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Production of a $^4_\Lambda$He hypernucleus in the $^4$He$(\pi, K)$ reactions reexamined

Toru Harada$^{1,2,*}$ and Yoshiharu Hirabayashi$^3$

$^1$Center for Physics and Mathematics, Osaka Electro-Communication University, Neyagawa, Osaka 572-8530, Japan
$^2$J-PARC Branch, KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), 203-1 Shirakata, Tokai, Ibaraki 319-1106, Japan
$^3$Information Initiative Center, Hokkaido University, Sapporo 060-0811, Japan

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We investigate theoretically production cross sections of the $J^p = 0^+$ ground state of a $^4_\Lambda$He hypernucleus in the $^4$He$(\pi, K)$ reaction with a distorted-wave impulse approximation using the optimal Fermi-averaged $\pi N \rightarrow K\Lambda$ t matrix. We demonstrate the sensitivity of the production cross sections to the $\Lambda$ wave functions obtained from 3N-$\Lambda$ potentials and to meson distorted waves in eikonal distortions. It is shown that the calculated laboratory cross sections of the $0^+$ ground state in $^4_\Lambda$He amount to $\sigma/(d\Omega)_{ab}$ $\approx 11$ $\mu b/sr$ at $p_\pi = 1.05$ GeV/$c$ in the $K$ forward direction because of an advantage of the use of an $s$-shell target nucleus such as $^4$He. The importance of the recoil effects and the energy dependence of the $\pi N \rightarrow K\Lambda$ cross sections is also discussed.

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I. INTRODUCTION

Recently, unexpected short lifetimes of a $^3_\Lambda$H hypernucleus were measured in heavy-ion production of high-energy heavy-ion collisions [1–3]; the world average lifetime of $\tau^{(03)}(3_\Lambda H) = 185^{+23}_{-29}$ ps is shorter than the free lifetime $\tau_{\Lambda} = 263.2 \pm 2.0$ ps by about 30%. However, the ALICE Collaboration [4] recently reported the result $\tau(3_\Lambda H) = 237^{+33}_{-36}$ ps, which is moderately closer to $\tau_{\Lambda}$. This is one of the most topical issues to study in hypernuclear physics [5] and is essential because $3_\Lambda H$ is the lightest hypernucleus which provides valuable information on interactions of a $\Lambda$ hyperon with nucleons [6]. In addition, the Hypernuclear Hydrodynamics Collaboration [1] suggested that the lifetime of a $^3_\Lambda$H hypernucleus, $\tau(3_\Lambda H) = 140^{+48}_{-45}$ ps, is shorter than the $\tau(3_\Lambda H) = 194^{+25}_{-28}$ ps measured in stopped $K^-$ experiments at KEK [7], whereas theoretical calculations [8,9] predicted $\tau(3_\Lambda H) = 196-264$ ps, which depends on the $\Lambda$ wave functions. To solve the lifetime puzzle, experimental measurements of the $3_\Lambda H$ lifetime have been planned in $(K^-, \pi^0)$ and $(\pi^-, K^0)$ reactions on $^3_\Lambda$He targets at J-PARC [10,11]. Therefore, it is important to investigate theoretically the production of $A = 3, 4$ hypernuclei via $K$, $\pi$ and $(\pi, K)$ reactions on $^3_\Lambda$He targets [12]. Especially, we believe that $\Lambda$ production studies on $^3$He target are useful for settling mysterious problems related to the $A = 4$ hypernuclei [13], e.g., the overbinding/underbinding anomaly [14,15] and charge symmetry breaking [16,17].

Many theoretical studies of hypernuclear spectroscopy have been performed in nuclear $(K^-, \pi^-)$, $(\pi^+, K^+)$, and $(\gamma, K^+)$ reactions [13]. Production cross sections of $\Lambda$ hypernuclei states for each reaction are usually characterized by a specific momentum transfer to a $\Lambda$ hyperon. The $(K^-, \pi^-)$ reactions on $^4$He [18] can produce a $\Lambda$ substitutional state in $^4$He under recoilless conditions. On the other hand, the $(\pi^+, K^+)$ reactions on $^4$He enable us to make large momentum transfers of $q \approx 300–500$ MeV/$c$ [19], populating a $\Lambda$ stretched state which seems to be disadvantageous to the $0s_N \rightarrow 0s_\Lambda$ transition in terms of a momentum matching law [20]; however, the production cross sections of $^4_\Lambda$He via the $(\pi^+, K^+)$ reaction on $^4$He were predicted theoretically [21] for the nonmesonic weak decay of $^4_\Lambda$He and $^3_\Lambda H$ at J-PARC [22]. Consequently, it is worth revisiting investigation of $^4_\Lambda$He production via the $(\pi, K)$ reactions on $^4$He because we have now achieved a good understanding of nuclear $(\pi, K)$ reactions [23].

In this paper, we investigate theoretically the production cross sections of the $J^p = 0^+$ ground state of a $^4_\Lambda$He hypernucleus via the $^4$He$(\pi^+, K^+)$ reaction in the distorted-wave impulse approximation. We demonstrate the differential cross sections at $p_\pi = 1.05$ GeV/$c$ in the $K^+$ forward direction in the laboratory system. We discuss medium effects of the $\pi N \rightarrow K\Lambda$ amplitude in nuclear $(\pi, K)$ reactions and recoil effects in a light hypernucleus such as $^4_\Lambda$He. Our results on $^4_\Lambda$He production via $^4$He$(\pi^+, K^+)$ reactions are expected to be identical to those on $^3_\Lambda H$ mirror production via $^4$He$(\pi^-, K^0)$ reactions because charge independence guarantees $f_{\pi^+ \pi^- \rightarrow K^+\Lambda} = -f_{\pi^- \pi^- \rightarrow K^0\Lambda}$ in the $(\pi, K)$ reactions on $T = 0$ target nuclei.

II. CALCULATIONS

A. Distorted-wave impulse approximation

Let us consider a calculation procedure of hypernuclear production for nuclear $(\pi, K)$ reactions in the laboratory frame. The double-differential cross sections within the distorted-wave impulse approximation (DWIA) [24,25] are

1harada@osakac.ac.jp

I. INTRODUCTION

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In this paper, we investigate theoretically the production cross sections of the $J^p = 0^+$ ground state of a $^4_\Lambda$He hypernucleus via the $^4$He$(\pi^+, K^+)$ reaction in the distorted-wave impulse approximation. We demonstrate the differential cross sections at $p_\pi = 1.05$ GeV/$c$ in the $K^+$ forward direction in the laboratory system. We discuss medium effects of the $\pi N \rightarrow K\Lambda$ amplitude in nuclear $(\pi, K)$ reactions and recoil effects in a light hypernucleus such as $^4_\Lambda$He. Our results on $^4_\Lambda$He production via $^4$He$(\pi^+, K^+)$ reactions are expected to be identical to those on $^3_\Lambda H$ mirror production via $^4$He$(\pi^-, K^0)$ reactions because charge independence guarantees $f_{\pi^+ \pi^- \rightarrow K^+\Lambda} = -f_{\pi^- \pi^- \rightarrow K^0\Lambda}$ in the $(\pi, K)$ reactions on $T = 0$ target nuclei.

