



Title	Proceedings of the 23rd Sapporo Symposium on Partial Differential Equations Sapporo
Author(s)	Giga, Yoshikazu
Citation	Hokkaido University technical report series in mathematics, 53, 1
Issue Date	1997-01-01
DOI	10.14943/731
Doc URL	<a href="http://hdl.handle.net/2115/750">http://hdl.handle.net/2115/750</a> ; <a href="http://eprints3.math.sci.hokudai.ac.jp/1232/">http://eprints3.math.sci.hokudai.ac.jp/1232/</a>
Type	bulletin (article)
Note	29 July-31 July,1998
File Information	53.pdf



[Instructions for use](#)

Proceedings of the 23rd Sapporo Symposium  
on Partial Differential Equations

Sapporo, 29 July - 31 July, 1998

Edited by Y. Giga

Series #53. July, 1998

HOKKAIDO UNIVERSITY  
TECHNICAL REPORT SERIES IN MATHEMATICS

- #24: K. Kubota (Ed.), 第 17 回偏微分方程式論札幌シンポジウム予稿集, 29 pages. 1992.
- #25: K. Takasaki, “非線型可積分系の数理” 1992.9.28~10.2 北海道大学での集中講義 講義録, 52 pages. 1993.
- #26: T. Nakazi (Ed.), 第 1 回関数空間セミナー報告集, 93 pages. 1993.
- #27: K. Kubota (Ed.), 第 18 回偏微分方程式論札幌シンポジウム予稿集, 40 pages. 1993.
- #28: T. Hibi (Ed.), 1992 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 108 pages. 1993.
- #29: I. Sawashima, T. Nakazi (Eds.), 第 2 回関数空間セミナー報告集, 79 pages. 1994.
- #30: Y. Giga, Y.-G. Chen, 動く曲面を追いかけて, 講義録, 62 pages. 1994.
- #31: K. Kubota (Ed.), 第 19 回偏微分方程式論札幌シンポジウム予稿集, 33 pages. 1994.
- #32: T. Ozawa (Ed.), 1993 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 113 pages. 1994.
- #33: Y. Okabe (Ed.), The First Sapporo Symposium on Complex Systems, 24 pages. 1994.
- #34: A. Arai, Infinite Dimensional Analysis on an Exterior Bundle and Supersymmetric Quantum Field Theory, 10 pages. 1994.
- #35: S. Miyajima, T. Nakazi (Eds.), 第 3 回関数空間セミナー報告集, 104 pages. 1995.
- #36: N. Kawazumi (Ed.), リーマン面に関連する位相幾何学, 63 pages. 1995.
- #37: I. Tsuda (Ed.), The Second & Third Sapporo Symposium on Complex Systems, 190 pages. 1995.
- #38: M. Saito (Ed.), 1994 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 100 pages. 1995.
- #39: S. Izumiya (Ed.), 接触幾何学と関連分野研究集会報告集, 186 pages. 1995.
- #40: H. Komatsu, A. Kishimoto (Eds.), 作用素論・作用素環論研究集会予稿集, 61 pages. 1995.
- #41: K. Okubo, T. Nakazi (Eds.), 第 4 回関数空間セミナー報告集, 103 pages. 1996.
- #42: R. Agemi (Ed.), 第 20 回偏微分方程式論札幌シンポジウム予稿集, 47 pages. 1996.
- #43: R. Agemi, Y. Giga and T. Ozawa(Eds.), Nonlinear Waves, Proceedings of the Fourth MSJ International Research Institute Vol I, 269 pages. 1996.
- #44: R. Agemi, Y. Giga and T. Ozawa(Eds.), Nonlinear Waves, Proceedings of the Fourth MSJ International Research Institute Vol II, 270 pages. 1996.
- #45: G. Ishikawa (Ed.), 1995 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 98 pages. 1996.
- #46: R. Agemi (Ed.), 第 21 回偏微分方程式論札幌シンポジウム予稿集, 34 pages. 1996.
- #47: N. Kawazumi (Ed.), リーマン面に関する位相幾何学, 61 pages. 1996.
- #48: S. Miyajima, J. Inoue (Eds.), 第 5 回関数空間セミナー報告集, 90 pages. 1997.
- #49: T. Ozawa (Ed.), Proceedings of the 22nd Sapporo Symposium on Partial Differential Equations, 67 pages. 1997.
- #50: H.-F. Yamada (Ed.), 1996 年度談話会・特別講演アブストラクト集 Colloquium Lectures, 99 pages. 1997.
- #51: N. Kawazumi (Ed.), リーマン面に関する位相幾何学, 121 pages. 1997.
- #52: J. Inoue (Ed.), 第 6 回関数空間セミナー報告集, 89 pages. 1998.

Proceedings of the 23rd Sapporo Symposium on  
Partial Differential Equations

Edited by Y. Giga

Sapporo, 1998

Partially supported by Grant-in-Aid for Scientific Research, Ministry of  
Education, Science, Sports, and Culture (Monbusho).

文部省科学研究費補助金 (基盤研究 A (1) 課題番号 09304012)

文部省科学研究費補助金 (基盤研究 A (2) 課題番号 10304010)

## PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on July 29 through July 31 in 1998 at Department of Mathematics, Hokkaido University.

This is the 23rd time of the symposium on Partial Differential Equations. The symposium was founded by Professor Taira Shirota more than 20 years ago. It has one of longest history among symposiums on partial differential equations in Japan.

We wish to dedicate this volume to Professors Rentaro Agemi and Kôji Kubota for their large contribution to the organization of the symposium for many years.

Y. Giga

# CONTENTS

## Programme

- J. Escher, Global existence and wave breaking for a shallow water equation
- T. Ogawa, Weak solvability and well-posedness for nonlinear dispersive systems
- K. Ono, On the asymptotic behavior of solutions for damped nonlinear wave equations of Kirchhoff type in unbounded domain
- P. N. Pipo, Smoothing effects for some derivative nonlinear Schrödinger equations
- K. Hidano, Small data scattering for wave equations with supercritical nonlinearity
- K. Nishihara, Asymptotic behavior of solutions to the  $p$ -system with linear damping
- K. Kato and T. Ogawa, Analyticity and regularizing effect for Korteweg de Vries equation
- T. Sideris, On the evolution of compactly supported planer vorticity
- Y. Shimizu,  $L^\infty$ -estimate of first space derivatives of Stokes flow in a half space
- M. Nakamura and T. Ozawa, Global solutions in the critical Sobolev space for the wave equations  
with nonlinearity of exponential growth
- S. Sakaguchi, When are the isothermal surfaces invariant with respect to the time variable?
- G. Nakamura, 連続体力学の逆問題 (Inverse problems in continuum mechanics)

## 第23回偏微分方程式論札幌シンポジウム

下記の要領でシンポジウムを行ないますのでご案内申し上げます。

代表者 儀我 美一

### 記

1. 日時 1998年7月29日(水)～7月31日(金)
2. 場所 北海道大学大学院理学研究科 数学教室
3. 講演

#### 7月29日(水)

- 9:30-10:30 J. Escher, University of Kassel  
Global existence and wave breaking for a shallow water equation
- 11:00-12:00 小川卓克 (T. Ogawa), 九大数理 (Kyushu U.)  
Weak solvability and well-posedness for nonlinear dispersive systems
- 13:30-14:00 \*
- 14:00-14:30 小野公輔 (K. Ono), 徳島大総科 (Tokushima U.)  
On the asymptotic behavior of solutions for damped nonlinear wave equations of Kirchhoff type in unbounded domain
- 15:00-15:30 P. N. Pipolo 東京理大理 (Science U. of Tokyo) / Paris  
Smoothing effects for some derivative nonlinear Schrödinger equations
- 15:45-16:15 肥田野久二男 (K. Hidano), 都立大理 (Tokyo Metropolitan U.)  
Small data scattering for wave equations with supercritical nonlinearity
- 16:30-17:00 \*

#### 7月30日(木)

- 9:30-10:30 西原健二 (K. Nishihara), 早大政経 (Waseda U.)  
Asymptotic behavior of solutions to the  $p$ -system with linear damping
- 11:00-12:00 加藤圭一 (K. Kato), 東京理大理 (Science U. of Tokyo)  
Analyticity and regularizing effect for Korteweg de Vries equation

- 13:30-14:00 \*
- 14:00-15:00 T. Sideris, U.C. Santa Barbara  
On the evolution of compactly supported planer vorticity
- 15:30-16:00 清水康之 (Y. Shimizu), 北大理院 (Hokkaido U.)  
 $L^\infty$ -estimate of first space derivatives of Stokes flow in a half space
- 16:15-16:45 中村 誠 (M. Nakamura), 北大理院 (Hokkaido U.)  
Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth
- 16:45-17:30 \*
- 18:00-20:00 懇親会 BANQUET

7月31日 (金)

- 9:30-10:30 坂口 茂 (S. Sakaguchi), 愛媛大理 (Ehime U.)  
When are the isothermal surfaces invariant with respect to the time variable?
- 11:00-12:00 中村 玄 (G. Nakamura), 群馬大 (Gunma U.)  
連続体力学の逆問題 (Inverse problems in continuum mechanics)
- 12:00-13:00 \*

\*この時間は講演者を囲んで自由な質問の時間とする予定です。

連絡先 060 札幌市北区北10条西8丁目  
北海道大学大学院理学研究科数学教室  
電話 (011)716-2111  
FAX (011)727-3705  
儀我美一 内2672  
小澤 徹 内3570

# GLOBAL EXISTENCE AND WAVE BREAKING FOR A SHALLOW WATER EQUATION

J. ESCHER, UNIVERSITY OF KASSEL

## 1. INTRODUCTION

Recently, R. Camassa and D. Holm [1] proposed a new model to describe unidirectional propagation of waves of the surface of a shallow layer of water:

$$(1.1) \quad \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, \cdot) = u_0. \end{cases}$$

The variable  $u(t, x)$  above represents the fluid's velocity at time  $t$  in the  $x$  direction in appropriate nondimensional units. Unlike the well-known Korteweg-de Vries equation (KdV), which is derived by an asymptotic expansion in the equation of motion, equation (1.1) is obtained in [1] by using an asymptotic expansion directly in the Hamiltonian for Euler's equation in the shallow water regime. Equation (1.1) was already derived earlier as a bi-Hamiltonian generalization of the KdV equation, cf. [8]. The novelty of Camassa and Holm's work was the physical derivation of (1.1) and the discovery that equation has solitary waves that retain their shape under interaction and eventually emerge with their original shape and speed. As for the KdV equation there is a spatial periodic version of the Camassa-Holm equation which is also of interest:

$$(1.2) \quad \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(t, x+1) = u(x, t) & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, \cdot) = u_0. \end{cases}$$

It is the aim of this note to present some recent results for both equations (1.1) and (1.2).

## 2. WELL-POSEDNESS AND GLOBAL EXISTENCE OF STRONG SOLUTIONS

It is convenient to reformulate equation (1.1) in terms of the new variable  $y := u - u_{xx}$ . Obviously, we have

$$uy_x + 2u_x y = 3uu_x - uu_{xxx} - 2u_x u_{xx},$$

so that we obtain from (1.1):

$$y_t + uy_x + 2u - xy = 0, \quad y(0) = y_0,$$

where  $y_0 := u_0 - (u_0)_{xx}$ . Writing  $Q := (1 - \partial_x^2)^{-1}$  and

$$A(y)z := (Qy)\partial_x z + 2(Qy)z,$$

with  $\text{dom}(A(y)) := \{z \in L_2(\mathbb{R}); (Qy)z \in H^1(\mathbb{R})\}$ , equation (1.1) is equivalent to the quasilinear evolution equation

$$\frac{d}{dt}y + A(y)y = 0, \quad y(0) = y_0.$$

Observe that  $A(y)$  represents a (possibly) degenerate first order operator, since  $Qy$  may vanish on an arbitrary closed subset of  $\mathbb{R}$ . Using some tools from harmonic analysis (in particular the so-called "T(1)" theorem), it is possible to verify the hypotheses of T. Kato's general theory for abstract quasilinear evolution equations of hyperbolic type (cf. [9]) to obtain the following result.

**Theorem 1.** [2, 3] (a) *Given  $u_0 \in H^3(\mathbb{R})$ , there exists a maximal  $T = T(u_0)$  and a unique solution*

$$u = u(\cdot, u_0) \in C([0, T], H^3(\mathbb{R})) \cap C^1([0, T], H^2(\mathbb{R}))$$

*to problem (1.1). The solution  $u$  depends continuously on the initial data  $u_0$ .*

(b) *Given  $u_0 \in H^3(\mathbb{R})$ , let  $y_0 := u_0 - (u_0)_{xx}$  and assume that  $y_0$  belongs to  $L_1(\mathbb{R})$  and does not change sign. Then the corresponding solution  $u(\cdot, u_0)$  exists globally.*

It is not difficult to see that the  $H^1(\mathbb{R})$ -norm of the solution is preserved during the evolution. However, in order to prove part (b) of Theorem 1, we need a priori bounds in  $H^3(\mathbb{R})$  for the solution. These estimates are obtained by using the hypotheses on the sign of the potential  $y_0$ . We shall see in Section 4 that the above result about global existence is close to optimal, i.e. we shall specify large classes of initial data for which  $y_0$  changes sign and for which the corresponding solution blows up in finite time.

### 3. WEAK SOLUTIONS

Observe that the class

$$(3.1) \quad C([0, T], H^3(\mathbb{R})) \cap C^1([0, T], H^2(\mathbb{R}))$$

is optimal to solve (1.1) in the space  $C([0, T], L_2(\mathbb{R}))$ . Solutions belonging to (3.1) are called **strong solutions**.

A particular feature of the Camassa-Holm equation is the soliton interaction of solitary waves with corners at their crests, discovered numerically in [1]. Obviously, such solutions do not belong to the space (3.1). To provide the mathematical framework for the study of soliton interaction we shall introduce the notion of weak solutions to (1.1). To do this, let us first rewrite (1.1) as a **conservation law**. More precisely, let  $p(x) := (1/2) \exp(-|x|)$  be the Fourier

transform of the Poisson kernel on  $\mathbb{R}$ . Then the resolvent  $(1 - \partial_x^2)^{-1}$  can be represented as the following convolution operator:

$$(1 - \partial_x^2)^{-1} f = p * f, \quad f \in L_2(\mathbb{R}).$$

Assume now  $u_0 \in H^3(\mathbb{R})$  and let  $u \in C([0, T], H^3(\mathbb{R})) \cap C^1([0, T], H^2(\mathbb{R}))$  be the corresponding strong solution to (1.1). Then using (1.1) it is not difficult to verify that

$$(1 - \partial_x^2)(u_t + uu_x) = -2uu_x - u_x u_{xx} = -\partial(u^2 + (1/2)u_x^2).$$

Hence

$$u_t + uu_x = -\frac{1}{2} \partial_x (p * (u^2 + \frac{1}{2} u_x^2)) \quad \text{in } C([0, T], H^1(\mathbb{R})).$$

Introducing the nonlinear operator

$$F(v) := -\frac{1}{2} (v^2 + p * (v^2 + \frac{1}{2} v_x^2)), \quad v \in H^1(\mathbb{R}),$$

equation (1.1) can be rewritten as the following nonlocal conservation law

$$(3.2) \quad u_t + F(u)_x = 0, \quad u(0, \cdot) = u_0$$

A function  $u \in L_\infty((0, \infty), H^1(\mathbb{R}))$  is called a **global weak solution** to (1.1) if and only if it satisfies equation (3.2) in the sense of distributions. It is not very difficult to verify that every strong solution is a weak solution, and that every weak solution belonging to (3.1) is a strong solution. The following result is more interesting:

**Theorem 2.** [4] *Let  $u_0 \in H^1(\mathbb{R})$  and assume that  $y_0 = u_0 - u_{0,xx}$  is a positive regular Borel measure on  $\mathbb{R}$  with bounded total variation. Then there exists a global weak solution of the initial value problem (1.1).*

The proof of Theorem 2 is based on an approximation procedure. More precisely, we first approximate appropriate initial data in  $H^1(\mathbb{R})$  by smooth functions producing a sequence of global strong solutions to (1.1). For this sequence we have uniform bounds in  $H^1(\mathbb{R})$  and therefore it is possible to extract a subsequence  $u_n$  converging weakly in  $H^1(\mathbb{R})$  to some  $u$ . The most delicate part of the proof is to show that  $u_{n,xx}^2$  converges weakly in  $L_2(\mathbb{R})$  to  $u_x^2$ . To overcome this difficulty we use the method of compensated compactness, cf. [4].

#### 4. WAVE BREAKING

As mentioned in Section 2, blow-up phenomena occur for certain initial profiles. In fact, in the case of the equation (1.2) on the circle  $\mathbb{S}$ , a quite detailed description of this phenomena is possible. Let us first state the following result:

**Theorem 3.** [2, 5] *Assume that  $u_0 \in H^3(\mathbb{S})$  satisfies one of the following conditions:*

- (a)  $\int_{\mathbb{S}} u_0 \, dx = 0$ ,
- (b)  $\int_{\mathbb{S}} (u_0^3 + u_0 u_{0,x}^2) \, dx = 0$ ,
- (c)  $\min_{\mathbb{S}} u_0' + \max_{\mathbb{S}} u_0' \leq -2\sqrt{3} \|u_0\|_{H^1(\mathbb{S})}$ .

*Then the maximal existence time of the corresponding solution is finite.*

It is worth to note that neither the smoothness nor the size in  $H^3(\mathbb{S})$ , nor the size of the support of the initial data influence the life-span, but the shape of the initial data. In addition, Theorem 3 implies that in general one can not expect to control by conservation laws the  $H^2(\mathbb{R})$ -norm of the solution  $u$  to (1.1). This is in striking contrast to the KdV-equation, for which the conservation laws immediately yield a priori estimates in any  $H^r(\mathbb{R})$ -norm,  $r \geq 0$ . To further investigate the structure of this blow-up phenomena the following result is very useful:

**Theorem 4.** [6] *Let  $T > 0$  and  $v \in C^1([0, T], H^2(\mathbb{S}))$ . Then for every  $t \in [0, T)$  there exists at least one point  $\xi(t) \in \mathbb{S}$  with*

$$m(t) := \min_{x \in \mathbb{S}} [v_x(t, x)] = v_x(t, \xi(t)),$$

*and the function  $m$  is almost everywhere differentiable with*

$$\frac{dm}{dt}(t) = v_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, T).$$

It can show that the maximal existence time of a solution to (1.2) is finite if and only if its slope becomes unbounded in finite time (while the solution itself remains bounded). This is classically referred to as wave breaking. Based on Theorem 4, it is even possible to determine the exact blow-up rate of this wave breaking:

**Theorem 5.** [7] *The maximal existence time  $T$  is finite if and only if*

$$\lim_{t \rightarrow T} \min_{x \in \mathbb{S}} u_x(t, x) = -\infty.$$

*In this case we have*

$$\lim_{t \rightarrow T} (T - t) \min_{x \in \mathbb{S}} u_x(t, x) = -2.$$

For a large class of odd initial data we can also describe the exact blow-up set, i.e. the points in  $\mathbb{S}$  where the slope of the solution becomes unbounded:

**Theorem 6.** [7] *Let  $u_0 \in H^3(\mathbb{S}) \setminus \{0\}$  be given and set  $y_0 := u_0 - u_{0,xx}$ . Assume that  $y_0$  is odd with  $y_0(x) \geq 0$  on  $[0, 1/2]$  and  $y_0(x) \leq 0$  on  $[1/2, 1]$ . Then the corresponding solution blows up in finite time  $T$ . We have that*

$$u_x(t, 0) = u_x(t, 1/2) = u_x(t, 1) \rightarrow -\infty \quad \text{as } t \rightarrow T,$$

while

$$\sup_{(t,x) \in [0,T) \times \mathbb{S}} |u(t,x)| < \infty$$

and

$$\sup_{t \in [0,T)} |u_x(t,x)| < \infty, \quad x \in (0, 1/2) \cup (1/2, 1).$$

#### REFERENCES

- [1] R. Camassa & D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Letters* **71**, 1661–1664 (1993).
- [2] A. Constantin & J. Escher, Well-posedness, global existence, and blow-up phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.* **51**, 475–504 (1998).
- [3] A. Constantin & J. Escher, Global existence and blow-up for a shallow water equation, *Annali Scuola Norm. Sup. Pisa*, to appear.
- [4] A. Constantin & J. Escher, Global weak solutions for a shallow water equation, *Indiana Univ. Math. J.*, to appear.
- [5] A. Constantin & J. Escher, On the structure of a family of quasilinear equations arising in shallow water theory, *Math. Ann.*, to appear.
- [6] A. Constantin & J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica*, to appear.
- [7] A. Constantin & J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, preprint.
- [8] A. Fokas & B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Physika 4D*, 47–66 (1981).
- [9] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, *Spectral Theory and Differential Equations, Lecture Notes in Mathematics* **448**, 25–70 (1975).

# WEAK SOLVABILITY AND WELL-POSEDNESS FOR NONLINEAR DISPERSIVE SYSTEMS

TAKAYOSHI OGAWA  
GRADUATE SCHOOL OF MATHEMATICS  
KYUSHU UNIVERSITY  
FUKUOKA 812-8581 JAPAN

## 1. INTERACTION EQUATIONS

This is a survey talk concerning the time local well-posedness of the system of dispersive equations which arose from the interaction phenomena appearing in the waver wave theory. Let  $\phi(t, x, y)$  be the fluid velocity potential and  $\eta(t, x)$  be the surface displacement. Then the motion of the fluid surface is described by the following system of equations.

$$\begin{aligned} \Delta\phi &= 0, & x \in \mathbb{R}, & \quad -h < y < \eta(x), \\ \frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial x} \frac{\partial\eta}{\partial x} &= 0, & y &= \eta(x), \\ \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \eta - \frac{C}{(1 + (\eta_x)^2)^{3/2}} \frac{\partial^2\eta}{\partial^2x} &= 0, \end{aligned}$$

where  $y = \eta(x)$  and  $t \geq 0$ .

The solvability and well-posedness are studied by Yoshihara [39] and Iguchi [17]. In order to extract interaction phenomena, so called multiple-scaling expansion is employed: Introducing scaled variables  $(t_k, x_k, y_k)$ , where  $t_k = \varepsilon^k t$ ,  $x_k = \varepsilon^k x$  ( $k = 1, 2, \dots$ ), unknown functions are expanded in the following forms:

$$\begin{aligned} \phi(t, x, y) &= \sum_n \varepsilon^{\mu(n)} \phi_n(t_1, t_2, \dots, x_1, x_2, \dots) \\ \eta(t, x) &= \sum_n \varepsilon^{\mu(n)} \eta_n(t_1, t_2, \dots, x_1, x_2, \dots) \end{aligned}$$

We then suppose that the first approximation of the surface displacement is given by

$$\eta_1(t_1, x_1) = S(t_1, x_1)e^{i(kx - \omega t)} + \bar{S}(t_1, x_1)e^{-i(kx - \omega t)} + L(t_1, x_1)$$

where the first two term describes highly oscillating wave and  $L(t, x)$  denotes the slowly drifting long wave. Comparing to the same order terms of  $\varepsilon^{\mu(n)}$  we obtain an interaction equation (Kaupman [18], Grimshaw [15]):

$$\begin{cases} i\partial_t S + \partial_x^2 S = \alpha L S, \\ \partial_t L + \partial_x L = \beta \partial_x (|S|^2), \end{cases}$$

(special case is the modified Zakharov system (Yajima-Oikawa [38]).

Slightly general form of the equation is obtained by Djordjevic-Redekopp [10] and Benney [6] :

$$\begin{cases} i(\partial_t S + c_g \partial_x S) + \partial_x^2 S = \alpha LS + \gamma |S|^2 S, \\ \partial_t L + c_l \partial_x L = \beta \partial_x (|S|^2), \end{cases}$$

$c_g, c_l, \alpha, \beta, \gamma$  are real constants. When the long wave  $L$  is governed by a dispersive equation, two kinds of models are suggested: Coupled Schrödinger-KdV equation (Kawahara-Sugimoto-Kakutani [20]):

$$\begin{cases} i(\partial_t S + c_g \partial_x S) + \partial_x^2 S = \alpha LS, \\ \partial_t L + c_l \partial_x L + \partial_x^3 L + \partial_x L^2 = \beta \partial_x (|S|^2), \end{cases}$$

and an interaction on surface between two phase flow (Funakoshi-Oikawa [11]):

$$\begin{cases} i\partial_t S + \partial_x^2 S = \alpha LS, \\ \partial_t L + \nu D_x \partial_x L = \beta \partial_x (|S|^2), \end{cases}$$

where  $\nu > 0$  and  $D_x = H\partial_x$  with  $Hu = \mathcal{F}^{-1}((-i)\text{sgn}(\xi)\hat{u})$  being the Hilbert transform.

Common structures among those equations are summarized as follows:

- Short wave envelope “ $S$ ” is governed by the Schrödinger type equation
- Long wave “ $L$ ” is subject to the dispersive or wave equation with the drift effect.
- Common coupling nonlinearities.

We note that the Benney’s equation is solvable by ”inverse scattering method” (Yajima-Oikawa [38], Ma [28] )

## 2. WELL-POSEDNESS

On a view of the theory of evolution equation, it is desirable to show the well-posedness of those systems. By using Galilei-Gauge transform

$$\begin{cases} u(t, x) = \sqrt{|\alpha\beta|} e^{it(c_g - c_l)/2 - i(c_g - c_l)x^2/4} S(t, x + c_l t), \\ v(t, x) = \beta L(t, x + c_l t), \end{cases}$$

Tsutsumi-Hatano [36] simplified the second system and consider the Cauchy problem in the following form:

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma |u|^2 u, & t, x \in \mathbb{R}, \\ \partial_t v = \partial_x (|u|^2), \\ u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x). \end{cases}$$

They showed the time local well-posedness Benney's' equation in the Sobolev space  $H^{m+1/2} \times H^m$  where  $m = 0, 1, 2, \dots$ . Analogous observation was done for the coupled Schrödinger-KdV equation (M.Tsutsumi [34]):

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma|u|^2 u, & t, x \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \partial_x v^2 = \partial_x(|u|^2) \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x). \end{cases}$$

Here we precisely define the meaning of "well-posedness" in the Sobolev space in  $H^s$ . For  $s \geq 0$  we let

$$H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}); \langle \xi \rangle^s \hat{u}(\xi) \in L^2\},$$

where  $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ .

To consider the weaker solution, we solve the corresponding integral equation to the Cauchy problem of interaction system. For example to Benney's equation,

$$(2.1) \quad u(t) = U(t)u_0 - i \int_0^t U(t-t') \left\{ u(t')v(t') + \gamma|u(t')|^2 u(t') \right\} dt',$$

$$(2.2) \quad v(t) = v_0 + \int_0^t \beta \partial_x |u(t')| dt',$$

where  $U(t) = e^{it\partial_x^2}$  is the free Schrödinger evolution group.

Definition. (Well posedness in  $H^s$ ) The equation (B) is (time locally) well-posed in  $H^s$  if for any  $(u_0, v_0) \in H^s \times H^{s-1/2}$ , there exists a time interval  $T = T(u_0, v_0)$  and unique pair of solutions  $(u, v)$  of the integral equations (2.1)-(2.2) such that

- $(u, v) \in C([0, T]; H^s) \cap X \times C([0, T]; H^{s-1/2}) \cap Y$ , where  $X$  and  $Y$  are properly chosen subspaces in  $C([0, T]; H^s)$  and  $C([0, T]; H^{s-1/2})$  respectively.
- $(u, v)$  is unique in the above space,
- $(u, v)$  is continuously depending on  $(u_0, v_0)$ .

