Title	Contiguous relations, Laplace's methods, and continued fractions for F-3(2)(1)
Author(s)	Ebisu, Akihito; Iwasaki, Katsunori
Citation	Ramanujan journal, 49(1), 159-213 https://doi.org/10.1007/s11139-018-0039-2
Issue Date	2019-05
Doc URL	http://hdl.handle.net/2115/75783
Rights	The final publication is available at Springer via https://dx.doi.org/10.1007/s11139-018-0039-2.
Type	article (author version)
File Information	Ramanujan J. 49(1) 159.pdf

Contiguous Relations, Laplace's Methods and Continued Fractions for $_3F_2(1)^*$

Akihito Ebisu† and Katsunori Iwasaki‡

April 16, 2018

Abstract

Using contiguous relations we construct an infinite number of continued fraction expansions for ratios of generalized hypergeometric series $_3F_2(1)$. We establish exact error term estimates for their approximants and prove their rapid convergences. To do so we develop a discrete version of Laplace's method for hypergeometric series in addition to the use of ordinary (continuous) Laplace's method for Euler's hypergeometric integrals.

1   Introduction

In 1813 Gauss [13] introduced a general continued fraction that represents the ratio of two $_2F_1$ hypergeometric functions. It is interesting because it contains a variety of continued fraction expansions of several important elementary functions and some of more transcendental ones. In 1901 Van Vleck [26] established a general result on its convergence. Gauss's continued fraction is derived from a three-term contiguous relation for $_2F_1$. In 1956, using other contiguous relations, Frank [12] constructed some more (eight or so) continued fractions of a similar sort and discussed their convergence. In 2005 Borwein, Choi and Pigulla [7] obtained an explicit bound for the error term in certain special cases of the Gauss continued fraction. In 2011, based on Gauss's continued fraction and other means, Colman, Cuyt and Van Deun [8] developed an efficient algorithm for the validated high-precision computation of certain $_2F_1$ functions.

The generalized hypergeometric series of unit argument $_3F_2(1)$ also admits three-term contiguous relations, among which the basic twelve relations were found by Wilson [27]; see also Bailey [5]. Thus it is feasible and interesting to discuss or utilize allied continued fractions for $_3F_2(1)$. For instance, Zhang [29] used contiguous relations for $_3F_2(1)$ to give new proofs of three of Ramanujan's elegant continued fractions for products and quotients of gamma functions, namely, entries 34, 36 and 39 in Ramanujan's second notebook [24, Chapter 12], or in its corrected version by Berndt, Lamphere and Wilson [6]. In a similar vein, Denis and Singh [9] dealt with entries 25 and 33 of the same notebook.

$$_3F_2\left(\begin{matrix} 1, & 1, & 1 \\ & 2, & 2 \end{matrix}\right) = \zeta(2) = \frac{\pi^2}{6} \qquad \text{(Euler 1735)},$$

$$_3F_2\left(\begin{matrix} 1, & \frac{1}{4}, & \frac{1}{2} \\ & \frac{5}{4}, & \frac{3}{2} \end{matrix}\right) = \frac{\pi + 2\log 2}{4},$$

$$_3F_2\left(\begin{matrix} 1, & \frac{1}{4}, & \frac{1}{4} \\ & \frac{5}{4}, & \frac{5}{4} \end{matrix}\right) = \frac{\pi^2 + 8\,G}{16} \qquad (G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \text{ is Catalan's constant}),$$

$$_3F_2\left(\begin{matrix} 1, & \frac{5}{6}, & \frac{4}{3} \\ & \frac{3}{2}, & \frac{11}{6} \end{matrix}\right) = \frac{5}{2}\sqrt{3}\log(2+\sqrt{3}) \qquad \text{(Watson 1918)},$$

$$_3F_2\left(\begin{matrix} 1, & 1, & \frac{3}{4} \\ & \frac{7}{6}, & \frac{11}{6} \end{matrix}\right) = \frac{2^{\frac{1}{2}}\cdot 5}{3^{\frac{5}{4}}}\log\left(\frac{3^{\frac{5}{4}} - 3^{\frac{3}{4}} + 2^{\frac{1}{2}}}{3^{\frac{5}{4}} - 3^{\frac{3}{4}} - 2^{\frac{1}{2}}}\right) - \frac{2^{\frac{3}{4}}\cdot 5}{3^{\frac{5}{4}}}\arccos\left(\frac{3^{\frac{5}{4}} + 3^{\frac{3}{4}}}{2\sqrt{5 + 3^{\frac{3}{2}}}}\right)$$

$$\text{(Asakura, Otsubo and Terasoma 2016 [3])},$$

$$_3F_2\left(\begin{matrix} 1, & 1, & \frac{2}{3} \\ & \frac{7}{6}, & \frac{11}{6} \end{matrix}\right) = \frac{5}{2^{\frac{5}{3}}\cdot 3^{\frac{1}{2}}}\log\left(\frac{2^{\frac{2}{3}} + 2^{\frac{1}{3}} - 1 + 3^{\frac{1}{2}}}{2^{\frac{2}{3}} + 2^{\frac{1}{3}} - 1 - 3^{\frac{1}{2}}}\right) - \frac{5}{2^{\frac{2}{3}}\cdot 3}\arctan\left(\frac{3}{3\cdot 2^{\frac{2}{3}} + 2^{\frac{1}{2}} + 3}\right)$$

$$\text{(Yabu 2017 [28])},$$

$$_3F_2\left(\begin{matrix} 1, & a, & b \\ & 1, & c \end{matrix}\right) = {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad \text{(Gauss 1812)},$$

$$_3F_2\left(\begin{matrix} 1, & a, & b \\ & 2, & c \end{matrix}\right) = \frac{c-1}{(a-1)(b-1)}\left\{\frac{\Gamma(c-1)\,\Gamma(c+1-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} - 1\right\}.$$

Table 1: Some special evaluations of the series $_3F_2(1, a_1, a_2; b_1, b_2)$.

To give a further motivation for $_3F_2(1)$ continued fractions, we look at the special case in which one of the numerator parameters, say a_0, is equal to one:

$$_3F_2\left(\begin{matrix} 1, & a_1, & a_2 \\ & b_1, & b_2 \end{matrix}\right) := \sum_{j=0}^{\infty} \frac{(a_1;\, j)\,(a_2;\, j)}{(b_1;\, j)\,(b_2;\, j)}, \qquad (a;\, j) := \frac{\Gamma(a+j)}{\Gamma(a)}, \tag{1}$$

where $\Gamma(a)$ is Euler's gamma function. This series is well defined and non-terminating if

$$a_1, a_2, b_1, b_2 \notin \mathbb{Z}_{\leq 0}, \tag{2}$$

in which case the series is absolutely convergent if and only if

$$\operatorname{Re} s > 0, \qquad s := b_1 + b_2 - a_1 - a_2 - 1. \tag{3}$$

This class of infinite sums are interesting because they contain a lot of special evaluations, some of which are presented in Table 1. Therefore it is important to establish a general framework for the precise and efficient computations of the series (1). Naturally, our approach here

$$q_0(n) := \frac{(3n + b_1 - 1)(3n + b_2 - 1) - (2n)(2n + a_2)}{(2n)(2n + a_1 - 1)} \qquad (n \geq 1),$$

$$q_1(n) := \frac{(3n + b_1)(3n + b_2) - (2n + 1)(2n + a_1)}{(2n + a_1)(2n + a_2)} \qquad (n \geq 0),$$

$$q_2(n) := \frac{(3n + b_1 + 1)(3n + b_2 + 1) - (2n + a_1 + 1)(2n + a_2 + 1)}{(2n + 1)(2n + a_2 + 1)} \qquad (n \geq 0),$$

$$r_0(n) := -\frac{(n + b_1 - a_2 - 1)(n + b_2 - a_2 - 1)}{(2n - 1)(2n + a_2 - 1)} \qquad (n \geq 1),$$

$$r_1(n) := -\frac{(n + b_1 - 1)(n + b_2 - 1)}{(2n)(2n + a_1 - 1)} \qquad (n \geq 1),$$

$$r_2(n) := -\frac{(n + b_1 - a_1)(n + b_2 - a_1)}{(2n + a_1)(2n + a_2)} \qquad (n \geq 0).$$

Table 2: Partial denominators and numerators of the continued fraction (6).

is based on three-term contiguous relations and allied continued fractions. As an illustration of a more general story to be developed in this article, we shall present a continued fraction expansion of the series (1) with an exact error term estimate for its approximants that exhibits an exponentially fast convergence (see Theorem 1.1).

To state Theorem 1.1, let $\{q(n)\}_{n=0}^{\infty}$ and $\{r(n)\}_{n=0}^{\infty}$ be infinite sequences defined by

$$q(n) := q_i((n - i)/3), \quad r(n) := r_i((n - i)/3), \quad \text{for} \quad n \equiv i \mod 3, \quad i = 0, 1, 2, \qquad (4)$$

where $q_i(n)$ and $r_i(n)$ are given by formulas in Table 2 and $q_0(0) := 1$, $r_0(0) := 1$, $r_1(0) = -1$. The modulo 3 structure in (4) is the reflection of a $\mathbb{Z}_3$-symmetry in the relevant contiguous relations (see §2.1). Under condition (2), all the $q(n)$ and $r(n)$ have non-vanishing denominators, while all the $r(n)$ have non-vanishing numerators if and only if the parameters satisfy

$$b_i - a_j \notin \mathbb{Z}_{\leq 0}, \qquad i, j = 1, 2. \qquad (5)$$

Thus the (formal) infinite continued fraction

$$\mathop{\mathbf{K}}_{j=0}^{\infty} \frac{r(j)}{q(j)} := \frac{r(0)}{q(0)} + \frac{r(1)}{q(1)} + \frac{r(2)}{q(2)} + \cdots\cdots \qquad (6)$$

makes sense, provided that the conditions (2) and (5) are satisfied.

