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**Proceedings of the 24th Sapporo Symposium on
Partial Differential Equations**

Edited by Y. Giga and T. Ozawa

Sapporo, 1999

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 4 through August 6 in 1999 at Faculty of Science, Hokkaido University.

This is the 24th time of the symposium on Partial Differential Equations. The symposium was founded by Professor Taira Shirota more than 20 years ago. It has one of longest history among symposiums on partial differential equations in Japan.

It is our great honor that Professor Emeritus Taira Shirota, the founder of the symposium, accepted our invitation and will deliver a lecture in this symposium. He recently got a medal from Japanese Government for his contribution to mathematical research and education. We take this opportunity to congratulate him as well as to thank him for his contribution to this symposium.

We wish to dedicate this volume to Professors Rentaro Agemi and Kôji Kubota for their large contribution to the organization of the symposium for many years.

Finally, we thank Springer-Verlag for permitting us to reproduce the first few pages from the paper 'On the initial-boundary-value problem for the linearized equations of magnetohydrodynamics' by M. Ohno and T. Shirota, Arch. Rational Mech. Anal., 144(1998), 259-299.

Y. Giga
T. Ozawa

CONTENTS

Programme

M. Bardi

Viscosity solutions of 1st and 2nd order Hamilton-Jacobi-Bellman equations

Y. Tonegawa

On phase interfaces in the van der Waals-Cahn-Hilliard theory

Y. Kohsaka

On a large-time existence of three-phase boundary motion by surface diffusion
- symmetric case -

Thai-Son Nguyen Duy

Hopf-type formulas for global solutions in the case of concave-convex
Hamiltonians

S. Kosugi

Semilinear elliptic equations on a thin network shaped domain

K. Tanaka

Four positive solutions for the semilinear elliptic equation:
 $-\Delta u + u = a(x)u^p + f(x)$ in \mathbf{R}^N

C. S. Lin

The Liouville theorem for the scalar curvature equation and its applications

K. Yokoyama

Global solutions of systems of wave equations with quadratic nonlinearities

K. Nakanishi

Scattering theory in the energy space for nonlinear Klein-Gordon and
Schrödinger equations

T. Yanagisawa

圧縮性流体の非粘性極限 (Inviscid limit for compressible fluids)

T. Shiota

On the role of MHD for confined plasma

第24回偏微分方程式論札幌シンポジウム

下記の要領でシンポジウムを行ないますのでご案内申し上げます。

代表者 儀我 美一, 小澤 徹
Organizers: Y. Giga and T. Ozawa

記

1. 日時 1999年8月4日(水)～8月6日(金)
2. 場所 北海道大学大学院 理学研究科 5号館 201号室, 202号室(数学教室の南向かい)
3. 講演

August 4, 1999 (水)

- 9:30-10:30 M. Bardi (Univ. of Padova, Italy)
Viscosity solutions of 1st and 2nd order Hamilton-Jacobi-Bellman equations
- 11:00-12:00 利根川 吉廣 (Y. Tonegawa), 慶応大 (Keio U.)
On phase interfaces in the van der Waals-Cahn-Hilliard theory
- 13:30-14:00 *
- 14:00-14:30 高坂 良史 (Y. Kohsaka), 北大理院 (Hokkaido U.)
On a large-time existence of three-phase boundary motion by surface diffusion - symmetric case -
- 15:00-15:30 Thai-Son Nguyen Duy (Hue Univ., Vietnam)
Hopf-type formulas for global solutions in the case of concave-convex Hamiltonians
- 15:45-16:15 小杉 聡史 (S. Kosugi), 北大理院 (Hokkaido U.)
Semilinear elliptic equations on a thin network shaped domain
- 16:30-17:00 *

August 5, 1999 (木)

- 9:30-10:30 田中 和永 (K. Tanaka), 早大 (Waseda U.)
Four positive solutions for the semilinear elliptic equation:
 $-\Delta u + u = a(x)u^p + f(x)$ in \mathbf{R}^N

- 11:00-12:00 C. S. Lin (Chung Cheng U., Taiwan)
The Liouville theorem for the scalar curvature equation and its applications
- 13:30-14:30 *
- 14:30-15:00 横山 和義 (K. Yokoyama), 北海道工大 (Hokkaido Inst. Tech.)
Global solutions of systems of wave equations with quadratic nonlinearities
- 15:30-16:30 中西 賢次 (K. Nakanishi), 東大数理 (U. of Tokyo)
Scattering theory in the energy space for nonlinear Klein-Gordon and Schrödinger equations
- 16:45-17:30 *
- 18:00-20:00 懇親会 BANQUET クラーク会館きゃら亭

August 6, 1999 (金)

- 9:30-10:30 柳沢 卓 (T. Yanagisawa), 奈良女大 (Nara Women's U.)
圧縮性流体の非粘性極限 (Inviscid limit for compressible fluids)
- 11:00-12:00 白田 平 (T. Shirota)
On the role of MHD for confined plasma
- 12:00-12:30 *

*この時間は講演者を囲んで自由な質問の時間とする予定です。

* indicates discussion time. Lecturers in each session are invited to stay in the coffee-tea room during discussion time.

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例年と会場が異なりますのでご注意ください。

なお 8 月 2 日, 3 日には, 札幌天神山国際ハウスにて粘性解小研究会を行う予定です。

シンポジウム終了後の 8 月 6 日 2:30-3:30 に数学教室にて田島慎一氏による 'Ehrenpreis の基本原理と Grothendieck の双対性' という題の特別講演があります。

Viscosity solutions of 1st and 2nd order Hamilton-Jacobi-Bellman equations

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Abstract. The theory of viscosity solutions [CIL, BCESS] provides the appropriate framework for studying fully nonlinear scalar PDEs of the general form

$$F(x, u, Du, D^2u) = 0$$

that are degenerate elliptic in the following very weak sense

$$F(x, r, p, X) \leq F(x, s, p, Y) \quad \text{if } r \leq s \quad \text{and } Y - X \leq 0.$$

I will focus on the Hamilton-Jacobi-Bellman (briefly, HJB) equations of the form

$$\sup_{\alpha \in A} \mathcal{L}^\alpha u = 0,$$

where for each α , \mathcal{L}^α is a linear nondivergence form operator with Lipschitz coefficients

$$\mathcal{L}^\alpha u := -a_{ij}^\alpha(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i^\alpha(x) \frac{\partial u}{\partial x_i} + c^\alpha(x)u - f^\alpha(x), \quad (1)$$

and the matrix a^α is nonnegative semidefinite for all x . These equations arise in the Dynamic Programming approach to the optimal control of diffusion processes. If $a^\alpha \equiv 0$ the PDE is of 1st order: this corresponds to the control of a deterministic system [BCD]. Many results hold as well for the Isaacs' equations

$$\sup_{\alpha \in A} \inf_{\beta \in B} \mathcal{L}^{\alpha, \beta} u = 0,$$

where $\mathcal{L}^{\alpha, \beta}$ are operators of the form (1). These PDEs arise in the theory of deterministic and stochastic differential games.

I will first recall the basic theory of viscosity solutions and the connection of the HJB equations with control theory. Then I will present some more recent results on generalized solutions of the Dirichlet problem. In view of the

degeneracy of equation the boundary conditions are not attained everywhere and must be interpreted in a weak sense. I will describe two different ways of doing this. One of them is related to the exit times of diffusion processes from an open set, the other corresponds to the exit times from a closed set. In both cases the resulting generalized viscosity solution is not continuous in general, as one expects from its control-theoretic interpretation. Some results are joint work with P. Goatin and H. Ishii.

[BCD] M. Bardi, I. Capuzzo Dolcetta: *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, 1997.

[BCESS] Bardi, M.; Crandall, M.G.; Evans, L.C.; Soner, H.M.; Souganidis, P.E.: *Viscosity solutions and applications*. Edited by I. Capuzzo Dolcetta and P.-L. Lions. Lecture Notes in Mathematics, 1660. Springer-Verlag, Berlin, 1997.

[CIL] M.G. Crandall, H. Ishii, P.-L. Lions: *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27 (1992), 1–67,

ON PHASE INTERFACES IN THE VAN DER WAALS-CAHN-HILLIARD THEORY

YOSHIHIRO TONEGAWA

1. INTRODUCTION

In my talk I briefly discuss some background materials and relevant results of the van der Waals-Cahn-Hilliard theory of phase transitions, and in the second half present a recent joint work with John Hutchinson (Australian National University) on general asymptotic behavior of critical points of the following energy functional which plays an important role in the theory. Since I do not have time to discuss the results in full, I describe the assumptions and statements of the results in this manuscript. The functional in question is

$$(1.1) \quad E_\varepsilon(u) = \int_U \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon},$$

where $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is the normalized density distribution of a two-phase fluid and W is a W -shaped potential with strict local minima at ± 1 . Such functional is derived via a mean field approximation in statistical mechanics to explain the macroscopic existence of ‘surface tension’ down from the intermolecular forces of the large number of particles ([21]). Very much related is the phase transitions in the context of the Ising model of ferromagnet, even though we do not deal with the phenomenon of ‘phase transitions’ in the usual sense in statistical mechanics. Our focus is, in a sense, within the domain of subcritical temperature with short range intermolecular forces.

Critical points of the functional (1.1) satisfy

$$(1.2) \quad \varepsilon \Delta u = \varepsilon^{-1} W'(u) - \lambda,$$

where λ is the Lagrange multiplier associated with a global volume constraint of the form $\int_U u = m$. Here, ε may be seen as the ‘thickness’ of the interface region, and one is interested in the geometric properties of the interface regions when ε is small. Thus we ask ourselves: what do we know, when given a sequence of solutions to (1.2), about its limit as ε goes to zero? How does that converge? What is the ‘worst thing’ to happen to the limit? For example, is it possible to have a configuration where lots of ‘bubbles’ of little interfaces are all over the domain? Can interface be smeared out?

For absolutely energy minimizing solutions with such a volume constraint, Modica [13] and Sternberg [18] used the technique of Γ -convergence [5] to show that (on passing to a subsequence) the limit of minimizers of (1.1) as $\varepsilon \rightarrow 0$ is a function with value ± 1 almost everywhere and with area minimizing interface in the appropriate class of competing functions. In particular, there is a known theory which deals with such questions as the regularity of the interface, and one concludes that the interface is a regular constant mean curvature hypersurface away from a closed singular set of codimension at least 7. There have been works by many authors on various generalizations and related problems in this direction, see for example [3, 7, 11, 14, 15, 19, 20].

1991 *Mathematics Subject Classification*. Primary: 49Q20; Secondary: 35J60, 80A22, 82B26.

The focus of our present results is on general critical points which may not be absolutely energy minimizing. A good understanding of such solutions is important in the study of dynamical problems such as the Allen-Cahn equation [2]

$$u_t = \Delta u - \frac{W'(u)}{\varepsilon^2}$$

and the Cahn-Hilliard equation [4]

$$u_t = \Delta \left(-\varepsilon \Delta u + \frac{W'(u)}{\varepsilon} \right)$$

in bounded domains, since it has been observed numerically that the solution often seems to undergo patterns similar to unstable equilibria before settling down to a stable pattern. One also observes various metastable patterns from which the interface typically moves extremely slowly, to the extent that the solution does not move in the numerical simulations. From a purely mathematical point of view, one can show the existence of unstable mountain-pass type solutions due to the non-convexity of the functional (1.1), and it is interesting to know the asymptotic limit of such solutions as $\varepsilon \rightarrow 0$ in this generality. On the other hand, Γ -convergence techniques essentially rely on energy minimality of solutions and thus do not deal with general non-minimizing (or even locally energy minimizing solutions).

Roughly speaking, for $\lambda = 0$ and any solution of (1.2), we show that as $\varepsilon \rightarrow 0$ the interface converges in the Hausdorff distance sense to a generalized minimal hypersurface. Moreover, the energy concentrates near the hypersurface and after division by twice the surface energy constant $\sigma = \int_{-1}^1 \sqrt{W(s)}/2 ds$, the energy density in the limit is an integer \mathcal{H}^{n-1} a.e. on the hypersurface. This integer multiplicity allows for “folding” of the interface as $\varepsilon \rightarrow 0$. When $\lambda \neq 0$, we prove that the hypersurface has locally constant mean curvature \mathcal{H}^{n-1} a.e., determined by λ and the interface multiplicity. As a corollary, we show that the additional assumption of local energy minimality implies no loss of energy in the limit, and the limit interface is a locally area minimizing hypersurface of multiplicity one. To describe our results in full, we state here some assumptions, notations and then our main results.

2. PRELIMINARIES AND RESULTS

2.1. Hypotheses and easy consequences. Except where stated otherwise we take the following as the starting point. These are satisfied for all interesting applications with some minor assumption on the growth of potential W at infinity and etc. Note that we do not assume any energy minimality for the u^i .

Assumptions.

A: The function $W : \mathbb{R} \rightarrow [0, \infty)$ is C^3 and $W(\pm 1) = 0$. For some $\gamma \in (-1, 1)$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1, \gamma)$. For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.

B: $U \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary ∂U . A sequence of $C^3(U)$ functions $\{u^i\}_{i=1}^\infty$ satisfies

$$(2.1) \quad \varepsilon_i \Delta u^i = \varepsilon_i^{-1} W'(u^i) - \lambda_i$$

on U . Here, $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, and we assume there exist c_0, λ_0 and E_0 such that $\sup_U |u^i| \leq c_0$, $|\lambda_i| \leq \lambda_0$ and

$$\int_U \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq E_0$$

for all i .

We next discuss a few immediate consequences of the assumptions. Let

$$\Phi(s) = \int_0^s \sqrt{W(s)/2} ds,$$

and define new functions

$$w^i = \Phi \circ u^i$$

for each i .

Since $|\nabla w^i| = \sqrt{W(u^i)/2} |\nabla u^i|$, it follows by the Cauchy-Schwartz inequality that

$$\int_U |\nabla w^i| \leq \frac{1}{2} \int_U \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq \frac{E_0}{2}.$$

We also have $\Phi(-c_0) \leq w^i \leq \Phi(c_0)$. By the compactness theorem for bounded variation functions, there exists a subsequence also denoted by $\{w^i\}$ and an a.e. pointwise limit w^∞ , such that

$$\lim_{i \rightarrow \infty} \int_U |w^i - w^\infty| = 0 \quad \text{and} \quad \int_U |Dw^\infty| \leq \liminf_{i \rightarrow \infty} \int_U |\nabla w^i|.$$

Here, $|Dw^\infty|$ is the total variation of the vector-valued Radon measure Dw^∞ .

Let Φ^{-1} be the inverse of Φ and define

$$u^\infty = \Phi^{-1}(w^\infty).$$

Then $u^i \rightarrow u^\infty$ a.e., and by the Lebesgue dominated convergence theorem

$$\int_U |u^i - u^\infty| \rightarrow 0.$$

Also by Fatou's Lemma and the energy bound, we have

$$\int_U W(u^\infty) = \int_U \lim_{i \rightarrow \infty} W(u^i) \leq \liminf_{i \rightarrow \infty} \int_U W(u^i) = 0.$$

This shows that $u^\infty = \pm 1$ a.e. on U , and the sets $\{u^\infty = \pm 1\}$ have finite perimeter in U , since

$$\|\partial\{u^\infty = 1\}\|(U) = \frac{1}{2} \int_U |Du^\infty| = \frac{1}{\sigma} \int_U |Dw^\infty| \leq \frac{E_0}{2\sigma},$$

where we define

$$\sigma = \int_{-1}^1 \sqrt{W(s)/2} ds,$$

and where $\|\partial A\|$ denotes the perimeter of A in the measure-theoretic sense (see [6]).

By the generalized Gauß-Green theorem for sets of finite perimeter ([6, page 209]), there exists an $(n-1)$ -rectifiable set M^∞ (the "reduced boundary") $\subset \text{supp}\|\partial\{u^\infty = 1\}\|$, and an \mathcal{H}^{n-1} measurable unit vector function ν^∞ defined on M^∞ (pointing into $\{u^\infty = 1\}$) such that

$$\int_{\{u^\infty = 1\}} \text{div} g = - \int_{M^\infty} \nu^\infty \cdot g d\mathcal{H}^{n-1},$$

for any $g \in C_c^1(U)$.

2.2. The associated varifolds. In this section we recall various definitions concerning varifolds and associate to each solution of (1.2) a varifold in a natural way. We refer to [1, 17] for a comprehensive treatment of varifolds.

Let $G(n, n-1)$ denote the Grassman manifold of unoriented $(n-1)$ -dimensional planes in \mathbb{R}^n . We also regard $S \in G(n, n-1)$ as the orthogonal projection of \mathbb{R}^n onto S , and write $S_1 \cdot S_2 = \text{trace}(S_1^t \cdot S_2)$. We say V is an $(n-1)$ -dimensional *varifold* in $U \subset \mathbb{R}^n$ if V is a Radon measure on $G_{n-1}(U) = U \times G(n, n-1)$. Let $V_{n-1}(U)$ denote the set of all $(n-1)$ -dimensional varifolds in U . *Convergence in the varifold sense* means convergence in the usual sense of measures. For $V \in V_{n-1}(U)$, we let the *weight* $\|V\|$ be the Radon measure in U defined by

$$\|V\|(A) = V(\{(x, S) \mid x \in A, S \in G(n, n-1)\})$$

for each Borel set $A \subset U$. If M is a $(n-1)$ -rectifiable subset of U we define $v(M) \in V_{n-1}(U)$ by

$$v(M)(E) = \mathcal{H}^{n-1}(\{x \in U \mid (x, \text{Tan}^{n-1}(\mathcal{H}^{n-1}|_M, x)) \in E\})$$

for each Borel set $E \in G_{n-1}(U)$, where $\text{Tan}^{n-1}(\mathcal{H}^{n-1}|_M, x)$ is the approximate tangent plane to M at x and so exists for \mathcal{H}^{n-1} a.e. $x \in M$. We say $V \in V_{n-1}(U)$ is an $(n-1)$ -dimensional *rectifiable varifold* if there exist positive real numbers $\{c_k\}_{k=1}^\infty$ and $(n-1)$ -rectifiable sets $\{M_k\}_{k=1}^\infty$ such that

$$V = \sum_{k=1}^{\infty} c_k v(M_k).$$

The *density* or *multiplicity function* θ for V is given by

$$\theta(x) = \sum \{c_k \mid x \in M_k\},$$

and then $\|V\| = \theta \mathcal{H}^{n-1}|_M$, where $M = \bigcup_k M_k$. If $\{c_k\}_{k=1}^\infty$ may be taken to be positive integers, we say V is an $(n-1)$ -dimensional *integral varifold*.

