\ f \text{ and } \frac{1}{|F|} \le \gamma\ Re\ \frac{1}{f} \text{ a.e. on } \partial D\right\}.$$

If $W$ is a positive function in $L^1$, then a contractive function $\alpha_W = \alpha_W(z)$ in $H^\infty$ is defined by

$$\frac{1+\alpha_W(z)}{1-\alpha_W(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} W(e^{i\theta})d\theta \quad (z \in D).$$

If $W$ is a Helson-Szegö weight, then $v$ is a real valued function such that $\|v\|_\infty < \frac{\pi}{2}$ and $\log W - \tilde{v} \in L^\infty$ if and only if

$$v = -\operatorname{Arg} \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad \text{a.e. on } \partial D,$$

where $\alpha = \alpha_W$ and a parameter $\beta$ is a contractive function in $H^\infty$ satisfying

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma\ \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial D \quad \text{for some positive constant } \gamma.$$

## 第１章　序文

開単位円板を $D$ で表し、単位円周を $\partial D$ で表す。条件：

$$\sup_{r<1}\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|d\theta < \infty$$

を満たす $D$ 上の解析関数 $f$ の全体からなる集合を $N$ で表す。もし $f \in N$ ならば $\lim_{r\to 1} f(re^{i\theta})$ が a.e. on $\partial D$ で存在し、それを $f(e^{i\theta})$ で表す。条件：

$$\lim_{r\to 1}\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|d\theta = \int_{-\pi}^{\pi} \log^+|f(e^{i\theta})|d\theta$$

を満たす $f \in N$ の全体からなる集合を $N_+$ で表す。$f$ の境界値関数の全体からなる集合も $N$ または $N_+$ で表す。$0 < p \le \infty$ に対し、Hardy 空間を $H^p = N_+ \cap L^p$ と定める。

$N_+$ の可逆元を outer 関数という。零でない $f \in H^1$ が条件：

$$\left\{g \in H^1\ ;\quad \frac{g}{f} > 0 \quad \text{a.e. on } \partial D\right\} = \{\gamma f\ ;\quad \gamma \text{ は正の数}\}$$

を満たすとき $f$ を strongly outer 関数という。$H^1$ の可逆元や実部が正値の $H^1$ 関数は strongly outer 関数であることが知られている。$H^1$ の閉球の exposed point は strongly outer 関数であり、extreme point は outer 関数であることもよく知られている。$|Q| = 1$ a.e. on $\partial D$ を満たす $Q \in H^\infty$ を inner 関数とい

う。もし $f \in N_+$ が outer 関数でないならば、$f$ は定数でない inner 関数 $Q$ を因数に持ち、左辺の集合は $g = \gamma f$ の他に $g = \gamma|Q-1|^2 f$ も含むから 左辺 $\neq$ 右辺、よって $f$ は strongly outer 関数でない。従って strongly outer 関数は outer 関数である。正値可積分関数 $W$ に対し、$\alpha_W = \alpha_W(z)$ を

$$\frac{1+\alpha_W(z)}{1-\alpha_W(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} W(e^{i\theta})d\theta \quad (z \in D).$$

で定める。このとき $\alpha_W \in \mathrm{ball}_1(H^\infty)$ かつ $|\alpha_W(e^{i\theta})| < 1$ a.e. on $\partial D$. ただし、$\mathrm{ball}_1(H^\infty)$ は $H^\infty$ の閉単位球を表す。$W$ が $u, v \in Re\ L^\infty$, $\|v\|_\infty < \frac{\pi}{2}$ により

$$W = e^{u+\tilde{v}} \quad \text{a.e. on } \partial D$$

と書けるとき、$W$ を Helson-Szegö 荷重という。ただし、$Re\ L^\infty$ は実数値 $L^\infty$ 関数の全体を表し $\tilde{v}$ は $v$ の Hilbert 変換を表す。$W$ がこのような表現を持つことと Muckenhoupt の $(A_2)$ 条件を満たすことは同値であることがよく知られている。このような $W$ が 1 つ与えられたとき、それに対する $u$, $v$ は一意的には定まらない。すなわち、

$$\|v\|_\infty < \frac{\pi}{2} \quad \text{かつ} \quad \log W = u + \tilde{v} \in Re\ L^\infty + (Re\ L^\infty)^\sim$$

という表現は一意的でない。正値可積分関数 $W$ に対するこのような $v$ の全体からなる集合を $\mathcal{E}_W$ で表す。よって

$$\mathcal{E}_W = \left\{ v \in Re\ L^\infty\ ;\quad \|v\|_\infty < \frac{\pi}{2} \quad \text{かつ} \quad \log W - \tilde{v} \in L^\infty \right\}.$$

よって $\mathcal{E}_W$ は $Re\ L^\infty$ の凸部分集合である。$W$ が Helson-Szegö 荷重であるための必要十分条件は $\mathcal{E}_W \neq \emptyset$ である。

**問題** Helson-Szegö 荷重 $W$ が与えられたとき、$\mathcal{E}_W$ をパラメター表示せよ。

この問題は、Adamyan-Arov-Krein の定理 (cf.[2]) を使って解くこともできるが、我々は outer 関数の実部のパラメター表示に着目して以下のように解く。最初に第 2 章において $W$ が strongly outer 関数の絶対値であるとき、$N_+$ の 2 つの部分集合：

$$M_W = \{f \in N_+\ ;\ f \text{ はouter関数、かつ} \quad W \leq Re\ f \quad \text{a.e.on } \partial D\}$$

$$N_W = \left\{ f \in N_+\ ;\ f \text{ はouter関数、かつ} \quad W \leq \gamma\ Re\ f \quad \text{かつ} \quad \frac{1}{W} \leq \gamma\ Re\ \frac{1}{f} \quad \text{a.e.on } \partial D \right\}$$

を $\alpha_W \in \mathrm{ball}_1(H^\infty)$ とパラメター $\beta \in \mathrm{ball}_1(H^\infty)$ を用いて表示する。$M_W$ は $N_+$ の凸部分集合である。定理 1 の (2) より、$N_W$ は $H^1$ の凸部分集合である。次に第 3 章において第 2 章の結果を使って上の問題を解く。

## 第 2 章　Outer 関数の実部のパラメター表示

もし $\alpha \in H^\infty$ が定数 $\gamma$ について $|1-\alpha| \leq \gamma(1-|\alpha|)$ a.e. on $\partial D$ を満たすならば、$\alpha$ を $\gamma$-Stolz 関数または単に Stolz 関数と呼ぶ。ただし、$\gamma$ は正の数を表す。従って、Stolz 関数は $\mathrm{ball}_1(H^\infty)$ の元である。このとき

$$|1-\alpha^2| \leq |1-\alpha| + |\alpha(1-\alpha)| \leq 2|1-\alpha| \leq 2\gamma(1-|\alpha|) \leq 2\gamma(1-|\alpha|^2).$$

従って $\alpha$ が Stolz 関数ならば $\alpha^2$ も Stolz 関数である。strongly outer 関数 $F \in H^1$ に対し $W = |F|$ と定めるとき、命題 2 と定理 1 により、それぞれ集合 $M_W$ と $N_W$ のパラメター表示を与える。

**命題 1** 零でない $F \in H^1$ と $\gamma \geq 1$ なる定数 $\gamma$ について (1) ~ (3) は同値である。

(1) $|F| \leq \gamma\ Re\ F$ a.e. on $\partial D$.

(2) $\alpha \in \mathrm{ball}_1(H^\infty)$ が存在して $\alpha^2$ は $\gamma$-Stolz 関数、かつ $F = \dfrac{1+\alpha}{1-\alpha}$ a.e. on $\partial D$.

(3) 実数値関数 $v$ と 正の数 $c$ が存在して

$$F = c\, e^{\tilde{v}-iv} \quad \text{かつ} \quad |v| \leq \cos^{-1}\left(\frac{1}{\gamma}\right) < \frac{\pi}{2} \quad \text{a.e. on } \partial D.$$

命題 1 より、もし零でない $F \in H^1$ が $|F| \leq \gamma\ Re\ F$ a.e. on $\partial D$ を満たすならば、(3) より、ある $p > 1$ が存在して $F, \dfrac{1}{F} \in H^p$ が成り立つ。正値可積分関数 $W$ が与えられたとき、命題 1 より、$W = c\, e^{\tilde{v}}$ が成り立つような正の数 $c$ と $\|v\|_\infty < \dfrac{\pi}{2}$ なる実数値関数 $v$ が存在するための必要十分条件は $\alpha^2$ が Stolz 関数であり、かつ $W = \left|\dfrac{1+\alpha}{1-\alpha}\right|$ が成り立つような $\alpha \in H^\infty$ が存在することである。このとき、ある実数値関数 $u \in L^\infty$ が存在して次が成り立つ：

$$W = \frac{|1-\alpha^2|}{1-|\alpha|^2}\frac{1-|\alpha|^2}{|1-\alpha|^2} = e^u\ \frac{1-|\alpha|^2}{|1-\alpha|^2} = e^u\ Re\ F \quad \text{a.e. on } \partial D.$$

**命題 2** strongly outer 関数 $F \in H^1$ が与えられたとき、$\alpha = \alpha_{|F|}$ と定めると、$f$ について (1)〜(3) は同値である。

(1) $|F| \leq Re\ f$ a.e. on $\partial D$ かつ $f$ は outer 関数。

(2) $\beta \in \mathrm{ball}_1(H^\infty)$ が存在して $f = \dfrac{1+\alpha}{1-\alpha} + \dfrac{1+\beta}{1-\beta}$ a.e. on $\partial D$.

(3) 実数値関数 $u, v$ と正の数 $c$ が存在して

$$|F| = e^{u+\tilde{v}}, \quad |v| < \frac{\pi}{2}, \quad e^u \leq c\ \cos v, \quad f = c\, e^{\tilde{v}-iv} \quad \text{a.e. on } \partial D.$$

命題 2 より、$W = |F|$ のとき、もし $f \in M_W$ ならば、(3) より、すべての $p < 1$ について $f, \dfrac{1}{f} \in H^p$ が成り立つ。$F \in H^1$ が零でないというだけの仮定の下で (3) ⇒ (2) ⇒ (1) は成り立つが、(1) ⇒ (3) は [4] の Theorem 6 より成り立たない。

**定理 1** strongly outer 関数 $F \in H^1$ が与えられたとき、$\alpha = \alpha_{|F|}$ と定めると、$f$ について (1)〜(4) は同値である。ただし、$\gamma_1, \ldots, \gamma_5$ は適当な正の数を表す。

(1) $|F| \leq \gamma_1\ Re\ f$ かつ $\dfrac{1}{|F|} \leq \gamma_1\ Re\ \dfrac{1}{f}$ a.e. on $\partial D$ かつ $f \in N_+$.

(2) $\dfrac{1}{\gamma_2} Re\ f \leq |F| \leq \gamma_2 Re\ f$ かつ $|f| \leq \gamma_2 Re\ f$ a.e. on $\partial D$ かつ $f \in H^1$.

(3) $\beta \in \mathrm{ball}_1(H^\infty)$ が存在して

$$\gamma_3 f = \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad \text{かつ} \quad \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \leq \gamma_4\ \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial D.$$

(4) 実数値関数 $u, v \in L^\infty$ と 正の数 $c$ が存在して

$$|F| = e^{u+\tilde{v}} \quad \text{かつ} \quad f = c\, e^{\tilde{v}-iv} \quad \text{a.e. on } \partial D \quad \text{かつ} \quad \|v\|_\infty \leq \cos^{-1}\gamma_5 < \frac{\pi}{2}.$$

定理 1 より、$W = |F|$ のとき、もし $f \in N_W$ ならば、(4) より、ある $p > 1$ について $f, \dfrac{1}{f} \in H^p$ が成り立つ。

## 第3章　Helson-Szegö 荷重の表現のパラメター表示

序文に書いた問題は Adamyan-Arov-Krein の定理を用いて解けるが、この章では第2章の結果を使って解く。以下の定理 2 がその解答になっている。

**定義**　$\alpha \in \mathrm{ball}_1(H^\infty)$ に対し、$\mathrm{ball}_1(H^\infty)$ の部分集合 $B^\alpha$ を次のように定める。

$$B^\alpha = \left\{ \beta \in \mathrm{ball}_1(H^\infty)\ ;\ \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma\ \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial D \quad \text{for some constant } \gamma \right\}.$$

**定理 2**　正値可積分関数 $W$ に対し $\alpha = \alpha_W$ と定めると、$\alpha \in \mathrm{ball}_1(H^\infty)$ が成り立ち、$\mathcal{E}_W$ を次のようにパラメター表示できる：

$$\mathcal{E}_W = \left\{ -\ \mathrm{Arg}\ \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\ ;\ \beta \in B^\alpha \right\}.$$

ただし、Arg は $-\pi < \theta \le \pi$ に限定された偏角を表す。

証明.「$\mathcal{E}_W \subset$ 右辺」を示す：$v \in \mathcal{E}_W$ とする。このとき、実数値関数 $u, v \in L^\infty$ が存在して　$W = e^{u+\tilde{v}}$,　$\|v\|_\infty \le \frac{\pi}{2} - \varepsilon$ がある正の数 $\varepsilon$ について成り立つ。よって $W$ は Helson-Szegö 荷重であり、$\cos v \ge \cos\left(\frac{\pi}{2} - \varepsilon\right) = \sin\varepsilon$ が成り立つ。正の数 $\gamma_1$ を $e^{\|u\|_\infty} \le \gamma_1 \sin\varepsilon$ が成り立つように定めると、$e^{\|u\|_\infty} \le \gamma_1 \cos v$ が成り立つから

$$W \le \gamma_1\ e^{\tilde{v}} \cos v \quad \text{かつ} \quad \frac{1}{W} \le \gamma_1\ e^{-\tilde{v}} \cos v$$

が成り立つ。従って $f = e^{\tilde{v} - iv}$ と定めると Zygmund の定理より $f \in H^1$,

$$W \le \gamma_1\ Re\ f \quad \text{かつ} \quad \frac{1}{W} \le \gamma_1\ Re\ f^{-1}.$$

Zygmund の定理より $W, \frac{1}{W} \in L^1$. よって outer 関数 $F$ が存在して $W = |F|$ かつ $F, F^{-1} \in H^1$ が成り立つ。よって $F$ は strongly outer 関数、$f \in N_+$ であり、

$$|F| \le \gamma_1\ Re\ f \quad \text{かつ} \quad \frac{1}{|F|} \le \gamma_1\ Re\ \frac{1}{f}.$$

定理 1 より、正の数 $\gamma_3, \gamma_4$ と $\beta \in \mathrm{ball}_1(H^\infty)$ が存在して

$$\gamma_3 f = \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad \text{かつ} \quad \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma_4\ \frac{1-|\alpha|^2}{|1-\alpha|^2}.$$

従って、$\beta \in B^\alpha$ かつ

$$v = -\ \mathrm{Arg} f = -\ \mathrm{Arg}\ \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}.$$

よって $v \in$ 右辺。逆に「$\mathcal{E}_W \supset$ 右辺」を示す：$v \in$ 右辺 のとき $f$ を

$$f = \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}$$

と定めると、ある正の数 $\gamma$ が存在して

$$v = -\ \mathrm{Arg}\ f \quad \text{かつ} \quad \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma\ \frac{1-|\alpha|^2}{|1-\alpha|^2},$$

$$W=\frac{1-|\alpha|^2}{|1-\alpha|^2}\le\frac{1-|\alpha|^2}{|1-\alpha|^2}+\frac{1-|\beta|^2}{|1-\beta|^2}=2\ Ref\le 2|f|\le 2\gamma\ \frac{1-|\alpha|^2}{|1-\alpha|^2}=2\gamma W.$$

よって $Re\ f\ge 0$ a.e. on $\partial D$ かつ $f\in H^1$. よって $f$ は strongly outer 関数である。$\log W\in L^1$ より、outer 関数 $F\in H^1$ が存在して $|F|=W$. 関数 $k$ が $k\in H^1$ かつ $\dfrac{k}{F}\ge 0$ a.e. on $\partial D$ を満たしているとする。このとき、$\dfrac{f}{F}\in H^\infty$ より $\dfrac{kf}{F}\in H^1$ が成り立つ。$f$ は strongly outer 関数であるから、ある正の数 $c$ が存在して $\dfrac{kf}{F}=cf$ と書ける。よって $k=cF$. 従って $F$ は strongly outer 関数であり、$\alpha=\alpha_W=\alpha_{|F|}$;

$$f=\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\quad かつ\quad \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|}\le\gamma\ \frac{1-|\alpha|^2}{|1-\alpha|^2}$$

が成り立つから、定理 1 より、実数値関数 $u,v_0\in L^\infty$ と正の数 $c_0$ が存在して

$$\|v_0\|_\infty<\frac{\pi}{2},\quad W=e^{u+\tilde{v}_0}\quad かつ\quad f=c_0\ e^{\tilde{v}_0-iv_0}.$$

従って

$$v_0=-\ \mathrm{Arg}\ f=-\ \mathrm{Arg}\ \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}=v.$$

よって

$$\|v\|_\infty<\frac{\pi}{2}\quad かつ\quad \log W-\tilde{v}\in L^\infty.$$

よって $v\in\mathcal{E}_W$. □

定理 2 より、$\alpha=\alpha_W$ のとき $\mathcal{E}_W\ne\emptyset\Leftrightarrow B^\alpha\ne\emptyset$ が成り立つ。更に

$$\alpha\ が\ \mathrm{Stolz}\ 関数\quad\Leftrightarrow\quad \alpha\in\mathrm{ball}_1(H^\infty)\quad かつ\quad 0\in B^\alpha$$

が成り立つ。$W$ が Helson-Szegö 荷重であることは $\mathcal{E}_W\ne\emptyset$ と同値であるから、もし $\alpha_W$ が Stolz 関数ならば $W$ は Helson-Szegö 荷重である。更に

$$W=\frac{1-|\alpha|^2}{|1-\alpha|^2}=\frac{1+|\alpha|}{|1-\alpha|}\frac{1-|\alpha|}{|1-\alpha|}$$

より、もし $\alpha_W$ が Stolz 関数ならば $\dfrac{1}{W}\in L^\infty$ も成り立つ。 $H^\infty$ の半径 $r$ の閉球を $\mathrm{ball}_r(H^\infty)$ で表す。もし $\alpha$ が $\gamma$-Stolz 関数かつ $\beta\in\mathrm{ball}_r(H^\infty)$ ならば、

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|}\le\frac{2}{(1-r)|1-\alpha|}\le\frac{2\gamma(1-|\alpha|^2)}{(1-r)|1-\alpha|^2}$$

が成り立つから $\beta\in B^\alpha$. 従って、

$$\alpha\ が\ \mathrm{Stolz}\ 関数\quad\Leftrightarrow\quad 0\in B^\alpha\quad\Rightarrow\quad \mathrm{ball}_r(H^\infty)\subset B^\alpha\quad(r<1)$$

が成り立つ。従って、もし $\alpha=\alpha_W$ が Stolz 関数ならば

$$\mathcal{E}_W\supset\left\{-\ \mathrm{Arg}\ \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\ ;\ \beta\in\mathrm{ball}_r(H^\infty)\right\}.$$

特に $W=1$ のとき $\alpha=0$ となるから、定理 2 より

$$\begin{aligned}\mathcal{E}_1&=\left\{v\in Re\ L^\infty\ ;\quad \|v\|_\infty<\frac{\pi}{2}\quad かつ\quad \tilde{v}\in L^\infty\right\}\\&=\left\{-\ \mathrm{Arg}\ \frac{1}{1-\beta}\ ;\ \beta\in\mathrm{ball}_1(H^\infty)\quad かつ\quad \frac{1}{1-\beta}\in L^\infty\right\}.\end{aligned}$$

もし $\partial D$ 上の2つの正値関数 $f, g$ に対して正の数 $\gamma$ が存在して $\frac{1}{\gamma}g \le f \le \gamma g$ a.e. on $\partial D$ が成り立つならば、$f \sim g$ と書く。

**補題** $\alpha, \beta \in \mathrm{ball}_1(H^\infty)$ について (1) ~ (5) は同値である。ただし、$\gamma_2, \gamma_3, \gamma_4$ は適当な正の数を表す。

(1) $\left\| \frac{\alpha - \bar{\beta}}{1 - \alpha\beta} \right\|_\infty < 1.$

(2) $|1 - \alpha\beta|^2 \le \gamma_2 (1 - |\alpha|^2)(1 - |\beta|^2)$ a.e. on $\partial D$.

(3) すべての正値関数 $t$ について

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \le \gamma_3 \left( t \, \frac{1 - |\alpha|^2}{|1 - \alpha|^2} + \frac{1}{t} \, \frac{1 - |\beta|^2}{|1 - \beta|^2} \right) \quad \text{a.e. on } \partial D.$$

(4) $\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \le \gamma_4 \min \left( \frac{1 - |\alpha|^2}{|1 - \alpha|^2}, \frac{1 - |\beta|^2}{|1 - \beta|^2} \right)$ a.e. on $\partial D$.

(5) $|1 - \alpha| \sim |1 - \beta|$ かつ $1 - |\alpha| \sim 1 - |\beta| \sim |1 - \alpha\beta|$.