II. CALCULATIONS

A. Distorted-wave impulse approximation

Let us consider a calculation procedure of hypernuclear production for nuclear $(\pi, K)$ reactions in the laboratory frame. The double-differential cross sections within the distorted-wave impulse approximation (DWIA) [24,25] are
given by (in units of $\hbar = c = 1$)

$$
\frac{d^2\sigma}{dE_K d\Omega_K} = \beta \frac{1}{[J_A]} \sum_{m_A} \sum_{B,m_B} |\langle \Psi_B | \hat{F} | \Psi_A \rangle|^2 
\times \delta(E_K + E_B - E_\pi - E_A),
$$  \hspace{1cm} (1)

where $[J] = 2J + 1$, and $E_K, E_\pi, E_B,$ and $E_A$ are the energies of outgoing $K$, incoming $\pi$, hypernuclear states, and the target nucleus, respectively. $\Psi_B$ and $\Psi_A$ are wave functions of hypernuclear final states and the initial state of the target nucleus, respectively. The kinematical factor $\beta$ [26,27] arising from a translation from a two-body meson-nucleon laboratory system to a meson-nucleus laboratory system [28] is given by

$$
\beta = \left(1 + \frac{p_{K_1}^{(0)} p_{K_2}^{(0)} - p_{\pi}^{(0)} \cos \theta_{\text{lab}}}{p_{K_1}^{(0)} p_{K_2}^{(0)} E_K^{(0)} E_B^{(0)}} \right),
$$  \hspace{1cm} (2)

where $p_{K_1}^{(0)}$ and $p_{K_2}^{(0)}$ are laboratory momenta of $\pi$ and $K$ (laboratory energies of $K$ and $\Lambda$) in the two-body $\pi N \rightarrow K \Lambda$ reaction, respectively. Here we have considered only the non-spin-flip amplitude and we are interested in the $\Delta S = 0$ cross sections in the $K$ forward direction. Thus an external operator $\hat{F}$ for the associated production $\pi N \rightarrow K \Lambda$ reactions is given by

$$
\hat{F} = \int dr \chi_\pi^{(-)}(p_{\pi}, r) \chi_\Lambda^{(+)}(p_{\Lambda}, r) \sum_{j=1}^A \tilde{J}_{\pi N \rightarrow K \Lambda} \delta(r - r_j) \hat{O}_j,
$$  \hspace{1cm} (3)

where we assume zero-range interaction for the $\pi N \rightarrow K \Lambda$ transitions. Distorted waves of $\chi_\pi^{(-)}$ and $\chi_\Lambda^{(+)}$ are obtained with the help of the eikonal approximation [24]. $\hat{O}_j$ is a baryon operator changing the $j$th nucleon into a $\Lambda$ hyperon in the nucleus, and $r$ is the relative coordinate between the mesons and the center-of-mass (c.m.) of the nucleus; $\tilde{J}_{\pi N \rightarrow K \Lambda}$ is the Fermi-averaged non-spin-flip amplitude for the $\pi N \rightarrow K \Lambda$ reactions in nuclei in the laboratory frame [23].

The energy and momentum transfer to the $\Lambda$ final state is given by

$$
\omega = E_\pi - E_K, \quad q = p_\pi - p_K,
$$  \hspace{1cm} (4)

where $E_\pi = (m_\pi^2 + p_{\pi}^2)^{1/2}$ and $E_K = (m_\Lambda^2 + p_K^2)^{1/2}$ are the laboratory energies of $\pi$ and $K$ in the nuclear reaction, respectively; $m_\pi$ and $m_\Lambda$ ($p_\pi$ and $p_K$) are the masses (laboratory momenta) of $\pi$ and $K$, respectively. Figure 1 displays the momentum transfer to the $0_{g.s.}^+$ final state in $^4$He,

$$
q = \left(\frac{p_\pi^2 + p_K^2}{2} - 2 p_\pi p_K \cos \theta_{\text{lab}}\right)^{1/2},
$$  \hspace{1cm} (5)

as a function of the incident laboratory momentum $p_\pi$.

The differential laboratory cross section of a $\Lambda$ hypernuclear state is obtained by the energy integration [29]

$$
\frac{d\sigma}{d\Omega_K} = \int dE_K \left(\frac{d^2\sigma}{dE_K d\Omega_K}\right),
$$  \hspace{1cm} (6)

around a corresponding peak in the inclusive $K$ spectrum. We often incorporate the effective number technique into the differential laboratory cross section within the DWIA [24, 27–30]. Thus the differential laboratory cross section of the $\Lambda$ bound state with $J^T$ can be written as

$$
\left(\frac{d\sigma}{d\Omega_K}\right)_{\text{lab}, \theta_{\text{lab}}} = \frac{1}{[J_A]} \sum_{m_A} |\langle \Psi_B | \tilde{J}_{\pi N \rightarrow K \Lambda} \chi_\pi^{(-)}(p_{\pi}, \frac{M_C}{M_B} r) \chi_\Lambda^{(+)}(p_{\Lambda}, \frac{M_C}{M_A} r) |\Psi_A \rangle|^2,
$$  \hspace{1cm} (7)

where $r$ is the relative coordinate between a $3N$-core nucleus and a nucleon or $\Lambda$ hyperon; the factors $M_C/M_B$ and $M_C/M_A$ take into account the recoil effects, where $M_A$, $M_B$, and $M_C$ are the masses of the target, the hypernucleus, and the core nucleus, respectively. The kinematical factor $\alpha$ [28] is related to $\beta$ in Eq. (2) as

$$
\alpha = \beta \left(1 + \frac{E_K p_\pi - p_\pi \cos \theta_{\text{lab}}}{E_B p_K} \right)^{-1}.
$$  \hspace{1cm} (8)

In Table I, we list the kinematic values for $^4$He production via the nuclear $(\pi, K)$ reaction on a $^4$He target at $p_\pi = 1.05$ GeV/$c$, $\theta_{\text{lab}} = 0^\circ$, $4^\circ$, $8^\circ$, and $12^\circ$, where we choose $B_\Lambda = 2.39$ MeV as the $\Lambda$ binding energy for $0_{g.s.}^+$ in $^4$He. Note that the value of $\alpha$ for a $^4$He target at $p_\pi = 1.05$ GeV/$c$ ($0^\circ$) amounts to 0.828, which is 25% larger than that for heavier nuclei; $\alpha \simeq 0.66$ for a $^{40}$Ca target [20].