For $V \in V_{n-1}(U)$, we define the *first variation* of V by

$$\delta V(g) = \int Dg(x) \cdot S dV(x, S)$$

for any vector field $g \in C_c^1(U; \mathbb{R}^n)$, and we say V is *stationary* if $\delta V(g) = 0$ for all such g . We also denote the total variation of δV by $\|\delta V\|$. If $\|\delta V\|$ is a Radon measure and is absolutely continuous with respect to $\|V\|$ on U , we define the *generalized mean curvature* $H(x)$ by

$$\delta V(g) = - \int g \cdot H d\|V\|,$$

where H is defined $\|V\|$ a.e. on U .

Finally we remark that if μ is a measure on U (e.g., $\|V\|$ or $\|\partial\{u^\infty = 1\}\|$) then by $\text{supp } \mu$ we will always denote the support of μ in U .

We associate to each function w^i a varifold V^i defined naturally as follows ([10, 16]). By Sard's theorem, $\{w^i = t\} \subset U$ is a C^3 hypersurface for L^1 almost all t . Define $V^i \in V_{n-1}(U)$ by

$$V^i(A) = \int_{-\infty}^{\infty} v(\{w^i = t\})(A) dt$$

for each Borel set $A \subset G_{n-1}(U)$. By the coarea formula ([6]), we have

$$\|V^i\|(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{w^i = t\} \cap A) dt = \int_A |\nabla w^i|$$

for each Borel set $A \subset U$. One may interpret the varifold V^i as a weighted averaging of the level sets of u^i , which is concentrated around the transition region. The first variation of V^i is given by (see [16, Section 2.1])

$$(2.2) \quad \delta V^i(g) = \int_U \left(\operatorname{div} g - \sum_{j,k=1}^n \frac{w_{x_j}^i}{|\nabla w^i|} \frac{w_{x_k}^i}{|\nabla w^i|} g_{x_k}^j \right) |\nabla w^i|$$

for each $g \in C_c^1(U; \mathbb{R}^n)$.

2.3. Main results. With the above terminology and assumptions A and B, we show the following.

Theorem 1. *Let V^i be the varifold associated with u^i (via w^i) as in Sections 2.1 and 2.2. On passing to a subsequence we can assume*

$$\lambda_i \rightarrow \lambda_\infty, \quad u^i \rightarrow u^\infty \text{ a.e.}, \quad V^i \rightarrow V \text{ in the varifold sense.}$$

Moreover,

(1) For each $\phi \in C_c(U)$,

$$\|V\|(\phi) = \lim_{i \rightarrow \infty} \int \phi \frac{\varepsilon_i |\nabla u^i|^2}{2} = \lim_{i \rightarrow \infty} \int \phi \frac{W(u^i)}{\varepsilon_i} = \lim_{i \rightarrow \infty} \int \phi |\nabla w^i|.$$

(2) $\operatorname{supp}|\partial\{u^\infty = 1\}| \subset \operatorname{supp}\|V\|$, and $\{u^i\}$ converges locally uniformly to ± 1 on $U \setminus \operatorname{supp}\|V\|$.

(3) For each $\tilde{U} \subset\subset U$ and $0 < b < 1$, $\{|u^i| \leq 1-b\} \cap \tilde{U}$ converges to $\tilde{U} \cap \operatorname{supp}\|V\|$ in the Hausdorff distance sense.

(4) $\sigma^{-1}V$ is an integral varifold. Moreover, the density $\theta(x) = \sigma N(x)$ of V satisfies

$$N(x) = \begin{cases} \text{odd} & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ \text{even} & \mathcal{H}^{n-1} \text{ a.e. } x \in \operatorname{supp}\|V\| \setminus M^\infty, \end{cases}$$

where M^∞ is the reduced boundary of $\{u^\infty = 1\}$.

(5) The generalized mean curvature H of V is given by

$$H(x) = \begin{cases} \frac{\lambda_\infty}{\theta(x)} \nu^\infty(x) & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ 0 & \mathcal{H}^{n-1} \text{ a.e. } x \in \operatorname{supp}\|V\| \setminus M^\infty, \end{cases}$$

where ν^∞ is the inward normal for M^∞ .

Heuristic interpretations may be given as follow. From (1), we see that in the limit the energy is equally divided between the two terms of the energy functional (1.1). Part (4) suggests that folding of the interface as $\varepsilon \rightarrow 0$ occurs locally as an integer multiple of 1-D traveling wave solutions, almost everywhere in the measure-theoretic sense. This can be seen more clearly in the proof of integrality in Section 5. Part (5) shows that whenever there is a cancellation of interface in the oriented sense the mean curvature is zero there. More generally, if odd N -folding occurs, the mean curvature decreases by that factor.

Without loss of generality, we may assume that $M^\infty \subset \operatorname{supp}|\partial\{u^\infty = 1\}|$. We were not able to prove or disprove that $\mathcal{H}^{n-1}(\operatorname{supp}|\partial\{u^\infty = 1\}| \setminus M^\infty) = 0$ in general. This is due to the lack of a uniform lower density estimate for the measure $|\partial\{u^\infty = 1\}|$ (as opposed to $\|V\|$) at \mathcal{H}^{n-1} a.e. x in the closure of M^∞ . On the other hand, if $N(x)$ is odd \mathcal{H}^{n-1} a.e. for $x \in \operatorname{supp}\|V\|$, the result (4) shows that $\mathcal{H}^{n-1}(\operatorname{supp}\|V\| \setminus M^\infty) = 0$ and $\operatorname{supp}\|V\| = \operatorname{supp}|\partial\{u^\infty = 1\}|$. If $N(x) = 1$ a.e., then $\sigma^{-1}\|V\| = |\partial\{u^\infty = 1\}|$ and V has constant mean curvature on U by (4) and (5). This last situation corresponds to “no energy loss”, since

$$\int |Dw^\infty| \phi = \sigma |\partial\{u^\infty = 1\}|(\phi) = \|V\|(\phi) = \lim_{i \rightarrow \infty} \int |\nabla w^i| \phi$$

for all $\phi \in C_c(U)$. The relation (5) between the Lagrange multiplier λ_∞ (or chemical potential in the two-phase fluid model) and the mean curvature of the limit interface, called the Gibbs-Thompson relation, was established by Luckhaus and Modica in [12] in the case of no energy loss.

It is well-known that the support of a rectifiable varifold with bounded mean curvature is locally a $C^{1,\alpha}$ graph on a relatively open dense subset \mathcal{O} ([1]). Moreover, the multiplicity of V on \mathcal{O} is locally constant, and hence the support has locally constant mean curvature by (5). Thus \mathcal{O} is in fact a C^∞ submanifold. On the other hand, we do not know if $\mathcal{H}^{n-1}(\text{supp}\|V\| \setminus \mathcal{O}) = 0$ in general.

If $N = 1$, \mathcal{H}^{n-1} a.e. on $\text{supp}\|V\|$, then the support is locally a C^∞ hypersurface of constant mean curvature, except for a closed set of \mathcal{H}^{n-1} measure zero. Such a situation occurs (away from the boundary) in the locally minimizing case discussed in Theorem 2. First we need the following.

Definition. For $\tilde{U} \subset\subset U$ we say $u \in H^1(U)$ is *locally energy minimizing* on \tilde{U} for E_ε if there exists a positive constant c such that $E_\varepsilon(u) \leq E_\varepsilon(\tilde{u})$ for all $\tilde{u} \in H^1(U)$ satisfying $\int_U |u - \tilde{u}| < c$ and $u - \tilde{u} = 0$ on $U \setminus \tilde{U}$. We may also (depending on the problem) impose the additional volume constraint $\int_U (u - \tilde{u}) = 0$.

Note that the definition is local in both the domain and the L^1 norm, which differs from the local minimality discussed in [11]. With this, we prove

Theorem 2. *In addition to assumptions A and B, suppose $\{u^i\}$ are locally energy minimizing on $\tilde{U} \subset\subset U$ for E_{ε_i} (with or without volume constraint). Then $N(x) = 1$, \mathcal{H}^{n-1} a.e. on $\tilde{U} \cap \text{supp}\|V\|$. The set $\partial\{u^\infty = 1\}$ on \tilde{U} has constant mean curvature $\frac{\lambda_\infty}{\sigma} \nu^\infty$ and no energy loss occurs on \tilde{U} .*

For absolutely energy minimizing solutions with a volume constraint, Modica [13] and Sternberg [18] showed that $\partial\{u^\infty = 1\}$ is an absolutely area minimizing hypersurface with the given volume constraint. For this case, with the additional Assumption A, Theorem 1.(3) gives a new result concerning convergence of the interface in the Hausdorff distance sense. We also prove a version of the Modica-Sternberg theorem for local minimizers, which was not known before.

Theorem 3. *Suppose that W satisfies Assumption A and U is a bounded open set with Lipschitz boundary. Suppose $c > 0$, $u^i \in H^1(U)$, $\varepsilon_i \rightarrow 0$, $m \in (-|U|, |U|)$ and $E_0 < \infty$ satisfy*

1. $\int_U u^i = m$ and $E_{\varepsilon_i}(u^i) \leq E_0$ for all i ,
2. $E_{\varepsilon_i}(u^i) \leq E_{\varepsilon_i}(\tilde{u})$ for all $\tilde{u} \in H^1(U)$ with $\int_U |u^i - \tilde{u}| < c$ and $\int_U \tilde{u} = m$.

Then the assumption B is satisfied. Moreover, with u^∞ as in Theorem 1, $\partial\{u^\infty = 1\}$ minimizes area locally; i.e. for any \tilde{u} satisfying $\tilde{u} = \pm 1$ \mathcal{L}^n a.e. on U , $\int_U \tilde{u} = m$ and $\int_U |u^\infty - \tilde{u}| < c$ (the same c in the assumption), we have

$$\|\partial\{u^\infty = 1\}\|(U) \leq \|\partial\{\tilde{u} = 1\}\|(U).$$

It is well-known that the support of a locally area minimizing perimeter is smooth except for a closed set of dimension at most $n - 8$ [8, 9].

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On a large-time existence of three-phase boundary motion by surface diffusion - symmetric case -

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1 Introduction

We study a sharp interface model for a three-phase boundary motion by surface diffusion proposed by H. Garcke and A. Novick-Cohen [1]. Let $\Omega \in \mathbf{R}^2$ be a bounded region. We consider a situation that a ternary alloy system with three phases in a non-equilibrium state is contained in Ω . These three phases are separated by evolving three interphase boundaries $\Gamma^i(t)$ ($i = 1, 2, 3$) with a triple junction $m(t) \in \Omega$ where $t \geq 0$ denotes the time variable. Here we set $\Gamma(t) = \bigcup_{i=1}^3 \Gamma^i(t)$. H. Garcke and A. Novick-Cohen [1] discussed a formal singular limit in a Cahn-Hilliard system with a concentration dependent mobility to derive a sharp interphase model in the following;

- (A) along interphase boundaries $\Gamma^i(t)$,

$$V^i = -l^i \sigma^i \kappa_{ss}^i \quad (: \text{ surface diffusion flow equation}),$$
- (B) at $\Gamma^i(t) \cap \partial\Omega$,
 (angle condition)

$$\Gamma^i(t) \perp \partial\Omega \quad (: \text{ contact angle is } \pi/2),$$
 (no flux condition)

$$\kappa_s^i = 0,$$
- (C) at a triple junction $m(t)$,
 (angle condition)

$$\angle(\Gamma^1(t), \Gamma^2(t)) = \theta^3, \quad \angle(\Gamma^2(t), \Gamma^3(t)) = \theta^1, \quad \angle(\Gamma^3(t), \Gamma^1(t)) = \theta^2,$$
 (Young's law)

$$\frac{\sigma^1}{\sin \theta^1} = \frac{\sigma^2}{\sin \theta^2} = \frac{\sigma^3}{\sin \theta^3}$$
 (the balance of fluxes)

$$l^1 \sigma^1 \kappa_s^1 = l^2 \sigma^2 \kappa_s^2 = l^3 \sigma^3 \kappa_s^3,$$
 (continuity of the chemical potential)

$$\sigma^1 \kappa^1 + \sigma^2 \kappa^2 + \sigma^3 \kappa^3 = 0,$$

with the initial condition

$$(D) \Gamma(0) = \Gamma_0 := \bigcup_{i=1}^3 \Gamma_0^i, \quad m(0) = m_0,$$

Here, V^i and κ^i stand for the normal velocity and the curvature of $\Gamma^i(t)$ respectively, and s denotes the arc-length parameter of $\Gamma^i(t)$. s runs from $m(t)$, at which $s = 0$, to the

point of intersection of $\Gamma^i(t)$ with $\partial\Omega$, at which $s = L[\Gamma^i(t)]$, where $L[\Gamma^i(t)]$ denotes the total length of $\Gamma^i(t)$. $V^i(t, s)$ and $\kappa^i(t, s)$ are computed to the direction of the unit normal $N^i(t, s)$ to $\Gamma^i(t)$ at s . Moreover l^i, σ^i , and θ^i are positive constants with $\theta^1 + \theta^2 + \theta^3 = 2\pi$, and $\angle(\Gamma^i(t), \Gamma^j(t))$ ($i, j = 1, 2, 3, i \neq j$) stands for the angle between $\Gamma^i(t)$ and $\Gamma^j(t)$. In [1] H. Garcke and A. Novick-Cohen also studied the problem (A)-(D) to obtain both a local existence result for $\Gamma_0 \in C^{4+\alpha}$ ($0 < \alpha < 1$) with a suitable compatibility condition and a uniqueness result in a geometric sense.

In this talk, our goal is to obtain a global solution $\Gamma(t)$ with triple junction $m(t)$ of the problem (A)-(D) for $\Gamma_0 \in C^3$ with a suitable compatibility condition in a symmetric framework and also to show its convergence to a stationary solution determined by the initial data as $t \rightarrow \infty$. As far as we know this is a first contribution to global results for the problem (A)-(D).

2 Symmetric framework

Let us explain our framework for the problem (A)-(D). Let $\Omega = \{(x, y) \in \mathbf{R}^2 ; -a < x < 0, -b < y < b\}$, where a and b are positive constants large enough. We consider the evolution such that $\Gamma^1(t)$ always stays a segment on x -axis, $\Gamma^2(t)$ and $\Gamma^3(t)$ are symmetric with respect to the x -axis, and $\Gamma^3(t)$ is in $\{(x, y) \in \Omega ; y \geq 0\}$. In addition, let $\theta \in (0, \pi/2)$, and set $\theta^1 = 2\theta, \theta^2 = \theta^3 = \pi - \theta$. For simplicity we put $\sigma^2 = \sigma^3 = 1$ and $l^1 = l^2 = l^3 = 1$. Then Young's law and the balance of fluxes condition are simplified to $\sigma^1 = 2 \cos \theta$ and $\kappa^1 = \kappa^2 = \kappa^3$, and by the symmetry the condition on the continuity of the chemical potential is automatically fulfilled. Moreover we set

$$\begin{aligned} \mu[\xi] &:= (-\xi, 0) : \text{triple junction,} \\ \Lambda[u, \xi] &:= \bigcup_{i=1}^3 \Lambda^i[u, \xi], \\ \Lambda^1[u, \xi] &:= \{(x, 0) ; -a \leq x \leq -\xi\}, \\ \Lambda^2[u, \xi] &:= \{(x, -u(x)) ; -\xi \leq x \leq 0\}, \\ \Lambda^3[u, \xi] &:= \{(x, u(x)) ; -\xi \leq x \leq 0\}. \end{aligned}$$

Then we define the following.

Definition 2.1 *We say that a curve Γ belongs to \mathcal{S}_θ , if there are $\xi \in (0, a)$ and non-negative function $u \in C^3[-\xi, 0]$ with $u(-\xi) = 0, u_x(-\xi) = \tan \theta, u_x(0) = 0$, and $\partial_x(u_{xx}(1 + u_x^2)^{-3/2}) = 0$ at $x = -\xi, 0$ such that $\Gamma = \Lambda[u, \xi]$ and $m = \mu[\xi]$.*

Definition 2.2 *Let $A > 0$ be a given constant. We say that a curve Γ belongs to $\mathcal{C}_{\theta, A}$, if there are $\xi \in (0, a)$ and non-negative function $u \in H^2(-\xi, 0)$ with $u(-\xi) = 0, u_x(-\xi) = \tan \theta, u_x(0) = 0$, and $\int_{-\xi}^0 u(x) dx = A$ such that $\Gamma = \Lambda[u, \xi]$ and $m = \mu[\xi]$.*

Note that a curve in \mathcal{S}_θ or in $\mathcal{C}_{\theta,A}$ is symmetric with respect to the x -axis. In view of the structure of the evolution problem (A)-(D) it can be expected that if $\Gamma_0 \in \mathcal{S}_\theta$, then the solution $\Gamma(t)$ of (A)-(D) also belongs to \mathcal{S}_θ for all $t > 0$ as long as it exists. So we consider the evolution problem (A)-(D) on \mathcal{S}_θ and set $\Gamma(t) = \Lambda[u(t, \cdot), \xi(t)]$ with $m(t) = \mu[\xi(t)]$ and $\Gamma_0 = \Lambda[u_0, \xi_0]$ with $m_0 = \mu[\xi_0]$. Then (u, ξ) is the unknown function to be looked for and the equations (A)-(C) for $t > 0$ is reduced to

$$\begin{cases} u_t = -\partial_x \left(\frac{1}{(1+u_x^2)^{1/2}} \partial_x \left(\frac{u_{xx}}{(1+u_x^2)^{3/2}} \right) \right), & -\xi(t) < x < 0, \\ u_x(t, -\xi(t)) = \tan \theta, & u_x(t, 0) = 0, \\ \partial_x \left(\frac{u_{xx}}{(1+u_x^2)^{3/2}} \right) = 0, & \text{at } x = -\xi(t) \text{ and } 0, \\ u(t, -\xi(t)) = 0, \end{cases} \quad (2.1)$$

and the initial condition (D) is reduced to

$$\xi(0) = \xi_0, \quad u(0, x) = u_0(x) \quad \text{for } -\xi_0 < x < 0. \quad (2.2)$$

Thus our task is reduced to solve the problem (2.1)-(2.2).