この補題より、命題 3 が成り立つ。

**命題 3** もし $\alpha \in \mathrm{ball}_1(H^\infty)$ ならば、

$$B^\alpha \supset \left\{ \beta \in \mathrm{ball}_1(H^\infty) \; ; \; \left\| \frac{\alpha - \bar{\beta}}{1 - \alpha\beta} \right\|_\infty < 1 \right\}.$$

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# A reduction of the problem of characterizing perfect semigroups

Torben Maack Bisgaard and Nobuhisa Sakakibara

## 1. Introduction

Suppose $(S, +, *)$ is an abelian semigroup equipped with an *involution*, that is, an involutory automorphism, written $s \mapsto s^*$. Such a structure will be called a *$*$-semigroup*, abbreviated "semigroup" when confusion is unlikely, such as when applying an adjective which is defined only for $*$-semigroups (e.g., "perfect semigroup"). Define $S+S = \{\, s+t \mid s,t \in S \,\}$ and define $\overbrace{S+\cdots+S}^{N}$ similarly for arbitrary $N \in \mathbf{N}$. A *positive definite function on $S$* is a function $\varphi\colon S+S \to \mathbf{C}$ such that

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k^*) \geq 0$$

for every choice of $n \in \mathbf{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbf{C}$. Denote by $\mathcal{P}(S)$ the set of all positive definite functions on $S$. A *character* on $S$ is a function $\sigma\colon S \to \mathbf{C}$, not identically zero, such that $\sigma(s^*) = \overline{\sigma(s)}$ and $\sigma(s+t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by $S^*$ the set of all characters on $S$. Denote by $\mathcal{A}(S^*)$ the least $\sigma$-ring of subsets of $S^*$ rendering the mapping $\sigma \mapsto \sigma(s)\colon S^* \to \mathbf{C}$ measurable for each $s \in S$. A function $\varphi\colon S+S \to \mathbf{C}$ is a *moment function* if there is a measure $\mu$ defined on $\mathcal{A}(S^*)$ such that

$$\varphi(s) = \int_{S^*} \sigma(s)\, d\mu(\sigma)$$

for all $s \in S+S$, and a moment function $\varphi$ is *determinate* if there is only one such $\mu$. (In writing an equation such as the preceding, it is understood that $\mu$ should integrate the integrands.) The semigroup $S$ is *perfect* if every positive definite function on $S$ is a determinate moment function. For positive definite functions and (Radon) moment functions on semigroups, we refer to [1], especially Section 6.5 on semigroups called "perfect" in that book but now called "Radon perfect".

A $*$-semigroup $H$ is *$*$-archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbf{N}$ such that $n(x + x^*) = y + z$. A *$*$-archimedean component* of a $*$-semigroup $S$ is a $*$-archimedean $*$-subsemigroup of $S$ which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components. It was shown in [2], Theorem 3.1, that a $*$-semigroup $S$ with zero is perfect if and only if $H \cup \{0\}$ is perfect for each $*$-archimedean component $H$ of $S$. We shall be concerned with extending this result to semigroups without zero. It is not true that a $*$-semigroup $S$, even with zero, is perfect if and only if each $*$-archimedean component of $S$ is perfect. For example, if $H = \mathbf{Q} \cap [1, \infty[$ and $S = H \cup \{0\}$ then $S$ is perfect ([4], Corollary 2) and $H$ is a $*$-archimedean component of $S$, yet $H$ is not perfect ([6], Remark 3.6). We shall define "quasi-perfect" semigroups in such a way that a $*$-semigroup $S$ satisfying $S = S+S$ is perfect if and only if each $*$-archimedean component of $S$ is quasi-perfect.

## 2. Reduction to the $*$-archimedean case

Say that a $*$-semigroup $H$ is *quasi-perfect of order $N \geq 2$* if for each $\varphi \in \mathcal{P}(H)$ there is a unique measure $\mu$ on $\mathcal{A}(H^*)$ such that

$$\varphi(x) = \int_{H^*} \eta(x)\, d\mu(\eta), \qquad x \in \overbrace{H+\cdots+H}^{N}.$$

Then $H$ is perfect if and only if $H$ is quasi-perfect of order 2.

**Theorem 1.** *For a $*$-semigroup $H$, possibly without zero, the following three conditions are equivalent :*

(i) *$H$ is quasi-perfect of every order greater than or equal to 3;*
(ii) *$H$ is quasi-perfect of some order greater than or equal to 3;*
(iii) *$H$ is quasi-perfect of order 3.*

**Proof.** (i)$\Rightarrow$(ii): Trivial.
(ii)$\Rightarrow$(iii): Suppose $H$ is quasi-perfect of order $N \geq 3$. We shall show that $H$ is quasi-perfect of order $M$ for $3 \leq M \leq N$ by backwards induction on $M$. Suppose $3 \leq M < N$ and that we have shown that $H$ is quasi-perfect of order $M+1$. Suppose $\varphi \in \mathcal{P}(H)$. Since $H$ is quasi-perfect of order $M+1$, there is a unique measure $\mu$ on $\mathcal{A}(H^*)$ such that

$$\varphi(x) = \int_{H^*} \eta(x)\, d\mu(\eta) \tag{1}$$

for $x \in \overbrace{H + \cdots + H}^{M+1}$. Clearly there is at most one measure with the corresponding property for $x$ in the larger set $\overbrace{H + \cdots + H}^{M}$. Thus we only have to show that (1) extends to all $x \in \overbrace{H + \cdots + H}^{M}$. Suppose $h \in H$. For $x_1, \ldots, x_n \in \overbrace{H + \cdots + H}^{M-1}$ and $c_1, \ldots, c_n \in \mathbf{C}$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\sum_{j=1}^{n} c_j \varphi(h + x_j)\right|^2 &\leq \varphi(h + h^*) \sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(x_j + x_k^*) \\ &= \varphi(h + h^*) \int_{H^*} \left|\sum_{j=1}^{n} c_j \eta(x_j)\right|^2 d\mu(\eta).\end{aligned}$$

We used the fact that for $j, k = 1, \ldots, n$ we have

$$x_j + x_k^* \in \overbrace{H + \cdots + H}^{M-1} + \overbrace{H + \cdots + H}^{M-1} = \overbrace{H + \cdots + H}^{2M-2} \subset \overbrace{H + \cdots + H}^{M+1}$$

since $2M - 2 \geq M + 1$ because of $M \geq 3$. The above inequality shows that the mapping

$$\left(\eta \mapsto \sum_{j=1}^{n} c_j \eta(x_j)\right) \mapsto \sum_{j=1}^{n} c_j \varphi(h + x_j)$$

is a well-defined bounded linear form on a linear subspace of $L^2(\mu)$. Extend this linear form to a bounded linear form $L_h$ on all of $L^2(\mu)$. Then there is a unique $\psi_h \in L^2(\mu)$ such that $L(f) = \int f \psi_h \, d\mu$ for all $f \in L^2(\mu)$. In particular,

$$\varphi(h + x) = \int_{H^*} \eta(x) \psi_h(\eta)\, d\mu(x) \tag{2}$$

for $x \in \overbrace{H + \cdots + H}^{M-1}$. For $x \in \overbrace{H + \cdots + H}^{M+1}$ we have by (1) and (2),

$$\int_{H^*} \eta(h)\eta(x)\, d\mu(\eta) = \int_{H^*} \eta(x)\psi_h(\eta)\, d\mu(\eta).$$

Since $H$ is quasi-perfect of order $M+1$ it follows that $\psi_h(\eta)\, d\mu(\eta) = \eta(h)\, d\mu(\eta)$ (cf. [1], Proposition 6.5.2), so (2) reduces to

$$\varphi(h + x) = \int_{H^*} \eta(h)\eta(x)\, d\mu(\eta).$$

This is the desired representation since each $y \in \overbrace{H + \cdots + H}^{M}$ can be written as $y = h + x$ with $h \in H$ and $x \in \overbrace{H + \cdots + H}^{M-1}$.
(iii)$\Rightarrow$(i): Suppose $N \geq 3$; we have to show that $H$ is quasi-perfect of order $N$. Suppose $\varphi \in \mathcal{P}(H)$. Since $H$ is quasi-perfect of order 3, there is a unique measure $\mu$ on $\mathcal{A}(H^*)$ such that

$$\varphi(x) = \int_{H^*} \eta(x)\, d\mu(\eta)$$

for $x \in H + H + H$. Since $\overbrace{H + \cdots + H}^{N} \subset H + H + H$, this integral representation holds, in particular, for $x \in \overbrace{H + \cdots + H}^{N}$, and the problem is to show that $\mu$ is uniquely determined by the latter property. Assume that $\mu_1$ and $\mu_2$ are measures on $\mathcal{A}(H^*)$ such that

$$\varphi(x) = \int_{H^*} \eta(x)\, d\mu_i(\eta)$$

for $x \in \overbrace{H + \cdots + H}^{N}$ for $i = 1, 2$; we have to show $\mu_1 = \mu_2$. For $h \in H$ and $x \in H + H + H$ we have $(N-3)h + x \in \overbrace{H + \cdots + H}^{N}$, so

$$\int_{H^*} \eta(h)^{N-3} \eta(x)\, d\mu_i(\eta) = \varphi\big((N-3)h + x\big)$$

for $i = 1, 2$. Since $H$ is quasi-perfect of order 3, the mapping

$$\lambda \mapsto \left( \int_{H^*} \eta(x)\, d\lambda(\eta) \right)_{x \in H+H+H}$$

is one-to-one on the space of complex measures on $\mathcal{A}(H^*)$ integrating the integrands, so from the preceding we can infer $\eta(h)^{N-3} d\mu_1(\eta) = \eta(h)^{N-3}\, d\mu_2(\eta)$. Hence $\mu_1 | G_h = \mu_2 | G_h$ where $G_h = \{\, \eta \in H^* \mid \eta(h) \neq 0 \,\}$. This being so for all $h \in H$, since $H^* = \bigcup_{h \in H} G_h$, and since every element of $\mathcal{A}(H^*)$ is, by definition, contained in the union of countably many $G_h$, it follows that $\mu_1 = \mu_2$, as desired. Q.E.D.

We call a $*$-semigroup $H$ *quasi-perfect* if the equivalent conditions of Theorem 1 are satisfied. An *ideal* of a $*$-semigroup $X$ is a nonempty $*$-stable subset $H$ of $X$ such that $X + H \subset H$. If $H$ is an ideal of $X$ then, in particular, $H + H \subset H$, so $H$ is a $*$-subsemigroup of $X$. In [4], Theorem, it was shown that if $S$ is a (Radon) perfect semigroup with zero and if $T$ is a $*$-subsemigroup of $S$ containing 0 and such that $T \setminus \{0\}$ is an ideal of $S$ then $T$ is likewise (Radon) perfect. The following result seems to be the closest analogue of this for semigroups without zero. The proof, like the proof of Theorem 1, is strongly inspired by the argument in [4].

**Theorem 2.** *If $X$ is a quasi-perfect semigroup and $H$ is an ideal of $X$ then $H$ is quasi-perfect.*
**Proof.** Suppose $\varphi \in \mathcal{P}(H)$. For $h, k \in H$ and $n = 0, 1, 2, 3$ define $\varphi_{h,k,n} \in \mathcal{P}(X)$ by

$$\varphi_{h,k,n}(x) = \varphi(h + h^* + x) + i^n \varphi(h + k^* + x) + i^{-n} \varphi(k + h^* + x) + \varphi(k + k^* + x)$$

for $x \in X + X$. Since $X$ is quasi-perfect, there is a unique measure $\lambda_{h,k,n}$ on $\mathcal{A}(X^*)$ such that

$$\varphi_{h,k,n}(x) = \int_{X^*} \xi(x)\, d\lambda_{h,k,n}(\xi)$$

for $x \in X + X + X$. We now introduce a subring $\mathcal{A}_0(X^*)$ of $\mathcal{A}(X^*)$ which generates the latter as a $\sigma$-ring, as follows. For $x \in X$ and $n \in \mathrm{N}$ define $G_{x,n} = \left\{ \xi \in X^* \,\middle|\, |\xi(x)| > 1/n \right\}$. Then let $\mathcal{A}_0(X^*)$ be the set of those elements of $\mathcal{A}(X^*)$ which are contained in the union of finitely many $G_{x,n}$. Clearly $\mathcal{A}_0(X^*)$ is a subring of $\mathcal{A}(X^*)$. This subring generates $\mathcal{A}(X^*)$ as a $\sigma$-ring since for each $A \in \mathcal{A}(X^*)$ there is a countable subset $Y$ of $X$ such that for each $\xi \in A$ there is some $y \in Y$ such that $\xi(y) \neq 0$. This is because the set of all subsets $A$ of $X^*$ with the property just described is a $\sigma$-ring of subsets of $X^*$ which renders the mapping $\xi \mapsto \xi(x)$ measurable for each $x \in X$, hence contains $\mathcal{A}(X^*)$ by the definition of $\mathcal{A}(X^*)$. Since $\mathcal{A}_0(X^*)$ generates $\mathcal{A}(X^*)$, every measure $\mu$ on $\mathcal{A}_0(X^*)$ which is *finite* in the sense that $\mu(A) < \infty$ for all $A \in \mathcal{A}_0(X^*)$ extends to a unique measure on $\mathcal{A}(X^*)$. Thus measures on $\mathcal{A}(X^*)$ can be identified with their restrictions to $\mathcal{A}_0(X^*)$. If $\mu$ is a measure on $\mathcal{A}(X^*)$ which integrates the function $\xi \mapsto \xi(x)$ for all $x \in \overbrace{X + \cdots + X}^{N}$ for some $N \in \mathrm{N}$ then $\mu|\mathcal{A}_0(X^*)$ is finite since for $x \in X$ and $n \in \mathrm{N}$ we have

$$\frac{\mu(G_{x,n})}{n^N} \leq \int_{G_{x,n}} |\xi(x)|^N \, d\mu(\xi) \leq \int_{X^*} |\xi(x)|^N \, d\mu(\xi) = \int_{X^*} |\xi(Nx)| \, d\mu(\xi) < \infty.$$

If we now define

$$\lambda_{h,k} = \frac{1}{4} \sum_{n=0}^{3} i^{-n} \lambda_{h,k,n}$$

(as a set function on $\mathcal{A}_0(X^*)$) then $\lambda_{h,k}$ is the unique complex measure on $\mathcal{A}(X^*)$ such that

$$\varphi(h + k^* + x) = \int_{X^*} \xi(x) \, d\lambda_{h,k}(\xi)$$

for $x \in X + X + X$. We see that for each $y \in H + H$ there is a unique complex measure $\lambda_y$ on $\mathcal{A}_0(X^*)$ such that

$$\varphi(x + y) = \int_{X^*} \xi(x) \, d\lambda_y(\xi) \tag{3}$$

for $x \in X + X + X$. Indeed, any $y \in H + H$ can be written as $y = h + k^*$ with $h, k \in H$, and then we have to define $\lambda_y = \lambda_{h,k}$. This definition of $\lambda_y$ is independent of the choice of $h$ and $k$ since if $h_1, h_2, k_1, k_2 \in H$ and $h_1 + k_1^* = h_2 + k_2^*$ then $\lambda_{h_1,k_1} = \lambda_{h_2,k_2}$ since these two measures represent the same function on $X + X + X$. If $y, z \in H + H$ then

$$\int_{X^*} \xi(x)\xi(y) \, d\lambda_z(\xi) = \varphi(x + y + z) = \int_{X^*} \xi(x)\xi(z) \, d\lambda_y(\xi)$$

for $x \in X + X + X$, and by the quasi-perfectness of $X$ it follows that

$$\xi(y) \, d\lambda_z(\xi) = \xi(z) \, d\lambda_y(\xi). \tag{4}$$

For $h \in H$ write $G_h = \{ \xi \in X^* \mid \xi(h) \neq 0 \}$. If for $y \in H + H$ we define a measure $\kappa_y$ on $G_y$ by

$$d\kappa_y(\xi) = \xi(y)^{-1} \, d\lambda_y(\xi)|G_y$$

then (4) shows that for $y, z \in H + H$ we have

$$\kappa_y|(G_y \cap G_z) = \kappa_z|(G_y \cap G_z).$$

Hence there is a unique measure $\kappa$ on the set

$$G = \bigcup_{h \in H} G_h = \left\{ \xi \in X^* \,\middle|\, \xi|H \neq 0 \right\}$$

such that

$$\kappa_y = \kappa|G_y$$

for all $y \in H + H$. More precisely, $\kappa$ is defined on the $\sigma$-ring $\mathcal{A}_*(G)$ consisting of those elements of $\mathcal{A}(X^*)$ which are contained in the union of countably many $G_h$. We claim that

$$d\lambda_y(\xi)|G = \xi(y) \, d\kappa(\xi) \tag{5}$$

for $y \in H+H$ where $|G$ denotes the operation of restriction of a measure to the $\sigma$-ring $\mathcal{A}_*(G)$. Since $G$ is the union of the sets $G_z$, $z \in H+H$, it suffices to verify $d\lambda_y(\xi)|G_z = \xi(y)\, d\kappa(\xi)|G_z$ for $z \in H+H$. But the right-hand side is equal to $\xi(y)\xi(z)^{-1}\, d\lambda_z(\xi)|G_z$, so the desired equality follows from (4). This proves (5). For $x \in H+H+H$ and $y \in H+H$, since characters outside $G$ vanish on $H$, by (3) and (5) we have

$$\varphi(x+y) = \int_{X^*} \xi(x)\, d\lambda_y(\xi) = \int_G \xi(x)\, d\lambda_y(\xi) = \int_G \xi(x)\xi(y)\, d\kappa(\xi).$$

Thus, for $x \in H+H+H+H+H$ we have

$$\varphi(x) = \int_G \xi(x)\, d\kappa(\xi).$$

Now $\kappa$ is uniquely determined by this property. To see this, note that if $\kappa_1$ and $\kappa_2$ are two measures with this property then $\varphi(3h+3h^*+x) = \int_G |\xi(h)|^6 \xi(x)\, d\kappa_i(\xi)$ for $h \in H$, $x \in X+X+X$, and $i = 1,2$, so by the quasi-perfectness of $X$ it follows that $|\xi(h)|^6\, d\kappa_1(\xi) = |\xi(h)|^6\, d\kappa_2(\xi)$, hence $\kappa_1|G_h = \kappa_2|G_h$. Since every measurable subset of $G$ is contained in the union of countably many $G_h$, it follows that $\kappa_1 = \kappa_2$.

Now the mapping $\xi \mapsto \xi|H: G \to H^*$ is a bijection. To see this, we first have to show that it is one-to-one. So suppose $\xi, \eta \in G$ and $\xi|H = \eta|H$; we have to show that $\xi = \eta$. Choose $h \in H$ such that $\xi(h) \neq 0$. Then for $x \in X$,

$$\xi(h)\xi(x) = \xi(h+x) = \eta(h+x) = \eta(h)\eta(x) = \xi(h)\eta(x),$$

and dividing by $\xi(h)$ we obtain $\xi(x) = \eta(x)$, as desired. Conversely, suppose $\eta \in H^*$; we have to show that there is some $\xi \in G$ such that $\xi|H = \eta$. Choose $h \in H$ such that $\eta(h) \neq 0$ and define $\xi \in X^*$ by $\xi(x) = \eta(h+x)/\eta(h)$ for $x \in X$. Then $\xi$ has the desired property. We leave it as an exercise to verify that the mapping $\xi \mapsto \xi|H$ is an isomorphism between the measurable spaces $\big(G, \mathcal{A}_*(G)\big)$ and $\big(H^*, \mathcal{A}(H^*)\big)$. Now if $\mu$ is the image measure of $\kappa$ under the mapping $\xi \mapsto \xi|H$ then $\varphi(x) = \int_{H^*} \eta(x)\, d\mu(\eta)$ for $x \in H+H+H+H+H$, and $\mu$ is uniquely determined by this property (since $\kappa$ is unique). Thus $H$ is quasi-perfect of order 5, that is, $H$ is quasi-perfect. Q.E.D.

It is not true that every ideal of a perfect semigroup is perfect. For example, if $H = \mathbf{Q} \cap [1, \infty[$ and $X = H \cup \{0\}$ then $X$ is perfect and $H$ is an ideal of $X$, yet $H$ is not perfect.

**Corollary 1.** *A $*$-semigroup $H$ is quasi-perfect if and only if the $*$-semigroup $X = H \cup \{0\}$ obtained by adjoining to $H$ a zero external to $H$ is perfect.*
**Proof.** If $X$ is perfect then $H$ is quasi-perfect by Theorem 2 since $H$ is an ideal of $X$. Conversely, if $H$ is quasi-perfect then the perfectness of $X$ follows just as in the proof of [6], Theorem 3.2, that perfectness of $H$ implies perfectness of $H \cup \{0\}$, only the argument with the Cauchy-Schwarz inequality has to be applied twice, first to get from the integral representation on $H+H+H+H$ ($\subset H+H+H$) to the integral representation on $H+H$ and a second time to get it on all of $H$. Q.E.D.

**Corollary 2.** *Every $*$-homomorphic image of a quasi-perfect $*$-semigroup is quasi-perfect.*
**Proof.** Suppose $h$ is a $*$-homomorphism of a quasi-perfect $*$-semigroup $H_1$ onto some $*$-semigroup $H_2$; we have to show that $H_2$ is quasi-perfect. For $i = 1,2$ let $S_i = H_i \cup \{0\}$ be the $*$-semigroup with zero obtained by adjoining to $H_i$ a zero external to $H_i$. By Corollary 1, $S_1$ is perfect. Extending $h$ to a $*$-homomorphism of $S_1$ onto $S_2$ by defining $h(0) = 0$, $S_2$ is a $*$-homomorphic image of the perfect $*$-semigroup $S_1$, hence perfect by [3], Theorem 1. By Corollary 1 it follows that $H_2$ is quasi-perfect. Q.E.D.

Suppose $(S_i)_{i \in I}$ is a family of $*$-semigroups with zero. The *direct sum* $S = \bigoplus_{i \in I} S_i$ is the set of all families $(s_i) \in \prod_{i \in I} S_i$ such that $s_i \neq 0$ for only finitely many $i \in I$. Addition and involution in $S$ are defined componentwise. It is known that if each $S_i$ is perfect, so is

$S$ ([3], Theorem 3). Now suppose $(H_i)_{i\in I}$ is a family of $*$-semigroups not necessarily having zeros. The *free sum* $H$ of the family $(H_i)$ is defined as follows. For $i \in I$ let $S_i = H_i \cup \{0\}$ be the $*$-semigroup with zero obtained by adjoining to $H_i$ a zero external to $H_i$. Define $S = \bigoplus_{i\in I} S_i$. Then $H = S \setminus \{0\}$.