### B. Fermi-averaged $(\pi, K)$ cross sections

It should be noted that a strong energy dependence of differential laboratory cross sections appears in the nuclear $(\pi, K)$ reactions, as discussed in Ref. [23]. To consider the $p_\pi$ dependence of elementary $\pi N \rightarrow K \Lambda$ laboratory cross sections, we average the $\pi N \rightarrow K \Lambda$ matrix in the laboratory frame over a Fermi-momentum distribution, where nuclear effects of a nucleon binding $E_N$ are naturally taken into account. This procedure is called “optimal Fermi averaging” under the
The non-spin-flip amplitudes in the $\pi N \to K \Lambda$ reaction are used.

<table>
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<tr>
<th>$\theta_{lab}$ (deg)</th>
<th>$p_K$ (MeV/c)</th>
<th>$q$ (MeV/c)</th>
<th>$q_{min}$ (MeV/c)</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$\alpha(d\sigma/d\Omega)_{\pi N \to K \Lambda}$ (Opt. (\mu b/sr))</th>
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<tr>
<td>0</td>
<td>690.1</td>
<td>359.9</td>
<td>271.1</td>
<td>0.684</td>
<td>0.828</td>
<td>574</td>
</tr>
<tr>
<td>4</td>
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<td>365.4</td>
<td>275.3</td>
<td>0.686</td>
<td>0.829</td>
<td>561</td>
</tr>
<tr>
<td>8</td>
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<td>381.3</td>
<td>287.3</td>
<td>0.692</td>
<td>0.834</td>
<td>525</td>
</tr>
<tr>
<td>12</td>
<td>684.6</td>
<td>406.2</td>
<td>306.1</td>
<td>0.703</td>
<td>0.843</td>
<td>472</td>
</tr>
</tbody>
</table>

*Effective momentum transfer to the $\Lambda$ final state, $q_{min} = (M_C/M_\Lambda)q$, given in Eq. (30).

Thus the $(\pi^+, K^+)$ and $(\pi^-, K^0)$ cross sections are identical to each other on the $T = 0$ nuclear targets as $^4$He. Here we have employed the elementary $\pi^- p \to K^0 \Lambda$ amplitudes analyzed by Sotona and Žofka [31].

FIG. 2. Optimal Fermi-averaged $\pi N \to K \Lambda$ laboratory cross sections of $\alpha(d\sigma/d\Omega)_{\pi N \to K \Lambda}^{\text{opt}}$ for the $\Delta S = 0$ nuclear ($\pi$, $K$) reactions in $^4$He target. The kinematics for a $^4$He target and the non-spin-flip amplitude at $K$ forward-direction laboratory angles of $\theta_{lab}$ are used. Elementary $\pi N \to K \Lambda$ laboratory cross sections of $\alpha(d\sigma/d\Omega)_{\pi N \to K \Lambda}^{\text{free}}$ for $\Delta S = 0$ in free space [31] are also drawn.

C. Wave functions

In our calculations, we have assumed the $3N 1/2^+$ ground state as a core nucleus in $\Lambda = 4$ systems [18], i.e., $\phi_{3N}$ is a wave function for the $3N$ eigenstates ($^3$He or $^3$H), and we have neglected the charge symmetry-breaking effects in $A = 4$ hypernuclei. Thus the wave function of the 0$^+$ ground state ($0^+_\text{ex}$) in $^4$He ($L_B = 0$, $S_B = 0$, $T_B = 1/2$) is written as

$$\Psi_B = \left[ \phi_{3N} \otimes \psi_{\ell/2}^{(3N)} \right]_{I_B} \otimes X_{I_B}^{B \Lambda N_{\Lambda \Lambda}} \otimes X_{I_B}^{B \Lambda N_{\Lambda \Lambda}},$$

where $\psi_{\ell/2}^{(3N)}$ is a relative wave function between $3N$ and $\Lambda$, and $X_{I_B}^{B \Lambda N_{\Lambda \Lambda}}$ is the isospin-spin function for $0^+$ in $^4$He; $\chi_{I_B}^{(3N)}$ and $\chi_{I_B}^{(\Lambda)}$ are the isospin-spin functions for $3N$ (isospin $I_B$, spin $S_B$) and $\Lambda$ ($I = 0$, $S = 1/2$), respectively. The explicit form of $X_{I_B}^{B \Lambda N_{\Lambda \Lambda}}$ is given in the Appendix. According to Ref. [32], we use $\phi_{\ell/2}^{(3N)}$, which is regarded as a spectroscopic amplitude obtained from a four-body $\Lambda N NN$ wave function using realistic central nucleon-nucleon ($NN$) and $\Lambda N$ potentials [14]. It should be noted that $\phi_{\ell/2}^{(3N)}$ includes the contribution of the $\Lambda N$ short-range correlations and also many-body correlations. Thus we consider that $\phi_{\ell/2}^{(3N)}$ satisfies the Schrödinger equation

$$\left( -\frac{\hbar^2}{2\mu} \nabla^2 + U_{\Lambda} \right) \phi_{\ell/2}^{(3N)} = -B_{\Lambda} \phi_{\ell/2}^{(3N)},$$

where $\mu$ is the $3N$-$\Lambda$ reduced mass and $B_{\Lambda}$ is the $\Lambda$ binding energy with respect to the $^3$He-$\Lambda$ threshold. The $3N$-$\Lambda$ potential $U_{\Lambda}$ is defined as

$$U_{\Lambda} = \langle \phi_{3N} | \mathcal{F}^{ex} \mathcal{F}^{ex} | \phi_{3N} \rangle,$$

where $\mathcal{F}^{ex}$ is the sum of isospin-spin averaged $\Lambda N$ potentials, and $\mathcal{F}^{ex}$ is an external operator on the basis of multiple scattering processes [33],

$$\mathcal{F}^{ex} = 1 + \frac{Q}{e} \mathcal{V}^{ex} \mathcal{F}^{ex},$$

where $Q = 1 - |\langle \phi_{3N} | \phi_{3N} \rangle|$ and $e = E - \hbar^2 \nabla^2 / 2\mu$. Therefore, $U_{\Lambda}$ can be derived from the four-body $\Lambda N NN$...
wave function, taking into account the $\Lambda N$ short-range correlations [32,33].