Theorem 1.1 *If conditions* (2), (3) *and* (5) *are fulfilled then continued fraction* (6) *converges to series* (1) *exponentially fast and there exists an exact error term estimate for its approximants:*

$${}_3F_2\left(\begin{matrix} 1, & a_1, & a_2 \\ & b_1, & b_2 \end{matrix}\right) - \mathop{\mathbf{K}}_{j=0}^{n} \frac{r(j)}{q(j)} = C\left(\begin{matrix} a_1, & a_2 \\ b_1, & b_2 \end{matrix}\right) \cdot \frac{(3n)^{\frac{1}{2}-s}}{2^{2n+a_1+a_2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\},$$

3

as $n \to +\infty$, where the constant $C(a_1, a_2; b_1, b_2)$ is given by

$$C\begin{pmatrix} a_1, & a_2 \\ b_1, & b_2 \end{pmatrix} := \frac{\pi^{\frac{3}{2}}\, \Gamma(b_1)\, \Gamma(b_2)\, \Gamma^2(s)}{\Gamma(a_1)\, \Gamma(a_2)\, \Gamma(b_1 - a_1)\, \Gamma(b_1 - a_2)\, \Gamma(b_2 - a_1)\, \Gamma(b_2 - a_2)}.$$

Theorem 1.1 is only a corollary to a specific example of infinitely many continued fractions with exact error estimates we shall establish in Theorems 3.2 and 3.3 (see Example 9.1). To generate infinitely many continued fractions, we naturally need infinitely many contiguous relations, so we then need a general theory, beyond the scopes of Bailey [5] and Wilson [27], that presides over all contiguous relations for $_3F_2(1)$. Our previous paper [10] develops such a theory and the present article relies substantially on the main results of that paper.

2 Contiguous and Recurrence Relations

The hypergeometric series of unit argument $_3F_2(1)$ with full five parameters is defined by

$$_3F_2\begin{pmatrix} a_0, & a_1, & a_2 \\ & b_1, & b_2 \end{pmatrix} := \sum_{j=0}^{\infty} \frac{(a_0; j)\, (a_1; j)\, (a_2; j)}{(1; j)\, (b_1; j)\, (b_2; j)}.$$

With the notation $\boldsymbol{a} = (a_0, a_1, a_2; a_3, a_4) = (a_0, a_1, a_2; b_1, b_2)$ this series is often denoted by $_3F_2(\boldsymbol{a})$. It is well defined and non-terminating as a formal sum if $\boldsymbol{a}$ satisfies

$$a_0,\ a_1,\ a_2,\ b_1,\ b_2\ \notin \mathbb{Z}_{\leq 0}, \tag{7}$$

in which case $_3F_2(\boldsymbol{a})$ is absolutely convergent if and only if

$$\operatorname{Re} s(\boldsymbol{a}) > 0, \qquad s(\boldsymbol{a}) := b_1 + b_2 - a_0 - a_1 - a_2, \tag{8}$$

where $s(\boldsymbol{a})$ is called the *parametric excess* for $_3F_2(\boldsymbol{a})$. We say that $\boldsymbol{a}$ is *balanced* if $s(\boldsymbol{a}) = 0$.

In order to discuss contiguous relations, however, we find it more convenient in many respects to replace $_3F_2(\boldsymbol{a})$ by the renormalized hypergeometric series defined by

$$_3f_2(\boldsymbol{a}) := \sum_{j=0}^{\infty} \frac{\Gamma(a_0 + j)\Gamma(a_1 + j)\Gamma(a_2 + j)}{\Gamma(1 + j)\Gamma(b_1 + j)\Gamma(b_2 + j)}.$$

This latter series is well defined and non-terminating as a formal sum, whenever

$$a_0,\ a_1,\ a_2\ \notin \mathbb{Z}_{\leq 0} \qquad \text{(compare this with condition (7))},$$

in which case series $_3f_2(\boldsymbol{a})$ is absolutely convergent if and only if (8) is satisfied. Note that

$$_3f_2(\boldsymbol{a}) = \frac{\Gamma(a_0)\, \Gamma(a_1)\, \Gamma(a_2)}{\Gamma(b_1)\, \Gamma(b_2)}\, _3F_2(\boldsymbol{a}), \tag{9}$$

as long as both sides of equation (9) make sense.

2.1 Contiguous Relations

It follows from [10, Theorem 1.1] that for any distinct integer vectors $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}^5$ different from $\boldsymbol{0}$ there exist *unique* rational functions $u(\boldsymbol{a}), v(\boldsymbol{a}) \in \mathbb{Q}(\boldsymbol{a})$ such that

$$ {}_3f_2(\boldsymbol{a}) = u(\boldsymbol{a}) \cdot {}_3f_2(\boldsymbol{a} + \boldsymbol{k}) + v(\boldsymbol{a}) \cdot {}_3f_2(\boldsymbol{a} + \boldsymbol{l}). \tag{10} $$

An identity of the form (10) is called a *contiguous relation* for ${}_3f_2(1)$. An algorithm to calculate $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ explicitly is given in [10, Recipe 5.4]. According to it, one calculates the connection matrix $A(\boldsymbol{a}; \boldsymbol{k})$ as in [10, formula (30)] and define $r(\boldsymbol{a}; \boldsymbol{k}) \in \mathbb{Q}(\boldsymbol{a})$ to be its $(1, 2)$-entry as in [10, formula (33)]. One also calculates $r(\boldsymbol{a}; \boldsymbol{l})$ as well as $r(\boldsymbol{a}; \boldsymbol{l} - \boldsymbol{k})$ in similar manners. If $\boldsymbol{k}$ and $\boldsymbol{l}$ are distinct then $r(\boldsymbol{a}; \boldsymbol{l} - \boldsymbol{k})$ is nonzero in $\mathbb{Q}(\boldsymbol{a})$ and the coefficients in (10) are represented as

$$ u(\boldsymbol{a}) = \frac{r(\boldsymbol{a}; \boldsymbol{l})}{\det A(\boldsymbol{a}; \boldsymbol{k}) \cdot r(\boldsymbol{a} + \boldsymbol{k}; \boldsymbol{l} - \boldsymbol{k})}, \qquad v(\boldsymbol{a}) = -\frac{r(\boldsymbol{a}; \boldsymbol{k})}{\det A(\boldsymbol{a}; \boldsymbol{k}) \cdot r(\boldsymbol{a} + \boldsymbol{k}; \boldsymbol{l} - \boldsymbol{k})}, \tag{11} $$

as in [10, Proposition 5.3], where according to [10, formula (32)] one has

$$ \det A(\boldsymbol{a}; \boldsymbol{k}) = \frac{(-1)^{k_0+k_1+k_2}(s(\boldsymbol{a}) - 1; s(\boldsymbol{k})) \prod_{i=0}^2 (a_i; k_i)}{\prod_{i=0}^2 \prod_{j=1}^2 (b_j - a_i; l_j - k_i)}. \tag{12} $$

In order to formulate our main results in §3.2, we need one more fact about the structure of $r(\boldsymbol{a}; \boldsymbol{k})$ which is not discussed in [10]. Given a vector $\boldsymbol{k} = (k_0, k_1, k_2; l_1, l_2) \in \mathbb{Z}^5$, let

$$ \langle \boldsymbol{a}; \boldsymbol{k} \rangle_\pm := \prod_{i=0}^2 \prod_{j=1}^2 (b_j - a_i; (l_j - k_i)_\pm), \qquad \|\boldsymbol{k}\|_+ := \sum_{i=0}^2 \sum_{j=1}^2 (l_j - k_i)_+, $$

where $m_\pm := \max\{\pm m, 0\}$. Note that $\prod_{i=0}^2 \prod_{j=1}^2 (b_j - a_i; l_j - k_i) = \langle \boldsymbol{a}; \boldsymbol{k} \rangle_+ / \langle \boldsymbol{a} + \boldsymbol{k}; \boldsymbol{k} \rangle_-$.

Lemma 2.1 *For any nonzero vector $\boldsymbol{k} \in \mathbb{Z}^5_{\geq 0}$ with $s(\boldsymbol{k}) = 0$ there exists a nonzero polynomial $\rho(\boldsymbol{a}; \boldsymbol{k}) \in \mathbb{Q}[\boldsymbol{a}]$ such that the rational function $r(\boldsymbol{a}; \boldsymbol{k})$ can be written*

$$ r(\boldsymbol{a}; \boldsymbol{k}) = -\frac{\{s(\boldsymbol{a}) - 1\}\rho(\boldsymbol{a}; \boldsymbol{k})}{\langle \boldsymbol{a}; \boldsymbol{k} \rangle_+}, \qquad \deg \rho(\boldsymbol{a}; \boldsymbol{k}) \leq \|\boldsymbol{k}\|_+ - 2. \tag{13} $$

Proof. A nonzero polynomial $p(\boldsymbol{a}) \in \mathbb{Q}[\boldsymbol{a}]$ is said to be a *denominator* of a rational function $r(\boldsymbol{a}) \in \mathbb{Q}(\boldsymbol{a})$ if the product $p(\boldsymbol{a}) r(\boldsymbol{a})$ becomes a polynomial. A denominator of the least degree, which is unique up to constant multiples, is referred to as the *reduced denominator*. Any denominator is divisible by the reduced denominator in $\mathbb{Q}[\boldsymbol{a}]$. A denominator of a matrix with entries in $\mathbb{Q}(\boldsymbol{a})$ is, by definition, a common denominator of those entries.

For $i = 0, 1, 2$, $\mu = 1, 2$, let $\boldsymbol{e}_\mu^i := (\delta_{0i}, \delta_{1i}, \delta_{2i}; \delta_{1\mu}, \delta_{2\mu})$, where δ_{**} is Kronecker's delta. A vector of this form is said to be basic. A product of contiguous matrices in [10, Table 2] yields

$$ A(\boldsymbol{a}; \boldsymbol{e}_\mu^i) = \frac{1}{(b_\mu - a_j)(b_\mu - a_k)} \begin{pmatrix} a_i(b_\mu - a_j - a_k) & s(\boldsymbol{a}) - 1 \\ a_i a_j a_k & (a_i + 1)b_\mu + a_j a_k - b_1 b_2 \end{pmatrix}, $$

where $\{i, j, k\} = \{0, 1, 2\}$. Any $\boldsymbol{k} = (k_0, k_1, k_2; l_1, l_2) \in \mathbb{Z}^5_{\geq 0}$ with $s(\boldsymbol{k}) = 0$ admits a decomposition $\boldsymbol{k} = \boldsymbol{v}_l + \cdots + \boldsymbol{v}_1$ with each $\boldsymbol{v}_i$ basic, so $A(\boldsymbol{a}; \boldsymbol{k})$ can be computed by the chain rule

$$ A(\boldsymbol{a}; \boldsymbol{k}) = A(\boldsymbol{a} + \boldsymbol{v}_{l-1} + \cdots + \boldsymbol{v}_1; \boldsymbol{v}_l) \cdots A(\boldsymbol{a} + \boldsymbol{v}_1; \boldsymbol{v}_2) A(\boldsymbol{a}; \boldsymbol{v}_1). \tag{14} $$

5

Thus $A(\boldsymbol{a}; \boldsymbol{k})$ has a denominator each irreducible factor of which is of the form $b_\mu - a_i +$ an integer. A factor of this form is said to be of type $b_\mu - a_i$ and the product of all factors of this type is referred to as the $b_\mu - a_i$ component of the denominator.