**Corollary 3.** *The free sum of an arbitrary family of quasi-perfect $*$-semigroups is quasi-perfect.*
**Proof.** With notation as above, for $i \in I$, from the quasi-perfectness of $H_i$ it follows that $S_i$ is perfect (Corollary 1). By the result from [3] cited above it follows that $S$ is perfect. By Corollary 1 it follows that $H$ is quasi-perfect. Q.E.D.

A *face* of a $*$-semigroup $X$ is a $*$-subsemigroup $H$ of $X$ such that if $x, y \in X$ and $x+y \in H$ then $x, y \in H$.

**Corollary 4.** *Suppose $H$ is a face of a $*$-semigroup $X$. Then $X$ is quasi-perfect if and only if both $H$ and $X \setminus H$ are quasi-perfect.*
**Proof.** This is clear from Theorem 2, Corollary 2, Corollary 3, and [5], Theorem 2.1. Q.E.D.

**Corollary 5.** *Suppose $H$ is a $*$-semigroup and $S = H \cup \{0\}$. Under the assumption that $H$ is quasi-perfect, the following two conditions are equivalent:*

(i) *$H$ is perfect;*

(ii) *if $\varphi \in \mathcal{P}(H)$ and $h, k \in H$ then the measure $\mu_{h,k}$ on $S^*$ defined as in* [6], (3.1), *satisfies $\mu_{h,k}(\{1_{\{0\}}\}) = 0$.*

**Proof.** This is clear from [6], Proposition 3.5. Q.E.D.

A $*$-semigroup $H$ is *$*$-divisible* if for each $x \in H$ there exist $y \in H$ and $m, n \in \mathbb{N}_0$ with $m + n \geq 2$ such that $x = my + ny^*$. It is known that every $*$-divisible $*$-semigroup with zero is perfect ([3], Theorem 4).

**Corollary 6.** *Every $*$-divisible $*$-semigroup is perfect.*
**Proof.** Suppose $H$ is a $*$-divisible $*$-semigroup. The $*$-semigroup $S = H \cup \{0\}$ is a $*$-divisible $*$-semigroup with zero, hence perfect. By Corollary 1 it follows that $H$ is quasi-perfect. But the $*$-divisibility of $H$ implies that $H = H + H$, so $H$ is perfect. Q.E.D.

For a $*$-semigroup $S$ we denote by $\mathcal{J}(S)$ the set of all $*$-archimedean components of $S$. For every nonempty subset $H$ of $S$ there is a least face of $S$ containing $H$, viz., the intersection of all faces of $S$ containing $H$, the set of such faces being nonempty since $S$ itself is such a face. If $H$ is a $*$-subsemigroup of $S$ then the least face $X$ of $S$ containing $H$ is the set of those $x \in S$ such that $x + y \in H$ for some $y \in S$. If $H$ is a $*$-archimedean component of $S$ then $X$ is the set of those $x \in S$ such that $x + H \subset H$. In particular, $X + H \subset H$.

**Theorem 3.** *A $*$-semigroup $S$ satisfying $S = S+S$ is perfect if and only if each $*$-archimedean component of $S$ is quasi-perfect.*
**Proof.** First suppose $S$ is perfect and $H \in \mathcal{J}(S)$. Let $X$ be the least face of $S$ containing $H$. Being a face of the perfect semigroup $S$, $X$ is perfect ([5], Theorem 2.1). Moreover, $X+H \subset H$. By Theorem 2 it follows that $H$ is quasi-perfect. Conversely, suppose each $H \in \mathcal{J}(S)$ is quasi-perfect. Then for $H \in \mathcal{J}(S)$ the semigroup $H \cup \{0\}$ is perfect (Corollary 1). By [3], Theorem 3, it follows that the direct sum $R = \bigoplus_{H\in\mathcal{J}(S)}(H \cup \{0\})$ is perfect. Now $S \cup \{0\}$ is perfect, being the image of $R$ under the $*$-homomorphism $(s_H)_{H\in\mathcal{J}(S)} \mapsto \sum_{H\in\mathcal{J}(S)} s_H$ ([3], Theorem 1). Since $S = S + S$, by [6], Theorem 3.2, it follows that $S$ is perfect. Q.E.D.

In the proof of Theorem 3, we did not apply the hypothesis in the proof of "only if". However, for "if", that hypothesis is necessary. For example, if $S = H = \mathbf{Q} \cap [1, \infty[$ then $H$ is the unique $*$-archimedean component of $S$, $H$ is quasi-perfect, and yet $S$ is not perfect.

## 3. Reduction to the rational case

A $*$-semigroup $S$ is *rational* if $S$ is isomorphic to a subsemigroup of a rational vector space carrying the identical involution. The condition is equivalent to saying that $S$ carries the identical involution, is cancellative, and the group $S - S$ is torsion-free. For an arbitrary $*$-semigroup $S$, denote by $\overline{S}$ the greatest rational $*$-homomorphic image of $S$, that is, the pair $(\overline{S}, s \mapsto \overline{s})$ consisting of a rational semigroup $\overline{S}$ and a $*$-homomorphism $s \mapsto \overline{s}$ of $S$ onto $\overline{S}$ such that for every rational semigroup $T$ and every $*$-homomorphism $f: S \to T$ there is a unique homomorphism $h: \overline{S} \to T$ such that $f(s) = h(\overline{s})$ for all $s \in S$. (This property is what we mean by greatest.) To see that such a pair $(\overline{S}, s \mapsto \overline{s})$ exists, let $\sim$ be the least congruence relation in $S$ such that $s^* \sim s$ for all $s \in S$, define $T = S/\sim$, and let $f: S \to T$ be the quotient mapping. Then $T$ is the greatest identical-involution $*$-homomorphic image of $S$. Now let $(G, g)$ be the pair—unique up to isomorphism—consisting of an abelian group $G$ and a homomorphism $g: T \to G$ such that for every abelian group $H$ and every homomorphism $h: T \to H$ there is a unique homomorphism $k: G \to H$ such that $h = k \circ g$. This construction is well-known from algebra. The semigroup $g(T)$ generates $G$ as a group, and for $x, y \in T$ we have $g(x) = g(y)$ if and only if $a + x = a + y$ for some $a \in T$. Finally, let $H$ be $G$ modulo the torsion of $G$ and denote by $\phi: G \to H$ the quotient mapping. Then $\overline{S}$ can be identified with $\phi \circ g \circ f(S)$, at the same time identifying the mapping $s \mapsto \overline{s}$ with the mapping $\phi \circ g \circ f$.

**Theorem 4.** *A $*$-semigroup $S$ satisfying $S = S + S$ is perfect if and only if $\overline{H \cup \{0\}}$ is perfect for each $*$-archimedean component $H$ of $S$.*

**Proof.** See [2] for the definition of the concept "Stieltjes perfect". We have the chain of bi-implications :

$$\begin{aligned} S \text{ is perfect } &\Leftrightarrow H \text{ is quasi-perfect for all } H \in \mathcal{J}(S) \\ &\Leftrightarrow H \cup \{0\} \text{ is perfect for all } H \in \mathcal{J}(S) \\ &\Leftrightarrow \overline{H \cup \{0\}} \text{ is Stieltjes perfect for all } H \in \mathcal{J}(S) \\ &\Leftrightarrow \overline{H \cup \{0\}} \text{ is perfect for all } H \in \mathcal{J}(S). \end{aligned}$$

The first bi-implication is by Theorem 3, the second by Corollary 1, and the last two by [2]. Q.E.D.

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Nandrupsvej 7 st. th.
DK-2000 Frederiksberg C
Denmark
e-mail:torben.bisgaard@get2net.dk

Faculty of Engineering
Ibaraki University
Nakanarusawa
Hitachi 316-8511
Japan
e-mail:sakaki@base.ibaraki.ac.jp

# On some generalizations of the von Neumann-Jordan constant

YASUJI TAKAHASHI and MIKIO KATO

**Abstract.** We consider some generalizations of the von Neumann-Jordan constant of Banach spaces.

## 1. Von Neumann-Jordan constant

The *von Neumann-Jordan* (NJ-) *constant* of a Banach (or normed) space $X$ is defined by

$$C_{NJ}(X) = \sup_{\|x_1\|^2+\|x_2\|^2\neq 0} \frac{\|x_1+x_2\|^2+\|x_1-x_2\|^2}{\|x_1\|^2+\|x_2\|^2} \tag{1}$$

(Clarkson [2]; cf.[8]).

**Proposition A** (Jordan-von Neumann [3]) *For any Banach space $X$*

$$1 \leq C_{NJ}(X) \leq 2; \tag{2}$$

*$C_{NJ}(X) = 1$ if and only if $X$ is a Hilbert space.*

**Theorem 1** (Kato-Takahashi [6, 9]). (i) *$C_{NJ}(X) < 2$ if and only if $X$ is uniformly non-square* ([9]).

(ii) *$\tilde{C}_{NJ}(X) < 2$ if and only if $X$ is super-reflexive, where $\tilde{C}_{NJ}(X)$ denotes the infimum of all the NJ-constants of equivalent norms to the original one of $X$* ([6]).

**Theorem 2** (Kato-Takahashi [6]). *Let $C_{NJ}(X) \leq 2^{2/p-1}$, $1 < p \leq 2$. Then $X$ is of type $r$ and of cotype $r'$ for any $1 \leq r < p$. The converse is not true.*

**Theorem 3** (Kato-Takahashi [6]). *Let $X$ be a Banach lattice and let $1 < p \leq 2$. Then the following are equivalent.*

(i) $\tilde{C}_{NJ}(X) \leq 2^{2/p-1}$.

(ii) *$X$ is of type $r$ and of cotype $r'$. for any $1 \leq r < p$.*

(iii) *$X$ and $X'$ are of type $r$ for any $1 \leq r < p$.*

(iv) *$X$ and $X'$ are of cotype $r'$ for any $1 \leq r < p$.*

(v) *$X$ is $r$-convex and $r'$-concave for any $1 \leq r < p$.*

## 2. $n$-th von Neumann-Jordan constant

The *$n$-th von Neumann-Jordan constant* of $X$ is defined by

$$C_{NJ}^{(n)}(X) = \sup_{\sum_{j=1}^n \|x_j\|^2 \neq 0} \frac{\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2}{2^n \sum_{j=1}^n \|x_j\|^2} \tag{3}$$

(Kato, Takahashi and Hashimoto [7]).

**Theorem 4** (Kato-Takahashi-Hashimoto [7]).
(i) *For any Banach space $X$*

$$1 \leq C_{NJ}^{(n)}(X) \leq n; \tag{4}$$

*$C_{NJ}^{(n)}(X) = 1$ for some resp., any, $n \geq 2$ if and only if $X$ is a Hilbert space.*
(ii) *$C_{NJ}^{(n)}(X) < n$ if and only if $X$ is uniformly non-$\ell_1^n$.*
(iii) *Let $1 \leq p < 2$. Then $X$ has type larger than $p$ if and only if $C_{NJ}^{(n)}(X) < n^{2/p-1}$ for some $n \geq 2$.*

We refer to [7] for some further results.

## 3. $A$-von Neumann-Jordan constant

Let $A = (a_{ij})$ be an $m \times n$, $\pm 1$ matrix, $m$, $n \geq 2$, i.e., $a_{ij} = 1$ or $-1$. We define $A$-NJ constant by

$$C_A(X) = \sup_{\sum_{j=1}^n \|x_j\|^2 \neq 0} \frac{\sum_{i=1}^m \left\| \sum_{j=1}^n a_{ij} x_j \right\|^2}{m \sum_{j=1}^n \|x_j\|^2} \tag{5}$$

We consider the following $\pm 1$ matrices as our typical examples.
(i) *Littlewood matrices $L_n = (\epsilon_{ij})$*:

$$L_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad L_{n+1} = \begin{pmatrix} L_n & L_n \\ L_n & -L_n \end{pmatrix} \quad (n = 1, 2 \ldots)$$

(ii) *Rademacher matrices $R_n = (r_{ij}^{(n)})$* ($2^n \times n$ matrices; cf. [9]):

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left( \begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array} \right) \quad (n = 1, 2 \ldots)$$

(iii) *Admissible matrices* $A_n = (\alpha_{ij})$:

$$A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & -1 \\ 1 & 1 & \cdots & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & \cdots & -1 & -1 \end{pmatrix} \quad (n = 1, 2...)$$

**Remark.** (i) Let $A = L_1$ (the first Littlewood matrix). Then $C_{L_1}(X) = C_{NJ}(X)$.

(ii) Let $A = R_n$ (the $n$-th Rademacher matrix). Then $C_{R_n}(X) = C_{NJ}^{(n)}(X)$.

**Theorem 5.** *Let $A = (a_{ij})$ be an $m \times n$, $\pm 1$ matrix. Then, for any Banach space $X$ ($\neq \{0\}$)*

$$\max\{1, \frac{n}{m}\} \leq C_A(X) \leq n. \tag{6}$$

**Theorem 6.** *A Banach space $X$ is a Hilbert space if and only if for any, resp. some, $n$ there exists an $m \times n$, $\pm 1$ matrix $A$ such that $C_A(X) = \max\{1, n/m\}$.*

**Remark.** (i) If $A$ is the $n$-th Rademacher matrix $R_n$ and $X$ is a Hilbert space, then $C_{R_n}(X) = 1$.

(ii) Assume that $m \geq n$ and $C_A(X) = 1$. Then $m$ is even, and for every $1 \leq j, k \leq n$ ($j \neq k$) we have

$$\sum_{i=1}^{m} a_{ij} a_{ik} = 0.$$

For the case $m \leq n$ we have a similar result.

As the case where $C_A(X)$ has a non-trivial value we have the following:

**Theorem 7.** *Let $X$ be a Banach space.*

(i) *$X$ is B-convex if and only if $C_A(X) < n$ for some $m \times n$, $\pm 1$ matrix $A$.*

(ii) *$X$ is uniformly non-$\ell_1^n$ ($B_n$-convex) if and only if $C_{R_n}(X) < n$, where $R_n$ is the $n$-th Rademacher matrix.*

(iii) *$X$ is uniformly non-square if and only if $C_{L_1}(X) < 2$, where $L_1$ is the first Littlewood matrix.*

(iv) *$X$ is super-reflexive if and only if $C_{A_n}(X) < n$ for some $n$, where $A_n$ is the $n$-th admissible matrix.*

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*Department of System Engineering,*
*Okayama Prefectural University,*
*Soja 719-1197, Japan*
*e-mail: takahasi@cse.oka-pu.ac.jp*

*Department of Mathematics,*
*Kyushu Institute of Technology,*
*Kitakyushu 804-8550, Japan*
*e-mail: katom@tobata.isc.kyuech.ac.jp*

# Weighted Djokovic and the Reversed Inequalities

Yasuji Takahashi (Okayama Prefectural University)
Sin-Ei Takahasi (Yamagata University)
Shuhei Wada (Kisarazu National College of Technology)

Abstract. In [6], the present authors considered an extension of Hlawka's inequality on a Hilbert space H in an integral form, and gave the following inequality as an application of their extension.

(WDI) Let $x_1, x_2, \ldots, x_n \in H$ $(n \geqq 3)$ and $\mu_j \geqq 1$ $(j = 1, 2, \ldots, n)$. Then

$$\sum_{j=1}^{n} \mu_j \|x_j\| + \left(\sum_{j=1}^{n} \mu_j - 2\right) \left\|\sum_{j=1}^{n} \mu_j x_j\right\| \geqq \sum_{j=1}^{n} \mu_j \left\|x_j - \sum_{i=1}^{n} \mu_i x_i\right\|$$

This inequality may be interpreted as an weighted version of Djokovic's inequality. (If $\mu_j = 1$ for all j, then this is called Djokovic's inequality.) Here we consider an extension of (WDI) and the reversed inequality on a Banach space isometric to a subspace of $L^1$.

Banach空間Xのノルム‖ ‖に関する不等式には様々なものがあり、空間の幾何学的な性質を特徴づけるものもあれば、任意のノルムで成立する（自明な不等式という）ものもある。例えば、$L^p$空間$(1<p<\infty)$における Clarkson 不等式やその精密化とみなされるHanner不等式からは、$L^p$空間の一様凸性が導かれる。これらは２要素のノルム不等式であるために、$L^1$空間では自明な不等式となっている。実際、$L^1$空間における２要素のノルム不等式は、任意の実Banach空間のノルムについても成立する。そのことは、任意の２次元実ノルム空間は$L^1$の部分空間とisometric になるという事実から分かる（cf.[5]）。しかしながら、Hanner不等式を３要素に拡張すると事態は一変する(cf.[3])。実際、$L^1$空間における３要素Hanner不等式からHlawka不等式を導くことができる（逆も真）。（Djokovic不等式でn=3 のときをHlawka不等式という。）Hlawka不等式が成り立つノルム空間をHlawka空間と呼ぶことにすれば、Hlawka空間においてDjokovic不等式が成立することは数学的帰納法を用いて示される。ところで、

Hilbert空間HにおけるHlawka不等式は極めて巧妙に証明されるのであるが、ここで別の証明方法を紹介しよう。まず、1次元実ノルム空間（実数空間）の場合には、絶対値の場合分けを考えると、比較的容易に示されることは誰でも分かるであろう。また、(WDI) のような不等式は、あるノルム空間Eで成立するならば、明らかに$L^1(E)$あるいはそのすべての部分空間において成立する。任意の可分な$L^p$空間$(1<p\leqq 2)$は$L^1[0,1]$ の部分空間とisometric である(cf.[1])から、Hilbert 空間や$L^p$空間でHlawka不等式が成立することが分かる。（$p>2$ のときは、3次元以上の$L^p$空間は$L^1$の部分空間とisometric にはならないが、Hlawka空間でないかどうかは不明。ただし、$p$ がある程度大きいときは Hlawka 空間ではない。）

ここで、小論の目的である重みつきDjokovic不等式とその逆不等式を考えよう。以下、Eは$L^1$の部分空間とisometric 、$n \geqq 3$ とする。すでに述べた考察から、Eにおいて(WDI) が成立することが分かる。このとき、$\mu_j \geqq 1$ の条件が必要であることに注意しよう。このことは、$x_j \neq 0$, $x_i = 0$ $(i \neq j)$ として(WDI) を考えれば容易に分かる。では、$\mu_j < 1$ なるものがあるとき、不等式(WDI) はいかなる形で成立するであろうか。

定理1．$\mu_j > 0$ $(j=1,2,...,n)$ とする。任意の $x_1, x_2, ....., x_n \in E$ に対し次の不等式が成り立つ。

$$\sum_{j=1}^{n} \mu_j \|x_j\| + \left(\sum_{j=1}^{n} \mu_j - 2\mu\right) \left\|\sum_{j=1}^{n} \mu_j x_j\right\| \geqq \sum_{j=1}^{n} \mu_j \|x_j - \sum_{i=1}^{n} \mu_i x_i\|$$

ただし、$\mu = \min\{1, \mu_1, \mu_2, ....., \mu_n\}$。

注意．$\mu_j \geqq 1$ $(j=1,2,...,n)$ のとき、$\mu = 1$ は最良定数である。$\mu > 1$ で不等式が成立することはない。ところで、$\mu_j < 1$ なるものがあるとき、$\mu \leqq \mu_j$ が示されるので、定理1における$\mu$はいずれの場合にも最良定数である。定理1は(WDI) の一般化であるが、その証明は(WDI) を用いてなされる。実際、$0 < \mu < 1$ のときに証明すればよいから、$\mu_j / \mu = \nu_j$ とおくと $\nu_j \geqq 1$ となり、この$\nu_j$ に対して(WDI) を適用し少し変形した後に、$0 < \mu < 1$ に注意すると定理1が得られる。結局のところ、(WDI) が成立する空間において定理1が成立する。
（Hlawka空間において(WDI) が成立するか否かは、いまのところ分からない。）

なお、$\mu = 0$ とすれば任意のノルムについて定理１が成立することは明らかであろう。$E = L^p$のとき、$1 \leqq p \leqq 2$であれば定理１は成立するが、$2 < p < \infty$ のときに最良定数$\mu$を決定することは容易でない。

次に、(WDI) の逆向きの不等式を考えたい。通例、Clarkson不等式、Hanner不等式あるいはより一般のノルム不等式においても、Banach空間Ｘにおいてある不等式が成立するとき、その逆不等式はＸの dual space $X^*$ で考えるのが普通である。有名な例としては、typeとcotypeのduality がある。しかしながら、ここで考えている不等式(WDI) は$L^1$空間で成立するものであるから、その dual space では自明な不等式しか得られないであろう。

まず、(WDI) の逆不等式が成り立つためのweight$\mu_j$ に関する条件を考える。すべての$\mu_j$ が等しいとき、すなわち$\mu_1=\mu_2= \cdots = \mu_n= C > 0$ のとき、Ｅにおいて(WDI) の逆不等式が成り立つための必要十分条件は $C\leqq 1/(n-1)$ であることが示される。このとき、$\Sigma \mu_i - \mu_j \leqq 1$ $(j=1,2,\ldots,n)$であるが、実は、この条件のもとで(WDI) の逆不等式が成り立つことが示される。

定理２．$\nu_j > 0$ $(j=1,2,\ldots,n)$ とする。 $\sum_{i=1}^{n} \nu_i - \nu_j \leqq 1$ $(j=1,2,\ldots,n)$ ならば、任意の $x_1, x_2,\ldots, x_n \in E$に対し、次の不等式が成り立つ。

$$\sum_{j=1}^{n} \nu_j \|x_j\| + \left(\sum_{j=1}^{n} \nu_j - 2\right) \left\|\sum_{j=1}^{n} \nu_j x_j\right\| \leqq \sum_{J=1}^{n} \nu_j \left\|x_j - \sum_{i=1}^{n} \nu_i x_i\right\|$$

注意．$\sum_{j=1}^{n} \nu_j \leqq 1$ならば、上記の不等式は任意のBanach空間で成立することが示される。また、(WDI) が成立するようなBanach空間において定理２が成立することも示される（逆も真）。

最後に、定理１に対する逆不等式を与える。

定理３．$\nu_j > 0$ $(j=1,2,\ldots,n)$ とする。任意の $x_1, x_2,\ldots, x_n \in E$ に対し次の不等式が成り立つ。

$$\sum_{j=1}^{n} \nu_j \|x_j\| + \left(\sum_{j=1}^{n} \nu_j - 2\nu\right) \left\|\sum_{j=1}^{n} \nu_j x_j\right\| \leqq \sum_{j=1}^{n} \nu_j \left\|x_j - \sum \nu_i x_i\right\|$$

ただし、$\sum_{i=1}^{n} \nu_i - \nu_j \leqq \nu \quad (j=1,2,\ldots,n), \quad \nu \geqq 1$。

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# Pick function which are defined implicitly and extensions of Löwner-Heinz inequality and Furuta inequality

Mitsuru Uchiyama
Fukuoka Univ. of Education

For selfadjoint operators $A, B$ $A \leq B$ if $(Ax, x) \leq (Bx, x)$ for every $x$.