Under this framework, Tsutsumi-Hatano established the local  $H^{m+1/2}$  well-posedness in [35] and [36]. In fact by observing three conservation laws:

- $\|u(t)\|_2 = \|u_0\|_2$ ,
- $P(u(t), v(t)) = P(u_0, v_0)$  (momentum),
- $E(u(t), v(t)) = E(u_0, v_0)$  (energy),

where

$$\begin{aligned} P(u, v) &= \|v(t)\|_2^2 + 2Im \int_{\mathbb{R}} u(t) \partial_x \bar{u}(t) dx, \\ E(u, v) &= \|\partial_x u\|_2^2 + \frac{\gamma}{2} \|u\|_4^4 + \int_{\mathbb{R}} v|u|^2 dx, \end{aligned}$$

they also proved the time global well-posedness to the Benney system for  $H^{3/2}$  initial data. Since the largest space where Tsutsumi-Hatano obtained the well-posedness was  $H^{1/2} \times L^2$ , the following natural question arose: Can we obtain the well-posedness in larger space than  $H^{1/2}$ ?

### 3. MAIN RESULTS

To cover all non-negative exponent  $s \geq 0$  and to treat the full system, we introduce the Fourier restriction norm used by Bourgain [7] as auxiliary function spaces  $X$  and  $Y$ .

$$\begin{aligned} \|u\|_{\Phi_b^s}^2 &\equiv \int_{\mathbb{R}^2} \langle \tau + \phi(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\hat{u}(\tau, \xi)|^2 d\xi d\tau = \|U(-t)f(t)\|_{H_t^b(\mathbb{R}; H_x^s(\mathbb{R}))}^2, \\ \|v\|_{\Psi_b^s}^2 &\equiv \int_{\mathbb{R}^2} \langle \tau + \psi(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\hat{v}(\tau, \xi)|^2 d\xi d\tau = \|V(-t)h(t)\|_{H_t^b(\mathbb{R}; H_x^s(\mathbb{R}))}^2, \end{aligned}$$

where  $\phi(\xi)$  and  $\psi(\xi)$  are real valued functions and  $U(t) = e^{it\phi(\partial_x)}$ ,  $V(t) = e^{it\psi(\partial_x)}$  are the unitary operators associated with the linear dispersive equations.

The basic argument due to Bourgain [7], Kenig-Ponce-Vega [25] give the well-posedness result for the Cauchy problem:

$$\begin{cases} \partial_t v + \partial_x^3 v + \partial_x(v^2) = 0 & t, x \in \mathbb{R}, \\ v(x, 0) = v_0(x), \end{cases} \quad (KdV)$$

We take  $\phi(\xi) = |\xi|^3$ ,  $U(t) = e^{-it\partial_x^3}$  and set  $\Phi_b^s = X_b^s$ .

**Theorem 0** ([7],[25]). Let  $s > -3/4$ . For  $v_0 \in H^s$  the KdV equation (KdV) is locally well-posed in  $H^s$  i.e.,  $\exists T = T(v_0) > 0$ ,  $b \in (1/2, 7/12)$  and  $\exists v \in C([0, T]; H^s) \cap X_b^s$ : unique solution of (KdV) and the map from  $(u_0, v_0)$  to  $(u, v)$  is Lipschitz continuous from  $H^s$  to  $C([0, T]; H^s) \cap X_b^s$ .

◦ Remark 1:

For the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = f(u) & t, x \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \quad (NLS)$$

where  $f(u)$  is either  $u^2$ ,  $\bar{u}^2$ ,  $u^3$  or  $\bar{u}^3$ , analogous result is obtained ([25]). However the gauge invariant version of (NLS) ( $f(u) = |u|^2 u$ ) is not improved by this method. Hence the regularity for the initial data is now obtained only  $u_0 \in L^2$ .

Also the single Benjamin-Ono equation, the regularity for the initial data is not improved by this method.

We now turn into the system equations: For the Benney's equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma|u|^2 u, & t, x \in \mathbb{R}, \\ \partial_t v + c\partial_x v = \partial_x(|u|^2), \\ u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x), \end{cases} \quad (B)$$

Let  $\phi(\xi) = |\xi|^2$ ,  $U(t) = e^{-t\partial_x^3}$ ,  $\psi(\xi) = \text{csgn}(\xi)|\xi|$ ,  $V(t) = e^{-ct\partial}$  and set  $X_b^s \equiv \Phi_b^s$  and  $Y_b^s \equiv \Psi_b^s$ . We have:

**Theorem 1** ([3]). Let  $s \geq 0$ . For  $(u_0, v_0) \in H^s \times H^{s-1/2}$  Benney's equation (B) is locally well-posed i.e.,  $\exists T = T(u_0, v_0) > 0$ ,  $b \in (1/2, 3/4)$  and  $\exists (u, v) \in C([0, T]; H^s) \times C([0, T]; H^{s-1/2})$ : unique solution of (B) with

$$u \in X_b^s \quad v \in Y_b^{s-1/2}$$

and the map from  $(u_0, v_0)$  to  $(u, v)$  is the Lipschitz continuous from  $H^s \times H^{s-1/2}$  to  $C([0, T]; H^s) \times C([0, T]; H^{s-1/2})$ .

For the two phase flow equations

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu, & t, x \in \mathbb{R}, \\ \partial_t v + \nu \partial_x D_x v = \partial_x(|u|^2), \\ u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x), \end{cases} \quad (FO)$$

we use  $\psi(\xi) = \nu \text{sgn}(\xi)|\xi|^2$  and  $Y_b^s \equiv \Psi_b^s$ .

**Theorem 2** ([3]). For  $s \geq 0$  and  $|\nu| < 1$ , (FO) is locally well-posed with  $(u_0, v_0) \in H^s \times H^{s-1/2}$ , i.e.,  $\exists T = T(u_0, v_0) > 0$ ,  $b \in (1/2, 3/4)$  and  $\exists (u, v) \in C([0, T]; H^s) \times C([0, T]; H^{s-1/2})$ : unique solution of (FO) and

$$u \in X_b^s \quad v \in Z_b^{s-1/2}.$$

◦ Remark 2:

Structure of the nonlinear terms requires the regularity difference  $\frac{1}{2}$ . Let  $(u, v)$  be a solution pair of the Benney's equation. Then setting

$$\begin{aligned} u_\lambda(x, t) &= \lambda^{3/2} u(\lambda x, \lambda^2 t), \\ v_\lambda(x, t) &= \lambda^2 v(\lambda x, \lambda^2 t), \end{aligned}$$

(ignoring  $\gamma|u|^2 u$ ).  $(u_\lambda, v_\lambda)$  solves Benney's equation with initial data

$$\begin{aligned} u_{\lambda 0} &= \lambda^{3/2} u_0(\lambda x), \\ v_{\lambda 0} &= \lambda^2 v_0(\lambda x). \end{aligned}$$

Now take the derivative of order  $s$  to  $u_\lambda$  and  $s - 1/2$  to  $v_\lambda$

$$\begin{aligned} \|D_x^s u_\lambda\|_2^2 &= \lambda^{2+2s} \|D_x^s u\|_2^2 \\ \|D_x^{s-1/2} v_\lambda\|_2^2 &= \lambda^{2+2s} \|D_x^{s-1/2} v\|_2^2 \end{aligned}$$

The difference of order  $1/2$  is required to keep them equivalent under the scaling.

◦ Remark 3:

A similar result for Schrödinger-KdV system is possible (Bekiranov-Ogawa-Ponce [2]):

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma|u|^2 u, & t, x \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \partial_x v^2 = \partial_x(|u|^2), \\ u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x), \end{cases} \quad (SK)$$

i.e.,  $H^s \times H^{s-1/2}$  local well-posedness ( $s \geq 0$ ). This improves the previous result by M. Tsutsumi [34].

◦ Remark 4:

An analogous result for the Zakharov system:

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu, & t, x \in \mathbb{R}, \\ \partial_t^2 v - \partial_x^2 v = \partial_x^2(|u|^2), \\ u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x), \partial_t v(x, 0) = v_1(x) \end{cases} \quad (Z)$$

is considered by Bourgain [8], Ginibre-Tsutsumi-Velo [13] and Takaoka [33].

◦ Remark 5:

For (FO), if  $|\nu| = 1$ , our method does not work well. A sort of cancellation prevents to establish the crucial estimate. We are expecting that if  $s > 0$  then a similar result hold for (FO) with  $\nu = \pm 1$ .

◦ Remark 6:

In view of Theorem 1 and 2, whether the second equation (long wave) is dispersive type or wave equation, it does not concern on the well-posedness result. In the other words, any smoothing effect in the second equation does not give any effect to obtain up to  $L^2 \times H^{-1/2}$  solutions.

We are expecting so far that the case  $s = 0$  is optimal under the method present here. In fact, we can show that the crucial estimate in our method does not hold for  $s < 0$ .

**Proposition 3.** If  $s < 0$  and  $\nu = \pm 1$  then there is a counter example of the following estimate:

$$\|uv\|_{X_b^{s-1}} \leq C\|u\|_{X_b^s}\|v\|_{Z_b^{s-1/2}}.$$

One application of the well-posedness for the equations is a limiting problem in the system (FO).

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu, & t, x \in \mathbb{R}, \\ \partial_t v + \nu \partial_x D_x v = \partial_x(|u|^2), \end{cases} \quad (FO)$$

Passing the parameter  $\nu \rightarrow 0$ , the solution strongly converges to the solution without the dispersive term  $\partial_x D_x v$ . Namely:

**Theorem 4** ([3]). As  $\nu \rightarrow 0$  in (FO) the  $L^2 \times H^{-1/2}$  solution converges to the solution of Benney's equation with  $c = 0$ . i.e., Let  $(u_\nu, v_\nu)$  be  $L^2 \times H^{-1/2}$  solution of (FO) and  $(u, v)$  be of (B) with  $c = 0$  with the same initial data. Then

$$\begin{aligned} \|u_\nu - u\|_{C(0,T;L^2)} &\rightarrow 0, \\ \|v_\nu - v\|_{C(0,T;H^{-1/2})} &\rightarrow 0 \end{aligned}$$

as  $\nu \rightarrow 0$ .

The system (FO) describes the model under the deep water flow, and (B) with  $c = 0$  is for the shallow flow, Theorem 4 states that the solution in the deep flow equation (FO), is approximately getting close to the shallow setting solution as parameter  $\nu \rightarrow 0$ .

In the regular case, the similar result is relatively easy to show. However our Theorem 4 proves that the system is stable even in the weaker space  $L^2 \times H^{-1/2}$ . This stability stems from the smoothing properties not only by free Schrödinger evolution operator but the nonlinear coupling term in the second equation. In fact, the nonlinear coupling  $\partial_x |u|^2$  has a better smoothing property itself than other term. One can observe that if we multiply the solution  $u$  with the complex conjugate of  $u$ , a sort of cancellation happens and the singularity is disappear. Therefore to obtain the weaker solution for (FO) or (SK), we do not need the dispersive properties for the second equation but only need the Schrödinger part and this special structure of nonlinearity.

#### REFERENCES

- [1] Bekiranov, D., Ogawa, T., Ponce, G., *On the well-posedness of Benney's interaction equation of short and long waves*, Advances Diff. Equations, **1** (1996) 919-937.
- [2] Bekiranov, D., Ogawa, T., Ponce, G., *Weak solvability and well posedness of the coupled Schrödinger Korteweg-de Vries equations in the capillary-gravity interaction waves*, Proc. Ameri. Math. Soc., **125** no. 10 (1997) 2907-2919.
- [3] Bekiranov, D., Ogawa, T., Ponce, G., *Interaction equations for short and long dispersive waves*, J. Funct. Anal. to appear.
- [4] Benilov, E.S., Burtsev, S.P., *To the integrability of the equations describing the Langmuir-wave-ion-wave interaction*, Phys. Let. **98** A (1983) 256-258.
- [5] Benjamin, T.B., *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. **29** A (1967) 559-592.
- [6] Benney, D. J., *Significant interactions between small and large scale surface waves*, Stud. Appl. Math., **55**, (1976), 93-106. *A general theory for interactions between short and long waves*, ibid. **56**, (1977), 81-94.
- [7] Bourgain, J., *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I Schrödinger equations*, Geometric and Funct. Anal. **3** (1993), 107-156. *Exponential sums and nonlinear Schrödinger equations*, ibid. **3** (1993), 157-178. *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II The KdV equation*, ibid. **3** (1993), 209-262.
- [8] Bourgain, J., *On the Cauchy and invariant measure problem for the periodic Zakharov system*, Duke Math. J. **76** (1994), 175-202.
- [9] Bourgain, J., *New global well-posedness results for nonlinear Schrödinger equations*, preprint.
- [10] Djordjevic, V.D., Redekopp, L.G., *On two-dimensional packet of capillary-gravity waves*, J. Fluid Mech. **79** (1977) 703-714.
- [11] Funakoshi, M., Oikawa, M., *The resonant interaction between a long internal gravity wave and a surface gravity wave packet*, J. Phys. Soc. Japan, **52** (1983) 1982-1995.
- [12] Ginibre, J., *Le problème de Cauchy pour des edp semi-linéaires périodiques en variables d'espace [d'après Bourgain]* Séminaire BOURBAKI, **47** (1994-5) 796, 1-23.
- [13] Ginibre, J., Tsutsumi, Y., Velo, G., *On the Cauchy problem for the Zakharov system* J. Funct. Anal. **151** (1997), 384-436.
- [14] Ginibre, J., Velo, G., *The global Cauchy problem for the nonlinear Schrödinger equations, revisited*, Ann. Inst. Henri Poincaré, Analyse nonlinéaire, **2** (1985) 309-327.
- [15] Grimshaw, R.H.J., *The modulation of an internal gravity-wave packet and the resonance with the mean motion*, Stud. Appl. Math. **56** (1977) 241-266.

- [16] Hayashi, N., *The initial value problem for the derivative nonlinear Schrödinger equation*, Nonlinear Anal. TMA. **20** (1993), 823-833.
- [17] Iguchi, T., *Well-posedness of the initial value problem for capillary-gravity waves*, preprint.
- [18] Karpman, V.I., *On the dynamics of sonic-Langmuir soliton*, Physica Scripta, **11** (1975) 263-265.
- [19] Kato, T. *On nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Physique théorique, **46** (1987), 113-129.
- [20] Kawahara, T., Sugimoto, N., Kakutani, T., *Nonlinear interaction between short and long capillary-gravity waves*, J. Phys. Soc. Japan, **39** (1975), 1379-1386
- [21] Kenig, C. E., Ponce, G., Vega, L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction mapping principle*, Comm. Pure Appl. Math. **46** (1993) 527-620.
- [22] Kenig, C. E., Ponce, G., Vega, L., *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993) 1-21.
- [23] Kenig, C. E., Ponce, G., Vega, L., *On the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc. **342** (1994) 155-172.
- [24] Kenig, C. E., Ponce, G., Vega, L., *A bilinear estimate with applications to the KdV equation*. J. Amer. Math. Soc. **9** (1996) 573-603.
- [25] Kenig, C. E., Ponce, G., Vega, L., *Quadratic Forms for the 1-D semilinear Schrödinger equation*, Trans. Amer. Math. Soc., **348** (1996), 3323-3353
- [26] Klainerman, S., Machedon, M., *Space-time estimates for null forms and the local existence theorem.*, *Comm. Pure Appl. Math.* **46** (1993) 1221-1268.
- [27] Laurençot, Ph., *On a nonlinear Schrödinger equation arising in the theory of water waves*, Nonlinear Anal. TMA **24** (1995), 509-527.
- [28] Ma, Yan-Chow, *The complete solution of the long-wave-short -wave resonance equations*, Stud. Appl. Math., **59** (1978), 201-221.
- [29] Ono, H., *Algebraic solitary waves in stratified fluids*, J. Phys Soc. Japan, **39**(1975) 1082-1091.
- [30] Ogawa, T., *Global well-posedness and conservation law of the water wave interaction equation*, Proc. Roy. Soc. Edinburgh, **127** ser A (1997) 369-384.
- [31] Ponce, G., *On the global well-posedness of the Benjamin-Ono equation*, Diff. Integral Equations, **4** (1991) 527-542.
- [32] Staffilani, G., *Quadratic forms for a 2-D semilinear Schrödinger equation*, Duke Math. J. **86** no.1 (1997), 79-107.
- [33] Takaoka, H., *Well-posedness for the Zakharov system with the periodic boundary condition*, preprint, Univ. Tokyo.
- [34] Tsutsumi, M., *Well-posedness of the Cauchy problem for a coupled Schrödinger-KdV equation*, Math. Sci. Appl. **2** (1993) 513-528.
- [35] Tsutsumi, M., Hatano, S., *Well-posedness of the Cauchy problem for long wave-short wave resonance equation*, Nonlinear Anal. TMA, **22** (1994), 155-171
- [36] Tsutsumi, M., Hatano, S., *Well-posedness of the Cauchy problem for Benney's first equations of long wave short wave interactions*, Funkcialaj Ekvacioj, **37** (1994), 289-316
- [37] Tsutsumi, Y.,  *$L^2$ - solutions for nonlinear Schrödinger equations and nonlinear group*, Funkcialaj Ekvacioj, **30** (1987), 115-125.
- [38] Yajima, N., Oikawa, M., *Formation and interaction of sonic-Langmuir soliton*, Progr. Theor. Phys. **56** (1976), 1719-1739.
- [39] Yoshihara, H., *Capillary-gravity waves for an incompressible ideal fluid*, J. Math. Kyoto Univ. **23-4** (1983), 649-694.

# On the asymptotic behavior of solutions for damped nonlinear wave equations of Kirchhoff type in unbounded domain

Kosuke ONO

Department of Mathematical and Natural Sciences, Faculty of Integrated Arts and Sciences, The University of Tokushima, Tokushima 770-8502, JAPAN.

E-mail : ono@ias.tokushima-u.ac.jp

第23回偏微分方程式札幌シンポジウム : 7月29日～7月31日

Abstract: We consider the global existence, asymptotic behavior, and global non-existence of solutions for the following damped nonlinear wave equations of Kirchhoff type in unbounded domain :

$$(E) \quad u_{tt} + M(\|A^{1/2}u\|^2)Au + \delta u_t = f(u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+$$

with initial data

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x),$$

where  $A = -\Delta \equiv \sum_{j=1}^N \partial_{x_j}^2$  is the usual Laplace operator,  $\|\cdot\|$  is the norm of  $L^2(\mathbb{R}^N)$ ,  $\delta$  is a non-negative constant,  $M(\cdot)$  is a nonlinear function satisfying

$$M(\|A^{1/2}u\|^2) \equiv a + b\|A^{1/2}u\|^{2\gamma}$$

with  $a, b \geq 0, a + b > 0$ , and  $\gamma \geq 1$ , and  $f(u)$  is a  $C^1$ -function satisfying

$$|f(u)| \leq k_1|u|^{\alpha+1} \quad \text{and} \quad |f'(u)| \leq k_2|u|^\alpha$$

with  $k_1, k_2 \geq 0$  and  $\alpha > 0$ . For typical example, we can take  $f(u) = C|u|^\alpha u$ .

When  $N = 1$ , Eq.(E) describes nonlinear vibrations of an elastic string. Kirchhoff (1883) firstly studied such integro-differential equations.

In the case of  $a > 0$  and  $b > 0$ , (E) is called non-degenerate type, and in the case of  $a = 0$  and  $b > 0$ , degenerate one.

When  $\delta = 0$ , the problem of the local well-posedness in suitable Sobolev spaces has been already studied by many authors. On the other hand, in such spaces there are no works related to the global-in-time solvability. For non-degenerate wave equations of Kirchhoff type with nonlinear perturbations

$$u_{tt} + (1 + \|A^{1/2}u\|^2)Au = f(u, u_t, \nabla u),$$

however, D'Ancona and Spagnolo (CPAM, 1994) have shown the existence of a global  $C^\infty$ -solution under small  $C_0^\infty$ -data. Their results cannot be applied to degenerate equations, and under weaker initial data belonging to  $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$  also, we cannot obtain any information.

To state our results, we define the following :

$$E(t) \equiv \|u_t(t)\|^2 + J(u(t)),$$

$$J(u) \equiv (a + b(1 + \gamma)^{-1} \|A^{1/2}u\|^{2\gamma}) \|A^{1/2}u\|^2 - \int_{\mathbb{R}^N} F(u) dx$$

with  $F(u) = 2 \int_0^u f(\eta) d\eta$ ,

$$E_1(t) \equiv \|u_t(t)\|^2 + (a + b\|A^{1/2}u(t)\|^{2\gamma}) \|A^{1/2}u(t)\|^2,$$

$$E_2(t) \equiv \|A^{1/2}u(t)\|^2 + (a + b\|A^{1/2}u(t)\|^{2\gamma}) \|Au(t)\|^2,$$

$$\mathcal{W}_* \equiv \{u \in H^1(\mathbb{R}^N); K(u) > 0\} \cup \{0\},$$

$$\mathcal{V}_* \equiv \{u \in H^1(\mathbb{R}^N); K(u) < 0\},$$

and

$$K(u) \equiv \begin{cases} a\|A^{1/2}u\|^2 - k_1\|u\|_{\alpha+2}^{\alpha+2} & \text{if } a > 0 \\ b\|A^{1/2}u\|^{2(\gamma+1)} - k_1\|u\|_{\alpha+2}^{\alpha+2} & \text{if } a = 0, \end{cases}$$

where we take  $k_1 = 1$  if  $f(u) = |u|^\alpha u$ .

### (0) Local Existence

Theorem. Suppose that  $\{u_0, u_1\} \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$  and  $M(\|A^{1/2}u_0\|^2) > 0$  (i.e.  $a > 0$  or  $\|A^{1/2}u_0\|^2 \neq 0$  if  $a = 0$ ) and  $\alpha \leq 2/(N-4)^+$ . Then, the problem (E) admits a unique local solution  $u(t)$  in the class  $\bigcap_{j=0}^2 C^j([0, T]; H^{2-j}(\mathbb{R}^N))$  for some  $T = T(\|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0$ .

### (I) Global Existence & Decay in Non-Degenerate Case ( $a = b = \delta = 1$ )

$$(E1) \quad u_{tt} + (1 + \|A^{1/2}u\|^{2\gamma})Au + u_t = f(u).$$

Theorem. Let  $N \leq 3$ . Suppose that  $\alpha \geq 4/N$  and  $u_0 \in \mathcal{W}_*$ . Then, there exists a certain open (unbounded) set  $\mathcal{S}$  in  $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$  including  $\{0, 0\}$  such that if  $\{u_0, u_1\} \in \mathcal{S}$ , the problem (E1) admits a unique global solution  $u(t) \in \mathcal{W}_*$  in the class  $\bigcap_{j=0}^2 C^j(\mathbb{R}_+; H^{2-j}(\mathbb{R}^N))$  satisfying

$$(1+t)E_1(t) + (1+t)^2E_2(t) \leq C \quad \text{for } t \geq 0.$$

Moreover, if  $\{u_0, u_1\} \in L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ , then

$$(1+t)^{N/2}\|u(t)\|^2 + (1+t)^{1+N/2}E_1(t) + (1+t)^{2+N/2}E_2(t) \leq C.$$

### (II) Global Existence & Decay in Degenerate Case ( $a = 0, b = \delta = 1$ )

$$(E2) \quad u_{tt} + \|A^{1/2}u\|^{2\gamma}Au + u_t = f(u).$$

Theorem. Let  $N \leq 3$ . Suppose that

$$\alpha \geq 4(\gamma + 1)/N,$$

and

$$\alpha > 2\gamma \quad \text{and} \quad \alpha + 4 > 4\gamma \quad \text{if } N = 3,$$

and  $u_0 \in \mathcal{W}_*$  with  $\|A^{1/2}u_0\|^2 \neq 0$ . Then, there exists a certain open (unbounded) set  $\mathcal{S}'$  in  $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$  including  $\{0, 0\}$  such that if  $\{u_0, u_1\} \in \mathcal{S}'$ , the problem (E2) admits a unique global solution  $u(t) \in \mathcal{W}_*$  in the class  $\bigcap_{j=0}^2(\mathbb{R}_+; H^{2-j}(\mathbb{R}^N))$  satisfying

$$(1 + t)E(t) + H(t) \leq C \quad \text{for } t \geq 0,$$

where we set

$$H(t) \equiv \|A^{1/2}u_t(t)\|^2 / \|A^{1/2}u(t)\|^{2\gamma} + \|Au(t)\|^2.$$

Moreover, if  $\gamma = 1$ , then we have

$$(1 + t)H(t) \leq C \quad \text{for } t \geq 0.$$

(III) Global Non-Existence in the case  $f(u) = |u|^\alpha u$

$$(E3) \quad u_{tt} + M(\|A^{1/2}u\|^2)Au + \delta u_t = |u|^\alpha u$$

Theorem. Suppose that  $\alpha \geq 2\gamma$  and either

$$E(0) < 0 \quad \text{or} \quad E(0) = 0 \quad \text{and} \quad (u_0, u_1) > 0.$$

Then, the local solution  $u$  of (E3) cannot be continued to some finite time  $T > 0$ .

Theorem. Let  $N \geq 3$  and  $\alpha = 4/(N - 2)$ . Suppose that

$$u_0 \in \mathcal{V}_* \quad \text{and} \quad E(0) < D_a$$

and either

$$\alpha \geq 2\gamma \quad \text{if } a > 0 \quad \text{or} \quad \alpha > 2\gamma \quad \text{if } a = 0.$$

Then, the local solution  $u$  of (E3) cannot be continued to some finite time  $T > 0$ , where we set

$$D_a \equiv \begin{cases} \frac{2}{N} \left( \frac{a}{c_*^2} \right)^{N/2} & \text{if } a > 0 \\ \frac{2-\gamma(N-2)}{N(\gamma+1)} \left( \frac{b}{c_*^{2(\gamma+1)}} \right)^{N/(2-(\gamma(N-2)))} & \text{if } a = 0 \end{cases}$$

with

$$c_* \equiv \sup\{\|u\|_{2N/(N-2)} / \|A^{1/2}u\|; u \in H^1(\mathbb{R}^N), u \neq 0\}.$$

# SMOOTHING EFFECTS FOR SOME DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

PATRICK-NICOLAS PIPOLO

Department of Applied Mathematics, Science University of Tokyo  
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162, JAPAN

and

Analyse Numérique et EDP,  
CNRS et Université Paris-Sud,  
Bâtiment 425, 91405 Orsay, FRANCE

## §1 INTRODUCTION

We study a smoothing property of solutions to the Cauchy problem for the derivative nonlinear Schrödinger equation of the following form

$$\begin{cases} iu_t + u_{xx} = \mathcal{N}(u), & x \in \mathbf{R}, \quad t \in \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}, \end{cases} \quad (1.1)$$

where the nonlinearity  $\mathcal{N}(u) = K_1 u + K_2 |u|^2 u_x + K_3 \bar{u} u_x^2$ , contains the derivatives of the unknown and the functions  $K_j = K_j(|u|^2)$ ,  $K_j(z) \in \mathbf{C}^{l+3}([0, \infty))$ . If  $K_3(z) = \frac{\lambda_3}{1+\mu z}$  and  $\lambda_3 = \mu = 1$ ,  $K_1 = K_2 = 0$ , then (1.1) appears in the classical pseudospin magnet model [16].

