STABILITY CRITERIA FOR THE SYSTEM OF DELAY DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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Abstract

In this paper, we consider the asymptotic stability for the system of linear delay differential equations. Because of the complicated interactions induced by the delay effects of the system, there are few results of the asymptotic stability for the system of the delay differential equations with multiple delays. Given this fact, we propose the new stability conditions for the system and apply these conditions to some mathematical models for the population dynamics and neural network system described by the system of delay differential equations.

Keywords: delay differential equation, asymptotic stability, population model, neural network model.

1 Introduction

We consider the system of linear delay differential equations:

$$
\begin{aligned}
u_1'(t) + a_1u_1(t) + b_nu_n(t) + \alpha_1u_1(t-\tau_{11}) + \beta_nu_n(t-\tau_{1n}) &= 0,\\
u_2'(t) + a_2u_2(t) + b_1u_1(t) + \alpha_2u_2(t-\tau_{22}) + \beta_1u_1(t-\tau_{21}) &= 0,\\
&\vdots\\
u_n'(t) + a_nu_n(t) + b_{n-1}u_{n-1}(t) + \alpha_nu_n(t-\tau_{nn}) + \beta_{n-1}u_{n-1}(t-\tau_{nn-1}) &= 0,
\end{aligned}
\tag{1.1}
$$

where $u(t) = (u_1, u_2, \cdots, u_n)^\top(t)$ denotes complex-valued unknown vector functions for $t \geq 0$. The coefficients a_j, α_j, b_j and β_j are complex-valued constants, and τ_{jk} are non-negative constants for $1 \leq j, k \leq n$. Here, we remark the parameters τ_{jk} describe the delay effects. By the coefficients of (1.1), we introduce the $n \times n$ constant matrices A and B that

$$
A := \begin{pmatrix} a_1 & 0 & \dots & 0 & b_n \\ b_1 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & \dots & b_{n-1} & a_n \end{pmatrix}, \quad B := \begin{pmatrix} \alpha_1 & 0 & \dots & 0 & \beta_n \\ \beta_1 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_{n-1} & 0 \\ 0 & 0 & \dots & \beta_{n-1} & \alpha_n \end{pmatrix}. \tag{1.2}
$$

Then, our system (1.1) can be expressed by the following form:

$$
u' + Au + f(u) = 0, \tag{1.3}
$$

where $f(u) := (f_1(u), \cdots, f_n(u))^\top$ with

$$
f_j(u) := \alpha_ju_j(t-\tau_{jj}) + \beta_{j-1}u_{j-1}(t-\tau_{jj-1})
$$

for $1 \leq j \leq n$, and $u_0(t) := u_n(t)$, $\beta_0 := \beta_n$ and $\tau_{10} := \tau_{1n}$. Moreover, if we assume $\tau_{jk} = \tau$ for all $1 \leq j, k \leq n$, then (1.3) can be written by

$$
u'(t) + Au(t) + Bu(t-\tau) = 0. \tag{1.4}
$$

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It is very important to study the delay differential equations because they often appear in the various fields of physics and engineering via mathematical models (e.g. population models of Lotka–Volterra type and prey-predator models, neural network models, chemical kinetics, and also traffic flow, see e.g. [4, 6, 12, 18, 19, 20] and also references therein).

It is not difficult to derive the global existence of solutions to (1.1) provided by suitable initial data (cf. Hale [7], Hale–Verduyn Lunel [8]). The purpose of this paper is to analyze the asymptotic profile of the solutions to (1.1). In particular, we introduce the useful conditions to get the asymptotic stability of the solutions.

Historically, there are several known results concerned with the asymptotic stability for the delay differential equations. In the middle of 1900s, Hayes [9] and Bellman–Cooke [1] studied the scalar differential equation with single delay and obtained the necessary and sufficient conditions for the asymptotic stability. In particular, Hayes [9] obtained an important result which characterize the asymptotic stability for the solutions by the coefficients of the equation. More precisely, he studied the following equation:

$$u'(t) + au(t) + \alpha u(t-\tau) = 0,$$

where $u(t)$ is a real-valued unknown scalar function, a and α are real-valued constants, and $\tau \geq 0$. For this equation, he showed that if either $a + \alpha > 0$ and $a - \alpha \geq 0$ or $a + \alpha > 0$, $a - \alpha < 0$ and $0 < \tau\sqrt{\alpha^2 - a^2} < \arccos(-a/\alpha)$ are satisfied, then the zero solution of the above equation is asymptotically stable. The proof of this result is based on the detailed analysis of the corresponding characteristic equation mentioned in Section 2. He succeeded to be clear the relation to the delay effect and asymptotic stability. On the other hand, there are few results for the system of delay differential equations. Furthermore, almost all of known results are focused on the case of $n = 2$. For example, Lu–Wang [12] studied the 2-dimensional system (1.1) with $\alpha_1 = \alpha_2 = 0$ and obtained the sufficient condition to get the asymptotic stability. Matsunaga [13] also studied the 2-dimensional system (1.1) with $\alpha_1 = \alpha_2$, $a_1 = a_2 = \beta_1 = \beta_2 = 0$ and $\tau_{11} = \tau_{22}$. Then he obtained the necessary and sufficient condition for the asymptotic stability. For the general multi-dimensional system, Suzuki–Matsunaga [21] studied the system (1.1) with $\alpha_j = b_j = 0$ for all $1 \leq j \leq n$, and succeeded to get the necessary and sufficient condition for the asymptotic stability. Their method is based on the analysis of the delay effects through the characteristic equation. However, it is difficult to apply the method used in [13, 21] to our general system (1.1) because our system is supposed more general situations and the corresponding characteristic equation is complicated. Furthermore, all of the above results are concerned with the system of real-valued coefficient matrices, and there are few results for the stability of complex-valued delay differential equations (e.g. [22] for the scalar equation and [2, 17] for the system with single delay). Their techniques are also difficult to apply to our problem, because (1.1) is a high-dimensional system with multiple delays. To overcome these difficulties we employ the different approach to get the sufficient condition for the asymptotic stability, and obtain the stability criteria for (1.1).

