REGULATORS OF K_2 OF HYPERGEOMETRIC FIBRATIONS

MASANORI ASAKURA

1. Introduction

In the paper [AO2], Otsubo and the author introduced a certain class of fibrations of algebraic varieties which we named *hypergeometric fibrations* (abbreviated HG fibrations, see §2.1 for the definition). In a series of joint papers [AO1]...[AO4], we studied K_1 of HG fibrations and the *Beilinson regulator*. Our main results are to describe the regulators via the generalized hypergeometric functions

$$ {}_pF_{p-1}\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_{p-1}\end{matrix};x\right)=\sum_{n=0}^\infty\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_{p-1})_n}\frac{x^n}{n!},\quad (\alpha)_n:=\Gamma(\alpha+n)/\Gamma(\alpha). $$

We refer the reader to [B], [E] and [Sl] for the fundamental theory on hypergeometric functions.

In this paper we study K_2 of HG fibrations. Let $f:X\to\mathbb{P}^1$ be a HG fibration defined in §2.1, and $X_t=f^{-1}(t)$ a smooth fiber, then we discuss the Beilinson regulator map

$$ \mathrm{reg}:K_2(X_t)\longrightarrow H^2_{\mathscr{D}}(X_t,\mathbb{Z}(2)) $$

to the Deligne-Beilinson cohomology group (e.g. [Sch]). We shall discuss the following cases.

- f is of Fermat type given in §2.2,
- f is of Gauss type given in §2.3,
- f is an elliptic fibration (e.g. the Legendre family).

In the above cases, there are nontrivial elements in $K_2(X_t)$. The main theorems are to give explicit descriptions of the regulators by linear combinations of the hypergeometric functions of the following types

$$ {}_3F_2\left(\begin{matrix}a,a,a\\ b,a+1\end{matrix};x\right),\quad {}_4F_3\left(\begin{matrix}a,b,1,1\\ 2,2,2\end{matrix};x\right). $$

The precise formulas are given in Theorems 3.2, 3.6, 3.4 in §3.2 for the Fermat type, and in Theorem 4.6 in §4.2 for the Gauss type.

In §5, we give similar formulas for some elliptic fibrations, such as the Legendre family. With the aid of MAGMA, we give some numerical examples verifying the Beilinson conjecture on $L(E,2)$, the L-function of an elliptic curve over $\mathbb{Q}$. In [RZ], Rogers and Zudilin proved certain formulas which describes $L(E,2)$ by special values of hypergeometric functions. Applying their result, we can obtain a "theorem" on the Beilinson conjecture for an elliptic curve of conductor 24. This seems a new approach toward the Beilinson conjecture for elliptic curves. The

Date: 2016.

2000 *Mathematics Subject Classification.* 14D07, 19F27, 33C20 (primary), 11G15, 14K22 (secondary).

Key words and phrases. Periods, Regulators, Hypergeometric functions.

author hopes that the study of the Beilinson conjecture by the hypergeometric functions will be developed more and bring a new progress.

Finally we note that there are previous works [O1], [O2] by Otsubo on hypergeometric functions and regulators on K_2 of Fermat curves. Although we were inspired a lot by his works, our results and methods are entirely different.

The author expresses sincere gratitude to Professor Noriyuki Otsubo for the stimulating discussion, especially on special values of L-functions of elliptic curves. He also expresses special thanks to Professor Wadim Zudilin for reading the first draft carefully and providing the proof of Lemma 3.5.

2. Hypergeometric Fibrations

Throughout this paper, we denote the fractional part of $x \in \mathbb{Q}$ by $\{x\}$:

$$\{x\} := x - \lfloor x \rfloor.$$

The Gaussian hypergeometric function

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)$$

is simply written by $F(a, b, c; x)$.

2.1. **Definition.** Let R be a finite-dimensional semisimple commutative $\mathbb{Q}$-algebra. Let $e : R \to E$ be a projection onto a number field E. For a R-module H, we write

$$H(e) := E \otimes_{e,R} H,$$

and call it the e-part of H.

Let X be a projective smooth variety over a field k. Let $f : X \to \mathbb{P}^1$ be a surjective morphism over k which is smooth over $U \subset \mathbb{P}^1$. Let $A = \mathrm{Pic}^0_f \to U$ be the Picard scheme over U. We say f is a *hypergeometric fibration with multiplication by* (R, e) (abbreviated HG fibration) if it is endowed with a ring homomorphism (called a *multiplication* by R)

$$R \longrightarrow \mathrm{End}_U(A) \otimes \mathbb{Q}$$

and the following conditions hold. We fix an inhomogeneous coordinate $t \in \mathbb{P}^1$.

- f is smooth outside $t = 0, 1, \infty$, hence we may take $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- Denote by $A(e) \to U$ the e-part of the abelian fibration which corresponds to the e-part $(R^1 f_* \mathbb{Q}_l)(e)$ of a l-adic sheaf,

$$T_l A(e) \otimes \mathbb{Q} \cong R^1 f_* \mathbb{Q}_l(e).$$

  Then $\dim(A(e)/U) = [E : \mathbb{Q}]$ or equivalently $\dim_{\mathbb{Q}_l}(R^1 f_* \mathbb{Q}_l)(e) = 2[E : \mathbb{Q}]$.
- The abelian fibration $A(e) \to U$ has a totally degenerate semistable reduction at $t = 1$.

The last condition is equivalent to say that the local monodromy T on $(R^1 f_* \mathbb{Q}_l)(e)$ at $t = 1$ is unipotent and the rank of log monodromy $N := \log(T)$ is maximal, namely $\mathrm{rank}(N) = \frac{1}{2} \dim_{\mathbb{Q}}(R^1 f_* \mathbb{Q}_l)(e)$ $(= [E : \mathbb{Q}]$ by the second condition).

Example 2.1 (Elliptic fibrations)**.** The simplest example of hypergeometric fibrations is an elliptic fibration $f : X \to \mathbb{P}^1$ which satisfies that f is smooth over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and has a multiplicative reduction at $t = 1$. In this case we take

$R = E = \mathbb{Q}$, $e = \mathrm{id}$. We refer the reader to [H] for the classification of elliptic fibrations over $\mathbb{P}^1$ which has singular fibers at most over 3 points. In particular, there are 14 cases of such fibrations which have at least one multiplicative reduction.

2.2. **HG fibration of Fermat type.** Suppose that the characteristic of k is 0. Let $f : X \to \mathbb{P}^1$ be the fibration such that the general fiber $X_t = f^{-1}(t)$ is a smooth projective curve defined by an equation

$$(x^n - 1)(y^m - 1) = 1 - t, \quad n, m \geq 2.$$

We call f a fibration of Fermat type[1]. One can show that the genus is given by $g(X_t) = (n-1)(m-1)$ (e.g. by the Hurwitz formula). Moreover f is smooth outside $t = 0, 1, \infty$, and f has a totally degenerate semistable reduction at $t = 1$. We denote by $\mu_k \subset \overline{k}^\times$ the group of k-th roots of unity. Suppose $\mu_n, \mu_m \subset k^\times$. The action $(x, y, t) \mapsto (\zeta_n x, \zeta_m y, t)$ for $(\zeta_n, \zeta_m) \in \mu_n \times \mu_m$ gives a multiplication by the group ring $R = \mathbb{Q}[\mu_n \times \mu_m]$. If $e : R \to E$ factors through projections $\mu_n \times \mu_m \to \mu_n$ or $\mu_n \times \mu_m \to \mu_m$, then $H^1(X_t)(e) = 0$. Therefore

$$H^1(X_t) = \bigoplus_e H^1(X_t)(e),$$

where e does not factor through projections $\mu_n \times \mu_m \to \mu_n$ or $\mu_n \times \mu_m \to \mu_m$.

Lemma 2.2. *Put*

$$\omega_{i,j} := x^{i-1} y^{j-1} \frac{m^{-1} dx}{y^{m-1}(x^n - 1)} = -x^{i-1} y^{j-1} \frac{n^{-1} dy}{x^{n-1}(y^m - 1)}$$

for $i, j \in \mathbb{Z}$. *Then* $\varGamma(X_t, \Omega^1_{X_t})$ *is* $(n-1)(m-1)$*-dimensional with basis* $\{\omega_{i,j} \mid 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$. *Hence*

$$\dim_E H^1(X_t)(e) = \begin{cases} 0 & e \text{ factoring through } \mu_n \times \mu_m \to \mu_n \text{ or } \mu_n \times \mu_m \to \mu_m \\ 1 & \text{others.} \end{cases}$$

f *is a HG fibration with multiplication by* (R, e) *if and only if* $\dim_E H^1(X_t)(e) = 1$, *and then*

$$\varGamma(X_t, \Omega^1_{X_t})(e) = \bigoplus_{(i,j) \in I_e} k \cdot \omega_{i,j}$$

$$I_e := \{([si_0]_n, [sj_0]_m) \mid s \in (\mathbb{Z}/nm\mathbb{Z})^\times\}, \tag{2.1}$$

where (i_0, j_0) *is a fixed index such that a homomorphism* $R \to k$, $(\zeta_n, \zeta_m) \mapsto \zeta_n^{i_0} \zeta_m^{j_0}$ *factors through* e, *and* $[a]_n$ *denotes the unique integer such that* $[a]_n \equiv a \bmod n$ *and* $0 \leq [a]_n < n$.

Proof. See [AO2] §3.3. □

Suppose that the base field is $\mathbb{C}$. Let $\varepsilon_1 \in \mu_n$ and $\varepsilon_2 \in \mu_m$, and let $P(\varepsilon_1, \varepsilon_2)$ denote the singular point $(x, y) = (\varepsilon_1, \varepsilon_2)$ of $f^{-1}(1)$. Let $\delta(\varepsilon_1, \varepsilon_2) \in H_1(X_t, \mathbb{Z})$ be the vanishing cycle at $t = 1$ which "converges to $P(\varepsilon_1, \varepsilon_2)$", namely it is a homology cycle characterized by

$$\frac{1}{(2\pi\sqrt{-1})^2} \oint_{t=1} \int_{\delta(\varepsilon_1, \varepsilon_2)} \omega = \mathrm{Res}_P(\omega), \quad \forall\, \omega \in H^2_{\mathrm{dR}}(\mathscr{X}^*)$$

[1]The reason why we call "Fermat type" is that the fiber over $t = 0$ is

$$(x^n - 1)(y^m - 1) = 1 \iff x^{-n} + y^{-m} = 1,$$

so that the Fermat curve appears in the degenerating fiber.

where $\mathscr{X}^*$ is the tubular neighborhood of $f^{-1}(1)$ and $\mathrm{Res}_P : H^2_{\mathrm{dR}}(\mathscr{X}^*) \to \mathbb{C}$ is the Poincare residue map at $P = P(\varepsilon_1, \varepsilon_2)$.