To state our main result we introduce some function spaces. The Lebesgue space is  $\mathbf{L}^p = \{\varphi \in \mathbf{S}' : \|\varphi\|_p < \infty\}$ , where  $\|\varphi\|_p = (\int |\varphi(x)|^p dx)^{1/p}$  if  $1 \leq p < \infty$  and  $\|\varphi\|_\infty = \text{ess.sup}\{|\varphi(x)|; x \in \mathbf{R}\}$  if  $p = \infty$ . For simplicity we let  $\|\varphi\| = \|\varphi\|_2$ . Weighted Sobolev space is  $\mathbf{H}_p^{m,s} = \{\varphi \in \mathbf{S}' : \|\varphi\|_{m,s,p} = \left\| (1+x^2)^{s/2} (1-\partial_x^2)^{m/2} \varphi \right\|_p < \infty\}$ ,  $m, s \in \mathbf{R}$ ,  $1 \leq p \leq \infty$ . For simplicity we write  $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$ . We denote also  $\mathbf{H}^{m,\infty} = \cap_{s=1}^\infty \mathbf{H}^{m,s}$ . We let  $\mathbf{C}(\mathbf{I}; \mathbf{B})$  be the space of continuous functions from a time interval  $\mathbf{I}$  to a Banach space  $\mathbf{B}$ .

Our main results are the following.

---

This is a joint work with N.Hayashi and P.I.Naumkin [22].

**Theorem 1.1.** *Let the initial data  $u_0 \in \mathbf{H}^{3,l}$ , with any  $l \in \mathbf{N}$ . Then for some time  $T > 0$  there exists a unique solution*

$$u \in \mathbf{C}([-T, T]; \mathbf{H}^{2,0}) \cap \mathbf{L}^\infty(-T, T; \mathbf{H}^{3,0}) \cap \mathbf{C}([-T, T] \setminus \{0\}; \mathbf{C}^{l+2}(\mathbf{R}))$$

of the Cauchy problem (1.1) such that

$$\sup_{t \in [-T, T]} |t|^k \left\| (1+x^2)^{-k/2} \partial_x^k u(t) \right\|_{2,0} < \infty \quad \text{for } 0 \leq k \leq l.$$

By virtue of Theorem 1.1, (1.1) and the Sobolev's embedding inequality (see [6]) we get

**Corollary 1.1.** *Let the initial data  $u_0 \in \mathbf{H}^{3,\infty}$ . Then for some time  $T > 0$  there exists a unique solution  $u \in \mathbf{C}^\infty([-T, T] \setminus \{0\}; \mathbf{C}^\infty(\mathbf{R}))$  of the Cauchy problem (1.1).*

Our method is applicable also to the case of the following general nonlinear term  $\mathcal{N}(u) = \mathcal{K}_1 u + \mathcal{K}_2 |u|^2 u_x + \mathcal{K}_3 \bar{u} u_x^2 + \mathcal{K}_4 u^2 \bar{u}_x + \mathcal{K}_5 u^3 \bar{u}_x^2$ , depending on  $\bar{u}_x$ . Here the functions  $K_j = K_j(|u|^2), K_j(z) \in \mathbf{C}^{l+3}([0, \infty))$ . However in this case we have to assume the smallness of the initial data in the norm of the weighted Sobolev space  $\mathbf{H}^{3,l}$ . Indeed we have the following result.

**Theorem 1.2.** *Let the initial data  $u_0 \in \mathbf{H}^{3,l}$  and the norm  $\|u_0\|_{3,l}$  be sufficiently small, where  $l \in \mathbf{N}$ . Then the same result as in Theorem 1.1 holds. Moreover if the initial data  $u_0 \in \mathbf{H}^{3,\infty}$  and the norm  $\|u_0\|_{3,l}$  is sufficiently small for any  $l \in \mathbf{N}$  then the result of Corollary 1.1 is valid.*

Smoothing effects of solutions to the nonlinear Schrödinger equation (1.1) with  $K_2 = K_3 \equiv 0$  was studied in [9] and the similar results to that of Theorem 1.1 were obtained by using the operator  $\mathcal{J} = x + 2it\partial_x$ , which commutes with the linear Schrödinger operator  $\mathcal{L} = i\partial_t + \partial_x^2$ . However there are no similar results on nonlinear Schrödinger equations of derivative type except the derivative nonlinear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = i(|u|^2 u)_x, & x \in \mathbf{R}, \quad t \in \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}. \end{cases} \quad (1.2)$$

The derivative nonlinear Schrödinger equation (1.2) can be translated to a system of nonlinear Schrödinger equations without derivatives of unknown functions. So the same results as in Theorem 1.1 were shown for the Cauchy problem (1.2) in [10].

The existence of solutions to the Cauchy problem (1.1) was established in [1] and [11] in the usual Sobolev space  $\mathbf{H}^{3,0}$ . Taking the function space

$$\{\phi \in \mathbf{C}([-T, T]; \mathbf{L}^2); \sup_{t \in [-T, T]} (\|\phi\|_{m,0} + \|\mathcal{J}^l \phi\|_{m,0}) < \infty\}$$

into consideration instead of the space  $\mathbf{C}([-T, T]; \mathbf{H}^{3,0})$  and using the method of [1], [11], [19] we can prove the following result.

**Proposition 1.1.** *Let the initial data  $u_0 \in \mathbf{H}^{m,l}$ , where  $m \geq 3, l \leq m, m, l \in \mathbf{N}$ . Then for some time  $T > 0$  there exists a unique solution  $u$  of the Cauchy problem (1.1) such that*

$$u \in \mathbf{C}([-T, T]; \mathbf{H}^{m-1,0}) \cap \mathbf{L}^\infty(-T, T; \mathbf{H}^{m,0}) \cap \mathbf{C}([-T, T] \setminus \{0\}; \mathbf{C}^{m+l-1}(\mathbf{R})).$$

This result also describes a smoothing property of solutions to the Cauchy problem (1.1), which is weaker than that of Corollary 1.1. In order to obtain the result we apply a smoothing effect of the linear Schrödinger equation. Smoothing properties of solutions to the linear Schrödinger equation were studied by many authors (see [2], [3], [5], [18], [21]) and later these results of [3],[18],[21] were improved in [14], [15]. However it seems that the method of the papers [14], [15] requires smallness and regularity assumptions on the initial data. To avoid these condition and to gain smoothing properties of solutions to the linear Schrödinger equation we use some pseudo-differential operator of order 0. The history of such operators starts from Doi, who discovered in [5] the following operator  $\exp\left(\int_{-\infty}^x (1+x'^2)^{-1} dx' \frac{D}{\langle D \rangle}\right)$ , where  $D = -i\partial_x$  and  $\langle D \rangle = (1 - \partial_x^2)^{1/2}$  which is useful to gain a smoothing property of solutions. Chihara [2] used the following modification of this pseudo-differential operator  $\exp\left(\int_{-\infty}^x |u(t, x')|^2 dx' \frac{D}{\langle D \rangle}\right)$ , to prove the local existence of solutions  $u$  to the Cauchy problem for the nonlinear Schrödinger equations in higher order Sobolev space. He made use of some well known results concerning pseudo-differential operators, such as the  $\mathbf{L}^2$  - boundedness theorem and the sharp Gårding inequality, and that is why the higher order Sobolev space was needed. In this paper we apply a more simple operator  $\mathcal{S}(\varphi) = \cosh(\varphi) + i \sinh(\varphi) \mathcal{H}$ , where  $\varphi = \int_{-\infty}^x |u(t, x')|^2 dx'$  and  $\mathcal{H}\phi = \frac{1}{\pi} \text{Pv} \int_{\mathbf{R}} \frac{\phi(x') dx'}{x-x'}$ , which enables us to avoid the use of the technique of the pseudo-differential operators and so by virtue of simple explicit computations we can treat the problem in the natural order Sobolev space  $\mathbf{H}^{3,0}$ . We note that by a different approach smoothing effects for the generalized KdV equation were studied in [4], [13].

In this note we use

*Notations.* We denote  $\partial_x = \frac{\partial}{\partial x}$ ,  $\partial_x^{-1} = \int_{-\infty}^x dx'$  and let  $\mathcal{F}\phi$  or  $\hat{\phi}$  be the Fourier transform of  $\phi(x)$ , namely  $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\xi x} \phi(x) dx$ . We denote by  $\mathcal{F}^{-1}\phi$  or  $\check{\phi}$  the inverse Fourier transform of the function  $\phi(\xi)$ , indeed  $\check{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi x} \phi(\xi) d\xi$ . In what follows we also use the following relation  $|\partial_x| = \mathcal{F}^{-1}|\xi|\mathcal{F} = -\mathcal{H}\partial_x$ . The Hilbert transformation  $\mathcal{H}$  with respect to the variable  $x$  is defined as follows

$$\mathcal{H}\phi(x) = \frac{1}{\pi} \text{Pv} \int_{\mathbf{R}} \frac{\phi(z)}{x-z} dz = -i\mathcal{F}^{-1} \frac{\xi}{|\xi|} \mathcal{F}\phi,$$

where *Pv* means the principal value of the singular integral. We widely use the fact that the Hilbert transformation  $\mathcal{H}$  is a bounded operator from  $\mathbf{L}^2$  to  $\mathbf{L}^2$ . The fractional derivative  $|\partial_x|^\alpha$ ,  $\alpha \in (0, 1)$  is equal to

$$|\partial_x|^\alpha \phi = \mathcal{F}^{-1} |\xi|^\alpha \mathcal{F}\phi = C \int_{\mathbf{R}} (\phi(x+z) - \phi(x)) \frac{dz}{|z|^{1+\alpha}}$$

and similarly we have

$$|\partial_x|^\alpha \mathcal{H}\phi = -i\mathcal{F}^{-1} \text{sign}\xi |\xi|^\alpha \mathcal{F}\phi = C \int_{\mathbf{R}} (\phi(x+z) - \phi(x)) \frac{dz}{z|z|^\alpha},$$

with some constant  $C$  ( see [20] for constants  $C$ ). Let  $\mathcal{J} = \mathcal{J}(t) = x + 2it\partial_x = M(t)(2it\partial_x)M(-t)$ , where  $M = M(t) = \exp(ix^2/4t)$ . We also freely use the following identities  $[\mathcal{J}, \partial_x] = -1$ ,  $[\mathcal{L}, \mathcal{J}] = 0$ , where  $\mathcal{L} = i\partial_t + \partial_x^2$ . Different positive constants might be denoted by the same letter  $C$  when they do not cause any confusion.

## §2 PROOF OF THEOREM 1.1

The main result is obtained by making use of the following three lemmas and the contraction mapping principle (for details of the proofs of Lemma 2.1-Lemma 2.3 and Theorem 1.1, see [22]).

The next lemma shows that the commutators  $[|\partial_x|^\alpha, \phi]$ , and  $[|\partial_x|^\alpha \mathcal{H}, \phi]$ , are continuous operators from  $\mathbf{L}^2$  to  $\mathbf{L}^2$ .

**Lemma 2.1.** *The following inequalities*

$$\| [|\partial_x|^\alpha, \phi] \psi \| \leq C \| \phi \|_{1,0,\infty} \| \psi \| \quad \text{and} \quad \| [|\partial_x|^\alpha \mathcal{H}, \phi] \psi \| \leq C \| \phi \|_{1,0,\infty} \| \psi \|$$

*are valid, provided that the right hand sides are bounded.*

We define a smoothing operator  $\mathcal{S}(\varphi) = \cosh(\varphi) + i \sinh(\varphi)\mathcal{H}$ , where the real-valued function  $\varphi(t, x) \in \mathbf{L}^\infty(0, T; \mathbf{H}_\infty^{2,0}) \cap \mathbf{C}^1([0, T]; \mathbf{L}^\infty)$  and is positive. From its

definition we easily see that the operator  $\mathcal{S}$  acts continuously from  $\mathbf{L}^2$  to  $\mathbf{L}^2$  with the following estimate  $\|\mathcal{S}(\varphi)\psi\| \leq 2 \exp(\|\varphi\|_\infty) \|\psi\|$ . The inverse operator  $\mathcal{S}^{-1}(\varphi) = (1 + i \tanh(\varphi)\mathcal{H})^{-1} \frac{1}{\cosh(\varphi)}$  also exists and is continuous

$$\|\mathcal{S}^{-1}(\varphi)\psi\| \leq (1 - \tanh(\|\varphi\|_\infty))^{-1} \|\psi\| \leq \exp(\|\varphi\|_\infty) \|\psi\|.$$

The operator  $\mathcal{S}$  helps us to obtain a smoothing property of the linear Schrödinger equation by virtue of the usual energy estimates. In the next lemma we prepare an energy estimate, involving the operator  $\mathcal{S}$ , in which we have an additional positive term giving us the norm of the half derivative of the unknown function  $u$ . We also assume that  $\varphi(x)$  is written as  $\varphi(x) = \partial_x^{-1}(\omega^2)$ , so that  $\omega(x) = \sqrt{(\partial_x \varphi)}$ . We consider the linear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = f, & x \in \mathbf{R}, \quad t \in \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}, \end{cases} \quad (2.1)$$

where the function  $f(t, x)$  is a given function. We have the following lemma for the solutions to (2.1).

**Lemma 2.2.** *The following inequality*

$$\begin{aligned} \frac{d}{dt} \|\mathcal{S}u\|^2 + \left\| \omega \mathcal{S} \sqrt{|\partial_x|} u \right\|^2 &\leq 2 |\operatorname{Im}(\mathcal{S}u, \mathcal{S}f)| \\ + C \|u\|^2 e^{2\|\varphi\|_\infty} (\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_{1,0,\infty} \|\omega\|_\infty + \|\varphi_t\|_\infty) \end{aligned}$$

is valid for the solution  $u$  of the Cauchy problem (2.1).

Lemma 2.2 is considered as a simple and explicit modification of smoothing effects of Doi [5].

**Lemma 2.3.** *We have the following estimates*

$$\begin{aligned} |(\mathcal{S}u, \mathcal{S}\phi\psi\partial_x v)| &\leq \left\| |\phi| \mathcal{S} \sqrt{|\partial_x|} u \right\|^2 + \left\| |\psi| \mathcal{S} \sqrt{|\partial_x|} v \right\|^2 \\ &+ C (\|u\|^2 + \|v\|^2) e^{6\|\varphi\|_\infty} (\|\phi\|_{1,0,\infty}^2 + \|\psi\|_{1,0,\infty}^2) (1 + \|\varphi\|_{1,0,\infty}^2), \end{aligned}$$

and

$$\begin{aligned} |(\mathcal{S}u, \mathcal{S}\phi\psi\partial_x \bar{v})| &\leq \left\| |\phi| \mathcal{S} \sqrt{|\partial_x|} u \right\|^2 + e^{4\|\varphi\|_\infty} \left\| |\psi| \mathcal{S} \sqrt{|\partial_x|} v \right\|^2 \\ &+ C (\|u\|^2 + \|v\|^2) e^{6\|\varphi\|_\infty} (\|\phi\|_{1,0,\infty}^2 + \|\psi\|_{1,0,\infty}^2) (1 + \|\varphi\|_{1,0,\infty}^2), \end{aligned}$$

provided that the right hand sides are bounded.

Lemma 2.3 is needed to obtain some estimates for nonlinearities.

## REFERENCES

1. H. Chihara, *Local existence for the semilinear Schrödinger equations in one space dimension*, J. Math. Kyoto Univ. **34** (1994), 353-367.
2. H. Chihara, *The initial value problem for the elliptic-hyperbolic Davey-Stewartson equation*, preprint.
3. P. Constantin and J.C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), 413-439.
4. W. Craig, T. Kappeler and W. Strauss, *Gain of regularity for equations of KdV type*, Ann. Inst. Henri Poincaré, Analyse non linéaire **9** (1992), 147-186.
5. S. Doi, *On the Cauchy problem for Schrödinger type equations and regularity of solutions*, J. Math. Kyoto Univ. **34** (1994), 319-328.
6. A. Friedman, *Partial Differential Equations*, Krieger, 1983.
7. N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equation in the energy space*, Nonlinear Analysis TMA **20** (1993), 823-833.
8. N. Hayashi and H. Hirata, *Global existence and asymptotic behavior in time of small solutions to the elliptic hyperbolic Davey-Stewartson system*, Nonlinearity **9** (1996), 1387-1409.
9. N. Hayashi, K. Nakamitsu and M. Tsutsumi, *On solutions of the initial value problem for nonlinear Schrödinger equations in one space dimension*, Math. Z. **192** (1986), 637-650.
10. N. Hayashi and T. Ozawa, *On the derivative nonlinear Schrödinger equations*, Physica D **55** (1992), 14-36.
11. N. Hayashi and T. Ozawa, *Remarks on nonlinear Schrödinger equations in one space dimension*, Diff. Integral Eqs. **7** (1994), 453-461.
12. S. Katayama and Y. Tsutsumi, *Global existence of solutions for nonlinear Schrödinger equations in one space dimension*, Commun. in P.D.E. **19** (1994), 1971-1997.
13. T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Stud. Appl. Math. Adv. in Math. Supplementary Studies **18** (1983), 93-128.
14. C.E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33-69.
15. C.E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré Analyse non linéaire **10** (1993), 255-288.
16. V.G. Makhankov and O.K. Pashaev, *Integrable pseudospin models in condensed matter*, Sov. Sci. Rev. C. Math. Phys. **9** (1992), 1-152.
17. T. Ozawa, *On the nonlinear Schrödinger equations of derivative type*, Indiana Univ. Math. J. **45** (1996), 137-163.
18. P. Sjölin, *Regularity of solutions to the Schrödinger equations*, Duke Math. J. **55** (1987), 699-715.
19. A. Soyeur, *The Cauchy problem for the Ishimori equations*, J. Funct. Anal. **105** (1992), 233-255.
20. E.M. Stein, *Singular Integral and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton Math. Series 30, 1970.
21. L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), 874-878.
22. N. Hayashi, P. I. Naumkin and P-N. Pipolo, *Smoothing effects for some derivative nonlinear Schrödinger equations*, in preparation.

# Small data scattering for wave equations with supercritical nonlinearity

Kunio Hidano

Department of Mathematics, Tokyo Metropolitan University  
Minami-Ohsawa 1-1, Hachioji-shi, Tokyo 192-0397, Japan

## 1 Introduction

Nonlinear small data scattering for the wave equation

$$(1.1) \quad \square u = \lambda |u|^{p-1} u, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^n$$

is considered. In this talk I would like to show that, in space dimension  $n = 2, 3, 4$ , the wave operators are defined for small data in a weighted Sobolev space under the condition  $p > p_0(n)$ , where  $p_0(n)$  is a positive root of  $(n-1)p^2 - (n+1)p - 2 = 0$  (i.e.,  $p_0(2) = (3 + \sqrt{17})/2$ ,  $p_0(3) = 1 + \sqrt{2}$ ,  $p_0(4) = 2$ ). Moreover, the asymptotic completeness of the wave operators is proved on a neighborhood of zero in the weighted Sobolev space, which settles the problem left open in the papers due to Kubota–Mochizuki [8], Tsutaya [18] and Pecher [15]. The modified energy method by Li–Zhou, together with the infinitesimal generators of the Lorentz group and the scaling operator, is employed to solve the associated integral equations.

## 2 Known results

In the nonlinear scattering theory we have two basic problems : existence of the wave operators and their asymptotic completeness. To prove the existence of the wave operator for negative time we usually try to solve the integral equation

$$(2.1) \quad u(t) = u_-(t) + \lambda \int_{-\infty}^t \frac{\sin \omega(t-\tau)}{\omega} |u(\tau)|^{p-1} u(\tau) d\tau,$$

where  $\omega = \sqrt{-\Delta}$  and  $u_-(t) = (\cos \omega t) f_- + (\omega^{-1} \sin \omega t) g_-$  for given data  $(f_-, g_-)$ . Put  $p_{conf}(n) := 1 + 4/(n-1)$ . Note that  $p_0(n) < p_{conf}(n)$  for all  $n \geq 2$ . Concerning the solvability of (2.1) in the case of  $p > p_{conf}(n)$ , we have the following result of Lindblad–Sogge which is an extension of the earlier ones due to Pecher [14], Ginibre–Velo [2].

**Theorem 2.1** ([10]) *Let  $n \geq 2$  and suppose that  $p \geq p_{conf}(n)$  for  $n = 2, 3$ ,  $p_{conf}(n) \leq p \leq (n+1)/(n-3)$  for  $n \geq 4$ . Put  $\mu = n/2 - 2/(p-1)$ . There exists a positive number  $\delta = \delta(n, p, \lambda)$  such that for any  $(f_-, g_-) \in \dot{H}^\mu \times \dot{H}^{\mu-1}$  with  $\|(f_-, g_-); \dot{H}^\mu \times \dot{H}^{\mu-1}\| < \delta$  the integral equation (2.1) has a unique solution satisfying*

$$(2.2) \quad (u, \partial_t u) \in C_b((-\infty, 0]; \dot{H}^\mu \times \dot{H}^{\mu-1}) \text{ and } u \in L^{(p-1)(n+1)/2}(\mathbf{R}^{n+1}),$$

$$(2.3) \quad \|(u(t) - u_-(t), \partial_t u(t) - \partial_t u_-(t)); \dot{H}^\mu \times \dot{H}^{\mu-1}\| \rightarrow 0 \quad (t \rightarrow -\infty).$$

Here and below  $\dot{H}^s$  means the homogeneous Sobolev space  $\dot{H}^s = (-\Delta)^{-s/2}L^2$ . A similar result for positive time also holds. As a consequence, the wave operators  $W_\pm : (f_\pm, g_\pm) \mapsto (u(0), \partial_t u(0))$  can be defined as mappings from a neighborhood of zero in  $\dot{H}^\mu \times \dot{H}^{\mu-1}$  into  $\dot{H}^\mu \times \dot{H}^{\mu-1}$ . In their proof a particular form<sup>1</sup> of the Strichartz-type inequalities and a kind of principle<sup>2</sup> play major roles. I also mention here that unnatural assumption  $p \leq (n+1)/(n-3)$  appearing in Theorem 2.1 has been recently relaxed by Nakamura–Ozawa [12].

Let us turn our attention to the case  $p < p_{conf}(n)$ . Though we have a more general form<sup>3</sup> of the Strichartz-type inequalities, such a generalized version does not seem to provide us with any result concerning the global solvability of (2.1) in the case of  $p < p_{conf}(n)$ . Nevertheless, the  $L^q - L^{q'}$ -estimate for the fixed time due to Pecher [13], which is a starting point of the proof of mixed space-time estimates, turns out useful to cover the case  $p < p_{conf}(n)$  at the expense of assuming more regularity and decay on data. Set  $p_1(n) := (n^2 + 3n - 2 + \sqrt{(n^2 + 3n - 2)^2 - 8n(n-1)})/2n(n-1)$ .

**Theorem 2.2** (Mochizuki–Motai [11]) *Let  $n \geq 2$  and suppose that  $p_1(n) < p \leq p_{conf}(n)$ . The wave operators  $W_\pm$  can be defined as mappings from a neighborhood of zero in  $(H^{1+s, q} \times H^{s, q}) \cap (H^{1, q/p} \times L^{q/p})$  into  $\dot{H}^1 \times L^2$ . Here  $q = 2(np - 1)/(n + 1)$ ,  $s = (n + 1)/q - (n - 1)/2$ .*

In view of the result of Sideris concerning the blow-up theorem for small data [16], we expect that the theory holds even for  $p_0(n) < p \leq p_1(n)$ . In fact, this is the

<sup>1</sup>This means a  $L^q L^q$ -estimate with the same exponents for the space and time variables.

<sup>2</sup>See, e.g., [1] on page 83–84

<sup>3</sup>This means a  $L^q L^r$ -estimate with different exponents for the space and time variables. Such an estimate is often referred to as “mixed space-time estimate”.

case and we have the following.

**Theorem 2.3** (Kubota–Mochizuki [8], Tsutaya [18], Pecher [15]) *Let  $n = 2, 3$  and assume that  $p > p_0(n)$ . Then the wave operators  $W_{\pm}$  can be defined as mappings from a neighborhood of zero in a weighted  $C^3 \times C^2$  into  $(C^2 \times C^1) \cap (\dot{H}^1 \times L^2)$ .*

In [6],[7] Kubo–Kubota have proved a similar result in higher dimension ( $n \geq 5$ ) at the expense of assuming spherical symmetry on data. While the strategies in the preceding papers [10], [11] and [12] are fairly general and in fact they have been employed to show similar results for other equations such as nonlinear Schrödinger equations and Klein–Gordon equations (see, e.g., [17]), the method employed in [8], [18], [15] and [6]–[7] is quite different and seems specific to the analysis of nonlinear wave equations. So let us review briefly the paper of Pecher [15] where three space dimensional case is studied. Writing the integral equation in the classical form

$$(2.4) \quad u(t, x) = \frac{\partial}{\partial t} \left( \frac{t}{4\pi} \int_{|\xi|=1} f_-(x + t\xi) dS \right) + \frac{t}{4\pi} \int_{|\xi|=1} g_-(x + t\xi) dS \\ + \frac{\lambda}{4\pi} \int_{-\infty}^t \frac{1}{t - \tau} \left( \int_{|y-x|=t-\tau} |u(\tau, y)|^{p-1} u(\tau, y) dS_y \right) d\tau$$

and assuming a suitable decay condition on the data  $(f_-, g_-)$ , Pecher solved (2.4) by a variant of iteration in the space with the norm

$$(2.5) \quad \|u\| = \sum_{|\alpha| \leq 2} \sup_{t, x} (1 + |t| + |x|)(1 + ||t| - |x||)^{p-2} |\partial_x^\alpha u(t, x)|.$$

This procedure reminds us of the idea of John [3], who strikingly proved that, in the case of  $n = 3$ , the Cauchy problem (1.1) with  $p > p_0(3)$  has a global  $C^2$ -solution for small smooth data of compact support.

Summing up, to solve the integral equation (2.1) or (2.4) and thus to define the wave operators, we have reviewed three available methods based on : (i) Strichartz-type estimates [10], [12] (ii)  $L^q - L^{q'}$ -estimate for the fixed time [11] (iii) John’s idea [8], [18], [15], [6], [7]. Each method of course has advantages and drawbacks of its own. While the case  $p_0(n) < p < p_{conf}(n)$  can not be treated by (i), this method has two big advantages. Theorem 2.1 is in fact proved under the assumption of minimal regularity on data. Another advantage lies in the fact that the asymptotic completeness of the wave operators can be shown on a neighborhood of zero in  $\dot{H}^\mu \times \dot{H}^{\mu-1}$ . Actually we have

**Theorem 2.4** ([10]. See also [12]) *Assume the same as in Theorem 2.1. Then there exists  $\delta = \delta(n, p, \lambda) > 0$  such that for any  $(f, g) \in \dot{H}^\mu \times \dot{H}^{\mu-1}$  with  $\|(f, g); \dot{H}^\mu \times \dot{H}^{\mu-1}\| < \delta$  the integral equation*

$$(2.6) \quad u(t) = u_0(t) + \lambda \int_0^t \frac{\sin \omega(t - \tau)}{\omega} |u(\tau)|^{p-1} u(\tau) d\tau$$

*has a unique solution satisfying*

$$(2.7) \quad (u, \partial_t u) \in C_b(\mathbf{R}; \dot{H}^\mu \times \dot{H}^{\mu-1}) \text{ and } u \in L^{(p-1)(n+1)/2}(\mathbf{R}^{n+1}).$$

*Here  $u_0(t) = (\cos \omega t)f + (\omega^{-1} \sin \omega t)g$ . Moreover, there exists a unique  $(f_\pm, g_\pm) \in \dot{H}^\mu \times \dot{H}^{\mu-1}$  such that the solution satisfies*

$$(2.8) \quad \|(u(t) - u_\pm(t), \partial_t u(t) - \partial_t u_\pm(t)); \dot{H}^\mu \times \dot{H}^{\mu-1}\| \rightarrow 0 \quad (t \rightarrow \pm\infty),$$

*where  $u_\pm(t) = (\cos \omega t)f_\pm + (\omega^{-1} \sin \omega t)g_\pm$ .*

Next let us turn our attention to the methods (ii), (iii). As I explained before, the method (ii) enables us to treat in part the case  $p < p_{conf}(n)$  and John's method (iii) reduces the lower bound on  $p$  to the optimal value  $p_0(n)$  (limit excluded). Compared with (i), the methods (ii) and (iii) however have a common drawback : the asymptotic completeness never follows from (ii), (iii).