3 Variational analysis for the energy

Let $A > 0$ satisfy $\xi_{\theta,A} < a$ where

$$\begin{aligned} \xi_{\theta,A} &= r_{\theta,A} \sin \theta, \\ r_{\theta,A} &= \left(\frac{2A}{\theta - \sin \theta \cos \theta} \right)^{1/2} \end{aligned}$$

For such A , we also set

$$u_{\theta,A}(x) = -r_{\theta,A} \cos \theta + (r_{\theta,A}^2 - x^2)^{1/2}, \quad x \in [-\xi_{\theta,A}, 0]$$

Then, if we define

$$\Gamma_{\theta,A} = \Lambda[u_{\theta,A}, \xi_{\theta,A}], \quad (3.1)$$

$\Gamma_{\theta,A}$ belongs to $\mathcal{C}_{\theta,A}$ and it consists of one segment $\Gamma_{\theta,A}^1 = \Lambda^1[u_{\theta,A}, \xi_{\theta,A}]$ on the x -axis and two circular arcs $\Gamma_{\theta,A}^i = \Lambda^i[u_{\theta,A}, \xi_{\theta,A}]$ ($i = 2, 3$), which are symmetric each other with respect to the x -axis. $\Gamma_{\theta,A}$ is also the unique stationary solution of (2.1) such that the area enclosed by $\Gamma_{\theta,A}$, the x -axis, and the y -axis is equal to A . In addition, $\Gamma_{\theta,A}$ has another important meaning. To explain this, as in [1], we introduce the associated energy E with the problem (2.1) defined by

$$E[\Gamma] := (a - \xi) \cos \theta + \int_{-\xi}^0 (1 + u_x^2)^{1/2} dx \quad \text{for } \Gamma = \Lambda[u, \xi] \in \mathcal{C}_{\theta,A}.$$

Then we obtain the following theorem.

Theorem 3.1 *The functional $E : C_{\theta,A} \rightarrow \mathbf{R}$ has a unique minimizer in $C_{\theta,A}$, which coincides with $\Gamma_{\theta,A}$ in (3.1).*

A similar argument can be valid for the length of $\Gamma^3 = \Lambda^3[u, \xi]$;

$$L[\Gamma^3] := \int_{-\xi}^0 (1 + u_x^2)^{1/2} dx.$$

That is, we get the following result.

Corollary 3.2 *The functional $L : \Gamma^3 \mapsto L[\Gamma^3]$ with $\Gamma = \bigcup_{i=1}^3 \Gamma^i \in C_{\theta,A}$ has a unique minimizer $\Gamma_{\theta,A}$ defined by (3.1).*

Now we turn our attention to the solution $\Gamma(t)$ of the problem (A)-(D) with the initial data $\Gamma_0 \in C^3$. Let $A(t)$ be the area enclosed by $\Gamma^3(t)$, the x -axis, and the y -axis, and also let A_0 be the area enclosed by Γ_0^3 , the x -axis, and the y -axis. First we mention the energy-decreasing and area-preserving properties found by H. Garcke and A. Novick-Cohen [1] in the following lemma.

Lemma 3.3 (H. Garcke and A. Novick-Cohen [1])

There hold for $t \geq 0$

$$\frac{d}{dt} E[\Gamma(t)] = - \int_0^{L[\Gamma^3(t)]} (\kappa_s^3(t, s))^2 ds,$$

$$A(t) \equiv A_0.$$

Then we have a priori estimate for $E[\Gamma(t)]$ and $L[\Gamma^3(t)]$.

Proposition 3.4 *There hold for $t \geq 0$*

$$\begin{aligned} E[\Gamma_{\theta,A}] &\leq E[\Gamma(t)] \leq E[\Gamma_0], \\ L[\Gamma_{\theta,A}^3] &\leq L[\Gamma^3(t)] \leq E[\Gamma_0]. \end{aligned}$$

4 Global existence and convergence to the minimizer of the energy

The purpose of this section is to obtain the global solution of (2.1) when the initial data Γ_0 and the stationary solution Γ_{θ,A_0} , which is the minimizer of the energy E , are sufficiently close to each other in some sense, and to derive that its global solution converges to Γ_{θ,A_0} as $t \rightarrow \infty$. To prove this, we shall first extend the result of H. Garcke and A. Novick-Cohen [1]. That is, we show a local-time existence result of (2.1)-(2.2) depending on the magnitude

of $\Gamma_0 \in C^{2+\alpha}$ ($0 < \alpha < 1$). This expansion is useful to obtain an a priori estimate for the solution $\Gamma(t)$ of (2.1)-(2.2).

To derive the local existence result, we prepare the following.

We shall first obtain the ξ ' equation. Assume that (u, ξ) satisfies (2.1)-(2.2). Then we differentiate the equation $u(t, -\xi(t)) = 0$. Using the equations $u_x(t, -\xi(t)) = \tan \theta$ and $\partial_x(u_{xx}/(1+u_x^2)^{3/2}) = 0$ at $x = -\xi(t)$, we get

$$\dot{\xi}(t) = -C_1(\theta)u_{xxxx}(t, -\xi(t)) + C_2(\theta)u_{xx}^3(t, -\xi(t)) \quad (4.1)$$

where $C_1(\theta) = 1/(1+\tan^2 \theta)^2 \tan \theta$, $C_2(\theta) = 3(1+5\tan^2 \theta)/(1+\tan^2 \theta)^4 \tan \theta$. Conversely, assume that (u, ξ) satisfies (2.1)-(2.2), which excludes the equation $u(t, -\xi(t)) = 0$, and (4.1), and that the initial data u_0 satisfies $u_0(-\xi_0) = 0$. Then the equation $u(t, -\xi(t)) = 0$ is obtained. Thus we use (4.1) with $u_0(-\xi_0) = 0$ instead of the equation $u(t, -\xi(t)) = 0$.

In addition, we perform the transformation

$$\eta = 1 + \frac{x}{\xi(t)}, \quad v(t, \eta) = u(t, -(1-\eta)\xi(t)),$$

carrying (2.1)-(2.2) into the form;

$$\left\{ \begin{array}{l} v_t = f(\eta, v_\eta, v_{\eta\eta}, v_{\eta\eta\eta}, v_{\eta\eta\eta\eta}, \xi, \dot{\xi}) \quad \text{for } (t, \eta) \in (0, T) \times (0, 1), \\ v_\eta(t, 0) = \xi(t) \tan \theta, \quad v_\eta(t, 1) = 0, \\ v_{\eta\eta\eta}(t, 0) = \frac{3 \tan \theta}{1 + \tan^2 \theta} \cdot \frac{v_{\eta\eta}^2(t, 0)}{\xi(t)}, \quad v_{\eta\eta\eta}(t, 1) = 0, \\ v(0, \eta) = v_0(\eta), \\ \dot{\xi}(t) = -C_1(\theta) \frac{v_{\eta\eta\eta\eta}(t, 0)}{\xi^4(t)} + C_2(\theta) \frac{v_{\eta\eta}^3(t, 0)}{\xi^6(t)} \end{array} \right. \quad (4.2)$$

with $\xi(0) = \xi_0$, where

$$\begin{aligned} & f(\eta, v_\eta, v_{\eta\eta}, v_{\eta\eta\eta}, v_{\eta\eta\eta\eta}, \xi, \dot{\xi}) \\ &= -\frac{1}{(\xi^2 + v_\eta^2)^2} v_{\eta\eta\eta\eta} + \frac{10v_\eta v_{\eta\eta}}{(\xi^2 + v_\eta^2)^3} v_{\eta\eta\eta} + \frac{3(\xi^2 - 5v_\eta^2)v_{\eta\eta}^3}{(\xi^2 + v_\eta^2)^4} - \frac{(1-\eta)\dot{\xi}v_\eta}{\xi}. \end{aligned}$$

Here, we set $I = [0, 1]$, $R_T = (0, T] \times I$, and $R_T^\delta = (\delta, T] \times I$ for any $\delta \in (0, T)$. In addition, we define the spaces as

$$\begin{aligned} \mathcal{Y}_{[0, T]}^{(0)} &:= C^{0, 2+\alpha}(\bar{R}_T) \cap C^{1/2}([0, T]; C^\alpha(I)) \\ \mathcal{Y}_{(0, T]}^{(1)} &:= \{v \in \mathcal{Y}_{[0, T]}^{(0)} \cap C^{1, 4+\alpha}(R_T); \\ & \quad \|v\|_{\mathcal{Y}^{(1)}} := \|v\|_{\mathcal{Y}^{(0)}} + \sup_{0 < \delta < T} \delta^{1/4} \|v_{\eta\eta\eta}\|_{C^{0, \alpha}(\bar{R}_T^\delta)} \end{aligned}$$

$$\begin{aligned}
& + \sup_{0 < \delta < T} \delta^{1/2} \|v_{\eta\eta\eta\eta}\|_{C^{0,\alpha}(\bar{R}_T^\delta)} \\
& + \sup_{0 < \delta < T} \delta^{1/2} \|v_t\|_{C^{0,\alpha}(\bar{R}_T^\delta)} < \infty\},
\end{aligned}$$

$$\mathcal{Y}_{(0,T)}^{(2)} := \{\xi \in C[0,T] \cap C^1(0,T);\}$$

$$\|\xi\|_{\mathcal{Y}^{(2)}} := \|\xi\|_{C[0,T]} + \sup_{0 < \delta < T} \delta^{1/2} \|\dot{\xi}\|_{C^1[\delta,T]} < \infty\},$$

for $\alpha \in (0,1)$. Here the spaces $C^\beta([t_1, t_2]; D)$, $C^{0,k+\beta}([t_1, t_2] \times I)$, $C^{1,k+\beta}([t_1, t_2] \times I)$ ($k = 1, 2, \dots$; $0 < \beta < 1$) used here are defined in [2] and [3].

Then, we obtain the following theorem.

Theorem 4.1 (*Local existence*) *Let us assume that $v_0 \in C^{2+\alpha}(I)$ and that v_0 satisfies $v_0(0) = 0$, $v_{0x}(0) = \xi_0 \tan \theta$, $v_{0x}(1) = 0$. Then, there exists a $T = T(\xi_0, 1/\|v_0\|_{C^{2+\alpha}(I)})$ such that the problem (4.2) with $\xi(0) = \xi_0$ has a unique solution $(v, \xi) \in \mathcal{Y}_{(0,T)}^{(1)} \times \mathcal{Y}_{(0,T)}^{(2)}$.*

Remark 4.2 *Existence time T obtained by 4.1 increases with ξ_0 and $1/\|v_0\|_{C^{2+\alpha}(I)}$. So, if we choose $(\xi(t_0), v(t_0, \cdot))$ (where $t_0 > 0$) as the initial data, existence time T is a increases with $\xi(t_0), 1/\|v(t_0, \cdot)\|_{C^{2+\alpha}(I)}$. Thus, as long as there exist constants ν_1, ν_2 such that*

$$0 < \nu_1 \leq \xi(t_0), \quad \|v(t_0, \cdot)\|_{C^{2+\alpha}(I)} \leq \nu_2 < \infty,$$

existence time T is bigger than t_0 . That is,

$$T = T(\xi(t_0), 1/\|v(t_0, \cdot)\|_{C^{2+\alpha}(I)}) \geq T(\nu_1, 1/\nu_2) > t_0.$$

Moreover we get the further regularity result.

Theorem 4.3 *Let (v, ξ) be the solution obtained from 4.1. Then,*

$$(v, \xi) \in C^{1,6+\alpha}((0,T] \times I) \times C^{1+(2+\alpha)/4}(0,T].$$

Next we shall derive a priori estimate for $\xi(t)$ and $\|v(t, \cdot)\|_{C^{2+\alpha}(I)}$. For simplicity we set

$$\kappa = \kappa^3, \quad \|\cdot\|_2 = \|\cdot\|_{L^2(\Gamma^3(t))}.$$

Fix $\delta_1 > 0$ so small so that $1 - \theta/\tan \theta - \delta_1 > 0$ and let us define

$$H[d, \lambda] = 2\left(1 - \frac{\theta}{\tan \theta} - \delta_1\right) - 3d^3\lambda^2 - d^{3/2}\left(5\theta + \frac{2}{\tan \theta}\lambda\right) \quad \text{for } d, \lambda > 0.$$

Then we obtain

$$\|\kappa_s(t)\|_2^2 + \int_0^t H[E[\Gamma_0], \|\kappa_s(\tau)\|_2] \cdot \|\kappa_{sss}(\tau)\|_2^2 d\tau \leq \|\kappa_{0,s}\|_2^2 + C_{\theta, A_0}(E[\Gamma_0] - E[\Gamma_{\theta, A_0}]), \quad (4.3)$$

where $C_{\theta, A_0} = C(\delta_1)\theta^4 L[\Gamma_{\theta, A_0}^3]^{-4}$.

To make the statement precise, we put

$$\rho_0^2 := \|\kappa_{0,s}\|_2^2 + C_{\theta, A_0}(E[\Gamma_0] - E[\Gamma_{\theta, A_0}]).$$

Then we say that Γ_0 and Γ_{θ, A_0} are close to each other if $H[E[\Gamma_0], \rho_0] > 0$.

Here we assume the following conditions on initial data Γ_0 ;

$$\begin{aligned} E[\Gamma_0] > E[\Gamma_{\theta, A_0}], \quad H[E[\Gamma_0], \rho_0] > 0, \\ \theta + E[\Gamma_0]^{3/2}\rho_0 < \pi/2, \quad L[\Gamma_{\theta, A_0}^3] \frac{\sin \theta}{\theta} - \frac{1}{2}E[\Gamma_0]^{5/2}\rho_0 > 0. \end{aligned} \quad (4.4)$$

By virtue of the inequality (4.3), we obtain a priori estimates for $\xi(t)$ and $\|v(t, \cdot)\|_{C^{2+\alpha}(I)}$.

Proposition 4.4

(i) *There holds for $t \geq 0$*

$$L[\Gamma_{\theta, A_0}^3] \frac{\sin \theta}{\theta} - \frac{1}{2}E[\Gamma_0]^{5/2}\rho_0 \leq \xi(t) \leq E[\Gamma_0] \frac{\sin \theta}{\theta} + \frac{1}{2}E[\Gamma_0]^{5/2}\rho_0.$$

(ii) *Let $\alpha \in (0, 1/2]$. Then we have*

$$\|v(t, \cdot)\|_{C^{2+\alpha}(I)} \leq N_0 \quad \text{for } t \geq 0,$$

where $N_0 > 0$ is a constant depending only on Γ_0 through ρ_0 , A_0 , and $E[\Gamma_0]$.

Remark 4.5 *Owing to the third assumption in (4.4), $\Gamma^3(t)$ is prevented from a graph-breaking. Moreover, owing to the fourth assumption in (4.4), it is assured that $\xi(t) > 0$ for $t \geq 0$.*

Consequently, by virtue of Remark 4.2 and Proposition 4.4, one can always solve the problem (4.2) with $\xi(0) = \xi_0$ on the time intervals $[0, T_0]$, $[T_0, 2T_0]$, $[2T_0, 3T_0]$, \dots , for a $T_0 > 0$ which is determined only on Γ_0 . So we arrive at the following global existence result.

Theorem 4.6 *Let $\alpha \in (0, 1/2]$ and assume (4.4). Then the problem (4.2) with $\xi(0) = \xi_0$ have a unique global solution $(v, \xi) \in \mathcal{Y}_{(0, \infty)}^{(1)} \times \mathcal{Y}_{(0, \infty)}^{(2)}$.*

Then we obtain the following theorem concerning the convergence as $t \rightarrow \infty$ for the global solution of (2.1)-(2.2) obtained by Theorem 4.6.

Theorem 4.7 *Let $\Gamma(t)$ be a solution obtained by Theorem 4.6 and let Γ_{θ, A_0} be the minimizer of the energy E obtained by Theorem 3.1. Then*

$$\Gamma(t) \rightarrow \Gamma_{\theta, A_0} \quad \text{as } t \rightarrow \infty.$$

In addition, we get the following result.

Theorem 4.8 *Let $\Gamma(t)$ be a solution obtained by Theorem 4.6 and let Γ_{θ, A_0} be the minimizer of the energy E obtained by Theorem 3.1. Then*

$$\begin{aligned} E[\Gamma(t)] &\rightarrow E[\Gamma_{\theta, A_0}] & \text{as } t \rightarrow \infty, \\ L[\Gamma^3(t)] &\rightarrow L[\Gamma_{\theta, A_0}^3] & \text{as } t \rightarrow \infty. \end{aligned}$$

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A Hopf-Type Formula for Global Solutions to First-order Nonlinear PDEs with Concave-Convex Hamiltonians

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§1. Introduction

Consider the Cauchy problem for the simplest Hamilton-Jacobi equation, namely,

$$\partial u / \partial t + f(\partial u / \partial x) = 0 \quad \text{in } \mathcal{D} \stackrel{\text{def}}{=} \{t > 0, x \in \mathbb{R}^n\}, \quad (1)$$

$$u(0, x) = \phi(x) \quad \text{on } \{t = 0, x \in \mathbb{R}^n\}. \quad (2)$$

Here, the notation $\partial/\partial x$ denotes the gradient $(\partial/\partial x_1, \dots, \partial/\partial x_n)$. Let $\text{Lip}(\overline{\mathcal{D}}) \stackrel{\text{def}}{=} \text{Lip}(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, where $\text{Lip}(\mathcal{D})$ is the set of all locally Lipschitz continuous functions $u = u(t, x)$ defined on \mathcal{D} . In this note, a function $u = u(t, x)$ in $\text{Lip}(\overline{\mathcal{D}})$ will be called a *global solution* of the Cauchy problem (1)-(2) if it satisfies (1) almost everywhere in \mathcal{D} and if $u(0, x) = \phi(x)$ for all $x \in \mathbb{R}^n$. A global solution of (1)-(2) is given by explicit formulas of Hopf [see Hopf, E., Generalized solutions of nonlinear equations of first-order, *J. Math. Mech.* 14 (1965), 951-973] in the following two cases: (a) $f = f(p)$ convex (or concave) and $\phi = \phi(x)$ largely arbitrary; and (b) $\phi = \phi(x)$ convex (or concave) and $f = f(p)$ largely arbitrary. It is unlikely that such restrictions, either on $f = f(p)$ or on $\phi = \phi(x)$, are really vital. A relevant solution is expected to exist under much wider assumptions. According to Hopf, that he has been unable to get further is doubtless due to a limitation in his approach: he uses the Legendre transformation globally, and this global theory has been carried through only in the case of convex (or concave) functions [Fenchel's theory of conjugate convex (or concave) functions].