The rest of this paper is organized as follows. In Section 2, we introduce the basic results for the characteristic equation of the system (1.1) to prove our main results. In addition, to explain our main results, we introduce the definitions for asymptotic stability, called the absolute stability and the conditional stability. In Section 3 and 4, we introduce the stability criterion and state our main results of the asymptotic stability for the zero solution of the system (1.1). Section 3 is devoted to the absolute stability, and Section 4 is devoted to the conditional stability. In Section 5, we consider the related problem of delay differential equations with distributed delay. Finally, as ones of examples, we apply our results to some mathematical models for the population dynamics and neural network system in Section 6.

2 Preliminaries

We introduce important properties for the characteristic equation in this section. The stability of the system (1.1) is completely determined by the roots of its associated characteristic equation. Now, let $(\lambda, \phi) \in \mathbb{C} \times \mathbb{C}^n$. Substituting $u(t) = e^{\lambda t}\phi$ into (1.1) (or (1.3)), we have the following eigenvalue problem:

$$\begin{pmatrix} \lambda+\gamma_1 & 0 & \dots & 0 & b_n+\beta_n e^{-\lambda\tau_{1n}} \\ b_1+\beta_1 e^{-\lambda\tau_{21}} & \lambda+\gamma_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda+\gamma_{n-1} & 0 \\ 0 & 0 & \dots & b_{n-1}+\beta_{n-1}e^{-\lambda\tau_{nn-1}} & \lambda+\gamma_n \end{pmatrix} \phi = 0,$$

where $\gamma_j := a_j + \alpha_j e^{-\lambda\tau_{jj}}$ for $1 \le j \le n$. Thus, the characteristic equation for the system (1.1) is given by

$$G(\lambda) = 0, \tag{2.1}$$

where $G(\lambda)$ is defined by

$$G(\lambda) := \det \begin{pmatrix} \lambda+\gamma_1 & 0 & \dots & 0 & b_n+\beta_n e^{-\lambda\tau_{1n}} \\ b_1+\beta_1 e^{-\lambda\tau_{21}} & \lambda+\gamma_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda+\gamma_{n-1} & 0 \\ 0 & 0 & \dots & b_{n-1}+\beta_{n-1}e^{-\lambda\tau_{nn-1}} & \lambda+\gamma_n \end{pmatrix}.$$

In particular, if we assume $\tau_{jk} = \tau$ for all $1 \le j, k \le n$, then $G(\lambda)$ can be written as the simple form $G(\lambda) = \det(\lambda I + A + Be^{-\lambda\tau})$ by using the coefficient matrices (1.2). Here I is the $n \times n$ identity matrix. For the system (1.1) and the characteristic equation (2.1), the following theorem plays an important role to analyze the stability of solutions (for the proof, see e.g. Hale [7] and Hale–Verduyn Lunel [8]).

Theorem 2.1. *If all of the roots of the characteristic equation* (2.1) *lie in the left half of the complex plane, then the zero solution of the system* (1.1) *is asymptotically stable. Moreover, the solutions satisfy the estimate* $|u(t)| \le Ce^{-ct}$ *for* $t \ge 0$*, where* C *and* c *are certain positive constants.*

By virtue of Theorem 2.1, our goal is to derive the sufficient conditions that the real parts of all of the characteristic roots are negative. In the rest of this section, we introduce the definitions of the absolute stability and conditional stability concerned with the asymptotic stability (cf. Ruan [20]).

Definition 2.2. *The equilibrium point of the system* (1.1) *is called absolutely stable if it is asymptotically stable for all delays* τ_{jk} $(1 \le j, k \le n)$*, and the equilibrium point is called conditionally stable if it is asymptotically stable for* τ_{jk} $(1 \le j, k \le n)$ *in some intervals, but not necessarily for all delays* τ_{jk} $(1 \le j, k \le n)$.

3 Absolute Stability

In this section, we show the new criterion of the absolute stability for the system (1.1). Our first main result is stated as follows.

Theorem 3.1. *If the coefficients of the system* (1.1) *satisfy*

$$\mathrm{Re}(a_j) - |\alpha_j| > 0, \quad 1 \le j \le n, \tag{3.1}$$

and

$$\prod_{j=1}^{n} (\mathrm{Re}(a_j) - |\alpha_j|) > \prod_{j=1}^{n} (|b_j| + |\beta_j|), \tag{3.2}$$

then the zero solution of the system (1.1) *is absolutely stable.*

Remark 3.2. In the case that $b_j = 0$ for all $1 \leq j \leq n$ in Theorem 3.1, this result was obtained by Kiri–Ueda [10].

Proof of Theorem 3.1. Let $\lambda = x + iy$ with $x, y \in \mathbb{R}$. Then, we shall derive the contradiction under the assumption $x \geq 0$. By the simple calculation, the characteristic equation (2.1) is rewritten by

$$\prod_{j=1}^{n} \left(\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}\right) + (-1)^{n+1} \left(b_n + \beta_n e^{-\lambda\tau_{1n}}\right) \prod_{j=1}^{n-1} \left(b_j + \beta_j e^{-\lambda\tau_{j+1j}}\right) = 0.$$

Namely, this gives

$$\prod_{j=1}^{n} \left|\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}\right| = \left|b_n + \beta_n e^{-\lambda\tau_{1n}}\right| \prod_{j=1}^{n-1} \left|b_j + \beta_j e^{-\lambda\tau_{j+1j}}\right|. \tag{3.3}$$

Firstly, we treat the right hand side of (3.3). Since $x \geq 0$ and $\tau_{jk} \geq 0$ for $1 \leq j, k \leq n$, we have $e^{-x\tau_{jk}} \leq 1$. Therefore, it follows that

$$\begin{aligned} \left|b_n + \beta_n e^{-\lambda\tau_{1n}}\right| \prod_{j=1}^{n-1} \left|b_j + \beta_j e^{-\lambda\tau_{j+1j}}\right| &\leq (|b_n| + |\beta_n| e^{-x\tau_{1n}}) \prod_{j=1}^{n-1} (|b_j| + |\beta_j| e^{-x\tau_{j+1j}}) \\ &\leq \prod_{j=1}^{n} (|b_j| + |\beta_j|). \end{aligned} \tag{3.4}$$