Hence my purpose is to show the asymptotic completeness of the wave operators on a neighborhood of zero in a weighted Sobolev space under the condition  $p > p_0(n)$ . I have succeeded in proving it in the case of  $n = 2, 3, 4$  by making use of the modified energy method by Li-Zhou [9] together with the infinitesimal generators of the Lorentz group and the scaling operator.

### 3 Main Results

Following Klainerman [4], [5], we introduce several partial differential operators as follows :  $\partial_0 = \partial/\partial t$ ,  $\partial_j = \partial/\partial x_j$ ,  $L_j = t\partial_j + x_j\partial_t$  ( $j = 1, \dots, n$ ),  $\Omega_{k\ell} = x_k\partial_\ell - x_\ell\partial_k$  ( $1 \leq k < \ell \leq n$ ),  $L_0 = t\partial_t + x_1\partial_1 + \dots + x_n\partial_n$ . These operators  $\partial_0, \dots, \partial_n, L_1, \dots, L_n, \Omega_{12}, \dots, \Omega_{n-1n}$  and  $L_0$  are denoted by  $\Gamma_0, \dots, \Gamma_\nu$  in this order, where  $\nu = (n^2 + 3n + 2)/2$ . For a multi-index  $\alpha = (\alpha_0, \dots, \alpha_\nu)$  we denote  $\Gamma_0^{\alpha_0} \dots \Gamma_\nu^{\alpha_\nu}$  by  $\Gamma^\alpha$ . As I referred to before, my proof is based on the modified energy method by

Li-Zhou. While the classical energy method consists of the usual energy inequality and the standard Sobolev estimates, the Sobolev-type estimate we instead use is

$$(3.1) \quad \|u(t, \cdot)\|_{L^q(|x| \leq \frac{1+|t|}{2})} \leq C(1+|t|)^{-n(1/p-1/q)} \sum_{|\alpha| \leq s} \|\Gamma^\alpha u(t, \cdot)\|_{L^p},$$

$$0 < s \leq n/p, \quad 1 \leq p \leq q < \infty, \quad 1/q \geq 1/p - s/n.$$

This inequality (3.1) tells us one of the advantages of the usage of the operators  $L_j$ ,  $\Omega_{k\ell}$  and  $L_0$ .

We next define the spaces of functions from which we take initial data. Let  $n = 2, 3, 4$  and set  $\theta = 1/2 - 1/p$  for  $p > p_0(n)$ . For integers  $N \geq 1$  and  $p > p_0(n)$  we define the Hilbert spaces  $X_{N,p}$  and  $Y_{N,p}$  as

$$X_{N,p} := \{ f \in \dot{H}^\theta \cap \dot{H}^{N+\theta} \mid \|f\|_{X_{N,p}}^2 := \sum_{j=0}^N \| \langle x \rangle^j \omega^{j+\theta} f \|_{L^2}^2 < \infty \},$$

$$Y_{N,p} := \{ g \in \dot{H}^{-1+\theta} \cap \dot{H}^{N-1+\theta} \mid \|g\|_{Y_{N,p}}^2 := \sum_{j=0}^N \| \langle x \rangle^j \omega^{j-1+\theta} g \|_{L^2}^2 < \infty \}.$$

Here  $\langle x \rangle := \sqrt{1 + |x|^2}$ . Note that  $\theta > 0$  because  $p_0(n) \geq 2$  for  $n = 2, 3$  and 4. Hereafter we simply denote  $X_{N,p}$ ,  $Y_{N,p}$  by  $X_N$ ,  $Y_N$  respectively. Now we are in a position to state our results.

**Theorem 3.1** *Let  $n = 2, 3, 4$  and assume  $p > p_0(n)$ . Put  $m = 1$  for  $n = 2$  and  $m = 2$  for  $n = 3, 4$ . There exists a constant  $\delta = \delta(n, p, \lambda) > 0$  with the following property : for any  $(f_-, g_-) \in X_m \times Y_m$  with  $\|(f_-, g_-)\|_{X_m \times Y_m} < \delta$  the integral equation (2.1) has a unique solution  $u(t, x)$  satisfying*

$$(3.2) \quad (\Gamma^\alpha u, \Gamma^\alpha \partial_t u) \in C_b((-\infty, 0]; \dot{H}^\theta \times \dot{H}^{-1+\theta}),$$

$$(3.3) \quad \|(\Gamma^\alpha u(t) - \Gamma^\alpha u_-(t), \Gamma^\alpha \partial_t u(t) - \Gamma^\alpha \partial_t u_-(t))\|_{\dot{H}^\theta \times \dot{H}^{-1+\theta}}$$

$$= O(|t|^{-(\frac{n-1}{2}p - \frac{n-1}{2} - \frac{1}{p})+1}) \quad \text{as } t \rightarrow -\infty$$

for any  $\alpha$  with  $|\alpha| \leq m$ , where  $u_-(t) = (\cos \omega t) f_- + (\omega^{-1} \sin \omega t) g_-$ . Moreover, this solution  $u$  satisfies

$$(3.4) \quad (u, \partial_t u) \in C((-\infty, 0]; X_m \times Y_m),$$

$$(3.5) \quad \|(u(0), \partial_t u(0))\|_{X_m \times Y_m} \leq C \|(f_-, g_-)\|_{X_m \times Y_m} \text{ for some constant } C > 0.$$

*Remark.* The corresponding result for positive time also holds.

**Theorem 3.2** *Let  $n = 2, 3, 4$  and assume  $p > p_0(n)$ . Let  $m$  be the same as in Theorem 3.1. There exists a constant  $\delta = \delta(n, p, \lambda) > 0$  with the following property : for any  $(f, g) \in X_m \times Y_m$  with  $\|(f, g)\|_{X_m \times Y_m} < \delta$  the integral equation (2.6) has a unique solution  $u(t, x)$  satisfying*

$$(3.6) \quad (\Gamma^\alpha u, \Gamma^\alpha \partial_t u) \in C_b(\mathbf{R}; \dot{H}^\theta \times \dot{H}^{-1+\theta}) \text{ for any } \alpha \text{ with } |\alpha| \leq m,$$

$$(3.7) \quad (u, \partial_t u) \in C(\mathbf{R}; X_m \times Y_m).$$

Moreover, there exists a unique pair of functions  $(f_+, g_+), (f_-, g_-) \in X_m \times Y_m$  satisfying

$$(3.8) \quad \|(f_\pm, g_\pm)\|_{X_m \times Y_m} \leq C \|(f, g)\|_{X_m \times Y_m} \text{ for some constant } C > 0,$$

$$(3.9) \quad \begin{aligned} & \|(\Gamma^\alpha u(t) - \Gamma^\alpha u_\pm(t), \Gamma^\alpha \partial_t u(t) - \Gamma^\alpha \partial_t u_\pm(t))\|_{\dot{H}^0 \times \dot{H}^{-1+\theta}} \\ & = O(|t|^{-(\frac{n-1}{2}p - \frac{n-1}{2} - \frac{1}{p})+1}) \text{ as } t \rightarrow \pm\infty \end{aligned}$$

for any  $\alpha$  with  $|\alpha| \leq m$ , where  $u_\pm(t) = (\cos \omega t) f_\pm + (\omega^{-1} \sin \omega t) g_\pm$ .

The outline of the proof of the above theorems will be given in my talk.

## References

- [1] T. Cazenave and F. B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations. *Math. Z.* **228** (1998), 83–120.
- [2] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of non-linear wave equations. *Comm. Math. Phys.* **123** (1989), 535–573.
- [3] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.* **28** (1979), 235–268.
- [4] S. Klainerman, Uniform decay estimate and the Lorentz invariance of the classical wave equations. *Comm. Pure Appl. Math.* **38** (1985), 321–332.
- [5] S. Klainerman, Remarks on the global Sobolev inequalities in the Minkowski space  $\mathbf{R}^{n+1}$ . *Comm. Pure Appl. Math.* **40** (1987), 111–117.

- [6] H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of  $\square u = |u|^p$  for super critical values  $p$  in odd space dimensions. *Hokkaido Math. J.* **22** (1995), 287–336.
- [7] H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of  $\square u = |u|^p$  for super critical values  $p$  in even space dimensions. *Japan. J. Math. (N.S.)* (to appear).
- [8] K. Kubota and K. Mochizuki, On small data scattering for 2-dimensional semilinear wave equations. *Hokkaido Math. J.* **22** (1993), 79–97.
- [9] T. T. Li and Y. Zhou, A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions. *Indiana Univ. Math. J.* **44** (1995), 1207–1248.
- [10] H. Lindblad and C. D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations. *J. Funct. Anal.* **130** (1995), 357–426.
- [11] K. Mochizuki and T. Motai, The scattering theory for the nonlinear wave equation II. *Publ. Res. Inst. Math. Sci.* **23** (1987), 771–790.
- [12] M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space. Preprint.
- [13] H. Pecher,  $L^p$ -Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen I. *Math. Z.* **150** (1976), 159–183.
- [14] H. Pecher, Nonlinear small data scattering for the wave and Klein–Gordon equation. *Math. Z.* **185** (1984), 261–270.
- [15] H. Pecher, Scattering for semilinear wave equations with small data in three space dimensions. *Math. Z.* **198** (1988), 277–289.
- [16] T. C. Sideris, Nonexistence of global solutions to semilinear wave equation in high dimensions. *J. Differential Equations* **52** (1984), 378–406.
- [17] W. A. Strauss, Nonlinear scattering theory at low energy. *J. Funct. Anal.* **41** (1981), 110–133.

- [18] K. Tsutaya, Scattering theory for semilinear wave equations with small data in two space dimensions. *Trans. Amer. Math. Soc.* **342** (1994), 598–618.

# Asymptotic behavior of solutions to the $p$ -system with linear damping

KENJI NISHIHARA

School of Political Science and Economics, Waseda University,  
Tokyo 169-8050, Japan

## 1 Introduction and the Cauchy Problem

In this talk we consider the Cauchy problem on the whole line  $\mathbf{R} = (-\infty, \infty)$  and the initial-boundary value problem on the half line  $\mathbf{R}_+ = (0, \infty)$  for a system of hyperbolic conservation laws with linear damping

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad x \in \mathbf{R} = (-\infty, \infty), \quad t > 0, \end{cases} \quad (1.1)$$

which is often called the  $p$ -system with damping. This system models a one-dimensional compressible flow through porous media. Here,  $v > 0$  is the specific volume,  $u$  is the velocity, the pressure  $p$  is a smooth function of  $v$  with  $p > 0, p' < 0$ , and  $\alpha$  is a positive constant.

Our concern is to investigate the asymptotic behavior of solutions to (1.1). In this section we consider the Cauchy problem, and the initial-boundary value problem will be treated in the next section.

Darcy's law  $p(v)_x = -\alpha u$  in fluid dynamics suggests that the solution  $(v, u)$  to (1.1) time-asymptotically behave as  $(\bar{v}, \bar{u})$ , called the diffusion wave, satisfying

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0 \\ p(\bar{v})_x = -\alpha \bar{u} \end{cases} \quad (1.2)$$

or

$$\bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \quad p(\bar{v})_x = -\alpha \bar{u}. \quad (1.2)'$$

In fact, this was shown by Hsiao and Liu [4, 5] in the case of the Cauchy problem with small data. Following them we remind the approach to the problem. Our interest is in the solutions of (1.1) whose Cauchy data have the limits at  $x = \pm\infty$ :

$$(v, u)(x, 0) = (v_0, u_0)(x) \rightarrow (v_{\pm}, u_{\pm}) \quad \text{as } x \rightarrow \pm\infty. \quad (1.3)$$

In the case of  $v_+ \neq v_-$ , by  $\bar{v}(x + x_0, t) = \tau(\frac{x+x_0}{\sqrt{t+1}})$  denote the similarity solution of (1.2)'<sub>1</sub> (first equation of (1.2)') with the same end state  $v_{\pm}$  at  $x = \pm\infty$ , and set  $\bar{u}$  by (1.2)'<sub>2</sub>. We now determine the shift  $x_0$  and introduce the auxiliary function  $(\hat{v}, \hat{u})$  because of  $\bar{u}(\pm\infty, 0) = 0 \neq u_{\pm}$ . Hinted by (1.1)<sub>2</sub> we suppose that

$$u(x, t) \rightarrow e^{-\alpha t} u_{\pm} \quad \text{as } x \rightarrow \pm\infty. \quad (1.4)$$

The integration of  $(v - \bar{v})_t - (u - \bar{u})_x = 0$  over  $\mathbf{R}$  yields

$$\frac{d}{dt} \int_{-\infty}^{\infty} (v - \bar{v}) dx = e^{-\alpha t} (u_+ - u_-) = \frac{d}{dt} \left( \frac{u_+ - u_-}{-\alpha} e^{-\alpha t} \cdot 1 \right)$$

and hence

$$\frac{d}{dt} \int_{-\infty}^{\infty} [v(x, t) - \bar{v}(x + x_0, t) - \frac{u_+ - u_-}{-\alpha} e^{-\alpha t} m_0(x)] dx = 0, \quad (1.5)$$

where  $m_0$  is a smooth function with compact support satisfying  $\int_{-\infty}^{\infty} m_0(x) dx = 1$  and  $x_0$  is a constant uniquely determined by

$$\int_{-\infty}^{\infty} [v_0(x) - \bar{v}(x + x_0, 0)] dx = \frac{u_+ - u_-}{-\alpha} \quad (1.6)$$

(Without loss of generality suppose that  $x_0 = 0$ ). Hence  $(\hat{v}, \hat{u})$  is defined by

$$(\hat{v}, \hat{u})(x, t) = \left( \frac{u_+ - u_-}{-\alpha} e^{-\alpha t} m_0(x), e^{-\alpha t} [u_- + (u_+ - u_-) \int_{-\infty}^x m_0(y) dy] \right) \quad (1.7)$$

or

$$\hat{v}_t - \hat{u}_x = 0, \quad \hat{u}_t = -\alpha \hat{u}, \quad (1.8)$$

Note that  $(\hat{v}, \hat{u})$  decays exponentially. Combination of (1.1), (1.2) and (1.8) gives

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0 \\ (u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x + \bar{u}_t + \alpha(u - \bar{u} - \hat{u}) = 0, \end{cases} \quad (1.9)$$

so that setting of the perturbation

$$(V, z)(x, t) = \left( \int_{-\infty}^x (v - \bar{v} - \hat{v})(y, t) dy, (u - \bar{u} - \hat{u})(x, t) \right) \quad (1.10)$$

yields, after integration of (1.9)<sub>1</sub> in  $x$  due to (1.5)-(1.7), the reformulated problem

$$\begin{cases} V_t - z = 0 \\ z_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \alpha z = -\bar{u}_t \\ (V, z)|_{t=0} := (V_0, z_0)(x) \\ = \left( \int_{-\infty}^x (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0)) dy, u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0) \right) \rightarrow 0 \quad (x \rightarrow \pm\infty) \end{cases} \quad (RP)$$

and the linearized problem around  $\bar{v}$

$$\begin{cases} V_t - z = 0 \\ z_t + (p'(\bar{v})V_x)_x + \alpha z = -F \\ (V, z)|_{t=0} = (V_0, z_0)(x) \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty, \end{cases} \quad (LP)$$

or the second order wave equation of  $V$  with damping

$$V_{tt} + (p'(\bar{v})V_x)_x + \alpha V_t = -F, \quad (1.11)$$

where

$$F = \bar{u}_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x. \quad (1.12)$$

In the special case  $v_+ = v_-$ ,  $\bar{v}(x, t)$  is defined by

$$\bar{v}(x, t) = v_- + \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-p'(v_{\pm})t)^2}{4t}} \int_{-\infty}^{\infty} (v_0(y) - v_{\pm}) dy \quad (1.13)$$

and  $\bar{u}, \hat{v}$  and  $\hat{u}$  are defined as above.

Denote the  $H^k$ -norm and  $L^2$ -norm by  $\|\cdot\|_k$  and  $\|\cdot\| = \|\cdot\|_0$ , respectively, and set  $\delta_0 = |v_+ - v_-| + |u_+ - u_-|$ . Then the following theorem was shown by Hsiao and Liu [4].

**Theorem 1.1 (Hsiao-Liu)** *Suppose that  $(V_0, z_0) \in H^3 \times H^2$  and  $\|V_0\|_3 + \|z_0\|_2 + \delta_0$  is suitably small. Then there exists a unique solution  $(V, z)$  of (RP) in  $C^0([0, \infty); H^3 \times H^2)$ , which satisfies*

$$\|(V_x, z)(\cdot, t)\|_{L^2 \cap L^\infty} = O(t^{-1/2}) \quad \text{as } t \rightarrow \infty. \quad (1.14)$$

Hinted by this theorem which implies that the system (1.1) essentially has a parabolic structure, the author in [11, 12] improved the decay rate (1.14).

**Theorem 1.2 (Nishihara)** *The solution  $(V, z)$  obtained in Theorem 1.1 satisfies*

$$\begin{aligned} & \sum_{k=0}^3 (1+t)^k \|\partial_x^k V(\cdot, t)\|^2 + \sum_{k=0}^2 (1+t)^{k+2} \|\partial_x^k z(\cdot, t)\|^2 + \\ & + \int_0^t \left[ \sum_{j=1}^3 (1+\tau)^{j-1} \|\partial_x^j V(\cdot, \tau)\|^2 + \sum_{j=0}^2 (1+\tau)^{j+1} \|\partial_x^j z(\cdot, \tau)\|^2 \right] d\tau \\ & \leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|), \end{aligned} \quad (1.15)$$

and

$$\begin{aligned} & (1+t)^4 \|z_t(\cdot, t)\|^2 + (1+t)^5 (\|z_{xt}(\cdot, t)\|^2 + \|z_{tt}(\cdot, t)\|^2) + \\ & + \int_0^t [(1+\tau)^4 \|z_{xt}(\cdot, \tau)\|^2 + (1+\tau)^5 \|z_{tt}(\cdot, \tau)\|^2] d\tau \\ & \leq C(\|V_0\|_3^2 + \|z_0\|_2^2 + |\delta_0|). \end{aligned} \quad (1.16)$$

From (1.15) and the Sobolev inequality

$$\|(V_x, z)(t)\|_{L^2(\mathbf{R})} = O(t^{-1/2}, t^{-1})$$

and

$$\|(V_x, z)(t)\|_{L^\infty(\mathbf{R})} = O(t^{-3/4}, t^{-5/4}),$$

latter of which is the same decay order as the  $L^\infty$ -norm of  $(\Gamma_x, \Gamma_t)$

$$|(\Gamma_x, \Gamma_t)(x, t)| = \left| \int_{-\infty}^{\infty} (E_x, E_t)(x-y, t) \Gamma_0(y) dy \right| = O(t^{-3/4}, t^{-5/4}),$$

where  $\Gamma$  is a solution of the parabolic equation  $\Gamma_t - \kappa \Gamma_{xx} = 0$  with the initial data  $\Gamma_0(x) \in L^2$  and  $E(x, t) = \frac{1}{\sqrt{4\kappa\pi t}} e^{-x^2/4\kappa t}$  is the Green function. In other words, we may say that the solution  $V(x, t)$  of the second order wave equation with damping time-asymptotically behaves as that of the corresponding parabolic equation

$$V_t + \frac{1}{\alpha} (p'(\bar{v}) V_x)_x = 0. \quad (1.17)$$

Similar considerations were taken into by Mizumachi [10] and Li [7].

From these observation, it may be expected that

$$\|(V, V_x, V_t)(\cdot, t)\|_{L^\infty} = O(t^{-1/2}, t^{-1}, t^{-3/2}) \quad (1.18)$$

provided that  $(V_0, z_0)$  is in  $L^1$ . However, we do not know nice estimates of the Green function for a parabolic equation with variable coefficient, likely (1.17), and the optimal convergence rate (1.18) is not known yet.

In the case of  $v_+ = v_-$ , since  $\bar{v}$  is given by (1.13), rewrite (1.11) as

$$V_t - \kappa V_{xx} = -\frac{1}{\alpha} \{V_{tt} + F + ((p'(\bar{v}) - p'(v_\pm)) V_x)_x\} := -\frac{1}{\alpha} V_{tt} - \frac{1}{\alpha} \tilde{F}, \quad (1.19)$$

where  $\kappa = -p'(\bar{v}_\pm)/\alpha$ . Hence

$$V(x, t) = \int_{-\infty}^{\infty} E(x-y, t) V_0(y) dy - \frac{1}{\alpha} \int_0^t \int_{-\infty}^{\infty} E(x-y, t-\tau) (V_{tt} + \tilde{F})(y, \tau) dy d\tau. \quad (1.20)$$

Since

$$\begin{aligned} & -\frac{1}{\alpha} \int_0^{t/2} \int_{-\infty}^{\infty} E(x-y, t-\tau) V_{tt}(y, \tau) dy d\tau \\ &= \int_{-\infty}^{\infty} E(x-y, t) \frac{z_0(y)}{\alpha} dy - \frac{1}{\alpha} \int_{-\infty}^{\infty} E(x-y, t/2) V_t(y, t/2) dy \\ & \quad - \frac{1}{\alpha} \int_0^{t/2} \int_{-\infty}^{\infty} E_t(x-y, t-\tau) V_t(y, \tau) dy d\tau, \end{aligned}$$

(1.20) is changed to

$$\begin{aligned}
V(x, t) &= \int_{-\infty}^{\infty} E(x-y, t) \left( V_0 + \frac{z_0}{\alpha} \right) (y) dy \\
&= -\frac{1}{\alpha} \int_{-\infty}^{\infty} E(x-y, t/2) V_t(y, t/2) dy \\
&\quad - \int_0^{t/2} \int_{-\infty}^{\infty} E_t(x-y, t-\tau) V_t(y, \tau) dy d\tau \\
&\quad - \frac{1}{\alpha} \int_{t/2}^t \int_{-\infty}^{\infty} E(x-y, t-\tau) V_{tt}(y, \tau) dy d\tau \\
&\quad - \frac{1}{\alpha} \left( \int_0^{t/2} + \int_{t/2}^t \right) \int_{-\infty}^{\infty} E(x-y, t-\tau) \tilde{F}(y, \tau) dy d\tau.
\end{aligned} \tag{1.21}$$

Hence, defining  $\phi(x, t)$  by the solution of

$$\phi_t - \kappa \phi_{xx} = 0, \quad \phi|_{t=0} = \left( V_0 + \frac{z_0}{\alpha} \right) (x), \tag{1.22}$$

we have the following theorem.

**Theorem 1.3** *Suppose that  $(V_0, z_0) \in L^1$  in addition to the assumptions in Theorem 1.1. Then the solution  $(V, z)$  of (RP) or (1.11) satisfies*

$$\| (V - \phi, (V - \phi)_x, (V - \phi)_t)(t) \|_{L^\infty} = O(t^{-1}, t^{-3/2}, t^{-2}). \tag{1.23}$$

*Remark 1.1* Decay order (1.23) shows that  $\phi(x, t)$  given by (1.22) is an asymptotic profile of the solution  $V$  to (1.11), because  $\| (\phi, \phi_x, \phi_t)(t) \|_{L^\infty} = O(t^{-1/2}, t^{-1}, t^{-3/2})$ . For the asymptotic profile of the damped wave equation of second order see Gallay and Raugel [1].

In [11, 12], the author treated the case  $u_+ = u_- = 0$  and  $\int_{-\infty}^{\infty} (v_0(y) - v_\pm) dy = 0$  not only  $v_+ = v_-$ , and obtained the same estimate as (1.23). Therefore, Theorem 1.3 generalized the results in [11, 12]. The proof of Theorem 1.3 is on the same line, which is done by the estimates of the right hand side in (1.21) using (1.15) and (1.16).

More general system than (1.1) has been considered by Hsiao and Serre [3], Hsiao and Luo [6], etc. See also the references therein and the book [2] by Hsiao.

## 2 The Initial-Boundary Value Problem on $\mathbf{R}_+$

In this section we consider

$$\begin{cases} v_t - u_x = 0, & x \in \mathbf{R}_+, t > 0 \\ u_t + p(v)_x = -\alpha u \end{cases} \tag{2.1}$$

with the initial data

$$(v, u)|_{t=0} = (v_0, u_0)(x) \rightarrow (v_+, u_+) \quad \text{as } x \rightarrow +\infty \quad (2.2)$$

and the null-Dirichlet boundary

$$u|_{x=0} = 0 \quad (2.3)$$

or the null-Neumann boundary

$$u_x|_{x=0} = 0, \quad (2.4)$$

the results of which are based on the joint work [13] of the author with T. Yang.

In both problems it is crucial how each problem is reformulated. Each reformulation below being taken the boundary effect into consideration does not yields the "boundary layer", and hence the results obtained and those proofs are much similar to the Cauchy problem in Sec. 1. Recently, corresponding to (2.4), Marcati and Mei have considered the problem with the Dirichlet boundary condition  $v|_{x=0} = g(t)$ . However, their reformulation problem has the "boundary layer" and it is necessary to assume  $g(t) \rightarrow v_+$ . We now show how our problems are reformulated and state our results.

For the Dirichlet boundary problem (2.1)-(2.3), expecting

$$(v, u)(x, t) \rightarrow (v_+, 0) \quad \text{as } t \rightarrow \infty, \quad (2.5)$$

we put  $u_t \equiv 0$  in (2.1)<sub>2</sub> by the Darcy law, so that we have (1.2)' with  $u|_{x=0} = -\frac{1}{\alpha}p'(v)v_x|_{x=0} = 0$ . Approximating this by the solution  $(\bar{v}, \bar{u})(x, t)$  of

$$\bar{v} - \kappa\bar{v}_{xx} = 0, \quad \bar{v}_x(0, t) = 0, \quad \bar{v}(+\infty, t) = v_+ \quad \text{and} \quad \bar{u} = \kappa\bar{v}_x$$

or explicitly

$$(\bar{v}, \bar{u})(x, t) = \left( v_+ + \frac{\delta_0 e^{-\frac{x^2}{4\kappa(t+1)}}}{\sqrt{4\pi\kappa(t+1)}}, \kappa\bar{v}_x \right).$$

where  $\kappa = -\frac{p'(v_+)}{\alpha}$  and

$$\delta_0 = 2\left(\int_0^\infty (v_0(x) - v_+)dx - \frac{u_+}{\alpha}\right). \quad (2.6)$$

Thus,  $(\bar{v}, \bar{u})$  satisfies

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, & p'(v_+)\bar{v}_x = -\alpha\bar{u} \\ (\bar{v}_x, \bar{u})|_{x=0} = 0, & (\bar{v}, \bar{u})|_{x=0} = (v_+, 0). \end{cases} \quad (2.7)$$

Auxiliary function  $(\hat{v}, \hat{u})(x, t)$  is defined by

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, & \hat{u}_t = -\alpha\hat{u} \\ \hat{u}|_{x=0} = 0, & (\hat{v}, \hat{u})|_{x=\infty} = (0, u_+e^{-\alpha t}) \end{cases} \quad (2.8)$$

or explicitly

$$(\hat{v}, \hat{u})(x, t) = \left( \frac{u_+m_0(x)}{-\alpha}e^{-\alpha t}, u_+ \int_x^\infty m_0(y)dy \cdot e^{-\alpha t} \right),$$

where  $m_0$  is smooth whose support is in  $\mathbf{R}_+$  and integration over  $\mathbf{R}_+$  is 1. Combining (2.1), (2.7), (2.8) with

$$\int_0^\infty (v - \bar{v} - \hat{v})(y, t) dy = \int_0^\infty (v_0(x) - v_+) dx - \frac{\delta_0}{2} - \frac{u_+}{\alpha} = 0$$

by (2.6), we reach the setting of the perturbation

$$(V, z)(x, t) = \left( - \int_x^\infty (v - \bar{v} - \hat{v})(y, t) dy, (u - \bar{u} - \hat{u})(x, t) \right) \quad (2.9)$$

and the reformulated problem

$$\left\{ \begin{array}{l} V_t - z = 0 \\ z_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \alpha z = -\bar{u}_t + (p'(v_+) - p'(\bar{v}))\bar{v}_x \\ (V, z)|_{t=0} = (V_0, z_0)(x) \\ \quad := \left( - \int_x^\infty (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0)) dy, u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0) \right) \\ z|_{x=0} = 0. \end{array} \right. \quad (DRP)$$

Both the energy method and the method of the Green function of parabolic equation are applicable to (DRP) and the theorem corresponding to Theorem 1.3 holds.