Figure 3 shows $U_\Lambda$ as a function of the distance between $3N$ and $\Lambda$. We recognize that this potential has a central repulsion and an attractive tail, so we call it the “Isle” potential [9,32–34]; the central repulsion originates predominantly from the $\Lambda N$ short-range correlations due to the repulsive core of the $\Lambda N$ potentials, and it plays an important role in describing the lifetime of $^5\Lambda$He in precise experimental studies on the mesonic weak decay of $\Lambda \to p\pi^-$ [79]. For convenience of use, we parametrize $U_\Lambda$ into a two-range Gaussian form as

$$U_\Lambda(r) = V_C \exp(-r/b_C^2) + V_A \exp(-r/b_A^2), \quad (14)$$

where $V_C = 91.61$ MeV, $V_A = -80.88$ MeV, $b_C = 1.14$ fm, and $b_A = 1.69$ fm, reproducing the experimental data of $\beta_N = 2.39 \pm 0.03$ MeV. To clearly see the effect of the central repulsion, we also introduce the single-range Gaussian (SG) potential $U_\Lambda(r) = -38.5 \exp(-r/b_A^2)$ with $b_A = 1.70$ fm, adjusting to $B_N = 2.39$ MeV. The SG potential is often used as a phenomenological one.

The wave function of the $0^+$ ground state in $^4$He ($L_\Lambda = 0$, $S_\Lambda = 0$, $T_\Lambda = 0$) is written as

$$\Psi_\Lambda = A\left[\phi_{3N}^{(N)} \otimes \phi_{\ell_\Lambda}^{(N)} \right]_{L_\Lambda} \otimes X_{S_{\Lambda},S_{\Lambda}}^{(3N)} \otimes \chi_{1/2,1/2}^{(N)}{0,0}, \quad (15)$$

where $A$ is the antisymmetrized operator for nucleons, and $\phi_{\ell_\Lambda}^{(N)}$ is a relative wave function between $3N$ and $N$; $X_{S_{\Lambda},S_{\Lambda}}^{(3N)}$ and $\chi_{1/2,1/2}^{(N)}$ are the isospin-spin functions for $^4$He and $N$ ($l = 1/2$, $S = 1/2$), respectively. Here we have used $\phi_{\ell_\Lambda}^{(N)}$ obtained from the $3N$-$N$ potential $U_N$, which was derived from a microscopic four-body calculation with a central $NN$ potential of Tamagaki’s C3G [33]. Therefore, $\phi_{\ell_\Lambda}^{(N)}$ includes the contribution of the $NN$ short-range correlations and also many-body correlations [32]. For convenience of use, this potential $U_N$ is parametrized into useful Gaussian forms as

$$U_N(r) = V_1 \exp(-r/b_1^2) + V_2 \exp(-r/b_2^2) + V_3 \exp(-r/b_3^2) \quad (16)$$

with $V_1 = 156.28$ MeV, $V_2 = -185.66$ MeV, $V_3 = -9.56$ MeV, $b_1 = 1.21$ fm, $b_2 = 1.58$ fm, and $b_3 = 2.82$ fm [35], making a fit to the experimental data of the binding energy $B_N = 20.6$ MeV and the nuclear root-mean-square (r.m.s.) distance of $(r_\Lambda^N)^{1/2} = 1.87$ fm between $^3$He and $n$.

Figure 4 shows that $3N$-$\Lambda$ density distributions $\rho_{\Lambda}(r)$ between $3N$ and $\Lambda$ for $0^+_{g.s.}$ in $^4$He, as a function of the relative distance, together with $3N$-$N$ density distributions $\rho_N(r)$ for $0^+_{g.s.}$ in $^4$He. We find that the $3N$-$\Lambda$ distribution obtained from the Isle potential differs considerably from that obtained from the SG potential around the nuclear center; the $3N$-$\Lambda$ distribution is considerably suppressed at the nuclear center, and it is pushed outside, as discussed in Refs. [32–35]. The relative r.m.s. distance between $3N$ and $\Lambda$ becomes $(r_\Lambda^N)^{1/2} = 3.57$ fm for Isle, which is 8% larger than the 3.31 fm for SG.

To see the properties of our $\phi_{\ell_\Lambda}^{(A)}$ obtained by $U_\Lambda$ in Fig. 3, we also consider a single-particle harmonic oscillator (HO) wave function with the size parameter $b_A = (\hbar/m_A\omega_A)^{1/2}$, which is used as a simple model calculation [20,36]. Here we choose $b_A = 2.233$ fm ($\hbar\omega_A = 7.0$ MeV), simulating the value of the relative r.m.s. distance of $(r_\Lambda^N)^{1/2} = 3.3-3.6$ fm for the $3N$-$\Lambda$ distribution obtained by four-body $\Lambda NNN$ calculations using the $NN$ and $\Lambda N$ potentials [14,37,38]; we have $(r_\Lambda^N)^{1/2} = 3.23$ fm for the HO. The relative $3N$-$\Lambda$ wave function with $\ell_\Lambda = 0$ for the HO is written as

$$\phi_{\ell_\Lambda}^{(A)}(r) = \left(\frac{4}{b_A^3 \sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{r^2}{2b_A^2}\right)Y_0^0(\hat{r}), \quad (17)$$
where the size parameter with the c.m. correction is denoted
\[ \tilde{b}_\Lambda = b_\Lambda \sqrt{1 + m_\Lambda / 3m_N} = 2.634 \text{ fm}, \]
where \( m_\Lambda \) and \( m_N \) are the masses of a \( \Lambda \) hyperon and a nucleon, respectively. We confirm \( r_{\Lambda}^2 \Lambda = 3.23 \text{ fm for the HO. Figure 4 also shows the } 3N-\Lambda \text{ distribution for the HO taking into account the c.m. correction, which is absolutely essential in light nuclei. We expect that the difference among the } 3N-\Lambda \text{ distributions obtained from several models in Fig. 4 is clearly observed in production cross sections of nuclear (} \pi, K \text{) reactions making a large momentum transfer to the } 3 \text{ final state. On the other hand, we confirm that the } 3N-\Lambda \text{ distribution for the HO using the size parameter } \tilde{b}_N = 1.329 \text{ fm (} h_\omega = 23.5 \text{ MeV) is in good agreement with that obtained from the C3G potential; the relative r.m.s. distance between } 3N \text{ and } N \text{ is } \langle r_{NN}^2 \rangle = \tilde{b}_N \sqrt{2/3} = 1.88 \text{ fm, which is quite close to the 1.87 fm for C3G, where } \tilde{b}_N = b_N \sqrt{2/3} = 1.535 \text{ fm.}