$f$ : real continuous on $I$ in $\mathbf{R}$. $f$ is called a *monotone operator (or operator monotone)* function
if $f(A) \leq f(B)$ for all $A, B$ on all Hilbert spaces such that $A \leq B$ and $\sigma(A), \sigma(B) \subseteq I$.
A holomorphic function which maps the upper half plane $\Pi_+$ into itself is called a **Pick function**; for instance,
$z^a$ $(0 < a < 1)$, $\log z$, $\tan z$, $-1/z$,
$((A - z)^{-1}x, x)$ $(x : vector)$, where $A$ is a selfadjoint operator.

**Löwner**
$f$ is a monotone operator function on an open interval $I$ if and only if $f$ has an analytic continuation $f(z)$ which is a Pick function

Especially if $f(t)$ is operator monotone on $(0, \infty)$

$$f(t) = a + bt - \int_0^\infty (\frac{1}{t+s} - \frac{s}{s^2+1}) dv(s) \text{ for } t > 0,$$

where $a$ and $b$ are real and $b \geq 0$, $v$ is a non-negative Borel measure such that

$$\int_0^\infty \frac{1}{1+s^2} dv(s) < \infty.$$

**Example**
$-1/(1+t)$, $t^\alpha (0 < \alpha \leq 1)$, $\log(1+t)$ are monotone operator functions on $[0, \infty)$.

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha \quad \text{for} \quad 0 < \alpha < 1,$$

that is,

$$A^p \geq B^p \quad \text{for} \quad p \geq 1, \quad A, B \geq 0 \quad \text{implies} \quad A \geq B, \tag{1}$$

which is called a **Löwner- Heinz inequality**.
But $A \geq B \geq 0$ does not generally imply $A^2 \geq B^2$; actually we have shown that if $A, B \geq 0$ and $(A + tB^n)^2 \geq A^2$ for every $t > 0$ and $n = 1, 2, \cdots$, then $AB = BA$ .

**Theorem. (A converse prop. of Löwner-Heinz theorem)**([19])
*Let $0 \leq A \leq B$, and let $B - A$ be of finite rank.*
*Then the following are equivalent:*
*(i) $A^s \leq B^s$ for some $s > 1$,*
*(ii) $\mathcal{N}(B - A)$ is invariant for $A$,*
*(iii) $\mathcal{N}(B - A)$ is invariant for $B$,*
*where $\mathcal{N}(T) := \{x : Tx = 0\}$.*
**Corollary.** *Let $0 \leq A$. Then the following are equivalent:*
*(i)$A^s \leq (A + x \otimes x)^s$ for some $s > 1$.*
*(ii) $A^s \leq (A + x \otimes x)^s$ for every $s > 1$.*
*(iii) $x$ is an eigenvector of $A$.*
**Example.** Let $A, B$ be $n \times n$ positive semi-definite matrices such that the all entries of $B - A$ are the same positive number. Then $A^s \leq B^s$ for some $s > 1$ if and only if $A^s \leq B^s$ for all $s > 1$.

Chan-Kwong had posed a conjecture:
*Does $A \geq B \geq 0$ imply $(BA^2B)^{1/2} \geq B^2$ ?*

**Furuta inequalities**

$$\begin{aligned} A \geq B \geq 0 \Longrightarrow \quad & (B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}, \\ & (A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}, \end{aligned} \tag{2}$$

where $r, p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

The essentially important part of Furuta inequlities are the case of $p > 1$ and $(1 + r)q = p + r$.

**Ando, Fujii-Furuta-Kamei** showed that for $p \geq 0,\ r \geq s \geq 0$

$$A \geq B \Longrightarrow \begin{array}{l} (e^{\frac{r}{2}B}e^{pA}e^{\frac{r}{2}B})^{\frac{s}{r+p}} \geq e^{sB} \\ e^{sA} \geq (e^{\frac{r}{2}A}e^{pB}e^{\frac{r}{2}A})^{\frac{s}{r+p}}. \end{array} \tag{3}$$

Recently we gave a simple proof of (3). The essentially important part of these inequalities are the case of $s = r$.

**Motivation**

$$A, B \geq 0, A^2 \geq B^2 \quad \text{implies} \quad (A+1)^2 \geq (B+1)^2,$$

because $A \geq B$ follows from (1).
But we can easily construct $2 \times 2$ matrices $A, B$ such that
$(A+1)^2 \geq (B+1)^2$, but $A^2 \ngeq B^2$;
for example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1.94 \end{pmatrix}.$$

The above results mean that $\phi(t) = (t^{1/2}+1)^2$ is operator monotone on $[0, \infty)$, but $\psi(t) = (t^{1/2}-1)^2$ is not on $[1, \infty)$. We may say that $\phi$ and $\psi$ are implicitely defined by $\phi(t^2) = (t+1)^2$ $(t \geq 0)$ and $\psi((t+1)^2) = t^2$ $(t \geq 0)$.

## 2. New operator monotone functions

In this paper we consider a function $u(t)$ represented as

$$u(t) = \prod_{i=1}^{k} (t + a_i)^{\gamma_i}, \quad (a_1 < a_2 < \cdots < a_k, \gamma_1 \geq 1, \gamma_i > 0) \tag{4}$$

We denote the restriction of $u(t)$ to $[-a_1, \infty)$ by $u_+(t)$, which is a non-negative increasing function.

**Theorem 2.1.**
Let us consider a function $s = u(t)$, where $u(t)$ is defined by (4). Then the inverse function $u_+^{-1}(s)$ is operator monotone on $[0, \infty)$.

**Example.**
(1). Set $u(t) = t^\gamma, (\gamma \geq 1)$. Then $u_+^{-1}(s) = s^{1/\gamma}$ is operator monotone (Löwner-Heinz).
(2). a map : $(t + n/\alpha)^n \to t$ is o.m. for $n \geq 1$.
Thus a map : $e^t \to t$ (i.e., $\log s$) is operator monotone .

To see the necessity of condition $\gamma_1 \geq 1$ we give
**Counter example.**

Set $u(t) = t^{1/2}(t+1)$. Then $u'(t) = \frac{1}{2}t^{-1/2}(3t+1)$.
$u''(t) = \frac{1}{4}t^{-3/2}(3t-1)$. $u''(t) < 0 \quad (0 < t < 1/3)$.
$(u_+^{-1})''(s) > 0 \quad (0 < s < \frac{4}{27})$.
$u_+^{-1}(s)$ is not operator monotone on $[0, \infty)$.

**Theorem 2.2.** Let $u(t)$ be represented as (4). Set

$$v(t) = \prod_{j=1}^{l}(t+b_j)^{\lambda_j}, \quad b_1 < b_2 < \cdots < b_l, 0 < \lambda_j \tag{5}$$

Then, if the following conditions

$$a_1 \leq b_1, \sum_{b_j < a_{i+1}} \lambda_j \leq \gamma_1 + \ldots + \gamma_i \text{ for } 1 \leq i \leq k, \quad \text{where } a_{k+1} = \infty, \tag{6}$$

are satisfied, a function $\phi$ defined on $[0, \infty)$ by

$$\phi(u(t)) = v(t) \quad (-a_1 \leq t), \quad \text{that is,} \quad \phi(s) = v(u_+^{-1}(s)) \quad (0 \leq s)$$

is an operator monotone function on $[0, \infty)$.

**Theorem 2.3.** Let $u(t), v(t)$ be functions defined by (4),(5). Suppose that condition (6) is satisfied. Then, if $0 \leq \beta \leq \alpha$, a function $\phi$ on $[0, \infty)$ defined by

$$\phi(u(t)e^{\alpha t}) = v(t)^r e^{\beta t} \quad (-a_1 \leq t < \infty, \quad 0 \leq r \leq 1))$$

is operator monotone on $[0, \infty)$.

**Corollary 2.4.** (1-parameter family) Let $u(t)$, $v(t)$ be functions given by (4),(5). Then $f(t) := v(t)^{\frac{1}{\Lambda}}$ is operator monotone, where $\Lambda = \lambda_1 + \ldots + \lambda_k$, and a function $\phi_r(s)$ on $[0, \infty)$ defined by

$$\phi_r(u(t)f(t)^r e^{\alpha t}) = f(t)^{c+r}$$

is operator monotone if $0 < r, \quad 0 \leq \alpha, \quad 0 \leq c \leq 1$.

$\varphi_r(s)$ defined by

$$\varphi_r(u(t)(v(t)e^{\alpha t})^r) = (v(t)e^{\alpha t})^r$$

is operator monotone.

## 3. Extension of Furuta inequality

**Lemma 3.1.** Let $\phi(t) \geq 0$ be an operator monotone function on $[0, \infty)$. Let $k(t)$ be a non-negative and increasing function on an interval $I \subseteq [0, \infty)$. Suppose

$$\phi(k(t)t) = t^2 \quad (t \in I).$$

Then

$$sp(A), sp(B) \subseteq I \quad A \geq B \implies \phi(B^{1/2}k(A)B^{1/2}) \geq B^2.$$

*Proof.* Let us assume that $sp(A), sp(B)$ are in the interior of $I$, that is, $A$ and $B$ are invertible. By making use of a connection $\sigma$ corresponding to $\phi$, we have

$$B^{-\frac{1}{2}}\phi(B^{\frac{1}{2}}k(A)B^{\frac{1}{2}})B^{-\frac{1}{2}} = B^{-1}\sigma k(A)$$

$$\geq A^{-1}\sigma k(A) = A^{-1}\phi(Ak(A)) = A \geq B.$$

**Lemma 3.2.** Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative functions on $[0, \infty)$. Let $f(t), h(t)$ be non-negative strictly increasing functions on an interval $J$. If for a fixed real number $c : 0 \leq c \leq 1$, the condition

$$\phi_r(h(t)f(t)^r) = f(t)^{c+r} \quad (t \in J; r > 0) \tag{7}$$

is satisfied, then

$$\phi_{c+2r}(s\phi_r^{-1}(s)) = s^2 \qquad (s \text{ in the range of } f(t)^{c+r}).$$

**Theorem 3.3.**(Essential inequality) Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$. Let $f(t), h(t)$ be non-negative strictly increasing functions on an interval $J$. Suppose that for a fixed $c$ condition (7) is satisfied and that $sp(A), sp(B) \subseteq J$. Then

$$f(A) \geq f(B) \implies \phi_r(f(B)^{r/2}h(A)f(B)^{r/2}) \geq f(B)^{c+r}. \tag{8}$$

*Proof.*For $0 < r \leq 1$ (8) follows in the same way as Lemma 3.1. We next assume (8) holds for all $r : 0 \leq r \leq n$. Take any $r : n < r \leq 2n$ and fix it. Because of $\frac{r-c}{2} \leq n$, we have

$$\phi_{\frac{r-c}{2}}(f(B)^{\frac{r-c}{4}}h(A)f(B)^{\frac{r-c}{4}}) \geq f(B)^{\frac{r+c}{2}}.$$

Here we simply denote the left hand side by $H$ and the right hand side by $K$; clearly $H \geq K$. Set $I := \{f(t)^{\frac{r+c}{2}} : t \in J\}$. Then $I \subseteq [0, \infty)$, $sp(K) \subseteq I$ and $sp(H) \subseteq I$. It follows from Lemma 3.2 that

$$\phi_r(s\phi_{\frac{r-c}{2}}^{-1}(s)) = s^2 \text{ for } s \in I.$$

Thus we can apply Lemma 3.1 to get

$$\phi_r(K^{1/2}\phi_{\frac{r-c}{2}}^{-1}(H)K^{1/2}) \geq K^2,$$

which means

$$\phi_r(f(B)^{\frac{r}{2}}h(A)f(B)^{\frac{r}{2}}) \geq f(B)^{c+r}. \qquad \square$$

The following is a generalization of Furuta inequality (2).

**Theorem 3.4.** Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$. Let $f(t), h(t)$ be non-negative strictly increasing functions on an interval $J$, and
let $f(t)$ operator monotone. If for a fixed $c$, condition (7) is satisfied then

$$\begin{aligned} sp(A), sp(B) \subseteq J,\ A \geq B \Longrightarrow \\ \phi_r(f(B)^{r/2}h(A)f(B)^{r/2}) \geq f(B)^{c+r}. \end{aligned} \tag{9}$$

We explain that the above theorem includes Furuta Inequality. Let $p \geq 1$, and define

$$f(t) = t, \quad h(t) = t^p \quad (0 \leq t < \infty).$$

Define a one-parameter family of operator monotone functions $\{\phi_r : r > 0\}$ by

$$\phi_r(t) = t^{\frac{1+r}{p+r}} \quad (0 \leq t < \infty).$$

Then

$$\phi_r(h(t)f(t)^r) = t^{1+r} = f(t)^{1+r}.$$

Thus (7) and other required conditions in Theorem 4.3 is satisfied. Therefore, from Theorem 4.3 it follows that

$$A \geq B \geq 0 \implies (B^{r/2}A^pB^{r/2})^{\frac{1+r}{p+r}} \geq B^{1+r}. \tag{10}$$

If $q(1+r) \geq p+r$, take $\lambda$ such that

$$\frac{1}{q} = \lambda\frac{1+r}{p+r}.$$

Then (10) implies

$$A \geq B \geq 0 \implies (B^{r/2}A^pB^{r/2})^{1/q} \geq B^{\frac{p+r}{q}}.$$

This is just the Furuta inequality.

**Corollary 3.5.**(See Cor. 2.4) Let $u(t)$, $v(t)$ be functions given by (4),(5). Set $f(t) = v(t)^{\frac{1}{\Lambda}}$, where $\Lambda = \lambda_1 + \ldots + \lambda_k$. Define a function $\phi_r(s)$ on $[0, \infty)$ by $\phi_r(u(t)f(t)^re^{\alpha t}) = f(t)^{c+r}$. If $0 < r, \quad 0 \leq \alpha, \quad 0 \leq c \leq 1$, then $A \geq B \geq -a_1 \implies$

$$\phi_r(f(B)^{\frac{r}{2}}u(A)e^{\alpha A}f(B)^{\frac{r}{2}}) \geq f(B)^{c+r}.$$

## 4. Extensions of exponential type operator inequality

**Theorem 4.1.** Let $\{\varphi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$. Let $f(t)$ and $h(t)$ be non-negative strictly increasing functions on an interval $J$. If $\log f(t)$ is a non-constant operator monotone function on the interior of $J$, and if the condition

$$\varphi_r(h(t)f(t)^r) = f(t)^r \quad (t \in J; r > 0) \tag{11}$$

is satisfied, then

$$\begin{aligned} &sp(A), sp(B) \subseteq J \text{ and } A \geq B \\ \Longrightarrow\ &\varphi_r(f(B)^{r/2}h(A)f(B)^{r/2}) \geq f(B)^r. \end{aligned} \tag{12}$$

**Corollary 5.2.** Let $0 \leq f(t)$ be a strictly increasing function on an interval $J$. If $\log f(t)$ is an operator monotone function on the interior of $J$, then for $r > 0, p > 0$

$$\begin{aligned} &sp(A), sp(B) \subseteq J, \quad A \geq B \Rightarrow \\ &(f(B)^{\frac{r}{2}}f(A)^p f(B)^{\frac{r}{2}})^{\frac{r}{p+r}} \geq f(B)^r. \end{aligned} \tag{13}$$

**Corollary 5.3.** (Concrete extension of (3)) If $\alpha, p, r > 0$, then $A \geq B \geq -a_1 \Rightarrow$

$$[(u(B)e^{\alpha B})^{\frac{r}{2}}\ (u(A)e^{\alpha A})^p\ (u(B)e^{\alpha B})^{\frac{r}{2}}]^{\frac{r}{p+r}} \geq (u(B)e^{\alpha B})^r.$$

**Corollary 5.4.** For $p \geq 1$ and $\alpha, \beta \geq 0$ let $\varphi_r(s)$ $(r > 0)$ be an operator monotone function on $[0, \infty)$ defined by $\varphi_r(u(t)v(t)^r e^{(\alpha+\beta r)}) = v(t)^r e^{\beta r}$ $(t \geq -a_1)$. Then $A \geq B \geq -a_1 \Rightarrow$

$$\varphi_r((v(B)e^{\beta B})^{\frac{r}{2}}\ (u(A)e^{\alpha A})\ (v(B)e^{\beta B})^{\frac{r}{2}}) \geq (v(B)e^{\beta B})^r.$$

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# On the characterization of $2 \times 2$ $\rho$-contraction matrices

Kazuyoshi Okubo (Hokkaido University of Education Sapporo)

Joint work with Ilya Spitkovsky (College of William and Mary)

**Abstruct**

We give an explicit description of all $\rho$-contractive (in Nagy-Foiaş sense) $2 \times 2$ matrices. This description leads to the formulas for $\rho$-numerical radii when the eigenvalues of such matrices either have equal absolute values or equal ( mod $\pi$) arguments. We also discuss (natural) generalizations to the case of decomposable operators with at most 2-dimensional blocks covering, in particular, the quadratic operators.

## 1. はじめに

$\rho > 0$ に対して $n \times n$ 複素行列 $A$ が $\rho$-縮小行列であるとは $\mathcal{K} \supset \mathbb{C}^n$ なるヒルベルト空間 $\mathcal{K}$ とその上のユニタリ作用素 $U$ があって、

$$A^n = \rho P U^n|_{\mathbb{C}^n} \quad (n = 1, 2, \cdots)$$

が成り立つこととする。ここで $P$ は $\mathcal{K}$ から $\mathbb{C}^n$ への直交射影作用素である。

この概念はS.-Z. Nagy and C. Foiaş によって導入された ([5,6] 参照)。そこで示されたように行列 $A$ が $\rho$-縮小であるための必要十分条件は $h \in \mathbb{C}^n$, $z \in \mathbb{D} = \{z: |z| < 1\}$ に対して、

$$(\rho - 2)||(I - zA)h||^2 + 2Re\langle (I - zA)h, h\rangle \geq 0 \tag{1.1}$$

である。条件 (1.1) から、$\sigma(A)$ を $A$ の固有値全体の集合とするとき、

$$A \text{ が } \rho - \text{縮小} \implies \sigma(A) \subset \overline{\mathbb{D}} \tag{1.2}$$

がいえる。Holbrook [3] によって、$n \times n$ 行列 $A$ に対して、$\rho$-半径 $w_\rho(A)$ が

$$w_\rho(A) = \inf\{r > 0 \ : \ \frac{1}{r}A \text{ が } \rho - \text{縮小行列}\} \tag{1.3}$$

によって定義された。(1.1) から、

$$w_\rho(U^*AU) = w_\rho(A) \text{ for any unitary } U \tag{1.4}$$

$$w_\rho(\xi A) = |\xi| w_\rho(A) \quad (\xi \in \mathbb{C})$$

がいえる。

Ando and Nishio[1] で、

$$w_\rho(A) \text{ は } \rho \in (0, \infty) \text{ に関して非増加関数} \tag{1.5}$$

$$\rho w_\rho(A) = (2 - \rho) w_{2-\rho}(A), \quad 0 < \rho < 2$$

が示された。

したがって、ここでは、$\rho \geq 1$ について考える。また、[3] で $w_1(A) = ||A||$, $w_2(A) = w(A)$ ($A$ の数域半径)、即ち、

$$w(A) = \sup\{|\langle Ah, h\rangle| \ : \ ||h|| = 1\}$$

そして、

$$w_\infty(A) := \lim_{\rho\to\infty} w_\rho(A) = r(A)$$

($r(A) = \max\{|\lambda|: \lambda \in \sigma(A)\}$ $A$ のスペクトル半径) が示されている。

## 2. $\rho$ 縮小 $2\times 2$ 行列

一般に $n \times n$ 行列が$\rho$-縮小行列であるかどうかを判定することは容易でないが、$2\times 2$ 行列に関しては次のことが分かっている (Nakazi and Okubo [7])。(1.2) 、(1.4) から、$2\times 2$ 行列は

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \tag{2.1}$$

$a, b \in \overline{\mathbb{D}}$ の形をしているとしてよい。

**Theorem.** $a, b \in \overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ とする。このとき、$\rho \geq 1$ とするとき、$A$ が $\rho$-縮小行列であるための必要十分条件は

$$|c|^2 + |a-b|^2 \leq \inf_{\zeta\in\mathbb{D}} \left|\frac{\{\rho + (1-\rho)\overline{a}\overline{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \overline{a}b|\zeta|^2}{\rho\zeta}\right|^2 \tag{2.2}$$

(2.2) の右辺の $\mathbb{D}$ は $\overline{\mathbb{D}}$ としてもよく、したがって、

$$(2.2)\text{ の右辺} = \min_{-\pi\leq\theta\leq\pi, 0\leq x\leq 1} |z(\theta) + \rho x^{-1} + \overline{a}b(\rho-2)x|^2$$

ただし $z(\theta) = (1-\rho)(be^{i\theta} + \overline{a}e^{-i\theta})$ となる。

**Problem.** 上の最小値は単位円周上でとるか、すなわち、 $|x| = 1$ で最小値をとるか。

このことは, 幾何学的には、原点を中心とする楕円 $E = \{z(\theta) \ : \ \pi \leq \theta \leq \pi\}$ を $H = \{\rho x^{-1} + \overline{a}b(\rho-2)x \ : \ x \in (0,1]\}$ に沿って動かすとき、原点からの最短距離が最後の楕円 ($x = 1$ のとき) 上にあるか、ということである。このことは肯定的で、次の結果を得る。