In the present note, we propose to examine a class of *concave-convex functions* as a more general framework where the discussion of the global Legendre transformation still makes sense. A Hopf-type formula for non-concave, non-convex Hamilton-Jacobi equations can thereby be considered.

We shall often suppose that $n \stackrel{\text{def}}{=} n_1 + n_2$ and that the variables $x, p \in \mathbb{R}^n$ are separated into two as $x \stackrel{\text{def}}{=} (x', x'')$, $p \stackrel{\text{def}}{=} (p', p'')$ with $x', p' \in \mathbb{R}^{n_1}$, $x'', p'' \in \mathbb{R}^{n_2}$.

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Accordingly, the zero-vector in \mathbb{R}^n will be $0 = (0', 0'')$, where $0'$ and $0''$ stand for the zero-vectors in \mathbb{R}^{n_1} and \mathbb{R}^{n_2} , respectively.

Recall that a function $f = f(p', p'')$ is called *concave-convex* if it is a concave function of $p' \in \mathbb{R}^{n_1}$ for each $p'' \in \mathbb{R}^{n_2}$ and a convex function of $p'' \in \mathbb{R}^{n_2}$ for each $p' \in \mathbb{R}^{n_1}$.

§2. Conjugate concave-convex functions

We use $|\cdot|$ and $\langle \cdot, \cdot \rangle$ to denote the Euclidean norm and scalar product, respectively, in \mathbb{R}^n . (It will cause no confusion if we use the same notation for the corresponding ones in \mathbb{R}^{n_1} or \mathbb{R}^{n_2} .) Let $f = f(p)$ be a differentiable real-valued function on an open nonempty subset A of \mathbb{R}^n . The *Legendre conjugate* of the pair (A, f) is defined to be the pair (B, g) , where B is the image of A under the gradient mapping $z = \partial f(p)/\partial p$, and $g = g(z)$ is the function on B given by the formula

$$g(z) \stackrel{\text{def}}{=} \langle z, (\partial f/\partial p)^{-1}(z) \rangle - f((\partial f/\partial p)^{-1}(z)).$$

It is not actually necessary to have $z = \partial f(p)/\partial p$ one-to-one on A in order that $g = g(z)$ be well-defined (i.e., single-valued). It suffices if

$$\langle z, p^1 \rangle - f(p^1) = \langle z, p^2 \rangle - f(p^2)$$

whenever $\partial f(p^1)/\partial p = \partial f(p^2)/\partial p = z$. Then the value $g(z)$ can be obtained unambiguously from the formula by replacing the set $(\partial f/\partial p)^{-1}(z)$ by any of the vectors it contains.

Passing from (A, f) to the Legendre conjugate (B, g) , if the latter is well-defined, is called the *Legendre transformation*. The important role played by the Legendre transformation in the classical local theory of nonlinear equations of first-order is well-known. The global Legendre transformation has been studied extensively for convex functions. In the case where $f = f(p)$ and A are convex, we can extend $f = f(p)$ to be a lower semicontinuous convex function on all of \mathbb{R}^n with A as the interior of its effective domain. If this extended $f = f(p)$ is proper, then the Legendre conjugate (B, g) of (A, f) is well-defined. Moreover, B is a subset of $\text{dom } f^*$ (namely the range of $\partial f/\partial p$), and $g = g(z)$ is the restriction of the Fenchel conjugate $f^* = f^*(z)$ to B .

All concave-convex functions $f = f(p', p'')$ under our consideration are assumed to be finite and to satisfy the following two "growth conditions."

$$\lim_{|p''| \rightarrow +\infty} \frac{f(p', p'')}{|p''|} = +\infty \quad \text{for each } p' \in \mathbb{R}^{n_1}. \quad (3)$$

$$\lim_{|p'| \rightarrow +\infty} \frac{f(p', p'')}{|p'|} = -\infty \quad \text{for each } p'' \in \mathbb{R}^{n_2}. \quad (4)$$

Let $f^{*2} = f^{*2}(p', z'')$ [resp. $f^{*1} = f^{*1}(z', p'')$] be, for each fixed $p' \in \mathbb{R}^{n_1}$ [resp. $p'' \in \mathbb{R}^{n_2}$], the Fenchel conjugate of a given p'' -convex [resp. p' -concave] function $f = f(p', p'')$. In other words,

$$f^{*2}(p', z'') \stackrel{\text{def}}{=} \sup_{p'' \in \mathbb{R}^{n_2}} \{ \langle z'', p'' \rangle - f(p', p'') \} \quad (5)$$

$$\text{[resp. } f^{*1}(z', p'') \stackrel{\text{def}}{=} \inf_{p' \in \mathbb{R}^{n_1}} \{ \langle z', p' \rangle - f(p', p'') \}] \quad (6)$$

for $(p', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ [resp. $(z', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$]. If $f = f(p', p'')$ is concave-convex, then the definition (5) [resp. (6)] actually implies the convexity [resp. concavity] of $f^{*2} = f^{*2}(p', z'')$ [resp. $f^{*1} = f^{*1}(z', p'')$] not only in the variable $z'' \in \mathbb{R}^{n_2}$ [resp. $z' \in \mathbb{R}^{n_1}$] but also in the whole variable $(p', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ [resp. $(z', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$]. Moreover, under the condition (3) [resp. (4)], the finiteness of $f = f(p', p'')$ clearly yields that of $f^{*2} = f^{*2}(p', z'')$ [resp. $f^{*1} = f^{*1}(z', p'')$] with

$$\lim_{|z''| \rightarrow +\infty} \frac{f^{*2}(p', z'')}{|z''|} = +\infty \quad \text{[resp.} \quad \lim_{|z'| \rightarrow +\infty} \frac{f^{*1}(z', p'')}{|z'|} = -\infty]$$

locally uniformly in $p' \in \mathbb{R}^{n_1}$ [resp. $p'' \in \mathbb{R}^{n_2}$].

If (4) [resp. (3)] is satisfied, then (5) [resp. (6)] gives

$$\frac{f^{*2}(p', z'')}{|p'|} \geq -\frac{f(p', 0'')}{|p'|} \rightarrow +\infty \quad \text{as} \quad |p'| \rightarrow +\infty \quad (7)$$

$$\text{[resp.} \quad \frac{f^{*1}(z', p'')}{|p''|} \leq -\frac{f(0', p'')}{|p''|} \rightarrow -\infty \quad \text{as} \quad |p''| \rightarrow +\infty] \quad (8)$$

uniformly in $z'' \in \mathbb{R}^{n_2}$ [resp. $z' \in \mathbb{R}^{n_1}$].

Now let $f = f(p', p'')$ be a concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Besides “partial conjugates” $f^{*2} = f^{*2}(p', z'')$ and $f^{*1} = f^{*1}(z', p'')$, we shall consider the following two “total conjugates” of $f = f(p', p'')$. The first one, which we denote by $\bar{f}^* = \bar{f}^*(z', z'')$, is defined as the Fenchel conjugate of the concave function $\mathbb{R}^{n_1} \ni p' \mapsto -f^{*2}(p', z'')$; more precisely,

$$\bar{f}^*(z', z'') \stackrel{\text{def}}{=} \inf_{p' \in \mathbb{R}^{n_1}} \{ \langle z', p' \rangle + f^{*2}(p', z'') \} \quad (9)$$

for each $(z', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The second, $\underline{f}^* = \underline{f}^*(z', z'')$, is defined as the Fenchel conjugate of the convex function $\mathbb{R}^{n_2} \ni p'' \mapsto -f^{*1}(z', p'')$; i.e.,

$$\underline{f}^*(z', z'') \stackrel{\text{def}}{=} \sup_{p'' \in \mathbb{R}^{n_2}} \{ \langle z'', p'' \rangle + f^{*1}(z', p'') \} \quad (10)$$

for $(z', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. By (5)-(6) and (9)-(10), we have

$$\bar{f}^*(z', z'') = \inf_{p' \in \mathbb{R}^{n_1}} \sup_{p'' \in \mathbb{R}^{n_2}} \{ \langle z', p' \rangle + \langle z'', p'' \rangle - f(p', p'') \}, \quad (11)$$

$$\underline{f}^*(z', z'') = \sup_{p'' \in \mathbb{R}^{n_2}} \inf_{p' \in \mathbb{R}^{n_1}} \{ \langle z', p' \rangle + \langle z'', p'' \rangle - f(p', p'') \}. \quad (12)$$

Therefore, $\bar{f}^* = \bar{f}^*(z', z'')$ and $\underline{f}^* = \underline{f}^*(z', z'')$ will usually be called the *upper* and *lower conjugates*, respectively, of $f = f(p', p'')$. [Of course, (11)-(12) imply $\bar{f}^*(z', z'') \geq \underline{f}^*(z', z'')$.] For any $z' \in \mathbb{R}^{n_1}$, the function

$$\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto h(p', z'') \stackrel{\text{def}}{=} \langle z', p' \rangle + f^{*2}(p', z'')$$

is convex. Thus (9) shows that $\bar{f}^* = \bar{f}^*(z', z'')$ as a function of z'' is the *image*

$$\mathbb{R}^{n_2} \ni z'' \mapsto (Ah)(z'') \stackrel{\text{def}}{=} \inf \{ h(p', z'') : A(p', z'') = z'' \}$$

of $h = h(p', z'')$ under the (linear) projection $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto A(p', z'') \stackrel{\text{def}}{=} z''$. It follows that $\bar{f}^* = \bar{f}^*(z', z'')$ is convex in $z'' \in \mathbb{R}^{n_2}$. On the other hand, by definition, $\bar{f}^* = \bar{f}^*(z', z'')$ is necessarily concave in $z' \in \mathbb{R}^{n_1}$. This upper conjugate is hence a concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The same conclusion may dually be drawn for the lower conjugate $\underline{f}^* = \underline{f}^*(z', z'')$.

We have previously seen that if the concave-convex function $f = f(p', p'')$ is finite on the whole $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and satisfies (3)-(4), its partial conjugates $f^{*2} = f^{*2}(p', z'')$ and $f^{*1} = f^{*1}(z', p'')$ must both be finite with (7)-(8) holding. Therefore, $\bar{f}^* = \bar{f}^*(z', z'')$ and $\underline{f}^* = \underline{f}^*(z', z'')$ are then also finite; and hence it is known that they coincide [for this see Corollary 37.1.2 in Rockafellar, R.T., *Convex analysis*, Princeton Univ. Press, 1970]. In this situation, the *conjugate*

$$f^* = f^*(z', z'') \stackrel{\text{def}}{=} \bar{f}^*(z', z'') = \underline{f}^*(z', z'') \quad (13)$$

of $f = f(p', p'')$ will simultaneously has the properties:

$$\lim_{|z''| \rightarrow +\infty} \frac{f^*(z', z'')}{|z''|} = +\infty \quad \text{for each } z' \in \mathbb{R}^{n_1}, \quad (3^*)$$

$$\lim_{|z'| \rightarrow +\infty} \frac{f^*(z', z'')}{|z'|} = -\infty \quad \text{for each } z'' \in \mathbb{R}^{n_2}. \quad (4^*)$$

A finite concave-convex function $f = f(p', p'')$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is said to be *strict* if its concavity in $p' \in \mathbb{R}^{n_1}$ and convexity in $p'' \in \mathbb{R}^{n_2}$ are both strict. It will then also be called a *strictly concave-convex function* on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We use the following technical preparations:

Lemma. *Let $f = f(p', p'')$ be a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with (3) [resp. (4)] holding. Then its partial conjugate $f^{*2} = f^{*2}(p', z'')$ [resp.*

$f^{*1} = f^{*1}(z', p'')$ defined by (5) [resp. (6)] is strictly convex [resp. concave] in $p' \in \mathbb{R}^{n_1}$ [resp. $p'' \in \mathbb{R}^{n_2}$] and everywhere differentiable in $z'' \in \mathbb{R}^{n_2}$ [resp. $z' \in \mathbb{R}^{n_1}$]. Besides that, the gradient mapping $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto \partial f^{*2}(p', z'')/\partial z''$ [resp. $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (z', p'') \mapsto \partial f^{*1}(z', p'')/\partial z'$] is continuous and satisfies the identity

$$\begin{aligned} f^{*2}(p', z'') &\equiv \langle z'', \partial f^{*2}(p', z'')/\partial z'' \rangle - f(p', \partial f^{*2}(p', z'')/\partial z'') \\ \text{[resp. } f^{*1}(z', p'') &\equiv \langle z', \partial f^{*1}(z', p'')/\partial z' \rangle - f(\partial f^{*1}(z', p'')/\partial z', p''). \end{aligned}$$

Proposition. Let $f = f(p', p'')$ be a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with both (3) and (4) holding. Then its conjugate $f^* = f^*(z', z'')$ defined by (9)-(13) is also a concave-convex function satisfying (3*)-(4*). Moreover, $f^* = f^*(z', z'')$ is everywhere continuously differentiable with

$$f^*(z', z'') \equiv \left\langle z', \frac{\partial f^*}{\partial z'}(z', z'') \right\rangle + \left\langle z'', \frac{\partial f^*}{\partial z''}(z', z'') \right\rangle - f\left(\frac{\partial f^*}{\partial z'}(z', z''), \frac{\partial f^*}{\partial z''}(z', z'')\right).$$

§3. A Hopf-type formula

We now consider the Cauchy problem

$$\partial u/\partial t + f(\partial u/\partial x) = 0 \quad \text{in } \mathcal{D} \stackrel{\text{def}}{=} \{t > 0, x = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}, \quad (14)$$

$$u(0, x) = \phi(x) \quad \text{on } \{t = 0, x = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}. \quad (15)$$

An explicit global solution $u = u(t, x) = u(t, x', x'')$ of the problem will be found under the following three standing hypotheses.

(I) The initial function $\phi = \phi(x) = \phi(x', x'')$ is of class C^0 and the Hamiltonian $f = f(p) = f(p', p'')$ is strictly concave-convex on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with (3)-(4) holding.

(II) The equality

$$\sup_{y' \in \mathbb{R}^{n_1}} \inf_{y'' \in \mathbb{R}^{n_2}} \zeta(t, x, y) = \inf_{y'' \in \mathbb{R}^{n_2}} \sup_{y' \in \mathbb{R}^{n_1}} \zeta(t, x, y)$$

is satisfied in \mathcal{D} , where

$$\zeta(t, x, y) \stackrel{\text{def}}{=} \phi(y) + t \cdot f^*((x - y)/t) \quad (16)$$

for $(t, x) = (t, x', x'') \in \mathcal{D}$, $y \stackrel{\text{def}}{=} (y', y'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Here, $f^* = f^*(z) = f^*(z', z'')$ denotes the conjugate defined by (9)-(13) of $f = f(p', p'')$.

(III) To each bounded subset V of \mathcal{D} there corresponds a positive number $N(V)$ so that

$$\begin{aligned} \min_{\substack{w'' \in \mathbb{R}^{n_2} \\ |w''| \leq N(V)}} \sup_{w' \in \mathbb{R}^{n_1}} \zeta(t, x, w', w'') &< \sup_{w' \in \mathbb{R}^{n_1}} \zeta(t, x, w', y''), \\ \max_{\substack{w' \in \mathbb{R}^{n_1} \\ |w'| \leq N(V)}} \inf_{w'' \in \mathbb{R}^{n_2}} \zeta(t, x, w', w'') &> \inf_{w'' \in \mathbb{R}^{n_2}} \zeta(t, x, y', w'') \end{aligned}$$

whenever $(t, x) \in V$, $y \stackrel{\text{def}}{=} (y', y'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $\min\{|y'|, |y''|\} > N(V)$.

The main result of this note reads as follows.

Theorem. Assume (I)-(III). Then the formula

$$u(t, x) \stackrel{\text{def}}{=} \sup_{y' \in \mathbb{R}^{n_1}} \inf_{y'' \in \mathbb{R}^{n_2}} \zeta(t, x, y) = \inf_{y'' \in \mathbb{R}^{n_2}} \sup_{y' \in \mathbb{R}^{n_1}} \zeta(t, x, y) \quad \text{for } (t, x) \in \mathcal{D} \quad (17)$$

determines a global solution of the Cauchy problem (14)-(15).

Remark 1. Our hypotheses imply that a ‘‘saddle-point’’ (y', y'') of the function $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni y = (y', y'') \mapsto \zeta(t, x, y)$ by (16) [with respect to maximizing over \mathbb{R}^{n_1} and minimizing over \mathbb{R}^{n_2}] exists. If $f = f(p', p'')$ has a special representation

$$f(p', p'') \equiv g_2(p'') - g_1(p') \quad \text{on } \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

with $g_1 = g_1(p')$ and $g_2 = g_2(p'')$ convex, then one can use the ‘‘index of non-convexity’’ and a classical minimax theorem to give sufficient conditions for (II) to hold. [Concerning this question, see Bardi, M. and Faggian, S., Hopf-type estimates and formulas for non-convex non-concave Hamilton-Jacobi equations, *SIAM J. Math. Anal.* **29** (1998), 1067-1086.]

Remark 2. If $n_1 = 0$ or $n_2 = 0$, the Hopf formulas for convex or concave Hamiltonians [see Hopf, E., Generalized solutions of nonlinear equations of first-order, *J. Math. Mech.* **14** (1965), 951-973] will be obtained from (17).

Corollary. Under Hypotheses (I)-(II), suppose that

$$\begin{aligned} \liminf_{|x''| \rightarrow +\infty} \frac{\phi(x', x'')}{|x''|} &> -\infty \quad \text{locally uniformly in } x' \in \mathbb{R}^{n_1}, \\ \limsup_{|x'| \rightarrow +\infty} \frac{\phi(x', x'')}{|x'|} &< +\infty \quad \text{locally uniformly in } x'' \in \mathbb{R}^{n_2}. \end{aligned}$$

Then (17) determines a global solution of the Cauchy problem (14)-(15).