D. Differential cross sections

We consider the differential laboratory cross sections of the hypernuclear bound state with } J^\pi \text{ on a closed-shell target nucleus with } J^\pi = 0^+, \text{ like } ^4\text{He, adapting the effective number technique into the DWIA [24,27–30]. Substituting Eqs. (15), (10), and (27) into Eq. (7), we obtain
\[ \left( \frac{d\sigma}{dQ}\right)_{\text{lab},\theta_\Lambda} = \alpha \frac{d\sigma}{dQ}_{\pi N-\Lambda = \Lambda} N_{\text{eff}}(\theta_\Lambda), \]
where \( \alpha \frac{d\sigma}{dQ}_{\pi N-\Lambda = \Lambda} \) is the optimal Fermi-averaged \( \pi N \rightarrow K \Lambda \) laboratory cross section, as discussed in Sec. II B. The effective number of nucleons } N_{\text{eff}} \text{ for } \Lambda \text{ production of the } J^\pi \text{ final state in the LS-coupling scheme is written as
\[ N_{\text{eff}}(\theta_\Lambda) = C_B(2L + 1)(2\ell_\Lambda + 1) \left( \begin{array}{ccc} \ell_\Lambda & L & \ell_N \\ 0 & 0 & 0 \end{array} \right)^2 |F(q)|^2, \]
where } \ell_\Lambda + \ell_\Lambda + \ell_N \text{ must be even due to the non-spin-flip reaction. Thus, only natural-parity states with } J^\pi = 0^+, 1^+, 2^+, 3^+, \ldots \text{ for the } 3N+\Lambda \text{ systems can be populated. } C_B \text{ is the isospin-spin spectroscopic amplitude between the } \Lambda \text{ final state of } ^4\text{He and the initial state of } ^3\text{He, which is given by
\[ C_B = \langle \chi^B_{\Lambda} | \sum_{j_1} | O_j | \chi^A_{\Lambda} \rangle. \]
The form factor } F(q) \text{ in Eq. (20) is given as
\[ F(q) = \int_0^{\infty} r^2 \rho_{\Lambda}(r)^2 j_L(q \tilde{r}) \left( \frac{M_C}{m_A} \right) r, \]
with the } N \rightarrow \Lambda \text{ transition densities
\[ \rho_{\Lambda}(r) = \phi_{\Lambda}(r) \phi_{\Lambda}(r) \]
and the distorted waves } j_L(q \tilde{r}), \text{ considering the nuclear distortions by mesons in the DW approximation, as we express in Eq. (28); } M_C/m_A \text{ is the so-called recoil factor for the momentum transfer } q. \text{ Here we have approximated to } M_C/m_A \approx M_C/m_B \text{ in Eq. (7). For } J^\pi = 0^+ \text{ in } ^4\text{He, we take } \ell_\Lambda = \ell_N = L = 0 \text{ and } C_B = \sqrt{2} \text{ in Eq. (20), then we obtain
\[ N_{\text{eff}}(\theta_\Lambda) = 2 |F(q)|^2, \]
which is expected to observe the hypernuclear fine structure because } F(q) \text{ is generally sensitive to the nature of the distribution of } \rho_{\Lambda}(r), \text{ as a function of } q.

E. Meson distorted waves

Full distorted waves of the } \pi \text{ and the } K \text{ nucleuses are important to reproduce the absolute values of the cross sections. Because the } (\pi, K) \text{ reaction requires a large momentum transfer with a high angular momentum, we simplify the computational procedure in the eikonal approximation to the distorted waves of the meson-nucleus states [20,24,27,28],
\[ \chi^{(\pi)}_{\Lambda}(p_K, r) \chi^{(+)}_{\Lambda}(p_\pi, r) = \exp \left( i q \cdot r \right) D(b, z), \]
with
\[ D(b, z) = \exp \left[ -\frac{\sigma_{\pi}(1 - is\pi)}{2} \int_0^z \rho(b, z')d'z' - \frac{\sigma_{\pi}(1 + is\pi)}{2} \int_{\infty}^z \rho(b, z')d'z' \right], \]
where } \sigma_{\pi} \text{ (} \sigma_K \text{) is the averaged total cross section in } \pi N \text{ (} K N \text{) elastic scatterings, and } \sigma_{\pi} \text{ (} \sigma_K \text{) is the ratio of the real and imaginary parts of the corresponding forward scattering amplitudes; } b \text{ is the impact parameter. } \rho(r) \equiv \rho(b, z) \text{ is a matter-density distribution fitting to the data on the nuclear charge density [39]. We assume } \sigma_{\pi} = \sigma_K = 0, \text{ which hardly affects the following results, and we study } \sigma_{\pi} = 20-30 \text{ mb and } \sigma_K = 10-30 \text{ mb [20,24]; } \sigma_{\pi}, \sigma_K = 30 \text{ mb, 15 mb) is chosen as a standard value [23]. Reducing the right-hand side of Eq. (25) by partial-wave expansion, we obtain
\[ \chi^{(\pi)}_{\Lambda}(p_K, r) \chi^{(+)}_{\Lambda}(p_\pi, r) = \sum_L \sqrt{4\pi(2L + 1)} j_L(q \tilde{r}) Y_L^0(\hat{r}), \]
with
\[ j_L(q \tilde{r}) = \int_{\infty}^{\infty} \frac{2\ell + 1}{2L + 1} j_L(q \tilde{r}) D_T(r), \]
where } D_T(r) \text{ is a distortion function [23] defined as
\[ D_T(r) = \frac{2\ell + 1}{2} \int_{-1}^{1} D(b, z) P_L(\cos \theta) d(\cos \theta). \]
Here } z = r \cos \theta, r^2 = b^2 + z^2, \text{ and } P_L(x) \text{ is a Legendre polynomial. If the distortion is switched off, } j_L(q \tilde{r}), \text{ which is a spherical Bessel function with } L. \text{ The production probability for } \tilde{o}_{\pi, K} \text{ in } ^3\text{He is expected to be only } 10^{-2}, \text{ which is roughly estimated as exp } (-(bjq)^2 / 2) \text{ due to the } 0\pi N \rightarrow 0\Lambda \text{ transition with } \Delta L = 0 \text{ in nuclear } (\pi, K) \text{ reactions because the } \Lambda \text{ continuum states can be populated predominately by the large angular momentum transfer. Figure 5 displays the distorted waves } j_0(q \tilde{r}), \text{ for } \pi

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\( q = 360 \text{ MeV/c} \)

\[ q_0 \approx A^{-1} q = 3 \times 360 = 271 \text{ MeV/c} \quad (30) \]

\( \sigma_{\text{eff}} = \frac{M_C}{M_A} q_{\text{eff}} = A \pi (n = 1, 2, \ldots) \),

so we have the point of \( r_1 = (M_A/M_C)\pi q = 2.29 \text{ fm} \) located on the outside of the nucleus having \( r_2^{1/2} = 1.88 \text{ fm} \) for \( ^4\text{He} \). If we omit the recoil effects \( M_C/M_A \rightarrow 1 \), we have \( r_1 \rightarrow \pi q = 1.72 \text{ fm} \), which is located on the inside of the nucleus. Therefore, the recoil effects must be taken into account for the light nuclear target in the \( (\pi, K) \) reactions providing the large momentum transfer to the \( \Lambda \) final state, as we discuss in Sec. III C.