**Theorem 2.1.** $\rho \geq 1$ とする。このとき、$A$ が $\rho$-縮小行列であるための必要十分条件は $|a|, |b| \leq 1$ でかつ、

$$|c|^2 + |a-b|^2 \leq \min_{-\pi\leq\theta\leq\pi} |z(\theta) + \rho + \overline{a}b(\rho-2)|^2$$

ただし、 $z(\theta) = (1-\rho)(be^{i\theta} + \overline{a}e^{-i\theta})$ とする。

## 3. 固有値が円周上にあるとき

固有値が円周上にあるとき、即ち、$|a| = |b|$ $(= R)$ となるとき、$R = 0$ ならば、$A$ が $\rho$–縮小であることは、$|c| \leq \rho$ と同値であることが知られている ([1,3] 参照)。このことは $A = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$ $(\rho \geq 1)$ に対して、$w_\rho(A) = |c|/\rho$ と同値である。したがって、$R > 0$ の場合を考えればよい。

Theorem 1 から次のことがいえる。

**Theorem 3.1.** $\rho \geq 1$ とする。$|a| = |b|(= R) > 0$ のとき、$A$ が $\rho$-縮小行列であるための必要十分条件は $R \leq 1$ で

$$|c|^2 + |a-b|^2 \leq \frac{1}{4}(\rho R^{-1} - (\rho-2)R)^2|a-b|^2 + \left(\max\{0, \frac{1}{2}(\rho R^{-1} + (\rho-2)R)|a+b| - 2(\rho-1)R\}\right)^2$$

が成り立つことである。

Theorem 3.1 の Corollary として次のことがいえる。

**Corollary 3.2.** $\rho \geq 1$ とする。$|a| = |b|(= R) > 0$ のとき、

*(i)* $|c| \cdot |a+b| < (\rho-1)|a-b|^2$ ならば

$$w_\rho(A) = \frac{R}{\rho}\left(\sqrt{1 + \frac{|c|^2}{|a-b|^2}} + \sqrt{(\rho-1)^2 + \frac{|c|^2}{|a-b|^2}}\right) \tag{3.1}$$

*(ii)* その他のとき

$$w_\rho(A) = \frac{Q + \sqrt{Q^2 - 4\rho(\rho-2)R^2}}{2\rho} \tag{3.2}$$

ただし、$Q = |c| + (\rho-1)|a+b|$ である。

特に$\rho = 2$ のときは、(3.1),(3.2) は Johnson, Spitkovsky and Gottlieb [4] によって得られており、次のようになる。

$$w_2(A) = \begin{cases} \dfrac{R}{|a-b|}\sqrt{|c|^2 + |a-b|^2} & \text{if } |c| \cdot |a+b| < |a-b|^2 \\ \dfrac{|c| + |a+b|}{2} & \text{その他} \end{cases}$$

## 4. 固有値が直線上にあるとき

**Theorem 4.1.** $\rho \geq 1$ として、$\overline{a}b \in \mathbb{R}$ とする。このとき、$A$ が $\rho$-縮小行列であるための必要十分条件は $|a|, |b| \leq 1$ で

$$|c|^2 + |a-b|^2 \leq (\rho + (\rho-2)\overline{a}b - (\rho-1)|a+b|)^2$$

が成り立つことである。

このことから、次の計算ができる。

**Corollary 4.2.** $\rho \geq 1$ として、$\overline{a}b \in \mathbb{R}$ とする。このとき、

$$w_\rho(A) = \frac{P + \sqrt{P^2 - 4\rho(\rho-2)\overline{a}b}}{2\rho}$$

ただし、$P=(\rho-1)|a+b|+\sqrt{|c|^2+|a-b|^2}$ である。

$\rho=2$ のとき、

$$w_2(A)=\frac{|a+b|+\sqrt{|c|^2+|a-b|^2}}{2}$$

となるが、このことは Johnson, Spitkovsky and Gottlieb [4] によって知られている。

## 5. 一般化

2-4 章はユニタリー不変の形で書くことができる。実際、(2.1) の形の行列 $A$ に対して次のことが一般的にいえる。

$$|a|^2+|b|^2+|c|^2=||A||^2+\frac{|ab|^2}{||A||^2} \tag{5.1}$$

したがって、(2.2) は

$$||A||^2+\frac{|ab|^2}{||A||^2}-2Re(\overline{a}b)\leq \min_{-\pi\leq\theta\leq\pi}|z(\theta)+\rho+\overline{a}b(\rho-2)|^2 \tag{5.2}$$

となり、仮に $\overline{a}b\in\mathbb{R}$ ならば、(5.2) は

$$||A||-\frac{\overline{a}b}{||A||}\leq\rho+(\rho-2)\overline{a}b-(\rho-1)|a+b|$$

となる。また、$|a|=|b|=R$ のとき、(5.1) から、$|c|=||A||-\dfrac{R^2}{||A||}$ となり、(3.1),(3.2) はこれを使って表すことができる。これらの話を、より一般的な形の行列に拡張しよう。

行列 $A$ がある複素数 $p,q$ で

$$A^2+pA+qI=0 \tag{5.3}$$

を満たすとき、$A$ は quadratic 行列とよばれる。quadratic 行列 $A$ に対して次のことがいえる。

**Theorem 5.1.** $A(\neq 0)$ を *quadratic* 行列として、$a$, $b$ を $z^2+pz+q=0$ の根とする。このとき、$\rho\geq 1$ に対して、$A$ が $\rho$-縮小行列であるための必要十分条件は $|a|,|b|\leq 1$ で

$$||A||^2+\frac{|q|^2}{||A||^2}\leq 2\mathrm{Re}(\overline{a}b)+\min_{-\pi\leq\theta\leq\pi}|\rho+\overline{a}b(\rho-2)-(\rho-1)(be^{i\theta}+\overline{a}e^{-i\theta})|^2$$

となることである。

(5.3) で係数 $p,q$ が実数ならば、その根 $a,b$ は実数か複素共役だから、3,4 章のことを一般化して次の$\rho$−半径 $w_\rho(\cdot)$ の式を得る。

**Theorem 5.2.** *$A$ ($\neq 0$) を (5.3) の係数 $p, q$ が実数である quadratic 行列とする。このとき、$\rho \geq 1$ に対して、*

*(i)* $p^2 \geq 4q$ または $p^2 \leq \left(||A|| + \frac{q}{||A||}\right)^2 - (\rho-1)^2\left(\frac{4q}{p} - p\right)^2$ のとき

$$w_\rho(A) = \frac{S_\rho + \sqrt{S_\rho^2 - 4\rho(\rho-2)q}}{2\rho} \tag{5.4}$$

ここで

$$S_\rho = (\rho-1)|p| + ||A|| - \frac{q}{||A||}$$

とする。

*(ii)* *(i)* 以外のとき、

$$w_\rho(A) = \frac{\sqrt{q}}{\rho}\left(\sqrt{1+T_\rho} + \sqrt{(\rho-1)^2 + T_\rho}\right) \tag{5.5}$$

ただし、

$$T_\rho = \frac{\left(||A|| - \frac{q}{||A||}\right)^2}{4q - p^2}$$

とする。

$S_\rho = (\rho-2)|p| + S_2$ だから、

$$p^2 \geq 4q \text{ または } p^2 \leq \left(||A|| + \frac{q}{||A||}\right)^2 - \max\{1, (\rho-1)^2\}\left(\frac{4q}{p} - p\right)^2$$

をみたす $p, q$ に対して

$$w_\rho(A) = \frac{w_2(A) + \frac{\rho-2}{2}|p| + \sqrt{\left(w_2(A) + \frac{\rho-2}{2}|p|\right)^2 - \rho(\rho-2)q}}{\rho}.$$

となる。特に

(i) $A^2 = sI,\ s \geq 0$ のとき

$$w_\rho(A) = \frac{w_2(A) + \sqrt{w_2(A)^2 + \rho(\rho-2)s}}{\rho}$$

(ii) $A^2 = pA (p \in \mathbb{R})$ のとき

$$w_\rho(A) = \frac{2w_2(A) + (\rho-2)|p|}{\rho} = \frac{||A|| + (\rho-1)|p|}{\rho}$$

がいえる。最後の２つの式は $s = 1$, $p = 1$ として Ando and Nishio [1] で得られている。

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# REVERSE WEIGHTED $L_p$–NORM INEQUALITIES IN CONVOLUTIONS AND STABILITY IN INVERSE PROBLEMS

Saburou Saitoh[1], Vu Kim Tuan[2] *and* Masahiro Yamamoto [3]

[1] *Department of Mathematics, Faculty of Engineering*
*Gunma University, Kiryu 376-8515, Japan*
*ssaitoh@eg.gunma-u.ac.jp*
[2] *Department of Mathematics and Computer Science*
*Faculty of Science, Kuwait University*
*P.O. Box 5969, Safat 13060, Kuwait*
*vu@sci.kuniv.edu.kw*
[3] *Department of Mathematical Sciences*
*The University of Tokyo*
*3-8-1 Komaba, Tokyo 153-8914, Japan*
myama@ms.u-tokyo.ac.jp

For the Fourier convolution, the Young's inequality

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q \tag{1}$$

for $f \in L_p(R^n), g \in L_q(R^n)$ and for $r^{-1} = p^{-1} + q^{-1} - 1 \quad (p, q, r > 0)$, is fundamental. Note, however, that for the typical case of $f, g \in L_2(R^n)$, inequality (1) is not valid. In a series of papers [4,5,6] (see also [1]) the first author obtained the following weighted $L_p(p > 1)$ inequality for convolution

**Proposition 1***([7]). For two nonvanishing continuous functions $\rho_j \in L_1(R) \quad (j = 1, 2)$ the following $L_p(p > 1)$ weighted convolution inequality*

$$\left\| \left( (F_1\rho_1) * (F_2\rho_2) \right) (|\rho_1| * |\rho_2|)^{\frac{1}{p}-1} \right\|_p \leq \|F_1\|_{L_p(R,|\rho_1|)} \, \|F_2\|_{L_p(R,|\rho_2|)} \tag{2}$$

*holds for $F_j \in L_p(R, |\rho_j|)(j = 1, 2)$. Equality holds if and only if*

$$F_j(x) = C_j e^{\alpha x}, \tag{3}$$

*where $\alpha$ is a constant such that $e^{\alpha x} \in L_p(R, |\rho_j|) \quad (j = 1, 2)$.*

Unlike the Young's inequality, inequality (2) holds also in case $p = 2$.

In many cases of interest, the convolution is given in the form

$$\rho_2(x) \equiv 1, \quad F_2(x) = G(x), \tag{4}$$

where $G(x - \xi)$ is some Green's function. Then inequality (2) takes the form

$$\| (F\rho) * G\|_p \leq \|\rho\|_p^{1-\frac{1}{p}} \, \|F\|_{L_p(R,|\rho|)} \, \|G\|_p, \tag{5}$$

where $\rho, F$, and $G$ are such that the right hand side of (5) is finite.

Inequality (5) enables us to estimate the output function

$$\int_{-\infty}^{\infty} F(\xi)\rho(\xi)G(x - \xi) \, d\xi \tag{6}$$

in terms of the input function $F$. In this paper we are interested in the reverse type inequality for (5), namely, we wish to estimate the input function $F$ by means of the output (6). This kind of estimates is important in inverse problems. Our estimate is based on the following reverse Hölder inequality

**Proposition 2** *([2], see also [3], pages 125-126). For two positive functions $f$ and $g$ satisfying*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{7}$$

*on the set $X$, and for $p, q > 0$, $p^{-1} + q^{-1} = 1$,*

$$\left(\int_X f d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \leq A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{8}$$

*if the right hand side integral converges. Here*

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{\frac{-1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

A general reverse weighted $L_p$ convolution inequality is given as follows:

**Theorem 1** *Let $F_1$ and $F_2$ be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leq F_1(x) \leq M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leq F_2(x) \leq M_2^{\frac{1}{p}} < \infty, \quad x \in R. \tag{9}$$

*Then for any two positive functions $\rho_1$ and $\rho_2$ belonging to $L_1(R)$ we have the reverse $L_p$-weighted convolution inequality*

$$\left\| \left((F_1\rho_1) * (F_2\rho_2)\right)(\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \geq \left\{ A_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right) \right\}^{-1} \|F_1\|_{L_p(R,\rho_1)} \|F_2\|_{L_p(R,\rho_2)}. \tag{10}$$

Inequality (10) and others should be understood in the sense that if the left hand side is finite, then so is the right hand side, and in this case the inequality holds.

Let $\rho_2 \equiv 1$, and $F_2(x-\xi) = G(x-\xi)$ be a certain Green's function. In formula (10) taking integration with respect to $x$ from $c$ to $d$ we arrive at the following inequality

$$\begin{aligned} &\int_c^d \left(\int_{-\infty}^{\infty} F(\xi)\,\rho(\xi)\,G(x-\xi)\,d\xi\right)^p dx \\ \geq &\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p} \left(\int_{-\infty}^{\infty} \rho(\xi)\,d\xi\right)^{p-1} \int_{-\infty}^{\infty} F^p(\xi)\,\rho(\xi)\,d\xi \int_{c-\xi}^{d-\xi} G^p(x)dx, \end{aligned} \tag{11}$$

for positive continuous functions $\rho$, $F$, and $G$, satisfying

$$0 < m^{\frac{1}{p}} \leq F(\xi)G(x-\xi) \leq M^{\frac{1}{p}}, \quad x \in [c,d], \quad \xi \in R. \tag{12}$$

Examples:

The first order differential equation. The solution $y(x)$ of the first order differential equation

$$y'(x) + \lambda\, y(x) = F(x), \quad y(0) = 0, \quad (\lambda > 0)$$

is represented in the form

$$y(x) = \int_0^x F(t) e^{-\lambda(x-t)} dt.$$

So we shall consider the integral transform

$$f(x) = \int_0^x F(t)\,\rho(t) e^{-\lambda(x-t)} dt.$$

The condition (12) reads

$$0 < m^{\frac{1}{p}} \leq F(t) e^{-\lambda(x-t)} \leq M^{\frac{1}{p}}. \tag{13}$$

It will be satisfied for $0 \le t \le x \le d < \infty$, if we have

$$0 < m^{\frac{1}{p}} e^{\lambda d - \lambda t} \le F(t) \le M^{\frac{1}{p}} e^{-\lambda t}, \qquad 0 < d < \frac{1}{p\lambda} \log \frac{M}{m}. \tag{14}$$

Take

$$G(x) = \begin{cases} e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases},$$

then we have

$$\int_{c-\xi}^{d-\xi} G^p(x)\, dx = \begin{cases} \frac{e^{-\lambda pc} - e^{-\lambda pd}}{\lambda p} e^{\lambda p \xi}, & \xi < c \\ \frac{1 - e^{-\lambda pd}}{\lambda p} e^{\lambda p \xi}, & \xi > c \end{cases}.$$

Thus the inequality (11) yields

$$\begin{aligned} &\int_c^d f^p(x) \left( \int_0^x \rho(t)\, dt \right)^{1-p} dx \\ \ge &\left\{ A_{p,q} \left( \frac{m}{M} \right) \right\}^{-p} \frac{1}{\lambda p} \left[ (e^{-\lambda pc} - e^{-\lambda pd}) \int_0^c F^p(\xi) \rho(\xi) e^{\lambda p \xi} d\xi \right. \\ &\left. + \int_c^d F^p(\xi) \rho(x) (1 - e^{-\lambda pd} e^{\lambda p \xi}) d\xi \right]. \end{aligned} \tag{15}$$

Here we assume that $\rho$ is a positive continuous function on $[0, d]$, and $F$ satisfies (14).

Picard transform. Note that $\frac{1}{2} e^{-|x-t|}$ is the Green's function for the boundary value problem

$$y'' - y = 0, \qquad \lim_{x \to \pm\infty} y(x) = 0.$$

So, we shall consider the Picard transform

$$f(x) = \frac{1}{2} \int_{-\infty}^{\infty} F(t) \rho(t) e^{-|x-t|} dt.$$

Since

$$e^{-a} e^{|t|} \le e^{|x-t|} \le e^a e^{|t|}, \quad |x| \le a,$$

we see that the condition (12)

$$0 < m^{\frac{1}{p}} \le F(t) e^{-|x-t|} \le M^{\frac{1}{p}}, \tag{16}$$

is valid if

$$0 < m^{\frac{1}{p}} e^a e^{|t|} \le F(t) \le M^{\frac{1}{p}} e^{-a} e^{|t|}, \quad t \in R, \qquad 0 < a < \frac{1}{2p} \log \frac{M}{m}. \tag{17}$$

We have

$$\int_{c-t}^{d-t} e^{-p|x|}\, dx = \begin{cases} \frac{e^{pt}}{p} \left[ e^{-pc} - e^{-pd} \right], & t < c \\ \frac{e^{-pt}}{p} \left[ e^{pd} - e^{pc} \right], & t > d \\ \frac{e^{pt-pc} + e^{pd-pt} - 2}{p}, & c < t < d \end{cases}.$$

Thus, for $-a \le c, d \le a$ the inequality (11) yields

$$\begin{aligned} &\int_c^d f^p(x)\, dx \ge \frac{1}{2^p p} \left( \int_{-\infty}^{\infty} \rho(t) dt \right)^{p-1} \left\{ A_{p,q} \left( \frac{m}{M} \right) \right\}^{-p} \\ &\left[ \left( e^{-pc} - e^{-pd} \right) \int_{-\infty}^{c} F^p(t) \rho(t) e^{pt} dt + \left( e^{pd} - e^{pc} \right) \int_d^{\infty} F^p(t) \rho(t) e^{-pt} dt \right. \\ &\left. + \int_c^d F^p(t) \rho(t) \left( e^{pt-pc} + e^{pd-pt} - 2 \right) dt \right], \end{aligned} \tag{18}$$

for a positive continuous function $\rho$, and for a function $F$ satisfying (17).

Heat equation. We consider the Weierstrass transform

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\xi)\rho(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi, \tag{19}$$

which gives the formal solution $u(x,t)$ of the heat equation

$$u_t = \Delta u \quad \text{on} \quad R_+ \times R,$$

subject to the initial condition

$$u(x,0) = F(x)\rho(x) \quad \text{on} \quad R.$$

Let

$$x \in [-a,a], \quad \xi \in [-b,b], \quad a+b \le \sqrt{\frac{4t}{p} \log \frac{M}{m}}.$$

From

$$1 \le \exp\left(\frac{(x-\xi)^2}{4t}\right) \le \exp\left(\frac{(a+b)^2}{4t}\right),$$

we have

$$0 < m^{\frac{1}{p}} \le F(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) \le M^{\frac{1}{p}},$$

if

$$m^{\frac{1}{p}} \exp\left(\frac{(a+b)^2}{4t}\right) \le F(\xi) \le M^{\frac{1}{p}}. \tag{20}$$

We have

$$\int_{c-\xi}^{d-\xi} e^{-\frac{px^2}{4t}} dx = \sqrt{\frac{\pi t}{p}} \left[\operatorname{erf}\frac{\sqrt{p}\,(d-\xi)}{2\sqrt{t}} - \operatorname{erf}\frac{\sqrt{p}\,(c-\xi)}{2\sqrt{t}}\right].$$

Therefore, for $-a \le c < d \le a$, the inequality (11) yields

$$\int_c^d u(x,t)^p dx \ge \frac{1}{2^p(\pi t)^{(p-1)/2}\sqrt{p}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p} \left(\int_{-b}^{b} \rho(\xi) d\xi\right)^{p-1}$$
$$\int_{-b}^{b} F^p(\xi)\rho(\xi) \left[\operatorname{erf}\frac{\sqrt{p}\,(d-\xi)}{2\sqrt{t}} - \operatorname{erf}\frac{\sqrt{p}\,(c-\xi)}{2\sqrt{t}}\right] d\xi, \tag{21}$$

for a positive continuous function $\rho$ on $[-b,b]$, and for a function $F$ satisfying (20).

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# Weak and Strong Convergence Theorems in Banach Spaces with Applications

Wataru Takahashi

Tokyo Institute of Technology
Department of Mathematical and Computing Sciences
Ohokayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail: wataru@is.titech.ac.jp

**Abstract** In this article, we first prove weak and strong convergence theorems for resolvents of accretive operators in Banach spaces. Next, we prove weak and strong convergence theorems for finite families of nonexpansive mappings in Banach spaces. Finally, using these results, we consider the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function and the feasibility problem by convex combinations of nonexpansive retractions.