Secondly, since $x \geq 0$ and (3.1), we compute that

$$|\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}| \geq |\lambda + a_j| - |\alpha_j e^{-\lambda\tau_{jj}}| \geq x + \mathrm{Re}(a_j) - |\alpha_j| e^{-x\tau_{jj}} \geq \mathrm{Re}(a_j) - |\alpha_j| > 0$$

for $1 \leq j \leq n$. Therefore, we obtain

$$\prod_{j=1}^{n} |\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}| \geq \prod_{j=1}^{n} (\mathrm{Re}(a_j) - |\alpha_j|). \tag{3.5}$$

Finally, combining (3.3), (3.4) and (3.5), we obtain

$$\prod_{j=1}^{n} (\mathrm{Re}(a_j) - |\alpha_j|) \leq \prod_{j=1}^{n} (|b_j| + |\beta_j|).$$

However, this inequality is a contradiction under the assumption (3.2). Consequently, we see that the real parts of all of the roots of the characteristic equation must be negative. Therefore, Theorem 2.1 tells us that the zero solution of the system (1.1) is asymptotically stable. This completes the proof. □

4 Conditional Stability

In the previous section, we derive the criterion of the absolute stability for the system (1.1). Theorem 3.1 can be applicable for a lot of mathematical models mentioned in Section 1. However, if the system (1.1) satisfies $\mathrm{Re}(a_j) \leq |\alpha_j|$ for some j, we can apply Theorem 3.1 no longer. Indeed, there is a typical example of population models which does not satisfy the condition (3.1). The detail will be discussed in Section 6. For this reason, we would like to propose the different stability criterion for the system (1.1). Actually, we can modify the proof given in the previous section and derive the different criterion concerned with the conditional stability.

Theorem 4.1. *If the coefficients of the system* (1.1) *satisfy*

$$0 \le \tau_{jj} < \frac{\mathrm{Re}\,(a_j+\alpha_j)}{|\alpha_j|\,(\mathrm{Re}\,(a_j+\alpha_j)+|a_j+\alpha_j|)}, \qquad 1 \le j \le n, \tag{4.1}$$

and

$$\prod_{j=1}^{n} \{\mathrm{Re}\,(a_j+\alpha_j)\,(1-|\alpha_j|\tau_{jj}) - |\alpha_j||a_j+\alpha_j|\tau_{jj}\} > \prod_{j=1}^{n} (|b_j|+|\beta_j|)\,, \tag{4.2}$$

then the zero solution of the system (1.1) *is conditionally stable.*

Proof. We note that (4.1) implies

$$\mathrm{Re}\,(a_j+\alpha_j) > 0$$

for $1 \le j \le n$. Let $\lambda = x+iy$ with $x, y \in \mathbb{R}$. Then, we also derive the contradiction under the assumption $x \ge 0$. We recall (3.3) which comes from the characteristic equation (2.1). We have already obtained the estimate of the right hand side of (3.3), which means (3.4). Let us estimate $|\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}|$ in the left hand side of (3.3) by a different way from the derivation of (3.5). Employing the mean value theorem, we obtain

$$e^{-\lambda\tau_{jj}} = 1 - \lambda\tau_{jj}\int_0^1 e^{-\theta\lambda\tau_{jj}}\,d\theta.$$

Thus, it follows that

$$\begin{aligned}
|\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}| &= |(\lambda + a_j + \alpha_j)(1-\alpha_j\tau_{jj}I) + \alpha_j(a_j+\alpha_j)\tau_{jj}I| \\
&\ge |\lambda + a_j + \alpha_j||1-\alpha_j\tau_{jj}I| - |\alpha_j||a_j+\alpha_j||I|\tau_{jj} \\
&\ge |x + \mathrm{Re}(a_j+\alpha_j)|\,(1-|\alpha_j|\tau_{jj}) - |\alpha_j||a_j+\alpha_j|\tau_{jj}
\end{aligned}$$

for $1 \le j \le n$, where I is defined by $I := \int_0^1 e^{-\theta\lambda\tau_{jj}}d\theta$, and we used the fact that $|I| \le 1$ obtained by the assumptions $x \ge 0$ and $\tau_{jj} \ge 0$. Furthermore, the assumption (4.1) gives

$$|\alpha_j|\tau_{jj} < \frac{\mathrm{Re}\,(a_j+\alpha_j)}{\mathrm{Re}\,(a_j+\alpha_j)+|a_j+\alpha_j|} < 1,$$

and this estimate and $x \ge 0$ lead to

$$|\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}| \ge \mathrm{Re}\,(a_j+\alpha_j)\,(1-|\alpha_j|\tau_{jj}) - |\alpha_j||a_j+\alpha_j|\tau_{jj} > 0.$$

Therefore, we obtain

$$\prod_{j=1}^{n} |\lambda + a_j + \alpha_j e^{-\lambda\tau_{jj}}| \ge \prod_{j=1}^{n} \{\mathrm{Re}\,(a_j+\alpha_j)\,(1-|\alpha_j|\tau_{jj}) - |\alpha_j||a_j+\alpha_j|\tau_{jj}\}\,. \tag{4.3}$$

Finally, combining (3.3), (3.4) and (4.3), this yields

$$\prod_{j=1}^{n} \{\mathrm{Re}\,(a_j+\alpha_j)\,(1-|\alpha_j|\tau_{jj}) - |\alpha_j||a_j+\alpha_j|\tau_{jj}\} \le \prod_{j=1}^{n} (|b_j|+|\beta_j|)\,.$$

However, this inequality is a contradiction under the assumption (4.2). Consequently, we see that the real parts of all of the roots of the characteristic equation must be negative. Therefore, because of Theorem 2.1, the zero solution of the system (1.1) is asymptotically stable, and this completes the proof. □

Remark 4.2. If it is possible to apply Theorem 3.1 to our problems, the time delays have no effect for breaking the stability of the zero solution. In other words, we can obtain the asymptotic stability to our problems for any time delays. On the other hand, Theorem 4.1 tells us the possibility to derive the asymptotic stability for our problems which do not satisfy the conditions in Theorem 3.1. In this situation, the smallness assumption for the time delays is the key to get the asymptotic stability. More precisely, we would like to emphasize that the assumptions (4.1) and (4.2) only depend on τ_{jj} for $1 \leq j \leq n$ and are independent of τ_{1n} and τ_{jj-1} for $2 \leq j \leq n$. Namely, Theorem 4.1 gives the conditional stability criterion for the system (1.1) with the dominant delays τ_{jj} for $1 \leq j \leq n$ and the harmless delays τ_{1n} and τ_{jj-1} for $2 \leq j \leq n$.