For later use, we here give a down-to-earth description of a path $\delta(\varepsilon_1, \varepsilon_2)$. For $(\zeta_1, \zeta_2) \in \mu_n \times \mu_m$, we denote by $\sigma(\zeta_1, \zeta_2)$ the automorphisms of X_t given by $(x, y) \mapsto (\zeta_1 x, \zeta_2 y)$. Suppose $|t - 1| \ll 1$ and fix $\sqrt[n]{t}$. Let $Q_1(x, y) = (1, \infty)$ and $Q_t(x, y) = (\sqrt[n]{t}, 0)$ be points of X_t. Define a (unique) path u from Q_t to Q_1 such that the projection onto the y-plane is a line $\arg(y) = -\pi/m$ from $y = 0$ to $y = \infty$. Put

$$\delta(1,1) := (1 - \sigma(1, e^{\frac{2\pi\sqrt{-1}}{m}}))u, \quad \delta(\varepsilon_1, \varepsilon_2) := \sigma(\varepsilon_1, \varepsilon_2)\delta(1,1). \tag{2.2}$$

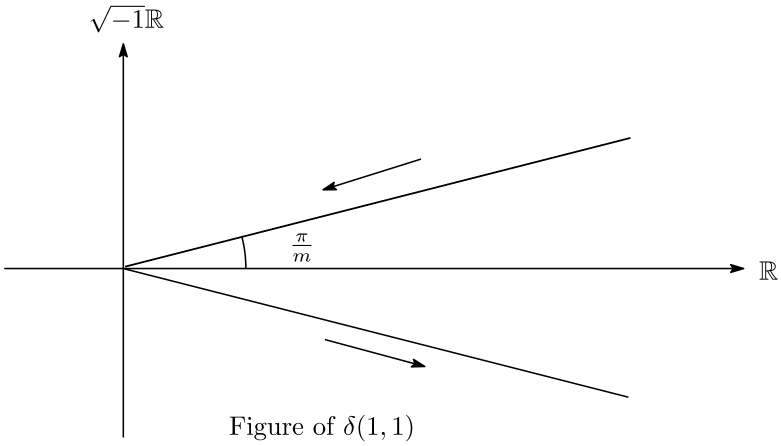


Figure of $\delta(1,1)$

Lemma 2.3.

$$\int_{\delta(\varepsilon_1,\varepsilon_2)} \omega_{i,j} \underset{\cdot}{=} -\frac{\varepsilon_1^i \varepsilon_2^j}{nm} \cdot 2\pi\sqrt{-1} F\left(1 - \frac{i}{n}, 1 - \frac{j}{m}, 1; 1 - t\right).$$

Proof. Since

$$\int_{\delta(\varepsilon_1,\varepsilon_2)} \omega_{i,j} = \int_{\delta(1,1)} \sigma(\varepsilon_1, \varepsilon_2)\omega_{i,j} = \varepsilon_1^i \varepsilon_2^j \int_{\delta(1,1)} \omega_{i,j}$$

we only need to show the case $\delta(1,1)$. Write $\zeta_m := e^{2\pi\sqrt{-1}/m}$ and $\zeta_{2m} := e^{\pi\sqrt{-1}/m}$

$$\begin{aligned}
\int_{\delta(1,1)} \omega_{i,j} &= (1 - \zeta_m^j) \int_u \omega_{i,j} \\
&= -(1 - \zeta_m^j) \int_u x^{i-1} y^{j-1} \frac{n^{-1} dy}{x^{n-1}(y^m - 1)} \\
&= (1 - \zeta_m^j) \int_u y^{j-1} \left(\frac{t - y^m}{1 - y^m}\right)^{\frac{i}{n} - 1} \frac{n^{-1} dy}{1 - y^m} \\
&= \frac{1 - \zeta_m^j}{n} \int_u y^{j-1} (1 - y^m)^{-\frac{i}{n}} (t - y^m)^{\frac{i}{n} - 1} dy \\
&= \frac{\zeta_{2m}^{-j} - \zeta_{2m}^j}{n} \int_0^\infty y^{j-1} (1 + y^m)^{-\frac{i}{n}} (t + y^m)^{\frac{i}{n} - 1} dy
\end{aligned}$$

$$\begin{aligned}
&= \frac{\zeta_{2m}^{-j} - \zeta_{2m}^{j}}{nm} \int_0^\infty y^{\frac{j}{m}-1}(1+y)^{-\frac{i}{n}}(t+y)^{\frac{i}{n}-1}dy \\
&= \frac{\zeta_{2m}^{-j} - \zeta_{2m}^{j}}{nm} \int_1^\infty (y-1)^{\frac{j}{m}-1}y^{-\frac{i}{n}}(t-1+y)^{\frac{i}{n}-1}dy \\
&= \frac{\zeta_{2m}^{-j} - \zeta_{2m}^{j}}{nm} \int_0^1 (y^{-1}-1)^{\frac{j}{m}-1}y^{\frac{i}{n}-2}(t-1+y^{-1})^{\frac{i}{n}-1}dy \\
&= \frac{\zeta_{2m}^{-j} - \zeta_{2m}^{j}}{nm} \int_0^1 (1-y)^{\frac{j}{m}-1}y^{-\frac{j}{m}}(1-(1-t)y)^{\frac{i}{n}-1}dy \\
&= \frac{\zeta_{2m}^{-j} - \zeta_{2m}^{j}}{nm} B\left(\frac{j}{m}, 1-\frac{j}{m}\right) F\left(1-\frac{i}{n}, 1-\frac{j}{m}, 1, 1-t\right) \\
&= -\frac{2\pi\sqrt{-1}}{nm} F\left(1-\frac{i}{n}, 1-\frac{j}{m}, 1, 1-t\right).
\end{aligned}$$

□

Lemma 2.4. *Let T_1 be the local monodromy at $t = 1$. There is a unique homology cycle $\gamma(\varepsilon_1, \varepsilon_2) \in H_1(X_t, \mathbb{Q})$ such that $(T_1 - 1)\gamma(\varepsilon_1, \varepsilon_2) = \delta(\varepsilon_1, \varepsilon_2)$ and*

$$\int_{\gamma(\varepsilon_1,\varepsilon_2)} \omega_{i,j} = \frac{\varepsilon_1^i \varepsilon_2^j}{nm} B\left(1-\frac{i}{n}, 1-\frac{j}{m}\right) F\left(1-\frac{i}{n}, 1-\frac{j}{m}, 2-\frac{i}{n}-\frac{j}{m}; t\right).$$

Proof. The uniqueness follows from the fact that the monodromy invariant part of $H_1(X_t)$ is trivial. We show the existence. Write $\mathrm{Ev} := \langle \delta(\varepsilon_1, \varepsilon_2) \mid (\varepsilon_1, \varepsilon_2) \in \mu_n \times \mu_m \rangle \subset H_1(X_t, \mathbb{Q})$. Then it follows from the last condition of HG fibration in Definition 2.1 that one has

$$N_1 := T_1 - 1 : H_1(X_t, \mathbb{Q})/\mathrm{Ev} \xrightarrow{\cong} \mathrm{Ev}.$$

Therefore there is a unique homology cycle $\gamma(\varepsilon_1, \varepsilon_2) \in H_1(X_t, \mathbb{Q})$ such that $(T_1 - 1)\gamma(\varepsilon_1, \varepsilon_2) = \delta(\varepsilon_1, \varepsilon_2)$ up to Ev. Let T_0 be the local monodromy at $t = 0$. Since $T_0 - 1 : \mathrm{Ev} \to \mathrm{Ev}$ is bijective, we can choose $\gamma(\varepsilon_1, \varepsilon_2)$ such that $(T_0 - 1)\gamma(\varepsilon_1, \varepsilon_2) = 0$ by replacing $\gamma(\varepsilon_1, \varepsilon_2)$ with $\gamma(\varepsilon_1, \varepsilon_2) + \delta_0$. Then we show that this gives the desired cycle. The monodromy of Gauss hypergeometric functions is well-known, in particular,

$$(T_1 - 1)B(a, b)F(a, b, a + b; t) = -2\pi\sqrt{-1}F(a, b, 1; 1 - t). \tag{2.3}$$

Therefore letting

$$f_1 := -2\pi\sqrt{-1}F\left(1-\frac{i}{n}, 1-\frac{j}{m}, 1; 1-t\right)$$

$$f_2 := B\left(1-\frac{i}{n}, 1-\frac{j}{m}\right) F\left(1-\frac{i}{n}, 1-\frac{j}{m}, 2-\frac{i}{n}-\frac{j}{m}; t\right),$$

we have

$$(T_1 - 1)\int_{\gamma(\varepsilon_1,\varepsilon_2)} \omega_{i,j} = \int_{\delta(\varepsilon_1,\varepsilon_2)} \omega_{i,j} = \frac{\varepsilon_1^i \varepsilon_2^j}{nm} f_1 = (T_1 - 1)\frac{\varepsilon_1^i \varepsilon_2^j}{nm} f_2.$$

On the other hand since $(T_0 - 1)\gamma(\varepsilon_1, \varepsilon_2) = 0$, we have

$$(T_0 - 1)\int_{\gamma(\varepsilon_1,\varepsilon_2)} \omega_{i,j} = 0 = (T_0 - 1)\frac{\varepsilon_1^i \varepsilon_2^j}{nm} f_2.$$

Thus

$$F := \int_{\gamma(\varepsilon_1,\varepsilon_2)} \omega_{i,j} - \frac{\varepsilon_1^i \varepsilon_2^j}{nm} f_2$$

is invariant under both local monodromies, and this means $F = 0$. □

2.3. **HG fibration of Gauss type.** Suppose that the characteristic of k is 0. Let $f : X \to \mathbb{P}^1$ be the fibration whose general fiber is the smooth completion of an affine curve

$$y^N = x^a(1-x)^b(1-tx)^{N-b}, \quad 0 < a, b < N, \ \gcd(N, a, b) = 1.$$

f is smooth over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. Suppose that $k^\times$ contains all N-th roots of unity, and denote by $\mu_N \subset k^\times$ the group of all N-th roots. The action $(x, y, t) \mapsto (x, \zeta_N y, t)$ for $\zeta_N \in \mu_N$ gives a multiplication by the group ring $R = \mathbb{Q}[\mu_N]$. Then f is a HG fibration with multiplication by (R, e) if and only if a projection $e : \mathbb{Q}[\mu_N] \to E$ satisfies $ad/N \notin \mathbb{Z}$ and $bd/N \notin \mathbb{Z}$ where $d := \sharp\mathrm{Ker}[e : \mu_N \to E^\times]$ ([AO2] §3.2).

Lemma 2.5. *Let $X_t = f^{-1}(t)$ denotes the general fiber. Put a 1-form*

$$\omega_n := \frac{x^{p_n}(1-x)^{q_n}(1-tx)^{n-1-q_n}}{y^n} dx, \quad p_n := \lfloor \frac{an}{N} \rfloor, \quad q_n := \lfloor \frac{bn}{N} \rfloor$$

for $n \in \{1, 2, \ldots, N-1\}$. Put $d := \sharp\mathrm{Ker}[e : \mu_N \to E^\times]$ and

$$I_e := \{n \in \mathbb{Z} \mid 1 \le n \le N-1, \ d|n, \ \gcd(n/d, N/d) = 1\}.$$

Then $\{\omega_n \mid n \in I_e\}$ forms a basis of the e-part $\Gamma(X_t, \Omega^1_{X_t})(e)$.

Proof. [Ar] (13), p.917. □

Lemma 2.6. *Suppose $k = \mathbb{C}$. Write $a_n := \{an/N\}$ and $b_n := \{bn/N\}$. There are points $P_0, P_1 \in X_t$ such that $x = 0, 1$ and a homology cycle*

$$u_0 \in H_1^B(X_t, \{P_0, P_1\}; \mathbb{Z})$$

such that

$$\int_{u_0} \omega_n = B(a_n, b_n) F(a_n, b_n, a_n + b_n; t) \quad \text{for } |t| \ll 1.$$

Moreover letting T_1 be the local monodromy at $t = 1$ and $u_1 := (1 - T_1)u_0$, we have

$$\int_{u_1} \omega_n = 2\pi\sqrt{-1} F(a_n, b_n, 1; 1-t).$$

The e-part $H_1^B(X_t, \mathbb{Q})(e)$ is spanned by

$$\gamma_0 := (1-\sigma)u_0, \quad \gamma_1 := (1-\sigma)u_1$$

as E-module where σ is an automorphism of X_t given by $(x, y) \mapsto (x, e^{\frac{2\pi\sqrt{-1}}{N}} y)$.

Proof. Define a path u_0 as

$$(x, y) = (s, s^{\frac{a}{N}}(1-s)^{\frac{b}{N}}(1-ts)^{1-\frac{b}{N}}), \quad s \in [0, 1]$$

in which $s^{\frac{a}{N}}$, $(1-s)^{\frac{b}{N}}$ take values in $\mathbb{R}_{\ge 0}$ and $(1-ts)^{1-\frac{b}{N}}$ takes values such that $|(1-ts)^{1-\frac{b}{N}} - 1| \ll 1$. Then

$$\int_{u_0} \omega_n = \int_0^1 x^{a_n - 1}(1-x)^{b_n - 1}(1-tx)^{-b_n} dx$$
$$= B(a_n, b_n) F(a_n, b_n, a_n + b_n; t).$$

Hence the assertion for u_1 follows from (2.3). The last assertion follows from the fact that $\dim_E H_1^B(X_t,\mathbb{Q})(e)=2$ and that γ_0 and γ_1 are E-linearly independent because their images by the map

$$H_1^B(X_t,\mathbb{Q})(e)\longrightarrow \mathscr{O}\omega_n^\vee=\mathrm{Hom}(\mathscr{O}\omega_n,\mathscr{O}),\quad \gamma\longmapsto\int_\gamma\omega_n$$

are $\mathbb{C}$-linearly independent. □

Lemma 2.7. *Let the notation be as in Lemma 2.5. Then*

$$\Gamma(X,\Omega^2_X(\log Y))(e)=\bigoplus_{n\in I_e}k\cdot\frac{dt}{t-1}\omega_n.$$

Proof. Let $\chi:R\to k$ be a homomorphism of $\mathbb{Q}$-algebra factoring through e. Write

$$\Gamma(X,\Omega^2_X(\log Y))(\chi):=k\otimes_{\chi,k\otimes_\mathbb{Q}R}\Gamma(X,\Omega^2_X(\log Y)).$$

Then the assertion is equivalent to that for any χ

$$\Gamma(X,\Omega^2_X(\log Y))(\chi)=k\cdot\frac{dt}{t-1}\omega_n,\tag{2.4}$$

where $n\in\{1,\dots,N-1\}$ such that $\chi(\zeta)=\zeta^{-n}$ for $\forall\zeta\in\mu_N$.