**Theorem 2.1 (Asymptotic profile for the Dirichlet boundary)** *Suppose that  $v_0 - v_+ \in L^1$ ,  $(V_0, z_0) \in H^3 \times H^2$  and  $\|V_0\|_3 + \|z_0\|_2 + |u_+|$  is small, and that  $(V_0, z_0)$  is in  $L^1 \times L^1$ . Then the unique time-global solution  $(V, z)$  satisfies*

$$\|(V - \phi, (V - \phi)_x, (V - \phi)_t)(\cdot, t)\|_{L^\infty} = O(t^{-1}, t^{-3/2}, t^{-2}) \quad \text{as } t \rightarrow \infty,$$

where  $\phi$  is defined by

$$\phi(x, t) = \int_0^\infty \frac{e^{-\frac{(x-y)^2}{4\kappa t}} - e^{-\frac{(x+y)^2}{4\kappa t}}}{\sqrt{4\pi\kappa t}} (V_0(y) + \frac{1}{\alpha} z_0(y)) dy.$$

In the case of the null-Neumann boundary problem (2.1), (2.2), (2.4), heuristic consideration to (2.1)<sub>1</sub> yields  $\frac{d}{dt}v(0, t) = u_x(0, t) = 0$  and hence  $v(0, t) = v_0(0)$  for any  $t > 0$ . Therefore, we can expect that  $(v, u)(x, t) \rightarrow (\bar{v}, 0)(x, t)$  as  $t \rightarrow \infty$ , where  $\bar{v}(x, t)$  is a diffusion wave connecting  $v_0(0)$  and  $v_+$ .

If  $v_0(0) \neq v_+$ , then putting  $u_t = 0$  in (2.1)<sub>2</sub> gives the definition of  $(\bar{v}, \bar{u})$  by  $(\bar{v}, \bar{u})(x, t) = (\psi(x/\sqrt{t+1})|_{x \geq 0}, -\frac{1}{\alpha}p(\bar{v})_x)$ , where  $\psi$  is a similarity solution of the parabolic equation

$$\left\{ \begin{array}{l} \tau_t + \frac{1}{\alpha}p(\tau)_{xx} = 0, \quad x \in \mathbf{R} = (-\infty, \infty), \quad t > 0 \\ \tau|_{x=\pm\infty} = v_\pm, \end{array} \right.$$

for a constant  $v_-$  satisfying  $(0 <)v_- < v_0(0) < v_+$  or  $v_+ < v_0(0) < v_- (< +\infty)$ . Since

$$\bar{v}_t|_{x=0} = \psi'(\frac{x}{\sqrt{t+1}})(-\frac{x}{2\sqrt{t+1}(t+1)})|_{x=0} = 0,$$

$(\bar{v}, \bar{u})$  satisfies

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, & p(\bar{v})_x = -\alpha\bar{u} \\ (\bar{v}, \bar{u}_x)|_{x=0} = (v_0(0), 0), & (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{cases} \quad (2.10)$$

The auxiliary function  $(\hat{v}, \hat{u})$  is defined by

$$(\hat{v}, \hat{u})(x, t) = (\frac{u_0(0) - u_+}{\alpha} m_0(x) e^{-\alpha t}, [(u_0(0) - u_+) \int_x^\infty m_0(y) dy + u_+] e^{-\alpha t})$$

or

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, & \hat{u}_t = -\alpha\hat{u} \\ (\hat{u}, \hat{u}_x)|_{x=0} = (u_0(0) e^{-\alpha t}, 0), & \hat{v}|_{x=0} = 0, & (\hat{v}, \hat{u})|_{x=\infty} = (0, u_+ e^{-\alpha t}). \end{cases} \quad (2.11)$$

Combination of (2.1), (2.10), (2.11) with the perturbation (2.9) yields

$$\begin{cases} V_t - z = 0 \\ z_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x = -\alpha z - \bar{u}_t \\ z_x|_{x=0} = 0 \quad (\text{or } V_x|_{x=0} = 0) \\ (V, z)|_{t=0} = (V_0, z_0)(x) \\ := (-\int_x^\infty (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0)) dy, u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0)). \end{cases} \quad (NRP)$$

The  $L^2$ -energy estimates give the theorem corresponding to Theorem 1.1.

**Theorem 2.2** (The Neumann problem in the case of  $v_0(0) \neq v_+$ ) *Suppose that  $v_0 - v_+$  is in  $L^1$  and both  $\|V_0\|_3 + \|z_0\|_2$  and  $\delta_0 = |(v_0(0) - v_+, u_+ - u_0(0))|$  are small. Then there exists a unique solution  $(V, z) \in C([0, \infty); H^3 \times H^2)$  of (NRP), which satisfies same estimates (1.15) and (1.16) in Theorem 1.1.*

Eventually if  $v_0(0) = v_+$ , the asymptotic profile of the solution to (2.1), (2.2), (2.4) is also obtained on the same line as that of Theorem 1.2. The details are omitted.

## References

- [1] Th. Gallay and G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, preprint.

- [2] L. Hsiao, "Quasilinear Hyperbolic Systems and Dissipative Mechanisms", World Scientific, Singapore, 1997.
- [3] L. Hsiao and D. Serre, Global existence of solutions for the system of compressible adiabatic flow through porous media, *SIAM J. Math. Anal.* 27(1996), 70-77.
- [4] L. Hsiao and T-P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Commun. Math. Phys.* 143(1992), 599-605.
- [5] L. Hsiao and T-P. Liu, Nonlinear diffusion phenomena of nonlinear hyperbolic system, *Chin. Ann. of Math.* 14B(1993), 465-480.
- [6] L. Hsiao and T. Luo, Nonlinear diffusion phenomena of solutions for the system of compressible adiabatic flow through porous media, *J. Differential Equations* 125(1996), 329-365.
- [7] T-T. Li, Nonlinear heat conduction with finite speed of propagation, *Proceedings of the China-Japan Symposium on Reaction-Diffusion Equations and their Applications and Computational Aspects*, 1996.
- [8] P. Marcati and M. Mei, Convergence to nonlinear diffusion waves for solutions of the initial boundary problem to the hyperbolic conservation laws with damping, preprint.
- [9] A. Matsumura, Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with the first-order dissipation, *Publ. RIMS, Kyoto Univ.* 13(1977), 349-379.
- [10] T. Mizumachi, Time decay of solutions to degenerate Kirchhoff type equation, preprint.
- [11] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, *J. Differential Equations* 131(1996), 171-188.
- [12] K. Nishihara, Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping, *J. Differential Equations* 137(1997), 384-395.
- [13] K. Nishihara and T. Yang, Boundary effect on asymptotic behavior of solutions to the  $p$ -system with linear damping, preprint.

# ANALYTICITY AND REGULARIZING EFFECT FOR THE KORTEWEG DE VRIES EQUATION

KEIICHI KATO AND TAKAYOSHI OGAWA

DEPARTMENT OF MATHEMATICS  
SCIENCE UNIVERSITY OF TOKYO  
SHINJYUKU-KU TOKYO 162-8601, JAPAN  
AND  
GRADUATE SCHOOL OF MATHEMATICS,  
KYUSHU UNIVERSITY,  
FUKUOKA 812-8581, JAPAN

## 1. INTRODUCTION AND THEOREM

We study the smoothing effect for the following initial-value problem of the Korteweg-de Vries equation (KdV equation):

$$(1.1) \quad \begin{cases} \partial_t v + \partial_x^3 v + \partial_x(v^2) = 0, & t, x \in \mathbb{R}, \\ v(0, x) = \phi(x). \end{cases}$$

Here the solution  $u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  denotes the surface displacement of the water wave. In this talk, we show that if the initial data  $\phi$  is Dirac's delta for example, the solution is real analytic for  $t \neq 0$ .

There are a lot of works for the study of KdV equation ([4], [5], [6], [8], [10], [14], [16], [21], [23], [26]). Among others, Kato [14] firstly extract the smoothing effect for the evolution operator of the linear part of the KdV equation  $e^{t\partial_x^3}$ . The Uniqueness result is also obtained by Kruzhkov-Faminskii [21], Ginibre-Y. Tsutsumi [8] in the subspace of  $H^1$ . Later on, Kenig-Ponce-Vega [16] extend the Kato type smoothing effect and they showed that the KdV equation is well-posed in the Sobolev space  $H^{3/4}$ .

Along the elegant method in the series of papers, Bourgain [2] obtained  $L^2$  well-posedness of the KdV equation in the periodic boundary condition. His argument also works for the Cauchy problem (1.1) and the global well-posedness is established. Furthermore, by refining the method by Bourgain, Kenig-Ponce-Vega [17] [18] proved some bilinear estimates involving the negative exponent Sobolev space and established the local well-posedness for the Cauchy problem in the negative Sobolev space  $H^s(\mathbb{R})$  where  $(-3/4 < s)$ .

On the other hand, a highly regular solution has also been studied by several authors. T.Kato-Masuda [20] obtained a global smooth solution and the analyticity for any point  $(t, x) \in \mathbb{R} \times \mathbb{R}$ . Hayashi-K.Kato [9] obtained the analytic smoothing effect for the nonlinear Schrödinger equation (see also K.Kato-Taniguchi [13]) and de Bouard-Hayashi-Kato [7] established the analyticity for KdV equations from the Gevrey initial data. Those results are basically obtained by using the commutation and almost commutation operators with the linear KdV equation.

Thanks to the paper [18], we have a time local solution of (1.1) with Dirac's delta as the initial function. Our problem in this note is to study the regularity of the solution with Dirac's delta as the initial function. In the following, we show that if the initial function is in some class which contains Dirac's delta, the solution is real analytic for  $t \neq 0$ .

More precisely, our result is the following:

**Theorem 1.1.** *Let  $-3/4 < s, b \in (1/2, 7/12)$ . Suppose that the initial data  $\phi \in H^s(\mathbb{R})$  and for some  $A_0 > 0$ , it satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty.$$

*Then there exists a unique solution  $v \in C((-T, T), H^s) \cap X_b^s$  of the KdV equation (1.1) in a certain time interval  $(-T, T)$  and the solution  $v$  is time locally well-posed, i.e. the solution continuously depends on the initial data. Moreover the solution  $v$  is analytic at any point  $(t, x) \in (-T, 0) \cup (0, T) \times \mathbb{R}$ , where we define*

$$\|f\|_{X_b^s} = \left( \iint \langle \tau - \xi^3 \rangle^{2b} \langle \xi \rangle^{2s} |\hat{f}(\tau, \xi)|^2 d\tau d\xi \right)^{1/2} = \|V(\cdot)f(\cdot)\|_{H_t^b(\mathbb{R}; H_x^s(\mathbb{R}))}$$

*and  $V(t) = e^{-t\partial_x^3}$  is the unitary group of the free KdV evolution.*

**Remark 1.** A typical example of the initial data satisfying the assumption of the above theorem is the Dirac delta measure, since  $(x\partial_x)^k \delta(x) = (-1)^k \delta(x)$ . The other example of the data is  $p.v. \frac{1}{x}$ , where  $p.v.$  denotes Cauchy's principal value. Any possible linear combination of those distributions with an analytic  $H^s$  data satisfying the assumption can be also the initial data. In this sense, Dirac's delta measure adding the soliton initial data can be taken as a initial data.

**Remark 2.** For a non-smooth initial data, it is known that the global in time solution has been obtained (see [5], [10]) by the inverse scattering method. Recently the analyticity for the inverse scattering solution with a weighted initial data was obtained by Tarama [24]. Since our method is based on the fact that the solution is in  $H^s$ , we don't know if our result is true globally in time.

By a almost similar argument of Theorem 1.1, one can also show the following corollary.

**Corollary 1.2.** *Let  $-3/4 < s$ ,  $b \in (1/2, 7/12)$ . Suppose that for some  $A_0 > 0$ , the initial data  $\phi \in H^s(\mathbb{R})$  and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^3} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

*then there exists a unique solution  $v \in C((-T, T), H^s) \cap X_b^s$  of the KdV equation (1.1) for a certain time interval  $(-T, T)$  and for any  $t \in (-T, 0) \cup (0, T)$   $v(t, \cdot)$  is analytic function in space variable and for  $x \in \mathbb{R}$ ,  $v(\cdot, x)$  is of Gevrey 3 as a time variable function.*

**Remark 3.** Both in Theorem and Corollary, the assumption on the initial data implies the analyticity and Gevrey 3 regularity except the origin respectively. In this sense, those results are stating that the singularity at the origin immediately disappear after  $t > 0$  or  $t < 0$  up to analyticity.

**Remark 4.** Some related results are obtained for the linear and nonlinear Schrödinger equations. For linear variable coefficient case, see Kajitani-Wakabayashi [11] and for nonlinear case, Chihara [3]. They are giving a global weighted uniform estimates of the solution with arbitrary order of derivatives in space variable. Even in our case, we expect that the similar uniform bounds are available.

## 2. METHOD OF THE PROOF.

Our method is based on the following observation. Firstly, we introduce the generator of the dilation  $P = 3t\partial_t + x\partial_x$  for the linear part of the KdV equation. Noting the commutation relation with the linear KdV operator  $L = \partial_t + \partial_x^3$ :

$$[L, P] = 3L,$$

it follows

$$(2.1) \quad LP^k = (P + 3)^k L,$$

$$(2.2) \quad (P + 3)^k \partial_x = \partial_x (P + 2)^k$$

for any  $k = 1, 2, \dots$ . Applying  $P = 3t\partial_t + x\partial_x$  to the KdV equation, we have

$$(2.3) \quad \begin{aligned} \partial_t(P^k v) + \partial_x^3(P^k v) &= (P + 3)^k Lv = (P + 3)^k (-\partial_x(v^2)) \\ &= -\partial_x(P + 2)^k v^2. \end{aligned}$$

We set  $v_k = P^k v$  and  $B_k(v, v) = \partial_x(P + 2)^k v^2$ . Then noting that

$$(2.4) \quad \begin{aligned} (P + 2)^l v &= (P + 2)^{l-1} P v + 2(P + 2)^{l-1} v = \dots \\ &= \sum_{j=0}^l \frac{l!}{j!(l-j)!} 2^{l-j} P^j v, \end{aligned}$$

we see

$$\begin{aligned}
B_k(v, v) &= \partial_x (P+2)^k (v^2) = \partial_x \sum_{l=0}^k \binom{k}{l} (P+2)^l v P^{k-l} v \\
&= \partial_x \sum_{l=0}^k \sum_{m=0}^l \binom{k}{l} \binom{l}{m} 2^{l-m} P^m v P^{k-l} v \\
&= \sum_{k=k_1+k_2+k_3} \frac{k!}{k_1!k_2!k_3!} 2^{k_1} \partial_x (v_{k_2} v_{k_3}).
\end{aligned}$$

The above nonlinear term maintains the bilinear structure like the original KdV equation, since the Leibniz law can be applicable for an operation of  $P$ . Now each  $v_k$  satisfies the following system of equations;

$$(2.5) \quad \begin{cases} \partial_t v_k + \partial_x^3 v_k + B_k(v, v) = 0, & t, x \in \mathbb{R}, \\ v_k(0, x) = (x \partial_x)^k \phi(x), & k = 0, 1, 2, \dots \end{cases}$$

Therefore we firstly establish the local well-posedness of the solution to the following infinitely coupled system of KdV equation in a suitable weak space:

$$(2.6) \quad \begin{cases} \partial_t v_k + \partial_x^3 v_k + B_k(v, v) = 0, & t, x \in \mathbb{R}, \\ v_k(0, x) = \phi_k(x), & k = 0, 1, 2, \dots \end{cases}$$

Then by taking  $\phi_k = (x \partial_x)^k \phi(x)$ , the uniqueness and local well-posedness yields that  $v_k = P^k v$  for all  $k = 0, 1, \dots$ .

According to Bourgain [2], we introduce the Fourier restriction space;

$$X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X_b^s} < \infty\},$$

where

$$\|f\|_{X_b^s}^2 = c \iint \langle \tau - \xi^3 \rangle^{2b} \langle \xi \rangle^{2s} |\hat{f}(\tau, \xi)|^2 d\tau d\xi = \|V(-t)f\|_{H_t^b(\mathbb{R}, H_x^s)}^2.$$

It has been proven that the KdV equation is well-posed in the above space  $X_b^s$  when  $s > -3/4$  with  $b > 1/2$ . The space where we solve the system (2.6) is infinite sum of copies of this space. Let  $f = (f_0, f_1, \dots, f_k, \dots)$  denotes the infinity series of distributions and define

$$\mathcal{A}_{A_0}(X_b^s) = \{f = (f_0, f_1, \dots, f_k, \dots), f_i \in X_b^s \quad (i = 0, 1, 2, \dots) \text{ such that } \|f\|_{\mathcal{A}_{A_0}} < \infty\},$$

where

$$\|f\|_{\mathcal{A}_{A_0}} \equiv \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|f_k\|_{X_b^s}.$$

The system (2.6) will be shown to be well-posed in the above space if  $s > -3/4$  and  $b > 1/2$  under the assumption for the initial data

$$\|\phi_k\|_{H^s} \leq C A_1^k k! \quad k = 0, 1, \dots$$

The well-posedness is derived by utilizing the contraction principle argument to the corresponding system of integral equations:

$$(2.7) \quad \psi(t)v_k(t) = \psi(t)V(t)\phi_k - \psi(t) \int_0^t V(t-t')\psi_T(t')B_k(v,v)(t')dt',$$

where  $\psi(t)$  is a smooth cut off function such that  $\psi(t) = 1$  for  $|t| < 1$  and  $\psi(t) = 0$  for  $|t| > 2$  and  $\psi_T(t) = \psi(t/T)$ .

The following estimates of linear and nonlinear part due to Bourgain [2] and refined by Kenig-Ponce-Vega [17] are our essential tools.

**Lemma 2.1** ([2],[17],[18]). *Let  $s \in \mathbb{R}$ ,  $a, a' \in (0, 1/2)$ ,  $b \in (1/2, 1)$  and  $\delta < 1$ . Then for any  $k = 0, 1, 2, \dots$ , we have*

$$(2.8) \quad \|\psi_\delta \phi_k\|_{X_{-a}^s} \leq C\delta^{(a-a')/4(1-a')} \|\phi_k\|_{X_{-a'}^s},$$

$$(2.9) \quad \|\psi_\delta V(t)\phi_k\|_{X_b^s} \leq C\delta^{1/2-b} \|\phi_k\|_{H^s},$$

$$(2.10) \quad \|\psi_\delta \int_0^t V(t-t')F_k(t')dt'\|_{X_b^s} \leq C\delta^{1/2-b} \|F_k\|_{X_{b-1}^s}.$$

**Lemma 2.2** ([2],[17],[18]). *Let  $s > -3/4$ ,  $b, b' \in (1/2, 7/12)$  with  $b < b'$ . Then for any  $k, l = 0, 1, 2, \dots$ , we have*

$$(2.11) \quad \|\partial_x(u_k v_l)\|_{X_{b'-1}^s} \leq C \|v_k\|_{X_b^s} \|v_l\|_{X_b^s}.$$

**Proof of Lemma 2.1 and 2.2.** See [17] and [18]. □

From Lemma 2.2, it is immediately obtained the bilinear estimate for the nonlinearity for the system.

**Corollary 2.3.** *Let  $s > -3/4$ ,  $b, b' \in (1/2, 7/12)$  with  $b < b'$ . Then, we have*

$$(2.12) \quad \|B_k(v,v)\|_{X_{b'-1}^s} \leq C \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X_b^s} \|v_{k_3}\|_{X_b^s}.$$

We construct a contraction map via the integral equations. Set a map  $\Phi : \{v_k\}_{k=0}^\infty \rightarrow \{v_k(t)\}_{k=0}^\infty$  such that  $\Phi = (\Phi_0, \Phi_1, \dots)$  and

$$\Phi_k(\phi_k) \equiv \psi V(t)\phi_k - \psi \int_0^t V(t-t')\psi_T(t')B_k(v,v)(t')dt'.$$

Then it is shown that  $\Phi_k : \mathcal{A}_{A_0}(H^s) \rightarrow \mathcal{A}_{A_1}(X_b^s)$  is contraction.

In fact, by using Lemma 2.1 and Lemma 2.2, we easily see that for any  $k \geq 0$ ,

$$(2.13) \quad \begin{aligned} \|\Phi_k(v_k)\|_{X_b^s} &\leq C_0 \|\phi_k\|_{H^s} + C_1 T^\mu \|B_k(v,v)\|_{X_{b'-1}^s} \\ &\leq C_0 \|\phi_k\|_{H^s} + C_1 T^\mu \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X_b^s} \|v_{k_3}\|_{X_b^s}. \end{aligned}$$

By taking a sum in  $k$ ,

$$\begin{aligned}
\|\Phi\|_{\mathcal{A}_{A_1}(X_b^s)} &= \sum_{k=0}^{\infty} \frac{A_1^k}{k!} \|v_k\|_{X_b^s} \\
&\leq C_0 \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|\phi_k\|_{H^s} + C_1 T^\mu \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{k!}{k_1!k_2!k_3!} \|v_{k_2}\|_{X_b^s} \|v_{k_3}\|_{X_b^s} \\
&\leq C_0 \|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 T^\mu \sum_{k=0}^{\infty} \sum_{k=k_1+k_2+k_3} 2^{k_1} \frac{A_0^{k_1}}{k_1!} \frac{A_0^{k_2}}{k_2!} \|v_{k_2}\|_{X_b^s} \frac{A_0^{k_3}}{k_3!} \|v_{k_3}\|_{X_b^s} \\
&\leq C_0 \|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 T^\mu \sum_{k_1=0}^{\infty} 2^{k_1} \frac{A_0^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!} \|v_{k_2}\|_{X_b^s} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!} \|v_{k_3}\|_{X_b^s}.
\end{aligned}$$

Hence we have

$$\|\Phi(v)\|_{\mathcal{A}_{A_1}(X_b^s)} \leq C_0 \|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 e^{2A_0} T^\mu \|v\|_{\mathcal{A}_{A_1}(X_b^s)}^2$$

and also we have the estimate for the difference

$$\|\Phi(v) - \Phi(\tilde{v})\|_{\mathcal{A}_{A_1}(X_b^s)} \leq C_1 e^{2A_0} T^\mu (\|v\|_{\mathcal{A}_{A_1}(X_b^s)} + \|\tilde{v}\|_{\mathcal{A}_{A_1}(X_b^s)}) \|v - \tilde{v}\|_{\mathcal{A}_{A_1}(X_b^s)}.$$

Choosing  $T$  small enough, the map  $\Phi$  is contraction from

$$X_T = \{f = (f_0, f_1, \dots); f_i \in X_b^s, \sum_0^\infty \frac{A_0^k}{k!} \|f_k\|_{X_b^s} \leq 2C_0 M_0\}$$

to itself, where  $M_0 = \|\phi\|_{\mathcal{A}_{A_0}(H^s)}$ . A similar argument in [1] gives us the uniqueness of the system of the solution. This shows the proof of well-posedness.

Hence under the assumption for the initial function

$$\|((x\partial_x)^k \phi)\|_{H^s} \leq C A_1^k k! \quad k = 0, 1, \dots,$$

the corresponding solution to (KdV) satisfies the following estimate

$$(2.14) \quad \|P^k v\|_{X_b^s} \leq C A_0^k k! \quad k = 0, 1, \dots.$$

Multiplying  $t$  to the both sides of the first equation of (2.5), we have

$$(2.15) \quad M^3 v_k = -\frac{1}{3} P v_k + \frac{1}{3} x \partial_x v_k + t B_k(v, v),$$

from which we gain the regularity of  $v$  with (2.14).

For a fixed point  $(t_0, x_0) \in (0, T) \times \mathbb{R}$ , we show that  $v$  is analytic near  $(t_0, x_0)$ . Let  $a(t, x) \in C_0^\infty(\mathbb{R}^2)$  be a cut-off function near  $(t_0, x_0)$  such that  $\text{supp } a \subset [t_0 - \varepsilon, t_0 + \varepsilon] \times [x_0 - \varepsilon^{1/3}, x_0 + \varepsilon^{1/3}]$ . First we show that

$$(2.16) \quad \|a P^k v\|_{L_{t,x}^2(\mathbb{R}^2)} \leq C_2 A_2^k k! \quad k = 0, 1, 2, \dots,$$

for some positive constants  $C_2$  and  $A_2$ . This is shown by using the following lemma:

**Lemma 2.4.** Let  $P = 3t\partial_t + x\partial_x$  be the generator of the dilation and  $D_{t,x}$  be an operator defined by  $\mathcal{F}_{t,x}^{-1}\langle |\tau| + |\xi| \rangle \mathcal{F}_{t,x}$ .

For a point  $(t_0, x_0) \in \mathbb{R}^2$  with  $t_0 \neq 0$ , we suppose that  $g \in H^{\mu-3}(\mathbb{R}^2)$  with  $\text{supp } g \subset B_{2\varepsilon}(t_0, x_0)$  and  $t\partial_x^3 g, P^3 g \in H^{\mu-3}(\mathbb{R}^2)$ . Then we have

$$(2.17) \quad \|(D_{t,x})^\mu g\|_{L^2(\mathbb{R}^2)} \leq C\{\|g\|_{H^{\mu-3}(\mathbb{R}^2)} + \|t\partial_x^3 g\|_{H^{\mu-3}(\mathbb{R}^2)} + \|P^3 g\|_{H^{\mu-3}(\mathbb{R}^2)}\},$$

where the constant  $C$  depends on  $(t_0, x_0)$  and  $\varepsilon$ .

From (2.16) and (2.15) we can show that

$$(2.18) \quad \|aP^k v\|_{H^{7/2}(\mathbb{R}^2)} \leq C_3 A_3^k k! \quad k = 0, 1, 2, \dots,$$

for some positive constants  $C_3$  and  $A_3$ . (2.18) gives us immediately that

$$(2.19) \quad \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \|P^k v\|_{H^1(x_0 - \varepsilon^{1/3}, x_0 + \varepsilon^{1/3})} \leq C_4 A_4^k k!,$$

for  $k = 0, 1, 2, \dots$  with some positive constants  $C_4$  and  $A_4$ . From this estimate (2.19) and (2.15) we have with some positive constants  $C_5$  and  $A_5$ ,

$$(2.20) \quad \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \|(t^{1/3} \partial_x)^l P^k v\|_{H^1(x_0 - \varepsilon^{1/3}, x_0 + \varepsilon^{1/3})} \leq C_5 A_5^{l+k} (l+k)!,$$

for  $m, k = 0, 1, 2, \dots$ . This is shown by induction with respect to  $l$ . From (2.20) and (2.15) we have with some positive constants  $C_6$  and  $A_6$ ,

$$(2.21) \quad \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \|\partial_t^m \partial_x^l v\|_{H^1(x_0 - \varepsilon^{1/3}, x_0 + \varepsilon^{1/3})} \leq C_6 A_6^{m+l} (m+l)!,$$

for  $m, l = 0, 1, 2, \dots$ , which shows that  $v$  is real analytic in  $(t, x)$  near  $(t_0, x_0)$ .