Claim. For each $i = 0, 1, 2$ and $\mu = 1, 2$ the matrix $A(\boldsymbol{a}; \boldsymbol{k})$ admits a denominator whose $b_\mu - a_i$ component is exactly the factorial function $(b_\mu - a_i; (l_\mu - k_i)_+)$.

To show the claim we may assume $i = 0$ and $\mu = 1$ without loss of generality.

(1) If $m_0 := k_0 - l_1 \geq 0$, then take the decomposition $\boldsymbol{k} = l_1 \boldsymbol{e}_1^0 + m_0 \boldsymbol{e}_2^0 + k_1 \boldsymbol{e}_2^1 + k_2 \boldsymbol{e}_2^2$.

(2) If $m_1 := l_1 - k_0 > 0$, then take the decomposition $\boldsymbol{k} = k_{12} \boldsymbol{e}_2^1 + k_{22} \boldsymbol{e}_2^2 + k_0 \boldsymbol{e}_1^0 + k_{11} \boldsymbol{e}_1^1 + k_{21} \boldsymbol{e}_1^2$, where k_{ij} are nonnegative integers such that $k_1 = k_{11} + k_{12}$, $k_2 = k_{21} + k_{22}$, $m_1 = k_{11} + k_{21}$ and $l_2 = k_{12} + k_{22}$; such k_{ij} exist thanks to $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^5$ and $s(\boldsymbol{k}) = 0$.

We use the fact that $A(\boldsymbol{a}; m\boldsymbol{e}_\mu^i)$ has a denominator $(b_\mu - a_j; m)(b_\mu - a_k; m)$, where $\{i, j, k\} = \{0, 1, 2\}$, which follows by induction on $m \in \mathbb{Z}_{\geq 0}$. In case (1) the decomposition of $\boldsymbol{k}$ and the chain rule (14) imply that $A(\boldsymbol{a}; \boldsymbol{k})$ has a denominator without $b_1 - a_0$ component. In case (2) the decomposition of $\boldsymbol{k}$ leads to the product $A(\boldsymbol{a}; \boldsymbol{k}) = A_2(\boldsymbol{a}; \boldsymbol{k}) A_1(\boldsymbol{a}; \boldsymbol{k})$ with

$$A_2(\boldsymbol{a}; \boldsymbol{k}) := A(\boldsymbol{a} + k_{11} \boldsymbol{e}_1^1 + k_{21} \boldsymbol{e}_1^2; k_{12} \boldsymbol{e}_2^1 + k_{22} \boldsymbol{e}_2^2 + k_1 \boldsymbol{e}_1^0), \quad A_1(\boldsymbol{a}; \boldsymbol{k}) := A(\boldsymbol{a}; k_{11} \boldsymbol{e}_1^1 + k_{21} \boldsymbol{e}_1^2).$$

Observe that $A_1(\boldsymbol{a}; \boldsymbol{k})$ has a denominator whose $b_1 - a_0$ component is $(b_1 - a_0; k_{11} + k_{21}) = (b_1 - a_0; m_1)$, while $A_2(\boldsymbol{a}; \boldsymbol{k})$ has a denominator without $b_1 - a_0$ component. So $A(\boldsymbol{a}; \boldsymbol{k})$ has a denominator whose $b_1 - a_0$ component is $(b_1 - a_0; m_1)$. The claim is thus verified.

For each entry of $A(\boldsymbol{a}; \boldsymbol{k})$ the Claim implies that for $i = 0, 1, 2$ and $\mu = 1, 2$ the $b_\mu - a_i$ component of its reduced denominator must divide the factorial $(b_\mu - a_i; (l_\mu - k_i)_+)$, so the reduced denominator itself must divide the product $\langle \boldsymbol{a}; \boldsymbol{k} \rangle_+ = \prod_{i=0}^2 \prod_{\mu=1}^2 (b_\mu - a_i; (l_\mu - k_i)_+)$. Thus one can take $\langle \boldsymbol{a}; \boldsymbol{k} \rangle_+$ as a denominator of $A(\boldsymbol{a}; \boldsymbol{k})$. The *index* of a rational function is the degree of its numerator minus that of its denominator. An induction on the length l of product (14) shows that the index $\leq i - j$ for the (i, j)-entry of $A(\boldsymbol{a}; \boldsymbol{k})$. Another induction shows that the $(1, 2)$-entry is divisible by $s(\boldsymbol{a}) - 1$. All these facts lead to expression (13) for $r(\boldsymbol{a}; \boldsymbol{k})$. $\square$

2.2 Symmetry and Dichotomy

Let $G = S_3 \times S_2$ be the group acting on $\boldsymbol{a} = (a_0, a_1, a_2; b_1, b_2)$ by permuting (a_0, a_1, a_2) and (b_1, b_2) separately. It is obvious that $_3 f_2(\boldsymbol{a})$ is invariant under this action, so that any element $\tau \in G$ transforms the contiguous relation (10) into a second one

$$_3 f_2(\boldsymbol{a}) = {}^\tau u(\boldsymbol{a}) \cdot {}_3 f_2(\boldsymbol{a} + \tau(\boldsymbol{k})) + {}^\tau v(\boldsymbol{a}) \cdot {}_3 f_2(\boldsymbol{a} + \tau(\boldsymbol{l})), \tag{15}$$

where $^\tau \varphi(\boldsymbol{a}) := \varphi(\tau^{-1}(\boldsymbol{a}))$ is the induced action of τ on a function $\varphi(\boldsymbol{a})$.

Take an element $\sigma \in G$ such that σ^3 is identity and set

$$\boldsymbol{l} := \boldsymbol{k} + \sigma(\boldsymbol{k}), \qquad \boldsymbol{p} := \boldsymbol{k} + \sigma(\boldsymbol{l}) = \boldsymbol{k} + \sigma(\boldsymbol{k}) + \sigma^2(\boldsymbol{k}). \tag{16}$$

Formula (15) with $\tau = \sigma$ followed by a shift $\boldsymbol{a} \mapsto \boldsymbol{a} + \boldsymbol{k}$ yields

$$_3 f_2(\boldsymbol{a} + \boldsymbol{k}) = {}^\sigma u(\boldsymbol{a} + \boldsymbol{k}) \cdot {}_3 f_2(\boldsymbol{a} + \boldsymbol{l}) + {}^\sigma v(\boldsymbol{a} + \boldsymbol{k}) \cdot {}_3 f_2(\boldsymbol{a} + \boldsymbol{p}), \tag{17}$$

and similarly formula (15) with $\tau = \sigma^2$ followed by another shift $\boldsymbol{a} \mapsto \boldsymbol{a} + \boldsymbol{l}$ gives

$$_3f_2(\boldsymbol{a} + \boldsymbol{l}) = {}^{\sigma^2}\!u(\boldsymbol{a} + \boldsymbol{l}) \cdot {}_3f_2(\boldsymbol{a} + \boldsymbol{p}) + {}^{\sigma^2}\!v(\boldsymbol{a} + \boldsymbol{l}) \cdot {}_3f_2(\boldsymbol{a} + \boldsymbol{p} + \boldsymbol{k}). \tag{18}$$

If $\boldsymbol{k}$ is nonzero, nonnegative $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^5$ and balanced $s(\boldsymbol{k}) = 0$, then so are $\boldsymbol{l} - \boldsymbol{k} = \sigma(\boldsymbol{k})$ and $\boldsymbol{l}$ by definition (16), hence Lemma 2.1 applies not only to $\boldsymbol{k}$ but also to $\sigma(\boldsymbol{k})$ and $\boldsymbol{l}$. Putting formulas (12) and (13) for these vectors into formula (11) we have

$$u(\boldsymbol{a}) = \frac{(-1)^{k_0+k_1+k_2} \cdot \rho(\boldsymbol{a}; \boldsymbol{l}) \cdot \langle \boldsymbol{a}; \boldsymbol{k} \rangle_+ \cdot \langle \boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k}) \rangle_+}{\rho(\boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k})) \cdot \langle \boldsymbol{a}; \boldsymbol{l} \rangle_+ \cdot \langle \boldsymbol{a} + \boldsymbol{k}; \boldsymbol{k} \rangle_- \prod_{i=0}^2 (a_i; k_i)}, \tag{19a}$$

$$v(\boldsymbol{a}) = -\frac{(-1)^{k_0+k_1+k_2} \cdot \rho(\boldsymbol{a}; \boldsymbol{k}) \cdot \langle \boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k}) \rangle_+}{\rho(\boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k})) \cdot \langle \boldsymbol{a} + \boldsymbol{k}; \boldsymbol{k} \rangle_- \prod_{i=0}^2 (a_i; k_i)}, \tag{19b}$$

Definition 2.2 For any nonzero vector $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^5$ with $s(\boldsymbol{k}) = 0$ we consider two cases.

(1) The case is said to be *of straight type* when σ is identity, $\boldsymbol{l} = 2\boldsymbol{k}$ and $\boldsymbol{p} = 3\boldsymbol{k}$.

(2) The case is said to be *of twisted type* when σ is a cyclic permutation of the upper parameters (a_0, a_1, a_2) that acts on the lower parameters (b_1, b_2) trivially,

$$\boldsymbol{k} = \begin{pmatrix} k_0, & k_1, & k_2 \\ & l_1, & l_2 \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} p, & p, & p \\ & 3l_1, & 3l_2 \end{pmatrix}, \tag{20}$$

with $p := k_0 + k_1 + k_2 = l_1 + l_2$, and if $\sigma(a_0, a_1, a_2; b_1, b_2) = (a_\lambda, a_\mu, a_\nu; b_1, b_2)$, then

$$\boldsymbol{l} = \begin{pmatrix} k_0 + k_\lambda, & k_1 + k_\mu, & k_2 + k_\nu \\ & 2l_1, & 2l_2 \end{pmatrix}, \tag{21}$$

where the index triple (λ, μ, ν) is either $(2, 0, 1)$ or $(1, 2, 0)$.