We may suppose $k=\mathbb{C}$. Put $Y_0=f^{-1}(0)$, $Y_\infty=f^{-1}(\infty)$, $S=\mathbb{P}^1\setminus\{0,1,\infty\}$ and $U:=X\setminus(Y\cup Y_0\cup Y_\infty)=f^{-1}(S)$. Let $\mathscr{H}=H^1_{\mathrm{dR}}(U/S)$ be a connection. Then

$$\begin{aligned}\Gamma(X,\Omega^2_X(\log Y+Y_0+Y_\infty))&=F^2H^2_{\mathrm{dR}}(U)\\&=F^2H^1_{\mathrm{dR}}(S,\mathscr{H})\\&=\Gamma(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}(\log(0+1+\infty))\otimes\mathscr{H}_e),\end{aligned}$$

where $\mathscr{H}_e\subset j_*\mathscr{H}$, $j:S\hookrightarrow\mathbb{P}^1$ is Deligne's canonical extension. Hence

$$\Gamma(X,\Omega^2_X(\log Y+Y_0+Y_\infty))(\chi)=\Gamma(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}(\log(0+1+\infty))\otimes\mathscr{H}_e(\chi)).$$

Note that X is a nonsingular rational surface (Lemma 4.2 below). The localization sequence induces an isomorphism

$$\mathrm{Res}:F^2H^2_{\mathrm{dR}}(U)\xrightarrow{\cong}F^1H_1^{\mathrm{dR}}(Y)\oplus F^1H_1^{\mathrm{dR}}(Y_0)\oplus F^1H_1^{\mathrm{dR}}(Y_\infty)$$

by the Poincare residue map. Since $a_n,b_n\notin\mathbb{Z}$, the local monodromy at $t=\infty$ on $H^1(X_t,\mathbb{Q})$ has no eigenvalue 1 by Lemma 2.6. This implies the composition $H^2_{\mathrm{dR}}(U)(e)\to H_1^{\mathrm{dR}}(Y_\infty)$ is zero. Hence $H_1^{\mathrm{dR}}(Y_\infty)(e)=0$ and

$$\Gamma(X,\Omega^2_X(\log Y+Y_0+Y_\infty))(\chi)=\Gamma(X,\Omega^2_X(\log Y+Y_0))(\chi).$$

Summing up the above we have

$$0\to\Gamma(X,\Omega^2_X(\log Y))(\chi)\to\Gamma(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}(\log(0+1+\infty))\otimes\mathscr{H}_e(\chi))\overset{\mathrm{Res}}{\to}H_1^{\mathrm{dR}}(Y_0)\to 0.$$

By a computation of the periods in Lemma 2.6, one can get a explicit description of $\mathscr{H}_e$ and then

$$\Gamma(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}(\log(0+1+\infty))\otimes\mathscr{H}_e(\chi))=\begin{cases}\langle\frac{dt}{t}\omega_n,\ \frac{dt}{t-1}\omega_n\rangle_\mathbb{C} & a_n+b_n\le 1\\ \langle\frac{dt}{t-1}\omega_n\rangle_\mathbb{C} & a_n+b_n>1.\end{cases}$$

The details are left to the reader because it is a tedious computation, but see the proof of [AO4] Lemma 3.7. Now (2.4) is immediate. □

3. Regulators of K_2 of HG fibration of Fermat type

In this section the base field is $\mathbb{C}$.

3.1. Let X be a smooth proper variety over $\mathbb{C}$. Let

$$\mathrm{reg} : H^p_{\mathscr{M}}(X,\mathbb{Z}(q)) \longrightarrow H^p_{\mathscr{D}}(X,\mathbb{Z}(q)) \tag{3.1}$$

be the *Beilinson regulator map* from the motivic cohomology group to the Deligne-Beilinson cohomology group (cf. [Sch]). If $p \leq q$ and $p \neq 2q$, then the right hand side is canonically isomorphic to $\mathrm{Hom}(H^B_{p-1}(X,\mathbb{Z}),\mathbb{C}/\mathbb{Z}(q))$ modulo torsion. For $\xi \in H^p_{\mathscr{M}}(X,\mathbb{Z}(q))$ and $\gamma \in H^B_{p-1}(X,\mathbb{Z})$, we write the pairing by

$$\langle \mathrm{reg}(\xi) \,|\, \gamma\rangle \in \mathbb{C}/\mathbb{Z}(q).$$

Proposition 3.1. *Let $f : U \to S$ be a smooth proper morphism onto a smooth curve S over $\mathbb{C}$. Let $U_t = f^{-1}(t)$ denotes a fiber. Suppose $p = q \geq 1$. Let $\xi \in H^p_{\mathscr{M}}(U,\mathbb{Z}(p))$ and $\gamma_t \in H^B_{p-1}(U_t,\mathbb{Z})$. We think of*

$$F = \langle \mathrm{reg}(\xi|_{U_t}) \,|\, \gamma_t\rangle$$

being a multi-valued function of variable t which is locally holomorphic on $t \in S$. Let

$$\mathrm{dlog}(\xi) = dt \wedge \omega \in \Gamma(U,\Omega^p_U).$$

Then

$$\frac{dF}{dt} = \pm \int_{\gamma_t} \omega.$$

If $p = q = 2$, let $\xi = \sum\{f,g\}$ be a K_2-symbol. Thanks to Beilinson's formula, one has

$$F = \sum \int_{\gamma_t} \log f \frac{dg}{g} - \log g(O) \frac{df}{f}$$

where O is the origin of a loop γ_t (e.g. [Ha] *Proposition 6.3). Then*

$$\frac{dF}{dt} = \int_{\gamma_t} \omega, \quad \textit{where } \sum \frac{df}{f} \wedge \frac{dg}{g} = dt \wedge \omega.$$

Proof. The regulator map (3.1) sits into a commutative diagram

$$\begin{array}{ccc} H^p_{\mathscr{M}}(U,\mathbb{Q}(p)) & \xrightarrow{\mathrm{reg}_S} & \mathrm{Ext}_S(\mathbb{Q}, R^{p-1}f_*\mathbb{Q}(p)) \\ \downarrow & & \downarrow \\ H^p_{\mathscr{M}}(U_t,\mathbb{Q}(p)) & \xrightarrow{\mathrm{reg}} & \mathrm{Ext}(\mathbb{Q}, H^{p-1}(U_t,\mathbb{Q}(p))) \end{array}$$

where Ext_S (resp. Ext) denotes the group of 1-extensions of admissible variations of MHS's (resp. MHS's), and the vertical arrows are the restriction maps. Let

$$0 \longrightarrow R^{p-1}f_*\mathbb{Q}(p) \longrightarrow \mathscr{V} \longrightarrow \mathbb{Q} \longrightarrow 0$$

be the corresponding 1-extension to $\mathrm{reg}_S(\xi)$. Let $e_{\mathrm{dR}} \in \mathscr{V}_{\mathrm{dR}} \cap F^0$ and $e_B \in \mathscr{V}_B$ be local liftings of $1 \in \mathbb{Q}$. Then $e_{\mathrm{dR}} - e_B \in R^{p-1}f_*\mathbb{Q}(p)$, and

$$F = \pm\langle e_{\mathrm{dR}} - e_B, \gamma_t\rangle,$$

where $\langle -,-\rangle : H^{p-1}(X_t,\mathbb{C})\otimes H^B_{p-1}(X_t,\mathbb{C})\to\mathbb{C}$ is the natural pairing. Fix a lifting $\widetilde{\gamma}_t\in\mathscr{V}_B^\vee\otimes\mathbb{Q}(p)$ via the surjective map $\mathscr{V}_B^\vee\otimes\mathbb{Q}(p)\to H^B_{p-1}(X_t,\mathbb{Q})$. Then one has

$$\pm F=\langle e_{\mathrm{dR}},\widetilde{\gamma}_t\rangle-\overbrace{\langle e_B,\widetilde{\gamma}_t\rangle}^{\mathbb{Q}(p)}$$

and hence

$$\pm\frac{dF}{dt}=\frac{d}{dt}\langle e_{\mathrm{dR}},\widetilde{\gamma}_t\rangle=\langle\nabla(e_{\mathrm{dR}}),\gamma_t\rangle,$$

where the last pairing is the natural pairing on $\Omega^1_S\otimes H^{p-1}_{\mathrm{dR}}(U/S)$ and $H^B_{p-1}(X_t,\mathbb{C})$. Note that $\nabla(e_{\mathrm{dR}})$ is the extension data of

$$0\longrightarrow H^{p-1}_{\mathrm{dR}}(U/S)\longrightarrow\mathscr{V}_{\mathrm{dR}}\longrightarrow\mathscr{O}_S\longrightarrow 0.$$

and this corresponds to the de Rham realization of ξ. Hence $\nabla(e_{\mathrm{dR}})=\operatorname{dlog}(\xi)$, and the former assertion follows. The latter assertion follows from this and the fact that $d(\int_\gamma\eta)=(\int_\gamma\omega)dt$ for 1-forms η and ω such that $d\eta=dt\wedge\omega$. □

3.2. Main Theorems. Let f be a HG fibration of Fermat type,

$$X_t=f^{-1}(t):(x^n-1)(y^m-1)=1-t,\quad n,\,m\geq 2$$

on which the group $\mu_n\times\mu_m$ acts where $\mu_n\subset\mathbb{C}^\times$ denotes the group of n-th roots of unity. We then discuss the Beilinson regulator map

$$\mathrm{reg}:H^2_{\mathscr{M}}(X_t,\mathbb{Q}(2))=K_2(X_t)^{(2)}\longrightarrow H^2_{\mathscr{D}}(X_t,\mathbb{Q}(2))=\operatorname{Hom}(H_1^B(X_t,\mathbb{Z}),\mathbb{C}/\mathbb{Q}(2)).$$

For $(\nu_1,\nu_2)\in\mu_n\times\mu_m$ such that $\nu_1,\nu_2\neq 1$, we consider a K_2-symbol

$$\xi=\left\{\frac{x-1}{x-\nu_1},\frac{y-1}{y-\nu_2}\right\}\in K_2(X\setminus f^{-1}(1)).\tag{3.2}$$

One immediately has

$$\operatorname{dlog}(\xi)=-\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{dt}{t-1}\omega_{i,j}.\tag{3.3}$$

The main theorems are formulas describing

$$\langle\mathrm{reg}(\xi)\,|\,\gamma\rangle=\langle\mathrm{reg}(\xi|_{X_t})\,|\,\gamma\rangle\in\mathbb{C}/\mathbb{Q}(2),\quad\gamma\in H_1^B(X_t,\mathbb{Q})$$

via the generalized hypergeometric functions.

Theorem 3.2. *Write $a_i:=1-i/n$ and $b_j:=1-j/m$. Let $\delta(\varepsilon_1,\varepsilon_2)$ be the homology cycle as in §2.2. Then for $|t-1|<1$*

$$\frac{1}{2\pi\sqrt{-1}}\langle\mathrm{reg}(\xi)\mid\delta(\varepsilon_1,\varepsilon_2)\rangle=C_0+C_1\log(1-t)+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\varepsilon_1^i\varepsilon_2^j}{nm}$$
$$\times\,a_ib_j(1-t)\,{}_4F_3\left(\begin{matrix}a_i+1,b_j+1,1,1\\2,2,2\end{matrix};1-t\right)$$

modulo $\mathbb{Q}(1) = 2\pi\sqrt{-1}\mathbb{Q}$ *where*

$$C_0 = \begin{cases} -\log(nm(1-\nu_1)(1-\nu_2)) & (\varepsilon_1,\varepsilon_2) = (1,1), (\nu_1,\nu_2) \\ \log(nm(1-\nu_1)(1-\nu_2)) & (\varepsilon_1,\varepsilon_2) = (1,\nu_2), (\nu_1,1) \\ \log\left(\frac{\varepsilon_2-1}{\varepsilon_2-\nu_2}\right) & \varepsilon_1 = 1 \text{ and } \varepsilon_2 \neq 1,\nu \\ \log\left(\frac{\varepsilon_1-1}{\varepsilon_1-\nu_1}\right) & \varepsilon_1 \neq 1,\nu_1 \text{ and } \varepsilon_2 = 1 \\ -\log\left(\frac{\varepsilon_2-1}{\varepsilon_2-\nu_2}\right) & \varepsilon_1 = \nu_1 \text{ and } \varepsilon_2 \neq 1,\nu \\ -\log\left(\frac{\varepsilon_1-1}{\varepsilon_1-\nu_1}\right) & \varepsilon_1 \neq 1,\nu_1 \text{ and } \varepsilon_2 = \nu_2 \\ 0 & \text{others} \end{cases}$$

$$C_1 = \begin{cases} 1 & (\varepsilon_1,\varepsilon_2) = (1,1), (\nu_1,\nu_2) \\ -1 & (\varepsilon_1,\varepsilon_2) = (1,\nu_2), (\nu_1,1) \\ 0 & \text{others.} \end{cases}$$

Remark 3.3. It is worth noting

$$C_0 = -\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\varepsilon_1^i\varepsilon_2^j}{nm}(2\psi(1)-\psi(a_i)-\psi(b_j)) \mod \mathbb{Q}(1)$$

$$C_1 = \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\varepsilon_1^i\varepsilon_2^j}{nm},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Hence we can rewrite

$$\frac{1}{2\pi\sqrt{-1}}\langle \mathrm{reg}(\xi) \mid \delta(\varepsilon_1,\varepsilon_2)\rangle = \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\varepsilon_1^i\varepsilon_2^j}{nm}$$
$$\times\left(-2\psi(1)+\psi(a_i)+\psi(b_j)+\log(1-t)+a_ib_j(1-t)\,{}_4F_3\left(\begin{matrix}a_i+1,b_j+1,1,1\\2,2,2\end{matrix};1-t\right)\right)$$

modulo $\mathbb{Q}(1)$.