### III. RESULTS AND DISCUSSION

Let us consider the \( \Lambda \) production for \( 0^+_{g.s.} \) in \( ^4\text{He} \) via the \( (\pi, K) \) reactions on the \( ^4\text{He} \) target at \( p_\pi = 1.05 \text{ GeV/c} \) in the \( K \) forward direction. We discuss the meson distortion effects, comparing the differential laboratory cross sections in the PWIA versus the DWIA, and we study the sensitivity of

<table>
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<th>( \mathcal{N}_{\text{eff}}^{\text{PW}} )</th>
<th>( \mathcal{N}_{\text{eff}}^{\text{DW}} )</th>
<th>( D_{\text{dis}} )</th>
<th>( \mathcal{N}_{\text{eff}}^{\text{DW}} )</th>
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</thead>
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</table>

\( D_{\text{dis}} = \mathcal{N}_{\text{eff}}^{\text{DW}} / \mathcal{N}_{\text{eff}}^{\text{PW}} \).

A. Differential cross sections for \( 0^+_{g.s.} \) in \( ^4\text{He} \)

1. PWIA vs DWIA

In Table II, we list the calculated PWIA and DWIA results of the differential laboratory cross sections for \( 0^+_{g.s.} \) in \( ^4\text{He} \) at \( p_\pi = 1.05 \text{ GeV/c} \) in the \( K \) forward-direction angles of \( \theta_{\text{lab}} = 0^\circ-12^\circ \). The differential laboratory cross section of \( 0^+_{g.s.} \) in \( ^4\text{He} \) accounts for

\[ \left( \frac{d\sigma}{d\Omega_K} \right)_{\text{lab}, 0^\circ} \approx 35.14 \mu\text{b}/\text{sr} \quad (\text{PWIA}) \]

at \( p_\pi = 1.05 \text{ GeV/c} \) \( (\theta_{\text{lab}} = 0^\circ) \), using the Isle potential. When we take into account the distortion with \( (\sigma_{\pi}, \sigma_K) = (30 \text{ mb}, 15 \text{ mb}) \), we obtain the differential laboratory cross section of \( 0^+_{g.s.} \) in \( ^4\text{He} \) as

\[ \left( \frac{d\sigma}{d\Omega_K} \right)_{\text{lab}, 0^\circ} \approx 11.73 \mu\text{b}/\text{sr} \quad (\text{DWIA}). \]

We confirm that the differential laboratory cross sections in DWIA are relatively reduced by a distortion factor which is defined as

\[ D_{\text{dis}} \equiv \mathcal{N}_{\text{eff}}^{\text{DW}} / \mathcal{N}_{\text{eff}}^{\text{PW}} \approx 0.3, \]

as listed in Table II. We find that the absolute values of the differential laboratory cross sections for SG (HO) are larger (smaller) than those for Isle. This is because the overlaps of wave functions between \( N \) and \( \Lambda \) are larger inside the nucleus in the order SG, Isle, HO, as shown in Fig. 4.
obtained from the Isle, SG, and HO potentials, respectively. Long-dashed curves denote the results in the eikonal-oscillator approximation with (b) the distorted-wave (DW) approximations, as a function of the momentum transfer $q$. Solid, dashed, and dotted curves denote the results obtained from the Isle, SG, and HO potentials, respectively. Long-dashed curves denote the results in the eikonal-oscillator approximation with $\sigma = 30$ mb. The half-tones denote the region of $q = 360-410$ MeV/c corresponding to $\theta_{lab} = 0^\circ-12^\circ$ at $p_\pi = 1.05$ MeV/c.

2. $N_{0+}^{0+} vs q$

To see the features of the $\Lambda$ production of the nuclear ($\pi, K$) reactions, we study the effective number of nucleons, $N_{0+}^{0+}$, as a function of the momentum transfer $q$, which is determined by the incident laboratory momentum $p_\pi$ and the $K$ forward-direction angles of $\theta_{lab}$.

Figure 6(a) displays the calculated results of $N_{0+}^{0+}$, using the $\Lambda$ wave functions obtained from the Isle, SG, and HO potentials. The essential physical features of the endothermic nuclear ($\pi, K$) reactions can be well understood in terms of $N_{0+}^{0+}$ in PWIA, as suggested by Dover et al. [20]. We confirm that the magnitudes of $N_{0+}^{0+}$ obtained from the SG, Isle, and HO potentials in PWIA are large in this order. Note that the nuclear ($\pi, K$) reactions at $p_\pi = 1.05$ GeV/c ($\theta_{lab} = 0^\circ-12^\circ$) provide $q = 360-410$ MeV/c, corresponding to the region of the half-tones drawn in Fig. 6, where the recoil effects are very important owing to $q_{eff} \sim 0.75q$ in $^4$He. On the other hand, exothermic nuclear ($K, \pi$) reactions with $p_\pi = 0.79$ GeV/c ($\theta_{lab} = 0^\circ-12^\circ$) have $q = 68-126$ MeV/c, which denote small momentum transfers, thus the recoil effects do not so affect the differential laboratory cross sections. In the region of $q > 600$ MeV/c, we find that the values of $N_{eff}$ obtained from the Isle potential fall off, and their slopes are steeper than those with the SG potential; a dip appears in the region of $q = 600-800$ MeV/c. This behavior is well known to occur due to high-momentum components in the $\Lambda N$ and $NN$ wave functions due to short-range correlations, as discussed in Ref. [19]. We also find a dip in $N_{0+}^{0+}$ for HO which includes only the $NN$ correlations.

Figure 6(b) shows the calculated results for $N_{0+}^{0+}$ with the DWIA. The distortions reduce the magnitudes of $N_{0+}^{0+}$ by about three times (see Table II), changing slightly the shapes of $N_{0+}^{0+}$ in the region of $q < 400$ MeV/c. The slopes of $N_{0+}^{0+}$ become gradually steeper with increasing $q$ as in the case of $q > 400$ MeV/c, and they develop a dip in the region of $q = 520-620$ MeV/c which can be achieved by $\theta_{lab} = 24^\circ-34^\circ$. These behaviors may originate from high-momentum components generated by meson distortions, because $D_0(r)$ significantly modifies $j_0(qr)$ inside the nucleus. Therefore, the distortion effects as well as the recoil effects are very important for large momentum transfer processes which can be realized in the ($\pi, K$) reactions.