Semilinear elliptic equations on a thin network-shaped domain

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We consider the following semilinear elliptic equation in a thin network-shaped domain $\Omega(\zeta) \subset \mathbf{R}^n$ with variable thickness (see Figure 1):

$$(1) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(\zeta), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta) \end{cases}$$

where ν denotes the unit outward normal vector on $\partial\Omega(\zeta)$ and f is a real valued smooth function on \mathbf{R} . We consider a situation that $\Omega(\zeta)$ approaches a certain geometric graph when ζ tends to zero (see Figure 2). In this situation, we deal with the asymptotic behavior of the solution of (1) as $\zeta \rightarrow 0$.

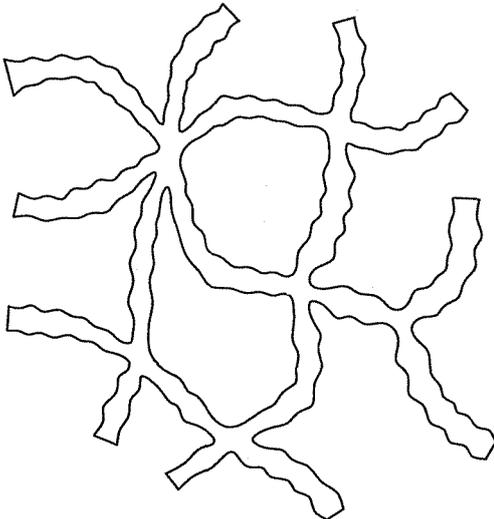


Figure 1

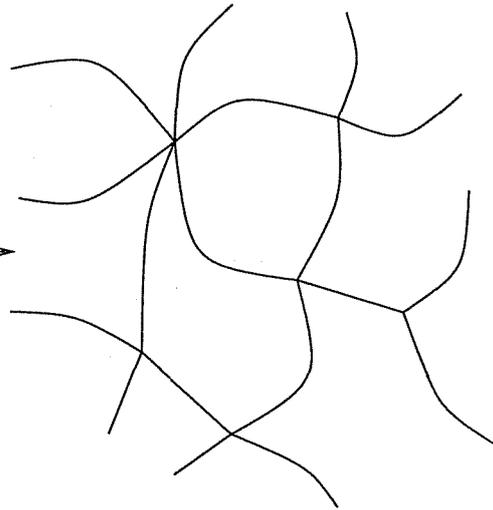


Figure 2

To simplify an argument, we consider a simple graph such that several line segments meet one point, that is, let $p_i(s)$ ($0 \leq s \leq l_i$, $i = 1, \dots, m$) be the arcs of the graph with $p_1(0) = \dots = p_m(0)$ where l_i denotes the length of the arc and

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

m is the number of the arcs. Let $S_i(\zeta, s)$ be the intersection of $\Omega(\zeta)$ and an $n - 1$ dimensional normal plane at $p_i(s)$ for $s \in (\zeta, l_i)$. We assume an $n - 1$ dimensional domain $D_i(s) = \zeta^{-1} S_i(\zeta, s)$ is invariable when ζ varies and let $a_i(s)$ be the volume of $D_i(s)$.

We consider that the associated limit equation on the graph is

$$(2) \quad \begin{cases} \frac{1}{a_i(s)} \frac{d}{ds} \left(a_i(s) \frac{du_i}{ds} \right) + f(u_i) = 0 & 0 \leq s \leq l_i, 1 \leq i \leq m, \\ u_1(0) = \cdots = u_m(0), \\ \sum_{i=1}^m a_i(0) \frac{du_i}{ds}(0) = 0, \\ \frac{du_i}{ds}(l_i) = 0 & 1 \leq i \leq m. \end{cases}$$

Our first purpose is to show that the solution of (1) converges uniformly to a solution of (2) as $\zeta \rightarrow 0$. This is proved if the solution of (1) is bounded independently of ζ . Conversely, the following problem occurs naturally. Let a solution of (2) be given. Can we show the existence of a solution of (1) which approaches the solution of (2)? Our second purpose is to show that if the eigenvalue problem of the linearized equation around the solution of (2) has no zero eigenvalue then there exists a solution of (1) which approaches the solution of (2). These results are stronger than the results of Kosugi [11] in the sense that the domain is not necessarily required to be constricted.

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Multiple positive solutions for semilinear elliptic equations *

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1. Introduction

In this talk, we study the following nonlinear elliptic problems:

$$\begin{cases} -\Delta u + u = a(x)u^p + f(x) & \text{in } \mathbf{R}^N, \\ u(x) > 0 & \text{in } \mathbf{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (*)_f$$

In particular, we would like to consider the continuous dependence of solutions to the data $a(x)$, $f(x)$, etc. For example, suppose that $f(x) \rightarrow 0$ in a suitable function space, then we would like to ask the following question:

“Does a solution $u(x; f)$ of $(*)_f$ converge to a solution $u(x; 0)$ of the following $(*)_0$ as $f \rightarrow 0$?”

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{in } \mathbf{R}^N, \\ u(x) > 0 & \text{in } \mathbf{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (*)_0$$

Such a question is important in various situations (not only for $(*)_f$). In general, it is a very delicate problem and here we give some examples.

2. Results

In what follows, we assume

$$1 < p < \frac{N+2}{N-2} \quad \text{if } N \geq 3, \quad 1 < p < \infty \quad \text{if } N = 1, 2$$

and

$$(A0) \quad a(x) \in C(\mathbf{R}^N, \mathbf{R}).$$

* Joint work with Shinji Adachi, Department of Mathematics, Waseda University, Japan

- (A1) $a(x) > 0$ for all $x \in \mathbf{R}^N$.
(A2) $a(x) \rightarrow 1$ as $|x| \rightarrow \infty$.
(A3) There exist $\delta > 0$ and $C > 0$ such that

$$a(x) \geq 1 - Ce^{-(2+\delta)|x|}$$

for all $x \in \mathbf{R}^N$.

Under these assumptions, the existence of at least one positive solution of $(*)_0$ is shown by Bahri-Li [2] (See also [7, 3]).

Now we add a small non-negative inhomogeneous term $f(x) \geq 0$. We consider the existence of positive solutions. This question is studied by [4, 5, 6, 8] under the assumption:

- (A4) $a(x) \geq 1$ for all $x \in \mathbf{R}^N$.

Their result is the following

Theorem 1 ([4, 5, 6, 8]). *Assume (A0)–(A3) and (A4). Then there exists a constant $M > 0$ such that if $f \geq 0$, $f \not\equiv 0$ and $\|f\|_{H^{-1}(\mathbf{R}^N)} \leq M$, then $(*)_f$ has at least 2 positive solutions in $H^1(\mathbf{R}^N)$.*

We can also observe that their solutions $u_1(x; f)$, $u_2(x; f)$ satisfy

$$u_1(x; f) \rightarrow 0, \tag{1}$$

$$u_2(x; f) \rightarrow u_*(x) \tag{2}$$

in $H^1(\mathbf{R}^N)$ as $\|f\|_{H^{-1}(\mathbf{R}^N)} \rightarrow 0$, where $u_*(x)$ is a solution of $(*)_0$.

Next we consider the case

- (A5) $a(x) \in (0, 1]$ for all $x \in \mathbf{R}^N$ and $a(x) \not\equiv 1$.

In this case, we have the following existence result.

Theorem 2 ([1]). *Assume (A0)–(A3) and (A5). Then there exists a constant $\delta_0 > 0$ such that if $f \geq 0$, $f \not\equiv 0$ and $\|f\|_{H^{-1}(\mathbf{R}^N)} \leq \delta_0$ then $(*)_f$ has at least 4 positive solutions.*

As to the behavior of our 4 solutions, we have

Theorem 3 ([1]). *Let $(f_n)_{n=1}^\infty$ be a sequence of such that*

$$f_n \geq 0, \quad f_n \not\equiv 0, \quad \|f_n\|_{H^{-1}(\mathbf{R}^N)} \rightarrow 0.$$

Moreover let $u_1(x; f_n)$, $u_2(x; f_n)$, $u_3(x; f_n)$, $u_4(x; f_n)$ be solutions obtained in Theorem 2. Then there exists a subsequence — still we denote by n — such that

- (i) $u_1(x; f_n)$, $u_2(x; f_n)$ behave as in (1), (2).

(ii) There exist 2 sequence $(y_n), (z_n) \subset \mathbf{R}^N$ such that

$$\begin{aligned} |y_n|, |z_n| &\rightarrow \infty, \\ \|u_3(x; f_n) - \omega(x - y_n)\|_{H^1(\mathbf{R}^N)} &\rightarrow 0, \\ \|u_4(x; f_n) - \omega(x - z_n)\|_{H^1(\mathbf{R}^N)} &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, where $\omega(x)$ is the unique positive radial solution of

$$-\Delta\omega + \omega = \omega^p \quad \text{in } \mathbf{R}^N.$$

In particular, $N = 1$, we have $y_n \rightarrow \infty, z_n \rightarrow -\infty$.

2. Radially symmetric case

As a special case, we study radially symmetric cases:

(R) $a(x), f(x)$ are radially symmetric.

Theorem 4. Assume (R) and $a_r \leq 0, f_r \leq 0$ in addition to the assumptions to Theorem 1. Then any solution of $(*)_f$ is radially symmetric and it tends to 0 if $\|f\|_{H^{-1}(\mathbf{R}^N)} \rightarrow 0$.

In contrary, under the condition (A5), we have the following theorem as a special case our Theorems 2 and 3.

Theorem 5. Assume (R) in addition to the assumptions of Theorem 2. For sufficiently small $\|f\|_{H^{-1}(\mathbf{R}^N)}$ (but not equal to 0), $(*)_f$ has a positive solution, which is not radially symmetric, and the conclusion of Theorem 3 holds.

Thus the shape of $a(x)$ plays an important role for the continuous dependence of solutions to $f(x)$.

Proofs of the above theorems are variational and we find solutions of $(*)_f$ as critical points of the following functional

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 + |u|^2 dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} dx - \int_{\mathbf{R}^N} f u dx : H^1(\mathbf{R}^N) \rightarrow \mathbf{R}.$$

An idea from concentration-compactness arguments will be used to find critical points of $I(u)$.

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On Liouville theorem and application to apriori estimates to scalarcurvature equations

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Abstract

Let (S^n, g_0) be the n -dimensional sphere of \mathbf{R}^{n+1} with the g_0 induced from the flat metric of \mathbf{R}^{n+1} . For a given C^1 function K on S^n , we want to seek a metric g conformal to g_0 such that K is the scalar curvature of the new metric g . Write $g = c_n \cdot u^{\frac{4}{n-2}} g_0$ for some suitable constant $c_n > 0$. Then u satisfies

$$(1) \quad \Delta u - \frac{n(n-2)}{4}u + Ku^{\frac{n+2}{n-2}} = 0 \text{ on } S^n.$$

In this talk, we want to prove some estimates of u in the region where $K \leq 0$. This can be done by proving some Liouville-type theorems.

In \mathbf{R}^n , we consider the equation

$$(2) \quad \begin{cases} \Delta u + Q(x)u^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbf{R}^n, \\ u > 0 \text{ in } \mathbf{R}^n, \end{cases}$$

where Q is a C^1 homogeneous function of degree $l > 0$. Assume that Q satisfies the nondegenerate condition:

$$(3) \quad c_1|x|^{l-1} \leq |\nabla Q(x)| \leq c_2|x|^{l-1} \text{ for } x \in \mathbf{R}^n \setminus \{0\}.$$

Then we have the following theorem.

Theorem. *Let Q be a C^1 homogenous function of degree $l > 0$ such that (3) holds. Then equation (2) possesses no positive solutions in \mathbf{R}^n .*

We should give a scatch of the proof of our Main Theorem and its application to apriori estimates.

Global Solutions of Systems of Wave Equations with Quadratic Nonlinearities

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Let $c_i > 0$ ($i = 1, 2$) and set $\partial_\alpha = \partial/\partial x^\alpha$ ($0 \leq \alpha \leq 3$) where $(x^0, x^1, x^2, x^3) \in \mathbf{R}^4$. We consider the Cauchy problem

$$\begin{cases} \partial_0^2 u^i - c_i^2 \sum_{j=1}^3 \partial_j^2 u^i = F^i(u, \partial u) & \text{in } [0, \infty) \times \mathbf{R}^3, \\ u^i(0, \cdot) = \varepsilon f^i, \partial_0 u^i(0, \cdot) = \varepsilon g^i & \text{in } \mathbf{R}^3 \quad (i = 1, 2), \end{cases} \quad (1)$$

where u^i ($i = 1, 2$) are real-valued unknown functions on $[0, \infty) \times \mathbf{R}^3$. We set $u = (u^1, u^2)$, $\partial u = (\partial u^1, \partial u^2)$ and $\partial u^i = (\partial_\alpha u^i)_{0 \leq \alpha \leq 3}$. We assume that $f^i, g^i \in C_0^\infty(\mathbf{R}^3)$ ($i = 1, 2$) and ε is a nonnegative small parameter.

In order to have a global solution for the Cauchy problem (1) for small initial data, we need the nonlinear terms $F^i(u, \partial u)$ ($i = 1, 2$) to be small enough at the origin, as it is seen typically in the problems $\partial_0^2 u - \sum_{i=1}^3 \partial_i^2 u = |u|^p$ or $\partial_0^2 u - \sum_{i=1}^3 \partial_i^2 u = |\partial u|^p$. In fact, they have critical powers $p = 1 + \sqrt{2}, 2$ respectively, and if p is smaller than or equal to the critical power, the solutions usually blow up in finite time ([6],[11],[4],[7]).

We suppose that the nonlinear terms $F^i(u, \partial u)$ ($i = 1, 2$) are quadratic forms of $(u, \partial u)$. Since $\partial_0^2 u - \sum_{i=1}^3 \partial_i^2 u = |\partial u|^p$ have the critical power $p = 2$, quadratic nonlinearities are the critical nonlinearities for global existence of small solutions when nonlinear terms depend only on ∂u . Though the nonlinearities are critical, if $F^i(u, \partial u)$ are independent of u and satisfy certain algebraic conditions which are often referred as the null conditions, we have a

unique global solution for (1) provided ε is sufficiently small ([2],[3],[8],[12]. See [1],[5] and references cited there for two dimensional case). Hence, even for critical nonlinearities there are some good forms for which we have global solutions.

In this talk we consider the case where the nonlinear terms are quadratic and depend also on u . As it was shown in the case where the nonlinear terms depend only on ∂u , discrepancy between the propagation speeds c_1, c_2 make a good possibility of global existence. By [3],[9] and [12], the Cauchy problem (1) for

$$F^1(u, \partial u) = \sum_{i,j=1}^2 \sum_{\alpha,\beta=0}^3 B_{\alpha\beta}^{ij} \partial_\alpha u^i \partial_\beta u^j$$

$$F^2(u, \partial u) = \sum_{i,j=1}^2 \sum_{\alpha,\beta=0}^3 D_{\alpha\beta}^{ij} \partial_\alpha u^i \partial_\beta u^j,$$

has a global solution if $c_1 \neq c_2$ and

$$B_{\alpha\beta}^{11} = 0, D_{\alpha\beta}^{22} = 0 \quad (\alpha, \beta = 0, \dots, 3).$$

So the question is whether the Cauchy problem (1) has a global solution even for the nonlinear terms $F^i(u, \partial u)$ containing u , if $c_1 \neq c_2$.

For quadratic forms

$$F^1(u, \partial u) = \sum_{i,j=1}^2 \sum_{\alpha=0}^3 A_\alpha^{ij} u^i \partial_\alpha u^j + \sum_{i,j=1}^2 \sum_{\alpha,\beta=0}^3 B_{\alpha\beta}^{ij} \partial_\alpha u^i \partial_\beta u^j \quad (2)$$

$$F^2(u, \partial u) = \sum_{i,j=1}^2 \sum_{\alpha=0}^3 C_\alpha^{ij} u^i \partial_\alpha u^j + \sum_{i,j=1}^2 \sum_{\alpha,\beta=0}^3 D_{\alpha\beta}^{ij} \partial_\alpha u^i \partial_\beta u^j, \quad (3)$$

we assume that

$$A_\alpha^{ii} = 0, B_{\alpha\beta}^{ii} = 0, C_\alpha^{ii} = 0, D_{\alpha\beta}^{ii} = 0 \quad (i = 1, 2; \alpha, \beta = 0, \dots, 3). \quad (4)$$

Then we have the following theorem.

Theorem. *Let $c_1 \neq c_2$. Assume (2)-(4). Then there exists a positive constant ε_0 such that the Cauchy problem (1) has a unique C^∞ global solution for arbitrary ε provided $0 \leq \varepsilon \leq \varepsilon_0$.*

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Scattering Theory in the Energy Space for Nonlinear Klein-Gordon and Schrödinger Equations

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1. INTRODUCTION

In this talk, we consider asymptotic behavior of solutions to the nonlinear Klein-Gordon equations (NLKG) and the nonlinear Schrödinger equations (NLS):

$$\ddot{u} - \Delta u + u + |u|^p u = 0, \quad (\text{NLKG}), \quad (1)$$

$$i\dot{u} - \Delta u + |u|^p u = 0, \quad (\text{NLS}), \quad (2)$$

where $u = u(t, x) : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$, $p > 1$ and $f : \mathbb{C} \rightarrow \mathbb{C}$. For any function $u(t, x)$, denote by $\text{eq}(u)$ the left hand side of the equation, denote by $\text{eq}_L(u) := \text{eq}(u) - |u|^p u$ the linear (free) part, and

$$\mathbf{u} := \begin{cases} (u, \sqrt{1 - \Delta}^{-1} \dot{u}), & (\text{for NLKG}) \\ u, & (\text{for NLS}) \end{cases} \quad (3)$$

We have the following conservation law of the energy:

$$E(u; t) := \int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 + |\mathbf{u}|^2 + \frac{2|u|^{p+1}}{p+1} = E(u; 0). \quad (4)$$

In this talk we deal with every solution with finite energy and investigate asymptotic behavior of the solutions by comparing with the solutions to the free equation $\text{eq}_L(v) = 0$. Let $u_{\pm}(t, x)$ and $v(t, x)$ satisfy $\text{eq}(u_{\pm}) = \text{eq}_L(v) = 0$ and

$$\lim_{t \rightarrow \pm\infty} \|\mathbf{v}(t) - \mathbf{u}_{\pm}(t)\|_{H^1} = 0. \quad (5)$$

Then the wave operators are given by the correspondences $W_{\pm} : \mathbf{v}(0) \mapsto \mathbf{u}_{\pm}(0)$, and the scattering operator is given by $S = W_+^{-1}W_-$. Our main result is the asymptotic completeness of the wave operators, namely that they are well-defined as homeomorphisms from H^1 into itself.