Proof of Theorem 3.2. Put

$$F := \frac{1}{2\pi\sqrt{-1}}\langle \mathrm{reg}(\xi) \mid \delta(\varepsilon_1,\varepsilon_2)\rangle.$$

By Proposition 3.1 and (3.3)

$$(t-1)\frac{dF}{dt} = -\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{1}{2\pi\sqrt{-1}}\int_{\delta(\varepsilon_1,\varepsilon_2)}\omega_{i,j}.$$

By Lemma 2.3

$$(t-1)\frac{dF}{dt} = \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\varepsilon_1^i\varepsilon_2^j}{nm}F(a_i,b_j,1;1-t). \quad (3.4)$$

This immediately implies the desired formula except the constant term "C_0". Therefore it is enough to show

$$F = C_0 + C_1\log(1-t) + o(t-1) \mod \mathbb{Q}(1) \quad (3.5)$$

for $|t-1| \ll 1$. Here "$o(t-1)$" denotes a continuous function which converges to 0 as $t \to 1$. To do this we use Beilinson's formula (e.g. [Ha] Proposition 6.3)

$$\langle \mathrm{reg}\{f,g\} \mid \gamma \rangle = \int_\gamma \log f \frac{dg}{g} - \log g(O) \frac{df}{f}$$

where O is the origin of a loop $\gamma \in \pi_1(X_t, O)$ (it is important to fix the origin in the above formula). We show (3.5) only in case $(\varepsilon_1, \varepsilon_2) = (1,1)$ (the others are proven in a similar way):

$$F = -\log(nm(1-\nu_1)(1-\nu_2)) + \log(1-t) + o(1-t). \tag{3.6}$$

Recall the loop $\delta := \delta(1,1)$ with the origin $y=0$ from (2.2). Beilinson's formula yields

$$F = \frac{1}{2\pi\sqrt{-1}} \int_\delta \left(\log \frac{x-1}{x-\nu_1} \mathrm{dlog} \frac{y-1}{y-\nu_2} - \log(\nu_2^{-1}) \mathrm{dlog} \frac{x-1}{x-\nu_1} \right).$$

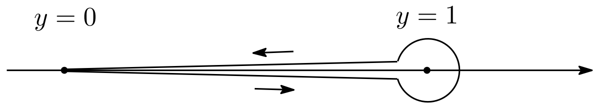
When $|t-1| \ll 1$, δ is a circle in a neighborhood of $(x,y) = (1,1)$ and

$$x - 1 = \frac{1-t}{y^m - 1}(1 + x + \cdots + x^{n-1})^{-1} = \frac{1-t}{nm(y-1)} + o(t-1)$$

on δ. Therefore

$$\begin{aligned}
\text{1st term of } F &= \frac{1}{2\pi\sqrt{-1}} \int_\delta \log\left(\frac{(1-t)(1-\nu_1)^{-1}}{nm(y-1)} \right) \mathrm{dlog} \frac{y-1}{y-\nu_2} + o(t-1) \\
&= \frac{1}{2\pi\sqrt{-1}} \oint_{y=1} \log\left(\frac{(1-t)(1-\nu_1)^{-1}}{nm(y-1)} \right) \frac{dy}{y-1} \\
&\quad - \frac{1}{2\pi\sqrt{-1}} \oint_{y=1} \log\left(\frac{(1-t)(1-\nu_1)^{-1}}{nm(y-1)} \right) \frac{dy}{y-\nu_2} + o(t-1) \\
&= \frac{-1}{2\pi\sqrt{-1}} \oint_{y=1} \log(y-1) \frac{dy}{y-1} + \log \frac{(1-t)}{nm(1-\nu_1)} \\
&\quad + \frac{1}{2\pi\sqrt{-1}} \oint_{y=1} \log(y-1) \frac{dy}{y-\nu_2} + o(t-1) \\
&= \log \frac{(1-t)}{nm(1-\nu_1)} + \frac{1}{2\pi\sqrt{-1}} \oint_{y=1} \log(y-1) \frac{dy}{y-\nu_2} + o(t-1) \mod \mathbb{Q}(1)
\end{aligned}$$

where "$\oint_{y=1}$" denotes the integral along a path with the origin $y=0$ which goes around $y=1$ in the counter-clockwise diretion and comes back to the origin (see figure).



Since

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}} \int_\delta \log(y-1) \frac{dy}{y-\nu_2} &= \log(-\nu_2) - \frac{1}{2\pi\sqrt{-1}} \oint_{y=1} \log(y-\nu_2) \frac{dy}{y-1} \\
&= \log(-\nu_2) - \log(1-\nu_2) \\
&= \log(\nu_2) - \log(1-\nu_2) \mod \mathbb{Q}(1),
\end{aligned}$$

we have

$$\text{1st term of } F = \log\left(\frac{1-t}{nm(1-\nu_1)(1-\nu_2)}\right) + \log(\nu_2) + o(t-1) \mod \mathbb{Q}(1).$$

On the other hand

$$\begin{aligned}\text{2nd term of } F &= \frac{-1}{2\pi\sqrt{-1}}\int_\delta \log(\nu_2^{-1})\mathrm{dlog}\left(\frac{1-t}{nm(y-1)}\right) + o(t-1)\\ &= \log(\nu_2^{-1}) + o(t-1).\end{aligned}$$

We thus have

$$F = \log\left(\frac{1-t}{nm(1-\nu_1)(1-\nu_2)}\right) + o(t-1).$$

the desired assertion (3.6). This completes the proof of Theorem 3.2. □

Theorem 3.4. *Put* $z := (1-t)^{-1}$ *and*

$$C_{a,b} := \frac{\sin(\pi a)}{\pi} B_{a,b} = \frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(b)}.$$

Then we have an alternative description of the regulators in Theorem 3.2

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\langle \mathrm{reg}(\xi) \mid \delta(\varepsilon_1,\varepsilon_2)\rangle &= -\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\varepsilon_1^i\varepsilon_2^j}{nm}\\ &\quad\times (a_i^{-1}C_{a_i,b_j}(-z)^{a_i}\, F_{a_i,b_j}(z) + b_j^{-1}C_{b_j,a_i}(-z)^{b_j}\, F_{b_j,a_i}(z))\end{aligned}$$

modulo $\mathbb{Q}(1) = 2\pi\sqrt{-1}\mathbb{Q}$.

Proof. This is immediate from Remark 3.3 and the following lemma due to W. Zudilin. □

Lemma 3.5 (Zudilin)**.** *Let* $z = (1-t)^{-1}$*. Then*

$$\begin{aligned}&\pi i + 2\psi(1) - \psi(a) - \psi(b) - \log(1-t) - ab(1-t)\,{}_4F_3\left({a+1, b+1, 1, 1 \atop 2, 2, 2}; 1-t\right)\\ &= a^{-1}C_{a,b}(-z)^a\,{}_3F_2\left({a, a, a \atop 1+a-b, a+1}; z\right) + b^{-1}C_{b,a}(-z)^b\,{}_3F_2\left({b, b, b \atop 1-a+b, b+1}; z\right),\end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ *is the digamma function.*

Proof. Apply $z\frac{d}{dz}$ on both sides. Then it turns out,

$${}_2F_1(a,b,1;1-t) = C_{a,b}(-z)^a F(a,a,1+a-b;z) + C_{b,a}(-z)^b F(b,b,1-a+b;z),$$

and this is valid ([NIST] 15.8.2). This proves Lemma 3.5 modulo constant. To remove 'modulo constant', we consider the limit $t \to 0^-$ so that $z = (1-t)^{-1} \to 1^-$.

Recall the formula (4.3.3) in Slater's book [Sl] (also on page 15 of Bailey's book [B]), which we write for the case $d = e = 1$ for the choice $+$ of the sign in the exponent:

$$\begin{aligned}
&\Gamma(a)\Gamma(b)\Gamma(c)\,{}_3F_2\left(\begin{matrix}a,\,b,\,c\\1,\,1\end{matrix}\,\middle|\,1\right)\\
&\quad= e^{\pi i a}\frac{\Gamma(a)\Gamma(b-a)\Gamma(c-a)}{\Gamma(1-a)^2}\,{}_3F_2\left(\begin{matrix}a,\,a,\,a\\1+a-b,\,1+a-c\end{matrix}\,\middle|\,1\right)\\
&\qquad+e^{\pi i b}\frac{\Gamma(b)\Gamma(a-b)\Gamma(c-b)}{\Gamma(1-b)^2}\,{}_3F_2\left(\begin{matrix}b,\,b,\,b\\1+b-a,\,1+b-c\end{matrix}\,\middle|\,1\right)\\
&\qquad+e^{\pi i c}\frac{\Gamma(c)\Gamma(a-c)\Gamma(b-c)}{\Gamma(1-c)^2}\,{}_3F_2\left(\begin{matrix}c,\,c,\,c\\1+c-a,\,1+c-b\end{matrix}\,\middle|\,1\right).
\end{aligned}$$

Dividing both sides by $\Gamma(a)\Gamma(b)$, putting the two summands on the right on one side and taking the limit as $c \to 0$ we get

$$\begin{aligned}
&a^{-1}e^{\pi i a}C_{a,b}\cdot{}_3F_2\left(\begin{matrix}a,\,a,\,a\\1+a-b,\,1+a\end{matrix}\,\middle|\,1\right)+b^{-1}e^{\pi i a}C_{b,a}\cdot{}_3F_2\left(\begin{matrix}b,\,b,\,b\\1+b-a,\,1+b\end{matrix}\,\middle|\,1\right)\\
&\quad=\lim_{c\to0}\Bigg(e^{\pi i c}\frac{\Gamma(c)\Gamma(a-c)\Gamma(b-c)}{\Gamma(a)\Gamma(b)\Gamma(1-c)^2}\,{}_3F_2\left(\begin{matrix}c,\,c,\,c\\1+c-a,\,1+c-b\end{matrix}\,\middle|\,1\right)\\
&\qquad\qquad-\Gamma(c)\,{}_3F_2\left(\begin{matrix}a,\,b,\,c\\1,\,1\end{matrix}\,\middle|\,1\right)\Bigg).
\end{aligned}$$

We have

$$\begin{aligned}
\lim_{c\to0}\Gamma(c)\left({}_3F_2\left(\begin{matrix}a,\,b,\,c\\1,\,1\end{matrix}\,\middle|\,1\right)-1\right)&=\lim_{c\to0}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n\Gamma(c+n)}{n!^3}=\sum_{n=1}^{\infty}\frac{(a)_n(b)_n\Gamma(n)}{n!^3}\\
&=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}n!}{(n+1)!^3}=ab\,{}_4F_3\left(\begin{matrix}1+a,\,1+b,\,1,\,1\\2,\,2,\,2\end{matrix}\,\middle|\,1\right).
\end{aligned}$$

We are left with the limit

$$L=\lim_{c\to0}\Gamma(c)\left(\frac{e^{\pi i c}\Gamma(a-c)\Gamma(b-c)}{\Gamma(a)\Gamma(b)\Gamma(1-c)^2}\,{}_3F_2\left(\begin{matrix}c,\,c,\,c\\1+c-a,\,1+c-b\end{matrix}\,\middle|\,1\right)-1\right).$$