This dichotomy is only due to the restriction of our attention to symmetries σ such that $\sigma^3 = 1$. Taking other symmetries from $S_3 \times S_2$ would lead to other patterns of twists. It is an interesting problem to treat some other cases or to exhaust all cases that are possible.

2.3 Recurrence Relations

In the situation of Definition 2.2, the shifts $\boldsymbol{a} \mapsto \boldsymbol{a} + n\boldsymbol{p}$, $n \in \mathbb{Z}_{\geq 0}$, in the contiguous relation (10) and its companions (17) and (18) induce a system of recurrence relations

$$f_0(n) = q_0(n) \cdot f_1(n) + r_1(n) \cdot f_2(n), \tag{22a}$$
$$f_1(n) = q_1(n) \cdot f_2(n) + r_2(n) \cdot f_0(n+1), \tag{22b}$$
$$f_2(n) = q_2(n) \cdot f_0(n+1) + r_0(n+1) \cdot f_1(n+1), \tag{22c}$$

for $n \in \mathbb{Z}_{\geq 0}$, where the sequences $f_i(n)$, $q_i(n)$ and $r_i(n)$ are defined by

$$\begin{aligned}
f_0(n) &:= {}_3f_2(\boldsymbol{a} + n\boldsymbol{p}), & q_0(n) &:= u(\boldsymbol{a} + n\boldsymbol{p}), & r_1(n) &:= v(\boldsymbol{a} + n\boldsymbol{p}), \\
f_1(n) &:= {}_3f_2(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{k}), & q_1(n) &:= {}^{\sigma}\!u(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{k}), & r_2(n) &:= {}^{\sigma}\!v(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{k}), \\
f_2(n) &:= {}_3f_2(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{l}), & q_2(n) &:= {}^{\sigma^2}\!u(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{l}), & r_0(n) &:= {}^{\sigma^2}\!v(\boldsymbol{a} + (n-1)\boldsymbol{p} + \boldsymbol{l}).
\end{aligned}$$

In view of the modulo 3 structure in (22) it is convenient to set

$$f(n) := f_i((n - i)/3), \tag{23a}$$
$$q(n) := q_i((n - i)/3), \qquad \text{for} \quad n \equiv i \mod 3, \quad i = 0, 1, 2. \tag{23b}$$
$$r(n) := r_i((n - i)/3). \tag{23c}$$

Then the system (22) is unified into a single three-term recurrence relation

$$f(n) = q(n) \cdot f(n + 1) + r(n + 1) \cdot f(n + 2), \qquad n \in \mathbb{Z}_{\geq 0}. \tag{24}$$

If $\boldsymbol{k}$ is nonnegative, $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^5$, then so are $\boldsymbol{l}$ and $\boldsymbol{p}$ by formula (16), hence all $f(n)$, $n \in \mathbb{Z}_{\geq 0}$, are well defined under single assumption (7). If moreover $\boldsymbol{k}$ is *balanced*, $s(\boldsymbol{k}) = 0$, then so are $\boldsymbol{l}$ and $\boldsymbol{p}$ again by formula (16), hence all $f(n)$, $n \in \mathbb{Z}_{\geq 0}$ have the same parametric excess. Thus all these series are convergent under the single assumption (8). In what follows we refer to $\boldsymbol{k}$ as the *seed* vector while $\boldsymbol{p}$ as the *shift* vector. We remark that $\boldsymbol{k}$ is primary in the sense that $\boldsymbol{l}$ and $\boldsymbol{p}$ are derived from $\boldsymbol{k}$ by the rule (16), but $\boldsymbol{p}$ is likewise important because it is $\boldsymbol{p}$ rather than $\boldsymbol{k}$ that is directly responsible for the asymptotic behavior of the sequence $f(n)$.

2.4 Simultaneousness

In place of the series $_3f_2(\boldsymbol{a})$ we consider another series

$$_3g_2(\boldsymbol{a}) = {}_3g_2 \begin{pmatrix} a_0, & a_1, & a_2 \\ & b_1, & b_2 \end{pmatrix} := {}_3f_2 \begin{pmatrix} a_0, & a_0 - b_1 + 1, & a_0 - b_2 + 1 \\ & a_0 - a_1 + 1, & a_0 - a_2 + 1 \end{pmatrix}. \tag{25}$$

Let $\boldsymbol{k}$, $\boldsymbol{l}$ and $\boldsymbol{p}$ be vectors as in (16) such that $s(\boldsymbol{k}) = 0$ and hence $s(\boldsymbol{l}) = s(\boldsymbol{p}) = 0$. By assertion (3) of [10, Theorem 1.1] the contiguous relation (10) for $_3f_2(\boldsymbol{a})$ is *simultaneously* satisfied by $_3h_2(\boldsymbol{a}) := \exp(\pi\sqrt{-1}\,s(\boldsymbol{a}))\,_3g_2(\boldsymbol{a})$, but the factor $\exp(\pi\sqrt{-1}\,s(\boldsymbol{a}))$ is irrelevant by $s(\boldsymbol{k}) = s(\boldsymbol{l}) = 0$, thus (10) is satisfied by $_3g_2(\boldsymbol{a})$ itself. Let $g_i(n)$ and $g(n)$ be defined from $_3g_2(\boldsymbol{a})$ in the same manner as $f_i(n)$ and $f(n)$ are defined from $_3f_2(\boldsymbol{a})$ in §2.3, that is, let

$$g_0(n) := {}_3g_2(\boldsymbol{a} + n\boldsymbol{p}), \quad g_1(n) := {}_3g_2(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{k}), \quad g_2(n) := {}_3g_2(\boldsymbol{a} + n\boldsymbol{p} + \boldsymbol{l}), \tag{26a}$$

$$g(n) := g_i((n - i)/3) \qquad \text{for} \quad n \equiv i \mod 3, \quad i = 0, 1, 2. \tag{26b}$$

Then the sequences $f(n)$ in (23a) and $g(n)$ in (26b) solve the *same* recurrence relation (24). With this observation we are now ready to consider continued fractions.

3 Continued Fractions

First we present a general principle to establish an exact error estimate for the approximants to a continued fraction. Next we announce the final goal of this article, Theorems 3.2 and 3.3, which will be achieved by the principle after a rather long journey of asymptotic analysis.

3.1 A General Error Estimate

Let $\{q(n)\}_{n=0}^{\infty}$ and $\{r(n)\}_{n=1}^{\infty}$ be sequences of complex numbers such that $r(n)$ is nonzero for every $n \in \mathbb{N} := \mathbb{Z}_{\geq 1}$. We consider a sequence of finite continued fractions

$$q(0) + \mathop{\mathbf{K}}_{j=1}^{n} \frac{r(j)}{q(j)} := q(0) + \frac{r(1)}{q(1)} + \cdots\cdots + \frac{r(n)}{q(n)}, \qquad n \in \mathbb{Z}_{\geq 0}. \tag{27}$$

The convergence of (27) can be described in terms of the three-term recurrence relation

$$x(n) = q(n) \cdot x(n+1) + r(n+1) \cdot x(n+2), \qquad n \in \mathbb{Z}_{\geq 0}. \tag{28}$$

A nontrivial solution $X(n)$ to equation (28) is said to be *recessive* if $X(n)/Y(n) \to 0$ as $n \to +\infty$ for any solution $Y(n)$ not proportional to $X(n)$. Recessive solution, if it exists, is unique up to nonzero constant multiples. Any non-recessive solution is said to be *dominant*.

Theorem 3.1 (Pincherle [23]) *Sequence* (27) *is convergent if and only if the recurrence equation* (28) *has a recessive solution* $X(n)$, *in which case* (27) *converges to the ratio* $X(0)/X(1)$.

We refer to Gil, Segura and Temme [15], Jones and Thron [17] and Gautschi [14] for more accessible sources on Pincherle's theorem. Let us make this theorem more quantitative. For any nontrivial solution $x(n)$ to equation (28) and any positive integer $m \in \mathbb{N}$ one has

$$\frac{x(0)}{x(1)} = q(0) + \mathop{\mathbf{K}}_{j=1}^{m-1} \frac{r(j)}{q(j)} + \frac{r(m)}{q(m) + \frac{r(m+1)}{\frac{x(m+1)}{x(m+2)}}}.$$

Thus if $x(n; m)$ is a nontrivial solution to (28) that vanishes at $n = m+2$, then

$$\frac{x(0; m)}{x(1; m)} = q(0) + \mathop{\mathbf{K}}_{j=1}^{m} \frac{r(j)}{q(j)}, \quad \text{or equivalently,} \quad \frac{x(1; m)}{x(0; m)} = \mathop{\mathbf{K}}_{j=0}^{m} \frac{r(j)}{q(j)}, \quad r(0) := 1.$$

One can express the solution $x(n; m)$ in the form

$$x(n; m) = X(n) - R(m) \cdot Y(n), \qquad R(m) := \frac{X(m+2)}{Y(m+2)}, \qquad m, n \in \mathbb{Z}_{\geq 0},$$

where $X(n)$ and $Y(n)$ are recessive and dominant solutions to (28) respectively, so that $R(m) \to 0$ as $m \to +\infty$. Hence if $X(0)$ is nonzero then so is $x(0; m)$ for every $m \gg 0$ and

$$\frac{X(1)}{X(0)} - \frac{x(1; m)}{x(0; m)} = \frac{X(1)}{X(0)} - \frac{X(1) - R(m) \cdot Y(1)}{X(0) - R(m) \cdot Y(0)} = \frac{\omega(0) \cdot R(m)}{X(0)^2 \{1 - R(m) \cdot Y(0)/X(0)\}},$$

where $\omega(n) := X(n) \cdot Y(n+1) - X(n+1) \cdot Y(n)$ is the Casoratian of $X(n)$ and $Y(n)$, thus

$$\frac{X(1)}{X(0)} - \mathop{\mathbf{K}}_{j=0}^{n} \frac{r(j)}{q(j)} = \frac{\omega(0) \cdot R(n)}{X(0)^2} \left\{ 1 + O\left(\frac{R(n) \cdot Y(0)}{X(0)} \right) \right\} \quad \text{as} \quad n \to +\infty. \tag{29}$$

In order to apply this general estimate to continued fractions for ${}_3f_2(1)$, we want to set up the situation in which the sequences $f(n)$ in (23a) and $g(n)$ in (26b) are recessive and dominant solutions, respectively, to the recurrence relation (24). We present in §4 a sufficient condition for $f(n)$ to be recessive, while we impose in §6 a further constraint that insures the dominance of $g(n)$. In fact, upon assuming those conditions, we deduce asymptotic representations for $f(n)$ and $g(n)$ showing that they are actually recessive and dominant respectively. The asymptotic analysis there is used not only to prove such a qualitative assertion but also to get a precise asymptotic behavior for the ratio $R(n) = f(n+2)/g(n+2)$. We have also to evaluate the initial term $\omega(0)$ for the Casoratian of $f(n)$ and $g(n)$; this final task is done in §7.