## 1 はじめに

$H$ を Hilbert 空間とし，$g$ を $H$ から $(-\infty,\infty]$ に値をとる proper で凸な下半連続関数とする．このとき，

$$\min\{g(x) : x \in H\} \tag{1}$$

という凸最小化問題を考える．このような $g$ に対して，$H$ 上の集合値写像 $\partial g$ を，$x \in H$ に対して，

$$\partial g(x) = \{x^* \in H : g(y) \geqq g(x) + (x^*, y-x),\ y \in H\}$$

で定義し，これを $g$ の劣微分と呼ぶ．$H$ 上の集合値写像 $A \subset H \times H$ は，任意の
$(x_1, y_1), (x_2, y_2) \in A$ に対して，

$$(x_1 - x_2,\ y_1 - y_2) \geqq 0$$

を満たすならば，増大であるといわれる．$A$ が増大であるならば，$\lambda > 0$ に対して，$A$ の resolvent が

$$J_\lambda = (I + \lambda A)^{-1}$$

で定義される．増大写像 $A$ が，すべての $\lambda > 0$ に対して，$R(I+\lambda A) = H$ を満たすならば，$m-$ 増大といわれる．ただし，$R(I+\lambda A)$ は $I+\lambda A$ の値域を表す．proper で凸な下半連続関数 $g : H \to (-\infty,\infty]$ に対して，その劣微分 $\partial g$ は $m-$ 増大になることが知られている．

(1) の解を求めるよく知られた方法として，Martinet[9] によって導入された proximal point algorithm というものがある．このアルゴリズムは，resolvent $J_\lambda$ に関係がある．すなわち，

$$J_\lambda x = \operatorname{argmin}\left\{g(z) + \frac{1}{2\lambda}\|z-x\|^2 : z \in H\right\}$$

である（Moreau[10] を参照せよ）．proximal point algorithm とは，$\{\lambda_n\} \subset (0,\infty)$ とするとき，$x_0 \in H$ を初期点とし，

$$x_{n+1} = J_{\lambda_n} x_n \quad (n = 0, 1, 2, \ldots)$$

で帰納的に点列 $\{x_n\}$ を生成し，(1) の解を求める点列的構成法のことである（Rockafellar[11] を参照せよ）．我々はまた制約可能性問題というつぎの問題を知っている．$\{g_1, g_2, \ldots, g_n\}$ を $H$ から $(-\infty, \infty]$ に値をとる proper で凸な下半連続関数とする．このとき，

$$g_i(x) \leq 0, \quad i = 1, 2, \ldots, n$$

を満たす $x \in H$ を求めることである．

一方，我々は，非拡大写像 $T$ の 2 つの不動点近似法を知っている．Halpern[3] によって導入された点列的近似法

$$x_0 = x \in H,\ x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \quad (n = 0, 1, 2, \ldots)$$

と，あとは Mann[8] によって導入された

$$x_0 = x \in H,\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \quad (n = 0, 1, 2, \ldots)$$

の近似法である．ただし，$\{\alpha_n\} \subset [0, 1]$ である．

ここでは，Halpern と Mann によって導入された点列的不動点近似法を用いて，まず初めに，Banach 空間における増大作用素の resolvents に対する弱及び強収束定理を証明することである．この後，有限個の非拡大写像に対する弱及び強収束定理を証明する．最後に，これらの結果を用いて，proper で凸な下半連続関数の minimizer を求める凸最小化問題と，非拡大レトラクトの凸結合によって解かれる制約可能性問題を考える．

## 2 準備

$E$ を Banach 空間とし，$E^*$ をその共役空間とする．$x \in E$ における $x^* \in E^*$ の値を $x^*(x)$ または $(x, x^*)$ で表す．$E$ における点列 $\{x_n\}$ が $x$ に強収束することを $x_n \to x$ で表し，弱収束することを $x_n \rightharpoonup x$ で表す．

$E$ の凸性の modulus $\delta$ は，$0 \leqq \varepsilon \leqq 2$ となる $\varepsilon$ に対して

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leqq 1,\ \|y\| \leqq 1,\ \|x - y\| \geqq \varepsilon\right\}$$

で定義される．Banach 空間 $E$ が一様凸であるとは，$\varepsilon > 0$ に対して，$\delta(\varepsilon) > 0$ がつねに成り立つときをいう．$E$ の元 $x$ に対して，

$$J(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$$

が定義されるが，この $J$ を $E$ 上の duality 写像という．$U = \{x \in E : \|x\| = 1\}$ としよう．このとき，$x, y \in U$ に対して，極限

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

を考えよう．$E$ のノルムが Gâteaux 微分可能であるとは，任意の $x, y \in U$ に対して，(2) がつねに存在するときをいう．$E$ のノルムが一様に Gâteaux 微分可能であるとは，任意の $y \in U$ に対して，(2) が $x \in U$ に関して一様に収束するときをいう．$E$ のノルムが Fréchet 微分可能であるとは，任意の $x \in U$ に対して，(2) が $y \in U$ に関して一様に収束するときをいう．$E$ が Gâteaux 微分可能なノルムをもてば，$E$ 上の duality 写像は一価写像になる．

$E$ を Banach 空間とし，$A \subset E \times E$ としよう．$A$ が増大作用素（accretive operator）であるとは，$(x_1, y_1), (x_2, y_2) \in A$ に対して，つねに $(y_1 - y_2, j) \geqq 0$ となる $j \in J(x_1 - x_2)$ が存在するときをいう．ただし，$J$ は $E$ の duality 写像である．$E$ を Banach 空間とし，$A \subset E \times E$ を増大作用素とする．このとき，すべての $\lambda > 0$ に対して $\overline{D(A)} \subset R(I + \lambda A)$ が成立するならば，$A$ は値域条件（range condition）を

満たすといわれる．このとき，$A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$ と $A$ の resolvent $J_r$ の不動点の集合 $F(J_r)$ の間には

$$F(J_r) = A^{-1}0$$

という関係がある．

## 3 Resolvents の収束定理

この節では，Halpern と Mann の不動点近似法のアイディアを用いて，resolvents の収束定理を述べる．

**定理 3.1**[6]　$E$ を一様 Gatêaux 微分可能なノルムをもつ一様凸な Banach 空間とし，$A \subset E \times E$ を値域条件を満たす増大作用素とする．$C$ を $E$ の空でない閉凸集合で

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$$

を満たすものとする．$x_0 = x \in C$ とし，

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n \qquad (n = 0, 1, 2, \ldots)$$

とする．ただし $\{\alpha_n\} \subset [0,1]$ と $\{r_n\} \subset (0,\infty)$ は $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ , $\lim_{n\to\infty} r_n = \infty$ を満たすものとする．このとき，$A^{-1}0 \neq \phi$ であるならば，$\{x_n\}$ は $A^{-1}0$ の元 $u$ に強収束する．ここで，$Px = u$ とおくと，$P$ は $C$ から $A^{-1}0$ の上への sunny nonexpansive retraction である．

**定理 3.2**[6]　$E$ を Fréchet 微分可能なノルムをもつ一様凸な Banach 空間とし，$A \subset E \times E$ を値域条件を満たす増大作用素とする．$C$ を $E$ の空でない閉凸集合で

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$$

を満たすものとする．$x_0 = x \in C$ とし，

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n \qquad (n = 0, 1, 2, \ldots)$$

とする．ただし，$\{\alpha_n\} \subset [0,1]$ と $\{r_n\} \subset (0,\infty)$ は $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} r_n > 0$ を満たすものとする．このとき，$A^{-1}0 \neq \phi$ であるならば，$\{x_n\}$ は $A^{-1}0$ のある元 $v$ に弱収束する．

定理 3.1 および定理 3.2 の直接的結果としてつぎの 2 つの定理が得られる．

**定理 3.3**　$H$ を Hilbert 空間とし，$A : H \to 2^H$ を極大単調作用素とする．$x_0 = x \in H$ とし，

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n \quad (n = 0, 1, 2, \ldots)$$

とする．ただし，$\{\alpha_n\} \subset [0,1]$ と $\{r_n\} \subset (0,\infty)$ は $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} r_n = \infty$ を満たすものとする．このとき，$A^{-1}0 \neq \phi$ であるならば，$\{x_n\}$ は $A^{-1}0$ の元 $u$ に強収束する．ここで，$Px = u$ とおくと．$P$ は $H$ から $A^{-1}0$ の上への metric projection である．

**定理 3.4**　$H$ を Hilbert 空間とし，$A : H \to 2^H$ を極大単調作用素とする．$x_0 = x \in H$ とし，

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n \quad (n = 0, 1, 2, \ldots)$$

とする．ただし，$\{\alpha_n\} \subset [0,1]$ と $\{r_n\} \subset (0,\infty)$ は $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} r_n > 0$ を満たすものとする．このとき，$A^{-1}0 \neq \phi$ であるならば，$\{x_n\}$ は $A^{-1}0$ の元 $u$ に弱収束する．

定理 3.3 および定理 3.4 を用いて，凸最小化問題の解を求める proximal point algorithm を議論することができる．

**定理 3.5** $H$ を Hilbert 空間とし，$f: H \to (-\infty, \infty]$ を proper で下半連続な凸関数とする．$x_0 = x \in H$ とし，

$$y_n = \arg\min_{z \in H} \left\{ f(z) + \frac{1}{2r_n}\|z - x_n\|^2 \right\},$$
$$x_{n+1} = \alpha_n x + (1-\alpha_n) y_n \quad (n = 0, 1, 2, \ldots)$$

とする．ただし，$\{\alpha_n\} \subset [0,1]$ と $\{r_n\} \subset (0,\infty)$ は $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} r_n = \infty$ を満たすものとする．このとき，$(\partial f)^{-1}0 \neq \phi$ であるならば，$\{x_n\}$ は $(\partial f)^{-1}0$ の元 $v$ に強収束する．ここで，$v$ は $x$ に一番近い $f$ の minimizer である．さらに，

$$f(x_{n+1}) - f(v) \leqq \alpha_n (f(x) - f(v)) + \frac{1-\alpha_n}{r_n}\|y_n - v\|\|y_n - x_n\|$$

が成り立つ．

**定理 3.6** $H$ を Hilbert 空間とし，$f: H \to (-\infty, \infty]$ を proper で下半連続な凸関数とする．$x_0 = x \in H$ とし，

$$y_n = \arg\min_{z \in H} \left\{ f(z) + \frac{1}{2r_n}\|z - x_n\|^2 \right\},$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n \quad (n = 0, 1, 2, \ldots)$$

とする．ただし，$\{x_n\} \subset [0,1]$ と $\{r_n\} \subset (0,\infty)$ は $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} r_n > 0$ を満たすものとする．このとき，$(\partial f)^{-1}0 \neq \phi$ であるならば，$\{x_n\}$ は $(\partial f)^{-1}0$ の元 $v$ に弱収束する．さらに，

$$f(x_{n+1}) - f(v) \leqq \alpha_n (f(x_n) - f(v)) + \frac{1-\alpha_n}{r_n}\|y_n - v\|\|y_n - x_n\|$$

が成り立つ．

# 4 有限個の非拡大写像に対する収束定理とその応用

高橋-下地 [16], 厚芝-高橋 [1] は, Crombez[2] や高橋-田村 [17] とは違う形の不動点近似法によって凸制約問題を研究した．彼らの定理を述べる前に，その問題を解くために重要となるある写像を定義しよう。$C$ を Banach 空間 $E$ の閉凸集合とし、$T_1, T_2, \ldots, T_r$ を $C$ 上の写像とする。また、$\alpha_1, \alpha_2, \ldots, \alpha_r$ を $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \ldots, r)$ となる実数とする。このとき、つぎのように定義される写像 $W$ は $T_1, T_2, \ldots, T_r$ と $\alpha_1, \alpha_2, \ldots, \alpha_r$ によって生成される $W$ 写像といわれる [15].

$$S_1 x = \alpha_1 T_1 x + (1-\alpha_1) x,$$
$$S_2 x = \alpha_2 T_2 S_1 x + (1-\alpha_2) x,$$
$$\vdots$$
$$S_{r-1} = \alpha_{r-1} T_{r-1} S_{r-2} x + (1-\alpha_{r-1}) x,$$
$$W x = \alpha_r T_r S_{r-1} x + (1-\alpha_r) x.$$

この $W$-写像に対してつぎの定理が証明できる．

**定理 4.1** $E$ を狭義凸な Banach 空間とし、$C$ を $E$ の閉凸集合とする．$T_1, T_2, \ldots, T_r$ を $\cap_{i=1}^{r} F(T_i) \neq \phi$ となる $C$ 上の非拡大写像とし，$\alpha_1, \alpha_2, \ldots, \alpha_r$ を $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, r-1)$, $0 < \alpha_r \leq 1$ となる実数とする．$W$ を $T_1, T_2, \ldots, T_r$ と $\alpha_1, \alpha_2, \ldots, \alpha_r$ によって生成される $W$ 写像とする．このとき

$$F(W) = \bigcap_{i=1}^{r} F(T_i)$$

が成り立つ.

高橋-下地 [16] は 定理 4.1 を用いてつぎの定理を証明した.

**定理 4.2** $E$ を一様凸で, Fréchet 微分可能なノルムをもつ Banach 空間とし、$C$ を $E$ の閉凸集合とする. $T_1, T_2, \ldots, T_r$ を $\cap_{i=1}^r F(T_i) \neq \phi$ となる $C$ 上の非拡大写像とし, $\alpha_1, \alpha_2, \ldots, \alpha_r$ を $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, r-1)$, $0 < \alpha_r \leq 1$ となる実数とする. $W$ を $T_1, T_2, \ldots, T_r$ と $\alpha_1, \alpha_2, \ldots, \alpha_r$ によって生成される $W$ 写像とする. このとき、任意の $x \in C$ に対して, $\{W^n x\}$ は $\cap_{i=1}^r F(T_i)$ の元に弱収束する.

この定理を凸制約問題に応用するとつぎの形になる.

**定理 4.3** $E$ を一様凸で, Fréchet 微分可能なノルムをもつ Banach 空間とし、$C$ を $E$ の閉凸集合とする. $C_1, C_2, \ldots, C_r$ を $\cap_{i=1}^r C_i \neq \phi$ となる非拡大 retracts とし、$P_i (i = 1, 2, \ldots, r)$ を $C$ から $C_i$ への非拡大 retractions とする. また $\alpha_1, \alpha_2, \ldots, \alpha_r$ を $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, r-1)$, $0 < \alpha_r \leq 1$ となる実数とする. $W$ を $P_1, P_2, \ldots, P_r$ と $\alpha_1, \alpha_2, \ldots, \alpha_r$ によって生成される $W$ 写像とする. このとき任意の $x \in C$ に対して $\{W^n x\}$ は $\cap_{i=1}^r C_i$ の元に弱収束する。

これに対して, 厚芝-高橋 [1] は, 定理 4.1 を用いてつぎの定理を証明した.

**定理 4.4** $E$ を一様凸で, 一様 Gâteaux 微分可能なノルムをもつ Banach 空間とし, $C$ を $E$ の閉凸集合とする. $T_1, T_2, \ldots, T_r$ を $\cap_{i=1}^r F(Ti) \neq \phi$ となる $C$ 上の非拡大写像とし, $\alpha_1, \alpha_2, \ldots, \alpha_r$ を $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, r-1)$, $0 < \alpha_r \leq 1$ となる実数とする. $W$ を $T_1, T_2, \ldots, T_r$ と $\alpha_1, \alpha_2, \ldots, \alpha_r$ によって生成される $W$ 写像とする. このとき, 任意の $x_1 = x \in C$ に対して

$$x_{n+1} = \beta_n x + (1 - \beta_n) W x_n \quad (n = 1, 2, \ldots),$$

$$0 \leq \beta_n \leq 1, \quad \lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \sum_{n=1}^{\infty} \beta_n = \infty$$

で定義される点列 $\{x_n\}$ は $\cap_{i=1}^r F(T_i)$ の点 $z$ に強収束する. ここで, $Px = z$ とすると, $P$ は $C$ から $\cap_{i=1}^r F(T_i)$ の上へのサニー非拡大 retraction である.

この定理を凸制約問題に応用するとつぎの形になる.

**定理 4.5** $E$ を 一様凸で, 一様 Gâteaux 微分可能なノルムをもつ Banach 空間とし, $C$ を $E$ の閉凸集合とする. $C_1, C_2, \ldots, C_r$ を $\cap_{i=1}^r C_i \neq \phi$ となる $C$ の非拡大 retracts とし, $P_1, P_2, \ldots, P_r$ をそれぞれ $C$ から $C_1, C_2, \ldots, C_r$ の上への非拡大 retractions とする. $\alpha_1, \alpha_2, \ldots, \alpha_r$ を $0 < \alpha_r < 1$ $(i = 1, 2, \ldots, r-1)$, $0 < \alpha_r \leq 1$ となる実数とし, $W$ を $P_1, P_2, \ldots, P_r$ と $\alpha_1, \alpha_2, \ldots, \alpha_r$ によって生成される $W$ 写像とする. このとき, 任意の $x_1 = x \in C$ に対して

$$x_{n+1} = \beta_n x + (1 - \beta_n) W x_n \quad (n = 1, 2, \ldots),$$

$$0 \leq \beta_n \leq 1, \quad \lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \sum_{n=1}^{\infty} \beta_n = \infty$$

で定義される点列 $\{x_n\}$ は $\cap_{i=1}^r C_i$ の点 $z$ に強収束する. ここで, $Px = z$ とすると, $P$ は $C$ から $\cap_{i=1}^r C_i$ の上へのサニー非拡大 retraction である.

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# On the property of supercyclic semigroups

Mai Matsui[1] and Fukiko Takeo[2]

[1] Doctoral Research Course in Human Culture
[2] Department of Information Sciences
Ochanomizu University, 2-1-1 Otsuka, Bunkyo-ku,Tokyo 112,Japan
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**Abstract**

We give a necessary and sufficient condition for a translation semigroup to be supercyclic in a weighted function space.

# 1 Introduction

For a strongly continuous semigroup $\{T(t)\}$ on a Banach space $X$, W. Desch et al.[1] investigated the conditions of a semigroup to be hypercyclic, where a semigroup is called to be hypercyclic if there exists $x \in X$ such that $\{T(t)x \mid t \geq 0\}$ is dense in $X$. In [1] they considered the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ ($I$ is an interval $[0, \infty)$ or $(-\infty, \infty)$ ) as a special case and they gave a necessary and sufficient condition to be hypercyclic. The hypercyclic semigroup is called to be chaotic if in addition the set of periodic points is dense in $X$. In [5], M. Yamada and F. Takeo gave a necessary and sufficient condition to be chaotic for the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$. When there exists $x \in X$ such that $\{cT(t)x \mid t \geq 0, c \in \mathbb{R}\}$ is dense in $X$, the semigroup is called to be supercyclic [4]. In this paper, we investigate the property of a supercyclic semigroup on a Banach space and show that the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ is always supercyclic if $I$ is an interval $[0, \infty)$. For $I = (-\infty, \infty)$, the semigroup is not always supercyclic, and so we give a necessary and sufficient condition to be supercyclic for the translation semigroup on a weighted function space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ for $I = (-\infty, \infty)$.

# 2 Preliminaries

We shall give some definitions and known results about hypercyclic and chaotic translation semigroups on a weighted function space.

**Definition 1.** ([4]) A strongly continuous semigroup $\{T(t)\}$ is called *supercyclic* (resp. *hypercyclic*) if there exists $x \in X$ such that $\{cT(t)x \mid t \geq 0, \quad c \in \mathbb{R}\}$ (resp. $\{T(t)x \mid t \geq 0\}$) is dense in $X$. A strongly continuous semigroup $\{T(t)\}$ is called *chaotic* if $\{T(t)\}$ is hypercyclic and the set of periodic points $X_p = \{x \in X \mid \exists t > 0 \ s.t. \ T(t)x = x\}$ is dense in $X$.

**Definition 2.** ([1]) Let $I$ be the interval $[0, \infty)$ or $(-\infty, \infty)$. By an *admissible weight function* on $I$ we mean a measurable function $\rho : I \to \mathbb{R}$ satisfying the conditions:
(i) $\rho(x) > 0$ for all $x \in I$;
(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \leq Me^{\omega t}\rho(t + x)$ for all $x \in I$ and $t > 0$.

With an admissible weight function, we construct the following function spaces:

$$L^p_\rho(I, \mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \ measurable, \int_I |u(\tau)|^p \rho(\tau) \ d\tau < \infty \right\}$$

$$\text{with } \|u\| = \left( \int_I |u(\tau)|^p \rho(\tau) \ d\tau \right)^{\frac{1}{p}},$$

$$C_{0,\rho}(I, \mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \ continuous, \lim_{\tau \to \pm\infty} \rho(\tau)u(\tau) = 0 \right\}$$

$$\text{with } \|u\| = \sup_{\tau \in I} |u(\tau)|\rho(\tau).$$

We consider a *translation semigroup* $\{T(t)\}$ with parameter $t > 0$ such as $[T(t)u](\tau) = u(\tau + t)$ for $u \in C_{0,\rho}(I)$ or $L^p_\rho(I)$. When $\rho(\tau) = 1$, weighted function spaces are equal to $L^p$ or $C_0$ and the hypercyclicity of the translation semigroup doesn't occur, since the norm of $T(t)$ is equal to 1 for all $t \geq 0$ in $L^p$ or $C_0$. Necessary and sufficient conditions for the translation semigroup in $L^p_\rho$ or $C_{0,\rho}$ to be hypercyclic and to be chaotic are known as follows.

**Theorem A [1].** *Let $X$ be $L^p_\rho(I)$ or $C_{0,\rho}(I)$ with an admissible weight function $\rho$. Then the following* (1) *and* (2) *are equivalent:*

(1) *The translation semigroup $\{T(t)\}$ in $X$ is hypercyclic;*

(2) (i) *If $I = [0, \infty)$, then $\liminf_{t \to \infty} \rho(t) = 0$ holds.*
(ii) *If $I = (-\infty, \infty)$, then for each $\theta \in \mathbb{R}$ there exists a sequence $\{t_j\}_{j=1}^\infty$ ($t_j \to \infty$ as $j \to \infty$) of positive real numbers such that*

$$\lim_{j \to \infty} \rho(t_j + \theta) = \lim_{j \to \infty} \rho(-t_j + \theta) = 0.$$

**Theorem B [5].** *Let $I = (-\infty, \infty)$ (resp. $I = [0, \infty)$) and let $X$ be $L^p_\rho(I)$. The translation semigroup $\{T(t)\}$ is chaotic if and only if for all $\epsilon > 0$ and for all $l > 0$, there exists $P > 0$ such that*

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \rho(l + nP) < \epsilon \quad \left( resp. \ \sum_{n=1}^{\infty} \rho(l + nP) < \epsilon \right)..$$

**Theorem C [5].** *Let $I = (-\infty, \infty)$ (resp. $I = [0, \infty)$) and let $X$ be $C_{0,\rho}(I)$. Then the following assertions are equivalent:*

(i) *the translation semigroup $\{T(t)\}$ in $X$ is chaotic;*

(ii) *for all $\epsilon > 0$ and for all $l > 0$, there exists $P > 0$ such that $\rho(l + nP) < \epsilon$ for all $n \in \mathbb{Z} \setminus \{0\}$ (resp. $n \in \mathbb{N}$);*

(iii) *there exists $\{l_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$ whose limit is infinity, such that for all $\epsilon > 0$ and for all $i \in \mathbb{N}$, there exists $P > 0$ such that $\rho(l_i + nP) < \epsilon$ for all $n \in \mathbb{Z} \setminus \{0\}$ (resp. $n \in \mathbb{N}$).*

# 3 Supercyclic semigroups

In this section, we shall give a necessary and sufficient condition for the translation semigroup to be supercyclic. In the first subsection we consider a semigroup on a Banach space, and in the next subsection we treat a translation semigroup on weighted function spaces.