We now use $\Gamma(c)=\Gamma(1+c)/c\sim 1/c$, $e^{\pi i c}=1+\pi i c+o(c)$, $\Gamma(x-c)=\Gamma(x)-\Gamma'(x)c+o(c)=\Gamma(x)(1-\psi(x)c)+o(c)$ for $x\in\{a,b,1\}$, and

$$\begin{aligned}
\frac{(c)_n^3}{n!\,(1+c-a)_n(1+c-b)_n}&=c^3\cdot\frac{(n-1)!}{n!\,(1-a)_n(1-b)_n}+o(c^3)\\
&=o(c^2)\quad\text{for } n=1,2,3,\ldots,
\end{aligned}$$

as $c \to 0$. This implies that

$$\begin{aligned}
L&=\lim_{c\to0}\frac{1}{c}\left(\frac{e^{\pi i c}\Gamma(a-c)\Gamma(b-c)}{\Gamma(a)\Gamma(b)\Gamma(1-c)^2}-1\right)\\
&=\lim_{c\to0}\frac{1}{c}\left(\frac{(1+\pi i c+o(c))(1-\psi(a)c+o(c))(1-\psi(b)c+o(c))}{(1-\psi(1)c+o(c))^2}-1\right)\\
&=\pi i-\psi(a)-\psi(b)+2\psi(1)
\end{aligned}$$

as required. □

Theorem 3.6. *Let $e:\mathbb{Q}[\mu_n\times\mu_m]\to E$ be a projection onto a number field E which does not factor through projections $\mu_n\times\mu_m\to\mu_n$ or $\mu_n\times\mu_m\to\mu_m$. We denote* *by* $\mathrm{reg}(\xi)(e) \in \mathrm{Hom}(H_1^B(X_t,\mathbb{Q})(e), \mathbb{C}/\mathbb{Q}(2))$ *the e-part, and* $\gamma(e) \in H_1^B(X_t)(e)$ *as well. Let* I_e *be the set of indices as in* (2.1)*. Put* $z := (1-t)^{-1}$ *and*

$$B_{a,b} := B(a, b-a) = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}, \quad F_{a,b}(z) := {}_3F_2\left({a, a, a \atop 1+a-b, a+1}; z \right).$$

Write $a_i := 1 - i/n$ *and* $b_j := 1 - j/m$*. Assume*

$$a_i \neq b_j \ (\Leftrightarrow\ i/n \neq j/m), \quad \forall\, (i,j) \in I_e \tag{3.7}$$

or equivalently the diagram

$$\begin{array}{ccc} \hat{\mathbb{Z}}(1) = \varprojlim_l \mu_l & \xrightarrow{\mathrm{can}} & \mu_n \cong \mu_n \times \{1\} \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle e} \\ \mu_m \cong \{1\} \times \mu_m & \xrightarrow{e} & E^\times \end{array}$$

does not commute. Then for $|t| < 1$

$$\begin{aligned} \langle \mathrm{reg}(\xi)(e) \mid \gamma(\varepsilon_1,\varepsilon_2)(e) \rangle = -\sum_{(i,j)\in I_e} &(1-\nu_1^{-i})(1-\nu_2^{-j}) \frac{\varepsilon_1^i \varepsilon_2^j}{nm} \\ &\times (a_i^{-1} B_{a_i,b_j} z^{a_i}\, F_{a_i,b_j}(z) + b_j^{-1} B_{b_j,a_i} z^{b_j}\, F_{b_j,a_i}(z)) \end{aligned}$$

modulo $\mathbb{Q}(2)$.

Proof. Put

$$F := \langle \mathrm{reg}(\xi)(e) \mid \gamma(\varepsilon_1,\varepsilon_2)(e) \rangle.$$

By Proposition 3.1 and (3.3)

$$(t-1)\frac{dF}{dt} = -\sum_{(i,j)\in I_e} (1-\nu_1^{-i})(1-\nu_2^{-j}) \int_{\gamma(\varepsilon_1,\varepsilon_2)} \omega_{i,j}.$$

By Lemma 2.4

$$(t-1)\frac{dF}{dt} = -\sum_{(i,j)\in I_e} (1-\nu_1^{-i})(1-\nu_2^{-j}) \frac{\varepsilon_1^i \varepsilon_2^j}{nm} B(a_i,b_j) F(a_i,b_j,a_i+b_j;t). \tag{3.8}$$

In particular F is holomorphic at $|t| < 1$. For $a \neq b$ there is a formula ([NIST] 15.8.3)

$$B(a,b)F(a,b,a+b;t) = B_{a,b} z^a F(a,a,1+a-b;z) + B_{b,a} z^b F(b,b,1-a+b;z). \tag{3.9}$$

Here we take the branches such that $F(a,b,a+b;t)$, $F(a,a,1+a-b;z)$ and $F(b,b,1-a+b;z)$ are holomorphic on the region $|t| < 1 < |1-t|$ ($\Leftrightarrow$ $|z| < 1$ and $\mathrm{Re}(z) > 1/2$) and z^a and z^b take the principal values on $|\arg(z)| < \pi$. Then (3.8) and (3.9) immediately imply the desired formula except the constant term:

$$F = C + \sum_{(i,j)\in I_e} (1-\nu_1^{-i})(1-\nu_2^{-j}) \frac{\varepsilon_1^i \varepsilon_2^j}{nm} (a_i^{-1} B_{a_i,b_j} z^{a_i}\, F_{a_i,b_j}(z) + b_j^{-1} B_{b_j,a_i} z^{b_j}\, F_{b_j,a_i}(z)).$$

To conclude $C \in \mathbb{Q}(2)$, we see the local monodromy T_∞ at $t = \infty$. We fix a loop T_∞ such that the origin is a point in the interval $-1 < t < 0$ and it goes along the negative real axis and turns around $t = \infty$.

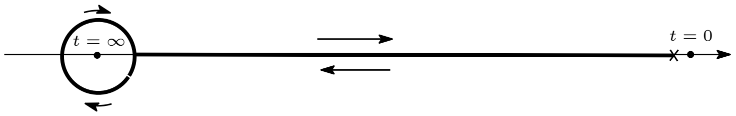


Figure of T_∞

The eigenvalues of T_∞ on $H_1^B(X_t)(e)$ are those of the hypergeometric function $F(a_i, b_j, a_i + b_j; t)$, and hence they are

$$p_i := \exp(2\pi\sqrt{-1}a_i), \quad q_j := \exp(2\pi\sqrt{-1}b_j), \quad (i,j) \in I_e.$$

Put a mondromy operator

$$Q := \prod_{(i,j)\in I_e} (T_\infty - p_i)(T_\infty - q_j) \in \mathbb{Z}[T_\infty].$$

We use the notation in the proof of Proposition 3.1

$$F = \langle e_{\mathrm{dR}} - e_B, \gamma_t\rangle = \langle e_{\mathrm{dR}}, \widetilde{\gamma}_t\rangle - \langle e_B, \widetilde{\gamma}_t\rangle, \quad \gamma_t := \gamma(\varepsilon_1, \varepsilon_2).$$

Note that $e_B = e_{B,t}$ may have nontrivial monodromy. Apply Q to the above, we then have

$$QF = \overbrace{\langle e_{\mathrm{dR}}, Q\widetilde{\gamma}_t\rangle}^{\mathbb{Q}(2)} - Q\overbrace{\langle e_{B,t}, \widetilde{\gamma}_t\rangle}^{\mathbb{Q}(2)} \in \mathbb{Q}(2).$$

On the other hand, since $F_{a_i,b_j}(z)$ and $F_{b_j,a_i}(z)$ are holomorphic along the loop T_∞ fixed above, one has

$$(T_\infty - p_i)(z^{a_i}F_{a_i,b_j}(z)) = 0, \quad (T_\infty - q_j)(z^{b_j}F_{b_j,a_i}(z)) = 0.$$

Therefore

$$QF = QC = \left(\prod_{(i,j)\in I_e} (1-p_i)(1-q_j)\right) C \in \mathbb{Q}(2).$$

Since $p_i, q_j \neq 1$ by definition, one concludes $C \in \mathbb{Q}(2)$. This completes the proof of Theorem 3.2. □

4. Regulators of K_2 of HG fibration of Gauss type

In this section the base field is $\mathbb{C}$.

4.1. Construction of elements of $H^2_{\mathscr{M}}(X_t, \mathbb{Q}(2))$. Let $f : X \to \mathbb{P}^1$ be a HG fibration defined with multiplication by (R, e). Let $Y = f^{-1}(1)_{\mathrm{red}}$ be the reduced fiber over $t = 1$. We assume that Y is a normal crossing divisor in X. Let $\partial_{\mathscr{M}} : H^2_{\mathscr{M}}(X \setminus Y, \mathbb{Q}(2)) \to H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2))$ be the boundary map arising from the localization sequence of motivic cohomology groups. Let $c_B : H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2)) \to H^3_Y(X, \mathbb{Q}(2)) \cap H^{0,0}$. be the Betti realization map. Let

$$\partial := c_B \circ \partial_{\mathscr{M}} : H^2_{\mathscr{M}}(X \setminus Y, \mathbb{Q}(2)) \longrightarrow H^3_Y(X, \mathbb{Q}(2)) \cap H^{0,0}$$

be the composition, which we call the boundary map. There is a natural injection

$$T := \mathrm{Coker}[T_1 - 1 : R^1 f_*\mathbb{Q}(2) \to R^1 f_*\mathbb{Q}(1)] \hookrightarrow H^3_Y(X, \mathbb{Q}(2))$$

where T_1 denotes the local monodromy at $t = 1$. One has $T \cap H^{0,0} = H^3_Y(X, \mathbb{Q}(2)) \cap H^{0,0}$. The ring R acts on T and hence on $T \cap H^{0,0}$. By the last condition of HG fibrations (see §2.1), $T(e) \cong E$ as E-module and the Hodge type is $(0, 0)$. Therefore $T(e) \cap H^{0,0} = T(e) \cong E$. We write by

$$\partial(e) : H^2_{\mathscr{M}}(X \setminus Y, \mathbb{Q}(2)) \longrightarrow (T \cap H^{0,0})(e) = T(e) \cap H^{0,0} \cong E \qquad (4.1)$$

the composition of ∂ with the projection onto the e-part.

Theorem 4.1. *Let f be a HG fibration of Gauss type*

$$X_t = f^{-1}(t) : y^N = x^a(1-x)^b(1-tx)^{N-b}, \quad 1 \le a, b < N, \ \gcd(N, a, b) = 1$$

with multiplication by $(\mathbb{Q}[\mu_N], e)$ such that the projection $e : \mathbb{Q}[\mu_N] \to E$ satisfies $ad/N, bd/N \notin \mathbb{Z}$, $d := \sharp\mathrm{Ker}(e : \mu_N \to E^\times)$. Suppose

$$\gcd(N, a) = \gcd(N, b) = 1. \tag{$*$}$$

Then the map $\partial(e)$ in (4.1) is surjective.

We do not know whether it is possible to remove the assumption $(*)$ in the above statement.

Before the proof of Theorem 4.1 we first show the following lemmas.