## REFERENCES

1. Bekiranov, D., Ogawa, T., Ponce, G., *Interaction Equations for Short and Long Dispersive Waves*, to appear in J. Funct. Anal.
2. Bourgain, J., Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I Schrödinger equations, Geometric and Funct. Anal. **3** (1993), 107-156. Exponential sums and nonlinear Schrödinger equations, ibid. **3** (1993), 157-178. Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II The KdV equation, ibid. **3** (1993), 209-262.
3. Chihara, H. *Analytic smoothing effect for the nonlinear Schrödinger equations*, preprint, Shinshu Univ.
4. P. Constantin and J.C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc., **1** (1988), 413-446.
5. Cohen, A., Kappeler, T., *Solutions to the Korteweg-de Vries equation with irregular initial profile in  $L^1_1(\mathbb{R}) \cap L^2_n(\mathbb{R})$* , SIAM J. Math. Anal., **18** (1987), 991-1025.
6. Craig, W., Kappeler, T., Strauss, W., *Gain of regularity for equations of KdV type*, Ann. Inst. Henri Poincaré Analyse Nonlinéaire, **9** (1992), 147-186.
7. de Bouard, A., Hayashi, N., Kato, K. *Regularizing effect for the (generalized) Korteweg de Vries equation and nonlinear Schrödinger equations* Ann. Inst. H. Poincaré, Analyse non linéaire **9** (1995) 673-725.
8. Ginibre, J., Y. Tsutsumi *Uniqueness for the generalized Korteweg -de Vries equations* SIAM J. Math. Anal., **20** (1989) 1388-1425.

9. Hayashi, N., Kato, K *Regularity in time of solution to nonlinear Schrödinger equations*, J. Funct. Anal. **128** (1995), 253–277,
10. Kappeler, T., *Solutions to the Korteweg-de Vries equation with irregular initial profile*, Comm. P.D.E., **11** (1986) 927–945.
11. Kajitani, K., Wakabayashi, S., *Analytically smoothing effect for Schrödinger type equations with variable coefficients* Preprint, Tsukuba University.
12. Kato, K., *Regularity of solutions to the wave equation with a non smoothing coefficient*, SUT J. Math. **33** no. 1 (1997), 105–113.
13. Kato, K., Taniguchi, K., *Gevrey regularizing effect for nonlinear Schrödinger equations* Osaka J. Math. **33** (1996) 863–880.
14. Kato, T. *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, in "Studies in Applied Mathematics", edited by V. Guilemin, Adv. Math. Supplementary Studies **18** Academic Press 1983, 93–128.
15. Kato, T., Masuda, K., *Nonlinear evolution equations and analyticity. I* Ann.Inst.Henri Poincaré. Analyse non linéaire **3** no. 6 (1986) 455–467.
16. Kenig, C.E., Ponce G., Vega, L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction mapping principle*, Comm. Pure Appl. Math., **46** (1993), 527–620.
17. Kenig, C. E., Ponce, G., Vega, L., *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993) 1–21.
18. Kenig, C. E., Ponce, G., Vega, L., *A bilinear estimate with applications to the KdV equation*. J. Amer. Math. Soc. **9** (1996) 573–603.
19. Kenig, C. E., Ponce, G., Vega, L., *Quadratic Forms for the 1-D semilinear Schrödinger equation*, to appear in Trans. Amer. Math. Soc.
20. Klainerman, S., Machedon, M., *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math., **46** (1993) 1221–1268.
21. Kruzhkov, S.N., Faminskii, A.V., *Generalized solutions of the Cauchy problem for the Korteweg- de Vries equation*, Math. USSR Sbornik, **48** (1984) 391–421
22. Kumano-go, H., *Pseudo Differential Operators*, Iwanami, Japan 1979
23. Sacks, B., *Classical solutions of the Korteweg- de Vries equation for non-smooth initial data via inverse scattering*, Comm.P.D.E., **10** (1985) 29–98.  
quadratic surfaces and
24. Tarama, S., *Analyticity of the solution for the Korteweg- de Vries equation*, Preprint
25. Taylor, M., *Pseudo Differential Operators*, Princeton Univ. Press 1978.
26. Tsutsumi, Y., *The Cauchy problem for the Korteweg- de Vries equation with measure as initial data*, SIAM J. Math. Anal, **20** (1989), 582–588.

# ON THE EVOLUTION OF COMPACTLY SUPPORTED PLANAR VORTICITY

Thomas C. Sideris  
University of California, Santa Barbara

*The work to be reported on in this talk was done jointly with Dragoş Iftimie of Université de Rennes 1.*

The evolution of ideal incompressible fluid vorticity preserves compactness of support. For planar fluids, the diameter of the support of nonnegative initial vorticity will be shown to grow no faster than  $\mathcal{O}[(t \log t)^{1/4}]$ , improving the bound of  $\mathcal{O}(t^{1/3})$  obtained by Marchioro. In addition, an example of an initial vorticity with indefinite sign will be given whose support grows unboundedly at a rate of  $\mathcal{O}(t)$ , the maximum rate possible.

The initial value problem for the 2d incompressible Euler equations is globally well-posed in variety of settings. Wolibner established the existence of classical solutions given initial vorticity in  $C^\alpha(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ , and Yudovitch gave the framework for weak solutions starting with initial vorticity in  $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ . Three basic elements of the classical existence theory are relevant for the present discussion. The divergence-free fluid velocity vector field  $v(t, x)$  generates a particle flow map  $\Phi(t, p)$  through the system of ODE's

$$(1) \quad \frac{d}{dt} \Phi(t, p) = v(t, \Phi(t, p)), \quad \Phi(0, p) = p,$$

such that the map  $p \mapsto \Phi(t, p)$  is a continuously varying family of area-preserving diffeomorphisms of the plane. The scalar vorticity  $\omega = \partial_1 v_2 - \partial_2 v_1$  is transported by this flow

$$(2) \quad D_t \omega = \partial_t \omega + v \cdot \nabla \omega = 0, \quad \omega(0, x) = \omega_0(x),$$

and the velocity is coupled to the vorticity through the Biot-Savart law

$$(3) \quad v(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t, y) dy.$$

Despite this successful existence theory, little can be said about the large time behavior of solutions. This is not surprising since point vortex approximations, even using small numbers of particles, can generate complex dynamics. Given that the vorticity is transported by a area-preserving flow (2), it follows that its  $L^p$  norms are constant in time. In the case of smooth data, Hölder regularity of the flow map is preserved in time, but the Hölder norm of the flow map is only known to be bounded by an expression of the form  $\exp(\exp Ct)$ . Clearly any growth in the Hölder norm of the flow map would be related to the evolution of compact regions under the flow.

If the initial vorticity is supported in a compact set  $\Omega \subset \mathbb{R}^2$ , then equation (2) shows that at time  $t > 0$  the vorticity is supported in  $\Omega(t) = \Phi(t, \Omega)$ . Nothing can be said about the geometry of  $\Omega(t)$ . However in the case of a vortex patch, Chemin proved that the regularity of the boundary is propagated. A simple estimate from (3) provides a uniform bound for the velocity, and so the support of the vorticity can grow at most linearly in time. For positive initial vorticity, Marchioro demonstrated that the conservation of the *moment of inertia*,  $\int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx$ , further acts to constrain the spreading of the support to a rate of  $\mathcal{O}(t^{1/3})$ .

Here it will be shown that Marchioro's bound for the growth rate of the support of positive vorticity can be improved to  $\mathcal{O}[(t \log t)^{1/4}]$  by taking into account not only the conservation of the moment of inertia but also the conservation of the center of mass,  $\int_{\mathbb{R}^2} x \omega(t, x) dx$ . Bounds for the flow map will come from an estimate for the radial component of the velocity starting from (3). The heart of the matter is to measure the vorticity in  $L^1$  outside of balls centered at the origin. The approach taken here is to derive a differential inequality for a certain smooth approximation to this  $L^1$  quantity. The analysis applies to weak solutions in  $L^1 \cap L^p$ ,  $2 < p \leq \infty$ .

There are a few examples of explicit solutions, but none of these exhibit any growth of support. Spherically symmetric initial vorticity gives rise to a stationary solution whose velocity vector field induces flow lines which follow circles about the origin. The support of the Kirchoff elliptical vortex patch rotates with constant angular velocity, although the velocity vector field has a nontrivial structure exterior to the support. The support of the so-called Batchelor couple, the continuous analog of a pair of oppositely charged point

vortices, moves by translation with speed  $\mathcal{O}(t)$ , without any change of shape. On the other hand, numerical simulations starting with a pair of positively charged vortex patches show homogenization of the patches simultaneous with the formation of long filaments.

Finally, an example will be presented for which the support of the vorticity grows at a rate of  $\mathcal{O}(t)$ . This rate is optimal since, as mentioned above, the growth can be at most linear in time. The initial vorticity is not positive, rather it consists of four blobs, identical except for alternating sign, located symmetrically in the four quadrants. The initial configuration is inspired by two examples. First, the discrete analog of this set-up can be integrated explicitly, and the point vortices are seen to spread at a rate of  $\mathcal{O}(t)$ . Secondly, at the other extreme, Bahouri and Chemin consider an example for which the initial vorticity is piecewise constant with alternating values  $\pm 1$  in the unit square of the four quadrants. There one finds rapid loss of Hölder regularity of the flow map. The motion in our example restricts to a solution of the Euler equations in the first quadrant with slip boundary conditions. The proof will show that the center of the mass located in the first quadrant moves at a rate of  $\mathcal{O}(t)$ . In this case, the conservation of the center of mass and moment of inertia are no longer useful since both quantities vanish. Instead, we shall use conservation of energy.

**$L^\infty$ -ESTIMATE OF FIRST SPACE DERIVATIVES  
OF STOKES FLOW IN A HALF SPACE**

YASUYUKI SHIMIZU

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

**§0 Introduction**

We consider the Stokes equation

$$(0.1) \quad \begin{aligned} u_t - \Delta u + \nabla p &= 0, \operatorname{div} u = 0 \text{ in } \Omega \times (0, \infty), \\ u &= u_0 \text{ at } t = 0, \\ u &= 0 \text{ on } \partial\Omega \times (0, \infty) \end{aligned}$$

in a domain  $\Omega$  in  $\mathbb{R}^n$  ( $n \geq 2$ ) with smooth boundary. Here  $u = (u^1, \dots, u^n)$  is unknown velocity field and  $p$  is unknown pressure field. Initial data  $u_0$  is assumed to satisfy a *compatibility condition*:  $\operatorname{div} u_0 = 0$  in  $\Omega$  and the normal component of  $u_0$  equals zero on  $\partial\Omega$ . This system is a typical parabolic equation and it has several properties resembling the heat equation.

Regularity-decay estimates like  $L^p - L^q$  estimates are well-studied especially for  $1 < p, q < \infty$  as explained in Giga-Matsui-Shimizu [8]. We reproduce its explanation for reader's convenience. If  $\Omega = \mathbb{R}^n$ ,  $u$  is reduced to a solution of the heat equation with initial data  $u_0$  because there is no boundary condition. For example regularity-decay estimate

$$(0.2) \quad \|\nabla u(t)\|_p \leq Ct^{-1/2} \|u_0\|_p \text{ for } t > 0$$

holds for all  $1 \leq p \leq \infty$  with  $C$  independent of  $t$  and  $u_0$ , where  $\|f(t)\|_p := (\int_\Omega |f(t, x)|^p dx)^{1/p}$  and  $\nabla$  denotes the gradient in space variables. If  $p = 2$ , the estimate (0.2) is still valid for any domain. Indeed, since the Stokes operator  $A$  is self-adjoint and nonnegative, the operator  $A$  generates an analytic semigroup  $e^{-tA}$ . This yields

$$\|A^{1/2} e^{-tA} u_0\|_2 \leq Ct^{-1/2} \|u_0\|_2.$$

Since  $u = e^{-tA} u_0$  and  $\|A^{1/2} u\|_2 = \|\nabla u\|_2$ , (0.2) follows for  $p = 2$ . (See Borchers and Miyakawa [3] for applications.) For  $1 < p < \infty$ , (0.2) is valid for bounded domains (Giga [7]) and for a half space (Ukai [14]). The estimate (0.2) is also valid for exterior domain with  $n \geq 3$ , with extra restriction  $1 < p < n$ . (See Borchers and Miyakawa [2], Giga and Sohr [9], Iwashita [11].)

However, there was few results for  $p = 1$  or  $p = \infty$  where the boundary of  $\Omega$  is not empty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in  $L^1$  and  $L^\infty$  type spaces, because it involves the singular integral operator such as Riesz operators. In the case  $p = 1$ , Giga-Matsui-Shimizu [8] showed (0.2) when  $\Omega$  is a half-space by estimating Ukai's representation of solutions in Hardy space  $\mathcal{H}^1$  which is a subspace of  $L^1$ . The proof of (0.2) for  $p = 1$  does not apply to get the results for  $p = \infty$  since many terms in Ukai's formula have singularity at  $x_n = 0$  when we estimate in  $L^\infty$  space. We are trying to derive  $L^\infty$  estimate from  $L^1$  estimate by duality argument but such method does not seem to apply since the projection is not bounded in  $L^1$  and  $L^\infty$ .

In this talk, we prove (0.2) for  $p = \infty$  where  $\Omega$  is a half space  $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n); x_n > 0\}$ .

**Theorem 0.1.** *Let  $u$  be the solution of the Stokes equation (0.1) in  $\Omega = \mathbb{R}_+^n$  with initial data  $u_0 \in L^\infty(\mathbb{R}^n)$ , which satisfies the compatibility condition. Then there is a constant  $C$  independent of  $u_0$  such that*

$$(0.3) \quad \|\nabla u(t)\|_\infty \leq Ct^{-1/2} \|u_0\|_\infty$$

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

for all  $t > 0$ .

The proof of this theorem is divided in three sections. First, we rearrange the solution formula of (0.1). The original formula obtained by Ukai [14] is not convenient for  $L^\infty$  estimate because many terms have singularity on  $x_n = 0$ . To overcome the difficulty, we change the way of extension of solution in  $x_n < 0$ . In our new formula, the terms which include Riesz operators  $\nabla' \Lambda^{-1}$  in tangential space does not have singularities at  $x_n = 0$ . Next we estimate the terms which don't include  $\nabla' \Lambda^{-1}$ . They can be obtained by  $L^1$ -estimate of the kernel, even though the terms has singularity on  $x_n = 0$ . Finally, we estimate the term which include  $\nabla' \Lambda^{-1}$ . We will show that the smoothness at  $x_n = 0$  of these terms is very useful to obtain estimates in the Hardy space  $\mathcal{H}^1$ .

#### REFERENCES

1. W. Borchers and T. Miyakaya,  $L^2$  decay for the Navier-Stokes flow in halfspaces, *Math. Ann.* **282** (1988), 139–155.
2. ———, *Algebraic  $L^2$  decay for Navier-Stokes flows in exterior domains*, *Acta Math.* **165** (1990), 189–227.
3. ———, *Algebraic  $L^2$  decay for Navier-Stokes flows in exterior domains, II*, *Hiroshima Math. J.* **21** (1991), 621–640.
4. A. Carpio, *Large time behavior in incompressible Navier-Stokes equations*, *SIAM J. Math. Anal.* **27** (1996), 449–475.
5. Z.-M. Chen, *Solution of the stationary and nonstationary Navier-Stokes equations in exterior domains*, *Pacific J. Math.* **159** (1993), 227–240.
6. C. Fefferman and E. Stein,  $\mathcal{H}^p$  spaces of several variables, *Acta Math.* **129** (1972), 137–197.
7. Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in  $L^r$  spaces*, *Math. Z.* **178** (1981), 297–329.
8. Y. Giga, S. Matsui and Y. Shimizu, *On Estimates in Hardy Spaces for the Stokes Flow in a Half Space*, To appear in *Math. Z.*, Hokkaido University Preprint Series in Mathematics **381** (1997).
9. Y. Giga and H. Sohr, *On the Stokes operator in exterior domains*, *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.* **36** (1989), 103–130.
10. Y. Giga and H. Sohr, *Abstract  $L^p$  estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, *J. of Functional Analysis* **102** (1991), 72–94.
11. H. Iwashita,  *$L_q - L_r$  estimate for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in  $L_q$  spaces*, *Math. Ann.* **285** (1989), 265–288.
12. T. Miyakawa, *Hardy spaces of solenoidal vector fields, with application to the Navier-Stokes equations*, *Kyushu J. Math.* **50** (1997), 1–64.
13. A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, 1986.
14. S. Ukai, *A solution formula for the Stokes equation in  $\mathbb{R}_+^n$* , *Comm. Pure Appl. Math.* **XL** (1987), 611–621.

# Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth

M.Nakamura\* and T.Ozawa

Department of Mathematics  
Hokkaido University  
Sapporo 060, Japan  
(e-mail: m-nakamu@math.sci.hokudai.ac.jp)

## 1 Introduction

We consider the existence of global solutions for nonlinear wave equations of the form

$$\begin{aligned} \partial_t^2 u(t, x) - \Delta u(t, x) &= f(u(t, x)) \\ u(0, \cdot) = \phi &\in \dot{H}^{n/2}(\mathbf{R}^n), \quad \partial_t u(0, \cdot) = \psi \in \dot{H}^{n/2-1}(\mathbf{R}^n), \end{aligned} \quad (1.1)$$

where  $u$  is a complex-valued function of  $(t, x) \in \mathbf{R} \times \mathbf{R}^n$ ,  $\partial_t = \partial/\partial t$ ,  $\Delta$  denotes the Laplacian of spatial variables,  $f$  is a complex function, and

$$\dot{H}^\mu(\mathbf{R}^n) \equiv (-\Delta)^{-\mu/2} L^2(\mathbf{R}^n)$$

is the homogeneous Sobolev space of order  $\mu$ . We prove the global solvability of (1.1) in the critical Sobolev space  $\dot{H}^{n/2}$  with nonlinearity  $f$  of exponential type, a typical example of which is given by  $u^2 e^{\lambda|u|^2}$  with  $\lambda > 0$ .

There is a large literature on the Cauchy problem for the equation (1.1), see for instance [2,5,7,9,10,12,13,16,17,21,22]. It is well-known that the Cauchy problem (1.1) is locally well-posed in the usual Sobolev space  $H^\mu(\mathbf{R}^n)$  if  $\mu > n/2$  and  $f$  is any smooth function with  $f(0) = 0$  [16], or if  $1/2 \leq \mu < n/2$  and  $f$  is given as a single power nonlinearity  $\lambda|u|^{p-1}u$  with  $\lambda \in \mathbf{C}$  and  $p \leq 1 + 4/(n-2\mu)$  [9, 13, 17, 22]. Moreover if  $p = 1 + 4/(n-2\mu)$  and  $1/2 \leq \mu < n/2$ , then we have global  $\dot{H}^\mu$ -solutions with the Cauchy data sufficiently small [13, 17]. The same situation happens for the nonlinear Schrödinger equations as well [3, 4, 11, 15, 23]. The critical power  $p = 1 + 4/(n-2\mu)$  at the level of  $\dot{H}^\mu$  naturally

---

\*JSPS Fellow

arises by the standard scaling argument and is consistent with the Sobolev embedding  $\dot{H}^\mu \hookrightarrow L^{n(p-1)/2}$  for  $\mu = n/2 - 2/(p-1)$ . The last embedding, however, breaks down for  $\mu = n/2$ . In view of the circumstantial evidence above, it seems natural to call  $n/2$  another critical exponent and accordingly, in this paper we are interested in the existence of solutions of (1.1) when  $\mu = n/2$  and especially in the admissible class of nonlinearities in that theory.  $\dot{H}^{n/2}$  solutions are of particular interest as finite energy and strong solutions when  $n = 2$  and  $4$ , respectively.

To state our theorem, first we clarify the class of nonlinear terms.

**Definition** Let  $n \geq 2$ . We define a class of functions  $G$  as follows. We say that  $f \in G$  if  $f$  satisfies  $f \in C^{[n/2]}(\mathbf{R}^2, \mathbf{C})$  and  $f(0) = 0$ , and there exists some positive number  $\lambda$  such that the following estimates hold for all  $z \in \mathbf{C}$ :

$$\begin{aligned} |f'(z)| &\leq \begin{cases} C|z|^4 e^{\lambda|z|^2} & \text{for } n = 2; \\ C|z|^2 e^{\lambda|z|^2} & \text{for } n = 3; \end{cases} \\ |f'(z)| &\leq C|z|^2 e^{\lambda|z|^2}, \quad |f''(z)| \leq C|z| e^{\lambda|z|^2} \quad \text{for } n = 4; \\ |f'(z)| &\leq C|z| e^{\lambda|z|^2}, \quad \sup_{2 \leq k \leq [n/2]} |f^{(k)}(z)| \leq C e^{\lambda|z|^2} \quad \text{for } n \geq 5; \end{aligned} \quad (1.2)$$

where  $f$  is regarded as a function of  $z$  and  $\bar{z}$ ,  $f'(z) = (\partial f / \partial z, \partial f / \partial \bar{z})$ , for  $k \geq 2$   $f^{(k)}$  denotes any of the derivatives of  $f$  of  $k$ -th order with respect to  $z$  and  $\bar{z}$ , and  $[s]$  denotes the integer part of  $s \in \mathbf{R}$ .

To show the existence of the global solutions of (1.1) with  $f$  in  $G$ , we use the standard fixed point argument on the corresponding integral equation. Let  $1/q_0 = (n-1)/2(n+1)$ . We introduce the following complete metric space

$$\begin{aligned} X(R) &\equiv \{u \in L^\infty(\mathbf{R}, \dot{H}^{n/2}) \cap L^{q_0}(\mathbf{R}, B_{q_0}^{(n-1)/2} \cap \dot{B}_{q_0}^0) \mid \\ &\quad \max\{\|u; L^\infty(\mathbf{R}, \dot{H}^{n/2})\|, \|u; L^{q_0}(\mathbf{R}, B_{q_0}^{(n-1)/2})\|, \|u; L^{q_0}(\mathbf{R}, \dot{B}_{q_0}^0)\|\} \leq R\}, \end{aligned} \quad (1.3)$$

endowed with the metric

$$d(u, v) \equiv \|u - v; L^{q_0}(\mathbf{R}, \dot{B}_{q_0}^0)\|, \quad (1.4)$$

where  $B_q^s$  [resp.  $\dot{B}_q^s$ ] denotes the [resp. homogeneous] Besov space and we made abbreviation such as  $B_q^s = B_{q,2}^s(\mathbf{R}^n)$ ,  $\dot{B}_q^s = \dot{B}_{q,2}^s(\mathbf{R}^n)$ ,  $H^{s,q} = H^{s,q}(\mathbf{R}^n)$ ,  $\dot{H}^{s,q} = \dot{H}^{s,q}(\mathbf{R}^n)$ . For the definitions and its properties of Sobolev and Besov spaces, we refer to [1, 6, 27]. For convenience of description, we define the following space of the Cauchy data by

$$Y \equiv (\dot{H}^{n/2} \times \dot{H}^{n/2-1}) \cap (\dot{H}^{1/2} \times \dot{H}^{-1/2}) \quad (1.5)$$

$$\|(\phi, \psi)\|_Y \equiv \max\{\|\phi; \dot{H}^{n/2}\|, \|\phi; \dot{H}^{1/2}\|, \|\psi; \dot{H}^{n/2-1}\|, \|\psi; \dot{H}^{-1/2}\|\} \quad (1.6)$$

Now we state our theorem. In the following, we use the operators

$$K(t) \equiv (\sin t\sqrt{-\Delta})/\sqrt{-\Delta}, \quad \dot{K}(t) \equiv \cos t\sqrt{-\Delta}.$$

**Theorem 1.1** *Let  $n \geq 2$ . Let  $f \in G$ . Let  $(\phi, \psi) \in Y$  be sufficiently small. Then there exists  $R > 0$  such that (1.1) has a unique global solution in  $X(R)$ . Moreover the solution  $u$  belongs to  $(C^1 \cap L^\infty)(\mathbf{R}, \dot{H}^{n/2} \cap \dot{H}^{1/2})$  and depends on the initial data continuously. In addition, there exists a pair  $(\phi_+, \psi_+) \in Y$  such that*

$$\|(u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+, \partial_t(u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+))\|_Y \longrightarrow 0 \text{ as } t \rightarrow \infty.$$

*Conversely, for sufficiently small  $(\phi_-, \psi_-) \in Y$ , there exists a solution  $u_-$  of (1.1) in  $X(R)$  such that*

$$\|(u_-(t) - \dot{K}(t)\phi_- - K(t)\psi_-, \partial_t(u_-(t) - \dot{K}(t)\phi_- - K(t)\psi_-))\|_Y \longrightarrow 0 \text{ as } t \rightarrow -\infty.$$

*Moreover the scattering operator  $(\phi_-, \psi_-) \mapsto (\phi_+, \psi_+)$  is continuous with respect to the  $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ -norm.*

**Remark 1.** We now comment on the relation between the dimension and the vanishing order of the functions of (1.2) at the origin. As we mentioned above, for the existence of the global solutions it is natural to require the vanishing order at least the conformal power  $1+4/(n-1)$  which is identical to the requirement in (1.2) when  $n = 2, 3, 5$ . As we see below, we need to differentiate the nonlinearity  $f$  at least  $(n-1)/2$ -times and as far as  $f$  is supposed to behave as a power  $u^p$  at the origin, we must impose  $p \geq (n-1)/2$  except  $p$  is an integer. Therefore, if  $p$  is not an integer, then we are restricted to the case  $n \leq 5$ . We keep ourselves away from the resulting technical difficulty and assume sufficient smoothness of  $f$  at the origin.

**Remark 2.** An analogous result has been proved by the same authors for the nonlinear Schrödinger equations in the critical Sobolev space  $H^{n/2}$  [18].

**Remark 3.** In (1.4), the exponent  $q_0$  of the auxiliary function space is that of the Strichartz space-time estimate in the diagonal case [25]. In Theorem 1.1, as for the local well-posedness, the smallness assumption on the Cauchy data may be removed if  $f$  satisfies

$$\sup_{0 \leq k \leq [n/2]} |f^{(k)}(z)| \leq C|z|^m e^{\lambda|z|^{2-\epsilon}}, \quad 0 < \epsilon \leq 2.$$

for  $m$  sufficiently large (see [16] for details).

We prove Theorem 1.1 in the next section. The proof depends on the Strichartz estimate [8, 13, 25] and on the estimates on the nonlinear terms in the form of the power series expansion given by the RHS of (1.2). To ensure the convergence of the corresponding series of norms on  $L^q$  or  $\dot{B}_q^0$  with  $q \rightarrow \infty$ ,

we use the sharp Gagliardo-Nirenberg inequalities [18, 19]. Those inequalities are closely related to Trudinger's inequality [14, 19, 20, 24, 26, 28] and in this sense the power 2 in the exponential functions on the RHS of (1.2) also seems critical.