5 Delay Differential Equations with Distributed Delay

We shall consider the related system of equations for (1.1) in this section. We study the asymptotic stability for the following system of linear differential equations with distributed delay:

$$\begin{aligned} u_1'(t) + a_1 u_1(t) + b_n u_n(t) + \alpha_1 \int_{t-\tau_{11}}^{t} u_1(s)ds + \beta_n \int_{t-\tau_{1n}}^{t} u_n(s)ds &= 0, \\ u_2'(t) + a_2 u_2(t) + b_1 u_1(t) + \alpha_2 \int_{t-\tau_{22}}^{t} u_2(s)ds + \beta_1 \int_{t-\tau_{21}}^{t} u_1(s)ds &= 0, \\ &\vdots \\ u_n'(t) + a_n u_n(t) + b_{n-1} u_{n-1}(t) + \alpha_n \int_{t-\tau_{nn}}^{t} u_n(s)ds + \beta_{n-1} \int_{t-\tau_{nn-1}}^{t} u_{n-1}(s)ds &= 0, \end{aligned} \tag{5.1}$$

where the coefficients a_j, α_j, b_j and β_j are complex-valued constants, and τ_{jk} are non-negative constants for $1 \leq j, k \leq n$. There are a lot of physical models described by (5.1), and the differential equations with distributed delay are studied in [6, 14, 15]. The characteristic equation for the system (5.1) is given by

$$\tilde{G}(\lambda) = 0, \tag{5.2}$$

where we define

$$\tilde{G}(\lambda) := \det \begin{pmatrix} \lambda + \tilde{\gamma}_1 & 0 & \dots & 0 & \tilde{\gamma}_{10} \\ \tilde{\gamma}_{21} & \lambda + \tilde{\gamma}_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda + \tilde{\gamma}_{n-1} & 0 \\ 0 & 0 & \dots & \tilde{\gamma}_{nn-1} & \lambda + \tilde{\gamma}_n \end{pmatrix},$$

and $\tilde{\gamma}_j = a_j + \alpha_j \int_{-\tau_{jj}}^{0} e^{\lambda s} ds$ and $\tilde{\gamma}_{jj-1} = b_{j-1} + \beta_{j-1} \int_{-\tau_{jj-1}}^{0} e^{\lambda s} ds$ for $1 \leq j \leq n$ with $b_0 := b_n$, $\beta_0 := \beta_n$ and $\tau_{10} := \tau_{1n}$. Then we have the following theorem for (5.1), which is the equivalent to Theorem 2.1 for (1.1) (cf. Hale [7] and Hale–Verduyn Lunel [8]).

Theorem 5.1. *If all of the roots of the characteristic equation* (5.2) *lie in the left half of the complex plane, then the zero solution of the system* (5.1) *is asymptotically stable. Moreover, the solutions satisfy the estimate* $|u(t)| \leq Ce^{-ct}$ *for* $t \geq 0$, *where* C *and* c *are certain positive constants.*

Using the similar arguments as in the previous sections, we can obtain the sufficient conditions to get the asymptotic stability for the solutions to (5.1), which conditions are relevant to the conditions appeared in Theorems 3.1 and 4.1.

Theorem 5.2. *Assume that the coefficients of the system* (5.1) *satisfy the following conditions* (i) *or* (ii)*:*

(i)

$$\mathrm{Re}(a_j) - |\alpha_j|\tau_{jj} > 0, \qquad 1 \le j \le n, \tag{5.3}$$

and

$$\prod_{j=1}^{n}(\mathrm{Re}(a_j) - |\alpha_j|\tau_{jj}) > (|b_n| + |\beta_n|\tau_{1n}) \prod_{j=1}^{n-1}(|b_j| + |\beta_j|\tau_{j+1j}). \tag{5.4}$$

(ii)

$$0 \le \frac{1}{2}\tau_{jj} < \frac{\mathrm{Re}\,(a_j + \alpha_j\tau_{jj})}{|\alpha_j|\tau_{jj}(\mathrm{Re}\,(a_j + \alpha_j\tau_{jj}) + |a_j + \alpha_j\tau_{jj}|)}, \qquad 1 \le j \le n, \tag{5.5}$$

and

$$\begin{aligned}&\prod_{j=1}^{n}\left\{\mathrm{Re}\,(a_j + \alpha_j\tau_{jj})\left(1 - \frac{1}{2}|\alpha_j|\tau_{jj}^2\right) - \frac{1}{2}|\alpha_j||a_j + \alpha_j\tau_{jj}|\tau_{jj}^2\right\}\\ &> (|b_n| + |\beta_n|\tau_{1n}) \prod_{j=1}^{n-1}(|b_j| + |\beta_j|\tau_{j+1j})\,.\end{aligned} \tag{5.6}$$

Then the zero solution of the system (5.1) *is asymptotically stable.*

Proof. Let $\lambda = x+iy$ with $x, y \in \mathbb{R}$. Then, we derive the contradiction under the assumption $x \ge 0$. The characteristic equation (5.2) is rewritten by

$$\begin{aligned}&\prod_{j=1}^{n}\left(\lambda + a_j + \alpha_j\int_{-\tau_{jj}}^{0} e^{\lambda s}ds\right)\\ &+ (-1)^{n+1}\left(b_n + \beta_n\int_{-\tau_{1n}}^{0} e^{\lambda s}ds\right)\prod_{j=1}^{n-1}\left(b_j + \beta_j\int_{-\tau_{j+1j}}^{0} e^{\lambda s}ds\right) = 0.\end{aligned}$$

This gives

$$\prod_{j=1}^{n}\left|\lambda + a_j + \alpha_j\int_{-\tau_{jj}}^{0} e^{\lambda s}ds\right| = \left|b_n + \beta_n\int_{-\tau_{1n}}^{0} e^{\lambda s}ds\right|\prod_{j=1}^{n-1}\left|b_j + \beta_j\int_{-\tau_{j+1j}}^{0} e^{\lambda s}ds\right|. \tag{5.7}$$