B. Comparison with the eikonal-oscillator approximation

We also consider the differential laboratory cross sections using the single-particle harmonic oscillator wave functions and the eikonal distortions by mesons, referring to this as the “eikonal-oscillator approximation” [20]. This is often employed in nuclear model calculations for several reactions [28,36]. When we use the HO wave functions for both the nucleon and $\Lambda$, we can express the eikonal distorted waves as

$$\chi_{K}^{(-)}(p_{K}, r)\chi_{\pi}^{(+)}(p_{\pi}, r) = \exp (iqz) \exp \left(-\frac{\sigma}{2} T(b) \right)$$  \hspace{1cm} (35)
at the $K$ forward-direction angle of $\theta_{lab}=0^\circ$; the nuclear thickness function for the $A=Z+N$ target nucleus is defined as
\[ T(b) \equiv \int_0^\infty \rho(r)dr, \quad \int T(b)db = A, \tag{36} \]
with the averaged total cross section $\bar{\sigma} = (\sigma_x + \sigma_K)/2$ for the $\pi N$ and $KN$ elastic scatterings. Thus we have the effective number of nucleons $N_{\text{eff}}^0$ for $0^\circ$, as in Eq. (19), which is rewritten as \cite{20,36}
\[ N_{\text{eff}}^0(\theta_{lab}) \simeq 2 \left( \frac{\bar{b}_N}{b_\Lambda} \right)^2 \left( \frac{1}{2} \left( \frac{\mu_b}{\mu_A} \right)^2 \right)^2 G_0(\bar{\sigma})^2, \tag{37} \]
where the distorted-wave integral \cite{20,36} is defined by
\[
G_0(\bar{\sigma}) = \int_0^\infty 2r \exp(-r^2) \exp \left( -\frac{\bar{\sigma}}{2} T(b)r \right) dt, 
\]
\[ G_0(\bar{\sigma} = 0) = 1. \tag{38} \]
Here the mean HO size parameter is denoted
\[ 1/b_0^2 = (1/b_\Lambda^2 + 1/b_N^2)/2. \tag{39} \]
These formulas provide good insight into the nuclear ($\pi, K$) reactions. The total effective number of nucleons $N_{\text{eff}}$ at $\theta_{lab} = 0^\circ$ is defined by the sum of all contributions of the $A$ final states. In the closure approximation, it can be easily written as
\[ N_{\text{eff}}^0(0^\circ) = \sum_{j=0}^{N-1} N_{\text{eff}}^j(0^\circ) \simeq \frac{N}{A} \int T(b) \exp(-\bar{\sigma}T(b))db, \tag{40} \]
so that the values of $N_{\text{eff}}^0(0^\circ)$ amount to 2.00, 1.22, and 0.99 for $\bar{\sigma} = 0$ mb (PW), 20 mb, and 30 mb, respectively. Figure 6 also displays the values of $N_{\text{eff}}^0$ calculated in the eikonal-oscillator approximation of Eq. (37). In the case of the PWIA, the magnitude of $N_{\text{eff}}^0$ is as large as that obtained from the HO potential, and it is smaller than that obtained from the Isol potential. In the case of the DWIA ($\bar{\sigma} = 30$ mb), the magnitude of $N_{\text{eff}}^0$ is as large as that for Isol, and these slopes are similar to each other in the region of $q < 400$ MeV/c. Thus we obtain $(d\sigma/d\Omega)^{\text{DW}}_{\text{lab}} = 10.41, 9.83, 6.90, \text{and } 3.64 \mu b/\text{sr}$ at $p_x = 1.05$ GeV/c for $\theta_{lab} = 0^\circ, 4^\circ, 8^\circ, \text{and } 12^\circ$, respectively; they are comparable to $(d\sigma/d\Omega)^{\text{PW}}_{\text{lab}} = 11.73, 10.99, 7.47, \text{and } 3.65 \mu b/\text{sr}$ for Isol. Because high-momentum components of neither the $N$ and $\Lambda$ wave functions nor the distorted waves for mesons are included in the eikonal-oscillator approximation, we confirm that the shape of $N_{\text{eff}}^0$ shows no dip with increasing $q$.

### C. Recoil effects

The recoil effects should be needed in the ($\pi, K$) reactions on a very light nuclear target such as $^4$He; the quantity of the recoil factor $M_C/M_A \lesssim 3/4 = 0.75$ characterizes the importance of the recoil effects in the nuclear systems. To see the sensitivity to the recoil effects quantitatively, we demonstrate the differential laboratory cross sections when we omit the recoil effects ($M_C/M_A \rightarrow 1$) in Eq. (7) using the Isol, SG, and HO potentials.

<table>
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<th>$\theta_{lab}$ (deg)</th>
<th>$N_{\text{eff}}^{\text{PW}}$ (x 10^{-2})</th>
<th>$(d\sigma/d\Omega)^{\text{PW}}_{\text{lab}}$ (\mu b/sr)</th>
<th>$D_{\text{eff}}$</th>
<th>$N_{\text{eff}}^{\text{DW}}$ (x 10^{-2})</th>
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In Table III, we list the calculated DWIA (PWIA) results of the differential laboratory cross sections omitting the recoil effects via the $^4$He($\pi, K$) reaction at $p_x = 1.05$ GeV/c. We find $(d\sigma/d\Omega)_{\text{lab}} = 0.382 \mu b/\text{sr}$ (4.113 $\mu b/\text{sr}$) at $\theta_{lab} = 0^\circ$. Surprisingly, this value is an order of magnitude smaller than $(d\sigma/d\Omega)_{\text{lab}} = 11.73 \mu b/\text{sr}$ (35.14 $\mu b/\text{sr}$), which is already listed in Table II. The recoil effects have a great influence on $(d\sigma/d\Omega)_{\text{lab}}$ depending on the radial behavior of the distorted waves for mesons. Here we consider the overlap function defined as
\[ \gamma(q;r) = r^2\rho_0(\pi)(r) \int_0^\infty \rho_m(q,q') \frac{M_C}{M_A} q', \tag{41} \]
which corresponds to the integrand in Eq. (22). Figure 7 displays the behaviors of $\gamma(q;r)$ for various types choosing the $\Lambda$ wave functions and the distortion parameters, as a function of the radial distance. When we omit the recoil effects ($M_C/M_A \rightarrow 1$), we find that the node point at $r_1$ where $\gamma(q;r_1) = 0$ must be shifted toward the nuclear interior. As a result, the integral value of $\int_0^\infty \gamma(q;r)dr = F(q)$ is significantly reduced by cancellation between the positive and the negative values over integration regions in $\gamma(q;r)$, as illustrated in Fig. 7.