Lemma 4.2. *X is a rational surface (without assumption $(*)$).*

Proof. Put $z := 1 - tx$. Then

$$y^N = x^a(1-x)^b(1-tx)^{N-b} \iff (y/z)^N = x^a(1-x)^b z^{-b}.$$

Let $y_1 := y/z$ and $z_1 := (1-x)/z$. Then $y_1^N = x^a z_1^b$. Let $z_2 := z_1/y$ then $y_1^{N-b} = x^a z_2^b$. If $N - b > b$, let $z_3 := z_2/y$ and then $y_1^{N-2b} = x^a z_3^b$. If $N - b < b$, let $y_2 := y/z_2$ and then $y_2^{N-b} = x^a z_3^{2b-N}$. Continuing this argument, we finally have a surface $y_0^d = x^a z_0^d$, $d := \gcd(N, b)$ which is birational to X. Note $\gcd(N, a, b) = \gcd(d, a) = 1$. Apply the same argument for the variables y_0 and x, we then have a surface $y_0' = x_0 z_0^d$ which is birational to X. Therefore X is a rational surface. □

Lemma 4.3. *Let $\mathrm{NS}(X)$ be the Neron-Severi group. The e-part $\mathrm{NS}(X)(e)$ is generated fibral divisors and a section of f (without assumption $(*)$). Here we say a divisor D is fibral if $f(D)$ is a point.*

Proof. Let $S := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $U := f^{-1}(S)$. Then

$$H^2(X)/\langle \text{fibral divisors} \rangle \cong W_2 H^2(U)$$

and there is an exact sequence

$$0 \longrightarrow H^1(S, R^2 f_*\mathbb{Q}) \longrightarrow H^2(U, \mathbb{Q}) \longrightarrow H^2(X_t, \mathbb{Q}) \longrightarrow 0.$$

Since the last term is spanned by the image of the cycle class of a section, it is enough to show $W_2 H^1(S, R^1 f_*\mathbb{Q}) = 0$. Let $j : S \hookrightarrow \mathbb{P}^1$. Since there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\mathbb{P}^1, j_* R^1 f_*\mathbb{Q}) & \longrightarrow & H^1(S, R^1 f_*\mathbb{Q}) & \longrightarrow & H^0(\mathbb{P}^1, R^1 j_* R^1 f_*\mathbb{Q}) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & W_2 H^1(\mathbb{P}^1, R^1 f_*\mathbb{Q}) & & & & & & \end{array}$$

it is enough to show that $H^1(S, R^1 f_*\mathbb{Q})(e) \to H^0(\mathbb{P}^1, R^1 j_* R^1 f_*\mathbb{Q})(e)$ is injective, or equivalently

$$\dim H^1(S, R^1 f_*\mathbb{Q})(e) \le \dim H^0(\mathbb{P}^1, R^1 j_* R^1 f_*\mathbb{Q})(e). \tag{4.2}$$

Since $H^0(S, R^1 f_*\mathbb{Q})(e) = 0$, one has

$$\dim H^1(S, R^1 f_*\mathbb{Q})(e) = -\chi(S, (R^1 f_*\mathbb{Q})(e)) = -\chi(S, \mathbb{Q}) \dim(H^1(X_t)(e)) = 2[E : \mathbb{Q}]$$

by the second condition of HG fibration in §2.1. On the other hand, letting T_P denotes the local monodromy at P,

$$H^0(\mathbb{P}^1, R^1j_*R^1f_*\mathbb{Q})(e) \cong \bigoplus_{P=0,1,\infty} \mathrm{Coker}[T_p - 1 : H^1(X_t)(e) \to H^1(X_t)(e)].$$

By Lemma 2.6 the eigenvalues of each T_P are known. In particular both of T_0 and T_1 have eigenvalue 1. This implies $\dim H^0(\mathbb{P}^1, R^1j_*R^1f_*\mathbb{Q})(e) \geq 2[E : \mathbb{Q}]$. Thus (4.2) follows. □

We prove Theorem 4.1. There are the localization sequences of the motivic cohomology groups and the Deligne-Beilinson cohomology groups which sit in a commutative diagram

$$\begin{array}{ccccc}
H^2_{\mathscr{M}}(X \setminus Y, \mathbb{Q}(2)) & \xrightarrow{\partial_{\mathscr{M}}} & H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2)) & \xrightarrow{i} & H^3_{\mathscr{M}}(X, \mathbb{Q}(2)) \\
\downarrow{\scriptstyle \mathrm{reg}_{X\setminus Y}} & & \downarrow{\scriptstyle \mathrm{reg}^Y_X} & & \cong \downarrow{\scriptstyle \mathrm{reg}_X} \\
H^2_{\mathscr{D}}(X \setminus Y, \mathbb{Q}(2)) & \longrightarrow & H^3_{\mathscr{D},Y}(X, \mathbb{Q}(2)) & \longrightarrow & H^3_{\mathscr{D}}(X, \mathbb{Q}(2)) \\
 & & \downarrow{\scriptstyle c_{\mathscr{D}}} & & \\
 & & H^3_Y(X, \mathbb{Q}(2)) \cap H^{0,0}. & &
\end{array}$$

where $c_{\mathscr{D}}$ is the canonical surjective map, and the bijectivity of reg_X follows from the fact that X is a smooth projective rational surface (Lemma 4.2). Note, $c_B = c_{\mathscr{D}} \circ \mathrm{reg}^Y_X$ is the Betti realization map, and hence $\partial = c_{\mathscr{D}} \circ \mathrm{reg}^Y_X \circ \partial_{\mathscr{M}}$ is the boundary map as above. Our goal is to show that there is a subspace $W \subset H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2))$ such that $i(W) = 0$ and W is onto $T(e)$ by $c_{\mathscr{D}} \circ \mathrm{reg}^Y_X$. Let $Y = \sum Y_i$ be the irreducible decomposition, and $T \subset Y$ the singular locus. Then there is the canonical isomorphism

$$H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2)) \cong \mathrm{Ker}\left[\bigoplus_i \mathbb{C}(Y_i)^\times \otimes \mathbb{Q} \xrightarrow{\mathrm{div}} \bigoplus_{P_i \in T} \mathbb{Q} \cdot P_i\right]$$

where div is the map of divisor. For $f \in \mathbb{C}(Y_i)^\times \otimes \mathbb{Q}$, we denote by

$$[f, Y_i] \in \bigoplus_i \mathbb{C}(Y_i)^\times \otimes \mathbb{Q}$$

an element of placed in the component of Y_i.

To show Theorem 4.1 we first describe Y in detail. Let $\mathscr{O} = \mathbb{C}[[t-1]]$ and

$$\hat{g} : X^* := \mathrm{Spec}\mathscr{O}[x, y]/(y^N - x^a(1-x)^b(1-tx)^{N-b}) \longrightarrow \mathrm{Spec}\mathscr{O}.$$

The surface X^* has two isolated singularities $(x, y, t) = (0, 0, 1), (1, 0, 1)$. Let $X_0 \to X^*$ be the blow-up at $(x, y, t) = (1, 0, 1)$, and $U \subset X_0$ an affine open set such that

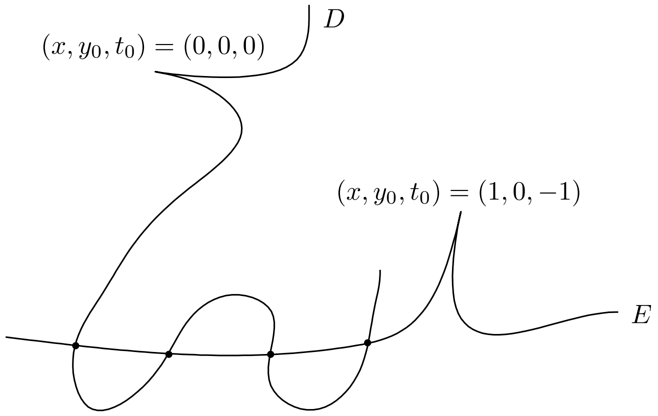
$$U = \mathrm{Spec}\mathscr{O}[x, y_0, t_0]/(y_0^N - x^a(1 + t_0x)^{N-b}) \subset X_0$$

where the morphism given by $y_0 = y/(1-x)$ and $t_0 = (1-t)/(1-x)$. Let $D \subset X_0$ be the proper transform of the central fiber of $\hat{f}_0$ over $t = 1$, and E the exceptional curve of the blow-up:

$$D \cap U = \{t_0 = 0,\, y_0^N = x^a\} \cong \mathrm{Spec}\mathscr{O}[x, y_1]/(y_0^N - x^a),$$

$$E \cap U = \{x = 1,\, y_0^N = (1 + t_0)^{N-b}\} \cong \mathrm{Spec}\mathscr{O}[y_0, t_0]/(y_0^N - (1 + t_0)^{N-b}).$$

By the assumption $(*)$, the curves D and E are irreducible.



Let $X \to X_0$ be a desingularization, then the fiber over $t = 1$ is Y. Hence there is an embedding of the nomalization of $D \cup E$ into Y.

- $\sigma D = D$, $\sigma E = E$ for any automorphism $\sigma \in \mu_N$.
- $D \cap U$ has a singular point $(x, y_0, t_0) = (0, 0, 0)$ unless $a = 1$. Let $\iota : D' \to D$ be the normalization, then $D' \cong \mathbb{P}^1$ and $\iota^{-1}(D \cap U) \cong \mathbb{A}^1$.
- $E \cap U$ has a singular point $(x, y_0, t_0) = (1, 0, -1)$ unless $b = N - 1$. Let $\iota : E' \to E$ be the normalization, then $E' \cong \mathbb{P}^1$ and $\iota^{-1}(E \cap U) \cong \mathbb{A}^1$.
- D and E intersect transversally and $D \cap E \subset U$. Moreover U is regular at each point of $D \cap E$.

We denote by u and v the affine coordinates of $\iota^{-1}(D \cap U)$ and $\iota^{-1}(E \cap U)$ respectively such that

$$(y_0, x)|_D = (u^a, u^N), \quad (y_0, 1 + t_0)|_E = (v^{N-b}, v^N).$$

The intersection points of $D \cap E$ consist of $\{u = \zeta \mid \zeta \in \mu_N\}$ or $\{v = \zeta \mid \zeta \in \mu_N\}$. A point $u = \zeta$ corresponds to $v = \zeta'$ if $\zeta^a = (\zeta')^{N-b} = (\zeta')^{-b}$. Thinking of D' and E' being components of $Y = f^{-1}(1)$, we consider elements

$$\Xi(\zeta_1, \zeta_2) := \left[\frac{u - \zeta_1}{u - \zeta_2}, D'\right] - \left[\frac{v - \zeta_1^{-a/b}}{v - \zeta_2^{-a/b}}, E'\right] \in H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2))$$

for $\zeta_1, \zeta_2 \in \mu_N$ in the motivic cohomology supported on Y. Define $W \subset H^3_{\mathscr{M},Y}(X, \mathbb{Q}(2))$ to be the subspace generated by $\Xi(\zeta_1, \zeta_2)$'s.

We first show $i(W) = 0$. Note $H^3_{\mathscr{M}}(X, \mathbb{Q}(2)) \cong (\mathbb{C}^\times \otimes \mathrm{NS}(X)) \otimes \mathbb{Q}$ since X is a rational surface (Lemma 4.2). Giving generators F_n's of $\mathrm{NS}(X) \otimes \mathbb{Q}$ which intersect with $D' \cup E'$ properly outside the points $u = \zeta_i$ or $v = \zeta_i^{-a/b}$, one has

$$i(\Xi(\zeta_1, \zeta_2)) = \sum_n c_n \otimes F_n,$$

$$c_n := \prod_{P \in F_n \cap D'} \left(\left.\frac{u - \zeta_1}{u - \zeta_2}\right|_P \right)^{m_P} \times \prod_{Q \in F_n \cap E'} \left(\left.\frac{v - \zeta_1^{-a/b}}{v - \zeta_2^{-a/b}}\right|_Q \right)^{-m_Q} \in \mathbb{C}^\times$$

where m_P, m_Q denote the intersection numbers. By Lemma 4.3, the e-part $\mathrm{NS}(X)(e)$ is generated by fibral divisors and a section. If F_n is a section of $x = \infty$, then P and Q are the points defined by $u = \infty$ and $v = \infty$ respectively. Therefore c_n is torsion. Suppose that F_n is an irreducible fibral divisor which is not D' or E'.

Then the intersection points of D' and F_n are at most $u=0$ or $u=\infty$. Therefore the first term of c_n is torsion. In the same way, the second term is also torsion, and hence so is c_n. Let $F_n=E'$. Replace E' with $E''=E'-\mathrm{div}(x-1)$. Let E''_0 be the image of E'' in X_0. Then any component of E''_0 is neither D or E. Moreover it intersects with $D\cap U$ (resp. $E\cap U$) at most at the singular point $(y_0,x)=(0,0)$ (resp. $(t_0,y_0)=(-1,0)$). Therefore the intersection points of $E''\cap D'$ or $E''\cap E'$ are at most $u=0,\infty$ or $v=0,\infty$. Hence c_n is torsion. Finally let $F_n=D'_k$. Then replace D'_k with $D'_k-\mathrm{div}(t-1)$, a fibral divisor without component D'_k. Hence this is reduced to the above. This completes the proof of $i(\Xi(\zeta_1,\zeta_2))=0$, and hence $i(W)=0$.