## 2 Proof of Theorem 1.1

In this section we prove Theorem 1.1. For  $(\phi, \psi) \in Y$  and  $u \in X(R)$  with  $R$  to be determined later, we define the operator  $\Phi$  by

$$\Phi(u)(t) \equiv \dot{K}(t)\phi + K(t)\psi + \int_0^t K(t-\tau)f(u(\tau))d\tau. \quad (2.7)$$

For the existence of the solutions of (1.1), it suffices to show that  $\Phi$  is a contraction map on  $X(R)$  for some  $R$ . By the Strichartz estimate and the standard duality argument, we have the following linear estimates

$$\begin{aligned} & \max\{\|\Phi(u); L^\infty(\mathbf{R}, \dot{H}^{n/2})\|, \|\Phi(u); L^{q_0}(\mathbf{R}, \dot{B}_{q_0}^{(n-1)/2})\|\} \\ & \leq C(\|\phi; \dot{H}^{n/2}\| + \|\psi; \dot{H}^{n/2-1}\| + \|f(u); L^{q_0'}(\mathbf{R}, \dot{B}_{q_0'}^{(n-1)/2})\|), \end{aligned} \quad (2.8)$$

$$\begin{aligned} & \|\Phi(u); L^{q_0}(\mathbf{R}, \dot{B}_{q_0}^0)\| \\ & \leq C(\|\phi; \dot{H}^{1/2}\| + \|\psi; \dot{H}^{-1/2}\| + \|f(u); L^{q_0'}(\mathbf{R}, \dot{B}_{q_0'}^0)\|), \end{aligned} \quad (2.9)$$

where  $1/q_0' \equiv 1 - 1/q_0$ , and  $C$  is independent of  $\phi, \psi, f$  and  $u$  (see also [8]). In (2.9), we may replace  $\dot{B}_{q_0}^0$  and  $\dot{B}_{q_0'}^0$  with  $L^{q_0}$  and  $L^{q_0'}$  respectively using the embedding  $\dot{B}_r^0 \hookrightarrow L^r$  for  $2 \leq r < \infty$ . Regarding the norms of  $f(u)$  on RHS in (2.8) and (2.9), we have the following lemma.

**Lemma 2.1** *Let  $f \in G$ . Then there exists a monotone increasing function  $\rho$  on  $\mathbf{R}$  such that for any  $u$  and  $v$  in  $X(R)$*

$$\|f(u); L^{q_0'}(\mathbf{R}, L^{q_0'})\| \leq C\rho(R)R^2, \quad (2.10)$$

$$\|f(u); L^{q_0'}(\mathbf{R}, \dot{B}_{q_0'}^{(n-1)/2})\| \leq C\rho(R)R^2, \quad (2.11)$$

$$\|f(u) - f(v); L^{q_0'}(\mathbf{R}, L^{q_0'})\| \leq C\rho(R)R\|u - v; L^{q_0}(\mathbf{R}, L^{q_0})\|, \quad (2.12)$$

where  $C$  is independent of  $u$  and  $v$ .

*Proof of Lemma 2.1*

We prove Lemma 2.1 for  $n \geq 5$ , since the lemma for  $n = 2, 3, 4$  could be shown quite analogously. First we prove (2.10). We recall the inequalities

$$\|u; L^q\| \leq Cq^{1/2+(r-2)/2q}\|u; \dot{H}^{n/2}\|^{1-r/q}\|u; L^r\|^{r/q}, \quad (2.13)$$

$$\|u; \dot{B}_q^0\| \leq Cq^{1/2+(r-2)/2q}\|u; \dot{H}^{n/2}\|^{1-r/q}\|u; \dot{B}_r^0\|^{r/q}, \quad (2.14)$$

for any  $1 < r \leq q < \infty$ , where  $C$  is independent of  $q$  and  $u$  (see [18, 19]). By expanding the RHS of (1.2) and estimating the resulting power series by the Hölder inequality in space and (2.13), we have

$$\begin{aligned} & \|f(u); L^{q_0'}\| \\ & \leq C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \|u; L^{r^*}\|^{2j+1} \|u; L^{q_0}\| \end{aligned} \quad (2.15)$$

$$\begin{aligned} & \leq C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} r^{*(1/2+(q_0-2)/2r^*)(2j+1)} \|u; \dot{H}^{n/2}\|^{(1-q_0/r^*)(2j+1)} \\ & \quad \cdot \|u; L^{q_0}\|^{1+(2j+1)q_0/r^*}, \end{aligned} \quad (2.16)$$

where  $r^* = (n+1)(2j+1)/2$  and  $C$  is independent of  $u$ . Therefore we have by the Hölder inequality in time

$$\|f(u); L^{q_0'}(\mathbf{R}, L^{q_0'})\| \leq C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} r^{*(1/2+(q_0-2)/2r^*)(2j+1)} R^{2j+2}. \quad (2.17)$$

Since the series in (2.17) converges for sufficiently small  $R$ , we shall regard it as  $\rho(R)R^2$ . This completes the proof of (2.10). The inequality (2.12) follows analogously if we use the equality

$$f(u) - f(v) = \int_0^1 f'(v + \theta(u-v))(u-v) d\theta.$$

Next we show (2.11) for  $n \geq 5$  odd. By the embedding  $\dot{H}_{q_0'}^{(n-1)/2} \hookrightarrow \dot{B}_{q_0'}^{(n-1)/2}$ , we have

$$\|f(u); \dot{B}_{q_0'}^{(n-1)/2}\| \leq C \sum_{k=1}^{(n-1)/2} \sum_{|\alpha|=(n-1)/2} \sum_{\substack{\beta_1+\dots+\beta_k=\alpha \\ |\beta_i| \geq 1}} \|f^{(k)}(u) \prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'}\|. \quad (2.18)$$

For  $k \geq 2$  and  $1 \leq i \leq k$ , let  $1/r^*$  and  $1/r_i^*$  be

$$1/r^* \equiv 2/(n+1)(2j+k-1), \quad 1/r_i^* \equiv (1-2|\beta_i|/(n-1))/r^* + 2|\beta_i|/(n-1)q.$$

Then we have by the interpolation of the Besov space

$$\|\partial^{\beta_i} u; L^{r_i^*}\| \leq C \|u; \dot{B}_{r^*}^0\|^{1-2|\beta_i|/(n-1)} \|u; \dot{B}_{q_0}^{(n-1)/2}\|^{2|\beta_i|/(n-1)} \quad \text{for } 1 \leq i \leq k, \quad (2.19)$$

where  $C$  is independent of  $j$  and  $u$  (see [6, Lemma A.1]). Therefore we have for  $k \geq 2$

$$\|f^{(k)}(u) \prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'}\| \leq C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \|u; \dot{B}_{r^*}^0\|^{2j+k-1} \|u; \dot{B}_{q_0}^{(n-1)/2}\|, \quad (2.20)$$

which corresponds to (2.15), and by the same argument as above the inequality (2.11) follows, where the estimate on the terms with  $k = 1$  is similar and simpler.

For  $n \geq 6$  even, we use the following equivalent norm on the homogeneous Besov space

$$\|u; \dot{B}_q^{(n-1)/2}\| \simeq \sum_{|\alpha|=(n-2)/2} \left\{ \int_0^\infty t^{-2} \sup_{|y|<t} \|\partial^\alpha u - \partial^\alpha \tau_y u; L^q\|^2 dt \right\}^{1/2}, \quad (2.21)$$

where  $\tau_y$  is the shift function by  $y \in \mathbf{R}^n$ . By (2.21), we have

$$\begin{aligned} & \|f(u); \dot{B}_{q_0'}^{(n-1)/2}\| \\ & \leq C \sum_{k=1}^{(n-2)/2} \sum_{|\alpha|=(n-1)/2} \sum_{\substack{\beta_1+\dots+\beta_k=\alpha \\ |\beta_i| \geq 1}} \left\{ \int_0^\infty t^{-2} \sup_{|y|<t} \|f^{(k)}(u) \prod_{i=1}^k \partial^{\beta_i} u \right. \\ & \quad \left. - f^{(k)}(\tau_y u) \prod_{i=1}^k \partial^{\beta_i} \tau_y u; L^{q_0'}\|^2 dt \right\}^{1/2}. \end{aligned} \quad (2.22)$$

Here we estimate (2.22) as follows. Let  $1/r^*$ ,  $1/\hat{r}$ ,  $1/r_i^*$  be

$$\begin{aligned} 1/r^* & \equiv 2/(n+1)(2j+k), \quad 1/\hat{r} \equiv (1 - 1/(n-1))/r^* + 1/(n-1)q_0, \\ 1/r_i^* & \equiv (1 - 2|\beta_i|/(n-1))/r^* + 2|\beta_i|/(n-1)q_0 \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

Then we have by the Hölder inequality and an estimate similar to (2.19)

$$\begin{aligned} & \| (f^{(k)}(u) - f^{(k)}(\tau_y u)) \prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'} \| \\ & \leq C \sum_{j=0}^\infty \frac{\lambda^j}{j!} \| |u|^{2j} (u - \tau_y u) \prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'} \| \\ & \leq C \sum_{j=0}^\infty \frac{\lambda^j}{j!} \|u; \dot{B}_{r^*}^0\|^{2j+k-(n-2)/(n-1)} \|u; \dot{B}_{q_0}^{(n-1)/2}\|^{(n-2)/(n-1)} \\ & \quad \cdot \|u - \tau_y u; L^{\hat{r}}\|. \end{aligned} \quad (2.23)$$

Therefore we have

$$\begin{aligned} & \left\{ \int_0^\infty t^{-2} \sup_{|y|<t} \| (f^{(k)}(u) - f^{(k)}(\tau_y u)) \prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'} \|^2 dt \right\}^{1/2} \\ & \leq C \sum_{j=0}^\infty \frac{\lambda^j}{j!} \|u; \dot{B}_{r^*}^0\|^{2j+k} \|u; \dot{B}_{q_0}^{(n-1)/2}\|. \end{aligned} \quad (2.24)$$

For  $k \geq 2$ , let  $1/r^* \equiv 2/(n+1)(2j+k-1)$  and let

$$\begin{aligned} 1/\hat{r} &\equiv (1 - (2|\beta_1| + 1)/(n-1))/r^* + (2|\beta_1| + 1)/(n-1)q_0, \\ 1/\tau_i^* &\equiv (1 - 2|\beta_i|/(n-1))/r^* + 2|\beta_i|/(n-1)q_0 \quad \text{for } 2 \leq i \leq k. \end{aligned}$$

Then we have by an estimate similar to (2.23)

$$\begin{aligned} & \|f^{(k)}(u)(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u) \prod_{i=2}^k \partial^{\beta_i}u; L^{q_0'}\| \\ & \leq C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \|u; \dot{B}_{r^*}^0\|^{2j+k-1-(n-2-2|\beta_1|)/(n-1)} \\ & \quad \cdot \|u; \dot{B}_{q_0}^{(n-1)/2}\|^{(n-2-2|\beta_1|)/(n-1)} \|\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u; L^{\hat{r}}\|. \end{aligned} \quad (2.25)$$

Therefore we have

$$\begin{aligned} & \left\{ \int_0^{\infty} t^{-2} \sup \|f^{(k)}(u)(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u) \prod_{i=2}^k \partial^{\beta_i}u; L^{q_0'}\|^2 dt \right\}^{1/2} \\ & \leq C \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \|u; \dot{B}_{r^*}^0\|^{2j+k-1} \|u; \dot{B}_{q_0}^{(n-1)/2}\|. \end{aligned} \quad (2.26)$$

The inequalities (2.24) and (2.26) correspond to (2.15), so that we have the required inequality (2.11), where the estimate on the terms with  $k = 1$  is again similar and simpler. This completes the proof of lemma 2.1.  $\square$

We now turn to the proof of Theorem 1.1. By (2.8), (2.9) and Lemma 2.1, we have

$$\begin{aligned} & \max\{\|\Phi(u); L^\infty(\mathbf{R}, \dot{H}^{n/2})\|, \|\Phi(u); L^q(\mathbf{R}, B_q^{(n-1)/2})\|, \|\Phi(u); L^q(\mathbf{R}, \dot{B}_q^0)\|\} \\ & \leq C\|(\phi, \psi)\|_Y + C\rho(R)R^2, \end{aligned} \quad (2.27)$$

$$d(\Phi(u), \Phi(v)) \leq C\rho(R)Rd(u, v), \quad (2.28)$$

for  $u, v \in X(R)$ , where  $C$  is independent of  $u$  and  $v$ . Therefore  $\Phi$  becomes a contraction map on  $X(R)$  if  $\|(\phi, \psi)\|_Y$  and  $R$  are sufficiently small. The existence of asymptotic states and the continuity of the scattering operator follow by the standard argument (see [17]).  $\square$

## References

- [1] J. Bergh, J. Löfström, "Interpolation Spaces," Springer-Verlag, Berlin-Heidelberg-New York, 1976.

- [2] J.Ginibre, *Scattering theory in the energy space for a class of nonlinear wave equation*, Advanced Studies in Pure Mathematics., **23**(1994), 83-103.
- [3] J.Ginibre, *An Introduction to Nonlinear Schrödinger Equations*, in *Nonlinear Waves*, Gakuto International Series, Mathematical Sciences and Applications, **10**(1997), 85-133.
- [4] J.Ginibre, T.Ozawa, G.Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations*, Ann.Inst.Henri Poincaré, Physique théorique, **60**(1994), 211-239.
- [5] J.Ginibre, A.Soffer, G.Velo, *The global Cauchy problem for the critical non-linear wave equation*, J. Funct. Anal., **110**(1992), 96-130.
- [6] J.Ginibre, G.Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math.Z., **189**(1985), 487-505.
- [7] J.Ginibre, G.Velo, *Regularity of solutions of critical and subcritical nonlinear wave equations*, Nonlinear Analysis, Theory, Methods & Applications, **22**(1994), No.1, 1-19.
- [8] J.Ginibre, G.Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal., **133**(1995), 50-68.
- [9] L.V.Kapitanski, *Weak and yet weaker solutions of semilinear wave equations*, COMMUN.IN PARTIAL DIFFERENTIAL EQUATIONS, **19**(1994), 1629-1676.
- [10] L.V.Kapitanski, *Global and unique weak solutions of nonlinear wave equations*, Mathematical Research Letters, **1**(1994), 211-223.
- [11] T.Kato, *On nonlinear Schrödinger equations II.  $H^s$ -solutions and unconditional well-posedness*, J.d'Anal.Math., **67**(1995), 281-306.
- [12] H.Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations*, Duke Math.J., **72**(1993), 503-539.
- [13] H.Lindblad, D.Sogge *On existence and scattering with minimal regularity for semilinear wave equations*, J.Funct.Anal., **130**(1995), 357-426.
- [14] J.Moser, *A sharp form of an inequality by N.Trudinger*, Indiana Univ.Math.J., **20**(1971), 1077-1092.
- [15] M.Nakamura, T.Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces*, Rev. Math. Phys., **9**(1997), No 3, 397-410.
- [16] M.Nakamura, T.Ozawa, *The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order*, (preprint).

- [17] M.Nakamura, T.Ozawa, *The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space*, Ann.Inst.Henri Poincaré, Physique théorique,(in press).
- [18] M.Nakamura, T.Ozawa, *Nonlinear Schrödinger equations in the Sobolev space of critical order*, (preprint).
- [19] T.Ozawa, *On critical cases of Sobolev's inequalities*, J.Funct.Anal., **127**(1995), 259-269.
- [20] T.Ozawa, *Characterization of Trudinger's inequality*, J.of Inequal.& Appl.,**1**(1997), 369-374.
- [21] H.Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math.Z., **185**(1984), 261-270.
- [22] H.Pecher, *Local solutions of semilinear wave equations in  $H^{s+1}$* , Mathematical Methods in the Applied Sciences, **19**(1996), 145-170.
- [23] H.Pecher, *Solutions of semilinear Schrödinger equations in  $H^s$* , Ann.Inst.Henri Poincaré, Physique théorique, **67**(1997), 259-296.
- [24] R.S.Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ.Math.J., **21**(1972), 841-842.
- [25] R.S.Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math.J., **44**(1977), 705-714.
- [26] M.Struwe, *Critical points of embeddings of  $H_0^{1,n}$  into Orlicz spaces*, Ann.Inst.Henri Poincaré, Analyse nonlinéaire,**5**(1988), 425-464.
- [27] H.Triebel, "Theory of Function Spaces," Birkhäuser, Basel, 1983.
- [28] N.S.Trudinger, *On imbeddings into Orlicz spaces and some applications*, J.Math.Mech.,**17**(1967), 473-483.

# WHEN ARE THE ISOTHERMAL SURFACES INVARIANT WITH RESPECT TO THE TIME VARIABLE?

SHIGERU SAKAGUCHI

DEPARTMENT OF MATHEMATICAL SCIENCES  
FACULTY OF SCIENCE  
EHIME UNIVERSITY  
MATSUYAMA 790-8577 JAPAN  
E-MAIL: SAKAGUCH@DPC.EHIME-U.AC.JP

**1. Introduction.** This note is based on the author's recent work [7]. There are some symmetry results of Alessandrini [1, 2], some of which proved a conjecture of Klamkin [3] ( see also [12] ). We quote a theorem of [2] ( see [2, Theorem 1.3, p. 254] ).

**Theorem A ( Alessandrini ).** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) with boundary  $\partial\Omega$  and let all of its boundary points be regular with respect to the Laplacian. Let  $\varphi \in L^2(\Omega)$  satisfy  $\varphi \not\equiv 0$  and let  $u = u(x, t)$  be the unique solution of*

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = \varphi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \quad (1.1)$$

*If there exists  $\tau > 0$  such that, for every  $t > \tau$ ,  $u(\cdot, t)$  is constant on every level surface  $\{ x \in \Omega ; u(x, \tau) = \text{const.} \}$  of  $u(\cdot, \tau)$  in  $\Omega$ , then one of the following two cases occurs.*

- (i)  $\varphi$  is an eigenfunction of  $-\Delta$  under the homogeneous Dirichlet boundary condition.
- (ii)  $\Omega$  is a ball,  $u(\cdot, t)$  is radially symmetric for each  $t \geq 0$ , and  $u$  never vanishes in  $\Omega \times [\tau, \infty)$ .

Klamkin's conjecture was that if all the spatial level surfaces of the solution  $u$  of (1.1) are invariant with respect to the time variable  $t$  for positive constant initial data, then the domain must be a ball. Therefore Theorem A proved the Klamkin's conjecture [3].

In this note we consider the similar problem under the homogeneous Neumann boundary condition. Our result is:

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

**Theorem 1.** Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) with boundary  $\partial\Omega$ , and let  $\varphi \in L^2(\Omega)$  satisfy  $\varphi \neq 0$  and  $\int_{\Omega} \varphi \, dx = 0$ . Let  $u = u(x, t)$  be the unique solution of the following initial-Neumann problem:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = \varphi(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \quad (1.2)$$

where  $\nu$  denotes the exterior normal unit vector to  $\partial\Omega$ . If there exists  $\tau > 0$  such that, for every  $t > \tau$ ,  $u(\cdot, t)$  is constant on every level surface  $\{x \in \Omega; u(x, \tau) = \text{const.}\}$  of  $u(\cdot, \tau)$  in  $\Omega$ , then one of the following two cases occurs.

- (i)  $\varphi$  is an eigenfunction of  $-\Delta$  under the homogeneous Neumann boundary condition.
- (ii) By a rotation and a translation of coordinates we have one of the following (a) and (b):

(a) There exists a finite interval  $(a, b)$  such that  $u$  is extended as a function of  $x_1$  and  $t$  only, say  $u = u(x_1, t)$  ( $(x_1, t) \in [a, b] \times (0, \infty)$ ), and there exist an integer  $n \geq 1$  and a finite sequence  $\{s_j\}_{j=0}^n$  satisfying

$$s_0 = a, \quad s_n = b, \quad \text{and} \quad s_{j+1} - s_j = \frac{b-a}{n} \quad \text{for } 0 \leq j \leq n-1,$$

such that  $\frac{\partial u}{\partial x_1}$  never vanishes in  $\bigcup_{j=0}^{n-1} (s_j, s_{j+1}) \times (\tau, \infty)$  and it vanishes on  $\{s_j\}_{j=0}^n \times (0, \infty)$ . When  $n \geq 2$ ,  $u$  is symmetric with respect to hyperplane  $\{x \in \mathbb{R}^N; x_1 = s_j\}$  for each  $1 \leq j \leq n-1$ . Furthermore, the boundary  $\partial\Omega$  consists of at most the following:

- (a-1) a part of hyperplane  $\{x \in \mathbb{R}^N; x_1 = b\}$ ,
- (a-2) a part of hyperplane  $\{x \in \mathbb{R}^N; x_1 = a\}$ ,
- (a-3) a part of hyperplane  $\{x \in \mathbb{R}^N; x_1 = s_j\}$  for each  $1 \leq j \leq n-1$  when  $n \geq 2$ ,
- (a-4) a collection of straight line segments  $\ell$ 's given by

$$\ell = \{x \in \mathbb{R}^N; x = (x_1, y') \text{ and } s_j \leq x_1 \leq s_{j+1}\},$$

where  $y'$  is a point in  $\mathbb{R}^{N-1}$  and  $0 \leq j \leq n-1$ .

Here (a-1), (a-2), and (a-4) are nonempty and there is a case where (a-3) is empty.

(b) There exist a finite interval  $(a, b)$  with  $a \geq 0$  and a natural number  $k$  with  $2 \leq k \leq N$  such that  $u$  is extended as a function of  $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$  and  $t$  only, say  $u = u(r, t)$  ( $(r, t) \in [a, b] \times (0, \infty)$ ), whose derivative  $\frac{\partial u}{\partial r}(r, t)$  never vanishes in  $(a, b) \times (\tau, \infty)$  and it vanishes on  $\{a, b\} \times (0, \infty)$ . Furthermore, when  $2 \leq k \leq N-1$ , the boundary  $\partial\Omega$  consists of the following:

- (b-1) a part of hypersurface  $\{x \in \mathbb{R}^N; r = b\}$ ,
- (b-2) a part of hypersurface  $\{x \in \mathbb{R}^N; r = a\}$  when  $a > 0$ ,

(b-3) a collection of straight line segments  $\ell'$ s given by

$$\ell = \{ x \in \mathbb{R}^N ; (x_1, \dots, x_k) = r\omega, a \leq r \leq b, \text{ and } (x_{k+1}, \dots, x_N) = y' \},$$

where  $y'$  is a point in  $\mathbb{R}^{N-k}$  and  $\omega$  is a point in the  $k-1$ -dimensional unit sphere  $S^{k-1}$  in  $\mathbb{R}^k$ .

When  $k = N$ , there exists a Lipschitz domain  $S$  in  $S^{N-1}$  ( $S$  can be the whole sphere  $S^{N-1}$ ) such that  $\Omega = \{ r\omega \in \mathbb{R}^N ; r \in (a, b) \text{ and } \omega \in S \}$  when  $a > 0$ , and either  $\Omega = \{ r\omega \in \mathbb{R}^N ; r \in (0, b) \text{ and } \omega \in S \}$  with  $S \neq S^{N-1}$  or  $\Omega = \{ x \in \mathbb{R}^N ; r < b \}$  when  $a = 0$ .

In particular, in case (ii), if  $\partial\Omega$  is  $C^1$ , then  $\Omega$  must be either a ball or an annulus.

Since any constant function is a trivial solution of the initial-Neumann problem (1.2) with constant initial data, and since adding any constant function to the solution  $u$  in Theorem 1 does not have any influence on the invariance condition of spatial level surfaces of  $u$ , so for simplicity we assumed that  $\varphi \neq 0$  and  $\int_{\Omega} \varphi \, dx = 0$  for initial data  $\varphi$ .

Alessandrini used an eigenfunction expansion and a special case of a well-known theorem of symmetry for elliptic equations of Serrin [10, Theorem 2, pp. 311-312] in order to prove Theorem A:

**Theorem S ( Serrin ).** *Let  $D$  be a bounded domain with  $C^2$  boundary  $\partial D$  and let  $v \in C^2(\overline{D})$  satisfy the following:*

$$\begin{cases} \Delta v = f(v) \text{ and } v > 0 \text{ in } D, \\ v = 0 \text{ and } \frac{\partial v}{\partial \nu} = c \text{ on } \partial D, \end{cases}$$

where  $f = f(s)$  is a  $C^1$  function of  $s$ ,  $c$  is a constant, and  $\nu$  denotes the exterior normal unit vector to  $\partial D$ . Then  $D$  is a ball and  $v$  is radially symmetric and decreasing in  $D$ .

Under the hypothesis that case (i) of Theorem A does not hold, Alessandrini showed that there exists a level set  $D = \{ x \in \Omega ; \psi(x) > s \}$  with  $s > 0$  of an eigenfunction  $\psi = \psi(x)$  of  $-\Delta$  under the homogeneous Dirichlet boundary condition such that the function  $v = \psi - s$  satisfies the overdetermined boundary conditions as in Theorem S. Then applying Theorem S to  $v$  implies that  $D$  is a ball and  $v$  is radially symmetric and decreasing in  $D$ . By a little more argument one gets case (ii) of Theorem A. In this proof essential is the fact that the boundary of  $D$  does not touch the boundary  $\partial\Omega$ . This fact comes from the homogeneous Dirichlet boundary condition of the eigenfunction  $\psi$ . Therefore, in our problem (1.2) we can not use Theorem S because of the homogeneous Neumann boundary condition. We overcome this obstruction by using the invariance condition of spatial level surfaces much more with the help of the classification theorem of *isoparametric hypersurfaces in Euclidean space* of Levi-Civita and Segre ( see [4, 9] ). Besides we can give another proof of Theorem A which does not depend on Theorem S.

In fact, the introduction of isoparametric surfaces was motivated by Somigliana [11] and Segre [8] in terms of similar questions of geometry of solutions of partial differential equations.

In Section 2 we give an outline of the proof of Theorem 1. For the details we refer to [7].

**2. Outline of the Proof of Theorem 1.** First of all, let us quote the classification theorem of isoparametric hypersurfaces in Euclidean space  $\mathbb{R}^N$ , which was proved by Levi-Civita [4] for  $N = 3$ , and by Segre [9] for arbitrary  $N$ . See [5, 6] for a survey of isoparametric surfaces.

**Theorem LcS ( Levi-Civita and Segre ).** *Let  $D$  be a bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and let  $f$  be a real-valued smooth function on  $D$  satisfying  $\nabla f \neq 0$  on  $D$ . Suppose that there exist two real-valued functions  $g = g(\cdot)$  and  $h = h(\cdot)$  of a real variable such that*

$$|\nabla f|^2 = g(f) \quad \text{and} \quad \Delta f = h(f) \quad \text{on } D. \quad (2.1)$$

*Then the family of level surfaces  $\{x \in D ; f(x) = s\}$  ( $s \in f(D)$ ) of  $f$  must be either parallel hyperplanes, concentric spheres, or concentric spherical cylinders.*

In this theorem the function  $f$  is called an *isoparametric function*, and the level surfaces of  $f$  are called isoparametric surfaces. For our use we assumed that the domain  $D$  is bounded.