The recoil effects are often omitted in several model calculations for nuclear reactions with large momentum transfers, e.g., ($\pi, K$) \cite{20}, ($\bar{K}$, $K$) \cite{28}, ($\bar{K}$, $\rho$) \cite{36}, and (stopped $\bar{K}$, $\pi$) \cite{40} reactions on $^{12}$C in which the recoil effects are not so important because $M_C/M_A \simeq (A-1)/A = 11/12 = 0.917$ for $A = 12$. But we must pay attention to the recoil effects when applying it to light nuclear systems such as $^4$He ($M_C/M_A \simeq 3/4 = 0.75$). We believe that the calculated cross sections \cite{28} or calculated production probabilities \cite{40} of $^4$He are perhaps underestimated by an order of magnitude due to the lack of the recoil effects \cite{41}.
D. Dependence on distortion parameters

Due to strong absorptions of mesons in nuclei, e.g., $\pi^+$ at $p_\pi = 1.0\text{–}1.5\text{ GeV}/c$ in the $N^*$ resonance region, the magnitude of the cross section may also be affected by meson distortions. To understand the distortion effects quantitatively, we demonstrate the differential laboratory cross sections of $0^+_{g.s.}$ in $^4\text{He}$, considering various eikonal distortions with parameters $(\sigma_\pi, \sigma_K)$. Table IV shows the calculated results of $(d\sigma/d\Omega)_{\text{lab}}$ at $p_\pi = 1.05\text{ GeV}/c$. The magnitudes of the cross sections are reduced upon increasing these parameters in the DWIA. Thus, the differential laboratory cross sections of $0^+_{g.s.}$ in $^4\text{He}$ at $\theta_{\text{lab}} = 0^\circ$ amount to $(d\sigma/d\Omega)_{\text{lab}} \approx 8\text{–}14\mu\text{b}/\text{sr}$, which depend on $\pi$ and $K$ distorted waves for $\sigma_\pi = 20\text{–}30\mu\text{b}$ and $\sigma_K = 10\text{–}30\mu\text{b}$. It should be noted that the parameter dependence gives an indication of the accuracy of our results within the eikonal distortion. Fully realistic distorted waves obtained from meson-nucleus optical potentials would be needed for a more quantitative discussion.

E. Dependence on the incident momentum

In Fig. 8, we display the differential laboratory cross sections of $0^+_{g.s.}$ in $^4\text{He}$ via the $^4\text{He}(\pi, K)$ reactions at $\theta_{\text{lab}} = 0^\circ, 4^\circ, 8^\circ$, and $12^\circ$, as a function of $p_\pi$. Here we have used the $\Lambda$ wave functions obtained from the Isle potential and the eikonal distortions with $(\sigma_\pi, \sigma_K) = (30\mu\text{b}, 15\mu\text{b})$. We find that the differential laboratory cross sections slightly increase with increasing $p_\pi$. This trend seems to be opposite to that of $\sigma/(d\sigma/d\Omega)_{\text{lab}}|_{\text{opt}}$ as shown in Fig. 2. This stems from the fact that the momentum transfers in this region decrease with increasing $p_\pi$ (see Fig. 1), together with the nature of $N^\pi_{\text{eff}}$, which must take into account the recoil effects.

IV. SUMMARY AND CONCLUSION

We have investigated theoretically the production cross sections of the $0^+_{\text{g.s.}}$ ground state of a $^4\text{He}$ hypernucleus in the $^4\text{He}(\pi, K)$ reaction with a distorted-wave impulse approximation using the optimal Fermi-averaged $\pi N \rightarrow K \Lambda \tau$ matrix. We have demonstrated the sensitivity of the production cross section to the $3\Lambda$ potentials and to the eikonal distorted waves for mesons. We have calculated the differential laboratory cross sections of $(d\sigma/d\Omega)_{\text{lab}}$ at $p_\pi = 1.05\text{ GeV}/c$ in the $K$ forward-direction angles of $\theta_{\text{lab}} = 0^\circ\text{–}12^\circ$. The results can be summarized as follows.

1. The calculated differential laboratory cross section of $0^+_{\text{g.s.}}$ in $^4\text{He}$ amounts to $(d\sigma/d\Omega)_{\text{lab}} \approx 11\mu\text{b}/\text{sr}$ at...
The authors would like to thank Prof. A. Sakaguchi, Prof. H. Tamura, and Dr. A. Feliciello for many valuable discussions. This work was supported by Grants-in-Aid for Scientific Research (KAKENHI) from the Japan Society for the Promotion of Science (Grant No. JP16K05363).

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APPENDIX: EXPLICIT FORMS OF ISOSPIN-SPIN FUNCTIONS FOR 0+; IN 4He AND 3He

The isospin-spin function \( X_{TA,SA}^A \) for \( 0^+ \) in \( 4\text{He} \) for \( \beta_{TA} = 0 \), \( m_{SA} = 0 \) in Eq. (15) is explicitly written as

\[
X_{TA,SA}^A = A\left[\chi_{I,SA}^{(3N)} \otimes \chi_{I/1/2,1/2}^{(N)}\right]_{0,0}
\]

where \( -p_+ \) and \( n_+ \) (\( p_\downarrow \) and \( n_\downarrow \)) denote spin states with \( m_\pi = +1/2 (-1/2) \) for a proton and a neutron, respectively. The isospin-spin function \( X_{TA,SA}^B \) for \( 0^+ \) in \( 3\text{He} \) for \( \beta_{TA} = 1/2 \), \( m_{SA} = 0 \) in Eq. (10) is explicitly written as

\[
X_{TA,SA}^B = \left[\chi_{I,SA}^{(3N)} \otimes \chi_{I/1/2,1/2}^{(N)}\right]_{0,0}
\]

where \( I_\uparrow \) (\( I_\downarrow \)) denotes a spin state with \( m_\pi = +1/2 (-1/2) \) for a hyperon.

[10] H. Asano et al., 3He and 4He mesonic weak decay lifetime measurement with \( 3\text{He} (K^+, \pi^0) \) reaction. Proposal for Nuclear and Particle Physics experiments at the J-PARC (2018); http://j-parc.jp/NuclPart/Proposal_e.html.
[11] M. Agnello et al., Direct measurement of the 3He and 4He lifetimes using the \( 3\text{He} (K^+, \pi^0) \) reaction. Proposal for Nuclear and Particle Physics experiments at the J-PARC (2019); http://j-parc.jp/NuclPart/Proposal_e.html.