Firstly, we estimate the right hand side of (5.7). Because of $x \ge 0$, we have $e^{xs} \le 1$ for $s \le 0$. Therefore, we compute

$$\begin{aligned}&\left|b_n + \beta_n\int_{-\tau_{1n}}^{0} e^{\lambda s}ds\right|\prod_{j=1}^{n-1}\left|b_j + \beta_j\int_{-\tau_{j+1j}}^{0} e^{\lambda s}ds\right|\\ &\le \left(|b_n| + |\beta_n|\int_{-\tau_{1n}}^{0} e^{xs}ds\right)\prod_{j=1}^{n-1}\left(|b_j| + |\beta_j|\int_{-\tau_{j+1j}}^{0} e^{xs}ds\right)\\ &\le (|b_n| + |\beta_n|\tau_{1n})\prod_{j=1}^{n-1}(|b_j| + |\beta_j|\tau_{j+1j}).\end{aligned} \tag{5.8}$$

Secondly, we estimate the left hand side of (5.7) provided by the conditions (i) or (ii), respectively.

Case (i): Using the assumptions $x \ge 0$ and (5.3), we estimate

$$\begin{aligned}\left|\lambda + a_j + \alpha_j\int_{-\tau_{jj}}^{0} e^{\lambda s}ds\right| &\ge |\lambda + a_j| - |\alpha_j|\int_{-\tau_{jj}}^{0} e^{xs}ds\\ &\ge x + \mathrm{Re}(a_j) - |\alpha_j|\tau_{jj}\\ &\ge \mathrm{Re}(a_j) - |\alpha_j|\tau_{jj} > 0\end{aligned}$$

for $1 \le j \le n$. Therefore, we obtain

$$\prod_{j=1}^{n} \left| \lambda + a_j + \alpha_j \int_{-\tau_{jj}}^{0} e^{\lambda s} ds \right| \ge \prod_{j=1}^{n} (\mathrm{Re}(a_j) - |\alpha_j| \tau_{jj}). \tag{5.9}$$

Eventually, combining (5.7), (5.8) and (5.9), we obtain

$$\prod_{j=1}^{n} (\mathrm{Re}(a_j) - |\alpha_j| \tau_{jj}) \le (|b_n| + |\beta_n| \tau_{1n}) \prod_{j=1}^{n-1} (|b_j| + |\beta_j| \tau_{j+1j}).$$

However, this inequality is a contradiction under the assumption (5.4).

Case (ii): We note that (5.5) implies

$$\mathrm{Re}\,(a_j + \alpha_j \tau_{jj}) > 0$$

for $1 \le j \le n$. To estimate $|\lambda + a_j + \alpha_j \int_{-\tau_{jj}}^{0} e^{\lambda s} ds|$, we also apply the mean value theorem to $e^{\lambda s}$. Then this yields

$$\int_{-\tau_{jj}}^{0} e^{\lambda s} ds = \int_{-\tau_{jj}}^{0} \left(1 + \lambda s \int_{0}^{1} e^{\theta \lambda s} d\theta \right) ds = \tau_{jj} + \lambda I,$$

where

$$I := \int_{-\tau_{jj}}^{0} s \int_{0}^{1} e^{\theta \lambda s} d\theta ds.$$

It is easy to estimate I that

$$|I| = \left| \int_{-\tau_{jj}}^{0} s \int_{0}^{1} e^{\theta \lambda s} d\theta \; ds \right| \le \int_{-\tau_{jj}}^{0} |s| \int_{0}^{1} \left| e^{\theta \lambda s} \right| d\theta \; ds \le \int_{-\tau_{jj}}^{0} |s| \; ds = \frac{1}{2} \tau_{jj}^2.$$

Thus, we obtain

$$\begin{aligned} \left| \lambda + a_j + \alpha_j \int_{-\tau_{jj}}^{0} e^{\lambda s} ds \right| &= |(\lambda + a_j + \alpha_j \tau_{jj})(1 + \alpha_j I) - \alpha_j (a_j + \alpha_j \tau_{jj}) I| \\ &\ge |\lambda + a_j + \alpha_j \tau_{jj}| |1 + \alpha_j I| - |\alpha_j| |a_j + \alpha_j \tau_{jj}| |I| \\ &\ge |x + \mathrm{Re}(a_j + \alpha_j \tau_{jj})| \left(1 - \frac{1}{2} |\alpha_j| \tau_{jj}^2 \right) - \frac{1}{2} |\alpha_j| |a_j + \alpha_j \tau_{jj}| \tau_{jj}^2 \end{aligned}$$

for $1 \le j \le n$. Furthermore, the assumption (5.5) gives

$$\frac{1}{2} |\alpha_j| \tau_{jj}^2 < \frac{\mathrm{Re}\,(a_j + \alpha_j \tau_{jj})}{\mathrm{Re}\,(a_j + \alpha_j \tau_{jj}) + |a_j + \alpha_j \tau_{jj}|} < 1.$$

Hence, this estimate and $x \ge 0$ lead to

$$\left| \lambda + a_j + \alpha_j \int_{-\tau_{jj}}^{0} e^{\lambda s} ds \right| \ge \mathrm{Re}\,(a_j + \alpha_j \tau_{jj}) \left(1 - \frac{1}{2} |\alpha_j| \tau_{jj}^2 \right) - \frac{1}{2} |\alpha_j| |a_j + \alpha_j \tau_{jj}| \tau_{jj}^2 > 0.$$

Therefore, we get

$$\begin{aligned} &\prod_{j=1}^{n} \left| \lambda + a_j + \alpha_j \int_{-\tau_{jj}}^{0} e^{\lambda s} ds \right| \\ &\ge \prod_{j=1}^{n} \left\{ \mathrm{Re}(a_j + \alpha_j \tau_{jj}) \left(1 - \frac{1}{2} |\alpha_j| \tau_{jj}^2 \right) - \frac{1}{2} |\alpha_j| |a_j + \alpha_j \tau_{jj}| \tau_{jj}^2 \right\}. \end{aligned} \tag{5.10}$$

Eventually, combining (5.7), (5.8) and (5.10), we arrive at

$$\prod_{j=1}^{n}\left\{\mathrm{Re}(a_j+\alpha_j\tau_{jj})\left(1-\frac{1}{2}|\alpha_j|\tau_{jj}^2\right)-\frac{1}{2}|\alpha_j||a_j+\alpha_j\tau_{jj}|\tau_{jj}^2\right\}$$
$$\leq (|b_n|+|\beta_n|\tau_{1n})\prod_{j=1}^{n-1}(|b_j|+|\beta_j|\tau_{j+1j})\,.$$

However, this inequality is also contradiction under the assumption (5.6).