2. KNOWN RESULTS

First we mention the known results on the asymptotic completeness.

1. [6, 7, 11, 12] Let $n \geq 3$ and $4/n < p < 4/(n-2)$. Then we have the asymptotic completeness for (NLKG) and (NLS) in the energy class H^1 .
2. [18] Let $n \geq 3$ and $p = 4/(n-2)$. Then we have the asymptotic completeness for (NLKG) in the energy class.
3. [10, 24, 8] Let $n \in \mathbb{N}$, $p \geq 8/(\sqrt{(n+2)^2 + 8n} + n - 2)$ and $(n-2)p < 4$. Then we have the asymptotic completeness for (NLS) in $\Sigma := \{\varphi \in H^1 \mid x\varphi \in L^2\}$.

3. MAIN RESULT

Theorem 1. *Let $n \in \mathbb{N}$ and $(n - 2)p < 4 < np$. Then we have the asymptotic completeness for (NLKG) and (NLS) in the energy class.*

The large data scattering of NLKG for $n \leq 2$ was one of the major open problems in [22, pp. 247]. Here we consider a single power nonlinearity for simplicity, but our proof can be applied to more general nonlinearity $f(u)$ satisfying

$$\exists F : \mathbb{R} \rightarrow \mathbb{R} \text{ s.t. } f(u) = 2F'(|u|)\frac{u}{|u|}, \quad F(0) = f(0) = 0, \quad (6)$$

$$|f(u) - f(v)| \leq C|u - v|(|u|^{p_1} + |v|^{p_1} + |u|^{p_2} + |v|^{p_2}), \quad (7)$$

for some $p_1 \leq p_2$ satisfying $(n - 2)p_2 < 4 < np_1$, and

$$G(u) := \Re(f(u)\bar{u}) - F(u) \geq 0. \quad (8)$$

In the preceding works, it was needed that

$$G(u) \geq C \min(|u|^2, |u|^{p_3}), \quad \exists p_3 > 2. \quad (9)$$

If we write $F(|u|) = V(|u|)|u|^2$, (8) is equivalent to $V' \geq 0$. We remark that if $V(r) < 0 = V(0)$ for some $r > 0$, there exist standing wave solutions to (NLKG) and (NLS) [2], so that the asymptotic completeness does not hold.

4. DIFFICULTIES IN LOW SPATIAL DIMENSIONS

There were two difficulties in proving the asymptotic completeness for $n < 3$. The first problem was that we can not prove the Morawetz estimate, which has been essentially the only a priori estimate for global space-time integral used to prove the asymptotic completeness. The second problem is on the decay order $t^{-n/2}$ of the free evolution. For $n \geq 3$, it is integrable on $(1, \infty)$, so that we can show, only by the boundedness of the solution, that the nonlinear interaction at a fixed time has little influence on the behavior of the solution in the distant future or past. Such an argument was essentially used in the preceding works, but it can not be used if $n < 3$. We overcome the first difficulty by a new Morawetz type estimate which holds in any spatial dimension and independent of the nonlinearity, so that we can also improve the generality of the nonlinearity (regarding (8)). To avoid the second difficulty, we employ a new idea inspired by Bourgain [5]. Separating localized energy into rapidly decreasing free solutions, we can reduce the problem to that for small energy data.

5. NEW MORAWETZ TYPE ESTIMATES

In this section, we deal with the equation with the general form of nonlinearity

$$\text{eq}(u) + f(u) = 0, \quad (10)$$

to show how we can replace the assumption (9) with (8). Assume (6) and (7). The Morawetz estimate is

$$\iint_{\mathbb{R}^{1+n}} \frac{G(u)}{|x|} dx dt \leq CE(u), \quad (11)$$

where u is any solution to (10) and $n \geq 3$. For $n = 2$, the following estimate can be proved so far only from the main estimate (25) of asymptotic completeness.

$$\iint_{\mathbb{R}^{1+2}} \frac{|u|^q}{|x|} dx dt \leq C(E(u)), \quad (12)$$

for any $q \geq 2 + 4/n$. If $n = 1$, (11) is obviously false. Our estimate is

Lemma 2. *Let $n \in \mathbb{N}$ and assume (6), (7) and (8). Then, for any finite energy solution u and p satisfying $p \geq 2 + 4/n$ and $(n-2)(p-2) \leq 4$, we have*

$$\iint_K \frac{t^2 |u|^p}{|(t, x)|^3} dx dt \leq C(p) E(u)^{p/2}, \quad (13)$$

where $K = \{(t, x) \mid |x| < |t|\}$ for (NLKG) and $K = \mathbb{R}^{1+n}$ for (NLS).

Any Morawetz-type estimate or conservation law is based on some integral identity derived by variations of the Lagrangian. We mention a general formula for such identities. First we have to prepare some notation.

$$\langle a, b \rangle := \Re(a\bar{b}), \quad \partial = (\partial_t, \nabla_x), \quad \mathcal{D} = \begin{cases} (-\partial_t, \nabla_x), & \text{(for NLKG)} \\ (-i/2, \nabla_x), & \text{(for NLS)} \end{cases} \quad (14)$$

$$2\ell(u) = \begin{cases} -|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + F(u), & \text{(for NLKG)} \\ (i\dot{u}, u) + |\nabla u|^2 + F(u), & \text{(for NLS)} \end{cases} \quad (15)$$

$\ell(u)$ is the Lagrangian density associated to the equation $\text{eq}(u) = 0$. The differential operator \mathcal{D} naturally appears from the variation of ℓ :

$$\delta_v \ell(u) := \lim_{\varepsilon \rightarrow 0} \frac{\ell(u + \varepsilon v) - \ell(u)}{\varepsilon} = \langle \text{eq}(u), v \rangle + \partial \cdot \langle \mathcal{D}u, v \rangle. \quad (16)$$

Using this identity, we can easily obtain the following formula.

$$\begin{aligned} \langle \text{eq}(u), h \cdot \mathcal{D}u + qu \rangle &= -\partial \cdot \langle \mathcal{D}u, h \cdot \mathcal{D}u + qu \rangle + \mathcal{D} \cdot \left(h\ell(u) + \frac{|u|^2}{2} \partial q \right) \\ &+ \langle \mathcal{D}u, (\partial h) \mathcal{D}u \rangle - \frac{|u|^2}{2} \mathcal{D} \cdot \partial q + (2q - \mathcal{D} \cdot h)\ell(u) + G(u)q, \end{aligned} \quad (17)$$

Now let

$$h := \frac{(t, x)}{|(t, x)|}, \quad q := \Re \frac{\mathcal{D} \cdot h}{2}, \quad (18)$$

and integrate the real part of (17) over

$$K_1 := \begin{cases} \{(t, x) \mid |t| > 1\}, & \text{(for NLKG)} \\ \{(t, x) \mid |t|^2 > |x|^2 + 1\}, & \text{(for NLS)} \end{cases}. \quad (19)$$

Then we have

$$\iint_{K_1} \langle \mathcal{D}u, (\partial_\beta h) \mathcal{D}u \rangle - \frac{|u|^2}{2} \Re \mathcal{D} \cdot \partial q + G(u)q dx dt \leq CE(u). \quad (20)$$

Since $q \geq 0$, $|\Re \mathcal{D} \cdot \partial q| \leq C/t^3$, we obtain

$$\iint_{K_1} \frac{|t \nabla u - x \mathcal{D}_0 u|^2}{|(t, x)|^3} dx dt \leq CE(u), \quad (21)$$

which comes from the first term in (20). Such an estimate was first derived in [17, Proposition 4.4] for (NLKG) with $n \geq 3$. Now we use the following Sobolev type inequality.

Lemma 3. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$. Then for any $u \in H^1(\mathbb{R}^n)$ and $p \geq 2 + 4/n$ satisfying $(n-2)(p-2) \leq 4$, we have*

$$\int_{\mathbb{R}^n} \chi^2 |u|^p dx \leq C \|u\|_{L^q}^{p-2} \int_{\mathbb{R}^n} \chi^2 |\nabla u + iVu|^2 dx + C \|\nabla \chi\|_{L^\infty}^2 \|u\|_{H^1}^p, \quad (22)$$

where $q = n(p-2)/2$ and $C > 0$ depend only on n and p .

We can apply this inequality directly to (21) in the NLS case. In the NLKG case, we apply it to the function $v(\tau, x) := u(\sqrt{\tau^2 + |x|^2}, x)$ and use the boundedness of the energy on the hyperboloids. Then we obtain the estimate on K_1 . Indeed, the estimate on $K \setminus K_1$ is trivial from the Hardy inequality: $\| |x|^{-\theta} u \|_{L^2} \leq C \|u\|_{H^1}$, where $0 \leq \theta \leq 1$ and $\theta < n/2$.

6. GLOBAL SPACE-TIME ESTIMATES AND ENERGY CONCENTRATION

For simplicity, in this section we consider (NLKG) for $n < 3$ and (NLS) for any n . For (NLKG) with $n \geq 3$, we have to change the exponents of the space-time norms below, though the arguments are essentially the same. The asymptotic completeness means that at time infinity any free solution can be approximated by a nonlinear solution and any nonlinear solution can be approximated by a free solution. That is possible because the nonlinear interaction term loses its effect as $|t|$ tends to infinity, since $|u|$ decays pointwise by the dispersion of wave. However, we can not expect any uniform decay estimate for the solutions because our setting is invariant under space-time translations and time inversion. Since the decay property of the solutions comes from the finiteness of the energy, it is natural that the decay property is also described in (space-time) integral forms. In fact, we know the Strichartz estimates, for example

$$\|v\|_{L_t^p(\mathbb{R}; B_{\rho,2}^\sigma(\mathbb{R}^n))} \leq C \|\mathbf{v}\|_{H^1}, \quad (23)$$

for any linear solution v , where $\rho := 2 + 4/n$, $\sigma = 1/2$ for (NLKG), $\sigma = 1$ for (NLS) and $B_{\rho,2}^\sigma$ is the inhomogeneous Besov space (cf. [3]). Moreover, let w be the solution for the linear inhomogeneous equation $\text{eq}_L(w) = -|v|^p v$ and $w(0) = 0$. We have also by the Strichartz estimate and well-known power estimates,

$$\|\mathbf{w}\|_{L_t^\infty((S,T); H^1)} \leq \|\mathbf{v}(S)\|_{H^1}^p \|v\|_{L_t^p((S,T); B_{\rho,2}^\sigma(\mathbb{R}^n))}. \quad (24)$$

Since $\|v\|_{L_t^p((S,T); B_{\rho,2}^\sigma(\mathbb{R}^n))}$ vanishes as $S \rightarrow \infty$ by (23), (24) means that the nonlinear interaction loses its effect for linear solutions. It is easy to construct the wave operators by a fixed point argument using such estimates as (24). Thus, the asymptotic completeness will immediately follow if we can prove that global space-time norms such as (23) are finite also for the nonlinear solutions:

$$\|u\|_{L_t^p(\mathbb{R}; B_{\rho,2}^\sigma(\mathbb{R}^n))} \leq C(E(u)), \quad (25)$$

which can be derived by a standard argument from the following weaker estimate:

$$\|u\|_{L^q(\mathbb{R}^{1+n})} \leq C(E(u)), \quad (26)$$

where $q := p(n + 2)/2$. Thus, our objective hereafter is (26). From now on, we denote

$$\|u\|_{(K;I)} := \|u\|_{L^p(I;B_{p,2}^\sigma(\mathbb{R}^n))}, \quad \|u\|_{(X;I)} := \|u\|_{L^q(I \times \mathbb{R}^n)}. \quad (27)$$

Indeed, it is the hardest step to prove the global estimate for the nonlinear solutions in the proof of asymptotic completeness, for we can not approximate the solution by one free solution as in the construction of local solutions and wave operators or as in the small data analysis. Of course we can divide the time axis into many intervals such that we can approximate the solution on each interval by a free solution. But how can we get any asymptotic information from those many free solutions? Bourgain [5] considered instead the space-time distribution of the energy density of u on each time interval using a standard local approximation by free solutions. More precisely, we have the following lemma essentially due to Bourgain (here the situation is simpler because we are considering the subcritical case).

Lemma 4 (Bourgain). *Let u be a nontrivial solution of (NLKG) or (NLS) with $E(u) = E < \infty$ and $\|u\|_I = \eta < \infty$ for some interval I . There exists a constant η_0 such that if $\eta \leq \eta_0$ then we have a subinterval $J \subset I$, a point $X \in \mathbb{R}^n$ and $R > 0$ such that for any $t \in J$ and $s \geq 1$ we have*

$$\int_{|x-X|<R} |u|^s dx \geq C(E, \eta) > 0, \quad (28)$$

$|J| > C(E, \eta) > 0$ and $R < C(E, \eta)$.

Outline of proof. Let v be the free solution with the same data at the top of I . Then, by the Strichartz estimate and the well-known power estimates we have

$$\|u\|_{(K;I)} \leq \|v\|_{(K;I)} + C\|u\|_{(X;I)}^p \|v\|_{(K;I)} \leq CE + C\eta^p \|u\|_{(K;I)}. \quad (29)$$

From this, we have $\|u\|_{(K;I)} \leq CE$ if η_0 is sufficiently small. By the interpolation inequality and the Sobolev embedding, we have

$$\eta = \|u\|_{(X)} \leq C\|u\|_{(B)}^{1-\rho/q} \|u\|_{(K)}^{\rho/q} \leq C(E)\|u\|_{(B)}^{1-\rho/q}, \quad (30)$$

where we denote $(B) := L^\infty(B_{\infty,\infty}^{1-n/2-\varepsilon})$ with a constant $\varepsilon > 0$ small enough for the above interpolation to hold. Thus we obtain $\|u\|_{(B)} > C(E, \eta)$, which means by the definition of (B) ,

$$2^{N(1-n/2-\varepsilon)} |(\varphi_N * u)(T, X)| > C(E, \eta), \quad (31)$$

for some $-1 \leq N \in \mathbb{Z}$, $T \in I$ and $X \in \mathbb{R}^n$, where $\{\varphi_j\}_{j=-1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ is a Paley-Littlewood partition of $\delta(x)$ satisfying $\varphi_j(x) = 2^{jn}\varphi_0(2^jx)$. On the other hand by the Sobolev embedding we have

$$2^{N(1-n/2)} \|\varphi_N * u(t)\|_{L^\infty} \leq C\|u(t)\|_{H^1} \leq C(E), \quad (32)$$

so that $N < C(E, \eta)$. Moreover we have by the equation,

$$\|\varphi_N * (u(t) - u(T))\|_{L^\infty} \leq 2^{(1+n/2)N} \|u(t) - u(T)\|_{H^{-1}} \leq C(E, \eta)|t - T|, \quad (33)$$

so that the estimate (31) remains valid for $T \in J$ with some interval J of length $> C(E, \eta)$. Since $|\varphi_N(x)|$ is sufficiently small for $|x| > C2^{-N}$, we obtain the desired result from (31). \square

Now let I be a finite interval and let us estimate $\|u\|_{(X;I)}$. First we divide I into subintervals $\{I_j\}_{j=1}^N$ such that $\|u\|_{(X;I_j)} = \eta_0$ on each subinterval. Applying the above lemma on each subinterval, we obtain $|J_j| > C(E)$, $X_j \in \mathbb{R}^n$ and $R < C(E)$ such that for any $t \in J_j \subset I_j$ and any $2 \leq s \leq q$ we have

$$\int_{|x-X_j|<R} |u|^s dx > \nu = \nu(E) > 0. \quad (34)$$

Let $T_j := \inf J_j$, $B_j := \{(T_j, x) \mid |x - X_j| < R\}$ and $K_j := \{(t, x) \mid t > T_j, |x - X_j| < M|t - T_j| + 3R\}$, where $M = 1$ for (NLKG) and we can choose $M = C(E)$ sufficiently large for (NLS) such that the loss of the L^2 norm inside K_j is at most $\nu/2$. We can choose $P \subset \{1, \dots, N\}$ such that

- (i). $k, j \in P, k \neq j \implies B_j \not\subset K_k$.
- (ii). $\forall j, \exists k \in P, B_j \subset K_k$.

From (i) and the energy propagation estimate, we have $E \geq \#P\nu/2$, so that $\#P < C(E)$. Using (ii) and the Morawetz type estimate, we have

$$C(E) \geq \sum_{k \in P} \int_{K_k} \frac{|u|^q}{|t - T_k|} dx dt \geq \sum_{j=1}^N \frac{\nu |J_j|}{|t - T_1|} \geq \frac{C(E) \log N}{\max_j |I_j|}, \quad (35)$$

so that $\max_j |I_j| \geq C(E) \log N \geq C(E) \log \|u\|_{(X;I)}$. Now assume that $\|u\|_{(X;I)}$ is very large. Then, there exists a very long I_j with $\|u\|_{(X;I_j)} = \eta_0$ fixed, which means that the mean density in I_j is very low. Nevertheless, we have $(T_j, X_j) \in I_j \times \mathbb{R}^n$, where there exists certain amount of energy $\nu > C(E)$ in a fixed radius $R < C(E)$. Thus in this interval there exist two waves whose scales are very different from each other. The smaller one comes from the energy localized around X_j at $t = T_j$. Now we seek the larger one which spreads very thinly around the smaller one. To this end, we divide $I_j =: (S, T)$ into further subintervals. We suppose that $T_j \leq (S + T)/2$. Otherwise the time direction should be reversed in the following argument. Let $\Lambda > 1$, $S_k := T_j - R + (M\Lambda)^k R$, $H_k := (S_k, S_{k+1})$. Let $A \in \mathbb{N}$ and assume that for $k \leq 3A$ we have $S_k \in I_j$. Then, there exists some $k < 3A$ such that $\|u\|_{(X;H_k)} \leq \eta_0/A^{1/q}$ and $\|\mathbf{u}\|_{(K;H_k)} \leq C(E)/A^{1/\rho}$. By the energy propagation estimate we have

$$\int_{|x-X_j|<R'} e(u; S_k) dx \geq \nu/2, \quad (36)$$

where $e(u; t)$ denotes the energy density and $R' := R + M(S_k - T_j) \leq M(M\Lambda)^k R \leq |H_k|/(\Lambda - 1)$. Thus, for any $\varepsilon > 0$, there exists $N = N(E, \varepsilon)$ such that if $\|u\|_{(X;I)} > N$ then we have a subinterval $J = (S, T) \subset I$, $X \in \mathbb{R}^n$ and $R > 0$ such that $\|u\|_{(X;J)} + \|\mathbf{u}\|_{(K;J)} < \varepsilon$, $R < \varepsilon|J|$ and $\int_{|x-X|<R} e(u; S) dx \geq \nu/2$. If ε is sufficiently small depending on ν, E , we may assume that $\|u/\langle x \rangle\|_{L^2} \ll \sqrt{\nu}$. Then, we can separate the energy around (S, X) by a free solution v such that

$$E(v; S) \leq C\nu, \quad E(u - v; S) \leq E - \nu/3, \quad \text{diam supp } \mathbf{v}(S) \leq CR. \quad (37)$$

Using the support property of $\mathbf{v}(S)$ and the decay estimate for the free evolution, we have for $t > T$,

$$\|v(t)\|_{B_{\infty,2}^{-n-3}} \leq C|J|^{-n/2} \|\mathbf{v}(S)\|_{B_{1,2}^{-1}} \leq C(R/|J|)^{n/2} \|\mathbf{v}(S)\|_{H^1} \leq C\varepsilon^{n/2} \sqrt{\nu}. \quad (38)$$

Interpolating with the Strichartz estimate, we obtain $\|v\|_{(X;(T,\infty))} \leq C(E)\varepsilon^\alpha$, where α is a positive constant. By the energy identity, if $\varepsilon < \sqrt{\nu}$ we have $E(u - v; T) \leq E(u - v; S) + C(E)\nu^{1+p/2}$. We may assume without loss of generality that $\nu(E)$ is so small that $C(E)\nu^{1+p/2} < \nu/12$. Thus we obtain $E(u - v; T) < E - \nu/4$. Then we can reduce the energy level by the following perturbation lemma essentially due to Bourgain.