Let us put  $u(x, \tau) = \psi(x)$  for  $x \in \overline{\Omega}$ . By the invariance condition of spatial level surfaces as in [1, (2.2), p. 231] we have:

$$u(x, t) = \mu(\psi(x), t) \quad \text{for any } (x, t) \in \overline{\Omega} \times [\tau, \infty) \quad (2.2)$$

for some function  $\mu = \mu(s, t) : \mathbb{R} \times [\tau, \infty) \rightarrow \mathbb{R}$  satisfying

$$\mu(s, \tau) = s \quad \text{for any } s \in \mathbb{R}. \quad (2.3)$$

Since  $\psi$  is not constant, there exist a point  $x_0 \in \Omega$  and an open ball  $B$  in  $\mathbb{R}^N$  centered at  $x_0$  such that

$$\nabla \psi \neq 0 \quad \text{on } \overline{B}(\subset \Omega). \quad (2.4)$$

Then by a standard difference quotient argument ( see [1, Lemma 1, p.232] and [2, Lemma 2.1, p. 255] ) we have:

**Lemma 2.1.** *There exists an interval  $I = [\psi(x_0) - \delta, \psi(x_0) + \delta]$  with some  $\delta > 0$  such that  $I \subset \psi(B)$  and  $\mu \in C^\infty(I \times [\tau, \infty))$ .*

In view of Lemma 2.1 we can substitute (2.2) into the heat equation and get

$$\mu_t = \operatorname{div} (\mu_s \nabla \psi) = \mu_s \Delta \psi + \mu_{ss} |\nabla \psi|^2 \quad \text{on } \psi^{-1}(I) \times [\tau, \infty), \quad (2.5)$$

where  $\psi^{-1}(I) = \{x \in \Omega ; \psi(x) \in I\}$ . Differentiating (2.5) with respect to  $t$  yields

$$\mu_{tt} = \mu_{st} \Delta \psi + \mu_{sst} |\nabla \psi|^2 \quad \text{on } \psi^{-1}(I) \times [\tau, \infty). \quad (2.6)$$

Let us introduce the function  $\mathfrak{D}$  by

$$\mathfrak{D} = \det \begin{pmatrix} \mu_s & \mu_{ss} \\ \mu_{st} & \mu_{sst} \end{pmatrix} \equiv \mu_s \mu_{sst} - \mu_{ss} \mu_{st}. \quad (2.7)$$

We distinguish the following two cases:

- (1)  $\mathfrak{D} \equiv 0$  on  $I \times [\tau, \infty)$ ,
- (2)  $\mathfrak{D} \neq 0$  on  $I \times [\tau, \infty)$ .

Remark that these cases are slightly different from the cases in the paper [1] where the time is fixed, that is,  $t = \tau$ .

**Case (1).** In this case let us show that the solution  $u$  must be a separable solution, which implies case (i) of Theorem 1. It follows from (2.3) that  $\mu_s(s, \tau) \equiv 1$ . Therefore there exists a time  $T_1 > \tau$  such that

$$\mu_s > 0 \quad \text{on } I \times [\tau, T_1].$$

Hence we have

$$(\log \mu_s)_{st} = \mathfrak{D}/(\mu_s)^2 = 0 \quad \text{on } I \times [\tau, T_1].$$

By solving this equation with  $\mu_s(s, \tau) \equiv 1$  and using the analyticity in  $x$  of  $u$  and the fact that  $\int_{\Omega} u(x, t) dx = 0$  for any  $t > 0$ , we can show that  $u$  is a separable solution.

**Case (2).** In this case, by supposing that case (i) of Theorem 1 does not hold, we show that case (ii) holds. It follows from the continuity of  $\mathfrak{D}$  that there exist a nonempty open subinterval  $J \subset I$  and a time  $t_0 \geq \tau$  such that  $\mathfrak{D} \neq 0$  on  $\bar{J} \times \{t_0\}$ . Hence we can solve equations (2.5) and (2.6) with respect to  $|\nabla\psi|^2$  and  $\Delta\psi$  for  $(x, t_0) \in \psi^{-1}(\bar{J}) \times \{t_0\}$ . Namely, there exists a nonempty bounded domain  $D \subset \psi^{-1}(\bar{J}) \subset \Omega$  in  $\mathbb{R}^N$  such that

$$|\nabla\psi|^2 = g(\psi) \quad \text{and} \quad \Delta\psi = h(\psi) \quad \text{on } D \tag{2.8}$$

for some functions  $g$  and  $h$  as in (2.1). Then, since  $\psi$  is analytic in  $\Omega$ , it follows from (2.2) and Theorem LcS that after a rotation and a translation of coordinates there exists a finite interval  $(a, b)$  such that we have one of the following (a) and (b):

- (a)  $u$  is extended as a function of  $x_1$  and  $t$  only, say  $u = u(x_1, t)$  ( $(x_1, t) \in [a, b] \times [\tau, \infty)$ ). Furthermore,  $\Omega \subset (a, b) \times \mathbb{R}^{N-1}$  with  $\partial\Omega \cap (\{a\} \times \mathbb{R}^{N-1}) \neq \emptyset$  and  $\partial\Omega \cap (\{b\} \times \mathbb{R}^{N-1}) \neq \emptyset$ .
- (b)  $a \geq 0$  and there exists a natural number  $k$  with  $2 \leq k \leq N$  such that  $u$  is extended as a function of  $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$  and  $t$  only, say  $u = u(r, t)$  ( $(r, t) \in [a, b] \times [\tau, \infty)$ ). Furthermore, when  $a > 0$ ,  $\Omega \subset \{(x_1, \dots, x_k) \in \mathbb{R}^k; a < r < b\} \times \mathbb{R}^{N-k}$  with  $\partial\Omega \cap (\{(x_1, \dots, x_k) \in \mathbb{R}^k; r = a\} \times \mathbb{R}^{N-k}) \neq \emptyset$  and  $\partial\Omega \cap (\{(x_1, \dots, x_k) \in \mathbb{R}^k; r = b\} \times \mathbb{R}^{N-k}) \neq \emptyset$ , and when  $a = 0$ ,  $\Omega \subset \{(x_1, \dots, x_k) \in \mathbb{R}^k; 0 \leq r < b\} \times \mathbb{R}^{N-k}$  with  $\bar{\Omega} \cap (\{0\} \times \mathbb{R}^{N-k}) \neq \emptyset$  and  $\partial\Omega \cap (\{(x_1, \dots, x_k) \in \mathbb{R}^k; r = b\} \times \mathbb{R}^{N-k}) \neq \emptyset$ . Here, when  $k = N$ ,  $\mathbb{R}^{N-k}$  is disregarded.

Here we have:

**Lemma 2.2.** *In case (b)  $u(r, \tau)(= \psi(r))$  is monotone on  $[a, b]$ , provided case (i) of Theorem 1 does not hold.*

Since  $\partial\Omega$  is only Lipschitz continuous, we need careful consideration to prove Theorem 1. Since problem (1.2) is solved by an eigenfunction expansion, we see that  $u = u(x_1, t)$  ( $(x_1, t) \in [a, b] \times [0, \infty)$ ) in case (a) and  $u = u(r, t)$  ( $(r, t) \in [a, b] \times [0, \infty)$ ) in case (b).

Let us consider case (b) first. Lemma 2.2 implies that either  $\psi' \geq 0$  or  $\psi' \leq 0$ . Consider the case where  $\psi' \geq 0$ . Since  $\psi$  is analytic and not constant, there exists a sequence of positive numbers  $\{\varepsilon_j\}_{j=1}^{\infty}$  with  $\varepsilon_j \downarrow 0$  as  $j \uparrow \infty$  such that

$$\psi'(a + \varepsilon_j) > 0 \quad \text{and} \quad \psi'(b - \varepsilon_j) > 0 \quad \text{for any } j \geq 1.$$

By continuity we see that for each  $j \geq 1$  there exists  $\tau_j > \tau$  satisfying

$$\partial_r u > 0 \quad \text{on} \quad \{a + \varepsilon_j, b - \varepsilon_j\} \times [\tau, \tau_j].$$

Hence, since  $\psi' \geq 0$ , it follows from the strong maximum principle that for each  $j \geq 1$   $\partial_r u > 0$  in  $[a + \varepsilon_j, b - \varepsilon_j] \times (\tau, \tau_j]$ . By dealing with the case where  $\psi' \leq 0$  similarly, we conclude that there exist two sequences  $\{\varepsilon_j\}_{j=1}^{\infty}$  and  $\{\tau_j\}_{j=1}^{\infty}$  with  $\varepsilon_j \downarrow 0$  as  $j \uparrow \infty$  and  $\tau_j > \tau$  such that for each  $j \geq 1$

$$\partial_r u \neq 0 \quad \text{in} \quad [a + \varepsilon_j, b - \varepsilon_j] \times (\tau, \tau_j]. \quad (2.9)$$

Since  $u$  satisfies the homogeneous Neumann boundary condition, this will determine the boundary  $\partial\Omega$ . If  $\partial\Omega \cap \{x \in \mathbb{R}^N; r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}} \in (a, b)\} = \emptyset$ , then  $k = N$  and  $\Omega$  must be either a ball or an annulus in  $\mathbb{R}^N$ . So, let us consider the case where  $\partial\Omega \cap \{x \in \mathbb{R}^N; r \in (a, b)\} \neq \emptyset$ . Take sufficiently large  $j \geq 1$  and take an arbitrary point  $x^* \in \partial\Omega$  with  $r = ((x_1^*)^2 + \cdots + (x_k^*)^2)^{\frac{1}{2}} \in (a + \varepsilon_j, b - \varepsilon_j)$ . Since  $\Omega$  is a bounded Lipschitz domain, we can find an orthogonal matrix  $\mathfrak{R} = (r_{ij})$  and a neighborhood  $V$  of  $x^*$  in  $\mathbb{R}^N$  with  $V \subset \{x \in \mathbb{R}^N; r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}} \in (a + \varepsilon_j, b - \varepsilon_j)\}$ , and by introducing the rotation of coordinates:  $z = \mathfrak{R}x$ , we may have in  $z$ -coordinates

$$\begin{cases} \partial\Omega \cap V = \{z = (\tilde{z}, \phi(\tilde{z})) \in \mathbb{R}^N; \tilde{z} = (z_1, \dots, z_{N-1}) \in \tilde{B}\}, \\ \Omega \cap V = \{z = (\tilde{z}, z_N) \in \mathbb{R}^N; c < z_N < \phi(\tilde{z}) \text{ and } \tilde{z} \in \tilde{B}\}, \\ V = \tilde{B} \times (c, d), \end{cases} \quad (2.10)$$

where  $\tilde{B}$  is an open ball in  $\mathbb{R}^{N-1}$ ,  $(c, d)$  is a bounded open interval, and  $\phi$  is a Lipschitz continuous function on  $\tilde{B}$ . Then, by Rademacher's theorem on the almost everywhere total differentiability of Lipschitz functions (see [13, Theorem 2.21, p. 50] for example) we see that the exterior unit normal vector  $\nu$  to  $\partial\Omega$  is given by

$$\nu(\tilde{z}, \phi(\tilde{z})) = (1 + |\nabla_{\tilde{z}}\phi(\tilde{z})|^2)^{-\frac{1}{2}}(-\nabla_{\tilde{z}}\phi(\tilde{z}), 1) \quad \text{for almost every } \tilde{z} \in \tilde{B}, \quad (2.11)$$

where  $\nabla_{\tilde{z}}\phi = (\partial_{z_1}\phi, \dots, \partial_{z_{N-1}}\phi)$ . Therefore it follows from the homogeneous Neumann boundary condition that for any  $t \in (\tau, \tau_j]$

$$(-\nabla_{\tilde{z}}\phi(\tilde{z}), 1) \cdot \nabla_z(u(\mathfrak{R}^*z, t)) = 0 \quad \text{for almost every } \tilde{z} \in \tilde{B}, \quad (2.12)$$

where  $z = (\tilde{z}, \phi(\tilde{z}))$  and  $\mathfrak{R}^*$  denotes the transposed matrix of  $\mathfrak{R}$ . Since  $u = u(r, t)$  with  $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ , so

$$\nabla_x u = \frac{\partial_r u(r, t)}{r}(x_1, \dots, x_k, 0, \dots, 0). \quad (2.13)$$

Then, by using  $\nabla_z = \mathfrak{R}\nabla_x$  we get

$$\nabla_z(u(\mathfrak{R}^*z, t)) = \frac{\partial_r u(r, t)}{r} \left( \sum_{\alpha=1}^k \sum_{j=1}^N \mathfrak{r}_{1\alpha} \mathfrak{r}_{j\alpha} z_j, \dots, \sum_{\alpha=1}^k \sum_{j=1}^N \mathfrak{r}_{N\alpha} \mathfrak{r}_{j\alpha} z_j \right). \quad (2.14)$$

Since  $\frac{\partial_r u(r, t)}{r} \neq 0$  for any  $t \in (\tau, \tau_j]$  and  $r \in (a + \varepsilon_j, b - \varepsilon_j)$ , we have from (2.12)

$$(-\nabla_{\tilde{z}} \phi(\tilde{z}), 1) \cdot \vec{a}(\tilde{z}, \phi(\tilde{z})) = 0 \quad \text{for almost every } \tilde{z} \in \tilde{B}, \quad (2.15)$$

where we put

$$\begin{aligned} \vec{a}(\tilde{z}, \phi(\tilde{z})) &= (a_1(z), \dots, a_N(z)) \\ &= \left( \sum_{\alpha=1}^k \sum_{j=1}^N \mathfrak{r}_{1\alpha} \mathfrak{r}_{j\alpha} z_j, \dots, \sum_{\alpha=1}^k \sum_{j=1}^N \mathfrak{r}_{N\alpha} \mathfrak{r}_{j\alpha} z_j \right) \quad \text{with } z_N = \phi(\tilde{z}). \end{aligned}$$

Equality (2.15) is regarded as a first-order quasilinear partial differential equation for a function  $\phi$ . We can solve this by the characteristic method. For each  $x \in \mathbb{R}^N$  with  $r = (x_1^2 + \dots + x_k^2)^{\frac{1}{2}} \in (a, b)$ , let  $z = z(s)$  ( $s \in \mathbb{R}$ ) be the curve satisfying

$$\begin{cases} \frac{d}{ds} z(s) = \vec{a}(z(s)) & \text{for } s \in \mathbb{R}, \\ z(0) = \mathfrak{R}x. \end{cases} \quad (2.16)$$

This curve is called a *characteristic curve*. By putting  $z(s) = \mathfrak{R}x(s)$ , we get

$$\frac{d}{ds} x(s) = (x_1(s), \dots, x_k(s), 0, \dots, 0).$$

Solving this yields

$$\mathfrak{R}^* z(s) = x(s) = (x_1(0)e^s, \dots, x_k(0)e^s, x_{k+1}(0), \dots, x_N(0)). \quad (2.17)$$

Namely,  $z = z(s)$  ( $s \in \mathbb{R}$ ) is a straight half-line through a point  $z(0) \in \mathbb{R}^N$  with direction  $\frac{1}{r_0} \mathfrak{R}(x_1(0), \dots, x_k(0), 0, \dots, 0)$ , where  $r_0 = (x_1^2(0) + \dots + x_k^2(0))^{\frac{1}{2}} \in (a, b)$ . We call such a line a *characteristic line*. Let  $\mathcal{L}$  be the set of all characteristic lines intersecting  $V$ . Since  $V$  is convex, for each  $\ell \in \mathcal{L}$  the intersection  $\ell \cap V$  is a line segment. By introducing the polar coordinates for the first  $k$  coordinates  $(x_1, \dots, x_k)$  in  $x$ -coordinates and using Fubini's theorem, we see that for almost every  $\left( \frac{1}{r_0}(x_1(0), \dots, x_k(0)), (x_{k+1}(0), \dots, x_N(0)) \right) \in S^{k-1} \times \mathbb{R}^{N-k}$ ,  $\phi$  has a total differential at almost every point  $z(s)$  on the intersection of  $V$  and the line  $z = z(s)$  with  $z(0) = \mathfrak{R}x(0)$ , provided the intersection is nonempty. Let  $\mathcal{G}$  be the set of such lines intersecting  $V$  and let  $\mathcal{B} = \mathcal{L} \setminus \mathcal{G}$ . We call an element of  $\mathcal{G}$  a *good line* and that of  $\mathcal{B}$  a *bad line*, respectively. In the above meaning almost all elements of  $\mathcal{L}$  are good lines.

First, let us show that if  $\ell$  is a good line intersecting  $\partial\Omega \cap V$ , then  $\ell \cap V$  is contained in  $\partial\Omega$ . Let  $\ell$  be a good line, given by  $z = z(s)$  ( $s \in \mathbb{R}$ ), intersecting  $\partial\Omega \cap V$ . Set

$$w(s) = z_N(s) - \phi(\tilde{z}(s)). \quad (2.18)$$

Then  $w(s_0) = 0$  for some  $s_0 \in \mathbb{R}$ , and for almost every  $s \in \mathbb{R}$  with  $z(s) \in V$  we have

$$\begin{aligned} \frac{d}{ds}w(s) &= \frac{d}{ds}z_N(s) - \nabla_{\tilde{z}}\phi(\tilde{z}(s)) \cdot \frac{d}{ds}\tilde{z}(s) \\ &= a_N(z(s)) - \nabla_{\tilde{z}}\phi(\tilde{z}(s)) \cdot (a_1(z(s)), \dots, a_{N-1}(z(s))) \\ &= (-\nabla_{\tilde{z}}\phi(\tilde{z}(s)), 1) \cdot \vec{a}(\tilde{z}(s), \phi(\tilde{z}(s)) + w(s)). \end{aligned}$$

Observe that

$$\vec{a}(\tilde{z}(s), \phi(\tilde{z}(s)) + w(s)) = \vec{a}(\tilde{z}(s), \phi(\tilde{z}(s))) + \left( \sum_{\alpha=1}^k \mathbf{r}_{1\alpha} \mathbf{r}_{N\alpha}, \dots, \sum_{\alpha=1}^k \mathbf{r}_{N\alpha} \mathbf{r}_{N\alpha} \right) w(s).$$

Therefore it follows from (2.15) that for almost every  $s \in \mathbb{R}$  with  $z(s) \in V$

$$\frac{d}{ds}w(s) = (-\nabla_{\tilde{z}}\phi(\tilde{z}(s)), 1) \cdot \left( \sum_{\alpha=1}^k \mathbf{r}_{1\alpha} \mathbf{r}_{N\alpha}, \dots, \sum_{\alpha=1}^k \mathbf{r}_{N\alpha} \mathbf{r}_{N\alpha} \right) w(s). \quad (2.19)$$

Hence, since the Lipschitz continuity of  $\phi$  implies that the absolute value of the right-hand side of this equality is bounded from above by  $K|w(s)|$  for some constant  $K > 0$ , by integrating this equality from  $s_0$  to  $s$ , we get from  $w(s_0) = 0$

$$|w(s)| \leq K \left| \int_{s_0}^s |w(s')| ds' \right|. \quad (2.20)$$

This implies that

$$w(s) = 0 \quad \text{for all } s \in \mathbb{R} \text{ with } z(s) \in V.$$

In view of the definition of  $w$  ( see (2.18) ), we see that  $\ell \cap V$  is contained in  $\partial\Omega$ .

Next, let us show that  $\partial\Omega \cap V$  consists of characteristic lines  $z = z(s)$  in  $V$ . Suppose that there exists a line  $\ell \in \mathcal{L}$  intersecting  $\partial\Omega \cap V$  such that  $\ell \cap V$  is not contained in  $\partial\Omega$ . Then  $\ell$  is a bad line. Let  $\ell$  be given by  $z = z(s)$  ( $s \in \mathbb{R}$ ). If necessary, by choosing another characteristic line sufficiently close to  $\ell$ , we may assume that there exists two numbers  $s_1$  and  $s_2$  satisfying

$$\begin{cases} z(s_i) \in V \text{ for } i = 1, 2, \\ z_N(s_1) < \phi(\tilde{z}(s_1)), \text{ and } z_N(s_2) > \phi(\tilde{z}(s_2)). \end{cases} \quad (2.21)$$

Since almost all elements of  $\mathcal{L}$  are good lines, from the continuity of  $\phi$  we can find a good line sufficiently close to  $\ell$ , which still satisfies (2.21). Therefore by continuity this good line must intersect  $\partial\Omega \cap V$ . This is a contradiction. Consequently, we see that  $\partial\Omega \cap V$  consists of characteristic lines  $z = z(s)$  in  $V$ .

Since we know that in original  $x$ -coordinates these characteristic lines are given by (2.17) globally, by some additional argument we can get the conclusion (ii) (b) of Theorem 1.

Next let us consider case (a). For  $u = u(x_1, t)$  consider the set  $\mathfrak{Z}$  given by

$$\mathfrak{Z} = \{ x_1 \in (a, b) ; \partial_{x_1} u(x_1, t) = 0 \text{ for all } t > 0 \}. \quad (2.22)$$

Since  $\mathfrak{Z}$  is contained in  $\{ s \in (a, b) ; \psi'(s) = 0 \}$  and  $\psi$  is an analytic non-constant function, so  $\mathfrak{Z} \cap (a + \varepsilon, b - \varepsilon)$  is at most a finite set for each  $\varepsilon > 0$ . Suppose that there exists a point  $s \in \mathfrak{Z} \cap (\frac{a+b}{2}, b)$ . Then by setting for any  $t > 0$

$$v(x_1, t) = \begin{cases} u(x_1, t) & \text{if } x_1 \in (a, s), \\ u(2s - x_1, t) & \text{if } x_1 \in [s, 2s - a), \end{cases}$$

we see that  $v$  also satisfies the one-dimensional heat equation on  $(a, 2s - a) \times (0, \infty)$ . Since  $2s - a > b$  and  $u \equiv v$  in  $(a, s) \times (0, \infty)$ , by analyticity we get  $u \equiv v$  in  $(a, b) \times (0, \infty)$ . Namely,  $u$  is symmetric with respect to  $x_1 = s$ . Also, by supposing that there exists a point  $s \in \mathfrak{Z} \cap (a, \frac{a+b}{2})$ , we get the same conclusion. These observations imply that  $\mathfrak{Z}$  itself is at most a finite set, and its elements are located at regular intervals if  $\mathfrak{Z} \neq \emptyset$ . Let

$$s_{\max} = \begin{cases} \max \mathfrak{Z} & \text{if } \mathfrak{Z} \neq \emptyset, \\ a & \text{if } \mathfrak{Z} = \emptyset. \end{cases}$$

Then  $s_{\max} < b$ . Since  $u$  satisfies the homogeneous Neumann boundary condition, we can determine  $\partial\Omega \cap \{ x \in \mathbb{R}^N ; s_{\max} < x_1 < b \}$  by the characteristic method as in the proof of case (b). Observe that the characteristic curves are the lines being parallel to  $x_1$ -axis. Then we see that  $\partial\Omega \cap \{ x \in \mathbb{R}^N ; s_{\max} < x_1 < b \}$  consists of lines being parallel to  $x_1$ -axis such that  $x_1$  varies from  $s_{\max}$  to  $b$ . Since these lines are parallel to the normal direction of hyperplane  $\{ x \in \mathbb{R}^N ; x_1 = b \}$ , then  $\partial\Omega \cap \{ x \in \mathbb{R}^N ; x_1 = b \}$  has positive area. Therefore it follows from the homogeneous Neumann boundary condition that

$$\partial_{x_1} u(b, t) = 0 \text{ for any } t \in (0, \infty). \quad (2.23)$$

By the same argument we get

$$\partial_{x_1} u(a, t) = 0 \text{ for any } t \in (0, \infty). \quad (2.24)$$

In view of (2.23) and (2.24), by using the above reflection argument for  $u$ , which was used to show that  $\mathfrak{Z}$  is at most a finite set, we see that elements of  $\mathfrak{Z} \cup \{a, b\}$  are located at regular intervals. Hence there exist an integer  $n \geq 1$  and a finite sequence  $\{s_j\}_{j=0}^n$  satisfying

$$s_0 = a, s_n = b, \text{ and } s_{j+1} - s_j = \frac{b-a}{n} \text{ for } 0 \leq j \leq n-1, \quad (2.25)$$

where  $\mathfrak{Z} \cup \{a, b\} = \{s_j\}_{j=0}^n$ . Remark that  $\mathfrak{Z} = \emptyset$  when  $n = 1$ . Hence it follows from the above reflection argument for  $u$  that  $u$  is symmetric with respect to hyperplane  $\{ x \in \mathbb{R}^N ; x_1 = s_j \}$  for each  $1 \leq j \leq n-1$ , when  $n \geq 2$ . Instead of Lemma 2.2 we have:

**Lemma 2.3.**  $u(x_1, \tau)(= \psi(x_1))$  is monotone on each interval  $[s_j, s_{j+1}]$  for any  $j = 0, \dots, n-1$ , provided case (i) of Theorem 1 does not hold.

In view of Lemma 2.3, (2.23), (2.24), and (2.25), by using the strong maximum principle we see that  $\partial_{x_1} u$  never vanishes in  $\bigcup_{j=0}^{n-1} (s_j, s_{j+1}) \times (\tau, \infty)$ . Finally, by the characteristic method we see that for each  $j = 0, \dots, n-1$   $\partial\Omega \cap \{x \in \mathbb{R}^N; s_j < x_1 < s_{j+1}\}$  consists of lines being parallel to  $x_1$ -axis such that  $x_1$  varies from  $s_j$  to  $s_{j+1}$ . This implies the conclusion (ii) (a) of Theorem 1.

#### REFERENCES

- [1] G. Alessandrini, *Matzoh ball soup: a symmetry result for the heat equation*, J. Analyse Math. 54 (1990), pp. 229–236.
- [2] G. Alessandrini, *Characterizing spheres by functional relations on solutions of elliptic and parabolic equations*, Applicable Anal. 40 (1991), pp. 251–261.
- [3] M. S. Klamkin, *A physical characterization of a sphere*, in Problems, SIAM Review 6 (1964), p. 61.
- [4] T. Levi-Civita, *Famiglie di superficie isoparametriche nell'ordinario spazio euclideo*, Atti Accad. naz. Lincei.Rend. Cl. Sci. Fis. Mat. Natur. 26 (1937), pp. 355–362.
- [5] K. Nomizu, *Elie Cartan's work on isoparametric families of hypersurfaces*, Proc. Symp. Pure Math. 27 (1975), pp. 191–200.
- [6] R. S. Palais and Chuu-lian Terng, *Critical Point Theory and Submanifold Geometry*, Springer Lecture Notes in Math. 1353, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1988.
- [7] S. Sakaguchi, *When are the spatial level surfaces of solutions of diffusion equations invariant with respect to the time variable?*, preprint.
- [8] B. Segre, *Una proprietà caratteristica di tre sistemi  $\infty^1$  di superfici*, Atti Reale Accad. Scienze Torino 59 (1923-1924), pp. 666–671.
- [9] B. Segre, *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*, Atti Accad. naz. Lincei.Rend. Cl. Sci. Fis. Mat. Natur. 27 (1938), pp. 203–207.
- [10] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. 43 (1971), pp. 304–318.
- [11] C. Somigliana, *Sulle relazioni fra il principio di Huygens e l'ottica geometrica*, Atti Reale Accad. Scienze Torino 54 (1918-1919), pp. 974–979.
- [12] L. Zalcman, *Some inverse problems of potential theory*, Contemp. Math. 63 (1987), pp. 337–350.
- [13] W. P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Math. 120, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1989.

## 連続体力学の逆問題

群馬大学工学部 中村 玄

固体力学の非破壊検査手法として超音波の散乱波の遠方場（散乱振幅）より，散乱源である介在物，空洞，亀裂，非均質性や残留応力 を同定する方法が知られている．固定されたエネルギーからの超音波散乱の逆問題と関連する数学上の未解決問題について言及する．また超音波の近接場を用いた材料特性の同定の逆問題についても言及したい．

(As a method of nondestructive evaluation of solid, the identification of inclusion, cavity, crack, inhomogeneity and residual stress by the farfield pattern (scattering amplitude) of ultrasound is known. The inverse scattering of ultrasound at fixed energy and the related unsolved mathematical problems will be discussed. Also, the inverse problem for identifying the physical characteristics of the material using the nearfield of ultrasound will be discussed.)