As a conclusion, we see that the real parts of all the characteristic roots must be negative under the conditions (i) or (ii). Therefore, because of Theorem 5.1, the zero solution of the system (5.1) is asymptotically stable under these conditions. This completes the proof. □

Remark 5.3. For the system of differential equations with distributed delay, it is difficult to derive the absolute stability. Indeed, as in Theorems 4.1 and 5.2, the smallness assumption on the time delays play an important role to get the asymptotic stability for the system (5.1). In particular, we emphasize that the assumptions (5.4) and (5.6) depend not only on τ_{jj} for $1 \leq j \leq n$ but also on τ_{1n} and τ_{jj-1} for $2 \leq j \leq n$, which fact is the one of the different properties between Theorems 4.1 and 5.1 on the conditional stability criteria.

Remark 5.4. For the one-dimensional case, Funakubo–Hara–Sakata [6] studied the following equation, and obtained the sufficient condition for the absolute stability:

$$u'(t)+au(t)+\alpha\int_{t-\tau}^{t}u(s)ds=0,$$

where $u(t)$ is a real-valued unknown scalar function and a, α, τ are positive constants. In [6], they showed that if $a^2 \geq 2\alpha$, then the zero solution of the above equation is absolutely stable. On the other hand, even if the 2-dimensional case, there are no results of the absolute stability for the system (5.1), except for the system reduced to the scalar equation (for the related results in the case of $n = 2$, see e.g. [14, 15]). These results imply that the asymptotic stability for the system with distributed delay is strongly affected by the time delay.

6 Applications

At the last section, we analyze the asymptotic stability of the equilibrium point for some mathematical models as applications for Theorems 3.1 and 4.1. In particular, we consider the two species population models such as the delayed Lotka–Volterra system and the delayed prey–predator system. Also, as an example for the high-dimensional system, we study a simple model of the neural network described by the system of delay differential equations.

Delayed Lotka–Volterra System: Firstly, we consider the Lotka–Volterra type competitive system with delay effects:

$$\begin{cases} x'(t)=x(t)\big(K_1-x(t)\big)-px(t-\tau_1)y(t), \\ y'(t)=y(t)\big(K_2-y(t)\big)-qx(t)y(t-\tau_2), \end{cases} \tag{6.1}$$

where K_1, K_2, p, q, τ_1 and τ_2 are positive constants. The real-valued unknown functions $x(t)$ and $y(t)$ are the population of each competitor, while K_1 and K_2 are called the carrying capacity of themselves. The system (6.1) has an equilibrium point

$$(x^*, y^*)=\left(\frac{pK_2-K_1}{pq-1},\frac{qK_1-K_2}{pq-1}\right).$$

We linearize the system (6.1) around (x^*, y^*), obtaining

$$\begin{cases} x'(t) = (K_1 - 2x^*)x(t) - py^*x(t-\tau_1) - px^*y(t), \\ y'(t) = -qy^*x(t) + (K_2 - 2y^*)y(t) - qx^*y(t-\tau_2). \end{cases} \tag{6.2}$$

Then the system (6.2) is rewritten as (1.3) with $u(t) = (x(t), y(t))^\top$,

$$A = \begin{pmatrix} 2x^* - K_1 & px^* \\ qy^* & 2y^* - K_2 \end{pmatrix}, \qquad B = \begin{pmatrix} py^* & 0 \\ 0 & qx^* \end{pmatrix},$$

and $\tau_{11} = \tau_1$, $\tau_{22} = \tau_2$, $\tau_{12} = \tau_{21} = 0$.

It is well known that the equilibrium point (x^*, y^*) of the nonlinear system (6.1) is also asymptotically stable if the zero solution of the linearized system (6.2) is asymptotically stable. Therefore, applying Theorems 3.1 and 4.1 to the linearized system (6.2), we derive the following results immediately.

Corollary 6.1. *If the coefficients of the system* (6.1) *satisfy*

$$2\frac{pK_2 - K_1}{pq-1} - K_1 - p\left|\frac{qK_1 - K_2}{pq-1}\right| > 0, \qquad 2\frac{qK_1 - K_2}{pq-1} - K_2 - q\left|\frac{pK_2 - K_1}{pq-1}\right| > 0$$

and

$$\begin{aligned} &\left(2\frac{pK_2 - K_1}{pq-1} - K_1 - p\left|\frac{qK_1 - K_2}{pq-1}\right|\right)\left(2\frac{qK_1 - K_2}{pq-1} - K_2 - q\left|\frac{pK_2 - K_1}{pq-1}\right|\right) \\ &\quad > pq\left|\frac{pK_2 - K_1}{pq-1}\right|\left|\frac{qK_1 - K_2}{pq-1}\right|, \end{aligned}$$

then the equilibrium point (x^*, y^*) *of the system* (6.1) *is absolutely stable.*

Corollary 6.2. *If the coefficients of the system* (6.1) *satisfy*

$$0 < \tau_1 < \frac{pq-1}{2p(qK_1 - K_2)}, \qquad 0 < \tau_2 < \frac{pq-1}{2q(pK_2 - K_1)},$$

$$\left(1 - \frac{2p(qK_1 - K_2)\tau_1}{pq-1}\right)\left(1 - \frac{2q(pK_2 - K_1)\tau_2}{pq-1}\right) > pq$$

and either

$$\frac{K_2}{q} < K_1 < pK_2 \quad \text{or} \quad pK_2 < K_1 < \frac{K_2}{q},$$

then the equilibrium point (x^*, y^*) *of the system* (6.1) *is conditionally stable.*

Remark 6.3. From the point of view of the population dynamics, it is natural to assume that the coefficients of (6.1) satisfy

$$\frac{pK_2 - K_1}{pq-1} > 0 \quad \text{and} \quad \frac{qK_1 - K_2}{pq-1} > 0$$

to guarantee $x^* > 0$ and $y^* > 0$. We note that if the conditions in Corollaries 6.1 or 6.2 are satisfied, then the above conditions are obtained automatically.