Lemma 5 (Bourgain). *Let $\text{eq}(u) = \text{eq}(w) = \text{eq}_L(v) = 0$ and $\mathbf{u}(0) = \mathbf{v}(0) + \mathbf{w}(0)$. Let $E(u), E(w) \leq E$ and $\|w\|_{(X;(0,\infty))} < M$. Then there exists $\varepsilon = \varepsilon(E, M) > 0$ such that if $\|v\|_{(X;(0,\infty))} < \varepsilon$, we have $\|u\|_{(X;(0,\infty))} < C(E, M)$.*

Now we prove the global estimate (26). It is well-known for solutions with sufficiently small energy. We use induction on the energy E . Suppose that for any solution u with $E(u) \leq E - \nu(E)/4$ we have $\|u\|_{(X;\mathbb{R})} < M$. Let u be a solution with $E(u) \leq E$. Take $\tau < \tau'$ such that $\|u\|_{(X;(-\infty,\tau))} = \|u\|_{(X;(\tau,\tau'))} = \|u\|_{(X;(\tau',\infty))}$. By the above argument, there exists $N = N(E, M)$ such that if $\|u\|_{(X;(\tau,\tau'))} > N$ then we have some $T \in (\tau, \tau')$ and a free solution v satisfying $E(u - v; T) < E - \nu/4$ and $\|v\|_{(X;(T,\infty))} < \varepsilon(E, M)$ or $\|v\|_{(X;(-\infty,T))} < \varepsilon$ (in the case where T_j is in the later half of I_j). Then, by the above lemma, we obtain $C(E, M) > \|u\|_{(X;(T,\infty))} > \|u\|_{(X;(\tau',\infty))}$ or $C(E, M) > \|u\|_{(X;(-\infty,\tau))}$. Thus we obtain $\|u\|_{(X;\mathbb{R})} \leq 3 \max(N(E, M), C(E, M))$ for any solution u with $E(u) \leq E$. Since it is obvious that we can take $\nu(E)$ depending continuously on E , by induction we obtain the desired estimate (26).

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INVISCID LIMIT OF COMPRESSIBLE FLUIDS

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This talk will present results obtained jointly by Zhouping Xin concerning asymptotic limiting behavior of solutions to the Navier-Stokes equations of 2-D isentropic compressible fluids, for small viscosity in the presence of boundaries. We mainly consider the linearized problem, in which the flow is linearized around an Euler solution. After constructing an approximate solution of the linearized problem by means of the asymptotic analysis with multiple scales, we show the pointwise estimates of the error term of the approximate solution by using the technique on energy methods. These estimates readily yield the uniform stability results for the linearized Navier-Stokes solution in the zero viscosity limit. The short time stability of viscous boundary layers of nonlinear Navier-Stokes solutions might be also discussed. The crux of such zero viscosity limit problems lies in the analysis on well-posedness (or ill-posedness) of the Prandtl solution which describes the motion of fluids in boundary layers.

On the role of MHD for a magnetically
confined plasma

by Taira SHIROTA

with a cooperator Masumi OHNO

1. Introduction and results

We will consider the well-posedness of the initial boundary value problem for the linearized equations of ideal MHD. The original system of equations takes the following form.

$$(1.1) \quad \begin{aligned} \rho_p(\partial_t + (u, \nabla))p + \rho \operatorname{div} u &= 0, \\ \rho(\partial_t + (u, \nabla))u &= -\nabla p + \mu_0(\nabla \times H) \times H, \\ \partial_t H - \nabla \times (u \times H) &= 0, \\ (\partial_t + (u, \nabla))s &= 0 \end{aligned} \quad \text{in } [0, T] \times \Omega.$$

The boundary condition is

$$(1.2) \quad (\nu, u) = 0 \quad \text{on } [0, T] \times \partial\Omega.$$

The constraint conditions

$$(1.3) \quad (\nu, H) = 0 \quad \text{on } [0, T] \times \partial\Omega,$$

$$(1.4) \quad \operatorname{div} H = 0 \quad \text{in } [0, T] \times \Omega$$

are also imposed. Here Ω is a bounded domain in \mathbb{R}^3 , T is a positive constant and $\nu = \nu(x) = {}^t(\nu_1, \nu_2, \nu_3)$ denotes the unit outward normal to the boundary at $x \in \partial\Omega$. Pressure $p = p(t, x)$, velocity $u = u(t, x) = {}^t(u_1, u_2, u_3)$, magnetic field $H = H(t, x) = {}^t(H_1, H_2, H_3)$ and entropy $s = s(t, x)$ are unknown functions. We suppose that density $\rho = \rho(p, s)$ is a smooth known function of $p > 0$ and s satisfying $\rho > 0$, $\rho_p = \partial\rho/\partial p > 0$. The magnetic permeability μ_0 is a positive constant.

In order to employ a useful symmetrization of (1.1), we introduce the new unknown vector valued function $U = {}^t(q, {}^t u, {}^t H, s)$ in place of ${}^t(p, {}^t u, {}^t H, s)$, where $q = p + \frac{1}{2}|H|^2$ is the total pressure. We linearize the equations (1.1) about a real vector valued function \bar{U} where $\bar{U} = {}^t(\bar{q}, {}^t \bar{u}, {}^t \bar{H}, \bar{s}) \in C^{l+1}([0, T] \times \bar{\Omega})$ is a solution of (1.1) which satisfies (1.2)–(1.4) with $\bar{p} > 0$ in $[0, T] \times \bar{\Omega}$ ($\bar{\Omega} = \Omega \cup \partial\Omega$).

(/)

Definition. The initial boundary value problem for the linearized equations is said to be well posed in the Sobolev space $H^l(\Omega)$, for an integer $l \geq 1$, if the following conditions are satisfied:

For any initial data $U_0 \in H^l(\Omega)$ satisfying

$$(1.5) \quad (\nu, H_0) = 0 \quad \text{on } \partial\Omega,$$

$$(1.6) \quad \operatorname{div} H_0 = 0 \quad \text{in } \Omega,$$

and the compatibility conditions of order $l-1$ for the linearized equations and the boundary condition (1.2), there exists a unique solution $U \in C([0, T_1]; H^l(\Omega))$ of the linearized equations such that it satisfies (1.2), (1.3), (1.4) with $T = T_1$ and the estimate

$$(1.7) \quad \|U(t)\|_{H^l(\Omega)} \leq C \|U_0\|_{H^l(\Omega)}$$

holds for any $t \in [0, T_1]$. Here C and $T_1 (\leq T)$ are positive constants independent of U_0 . (For $\partial_t U$, see, e.g., R. Temam [15], ch II.3.)

Let $\partial\Omega \in C^{l+3}$, $l \geq 1$, then main results of the present paper are the following two theorems.

Theorem I. *The initial boundary value problem for the linearized equations (2.2) with (1.2), (1.3), and (1.4) is well posed in $H^1(\Omega)$.*

Theorem II. *Let $\bar{H} \not\equiv 0$ on $[0, T] \times \partial\Omega$. Then the above problem is not well posed in $H^l(\Omega)$ for $l \geq 2$.*

Theorem I has the following significance. First it releases us from troubles with compatibility conditions, since one of order zero is the boundary condition itself and also it follows the well posedness in $H^0(\Omega)$ in more precise sense than J. Rauch's result (cf. [9]) under the condition of Theorem I. As a special case, where \bar{U} is a static equilibrium defined over $\bar{\Omega}$ whose boundary is a magnetic surface, i.e., a surface where $(\nu, \bar{H}) = 0$, contained in plasma region, these facts above mentioned *is related* to the linearized internal (local) stability second-order system. (See I. B. Bernstein et al. [1] and J. P. Freidberg [3]. For equilibrium, see R. Temam [12], [14] and A. Friedman & Y. Liu [4]. For the existence of solutions, see R. Temam [15], ch II.4.) \ast)

Theorem I, which proves the non-existence of "loss of regularity" of solutions in $H^1(\Omega)$, has been found by us after the completion of the proof of Theorem II

Theorem II implies that, for any Ω with smooth boundary, the regularity loss of solutions of the linearized problem always arises in $H^l(\Omega)$ ($l \geq 2$). The initial data are to be taken in a way such that their supports are sufficiently small and intersect with $\partial\Omega$. Obviously Theorem II is also valid in the case where the linearized equations are such that the equation

$$\partial_t H + (\bar{u}, \nabla)H - (\bar{H}, \nabla)u + \bar{H}(\operatorname{div}u) = \text{a certain terms of lower order}$$

guarantees that $(\nu, H)|_{\partial\Omega}(t) = 0$ for $t \in [0, T_1]$ whenever $(\nu, H_0)|_{\partial\Omega} = 0$ and where the condition $\operatorname{div}H = 0$ in Ω is neglected as usual.

But Theorem II seems to reflect a certain roughness^{**)} of the physical derivation of a model (1.1) ~ (1.4) for the magnetically confined plasma.

We do remark that (linearized) classical physical examples of initial boundary value problem for hyperbolic systems fall into a category where there is no reflective bicharacteristic line associated with unbounded characteristic root $\lambda^+(\zeta, \xi_2, \xi_3)$ w.r.t. ξ_1 . (Cf. Section 4.)

2. Linearized problem

Using the unknown vector valued function $U = {}^t(q, {}^t u, {}^t H, s)$ we rewrite (1.1) as follows.

$$(2.1) \quad \begin{aligned} \alpha(\partial_t + (u, \nabla))q - \alpha(H, \partial_t H + (u, \nabla)H) + \operatorname{div}u &= 0, \\ \rho(\partial_t + (u, \nabla))u + \nabla q - (H, \nabla)H &= 0, \\ \partial_t H + (u, \nabla)H - (H, \nabla)u + H(\operatorname{div}u) - (\operatorname{div}H)u &= 0, \\ (\partial_t + (u, \nabla))s &= 0 \end{aligned} \quad \text{in } [0, T] \times \Omega.$$

(3)

Here we put $\mu_0 = 1$, for simplicity and $\alpha = \rho_p/\rho$. Then we linearize (2.1) about a solution $\bar{U} \in C^{l+1}([0, T] \times \bar{\Omega})$ to (2.1) with (1.2)–(1.4). The resulting equations are the following.

$$(2.2) \quad \begin{aligned} & \bar{\alpha}(\partial_t + (\bar{u}, \nabla))q - \bar{\alpha}(\bar{H}, \partial_t H + (\bar{u}, \nabla)H) + \operatorname{div} u = l_1, \\ & \bar{p}(\partial_t + (\bar{u}, \nabla))u + \nabla q - (\bar{H}, \nabla)H = l_2, \\ & \partial_t H + (\bar{u}, \nabla)H - (\bar{H}, \nabla)u + \bar{H}(\operatorname{div} u) = l_3, \\ & (\partial_t + (\bar{u}, \nabla))s = l_4 \end{aligned} \quad \text{in } [0, T] \times \Omega.$$

We observe that the terms of lower order l_i , $i = 1, \dots, 4$, are linear combinations of the components of U with coefficients depending smoothly on the components of \bar{U} and their derivatives of the first order with respect to x and t . In particular, we have

$$l_3 = -(u, \nabla)\bar{H} + (H, \nabla)\bar{u} - H(\operatorname{div}\bar{u})$$

and $\bar{\alpha} = \alpha(\bar{q}, \bar{H}, \bar{s})$, etc. We obtain (2.2)₃ by subtracting $\bar{u}(\operatorname{div}H) + u(\operatorname{div}\bar{H})$ from the third equations of the linearization of (2.1). For simplicity of the description we omit s in (2.2) without loss of generality. In the following, assuming that p and ρ are independent of s , we may set U and \bar{U} to be ${}^t(q, {}^t u, {}^t H)$ and ${}^t(\bar{q}, {}^t \bar{u}, {}^t \bar{H})$, respectively, which are all real vector valued functions.

Adding (2.2)₁ $\times (-\bar{H})$ to (2.2)₃, we get the following system which is a symmetization of (2.2).

$$(2.3) \quad \begin{aligned} & \bar{\alpha}(\partial_t + (\bar{u}, \nabla))q - \bar{\alpha}(\bar{H}, \partial_t H + (\bar{u}, \nabla)H) + \operatorname{div} u = l_1, \\ & \bar{p}(\partial_t + (\bar{u}, \nabla))u + \nabla q - (\bar{H}, \nabla)H = l_2, \\ & \partial_t H + (\bar{u}, \nabla)H - (\bar{H}, \nabla)u - \bar{\alpha}\bar{H}\{(\partial_t + (\bar{u}, \nabla))q - (\bar{H}, \partial_t H + (\bar{u}, \nabla)H)\} \\ & \quad = l_3 - l_1\bar{H} \end{aligned} \quad \text{in } [0, T] \times \Omega.$$

We write equations of our problem in the following form.

$$(2.4) \quad \begin{aligned} & A_0(\bar{U})\partial_t U + \sum_{j=1}^3 A_j(\bar{U})\partial_j U + B(\bar{U})U = 0 & \text{in } [0, T] \times \Omega, \\ & MU = 0 & \text{on } [0, T] \times \partial\Omega, \\ & NU = 0 & \text{on } [0, T] \times \partial\Omega, \\ & \operatorname{div} H = 0 & \text{in } [0, T] \times \Omega, \\ & U(0, x) = U_0(x) & \text{for } x \in \Omega, \end{aligned}$$

where $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$,

$$A_0(\bar{U}) = \left(\begin{array}{c|ccc|ccc} \bar{\alpha} & 0 & 0 & 0 & -\bar{\alpha}\bar{H}_1 & -\bar{\alpha}\bar{H}_2 & -\bar{\alpha}\bar{H}_3 \\ \hline 0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\rho} & 0 & 0 & 0 \\ \hline -\bar{\alpha}\bar{H}_1 & 0 & 0 & 0 & 1 + \bar{\alpha}\bar{H}_1^2 & \bar{\alpha}\bar{H}_1\bar{H}_2 & \bar{\alpha}\bar{H}_1\bar{H}_3 \\ -\bar{\alpha}\bar{H}_2 & 0 & 0 & 0 & \bar{\alpha}\bar{H}_1\bar{H}_2 & 1 + \bar{\alpha}\bar{H}_2^2 & \bar{\alpha}\bar{H}_2\bar{H}_3 \\ -\bar{\alpha}\bar{H}_3 & 0 & 0 & 0 & \bar{\alpha}\bar{H}_1\bar{H}_3 & \bar{\alpha}\bar{H}_2\bar{H}_3 & 1 + \bar{\alpha}\bar{H}_3^2 \end{array} \right),$$

$$B(\bar{U})U = - \begin{pmatrix} l_1 \\ l_2 \\ l_3 - l_1\bar{H} \end{pmatrix},$$

$$A_\nu(\bar{U}) = \sum_{j=1}^3 \nu_j A_j(\bar{U}) = \left(\begin{array}{c|ccc|ccc} 0 & & & & \nu & & & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \text{on } \partial\Omega,$$

$M =$ all elements are equal to zero except that the (2,2),(2,3),(2,4) entries are equal to ν ,

$N =$ all elements are equal to zero except that the (5,5),(5,6),(5,7) entries are equal to ν ,

and $B(\bar{U}) = B(\bar{U}, \partial_i \bar{U}, \partial_j \bar{U}; 1 \leq j \leq 3)$.

The resulting system (2.4)₁, (2.4)₂ is a symmetric hyperbolic system with characteristic boundary of constant multiplicity in the sense of J. Rauch [9]. Note that $A_0(\bar{U})$ is positive definite, although $A_0(\bar{U}) \neq I$. The boundary condition (2.4)₂ is maximal nonnegative. Actually, the boundary matrix $A_\nu = \sum_{j=1}^n \nu_j A_j$ is of a constant rank 2 on $\partial\Omega$ and $\text{Ker } A_\nu \subset \text{Ker } M$ on $\partial\Omega$ which is maximal nonnegative subset of A_ν . Now we give a lemma which will be useful in the proofs of theorems.

Lemma

- (i) Let \bar{U} be a solution $\in C^{l+1}([0, T] \times \bar{\Omega})$ of (1.1)–(1.4). Then the assumption in Theorem II, i.e., $\bar{H} \neq 0$ on $[0, T] \times \partial\Omega$, implies that $\bar{H} \neq 0$ on $\{t = 0\} \times \partial\Omega$.