## 3.1 Supercyclic semigroup on a Banach space

The next lemma is useful in proving the following lemma.

**Lemma 1.** *Let $X$ be a separable infinite dimensional Banach space. Suppose that $\{T(t)\}$ is supercyclic, i.e. there exists $x \in X$ such that the set $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ is dense in $X$. Then the set $\{cT(t)x \mid t \geq s,\ c \in \mathbb{R}\}$ is also dense in $X$ for all $s \geq 0$.*

*Proof.* Assume there exists $s_0 \geq 0$ such that $A = \{cT(t)x \mid t \geq s_0,\ c \in \mathbb{R}\}$ is not dense in $X$. By the assumption, there exists a bounded open set $U$ such that $U \cap \overline{A} = \phi$. Then we have

$$\begin{aligned} X &= \overline{\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}} \\ &= \overline{\{cT(t)x \mid t \geq s_0,\ c \in \mathbb{R}\}} \cup \overline{\{cT(t)x \mid 0 \leq t \leq s_0,\ c \in \mathbb{R}\}} \end{aligned}$$

Note that $U \subset \overline{\{cT(t)x \mid 0 \leq t \leq s_0,\ c \in \mathbb{R}\}}$ because $U$ has no intersection with $\overline{A}$. By the definition of semigroup, if there exists $t_0 > 0$ such that $T(t_0)x = 0$ then $T(t)x = 0$ for all $t \geq t_0$. So we have $T(t)x \neq 0$ for all $t > 0$ since the set $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ is dense in $X$. Considering that $T(t)x$ is continuous with $t$ and $T(t)x \neq 0$ for all $t > 0$, there exists $m_1, m_2 \in \mathbb{R}$ such that $0 < m_1 \leq \{||T(t)x|| \mid 0 \leq t \leq s_0\} \leq m_2$. Since $U$ is bounded, there exists $M \geq 0$ such that $||y|| \leq M$ for any $y \in U$. We have $U \subset \left\{cT(t)x \mid 0 \leq t \leq s_0, |c| \leq \frac{M}{m_1}\right\}$. Consider the map $(t, c) \in [0, s_0] \times [-\frac{M}{m_1}, \frac{M}{m_1}] \mapsto cT(t)x \in X$. Then the image is compact since the set $[0, s_0] \times [-\frac{M}{m_1}, \frac{M}{m_1}]$ is compact. It means $\overline{U}$ is compact, i.e. $X$ is finite dimensional, which contradicts that $X$ is infinite dimensional. □

**Lemma 2.** *Let $\{T(t) \mid t \geq 0\}$ be a strongly continuous linear semigroup on a separable Banach space $X$. Then the following are equivalent:*
(1) *$\{T(t)\}$ is supercyclic;*

(2) *for all* $y, z \in X$ *and all* $\varepsilon > 0$, *there exists* $v \in X$, $t > 0$ *and* $c \in \mathbb{R}$ *such that* $||y - v|| < \varepsilon$ *and* $||z - cT(t)v|| < \varepsilon$;
(3) *for all* $y, z \in X$, *all* $\varepsilon > 0$ *and for all* $l \geq 0$, *there exists* $v \in X$, $t > l$ *and* $c \in \mathbb{R}$ *such that* $||y - v|| < \varepsilon$ *and* $||z - cT(t)v|| < \varepsilon$;
(4) *for all* $\varepsilon > 0$ *there exists a dense subset* $D \subset X$ *such that for all* $z \in D$ *there exists a dense subset* $D' \subset X$ *such that for all* $y \in D'$ *there exists* $v \in X$, $t > 0$ *and* $c \in \mathbb{R}$ *such that* $||y - v|| < \varepsilon$ *and* $||z - cT(t)v|| < \varepsilon$.

*Proof.* (1) implies (3): Let $\{cT(t)x \mid t \geq 0,\ c \in \mathbb{R}\}$ be dense in $X$. By Lemma 1, for any $y, z \in X$ and any $l \geq 0$, there exists $s > 0$ and $c_1 \in \mathbb{R}$ such that $||y - c_1T(s)x|| < \varepsilon$, and there exists $u > s + l$ and $c_2 \in \mathbb{R}$ such that $||z - c_2T(u)x|| < \varepsilon$. Put $v = c_1T(s)x$. Then we have the first inequality. Put $t = u - s > l$ and $c = \frac{c_2}{c_1}$. Then we have the second inequality.
(3) implies (2): It is obvious.
(2) implies (1): The proof is similar to the proof in the case of hypercyclic. Let $\{z_1, z_2, z_3, \cdots\}$ be a dense sequence in $X$ and we construct sequences $\{y_1, y_2, y_3, \cdots\} \subset X$, $\{t_1, t_2, t_3, \cdots\} \subset [0, \infty)$ and $\{c_1, c_2, c_3, \cdots\} \subset (-\infty, \infty)$ inductively:

- put $y_1 = z_1, t_1 = 0, c_1 = 1$;
- for $n > 1$ find $y_n, t_n c_n$ such that

$$||y_n - y_{n-1}|| \leq \frac{2^{-n}}{\sup\{||c_jT(t_j)|| \mid 1 \leq j < n\}} \quad \text{and} \tag{1a}$$

$$||z_n - c_nT(t_n)y_n|| \leq 2^{-n}. \tag{1b}$$

Since (1a) implies $||y_n - y_{n-1}|| \leq 2^{-n}$, the sequence $\{y_n\}$ has a limit $v$. By (1a) and (1b), we have the following:

$$\begin{aligned} ||z_n - c_nT(t_n)v|| &\leq ||z_n - c_nT(t_n)y_n|| + ||c_nT(t_n)||||y_n - v|| \\ &\leq ||z_n - c_nT(t_n)y_n|| + ||c_nT(t_n)||\Sigma_{i=n+1}^{\infty}||y_i - y_{i-1}|| \\ &\leq 2^{-n} + ||c_nT(t_n)||\Sigma_{i=n+1}^{\infty}\frac{2^{-i}}{\sup_{j<i}||c_jT(t_j)||} \\ &\leq 2^{-n} + \Sigma_{i=n+1}^{\infty}2^{-i} = 2^{-n+1} \end{aligned}$$

Given $z \in X$ and $\varepsilon > 0$, there exists an arbitrarily large $n$ such that $||z_n - z|| < \frac{\varepsilon}{2}$. Choosing $n$ large enough satisfying $2^{-n+1} < \frac{\varepsilon}{2}$, we have

$$||c_nT(t_n)v - z|| \leq ||z - z_n|| + ||z_n - c_nT(t_n)v|| < \varepsilon.$$

Therefore $\{cT(t)v \mid t \geq 0, c \in \mathbb{R}\}$ is dense in X.
(2) implies (4): It is obvious.
(4) implies (2): The proof is similar to the case of hypercyclic. Let $\varepsilon > 0$ and $y, z \in X$. Since $D$ and $D'$ is dense, we can pick $z' \in D$ and $y' \in D'$ such that $||z - z'|| < \frac{\varepsilon}{2}$ and $||y - y'|| < \frac{\varepsilon}{2}$. We choose $v$, $t$ and $c$ according to (4) with $\frac{\varepsilon}{2}$, $y'$, $z'$ instead of $\varepsilon$, $y$, $z$ and obtain

$$\begin{aligned} ||cT(t)v - z|| &\leq ||cT(t)v - z'|| + ||z' - z|| < \varepsilon, \\ ||v - y|| &\leq ||v - y'|| + ||y' - y|| < \varepsilon. \end{aligned}$$

□

## 3.2 Supercyclic translation semigroups on a separable Banach space, $L^p_\rho$ and $C_{0,\rho}$

Let $I$ be the interval $[0,\infty)$ or $(-\infty,\infty)$ and $\rho$ be an admissible weight function. We shall give a necessary and sufficient condition for the translation semigroup in $L^p_\rho(I)$ and $C_{0,\rho}(I)$ to be supercyclic.

At first we shall quote the lemma, which is needed to prove the following theorem.

**Lemma 3.** [1] *Let $I$ be the interval $(-\infty,\infty)$ or $[0,\infty)$ and $\rho$ be an admissible weight function on $I$. If $\omega > 0$, then there exists a constant $M_1 > 0$ such that the inequality $\frac{1}{M_1}\rho(\sigma) \le \rho(\tau) \le M_1\rho(\sigma+l)$ holds for any $l > 0$, any $\sigma \in I$ and any $\tau \in [\sigma, \sigma+l]$.*

By using the lemma, we give the following necessary and sufficient condition for supercyclic.

**Theorem 1.** *Let $X$ be the space $L^p_\rho(I)$ or $C_{0,\rho}(I)$ and $\rho$ be an admissible weight function. Let $\{T(t)\}$ be a translation semigroup on $X$. Then the following assertions hold:*

(1) *If $I = [0,\infty)$, then $\{T(t)\}$ is supercyclic;*

(2) *If $I = (-\infty,\infty)$, then $\{T(t)\}$ is supercyclic if and only if there exists a sequence $\{t_j\}_{j=1}^\infty$ $(t_j \to \infty)$ such that $\lim_{j\to\infty} \rho(t_j+\theta)\rho(-t_j+\theta) = 0$ for each $\theta \in \mathbb{R}$,*

*Proof.* (1) Let $X_0$ be the set of all $x \in X$ such that the support of $x$ is compact. For $l \ge 0$, let $X_l$ be the set of all $x \in X$ such that for each $\varepsilon > 0$, there exists some $\omega \in X$ and $t > l$ such that $||T(t)\omega - x|| < \varepsilon$. In the both cases $X = L^p_\rho(I)$ and $X = C_{0,\rho}(I)$, we can prove that $X_0$ is dense in $X$. For any $l \ge 0$, $X = X_l$ holds. We use Lemma 2(4) with $D = X_l$ and $D' = X_0$. For any $y \in X_0$, there exists $t_1 > 0$ such that for any $s > t_1$ $||T(s)y|| = 0$. Put $l = t_1$ and pick any $z \in X_l$. For $\varepsilon > 0$, there exists $\omega' \in X$ and $t > t_1$ such that $||T(t)\omega' - z|| < \varepsilon$. Put $c = \frac{||\omega'||}{\varepsilon}$ and $\omega = \frac{\varepsilon\omega'}{||\omega'||}$. Then $||T(t)\omega' - z|| = ||cT(t)\omega - z|| < \varepsilon$ and $||\omega|| = \varepsilon \le \varepsilon$ hold. Put $v = y + \omega$. We have $||y - v|| = ||\omega|| \le \varepsilon$ and $||z - cT(t)v|| \le ||z - cT(t)\omega|| + ||cT(t)\omega - cT(t)v|| < \varepsilon + ||cT(t)y|| = \varepsilon$. By Lemma 2(4), $\{T(t)\}$ is supercyclic.
(2) We suppose $X = L^p_\rho(I,\mathbb{C})$. In the case $X = C_{0,\rho}(I,\mathbb{C})$, we can prove in a similar way.
($\Leftarrow$) Let $\{T(t)\}$ be supercyclic. We will show $\lim_{j\to\infty} \rho(t_j+\theta)\rho(-t_j+\theta) = 0$.
Fix any $\theta \in \mathbb{R}$. Let $y, z \in X$ be functions with compact support $\subset [\theta, \theta+l]$ $(l > 0)$ and $y \ge 0, z \le 0, ||y|| = ||z|| = 1$. By Lemma 2, for any $\varepsilon > 0$ there exists $v_\varepsilon \in X$, $t_\varepsilon > 0$ and $c_\varepsilon > 0$ such that $||c_\varepsilon T(t_\varepsilon)v_\varepsilon - z|| < \varepsilon$ and $||v_\varepsilon - y|| < \varepsilon$. Put $\omega_1 = v^+_{\varepsilon||[\theta,\theta+l]}$ and $\omega_2 = v^-_{\varepsilon||[\theta+t_\varepsilon,\theta+l+t_\varepsilon]}$.
We can show the following:

$$\begin{cases} ||c_\varepsilon T(t_\varepsilon)\omega_1|| < \varepsilon & \text{(a)} \\ ||y|| - ||\omega_1|| < \varepsilon & \text{(b)} \\ ||\omega_2|| < \varepsilon & \text{(c)} \\ ||z|| - ||c_\varepsilon T(t_\varepsilon)\omega_2|| < \varepsilon. & \text{(d)} \end{cases}$$

By Lemma 3 and the above (a)-(d), there exists some constant $M$ such that $\varepsilon^{2^p} > \frac{(1-\varepsilon)^{2p}\rho(\theta+t_\varepsilon)\rho(\theta-t_\varepsilon)}{M^2\rho^2(\theta+l)}$. If $\varepsilon$ tends to 0 then $\rho(\theta-t_\varepsilon)\rho(\theta+t_\varepsilon)$ tends to 0.
($\Rightarrow$) Assume for each $\theta \in \mathbb{R}$, there exists a sequence $\{t_j\}$ whose limit tends to $\infty$ such that $\lim_{j\to\infty}\rho(t_j+\theta)\rho(-t_j+\theta)=0$. Let $y$ and $z$ be any functions with compact support, $[\theta-l,\theta]$ ( $\theta\in\mathbb{R},\ l>0$). By assumption and Lemma 3, for any $\varepsilon>0$, there exists $t_j>l$ and the constants $M>0$ such that $\rho(t_j+\theta)\rho(-t_j+\theta) < \frac{\rho^2(\theta-l)\varepsilon^2}{M^4||z||^p||y||^p}$.
Put

$$v_j(\tau) = \begin{cases} y(\tau) & \tau\in[\theta-l,\theta] \\ A_j\cdot z(\tau-t_j) & \tau\in[t_j+\theta-l,t_j+\theta] \\ 0 & \text{otherwise} \end{cases}$$

and $c_j=\frac{1}{A_j}$, where $A_j=(\frac{\varepsilon\rho(\theta-l)}{||z||^pM^2\rho(t_j+\theta)})^{\frac{1}{p}}$.
By Lemma 3 and the above inequality, we have $||v_j-y||^p<\varepsilon$ and $||c_jT(t_j)v_j-z||^p<\varepsilon$, therefore $\{T(t)\}$ is supercyclic by Lemma 2. □

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# Convergence to the limit set of linear cellular automata

**Mie Matsuto and Fukiko Takeo**

**Ochanomizu University**

## 1 Introduction

A cellular automaton consists of a finite-dimensional lattice of sites, each of which takes an element of a finite set $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$ ($M \in \mathbb{Z}_+$) of integers at each time step and the value of each site at time $t+1 \in \mathbb{Z}_+$ is determined as a function $L$ of the values of the neighbouring sites at time $t$.

Concerning a linear cellular automata rule $L$ with $M = 2$, S. J. Willson considered the subset of a product space $\mathbb{Z}^d \times \mathbb{Z}_+$ which consists of sites taking the state 1 up to time $2^n$ with an initial configuration $\omega$, that is,

$$K(n, \omega) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq 2^n,\ (L^t\omega)(x) = 1\}.$$

He showed that a limit of $K(n, \omega)$ contracted with rate $1/2^n$ exists in the sense of the Kuratowski limit and that the limit set does not depend on an initial configuration. As an extension of the above result, S. Takahashi considered the limit set in the case $M = p^r$ ($p$ is prime and $r \in \mathbb{Z}_+$). By considering the set $K(n, \delta)$ (the configuration $\delta$ takes 1 at the origin and takes 0 at other sites) of sites taking nonzero state up to $p^n - 1$, that is,

$$K(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ (L^t\delta)(x) \neq 0\}.$$

He showed the existence of the limit set of $\{\frac{K(n,\delta)}{p^n}\}$. Moreover he investigated the set of sites which takes the state $b$ ($b = 1, 2, \ldots, p^r - 1$)

$$K_b(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ (L^t\delta)(x) = b\}$$

and showed the existence of a limit set of $\{\frac{K_b(n,\delta)}{p^n}\}$ [2].

In [1], we consider series $L^t\omega$ up to time $p^n$ as a function $\psi_n(\omega)$ with $M = p$, where $p$ is a prime integer and show the convergence of $\psi_n(\omega)$ in the pointwise topology. Let $T$ be an operator from $\psi_n(\omega)$ to $\psi_{n+1}(\omega)$ and $g$ be a limit function of $\psi_n(\omega)$ in the pointwise topology. We verify that $g$ is $T$-invariant and that $\hat{g}$ (an upper envelope of $g$) is a scalar multiple of a characteristic function of the limit set of $\{K(n, \omega)/p^n\}$. In order to investigate the topology which $\{\psi_n(\omega)\}$ converges to $\hat{g}$, we consider the space $USC$ of all $\mathbb{Z}_p$-valued upper semi-continuous functions and the metric $D_f$ on $USC$ and verify that $\{\psi_n(\omega)\}$ converge to $\hat{g}$ with respect to $D_f$.

We extend the result above to the case mod $p^r$ and we show the existence of the limit function of $\{\psi_n(\omega)\}$ with respect to $D_f$ and the relation between the limit set of $\{K_b(n, \omega)/p^n\}$ and the limit function.

## 2 The convergence in the pointwise topology

We define a $d$-dimensional $p^r$-state *linear cellular automata* (LCA) as follows:

Let $p$ be a prime number and let $\mathcal{P}$ be the set of all configurations $a : \mathbb{Z}^d \to \mathbb{Z}/p^r$ with compact support. We define $\delta \in \mathcal{P}$ as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Let $L{:}\mathcal{P} \to \mathcal{P}$ mod $p^r$ be a linear transition rule as follows:

$$(La)(x) = \sum_{j \in G} \alpha_j a(x + k_j) \qquad \text{for } a \in \mathcal{P}, \tag{2.1}$$

where $G$ is a finite subset of $\mathbb{Z}$ with $\sharp G \geq 2$, $k_j \in \mathbb{Z}^d$ $(j \in G)$ is a neighbouring site of origin, $\alpha_k \in \mathbb{Z}_{p^r} \backslash \{0\}$ and the summation $\sum$ is taken as the summation with mod $p^r$.

Let

$$X_n = \{(\frac{x}{p^n}, \frac{t}{p^n}) \in \mathbb{R}^d \times [0,1] \mid x \in \mathbb{Z}^d, t \in \mathbb{Z}_+, 0 \leq t \leq p^n\}$$

for $n \in \mathbb{Z}_+$ and put

$$G_j = \{\ell \in \mathbb{Z}^d \mid (L^j \delta)(\ell) \neq 0\}$$

for $j \in \mathbb{Z}_+$.

Define a map $\psi_n$ from $\mathcal{P}$ to the function space on $\mathbb{R}^d \times [0,1]$ for $a \in \mathcal{P}$ and $n \in \mathbb{Z}_+$ by

$$(\psi_n(a))(\frac{x}{p^n}, \frac{t}{p^n}) = \begin{cases} (L^t a)(x) & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in X_n, \\ 0 & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in (\mathbb{R}^d \times [0,1]) \setminus X_n \end{cases}$$

and a map $S_{\ell,j} \colon \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [\frac{j}{p}, \frac{j+1}{p}]$ by

$$S_{\ell,j}(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{\ell}{p^r}, \frac{j}{p}).$$

For a function $g$ on $\mathbb{R}^d \times [0,1]$, by using maps $S_{\ell,j}$ define a function $Tg$ on $\mathbb{R}^d \times [0,1]$ by

$$Tg(y,q) = \sum_{\ell \in G_{jp^{r-1}}} (L^{jp^{r-1}}\delta)(\ell) g(S_{\ell,j}^{-1}(y,q))$$

for $\frac{j}{p} < q \leq \frac{j+1}{p}$ with $0 \leq j \leq p-1$ and

$$Tg(y,0) = g(py,0).$$

We can show the following theorem.

**Theorem 2.1.** *For $a \in \mathcal{P}$ with $a(0) \neq 0$, we have the following assertions:*

(1) *The sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.*

(2) *The limit function $g_a$ of the sequence $\{\psi_n(a)\}$ in the pointwise topology is $T$-invariant, that is, $Tg_a = g_a$.*

(3) *As for the limit functions $g_\delta$ and $g_a$ of $\{\psi_n(\delta)\}$ and $\{\psi_n(a)\}$ respectively, we have $a(0)g_\delta = g_a$.*

# 3 The space of $\mathbb{Z}_{p^r}$-valued upper semi continuous functions

In this section, we shall introduce two metrics $d_f$, $D_f$ in the space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on a compact subset of $\mathbb{R}^d \times [0,1]$. Let $USC$ be the space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, where $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions mean upper semi-continuous functions embedded in $\mathbb{R}$-valued function spaces. For functions $f$, $g \in USC$, the order $f \geq g$ is defined by $f(y,q) \geq g(y,q)$ for any $(y,q) \in \mathbb{R}^d \times [0,1]$ by considering $\mathbb{Z}_{p^r}$ as a subset of $\mathbb{R}$. For functions $\{f_\lambda\}_{\lambda \in \Lambda} \subset USC$ having an upper bound, let

$$g_1(y,q) = \inf\{g(y,q) \mid g \in USC, g \geq f_\lambda \text{ for any } \lambda \in \Lambda\}$$

and

$$g_2(y,q) = \inf\{f_\lambda(y,q) \mid \lambda \in \Lambda\}.$$

Then $g_1$ and $g_2$ belong to $USC$ and $g_1$ is the least upper bound function $\bigvee f_\lambda$ and $g_2$ is the greatest lower bound function $\bigwedge f_\lambda$ in $USC$. So the space $USC$ is an order complete lattice.