For example, let $K_1 = K_2 = 1$ and $p = q = 1/4$ for (6.1). The equilibrium point is $(x^*, y^*) = (12/15, 12/15)$. In this situation, it is easy to check that these coefficients satisfy the conditions in Corollary 6.1. Therefore, we conclude that this equilibrium point is absolutely stable.

On the other hand, let $K_1 = K_2 = 1$ and $p = q = 1/2$ for (6.1). The equilibrium point is $(x^*, y^*) = (2/3, 2/3)$. Unfortunately, these coefficients do not satisfy the conditions in Corollary 6.1. Then the conditions in Corollary 6.2 are reduced to

$$0 < \tau_1 < \frac{3}{2}, \qquad 0 < \tau_2 < \frac{3}{2}, \qquad \left(1 - \frac{2}{3}\tau_1\right)\left(1 - \frac{2}{3}\tau_2\right) > \frac{1}{4}.$$

In particular, $\tau_1 = \tau_2 = 1/2$ satisfies the above inequalities. This fact tells us that the equilibrium point is conditionally stable.

Delayed Prey–Predator System: Secondly, we consider the prey–predator system with delay effects, which system has the different type of the delay effects from the system (6.1).

$$\begin{cases} x'(t) = x(t)\big(K_1 - x(t-\tau_1)\big) - px(t)y(t), \\ y'(t) = y(t)\big(- K_2 - y(t-\tau_2)\big) + qx(t)y(t), \end{cases} \tag{6.3}$$

where K_1, K_2, p, q, τ_1 and τ_2 are positive constants. The real-valued unknown functions $x(t)$ and $y(t)$ denote the population of prey and predator, respectively. The system (6.3) has an equilibrium point

$$(x^*, y^*) = \left(\frac{K_1 + pK_2}{pq+1}, \frac{qK_1 - K_2}{pq+1}\right).$$

We linearize the system (6.3) around (x^*, y^*), obtaining

$$\begin{cases} x'(t) = -x^*x(t-\tau_1) - px^*y(t), \\ y'(t) = qy^*x(t) - y^*y(t-\tau_2). \end{cases} \tag{6.4}$$

Then the system (6.4) is also rewritten as (1.3) with $u(t) = (x(t), y(t))^\top$,

$$A = \begin{pmatrix} 0 & px^* \\ -qy^* & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} x^* & 0 \\ 0 & y^* \end{pmatrix}, \tag{6.5}$$

and $\tau_{11} = \tau_1$, $\tau_{22} = \tau_2$, $\tau_{12} = \tau_{21} = 0$. The coefficient matrices (6.5) tell us that Theorem 3.1 is not applicable for this system. Therefore, we try to apply Theorem 4.1 to the linearized system (6.4), and obtain the following result.

Corollary 6.4. *If the coefficients of the system* (6.3) *satisfy*

$$0 < \tau_1 < \frac{pq+1}{2(K_1 + pK_2)}, \qquad 0 < \tau_2 < \frac{pq+1}{2(qK_1 - K_2)}$$

and

$$\left(1 - \frac{2(K_1 + pK_2)\tau_1}{pq+1}\right)\left(1 - \frac{2(qK_1 - K_2)\tau_2}{pq+1}\right) > pq,$$

then the equilibrium point (x^*, y^*) *of the system* (6.3) *is conditionally stable.*

Remark 6.5. As in the case of Corollaries 6.1 and 6.2, $x^* > 0$ and $y^* > 0$ are always satisfied under the assumptions in Corollary 6.4.

For example, let $K_1 = 2/3$, $K_2 = 1/3$ and $p = q = 2/3$ for (6.3). The equilibrium point is $(x^*, y^*) = (8/13, 1/13)$. Then the conditions in Corollary 6.4 are reduced to

$$0 < \tau_1 < \frac{13}{16}, \qquad 0 < \tau_2 < \frac{13}{2}, \qquad \left(1 - \frac{16}{13}\tau_1\right)\left(1 - \frac{2}{13}\tau_2\right) > \frac{4}{9}.$$

In particular, $\tau_1 = \tau_2 = 13/48$ satisfies the above inequalities. This fact tells us that the equilibrium point is conditionally stable.

For the other related results to the stability and the bifurcation of solutions to the population models (6.1) and (6.3), we refer [11, 12, 16, 20] to readers.

Neural Network Models with Multiple Time Delays: In the rest of this article, as an example of the high-dimensional system, we consider the Hopfield network of arbitrary size with multiple time delays studied by Campbell in [3, 4]. We assume that the network consists of a ring of neurons where the j-th element receives two time delayed inputs: one from itself with delay τ_{jj}, one from the previous element with delay τ_{j-1}. For the structure of this system, we refer to Figure 1 in [4]. The model system is given by the following delay differential equations:

$$\begin{aligned}
C_1 u_1'(t) &= -\frac{1}{R_1} u_1(t) + F_1(u_1(t-\tau_{11})) + G_1(u_n(t-\tau_n)),\\
C_2 u_2'(t) &= -\frac{1}{R_2} u_2(t) + F_2(u_2(t-\tau_{22})) + G_2(u_1(t-\tau_1)),\\
&\ \vdots\\
C_n u_n'(t) &= -\frac{1}{R_n} u_n(t) + F_n(u_n(t-\tau_{nn})) + G_n(u_{n-1}(t-\tau_{n-1})),
\end{aligned} \tag{6.6}$$

where C_j and R_j are positive constants representing, respectively, the capacitances and resistances of the individual neurons, while F_j and G_j are smooth nonlinear functions representing, respectively, the feedback from j-th neuron to itself, and the connection from $j-1$-th neuron to j-th neuron. We normalize (6.6) and get the following system:

$$\begin{aligned}
u_1'(t) &= -d_1 u_1(t) + f_1(u_1(t-\tau_{11})) + g_1(u_n(t-\tau_n)),\\
u_2'(t) &= -d_2 u_2(t) + f_2(u_2(t-\tau_{22})) + g_2(u_1(t-\tau_1)),\\
&\ \vdots\\
u_n'(t) &= -d_n u_n(t) + f_n(u_n(t-\tau_{nn})) + g_n(u_{n-1}(t-\tau_{n-1})),
\end{aligned} \tag{6.7}$$

where $d_j = 1/(C_j R_j)$, $f_j(u_j) = F_j(u_j)/C_j$ and $g_j(u_{j-1}) = G_j(u_{j-1})/C_j$. Assume that the system (6.7) has the equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^\top$ which satisfies

$$d_j u_j^* - f_j(u_j^*) = g_j(u_{j-1}^*), \qquad j = 1, \cdots, n.$$

We linearize the system (6.7) around $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^\top$, obtaining

$$\begin{aligned}
u_1'(t) &= -d_1 u_1(t) + f_1'(u_1^*) u_1(t-\tau_{11}) + g_1'(u_n^*) u_n(t-\tau_n),\\
u_2'(t) &= -d_2 u_2(t) + f_2'(u_2^*) u_2(t-\tau_{22}) + g_2'(u_1^*) u_1(t-\tau_1),\\
&\ \vdots\\
u_n'(t) &= -d_n u_n(t) + f_n'(u_n^*) u_n(t-\tau_{nn}) + g_n'(u_{n-1}^*) u_{n-1}(t-\tau_{n-1}).
\end{aligned} \tag{6.8}$$

Then the system (6.8) is rewritten as (1.3) with

$$A = \begin{pmatrix} d_1 & 0 & \dots & 0\\ 0 & d_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & d_n \end{pmatrix}, \quad B = -\begin{pmatrix} f_1'(u_1^*) & 0 & \dots & 0 & g_1'(u_n^*)\\ g_2'(u_1^*) & f_2'(u_2^*) & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & f_{n-1}'(u_{n-1}^*) & 0\\ 0 & 0 & \dots & g_n'(u_{n-1}^*) & f_n'(u_n^*) \end{pmatrix}.$$

Therefore, applying Theorems 3.1 and 4.1 to the linearized system (6.8), we derive the following results immediately.

Corollary 6.6. *If the coefficients of the system* (6.7) *satisfy*

$$d_j - |f_j'(u_j^*)| > 0, \qquad 1 \le j \le n,$$

and

$$\prod_{j=1}^{n} (d_j - |f_j'(u_j^*)|) > |g_1'(u_n^*)| \prod_{j=2}^{n} |g_j'(u_{j-1}^*)|,$$

then the equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^\top$ *of the system* (6.7) *is absolutely stable.*

Corollary 6.7. *If the coefficients of the system* (6.7) *satisfy*

$$d_j - f_j'(u_j^*) > 0, \qquad 0 \le \tau_{jj} < \frac{1}{2|f_j'(u_j^*)|}, \qquad 1 \le j \le n, \tag{6.9}$$

and

$$\prod_{j=1}^{n} \left(d_j - f_j'(u_j^*)\right) (1 - 2|f_j'(u_j^*)|\tau_{jj}) > |g_1'(u_n^*)| \prod_{j=2}^{n} |g_j'(u_{j-1}^*)|, \tag{6.10}$$

then the equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^\top$ *of the system* (6.7) *is conditionally stable.*

Remark 6.8. In Theorems 4.1 and 4.2 in [3], some sufficient conditions for the absolute stability and the conditional stability for the system (6.7) had already been derived. Concerned with the absolute stability, Corollary 6.6 is same as Theorem 4.1 in [3]. On the other hand, for the conditional stability, Corollary 6.7 is a different statement from Theorem 4.2 in [3]. Precisely, it was shown in Theorem 4.2 in [3] that if the coefficients of the system (6.7) satisfy

$$\begin{gathered} d_j - f_j'(u_j^*) > 0, \qquad 1 \le j \le n, \\ 0 \le \tau_{jj} < \frac{1}{d_j}\left(\sqrt{1 + \frac{d_j}{|f_j'(u_j^*)|}} - 1\right) \qquad \text{for } j \text{ s.t. } f_j'(u_j^*) < 0, \end{gathered} \tag{6.11}$$

and

$$\prod_{j=1}^{n} (d_j - f_j'(u_j^*)) > |g_1'(u_n^*)| \prod_{j=2}^{n} |g_j'(u_{j-1}^*)|, \tag{6.12}$$

then the equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^\top$ of the system (6.7) is asymptotically stable. We note that there is no inclusion relation between the above stability condition and the one appeared in Corollary 6.7. To explain this fact, let us consider a simple case of $d_j = d > 0$, $-f_j'(u_j^*) = c > 0$ and $\tau_{jj} = \tau > 0$ for all $1 \le j \le n$. Then the conditions (6.9) and (6.10) are reduced to

$$\tau < \frac{1}{2c}, \qquad (d + c)(1 - 2\tau c) > \gamma. \tag{6.13}$$

On the other hand, the conditions (6.11) and (6.12) are also reduced to

$$\tau < \frac{1}{d}\left(\sqrt{1 + \frac{d}{c}} - 1\right), \qquad d + c > \gamma. \tag{6.14}$$

Here we define

$$\gamma = \left(|g_1'(u_n^*)| \prod_{j=2}^{n} |g_j'(u_{j-1}^*)|\right)^{1/n}.$$

In particular, letting $\tau = c = 1/2$, the conditions (6.13) and (6.14) are described as

$$\frac{1}{2}\left(d + \frac{1}{2}\right) > \gamma \tag{6.15}$$

and

$$\frac{1}{2} < \frac{1}{d}\left(\sqrt{1+2d}-1\right), \qquad d+\frac{1}{2} > \gamma, \tag{6.16}$$

respectively. Then $d = 4$ and $\gamma = 2$ satisfies (6.15) but not (6.16). On the contrary $d = \gamma = 1$ satisfies (6.16) but not (6.15). Therefore, we conclude that there is no inclusion relation between (6.15) and (6.16).

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