- (ii) Assume that $\bar{U} \in C^{l+1}([0, T] \times \bar{\Omega})$ satisfies (1.2), (1.3) and $\bar{p} > 0$ in $[0, T] \times \bar{\Omega}$. This implies that \bar{U} satisfies neither (1.1) nor (1.4). Then if (1.5) holds for $U(0)$, the solution $U(t)$ of (2.4)₁ that belongs to $C([0, T_1], H^2(\Omega))$ satisfies (1.3) in $[0, T_1] \times \partial\Omega$.
- (iii) Let $\bar{U} \in C^{l+1}([0, T] \times \bar{\Omega})$ satisfy (1.2)-(1.4). Then if (1.6) holds for $U(0)$, the solution $U(t)$ of (2.4)₁ that belongs to $C([0, T_1]; H^1(\Omega))$ also satisfies (1.4), i.e., (2.4)₄, in $[0, T_1] \times \Omega$.

Lemma 2.1, tell us that important properties of MHD “(1.3) and (1.4) are derived from (1.5) and (1.6) respectively” persist for its linearized equations (2.2) or (2.3).

In the following we always assume that \bar{U} satisfy the assumption of Lemma 2.1 (ii). By virtue of Lemma 2.1 (iii), we consider solutions omitting (1.4) and (1.6) in localized problem used in Proof of Theorem I.

3. Outline of Proof of Theorem I

Taking account of the finiteness of the speed of propagation for the solution, we use a suitable finite partition of unity $\{\phi_\alpha\}$ of $\bar{\Omega}$ where $\sum_\alpha \phi_\alpha = 1$ and diffeomorphisms. Then we are reduced to the problem in the half space. We fix $p \in \partial\Omega$ arbitrarily. We assume that $\partial\Omega \in C^{l+3}$. Then there exists a C^{l+2} -admissible boundary coordinate system $(y(x))$ which maps p to the origin. We have

$$(3.1) \quad \mathbb{P} = \begin{pmatrix} \frac{\partial x_i}{\partial y_j} \end{pmatrix}, \quad {}^t\mathbb{P}\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & G & \\ 0 & & \end{pmatrix} \quad \text{on } \{y_1 = 0\},$$

$\mathbb{P} = (\delta_{i,j})$ at the origin,
where the G is a certain 2×2 matrix (cf. p301 of [6]).

(6)

Let us denote the inverse map of $y(x)$ by ψ . Then the known and unknown functions are changed as follows: for $x = \psi(y)$

$$\begin{aligned}\tilde{u}(t, y) &= \mathbb{P}^{-1}u(t, x), & \tilde{H}(t, y) &= \mathbb{P}^{-1}H(t, x), & \tilde{q}(t, y) &= q(t, x), \\ \tilde{\rho}(t, y) &= \rho(t, x), & \tilde{\tilde{u}}(t, y) &= \mathbb{P}^{-1}\tilde{u}(t, x), & \tilde{\tilde{H}}(t, y) &= \mathbb{P}^{-1}\tilde{H}(t, x), \\ \tilde{\tilde{q}}(t, y) &= \tilde{q}(t, x), & \tilde{\tilde{\alpha}}(t, y) &= \tilde{\alpha}(t, x), & \tilde{\tilde{\rho}}(t, y) &= \tilde{\rho}(t, x).\end{aligned}$$

Our problem in Theorem I is reduced to find the solutions to the following localized system of equations. For $T_1 \ll 1$,

$$(3.2) \quad \begin{aligned}\tilde{A}_0(\tilde{U})\partial_t\tilde{U} + \sum_{j=1}^3 \tilde{A}_j(\tilde{U})\partial_j\tilde{U} + \tilde{B}(\tilde{U})\tilde{U} &= 0 & \text{in } [0, T_1] \times \{y_1 > 0\}, \\ \tilde{M}\tilde{U} &= 0 & \text{on } [0, T_1] \times \{y_1 = 0\}, \\ \tilde{N}\tilde{U} &= 0 & \text{on } [0, T_1] \times \{y_1 = 0\}, \\ \tilde{U}(0) &= \tilde{\phi}_\alpha \tilde{U}_0 & \text{for a certain } \alpha & \text{in } \{y_1 > 0\},\end{aligned}$$

The concrete form of $(3.2)_1$ is as follows.

$$(3.3) \quad \begin{aligned}\tilde{\alpha}\{\partial_t\tilde{q} + (\tilde{u}, \nabla_y)\tilde{q} - ({}^t\mathbb{P}\tilde{H}, \partial_t\tilde{H} + (\tilde{u}, \nabla_y)\tilde{H})\} + \operatorname{div}\tilde{u} &= \hat{l}_1, \\ \tilde{\rho}({}^t\mathbb{P}(\partial_t\tilde{u} + (\tilde{u}, \nabla_y)\tilde{u}) + \nabla_y\tilde{q} - {}^t\mathbb{P}(\tilde{H}, \nabla_y)\tilde{H}) &= \hat{l}_2, \\ {}^t\mathbb{P}[\partial_t\tilde{H} + (\tilde{u}, \nabla_y)\tilde{H} - (\tilde{H}, \nabla_y)\tilde{u}] & \\ + \tilde{\tilde{\alpha}}\tilde{\tilde{H}}\{-\partial_t\tilde{q} - (\tilde{u}, \nabla_y)\tilde{q} + ({}^t\mathbb{P}\tilde{H}, \partial_t\tilde{H} + (\tilde{u}, \nabla_y)\tilde{H})\} &= \hat{l}_3 \\ & \text{in } [0, T_1] \times \{y_1 > 0\},\end{aligned}$$

where \hat{l}_i , $i = 1, 2, 3$, denote terms of lower order.

For some $r_0 > 0$ we set $B(0) = \{y; \operatorname{dist}(0, y) < r_0 \text{ and } y_1 \geq 0\}$. Assume that $\operatorname{supp} \tilde{\phi}_\alpha \subset\subset B(0)$.

Furthermore, let \tilde{U}^δ be vector valued functions $\in C^{1,0}([0, T_1] \times \overline{B(0)})$ such that

$$\begin{aligned}\tilde{U}^\delta &\rightarrow \tilde{U} & \text{in } C^2([0, T_1] \times \overline{B(0)}) & \text{as } \delta \rightarrow 0, \\ \tilde{M}\tilde{U}^\delta &= \tilde{N}\tilde{U}^\delta = 0 & \text{on } [0, T_1] \times (B(0) \cap \{y_1 = 0\}).\end{aligned}$$

Assuming that $\partial\Omega \in C^4$, we consider the approximate problem: for $T_1 \ll 1$ and for sufficiently small $\delta > 0$ ($\tilde{U}^\delta \in C([0, T_1]; H^2(B(0)))$)

$$(3.4) \quad \begin{aligned}\tilde{A}_0(\tilde{U}^\delta)\partial_t\tilde{U}^\delta + \sum_{j=1}^3 \tilde{A}_j(\tilde{U}^\delta)\partial_j\tilde{U}^\delta + \tilde{B}(\tilde{U}^\delta)\tilde{U}^\delta &= 0 & \text{in } [0, T_1] \times \{y_1 > 0\}, \\ \tilde{M}\tilde{U}^\delta &= 0 & \text{on } [0, T_1] \times \{y_1 = 0\}, \\ \tilde{N}\tilde{U}^\delta &= 0 & \text{on } [0, T_1] \times \{y_1 = 0\}, \\ \tilde{U}^\delta(0) &= \tilde{f}^\delta & \text{with compact support} & \text{on } \{y_1 \geq 0\}.\end{aligned}$$

Here $\tilde{f}^\delta \rightarrow \tilde{\phi}_\alpha \tilde{U}_0$ in $H^4(B(0))$ ($\delta \rightarrow 0$), $\tilde{N}\tilde{f}^\delta = 0$ on $\{y_1 = 0\}$ and \tilde{f}^δ satisfies c.c. of order 4 for (3.4)₄, (3.4)₂.

(7)

Lemma. The solution \tilde{U}^δ of the problem (3.4) satisfies the following estimate

$$(3.5) \quad \|\tilde{U}^\delta(t)\|_{H^1(\mathbb{R}_+^3)} \leq C \|\tilde{U}^\delta(0)\|_{H^1(\mathbb{R}_+^3)} \quad \text{for } t \in [0, T_1],$$

Proof. We omit simply the indices δ and tilde in the proof.

Since

$$\begin{aligned} & (A_0 \partial_1 U, \partial_1 U)(t) - (A_0 \partial_1 U, \partial_1 U)(t') \\ &= \int_{t'}^t \int_{\mathbb{R}_+^3} \partial_\tau (A_0 \partial_1 U, \partial_1 U)(\tau, y) dy d\tau, \quad (t' = 0) \end{aligned}$$

we have from (3.4) that for a constant $C > 0$ depending only on \bar{U} , \mathbb{P} and their derivatives up to the second order

$$(3.6) \quad \begin{aligned} & \text{The right hand side of the above equality} \\ & \leq - \int_{t'}^t \int_{\mathbb{R}_+^3} \sum_{j=1}^3 \partial_j (A_j \partial_1 U, \partial_1 U) dy d\tau + C \int_{t'}^t \|\partial_1 U\| \cdot \|U\|_{H^1(\mathbb{R}_+^3)} d\tau. \end{aligned}$$

Using (3.4)₂, (3.4)₃, and (3.1), we see from the corresponding form to (3.3) that

$$(3.7) \quad \begin{aligned} & \partial_1 q|_{y_1=0} = \hat{l}_{21}|_{y_1=0}, \\ & \partial_1 u_1|_{y_1=0} = [-\bar{\alpha} \{ \partial_t q + \bar{u}_2 \partial_2 q + \bar{u}_3 \partial_3 q - ({}^t \mathbb{P} \mathbb{P} \bar{H})_2 \partial_t H_2 - ({}^t \mathbb{P} \mathbb{P} \bar{H})_3 \partial_t H_3 \\ & \quad - ({}^t \mathbb{P} \mathbb{P} \bar{H})_2 (\bar{u}_2 \partial_2 + \bar{u}_3 \partial_3) H_2 - ({}^t \mathbb{P} \mathbb{P} \bar{H})_3 (\bar{u}_2 \partial_2 + \bar{u}_3 \partial_3) H_3 \} \\ & \quad - \partial_2 u_2 - \partial_3 u_3 + \hat{l}_1]|_{y_1=0} \quad \text{for } t \in [0, T_1]. \end{aligned}$$

Note that $\bar{U} \in C^3$ and $\mathbb{P} \in C^3$ on a neighborhood of $\text{supp } U$. The first term on the right hand side of (3.6)

$$\begin{aligned} & \int_{t'}^t \int_{\partial \mathbb{R}_+^3} (A_1 \partial_1 U, \partial_1 U)(\tau, 0, y') dy' d\tau \\ &= 2 \int_{t'}^t \int_{\partial \mathbb{R}_+^3} (\partial_1 q, \partial_1 u_1)(\tau, 0, y') dy' d\tau. \end{aligned}$$

Here using (3.7) essentially, we obtain (3.5). Then by a limit process ($\delta \rightarrow 0$) we see the validity of Theorem I.

(8)

4. Outline of Proof of Theorem 2.

The linearized problem (2.4) is reduced to the problem with frozen coefficients at the origin:

$$(4.1) \quad \begin{aligned} \bar{\alpha}(\partial_t q - (\bar{H}, \partial_t H)) + \operatorname{div} u &= 0, \\ \bar{\rho} \partial_t u - (\bar{H}, \nabla_x) H + \nabla_x q &= 0, \\ \partial_t H - (\bar{H}, \nabla_x) u - \bar{\alpha} \bar{H} [\partial_t q - (\bar{H}, \partial_t H)] &= 0 \quad \text{in } [0, \infty) \times \{x_1 > 0\}. \end{aligned}$$

Here, $\bar{\alpha}, \bar{H} = {}^t(0, \bar{h}, 0)$ are constant, $\bar{h} > 0$ and

$$(4.2) \quad \begin{aligned} u_1 = H_1 &= 0 && \text{on } [0, \infty) \times \{x_1 = 0\}, \\ u_1 = H &\equiv 0 && \text{on } \{t=0\} \times \{x_1 \geq 0\}. \end{aligned}$$

Let a and b be given by

$$a \equiv (\bar{\rho}_p)^{\frac{1}{2}} = (\bar{\delta}_p)^{\frac{1}{2}} = (\bar{\alpha} \bar{p})^{-\frac{1}{2}}, \quad b \equiv \bar{h} (\bar{p})^{-\frac{1}{2}}.$$

After we prove lemmas, we obtain the following: the well posedness in $H_l(\Omega)$ w.r.t. (2.4) implies that in $H_l(\mathbb{R}^3)$ w.r.t. (4.1) and (4.2) such that for the solution $U(t) = {}^t(q, {}^t u, {}^t H)(t)$ and for $t \in [0, \infty)$

$$(4.3) \quad \|U(t)\|_{H_l} \leq C_l \|U(0)\|_{H_l} \quad (l \geq 2).$$

Now, denoting

$$\hat{u}(\tau, \xi_1, \xi_2, \xi_3) = \int e^{-i(\tau t + \langle \xi, x \rangle)} u(t, x) dt dx$$

for $\tau = \eta - i\delta$, where $\eta, \xi = (\xi_1, \xi_2, \xi_3)$ are real and $\delta \geq 0$, we see that the characteristic equation is the following:

$$(4.4) \quad \tau \left(\tau^2 - \frac{(\bar{H} \cdot \xi)^2}{\bar{p}} \right) \left(\tau^4 - (a^2 + b^2) |\xi|^2 \tau^2 + \frac{a^2}{\bar{p}} (\bar{H} \cdot \xi)^2 |\xi|^2 \right) = 0,$$

Let $\lambda^\pm(\eta, \xi_2, \xi_3)$ be the characteristic roots of (4.4) w.r.t. ξ_1 .

Here we assume that for $\gamma > 0$

$$\text{Im } \lambda^+ > 0, \quad \text{Im } \lambda^- < 0.$$

Since $\lambda^\pm(\eta - i\gamma, \xi_2, \xi_3) = \lambda^\pm(\eta, \xi_2, \xi_3) - \lambda_z^\pm(i\gamma) + \dots$,

if $\lambda^\pm(\eta, \xi_2, \xi_3)$ is real

$$\rightarrow \text{Im } \lambda^+ > 0 \iff \lambda_z^+(\eta, \xi_2, \xi_3) < 0.$$

Then the bicharacteristic line associated with the characteristic root $\xi_1 = \lambda^+(\eta, \xi_2, \xi_3)$

is defined as follows:

$$\frac{\partial t}{\partial x_1} = -\lambda_z^+(\eta, \xi_2, \xi_3), \quad \frac{\partial x_2}{\partial x_1} = -\lambda_{\xi_2}^+(\eta, \xi_2, \xi_3).$$

Therefore the line above and one corresponding to λ^- are reflective and incident respectively.

Setting that $\lambda = \xi_1$, $\sigma_2 = \xi_2$ and $\sigma_3 = \xi_3$, we see that if $\tau = 1$, $\sigma_2 \uparrow c^{-1}$, $\lambda^+(\tau, \sigma_2, \sigma_3) \rightarrow +\infty$.

In fact we have that for $\sigma = (\sigma_2, \sigma_3) \in \Sigma c(\gamma_0)$

$$\begin{aligned} & (\lambda^\pm(\tau(\sigma, \gamma_0), \sigma))^{-2} \\ &= \frac{(c\sigma_2 - i\gamma_0)^4 - (a^2 + b^2)|\sigma|^2((c\sigma_2 - i\gamma_0)^2 - (c\sigma_2)^2)}{(a^2 + b^2)((c\sigma_2 - i\gamma_0)^2 - (c\sigma_2)^2)} \sim \frac{(c\sigma_2)^3}{2(a^2 + b^2)\gamma_0} i. \end{aligned}$$

Here

$$\tau = \tau(\sigma, \gamma_0) = c\sigma_2 - i\gamma_0, \quad c = \left(\frac{a^2 b^2}{a^2 + b^2} \right)^{\frac{1}{2}}, \quad \gamma_0 > 0.$$

(/o)

$$\Sigma_{C(\gamma_0)} \equiv \{(\sigma_2, \sigma_3) \mid 0 < 2^{-1}|\sigma_3| < |\sigma_2| < 2|\sigma_3|, \sigma_2 \cdot \sigma_3 > 0, |\sigma| \in C(\gamma_0)\}, \quad \epsilon(\gamma_0) \gg 1.$$

Therefore we see that for such σ

$$(4.5) \quad \begin{aligned} \tau &= O(|\sigma|), \quad |\lambda^\pm| = O(|\sigma|^{\frac{3}{2}}), \quad \text{Im} \lambda^\pm = O(|\sigma|^{\frac{3}{2}}), \\ X &= \tau^2 - b^2 \sigma_2^2 = O(|\sigma|^2), \quad Y = \frac{a^2 + b^2}{a^2} (\tau^2 - c^2 \sigma_2^2) = O(|\sigma|), \\ \frac{\bar{\rho} X}{2\tau \lambda^\pm} &= O(|\sigma|^{-\frac{1}{2}}), \quad \tau \sigma_2 X = O(|\sigma|^4), \quad \tau \sigma_3 Y = O(|\sigma|^3). \end{aligned}$$

Now we can represent the solutions $\{g, v, H\}$ of (4.1) in the following form, for example

$$\begin{aligned} \mathcal{L}\hat{u}_1(\tau, x_1, \sigma) &= O(|\sigma|) \left(\int_0^\infty e^{i\lambda^+(x_1+y)} \psi(y, \sigma) dy + \int_0^{x_1} e^{i\lambda^+(x_1-y)} \psi(y, \sigma) dy \right. \\ &\quad \left. - \int_{x_1}^\infty e^{-i\lambda^+(x_1-y)} \psi(y, \sigma) dy \right) \\ &\quad + O(1) \left(\int_0^\infty e^{i\lambda^+(x_1+y)} \varphi(y, \sigma) dy + \int_0^{x_1} e^{i\lambda^+(x_1-y)} \varphi(y, \sigma) dy \right. \\ &\quad \left. - \int_{x_1}^\infty e^{-i\lambda^+(x_1-y)} \varphi(y, \sigma) dy \right) \quad \text{for } \sigma \in \Sigma_{\epsilon} C(\gamma_0). \end{aligned}$$

Assuming (4.3) and by means of precise, careful calculation used (4.5) we arrive at a contradiction.

(For details of proofs, see Arch. Rational Mech. and Anal. 144 (1998) 259-299.)

(11)

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(ii)