3.2 Main Results on Continued Fractions

Let $\{q(n)\}_{n=0}^{\infty}$ and $\{r(n)\}_{n=1}^{\infty}$ be sequences (23b) and (23c) derived from $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ as in formula (19). Consider the continued fraction $\mathbf{K}_{j=0}^{\infty} r(j)/q(j)$, where $r(0) := 1$ by convention. It is said to be *well defined* if $q(j)$ and $r(j)$ take finite values with $r(j)$ nonzero for every $j \geq 0$.

Let $\mathcal{S}(\mathbb{R})$ be the set of all real vectors $\boldsymbol{p} = (p_0, p_1, p_2; q_1, q_2) \in \mathbb{R}^5$ such that

$$s(\boldsymbol{p}) = 0; \qquad p_1, p_2 \leq p_0 < q_1 \leq q_2 < p_1 + p_2. \tag{30}$$

Note that (30) in particular implies $p_1, p_2 > 0$ and that $\mathcal{S}(\mathbb{R})$ is a 4-dimensional polyhedral convex cone defined by a linear equation and a set of linear inequalities. It is the space to which the shift vector $\boldsymbol{p}$ in (16) should belong; or rather as an integer vector it should lie on

$$\mathcal{S}(\mathbb{Z}) := \mathcal{S}(\mathbb{R}) \cap \mathbb{Z}^5. \tag{31}$$

The following functions of $\boldsymbol{p} \in \mathcal{S}(\mathbb{R})$ play important roles in several places of this article:

$$D(\boldsymbol{p}) := \frac{(-1)^{q_1+q_2} p_0^{p_0} p_1^{p_1} p_2^{p_2}}{\prod_{i=0}^{2} \prod_{j=1}^{2} (q_j - p_i)^{q_j - p_i}}, \tag{32}$$

$$\Delta(\boldsymbol{p}) := e_1^2 e_2^2 + 18\, e_1 e_2 e_3 - 2\, e_2^3 - 8\, e_1^3 e_3 - 27\, e_3^2, \tag{33}$$

where $e_1 := p_0 + p_1 + p_2 = q_1 + q_2$, $e_2 := p_0 p_1 + p_1 p_2 + p_2 p_0 + q_1 q_2$ and $e_3 := p_0 p_1 p_2$. We remark that $\Delta(\boldsymbol{p})$ is the discriminant (up to a positive constant multiple) of the cubic equation

$$(x - p_0)(x - p_1)(x - p_2) + x(x - q_1)(x - q_2) = 0,$$

which plays an important role in §6.2. Moreover, for $\boldsymbol{k} = (k_0, k_1, k_2; l_1, l_2) \in \mathbb{Z}^5$ we put

$$\gamma(\boldsymbol{a}; \boldsymbol{k}) := \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma^2(s(\boldsymbol{a}))}{\prod_{i=0}^{2} \prod_{j=1}^{2} \Gamma(b_j - a_i + (l_j - k_i)_+)}. \tag{34}$$

We are now able to state the main results of this article; they are stated in terms of the seed vector $\boldsymbol{k}$, but a large part of their proofs will be given in terms of the shift vector $\boldsymbol{p}$. For continued fractions of straight type in Definition 2.2 we have the following theorem.

Theorem 3.2 (Straight Case) *If $\boldsymbol{k} = (k_0, k_1, k_2; l_1, l_2) \in \mathcal{S}(\mathbb{Z})$ satisfies either*

$$\text{(a)} \quad \Delta(\boldsymbol{k}) \leq 0 \qquad or \qquad \text{(b)} \quad 2l_1^2 - 2(k_1 + k_2)l_1 + k_1 k_2 \geq 0, \tag{35}$$

then $|D(\boldsymbol{k})| > 1$ and there exists an error estimate of continued fraction expansion

$$\frac{{}_3f_2(\boldsymbol{a} + \boldsymbol{k})}{{}_3f_2(\boldsymbol{a})} - \mathop{\mathbf{K}}_{j=0}^{n} \frac{r(j)}{q(j)} = \frac{c_{\mathrm{s}}(\boldsymbol{a}; \boldsymbol{k})}{{}_3f_2(\boldsymbol{a})^2} \cdot D(\boldsymbol{k})^{-n} \cdot n^{-s(\boldsymbol{a}) + \frac{1}{2}} \cdot \left\{ 1 + O(n^{-\frac{1}{2}}) \right\}, \tag{36}$$

as $n \to +\infty$, provided that $\operatorname{Re} s(\boldsymbol{a})$ is positive, ${}_3f_2(\boldsymbol{a})$ is nonzero and the continued fraction $\mathbf{K}_{j=0}^{\infty} r(j)/q(j)$ is well defined, where $D(\boldsymbol{k})$ is defined in (32) with $\boldsymbol{p}$ replaced by $\boldsymbol{k}$, while

$$c_{\mathrm{s}}(\boldsymbol{a}; \boldsymbol{k}) := \rho(\boldsymbol{a}; \boldsymbol{k}) \cdot e_{\mathrm{s}}(\boldsymbol{a}; \boldsymbol{k}) \cdot \gamma(\boldsymbol{a}; \boldsymbol{k}),$$

with $\rho(\boldsymbol{a}; \boldsymbol{k}) \in \mathbb{Q}[\boldsymbol{a}]$ being the polynomial in (13), explicitly computable from $\boldsymbol{k}$,

$$e_{\mathrm{s}}(\boldsymbol{a}; \boldsymbol{k}) := (2\pi)^{\frac{3}{2}} \frac{\prod_{i=0}^{2} \prod_{j=1}^{2} (l_j - k_i)^{2(l_j - k_i) + b_j - a_i - \frac{1}{2}}}{s_2(\boldsymbol{k})^{2s(\boldsymbol{a}) - 1} \prod_{i=0}^{2} k_i^{2k_i + a_i - \frac{1}{2}}}, \tag{37}$$

with $s_2(\boldsymbol{k}) := k_0 k_1 + k_1 k_2 + k_2 k_0 - l_1 l_2$ and $\gamma(\boldsymbol{a}; \boldsymbol{k})$ defined by formula (34).

10

A numerical inspection shows that about 43 % of the vectors in $\mathcal{S}(\mathbb{Z})$ satisfy condition (35) (see Remark 6.2). In the straight case with $\boldsymbol{k} \in \mathcal{S}(\mathbb{Z})$ formulas (19) become simpler:

$$u(\boldsymbol{a}) = \frac{(-1)^{l_1+l_2}\,\rho(\boldsymbol{a};2\boldsymbol{k})}{\rho(\boldsymbol{a}+\boldsymbol{k};\boldsymbol{k})\prod_{i=0}^{2}(a_i;\,k_i)}, \qquad v(\boldsymbol{a}) = -\frac{(-1)^{l_1+l_2}\,\rho(\boldsymbol{a};\boldsymbol{k})\cdot\langle\boldsymbol{a}+\boldsymbol{k};\boldsymbol{k}\rangle_+}{\rho(\boldsymbol{a}+\boldsymbol{k};\boldsymbol{k})\prod_{i=0}^{2}(a_i;\,k_i)}. \tag{38}$$

We turn our attention to continued fractions of twisted type in Definition 2.2.

Theorem 3.3 (Twisted Case) *If $\boldsymbol{k} = (k_0,k_1,k_2;l_1,l_2) \in \mathbb{Z}_{\geq 0}^5$ satisfies the condition*

$$k_0 + k_1 + k_2 = l_1 + l_2, \qquad l_1 \leq l_2 \leq \tau\,l_1, \quad \tau := (1+\sqrt{3})/2 = 1.36602540\cdots, \tag{39}$$

then there exists an error estimate of continued fraction expansion

$$\frac{{}_3f_2(\boldsymbol{a}+\boldsymbol{k})}{{}_3f_2(\boldsymbol{a})} - \mathop{\mathbf{K}}_{j=0}^{n}\frac{r(j)}{q(j)} = \frac{c_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k})}{{}_3f_2(\boldsymbol{a})^2}\cdot E(l_1,l_2)^{-n}\cdot n^{-s(\boldsymbol{a})+\frac{1}{2}}\cdot\left\{1 + O(n^{-\frac{1}{2}})\right\}, \tag{40}$$

as $n \to +\infty$, provided that $\operatorname{Re} s(\boldsymbol{a})$ is positive, ${}_3f_2(\boldsymbol{a})$ is nonzero and the continued fraction $\mathbf{K}_{j=0}^{\infty} r(j)/q(j)$ is well defined, where $E(l_1,l_2)$ and $c_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k})$ are given by

$$E(l_1,l_2) := \frac{(-l_1-l_2)^{l_1+l_2}}{(2l_1-l_2)^{2l_1-l_2}(2l_2-l_1)^{2l_2-l_1}}, \qquad |E(l_1,l_2)| > 1, \tag{41}$$

$$c_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k}) := \rho(\boldsymbol{a};\boldsymbol{k})\cdot e_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k})\cdot\gamma(\boldsymbol{a};\boldsymbol{k}),$$

with $\rho(\boldsymbol{a};\boldsymbol{k}) \in \mathbb{Q}[\boldsymbol{a}]$ being the polynomial in (13), explicitly computable from $\boldsymbol{k}$,

$$e_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k}) := (2\pi)^{\frac{3}{2}}\frac{(2l_1-l_2)^{2(2l_1-l_2)+2b_1-b_2+s(\boldsymbol{a})-\frac{3}{2}}\cdot(2l_2-l_1)^{2(2l_2-l_1)+2b_2-b_1+s(\boldsymbol{a})-\frac{3}{2}}}{3^{s(\boldsymbol{a})-\frac{1}{2}}\cdot(l_1+l_2)^{2(l_1+l_2)+a_0+a_1+a_2-\frac{3}{2}}\cdot(l_1^2-l_1l_2+l_2^2)^{2s(\boldsymbol{a})-1}}, \tag{42}$$

and $\gamma(\boldsymbol{a};\boldsymbol{k})$ being defined by formula (34).