There remains to show that W is onto $T(e)$. Let $D^*:=D'\setminus\{u^N=1\}$, and let

$$T\cong H^3_{D'+E'}(X,\mathbb{Q}(2))\cap H^{0,0}\overset{\subset}{\longrightarrow} H^1(D^*,\mathbb{Q}(1))\overset{\subset}{\longrightarrow}\bigoplus_{\zeta\in\mu_N}\mathbb{Q}\cdot(u=\zeta)$$

be the composition of the Poincare residue maps. An automorphism $\sigma\in\mu_N$ such that $\sigma(y)=\zeta_N y$ acts on the last term by $\sigma(u)=\zeta_N^{1/a}u$. The above map induces an isomorphism $T(e)\cong H^1(D^*,\mathbb{Q}(1))(e)$ on the e-part. Under this identification, one directly has

$$\partial(\Xi(\zeta_1,\zeta_2))=(u=\zeta_1)-(u=\zeta_2).$$

This means that W is onto $H^1(D^*,\mathbb{Q}(1))$ and hence onto $T(e)\cong H^1(D^*,\mathbb{Q}(1))(e)$. This completes the proof. □

Problem 4.4. *Find explicit descriptions of the K_2-symbols in $K_2(X\setminus Y)$ constructed in Theorem 4.1.*

If f is a HG fibration defined by

$$y^N=x(1-x)^{N-1}(1-tx),$$

then one finds K_2-symbols

$$\left\{\frac{y-\zeta_1(1-x)}{y-\zeta_2(1-x)},\frac{(1-x)^2}{x^2(1-t)}\right\}\in K_2(X\setminus Y),\quad \zeta_i\in\mu_N, \tag{4.3}$$

and shows that their boundary span $T(e)$ (hence we do not need Theorem 4.1 in this case). We do not know how to find such symbols for general $y^N=x^a(1-x)^b(1-tx)^{N-b}$.

Corollary 4.5. *Let f be a HG fibration of Gauss type as in Theorem 4.1. Let* $\mathrm{Res}:\varGamma(X,\Omega^2_X(\log Y))\to H_1^{\mathrm{dR}}(Y)\cong H_1^B(Y,\mathbb{C})$ *be the Poincare residue map at $t=1$. Let*

$$\mathrm{dlog}:H^2_{\mathscr{M}}(X\setminus Y,\mathbb{Q}(2))\longrightarrow\varGamma(X,\Omega^2_X(\log Y))(e)=\bigoplus_{n\in I_e}\mathbb{C}\cdot\frac{dt}{t-1}\omega_n$$

be the dlog map (see Lemma 2.7 for the right hand side). Then the dlog map is onto a set of 2-forms

$$V:=\left\{\sum_{n\in I_e}\lambda_n\left(\frac{dt}{t-1}\omega_n\right)\ \textit{s.t.}\ \sum_{n\in I_e}\lambda_n\mathrm{Res}\left(\frac{dt}{t-1}\omega_n\right)\in H_1^B(Y,\mathbb{Q})(e)\right\}$$

where ω_n and I_e are as in Lemma 2.5.

We note

$$\sum_{n\in I_e} \lambda_n \mathrm{Res}\left(\frac{dt}{t-1}\omega_n\right) \in H_1^B(Y,\mathbb{Q})(e) \iff \sum_{n\in I_e} \lambda_n\zeta^n \in \mathbb{Q},\, \forall\zeta\in\mu_N. \quad (4.4)$$

Proof. Obviously $\mathrm{Im}(\mathrm{dlog}) \subset V$. Since $\dim_{\mathbb{Q}} \mathrm{Im}(\mathrm{dlog}) = [E:\mathbb{Q}]$ by Theorem 4.1, it is enough to show $\dim_{\mathbb{Q}} V \le [E:\mathbb{Q}]$. However this is immediate from (4.4). □

4.2. Main Theorem. Let f be a HG fibration of Gauss type

$$X_t = f^{-1}(t) : y^N = x^a(1-x)^b(1-tx)^{N-b}, \quad 1\le a,b<N,\, \gcd(N,a,b)=1$$

with multiplication by $(\mathbb{Q}[\mu_N],e)$ such that the projection $e:\mathbb{Q}[\mu_N]\to E$ satisfies $ad/N, bd/N \notin \mathbb{Z}$, $d := \sharp\mathrm{Ker}(e:\mu_N\to E^\times)$. Let $\xi \in H^2_{\mathscr{M}}(X\setminus Y,\mathbb{Q}(2))(e)$ be an element of the e-part, and let

$$\mathrm{dlog}(\xi) = \sum_{n\in I_e} \lambda_n\left(\frac{dt}{t-1}\omega_n\right). \quad (4.5)$$

Note λ_n's satisfy the condition (4.4). Conversely if $\gcd(N,a)=\gcd(N,b)=1$, then it follows from Theorem 4.1 that, for any λ_n's satisfying (4.4) there exists ξ such that (4.5) holds.

Theorem 4.6. *Suppose $a\ne b$. Let $\gamma_0=(1-\sigma)u_0$ and $\gamma_1=(1-\sigma)u_1$ be the homology cycles as in Lemma 2.6. Write $a_n := \{an/N\}$, $b_n := \{bn/N\}$, $z := (1-t)^{-1}$ and*

$${}_4F_3^{(n)}(t) := {}_4F_3\left(\begin{matrix} a_n+1, b_n+1, 1, 1 \\ 2,2,2 \end{matrix}; t\right).$$

Then

$$\begin{aligned}&\frac{1}{2\pi\sqrt{-1}}\langle \mathrm{reg}(\xi) \mid \gamma_1\rangle \\ &= \sum_{n\in I_e}(1-\zeta_N^n)\lambda_n[2\psi(1)-\psi(a_n)-\psi(b_n)-\log(1-t)-a_nb_n(1-t)\,{}_4F_3^{(n)}(1-t)] \\ &= \sum_{n\in I_e}(1-\zeta_N^n)\lambda_n[a_n^{-1}C_{a_n,b_n}(-z)^{a_n}F_{a_n,b_n}(z)+b_n^{-1}C_{b_n,a_n}(-z)^{b_n}F_{b_n,a_n}(z)]\end{aligned}$$

and

$$\langle \mathrm{reg}(\xi)\mid\gamma_0\rangle = \sum_{n\in I_e}(1-\zeta_N^n)\lambda_n[a_n^{-1}B_{a_n,b_n}z^{a_n}F_{a_n,b_n}(z)+b_n^{-1}B_{b_n,a_n}z^{b_n}F_{b_n,a_n}(z)],$$

where $B_{a,b}$, $C_{a,b}$ and $F_{a,b}(z)$ are as in Theorems 3.6 and 3.4.

Proof. The same proof as Theorems 3.2, 3.4 and 3.6. □

It is worth noting that Theorem 4.6 is proven without knowledge of explicit description of the K_2-symbol ξ (cf. Problem 4.4).

Conjecture 4.7. *The first equality in Theorem 4.6 is valid even when $a=b$.*

5. Real regulators of K_2 of Elliptic fibrations and the Beilinson conjecture

For an elliptic curve E over $\mathbb{R}$ we discuss the *real regulator map*

$$\mathrm{reg}_{\mathbb{R}} : H^2_{\mathscr{M}}(E, \mathbb{Z}(2)) \longrightarrow \mathbb{R}$$

which is defined in the following way. Let $F_\infty : E(\mathbb{C}) \to E(\mathbb{C})$ be the infinite Frobenius map of real manifolds. We denote by "$F_\infty = 1$" (resp. "$F_\infty = -1$") the fixed part (resp. anti-fixed part). Then $\mathrm{reg}_{\mathbb{R}}$ is defined as the composition

$$\begin{aligned} H^2_{\mathscr{M}}(E, \mathbb{Z}(2)) &\xrightarrow{\mathrm{reg}} \mathrm{Hom}(H_1^B(E(\mathbb{C}), \mathbb{Z})^{F_\infty=-1}, \mathbb{C}/\mathbb{Z}(2)) \\ &\longrightarrow \mathrm{Hom}(H_1^B(E(\mathbb{C}), \mathbb{Q})^{F_\infty=-1}, \mathbb{R}(1)) \\ &\xrightarrow{\cong} \mathbb{R}(1) \\ &\xrightarrow{\cong} \mathbb{R} \end{aligned}$$

where the 2nd arrow is given by the projection $\mathbb{C}/\mathbb{Z}(2) \to \mathbb{R}(1) = \sqrt{-1}\mathbb{R}$, the 3rd given by a fixed base of $H_1^B(E(\mathbb{C}), \mathbb{Q})^{F_\infty=-1} \cong \mathbb{Q}$, and the 4th arrow given by multiplication by $(2\pi\sqrt{-1})^{-1}$.

Conjecture 5.1 (Beilinson conjecture for an elliptic curve over $\mathbb{Q}$, cf. [Sch], [DW])**.** *Let E be an elliptic curve over $\mathbb{Q}$, and $L(E, s)$ the motivic L-function of E. Then there is an integral element $\xi \in H^2_{\mathscr{M}}(E, \mathbb{Z}(2))$ in the sense of Scholl* [S] *such that*

$$\mathrm{reg}_{\mathbb{R}}(\xi) \sim_{\mathbb{Q}^\times} \pi^{-2} L(E, 2)$$

where $x \sim_{\mathbb{Q}^\times} y$ means $xy \neq 0$ and $xy^{-1} \in \mathbb{Q}^\times$.

Beilinson further conjectures that the space $H^2_{\mathscr{M}}(E, \mathbb{Q}(2))_{\mathbb{Z}}$ of integral elements is 1-dimensional, spanned by ξ in the above, though we will not discuss this issue.

5.1. Legendre family. Let $f : X \to \mathbb{P}^1$ be the Legendre family of elliptic curves given by an affine equation

$$X_t = f^{-1}(t) : y^2 = x(1-x)(1-tx).$$

Consider a K_2-symbol

$$\xi := \left\{ \frac{y - x + 1}{y + x - 1}, \frac{(x-1)^2}{x^2(t-1)} \right\} \in K_2(X \setminus Y), \quad Y := f^{-1}(1). \tag{5.1}$$

One immediately has

$$\mathrm{dlog}\xi = \frac{dx}{y} \wedge \frac{dt}{t-1}.$$

Theorem 5.2. *Write $\xi_t := \xi|_{X_t} \in K_2(X_t)$ for $t \in \mathbb{R} \setminus \{0, 1\}$.*

(1) *If $t > 0$, then*

$$\mathrm{reg}_{\mathbb{R}}(\xi_t) = \mathrm{Re}\left[-\log 16 + \log(1-t) + \frac{1-t}{4} {}_4F_3\left({\frac{3}{2}, \frac{3}{2}, 1, 1 \atop 2, 2, 2} ; 1-t \right) \right].$$

(2) *If $t < 0$, then*

$$\mathrm{reg}_{\mathbb{R}}(\xi_t) = z^{\frac{1}{2}} {}_3F_2\left({\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop 1, \frac{3}{2}} ; z \right), \quad z := (1-t)^{-1}.$$

We can prove Theorem 5.2 in a similar way to §3, on noting the following.

Case $t>0$**:** $H_1^B(X_t(\mathbb{C}),\mathbb{Q})^{F_\infty=-1}$ is spanned by a homology cycle δ_t going around the interval from $x=1$ to $x=t^{-1}$, and

$$\int_{\delta_t}\frac{dx}{y}=2\pi\sqrt{-1}F\left(\frac{1}{2},\frac{1}{2},1;1-t\right),$$

Case $t<0$**:** ——— γ_t ——— from $x=0$ to $x=t^{-1}$, and

$$\int_{\gamma_t}\frac{dx}{y}=2\pi z^{\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2},1;z\right),\quad z=(1-t)^{-1}.$$

Corollary 5.3. *Let* $t\in\mathbb{Q}\setminus\{0,1\}$ *such that* ξ_t *is integral. Then we have an equivalence*

$$\text{The Beilinson Conjecture 5.1 for } X_t \iff \tag{5.2}$$

$$\pi^{-2}L(X_t,2)\sim_{\mathbb{Q}^\times}\begin{cases}\mathrm{Re}\left[-\log 16+\log(1-t)+\frac{1-t}{4}{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};1-t\right)\right] & t>0\\ z^{\frac{1}{2}}{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,\frac{3}{2}};z\right) & t<0.\end{cases} \tag{5.3}$$

Corollary 5.4. *The Beilinson Conjecture 5.1 is true for* X_{-3}*.*

Proof. Rogers and Zudilin show

$$\frac{\pi^2}{12}\,{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop \frac{3}{2},1};\frac{1}{4}\right)=L(E_{24},2)$$

where E_{24} is an elliptic curve over $\mathbb{Q}$ of conductor 24 ([RZ] Theorem 2, p.399 and (6), p.386). There is only one elliptic curve of conductor 24 up to isogeny, and X_{-3} is the one. Hence (5.3) holds. □

5.2. Elliptic fibration $3y^2=2x^3-3x^2+t$**.** Let $f:X\to\mathbb{P}^1$ be en elliptic fibration defined by $3y^2=2x^3-3x^2+t$. Put

$$\xi:=\left\{\frac{y-x+1}{y+x-1},\frac{1-t}{2(x-1)^3}\right\}\in K_2(X\setminus Y),\quad Y:=f^{-1}(1). \tag{5.4}$$

In a similar way to Theorem 5.2 we have the following theorem.