Let $K$ be a compact subset of $\mathbb{R}^d \times [0,1]$ and $(y_0, q_0)$ be a point of $(\mathbb{R}^d \times [0,1]) \setminus K$. Let

$$USC|_K = \{g \in USC \mid \text{ support of } g \subset K\}.$$

By using the Hausdorff distance $D(A,B)$ of non-empty compact sets $A$ and $B$ in $\mathbb{R}^d \times [0,1]$, we shall define the pseudodistance $D_0(A,B)$ of $A$ and $B$ in $\mathbb{R}^d \times [0,1]$ by

$$D_0(A,B) = D(A \cup \{(y_0,q_0)\}, B \cup \{(y_0,q_0)\})$$

and metrics $d_f$, $D_f$ in $USC|_K$ as follows:

$$d_f(g_1,g_2) = \max_{1 \leq j \leq p^r-1} D_0(\overline{g_1^{-1}(j)}, \overline{g_2^{-1}(j)}),$$

$$D_f(g_1,g_2) = \max_{1 \leq s \leq p^r-1} D_0(g_1^{-1}[s+], g_2^{-1}[s+]),$$

for $g_1, g_2 \in USC|_K$, where $g^{-1}[s+] = \{(x,t) \mid g(x,t) \geq s\}$ and $\overline{g_1^{-1}(j)}$ is the closure of the set $g_1^{-1}(j) = \{(x,t) \mid g(x,t) = j\}$. It is easy to see that $d_f$ and $D_f$ satisfy the axioms of metric in $USC|_K$. Then the following theorem holds.

**Theorem 3.1 ([1]).** *For $\{f_n\} \subset USC|_K$, suppose $d_f(f_n, f_m) \to 0$ as $n, m \to \infty$. Let $g = \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} f_n$. Then we have*

$$D_f(f_n, g) \to 0 \text{ as } n \to \infty.$$

# 4 Results

Using the metrics $d_f$ and $D_f$ in Section 3, we consider the convergence of $\{\psi_n(a)\}_{n=r}^{\infty}$ to the limit set.

**Definition 1.** An element $j \in G$ is *prime* if $\alpha_j/p \notin \mathbb{Z}_+$.

We have the following theorem by Theorem 3.1.

**Theorem 4.1.** *Let the set $G$ in (2.1) with mod $p^r$ have at least two prime elements. For a nonzero $a \in \mathcal{P}$, we have*

(1) $d_f(\psi_n(a), \psi_m(a)) \to 0$ *as* $n, m \to \infty$.

(2) *Put* $f_0 = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_{n+r-1}(a)$, *where* $\bigwedge$ *and* $\bigvee$ *are lattice operations in USC. Then we have*

$$D_f(\psi_n(a), f_0) \to 0 \qquad as \qquad n \to \infty.$$

Put

$$K^f(n, \delta) = \{(x, t) \in \mathbb{Z} \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ L^t\delta(x) \neq 0 \pmod{p^f}\}$$

for $f \in \{1, 2, \ldots, r\}$ and

$$Y_f = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n \geq k} \frac{K^f(n, \delta)}{p^n}},$$

which is the limit set in the sense of the Kuratowski limit. Let $\hat{g}$ be the upper envelope of $g$, that is,

$$\hat{g}(x, t) = \inf\{\phi(x, t) | \phi \in USC, \phi(x, t) \geq g(x, t)\}.$$

The following theorem shows the relation among the limit set $Y_f$, $g_\delta$ in Theorem 2.1 and $\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(\delta)$, which is the limit function with respect to $D_f$.

**Theorem 4.2.** *Suppose that the set $G$ defined as (2.1) has at least two prime elements. Let the function $g_\delta$ be defined by $g_\delta(y, q) = \lim_{n \to \infty}(\psi_n(\delta))(y, q)$.*

*Then*

$$\hat{g}_\delta = \sum_{1 \leq f \leq r} (p^r - p^{f-1}) 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i}$$

*and*

$$\hat{g}_\delta = \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(\delta).$$

For $a = a(x) \in \mathcal{P}$, put

$$G_a = \{x \in \mathbb{Z} \mid a(x) \neq 0\}.$$

Let $\tau_x \colon \mathcal{P} \to \mathcal{P}$ be a shift operator such that

$$\tau_x a(y) = a(y - x).$$

The following theorem shows the relation among the limit set $Y_f$, $g_a$ in Theorem 2.1 and $\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a)$. While the upper envelope of $g_a$ depends on only the value $a(0)$, $\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a)$ depends on all values $a(x)(x \in \mathbb{Z})$. So $g_a$ is not necessarily equal to $\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a)$ and we have the following

**Theorem 4.3.** *Suppose that the set $G$ defined as (2.1) has at least two prime elements. For $a \in \mathcal{P}$ with $a(0) = kp^j$ for $k/p \notin \mathbb{Z}_+$ and $j \in \{0, 1, \ldots, r-1\}$, put $g_a(y, q) = \lim_{n \to \infty}(\psi_n(a))(y, q)$.*

*Then*

$$\hat{g}_a = \sum_{1 \leq f \leq r-j} (p^r - p^{f-1+j}) 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i}$$

*and*

$$\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) = \bigvee_{x \in G_a} \hat{g}_{\tau_x(a)}.$$

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# Prime ideals in $H^\infty + C$

Keiji Izuchi

Niigata University

ABSTRACT. It is given an affirmative answer to the problem on prime ideals of $H^\infty + C$ posed by Gorkin and Mortini.

This is an abstract of the paper [5].

Let $D$ be the open unit disk in the complex plane. Let $L^\infty$ and $H^\infty$ denote the usual Banach algebras on the unit circle $\partial D$. A closed subalgebra $B$ of $L^\infty$ containing $H^\infty$ is called a Douglas algebra. We denote by $M(B)$ the maximal ideal space of $B$ and can be viewed as a compact subset of $M(H^\infty)$. Identifying each point of $D$ with the point evaluation, we think $D$ as a subset of $M(H^\infty)$. The smallest Douglas algebra, except $H^\infty$, is $H^\infty + C$, where $C$ denotes the space of continuous functions on $\partial D$, and it is known that $M(H^\infty + C) = M(H^\infty) \setminus D$, see [1]. For a subset $E$ of $M(H^\infty + C)$, we denote by $\overline{E}$ and $int\, E$, the closure and the interior of $E$ in $M(H^\infty + C)$, respectively. For a function $f$ in $H^\infty + C$, we put

$$Z(f) = \{\zeta \in M(H^\infty + C); f(\zeta) = 0\}$$

and

$$\{|f| < 1\} = \{\zeta \in M(H^\infty + C); |f(\zeta)| < 1\}.$$

Similary, we define the set $\{f \neq 0\}$ in the space $M(H^\infty + C)$.

For a sequence $\{z_j\}_j$ in $D$ such that $\sum_{j=1}^\infty (1 - |z_j|) < \infty$, there is the associated Blaschke product $b$ given by

$$b(z) = \prod_{j=1}^\infty \frac{-\overline{z}_j}{|z_j|} \frac{z - z_j}{1 - \overline{z}_j z}, \quad z \in D.$$

Then $\{z_j\}_j$ is the zeros of $b$. It is known that $|b| = 1$ on $M(L^\infty)$, see [3].

In this paper, we study topological properties of $Z(f), f \in H^\infty + C$, and apply them to study prime ideals of $H^\infty + C$.

**Theorem 1.** *Let $\{b_n\}_n$ be a sequence of infinite Blaschke products. Let $E$ be a $G_\delta$-subset of $M(H^\infty + C)$ such that $\overline{\bigcup_{k=1}^\infty \{|b_k| < 1\}} \subset E$. Then there exists a Blaschke product $B$ such that $\overline{\bigcup_{k=1}^\infty \{|b_k| < 1\}} \subset Z(B) \subset \{|B| < 1\} \subset E$.*

Applying Theorem 1, we get similar results obtained in [4] and we have the following theorem which is the key of this paper.

**Theorem 2.** *Let $f \in H^\infty + C, f \neq 0$ and $int\, Z(f) \neq \emptyset$. Let $E$ be an $F_\sigma$-subset of $M(H^\infty + C)$ such that $E \subset int\, Z(f)$. Then $\overline{E} \cap \overline{\{f \neq 0\}} = \emptyset$.*

Applying Theorem 2, we answer the problem posed by Gorkin and Mortini [2, Q4]. Let $x \in M(H^\infty + C)$. We denote by $J(x)$ the ideal of functions in $H^\infty + C$ which vanishes in a neighborhood of $x$. In [2], they studied these ideals and posed the problem whether $J(x)$ is a prime ideal of $H^\infty + C$ or not.

**Theorem 3.** *For every $x \in M(H^\infty + C)$, $J(x)$ is a prime ideal of $H^\infty + C$.*

*Proof.* Let $f, g \in H^\infty + C$ such that $fg \in J(x)$. Then there is an open $F_\sigma$-subset $U$ of $M(H^\infty + C)$ such that $x \in U$ and

$$fg = 0 \quad \text{on } U. \tag{1}$$

To prove our theorem, suppose not, that is,

$$f \notin J(x) \quad \text{and} \quad g \notin J(x). \tag{2}$$

If $f(x) \neq 0$, then by (1) we have $g \in J(x)$. So we may assume that

$$f(x) = 0 \quad \text{and} \quad g(x) = 0. \tag{3}$$

Let

$$U_f = \{\zeta \in U; f(\zeta) \neq 0\} \quad \text{and} \quad U_g = \{\zeta \in U; g(\zeta) \neq 0\}. \tag{4}$$

Then by (2) and (3),

$$x \in \overline{U_f} \cap \overline{U_g}. \tag{5}$$

Since $U$ ia an $F_\sigma$-set, $U_f$ is an $F_\sigma$-subset of $M(H^\infty + C)$, and by (1) and (4), $U_f \subset Z(g)$. Then by Theorem 2, $\overline{U_f} \cap \overline{\{g \neq 0\}} = \emptyset$. Since $U_g \subset \{g \neq 0\}$, $\overline{U_f} \cap \overline{U_g} = \emptyset$. This contradicts (5).

Let $x \in M(H^\infty + C)$. Then by Newman's theorem [6], $J(x) \cap H^\infty = \{0\}$ if and only if $x \in M(L^\infty)$. By Theorem 3, we have the following corollary.

**Corollary 4.** *Let $x \in M(H^\infty + C) \setminus M(L^\infty)$. Then $J(x) \cap H^\infty$ is a prime ideal of $H^\infty$.*

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Department of Mathematics, Niigata University, Niigata 950-2181, Japan

*E-mail address*: izuchi@math.sc.niigata-u.ac.jp

# On Closedness of the Range of the Generator of a $C_0$-semigroup

Shizuo Miyajima
Dept. of Math., Faculty of Science, Science Univ. of Tokyo

**Abstract.** An elementary proof of the following known theorem is presented to draw attention to the characterization problem of the closedness of the range of a generator: "a bounded $C_0$-semigroup is uniformly mean ergodic if and only if the range of its generator is closed."

初めに，この報告に於いては有界線型作用素のなす $C_0$-半群（強連続半群）のみ扱い，それらを単に「半群」と書く場合もあることをお断りしておきたい．

さて，Banach 空間 $X$ 上の半群理論に於いては，$C_0$-半群 $\{T(t)\}_{t\geq 0}$ とその generator（生成作用素）$A$ またはその resolvent $R(\lambda, A)$ は緊密に結びついている (Hille–Yosida の理論)．そして，半群 $\{T(t)\}_{t\geq 0}$ の性質を $A$ または $R(\lambda, A)$ の性質で特徴付けることがよく行われる．例えば，$\{T(t)\}_{t\geq 0}$ が縮小半群（$\|T(t)\| \leq 1, \forall t \geq 0$）であることは，$A$ が $m$-dissipative であることや，$\|R(\lambda, A)\| \leq 1/\lambda$ $(\forall \lambda > 0)$ で特徴付けられる．また，$T(t)$ が Banach lattice 上の順序保存作用素 (positivity preserving operator) となることが $A$ についての abstract Kato's inequality を含む条件で特徴付けられること (W. Arendt) も知られている．これらの事柄については [1] や [3] を参照していただきたい．

今述べたことは，半群の性質から出発して，それを生成作用素についての性質で置き換えるという方向である．これは，半群が発展方程式 $du(t)/dt = Au(t)$ の初期値問題の解を与えるということからすれば，この方程式の解についての性質から出発するということであり，自然なものである．しかしながら，理論的には半群と生成作用素は 1 : 1 に対応しており，どちらが主体ということもないので，$A$ の作用素論的な性質を半群の性質で特徴付けるという問題も同様に考えられてよい．

そこで，手始めに $A$ が単射であるという性質を考えてみよう．

$\mathcal{N}(A)$ で $A$ の核 (null space) を表すと，$x \in \mathcal{N}(A)$ は $T(t)x = x$ $(\forall t \geq 0)$ と同等であることが分かっているので，$A$ が単射であることは「$\{T(t)\}_{t\geq 0}$ の不動点が 0 だけである」という半群の性質で特徴付けられることになる．

それでは次に $A$ が全射であることをもう少し一般化した，「$A$ が閉値域を持つ」という性質に対応する半群の性質は何かという問題を考えよう．

$\{T(t)\}_{t\geq 0}$ は最初に述べたとおり $A$ を生成作用素とする $X$ 上の $C_0$-半群とするが，記号を整理しておこう．まず，$A$ の定義域 $\mathcal{D}(A)$ は

$$\mathcal{D}(A) := \{x \in X \mid \exists \lim_{t\downarrow 0}(T(t)x - x)/t\} \tag{1}$$

で与えられ，$x \in \mathcal{D}(A)$ に対して

$$Ax := \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \tag{2}$$

である．そして，$x \in \mathcal{D}(A)$ ならば $t \geq 0$ に対して $T(t)x \in \mathcal{D}(A)$ であり，

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \tag{3}$$

が成り立つことは容易に分かる．$\mathcal{N}(A)$ は前に述べた通り $A$ の null space を表し，$\mathcal{R}(A)$ で $A$ の値域 (range) を表す．また，$\mathcal{N}(A^*)$, $\mathcal{R}(A^*)$ でそれぞれ $A$ の conjugate operator $A^*$ の null space と値域を表す．

$x = Ay \in \mathcal{R}(A)$ であることの $\{T(t)\}_{t\geq 0}$ にとっての意味を考える上で重要なのは，(3) から導かれる次の式である：

$$\frac{1}{t}\int_0^t T(s)x\,ds = \frac{1}{t}\int_0^t T(s)Ay\,ds = \frac{T(t)y - y}{t}. \tag{4}$$

この式から，もし $\{T(t)\}_{t\geq 0}$ が有界（i.e., $\sup_{t\geq 0}\|T(t)\| < \infty$）ならば，$x \in \mathcal{R}(A)$ のとき $T(t)x$ の Cesàro 平均 $(1/t)\int_0^t T(s)x\,ds$ は $t \to \infty$ で 0 にノルム収束する，という著しい性質を持つことが分かる．このことから，以下に於いては $\{T(t)\}_{t\geq 0}$ は有界と仮定しておくことにする．

さて，生成作用素 $A$ の閉値域性について，講演時には検索できていなかったのであるが，その後次の結果が知られていることが判明した．

**Theorem 1** ( [2] Chapter 5, §4) Banach 空間 $X$ 上の有界な $C_0$-半群 $\{T(t)\}_{t\geq 0}$ の生成作用素を $A$ とすると，$A$ の値域 $\mathcal{R}(A)$ が閉であることは Cesàro 平均 $\frac{1}{t}\int_0^t T(s)\,ds$ がある有界作用素にノルム収束すること (uniform mean ergodicity) に同値である．

この定理から次の系が容易に得られる．

**Corollary 2** 上の定理と同じ仮定の下で，0 が $A$ の resolvent set に属するための必要十分条件は，$\{T(t)\}_{t\geq 0}$ の不動点が 0 だけであり，かつ Cesàro 平均が 0 にノルム収束することである．

**Remark.** 後の Lemma 6 を用いれば，$A$ が全射であるための条件は上の系の条件と同一であることが分かる．

これらの結果において，$\{T(t)\}_{t\geq 0}$ の有界性の仮定は crucial であるが，この仮定なしで一般的に生成作用素 $A$ の閉値域性を $\{T(t)\}_{t\geq 0}$ の性質で特徴付ける問題はなかなか困難で，まだ未解決と思われる．

半群が解析的な場合についての部分的な結果以外に，一般の場合についてはまだ報告できるようなことは得られていないので，ここでは Theorem 1 の，Banach 空間論の基本定理にのみ基づく，elementary proof を与えることにする．([2] に載っている Theorem 1 の証明は，$\mathcal{N}(A) + \mathcal{R}(A)$ が閉であることを使っているが，証明はされていないようである．)

有界な半群 $\{T(t)\}_{t\geq 0}$ の生成作用素 $A$ について次の補題を準備しておく．

**Lemma 3** $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$.

**Proof.** $x \in \mathcal{N}(A) \cap \mathcal{R}(A)$ とすると，$x \in \mathcal{N}(A)$ より $x$ は $T(t)x(= x)$ の Cesàro 平均に常に等しいが，一方 (4) により，この平均は $t \to \infty$ で 0 に収束するから，$x = 0$ となる．□

**Lemma 4** $\mathcal{N}(A^*) \cap \mathcal{R}(A^*) = \{0\}$.

**Proof.** $f = A^*g$ $(f, g \in \mathcal{D}(A^*))$ に対して，weak* integral の意味で，(4) と同様な

$$\frac{1}{t}\int_0^t T^*(s)f\,ds = \frac{1}{t}\int_0^t T^*(s)A^*g\,ds = \frac{T(t)^*g - g}{t}.$$

が成り立つので，前補題と同様にして示される．□

**Lemma 5** *$\mathcal{R}(A)$ closed $\iff$ $\mathcal{R}(A) + \mathcal{N}(A)$ closed.*

**Proof.** 幾何学的に述べれば，$\mathcal{N}(A)$ と $\mathcal{R}(A)$ が 0 でない角度で交わっている，ということなのであるが，$M := \sup_{t\geq 0}\|T(t)\|$ として，$x_1 \in \mathcal{N}(A)$, $x_2 \in \mathcal{R}(A)$ として

$$\|x_1\| \leq M\,\|x_1 + x_2\|$$

が成り立つことを使えば容易に示される．この不等式は $T(s)(x_1+x_2) = x_1 + T(s)x_2$ から導かれる

$$\|x_1\| \leq \frac{1}{t}\left\|\int_0^t T(s)(x_1+x_2)\,ds\right\| + \frac{1}{t}\left\|\int_0^t T(s)x_2\,ds\right\|$$

と (4) を使えば得られる．□

**Lemma 6** *$\mathcal{R}(A)$ closed* $\iff$ $\mathcal{R}(A) \oplus \mathcal{N}(A) = X$.

**Proof.** 前補題から $\Leftarrow$ は明らかなので，$\Rightarrow$ のみ示せばよい．$\mathcal{R}(A)$ が閉とすると，閉値域定理から $\mathcal{R}(A^*)$ も閉で，$\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$ が成り立つ．従って，Lemma 4 より

$$(\mathcal{R}(A) + \mathcal{N}(A))^\perp \subset \mathcal{R}(A)^\perp \cap \mathcal{N}(A)^\perp = \mathcal{N}(A^*) \cap \mathcal{R}(A^*) = 0$$

が分かる．前補題から $\mathcal{N}(A) + \mathcal{R}(A)$ は閉だったので，$\mathcal{N}(A) + \mathcal{R}(A) = X$ が示された．□

Lemma 6 が得られれば Theorem 1 の証明は容易である．実際，$A$ が閉値域として $\{T(t)\}_{t\geq 0}$ が uniformly mean ergodic であることを示すには，$\mathcal{N}(A) \oplus \mathcal{R}(A) = X$ と開写像定理によって，ある定数 $C > 0$ があって，任意の $x \in \mathcal{R}(A)$ に対して $\|y\| \leq C\|x\|$ をみたす $y \in \mathcal{R}(A)$ で $x = Ay$ となるものが存在することに注意して (4) を使えばよい．逆に $\{T(t)\}_{t\geq 0}$ が uniformly mean ergodic とし，Cesàro 平均の極限を $P$ としよう．このとき，$\mathcal{N}(P)$ と $\mathcal{R}(P)$ がともに $\{T(t)\}_{t\geq 0}$ について不変なので，問題をこれらの空間のおのおので考えればよい．しかし $\mathcal{R}(P)$ 上では $T(t) = I$ $(\forall t \geq 0)$ なので，こちらの空間上では生成作用素の値域は $\{0\}$．従って，$\mathcal{N}(P)$ 上でのみ考えればよい．これは最初から $P = 0$ と仮定して $\mathcal{R}(A)$ が閉であることを示せばよいということである．そこでそのように仮定しておくと，ある $t_0 > 0$ で $\left\|(1/t_0)\int_0^{t_0} T(s)\,ds\right\| < 1/2$ をみたすものが存在することになる．このとき $x \in \mathcal{D}(A)$ に対して

$$\frac{1}{2}\|x\| \leq \left\|\frac{1}{t_0}\int_0^{t_0} T(s)x\,ds - x\right\| = \frac{1}{t_0}\left\|\int_0^{t_0}(T(s)x - x)\,ds\right\|$$

であるが，

$$\begin{aligned}\left\|\int_0^{t_0}(T(s)x - x)\,ds\right\| &= \left\|\int_0^{t_0}\int_0^s T(\tau)Ax\,d\tau\,ds\right\| \\ &\leq \int_0^{t_0}\int_0^s M\|Ax\|\,d\tau\,ds \\ &= \frac{t_0^2 M\,\|Ax\|}{2}\end{aligned}$$

であることも容易に分かる ($M := \sup_{t\geq 0}\|T(t)\|$)．よって，$x \in \mathcal{D}(A)$ に対して $\|x\| \leq t_0 M\,\|Ax\|$ が得られて，$A$ の閉値域性が示される．

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