The proofs of Theorems 3.2 and 3.3 will be completed at the end of §7.

4 Continuous Laplace Method

We shall find a class of directions $\boldsymbol{p} = (p_0,p_1,p_2;q_1,q_2) \in \mathbb{R}^5$ in which the sequence

$$f(n) = {}_3f_2(\boldsymbol{a}+n\boldsymbol{p}) = {}_3f_2\begin{pmatrix} a_0+p_0 n, & a_1+p_1 n, & a_2+p_2 n \\ & b_1+q_1 n, & b_2+q_2 n \end{pmatrix}, \qquad n \in \mathbb{Z}_{\geq 0}, \tag{43}$$

behaves like n^α as $n \to +\infty$ for some $\alpha \in \mathbb{R}$, where we assume $s(\boldsymbol{p}) = 0$ so that the parametric excesses for $f(n)$ are independent of n, always equal to $s(\boldsymbol{a})$. We remark that the current $f(n)$ corresponds to the sequence $f_0(n)$ in §2.3, not to $f(n)$ in formula (23a).

In terms of the series ${}_3f_2(\boldsymbol{a})$, Thomae's transformation [1, Corollary 3.3.6] reads

$${}_3f_2\begin{pmatrix} a_0, & a_1, & a_2 \\ & b_1, & b_2 \end{pmatrix} = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_1-a_0)\Gamma(b_2-a_0)}\,{}_3f_2\begin{pmatrix} s(\boldsymbol{a}), & b_1-a_0, & b_2-a_0 \\ & s(\boldsymbol{a})+a_1, & s(\boldsymbol{a})+a_2 \end{pmatrix}. \tag{44}$$

To investigate the asymptotic behavior of $f(n)$, take Thomae's transformation of (43) to have

$$f(n) = \psi_1(n) \cdot f_1(n), \tag{45a}$$

$$\psi_1(n) := \frac{\Gamma(a_1 + p_1 n)\Gamma(a_2 + p_2 n)}{\Gamma(b_1 - a_0 + (q_1 - p_0)n)\Gamma(b_2 - a_0 + (q_2 - p_0)n)}, \tag{45b}$$

$$f_1(n) = {}_3f_2\left(\begin{array}{ccc} s(\boldsymbol{a}), & b_1 - a_0 + (q_1 - p_0)n, & b_2 - a_0 + (q_2 - p_0)n \\ & s(\boldsymbol{a}) + a_1 + p_1 n, & s(\boldsymbol{a}) + a_2 + p_2 n \end{array}\right), \tag{45c}$$

and then apply ordinary Laplace's method to the Euler integral representation for (45c). Since this analysis is not limited to ${}_3f_2(1)$, we shall deal with more general ${}_{p+1}f_p(1)$ series.

4.1   Euler Integral Representations

The renormalized generalized hypergeometric series ${}_{p+1}f_p(z)$ is defined by

$$_{p+1}f_p\left(\begin{array}{cccc} a_0, & a_1, & \ldots, & a_p \\ & b_1, & \ldots, & b_p \end{array}; z\right) := \sum_{k=0}^{\infty} \frac{\Gamma(a_0 + k)\Gamma(a_1 + k)\cdots\Gamma(a_p + k)}{\Gamma(1 + k)\Gamma(b_1 + k)\cdots\Gamma(b_p + k)} z^k, \tag{46}$$

where $\boldsymbol{a} = (a_0, \ldots, a_p; b_1, \ldots, b_p) \in \mathbb{C}^{p+1} \times \mathbb{C}^p$ are parameters such that none of $a_0, \ldots, a_p$ is a negative integer or zero. Then (46) is absolutely convergent on the open unit disk $|z| < 1$.

It is well known that if the parameters $\boldsymbol{a}$ satisfy the condition

$$\operatorname{Re} b_i > \operatorname{Re} a_i > 0 \qquad (i = 1, \ldots, p), \tag{47}$$

then the improper integral of Euler type

$$E_p(\boldsymbol{a}; z) := \int_{I^p} \phi_p(\boldsymbol{t}; \boldsymbol{a}; z)\, d\boldsymbol{t}, \qquad \phi_p(\boldsymbol{t}; \boldsymbol{a}; z) := \frac{\prod_{i=1}^{p} t_i^{a_i - 1}(1 - t_i)^{b_i - a_i - 1}}{(1 - z\, t_1 \cdots t_p)^{a_0}}$$

is absolutely convergent and the series (46) admits an integral representation

$$_{p+1}f_p(\boldsymbol{a}; z) = \frac{\Gamma(a_0) \cdot E_p(\boldsymbol{a}; z)}{\prod_{i=1}^{p} \Gamma(b_i - a_i)} \qquad \text{on the open unit disk} \quad |z| < 1. \tag{48}$$

where $I = (0, 1)$ is the open unit interval, $\boldsymbol{t} = (t_1, \ldots, t_p) \in I^p$ and $d\boldsymbol{t} = dt_1 \cdots dt_p$.

We are more interested in ${}_{p+1}f_p(1)$, that is, in the series (46) at unit argument $z = 1$:

$$_{p+1}f_p(\boldsymbol{a}) = {}_{p+1}f_p(\boldsymbol{a}; 1) := \sum_{k=0}^{\infty} \frac{\Gamma(a_0 + k)\Gamma(a_1 + k)\cdots\Gamma(a_p + k)}{\Gamma(1 + k)\Gamma(b_1 + k)\cdots\Gamma(b_p + k)}. \tag{49}$$

It is well known that series (49) is absolutely convergent if and only if

$$\operatorname{Re} s(\boldsymbol{a}) > 0, \qquad s(\boldsymbol{a}) := b_1 + \cdots + b_p - a_0 - a_1 - \cdots - a_p, \tag{50}$$

in which case we have ${}_{p+1}f_p(\boldsymbol{a}; z) \to {}_{p+1}f_p(\boldsymbol{a})$ as $z \to 1$ within the open unit disk $|z| < 1$.

Lemma 4.1 *If conditions* (47) *and* (50) *are satisfied, then the integral*

$$E_p(\boldsymbol{a}) := \int_{I^p} \phi_p(\boldsymbol{t}; \boldsymbol{a}) \, d\boldsymbol{t}, \qquad \phi_p(\boldsymbol{t}; \boldsymbol{a}) := \frac{\prod_{i=1}^p t_i^{a_i-1}(1-t_i)^{b_i-a_i-1}}{(1-t_1 \cdots t_p)^{a_0}} \tag{51}$$

is absolutely convergent and the series (49) *admits an integral representation*

$$_{p+1}f_p(\boldsymbol{a}) = \frac{\Gamma(a_0) \cdot E_p(\boldsymbol{a})}{\prod_{i=1}^p \Gamma(b_i - a_i)}. \tag{52}$$

Proof. If r denotes the distance of $\boldsymbol{t}$ from $\boldsymbol{1} := (1, \ldots, 1)$ then one has

$$\phi_p(\boldsymbol{t}; \boldsymbol{a}) = O(r^{s(\boldsymbol{a})-p}) \qquad \text{as} \quad I^p \ni \boldsymbol{t} \to \boldsymbol{1}, \tag{53}$$

The absolute convergence of integral (51) off a neighborhood U of $\boldsymbol{1}$ is due to condition (47), while that on U follows from condition (50) and estimate (53). In view of

$$\lim_{I \ni z \to 1} \phi_p(\boldsymbol{t}; \boldsymbol{a}; z) = \phi_p(\boldsymbol{t}; \boldsymbol{a}), \qquad |\phi_p(\boldsymbol{t}; \boldsymbol{a}; z)| \leq \begin{cases} \phi_p(\boldsymbol{t}; \operatorname{Re} \boldsymbol{a}; 0) & (\operatorname{Re} a_0 \leq 0, \, z \in I), \\ \phi_p(\boldsymbol{t}; \operatorname{Re} \boldsymbol{a}) & (\operatorname{Re} a_0 > 0, \, z \in I), \end{cases}$$

formula (52) is derived from formula (48) by Lebesgue's convergence theorem. $\qquad\square$

The series (49) is symmetric in $a_0, a_1, \ldots, a_p$, but the integral representation (52) is symmetric only in $a_1, \ldots, a_p$. This fact is efficiently used in the next subsection.

4.2 Asymptotic Analysis of Euler Integrals

Observing that the 0-th numerator parameter of the sequence $f_1(n)$ in (45c) is independent of n, we consider a sequence of the form

$$f_1(n) := {}_{p+1}f_p \begin{pmatrix} a_0, & a_1 + k_1 n, & \ldots, & a_p + k_p n \\ & b_1 + l_1 n, & \ldots, & b_p + l_p n \end{pmatrix}, \qquad n \in \mathbb{Z}_{\geq 0}.$$

The associated Euler integrals have an almost product structure which allows a particularly simple treatment in applying Laplace's approximation method.