Theorem 5.5. *Let* $t\in\mathbb{R}\setminus\{0,1\}$*. If* $|t-1|<1$*, then*

$$\mathrm{reg}_{\mathbb{R}}(\xi_t)=\log 432-\log(1-t)-\frac{5}{36}(1-t)\,{}_4F_3\left({\frac{7}{6},\frac{11}{6},1,1 \atop 2,2,2};1-t\right).$$

If $|t-1|>1$*, then*

$$\mathrm{reg}_{\mathbb{R}}(\xi_t)=\pi^{-1}\left[\frac{3}{2}B\left(\frac{1}{6},\frac{1}{6}\right)z^{\frac{1}{6}}\,{}_3F_2\left({\frac{1}{6},\frac{1}{6},\frac{1}{6} \atop \frac{1}{3},\frac{7}{6}};z\right)+\frac{3}{10}B\left(\frac{5}{6},\frac{5}{6}\right)z^{\frac{5}{6}}\,{}_3F_2\left({\frac{5}{6},\frac{5}{6},\frac{5}{6} \atop \frac{5}{3},\frac{11}{6}};z\right)\right]$$

where $z:=(1-t)^{-1}$*.*

5.3. Elliptic fibration $y^2 = x^3 + (3x + 4t)^2$. Let $f : X \to \mathbb{P}^1$ be en elliptic fibration defined by $y^2 = x^3 + (3x + 4t)^2$. Put

$$\xi := \left\{ \frac{y - 3x - 4t}{-8t}, \frac{y + 3x + 4t}{8t} \right\} \in K_2(X \setminus Y), \quad Y := f^{-1}(1). \tag{5.5}$$

Theorem 5.6. *Let $t \in \mathbb{R} \setminus \{0, 1\}$. If $0 < |t| < 1$, then*

$$\mathrm{reg}_{\mathbb{R}}(\xi_t) = \log 27 - \log t - \frac{2t}{9}\, {}_4F_3\left(\begin{matrix} \frac{4}{3}, \frac{5}{3}, 1, 1 \\ 2, 2, 2 \end{matrix} ; t \right).$$

If $|t| > 1$, then

$$\mathrm{reg}_{\mathbb{R}}(\xi_t) = \sqrt{3}\pi^{-1} \left[B\left(\frac{1}{3}, \frac{1}{3}\right) t^{-\frac{1}{3}}\, {}_3F_2\left(\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ \frac{2}{3}, \frac{4}{3} \end{matrix} ; t^{-1} \right) + \frac{1}{2} B\left(\frac{2}{3}, \frac{2}{3}\right) t^{-\frac{2}{3}}\, {}_3F_2\left(\begin{matrix} \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \\ \frac{4}{3}, \frac{5}{3} \end{matrix} ; t^{-1} \right) \right].$$

5.4. Numerical verification of the Beilinson conjecture by MAGMA. As an application of Theorems 5.2, 5.5 and 5.6, we have numerical examples verifying the Beilinson conjecture.

Case : $y^2 = x(1-x)(1-tx)$. By definition of the symbol ξ in (5.1), ξ_t is integral if X_t does not have a multiplicative reduction at any prime p such that $\mathrm{ord}_p(1-t) \neq 0$. In more practical way, ξ_t is integral if

$$\mathrm{ord}_p(j(X_t)) = \mathrm{ord}_p\left(\frac{256(t^2 - t + 1)^3}{t^2(1-t)^2} \right) \geq 0 \text{ for any } p \text{ s.t. } \mathrm{ord}_p(1-t) \neq 0$$

$$\iff \quad t = -1, -3, -7, -15, 2, 3, 5, 9, 17, \frac{1}{2}, \frac{3}{2}, \frac{7}{8}, \frac{9}{8}, \frac{3}{4}, \frac{5}{4}, \frac{15}{16}, \frac{17}{16}.$$

Put

$$R_t := \mathrm{reg}_{\mathbb{R}}(\xi_t)/(\pi^{-2} L(X_t, 2)).$$

Here is the list of numerical verification of the Beilinson Conjecture 5.1 for above t's with the aid of MAGMA (digits of precision is at least 20).

t	-1	-3	-7	-15	2	3	5	9	17
R_t	8	6	$\frac{7}{2}$	$\frac{15}{4}$	-32	-24	-20	-18	-17

t	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{8}$	$\frac{9}{8}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{15}{16}$	$\frac{17}{16}$
R_t	-32	-48	-56	-48	-48	-40	$-\frac{165}{2}$	-68

Case : $3y^2 = 2x^3 - 3x^2 + t$. Let $n \geq 2$ be an integer and let $t = 1 - 1/n$. Then $j(X_t) = 432n^2/(n-1)$ and

$$\xi_t = \left\{ \frac{y - x + 1}{y + x - 1}, \frac{1}{2n(x-1)^3} \right\}.$$

Therefore if the denominator of $432n^2/(n-1)$ is prime to $6n$, then ξ_t is integral. There exist infinitely many such n's (e.g. $n \geq 2$ such that $n \equiv 0, 2 \bmod 6$).

n	2	3	4	5	6	7	8	9	10	11
R_t	72	$\frac{486}{7}$	81	$\frac{135}{2}$	$\frac{4860}{67}$	$\frac{189}{2}$	$\frac{1512}{19}$	81	90	$\frac{165}{2}$

n	12	13	14	15	16	17	18	19	20	21
R_t	$\frac{2673}{28}$	$\frac{13689}{176}$	$\frac{34398}{443}$	$\frac{1701}{19}$	$\frac{405}{8}$	$\frac{2601}{23}$	$\frac{2754}{29}$	$\frac{3249}{40}$	$\frac{171}{2}$	$\frac{8505}{104}$

Case : $y^2 = x^3 + (3x+4t)^2$. Let $n \geq 1$ be an integer, and let $t = \frac{1}{6n}$. Then $j(X_t) = 1296(4-27n)^3 n/(6n-1)$ and

$$\xi_t = \left\{ -\frac{3n}{4}\left(y - 3x - \frac{2}{3n}\right), \frac{3n}{4}\left(y + 3x + \frac{2}{3n}\right) \right\}.$$

Since the denominator of $j(X_t)$ is prime to $6n$, ξ_t is integral for all $n \geq 1$.

n	1	2	3	4	5	6	7	8	9	10
R_t	$\frac{405}{8}$	$\frac{891}{16}$	$\frac{1377}{20}$	$\frac{5589}{88}$	$\frac{19575}{256}$	$\frac{135}{2}$	$\frac{54243}{776}$	$\frac{1269}{16}$	$\frac{477}{8}$	$\frac{13275}{166}$

n	11	12	13	14	15	16	17	18	19	20
R_t	$\frac{70785}{1016}$	$\frac{5751}{64}$	$\frac{10647}{128}$	$\frac{15687}{230}$	$\frac{20025}{248}$	$\frac{2565}{32}$	$\frac{788103}{10172}$	$\frac{321}{4}$	$\frac{1101411}{14216}$	$\frac{80325}{872}$

5.5. Remark on the Elliptic dilogarithms. Recall the *Bloch-Wigner function*

$$D(x) := \mathrm{Im}(\ln_2(x)) + \log|x| \arg(1-x).$$

For $q \neq 0$, the *elliptic dilogarithms* is defined to be

$$D_q(x) := \sum_{n \in \mathbb{Z}} D(xq^n),$$

satisfies $D_q(qx) = D_q(x)$ and $D_q(x^{-1}) = -D_q(x)$ ([Bl], [GL]).

Recall Bloch's formla which decsribes the real regulator via the elliptic dilogarithm (cf. [GL] p.416–417). Noting that the K_2-symbols (5.1) and (5.5) are defined by rational functions supported on torsion points, Bloch's formula implies the following.

Theorem 5.7. *If $-1 < t < 0$ then*

$$\frac{\pi}{4}(1-t)^{-\frac{1}{2}} {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} ; (1-t)^{-1} \right) = D_q(i) + D_q(iq^{\frac{1}{2}}) = D_{q^{\frac{1}{2}}}(i).$$

If $0 < t < 1$ then

$$-\frac{\pi}{8}\left(\log\frac{1-t}{16} + \frac{1-t}{4} {}_4F_3\left(\begin{matrix} \frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2 \end{matrix} ; 1-t \right) \right) = D_q(i) + D_q(iq^{\frac{1}{2}}).$$

where $i = \sqrt{-1}$ and we put

$$q := \exp\left(-2\pi \frac{F(\frac{1}{2}, \frac{1}{2}, 1; 1-t)}{F(\frac{1}{2}, \frac{1}{2}, 1; t)} \right).$$

Theorem 5.8. *If $1 < t < 2$ then*

$$B\left(\frac{1}{3}, \frac{1}{3}\right) t^{-\frac{1}{3}} {}_3F_2\left(\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ \frac{2}{3}, \frac{4}{3} \end{matrix} ; t^{-1} \right) + \frac{1}{2} B\left(\frac{2}{3}, \frac{2}{3}\right) t^{-\frac{2}{3}} {}_3F_2\left(\begin{matrix} \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \\ \frac{4}{3}, \frac{5}{3} \end{matrix} ; t^{-1} \right)$$
$$= 6\sqrt{3} D_q(e^{2\pi i/3})$$

where

$$q := \exp\left(\frac{-2\pi}{\sqrt{3}} \frac{F(\frac{1}{3}, \frac{2}{3}, 1; t)}{F(\frac{1}{3}, \frac{2}{3}, 1; 1-t)} \right).$$

References

[Ar] N. Archinard, Hypergeometric abelian varieties, Canad. J. Math. **55** (2003), 897-932.

[AO1] M. Asakura and N. Otsubo, *CM periods, CM regulators and hypergeometric functions, I*, To appear in Canad. J. Math. (arXiv:1503.07962).

[AO2] M. Asakura and N. Otsubo, *CM periods, CM regulators and hypergeometric functions, II*, To appear in Math. Z. (arXiv:1503.08894).

[AO3] M. Asakura and N. Otsubo, *Regulators on K_1 of hypergeometric fibrations*, preprint, arXiv:1709.04144.

[AO4] M. Asakura and N. Otsubo, *Regulators on K_1 hypergeometric fibrations and a logarithmic formula on ${}_3F_2(1,1,q;a,b;x)$*, preprint.

[B] W.N. Bailey, *Generalized Hypergeometric series.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 32 Stechert-Hafner, Inc., New York 1964.

[Bl] S. Bloch,*Higher regulators, algebraic K -theory, and zeta functions of elliptic curves.* CRM Monograph Series, 11. American Mathematical Society, Providence, RI, 2000.

[DW] C. Deninger and K. Wingberg: *On the Beilinson conjectures for elliptic curves with complex multiplications.* In Beilinson's Conjectures on Special Values of L-Functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math. Vol.4, 249–272, 1988.

[E] A. Erdélyi et al. ed., *Higher transcendental functions*, Vol. 1, California Inst. Tech, 1981.

[GL] A. Goncharov and A. Levin, *Zagier's conjecture on $L(E,2)$* . Invent. Math. 132 (1998), no. 2, 393–432.

[Ha] R. Hain, *Classical polylogarithms.* (English summary) Motives (Seattle, WA, 1991), 3–42, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[H] Schmickler-Hirzebruch, U.: *Elliptische Flächen über $\mathbb{P}^1(\mathbb{C})$ mit drei Ausnahmefasern und die hypergeometrische Differentialgleichung.* Schriftreihe des Mathematischen Instituts der Universität Münster, Februar 1985.

[O1] N. Otsubo, *On the regulator of Fermat motives and generalized hypergeometric functions.* J. reine angew. Math. **660** (2011), 27-82.

[O2] N. Otsubo, *Certain values of Hecke L-functions and generalized hypergeometric functions.* J. Number Theory **131** (2011), 648-660.

[RZ] M. Rogers and W. Zudilin, *From L-series of elliptic curves to Mahler measures.* Compos. Math. **148** (2012), no. 2, 385–414.

[Sch] P. Schneider: *Introduction to the Beilinson conjectures.* In Beilinson's Conjectures on Special Values of L-Functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math. Vol.4, 1–35, 1988.

[S] A. Scholl. *Integral elements in K-theory and products of modular curves.* In: Gordon, B. B., Lewis, J. D., Müller-Stach, S., Saito, S., Yui, N. (eds.) The arithmetic and geometry of algebraic cycles, Banff, 1998, (NATO Sci. Ser. C Math. Phys. Sci., 548), pp. 467–489, Dordrecht, Kluwer, 2000.

[Sl] L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge 1966.

[NIST] *NIST Handbook of Mathematical Functions.* Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark. Cambridge Univ. Press, 2010.

Department of Mathematics, Hokkaido University, Sapporo, 060-0810 Japan
E-mail address: asakura@math.sci.hokudai.ac.jp