Proposition 4.2 *If* $\boldsymbol{k} = (0, k_1, \ldots, k_p; l_1, \ldots, l_p) \in \mathbb{R}^{2p+1}$ *is a real vector such that*

$$l_i > k_i > 0 = k_0, \qquad i = 1, \ldots, p, \tag{54}$$

then $E_p(\boldsymbol{a} + n \boldsymbol{k})$ *admits an asymptotic representation as* $n \to +\infty$,

$$E_p \begin{pmatrix} a_0, & a_1 + k_1 n, & \ldots, & a_p + k_p n \\ & b_1 + l_1 n, & \ldots, & b_p + l_p n \end{pmatrix} = C \cdot \Phi_{\max}^n \cdot n^{-\frac{p}{2}} \cdot \{1 + O(1/n)\}, \tag{55}$$

uniform for $\boldsymbol{a} = (a_0, \ldots, a_p; b_1, \ldots, b_p)$ *in any compact subset of* $(\mathbb{C} \setminus \mathbb{Z}_{\leq 0}) \times \mathbb{C}^p \times \mathbb{C}^p$, *where*

$$\Phi_{\max} := \prod_{i=1}^p \frac{k_i^{k_i}(l_i - k_i)^{l_i - k_i}}{l_i^{l_i}}, \tag{56a}$$

$$C := (2\pi)^{\frac{p}{2}} \left(1 - \frac{k_1 \cdots k_p}{l_1 \cdots l_p}\right)^{-a_0} \prod_{i=1}^p \frac{k_i^{a_i - \frac{1}{2}}(l_i - k_i)^{b_i - a_i - \frac{1}{2}}}{l_i^{b_i - \frac{1}{2}}}. \tag{56b}$$

13

Proof. The proof is an application of the standard Laplace method to the integral (52), so only an outline of it is presented below. Replacing $\boldsymbol{a}$ with $\boldsymbol{a} + n\,\boldsymbol{k}$ in definition (51), we have

$$E_p(\boldsymbol{a} + n\,\boldsymbol{k}) = \int_{I^p} \Phi(\boldsymbol{t})^n \cdot u(\boldsymbol{t})\,d\boldsymbol{t} = \int_{I^p} e^{-n\,\phi(\boldsymbol{t})} \cdot u(\boldsymbol{t})\,d\boldsymbol{t},$$

where $\Phi(\boldsymbol{t})$, $\phi(\boldsymbol{t})$ and $u(\boldsymbol{t})$ are defined by

$$\Phi(\boldsymbol{t}) := \prod_{i=1}^{p} t_i^{k_i}(1 - t_i)^{l_i - k_i}, \qquad \phi(\boldsymbol{t}) := -\log \Phi(\boldsymbol{t}), \qquad u(\boldsymbol{t}) := \phi_p(\boldsymbol{t}; \boldsymbol{a}).$$

Observe that $\phi(\boldsymbol{t})$ attains a unique minimum at $\boldsymbol{t}_0 := (k_1/l_1, \ldots, k_p/l_p)$ in the interval I^p, since

$$\frac{\partial \phi}{\partial t_i} = -\frac{k_i}{t_i} + \frac{l_i - k_i}{1 - t_i} = \frac{l_i t_i - k_i}{t_i(1 - t_i)}, \qquad \frac{\partial^2 \phi}{\partial t_i^2} = \frac{k_i}{t_i^2} + \frac{l_i - k_i}{(1 - t_i)^2} > 0, \qquad \frac{\partial^2 \phi}{\partial t_i \partial t_j} = 0 \quad (i \neq j).$$

The standard formula for Laplace's approximation then leads to

$$\int_{I^p} e^{-n\,\phi(\boldsymbol{t})} \cdot u(\boldsymbol{t})\,d\boldsymbol{t} = \frac{u(\boldsymbol{t}_0)}{\sqrt{\mathrm{Hess}(\phi; \boldsymbol{t}_0)}} \left(\frac{2\pi}{n}\right)^{\frac{p}{2}} \exp(-n\,\phi(\boldsymbol{t}_0))\,\{1 + O(1/n)\}$$

$$= C \cdot \Phi_{\max}^n \cdot n^{-\frac{p}{2}}\,\{1 + O(1/n)\} \qquad \text{as} \quad n \to \infty,$$

where $\mathrm{Hess}(\phi; \boldsymbol{t}_0)$ is the Hessian of ϕ at $\boldsymbol{t}_0$ while $\Phi_{\max}$ and C are given by formulas (56). $\qquad \square$

4.3 Recessive Sequences

We return to the special case of $_3f_2(1)$ series and prove the following.

Theorem 4.3 *If* $\boldsymbol{p} = (p_0, p_1, p_2; q_1, q_2) \in \mathbb{R}^5$ *is balanced,* $s(\boldsymbol{p}) = 0$, *and*

$$p_1 > q_1 - p_0 > 0, \qquad p_2 > q_2 - p_0 > 0, \tag{57}$$

then the sequence $f(n) = {}_3f_2(\boldsymbol{a} + n\boldsymbol{p})$ *in (43) admits an asymptotic representation*

$${}_3f_2(\boldsymbol{a} + n\boldsymbol{p}) = \Gamma(s(\boldsymbol{a})) \cdot s_2(\boldsymbol{p})^{-s(\boldsymbol{a})} \cdot n^{-2s(\boldsymbol{a})} \cdot \{1 + O(1/n)\}, \quad \text{as} \quad n \to +\infty, \tag{58}$$

uniform in any compact subset of $\mathrm{Re}\, s(\boldsymbol{a}) > 0$, *where* $s_2(\boldsymbol{p}) := p_0 p_1 + p_1 p_2 + p_2 p_0 - q_1 q_2$.

Proof. By formulas (45) and (52) the sequence (43) can be written $f(n) = \psi_2(n)\,e_2(n)$ with

$$\psi_2(n) := \frac{\Gamma(s(\boldsymbol{a}))\,\Gamma(a_1 + p_1 n)\,\Gamma(a_2 + p_2 n)}{\prod_{j=1}^{2} \prod_{i=0,j} \Gamma(b_j - a_i + (q_j - p_i)n)}, \tag{59a}$$

$$e_2(n) := E_2 \begin{pmatrix} s(\boldsymbol{a}), & b_1 - a_0 + (q_1 - p_0)n, & b_2 - a_0 + (q_2 - p_0)n \\ & s(\boldsymbol{a}) + a_1 + p_1 n, & s(\boldsymbol{a}) + a_2 + p_2 n \end{pmatrix}. \tag{59b}$$

Conditions $s(\boldsymbol{p}) = 0$ and (57) imply that $p_1,\, p_2 > 0$ and $q_j - p_i > 0$ for every $j = 1,\, 2$ and $i = 0,\, j$, so Stirling's formula applied to (59a) yields an asymptotic representation

$$\psi_2(n) = B \cdot A^n \cdot n^{1 - 2s(\boldsymbol{a})}\,\{1 + O(1/n)\}, \tag{60}$$

14

as $n \to +\infty$, where A and B are given by

$$A := \frac{p_1^{p_1} \, p_2^{p_2}}{\prod_{j=1}^2 \prod_{i=0,j} (q_j - p_i)^{q_j - p_i}}, \qquad B := \frac{\Gamma(s(\boldsymbol{a})) \cdot p_1^{a_1 - \frac{1}{2}} \, p_2^{a_2 - \frac{1}{2}}}{2\pi \, \prod_{j=1}^2 \prod_{i=0,j} (q_j - p_i)^{b_j - a_i - \frac{1}{2}}}.$$

When $p = 2$, $k_1 = q_1 - p_0$, $k_2 = q_2 - p_0$, $l_1 = p_1$, $l_2 = p_2$, condition (54) becomes (57), so Proposition 4.2 applies to the sequence (59b). In this situation we have $\Phi_{\max} = A^{-1}$ in (56a) and $C = B^{-1} \cdot \Gamma(s(\boldsymbol{a})) \cdot \{p_1 p_2 - (q_1 - p_0)(q_2 - p_0)\}^{-s(\boldsymbol{a})}$ in (56b), where we have $p_1 p_2 - (q_1 - p_0)(q_2 - p_0) = s_2(\boldsymbol{p})$ from $s(\boldsymbol{p}) = 0$. Thus formula (55) reads

$$e_2(n) = B^{-1} \cdot \Gamma(s(\boldsymbol{a})) \cdot s_2(\boldsymbol{p})^{-s(\boldsymbol{a})} \cdot A^{-n} \cdot n^{-1} \{1 + O(1/n)\} \quad \text{as} \quad n \to +\infty. \tag{61}$$

Combining formulas (60) and (61) we have the asymptotic representation (58). $\qquad \square$

Thomae's transformation (44) rewrites $_3f_2(\boldsymbol{a})$ so that the parametric excess $s(\boldsymbol{a})$ appears as an upper parameter and the invariance $s(\boldsymbol{a}) = s(\boldsymbol{a} + n\boldsymbol{p})$, $n \in \mathbb{Z}_{\geq 0}$, for balanced $\boldsymbol{p}$ facilitates the analysis leading to Theorem 4.3. Note that (44) is only one of an order 120 group of transformations for $_3F_2(1)$ (see [19, Theorem 3] for an impressive account). We wonder if other transformations of the group could be applied to cover some non-balanced cases.

Remark 4.4 We take this opportunity to review some existing results on the large-parameter asymptotics of $_2F_1$ and $_3F_2$. For the former we refer to a classical book of Luke [20, Chap. 7] and more recent articles of Temme [25], Paris [22], Farid Khwaja and Olde Daalhuis [18], Aoki and Tanda [2] and Iwasaki [16], where much work has used the traditional (continuous) version of Laplace's method, while [2] employs exact WKB analysis. For the latter, there are very few to cite; some results are mentioned in [20, §7.4], but most work has focused on the asymptotics of terminating series such as the behavior as $n \to \infty$ of the 'extended Jacobi' polynomials $_3F_2(-n, n + \lambda, a_3; b_1, b_2; z)$, to which one can apply very different techniques such as ones based on generating series; see e.g. Fields [11]. Temme [25] comments on the difficulty of obtaining large-parameter asymptotics of $_3F_2$ functions, even in the terminating cases. As an attempt to overcome this difficulty we shall introduce a discrete version of Laplace's method.

5 Discrete Laplace Method

When a solution to a recurrence equation is given in terms of hypergeometric series, we want to know its asymptotic behavior and thereby to check whether it is actually a dominant solution. To this end, regarding the series as a "discrete" integral, we develop a *discrete Laplace method* as an analogue to the usual (continuous) Laplace method for ordinary integrals. While Theorems 3.2 and 3.3 on continued fractions are the final goal of this article, the main result of this section, Theorem 5.2, and the method leading to it are the methodological core of the article.

5.1 Formulation

Let $\boldsymbol{\sigma} = (\sigma_i) \in \mathbb{R}^I$, $\boldsymbol{\lambda} = (\lambda_i) \in \mathbb{R}^I$, $\boldsymbol{\tau} = (\tau_j) \in \mathbb{R}^J$, $\boldsymbol{\mu} = (\mu_j) \in \mathbb{R}^J$ be *real* numbers indexed by finite sets I and J. Suppose that the pairs $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ and $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are *balanced* to the effect that

$$\sum_{i \in I} \sigma_i = \sum_{j \in J} \tau_j, \qquad \sum_{i \in I} \lambda_i = \sum_{j \in J} \mu_j. \tag{62}$$

Let $\boldsymbol{\alpha}(n) = (\alpha_i(n)) \in \mathbb{C}^I$ and $\boldsymbol{\beta}(n) = (\beta_j(n)) \in \mathbb{C}^J$ be sequences in $n \in \mathbb{N}$ of complex numbers indexed by $i \in I$ and $j \in J$. Suppose that they are *bounded*, that is, for some constant $R > 0$,

$$|\alpha_i(n)| \le R \quad (i \in I); \qquad |\beta_j(n)| \le R \quad (j \in J), \qquad {}^{\forall}n \in \mathbb{N}. \tag{63}$$

In practical applications $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ will typically be independent of n, however allowing such a moderate dependence upon n as in (63) is quite helpful in developing the theory.

Given $0 \le r_0 < r_1 \le +\infty$, we consider the sum of gamma products

$$g(n) := \sum_{k=\lceil r_0 n \rceil}^{\lceil r_1 n \rceil - 1} G(k; n), \quad G(k; n) := \frac{\prod_{i \in I} \Gamma(\sigma_i\, k + \lambda_i\, n + \alpha_i(n))}{\prod_{j \in J} \Gamma(\tau_j\, k + \mu_j\, n + \beta_j(n))}, \quad n \in \mathbb{N}, \tag{64}$$

where $\lceil x \rceil := \min\{m \in \mathbb{Z} : x \le m\}$ denotes the ceiling function. We remark that the *reflection* of discrete variable $k \mapsto \lceil r_0 n \rceil + \lceil r_1 n \rceil - 1 - k$ in (64) induces an involution

$$\sigma_i' = -\sigma_i, \qquad \lambda_i' = \lambda_i + \sigma_i(r_0 + r_1), \qquad \alpha_i'(n) = \alpha_i(n) - \sigma_i\, r(n), \tag{65a}$$

$$\tau_j' = -\tau_j, \qquad \mu_j' = \mu_j + \tau_j(r_0 + r_1), \qquad \beta_j'(n) = \beta_j(n) - \tau_j\, r(n), \tag{65b}$$

where $r(n) := (r_0 + r_1)n + 1 - \lceil r_0 n \rceil - \lceil r_1 n \rceil$ and the resulting data are indicated with a prime, while the reflection leaves r_0 and r_1 unchanged. Since $-1 < r(n) \le 1$, if $\alpha_i(n)$ and $\beta_j(n)$ are bounded then so are $\alpha_i'(n)$ and $\beta_j'(n)$. This reflectional symmetry is helpful in some occasions. Moreover, for any integer $s \le r_0$ the *shift* $k \mapsto k + sn$ in (64) results in the translations

$$r_0 \mapsto r_0 - s, \quad r_1 \mapsto r_1 - s; \quad \sigma_i \mapsto \sigma_i, \quad \lambda_i \mapsto \lambda_i + \sigma_i s; \quad \tau_j \mapsto \tau_j, \quad \mu_j \mapsto \mu_j + \tau_j s. \tag{66}$$

Taking $s = \lfloor r_0 \rfloor$ we may assume $0 \le r_0 < 1$, where $\lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \le x\}$ is the floor function. This normalization is also sometimes convenient.

It is insightful to rewrite the gamma product $G(k; n)$ as

$$G(k; n) = H\left(k/n; n\right), \qquad H(x; n) := \frac{\prod_{i \in I} \Gamma(l_i(x)\, n + \alpha_i(n))}{\prod_{j \in J} \Gamma(m_j(x)\, n + \beta_j(n))}, \tag{67}$$

where $l_i(x)$ and $m_j(x)$ are affine functions defined by

$$l_i(x) := \sigma_i x + \lambda_i \quad (i \in I), \qquad m_j(x) := \tau_j x + \mu_j \qquad (j \in J).$$

We remark that condition (62) is equivalent to the balancedness of affine functions

$$\sum_{i \in I} l_i(x) = \sum_{j \in J} m_j(x), \qquad {}^{\forall}x \in \mathbb{R}. \tag{68}$$

The sum $g(n)$ is said to be *admissible* if

$$\sigma_i \ne 0; \qquad l_i(r_0) \ge 0, \qquad l_i(r_1) \ge 0 \qquad (i \in I), \tag{69a}$$

$$\tau_j \ne 0; \qquad m_j(r_0) \ge 0, \qquad m_j(r_1) \ge 0 \qquad (j \in J), \tag{69b}$$

where if $r_1 = +\infty$ then by $l_i(r_1) \ge 0$ and $m_j(r_1) \ge 0$ we mean $\sigma_i > 0$ and $\tau_j > 0$. Condition (69) says that $l_i(x)$ and $m_j(x)$ are non-constant affine functions taking nonnegative values at both ends of the interval $[r_0, r_1]$, so they must be positive in its interior, that is,

$$l_i(x) > 0 \quad (i \in I); \quad m_j(x) > 0 \quad (j \in J), \qquad r_0 < {}^{\forall}x < r_1. \tag{70}$$

To work near the endpoints of the interval we introduce four index subsets

$$I_0 := \{i \in I \,:\, l_i(r_0) = 0\}, \qquad I_1 := \{i \in I \,:\, l_i(r_1) = 0\}, \tag{71a}$$
$$J_0 := \{j \in J \,:\, m_j(r_0) = 0\}, \qquad J_1 := \{j \in J \,:\, m_j(r_1) = 0\}. \tag{71b}$$

Then there exists a positive constant $c > 0$ such that

$$l_i(x) \geq c \quad (i \in I \setminus (I_0 \cup I_1)); \quad m_j(x) \geq c \quad (j \in J \setminus (J_0 \cup J_1)), \quad r_0 \leq {}^\forall x \leq r_1. \tag{72}$$

This "uniformly away from zero" property will be important in applying a version of Stirling's formula which is given later in (90), especially when $I_0 \cup I_1 \cup J_0 \cup J_1 = \emptyset$ (regular case).

Lemma 5.1 *We have $\sigma_i > 0$ for $i \in I_0$ while $\sigma_i < 0$ for $i \in I_1$, in particular $I_0 \cap I_1 = \emptyset$. Similarly we have $\tau_j > 0$ for $j \in J_0$ while $\tau_j < 0$ for $j \in J_1$, in particular $J_0 \cap J_1 = \emptyset$. If*

$$\alpha_i^{(\nu)}(n) := \alpha_i(n) + \sigma_i(\lceil r_\nu n \rceil - r_\nu n) \notin \mathbb{Z}_{\leq 0} + |\sigma_i|\,\mathbb{Z}_{\leq -\nu}, \quad i \in I_\nu, \ \nu = 0, 1, \ n \in \mathbb{N}, \tag{73}$$

then the sum $g(n)$ is well defined, that is, every summand $G(k; n) = H(k/n; n)$ in (64) takes a finite value for any $n \geq (R + 1)/c$ with R and c given in (63) and (72).

Proof. By condition (69a), if $i \in I_0$ then $0 \leq l_i(r_1) = l_i(r_1) - l_i(r_0) = (r_1 - r_0)\sigma_i$ with $r_1 - r_0 > 0$ and $\sigma_i \neq 0$, which forces $\sigma_i > 0$, while if $i \in I_1$ then $0 \leq l_i(r_0) = l_i(r_0) - l_i(r_1) = (r_0 - r_1)\sigma_i$ with $r_0 - r_1 < 0$ and $\sigma_i \neq 0$, which forces $\sigma_i < 0$. A similar argument using (69b) leads to the assertions for J_0 and J_1. The sum $g(n)$ fails to make sense only when the argument of an upper gamma factor of a summand $G(k; n)$ takes a negative integer value or zero, that is,

$$\sigma_i k + \lambda_i n + \alpha_i(n) = l_i(k/n)\,n + \alpha_i(n) \in \mathbb{Z}_{\leq 0}, \qquad {}^\exists i \in I, \quad \lceil r_0 n \rceil \leq {}^\exists k \leq \lceil r_1 n \rceil - 1.$$

This cannot occur for $i \in I \setminus (I_0 \cup I_1)$ and $n \geq (R + 1)/c$, since (63) and (72) imply that $l_i(k/n)\,n + \mathrm{Re}\,\alpha_i(n) \geq cn - R \geq 1$ for any $k \in \mathbb{Z}$ such that $r_0 \leq k/n \leq r_1$. Observe that

$$\sigma_i k + \lambda_i n + \alpha_i(n) = \sigma_i\,l + l_i(r_0)n + \alpha_i^{(0)}(n) = \sigma_i\,l + \alpha_i^{(0)}(n), \qquad i \in I_0,$$

where $l := k - \lceil r_0 n \rceil$ ranges over $0, 1, \ldots, \lceil r_1 n \rceil - \lceil r_0 n \rceil - 1$. This cannot be a negative integer or zero, if condition (73) is satisfied for $\nu = 0$. A similar argument can be made for $\nu = 1$, since condition (73) for $\nu = 1$ is obtained from that for $\nu = 0$ by applying reflectional symmetry (65). Thus if (73) is satisfied then $g(n)$ is well defined for $n \geq (R + 1)/c$. $\qquad\square$

To carry out analysis it is convenient to quantify condition (73) by writing

$$\delta_\nu(n) := \min\left\{1, \prod_{i \in I_\nu} \mathrm{dist}(\alpha_i^{(\nu)}(n), \mathbb{Z}_{\leq 0} + |\sigma_i|\mathbb{Z}_{\leq -\nu})\right\} > 0, \quad \nu = 0, 1, \ n \in \mathbb{N}, \tag{74}$$

where $\mathrm{dist}(z, Z)$ stands for the distance between a point z and a set Z in $\mathbb{C}$, and cut off by 1 is simply to make $\delta_\nu(n) \leq 1$ as it really works only when $0 < \delta_\nu(n) \ll 1$. Condition (74) or (73) is referred to as the *genericness* for the data $\boldsymbol{\alpha}(n)$.

5.2 Main Results on Discrete Laplace Method

To state the main result of this section we introduce the following quantities:

$$\Phi(x) := \prod_{i \in I} l_i(x)^{l_i(x)} \prod_{j \in J} m_j(x)^{-m_j(x)}, \tag{75a}$$

$$u(x; n) := (2\pi)^{\frac{|I|-|J|}{2}} \prod_{i \in I} l_i(x)^{\alpha_i(n)-\frac{1}{2}} \prod_{j \in J} m_j(x)^{\frac{1}{2}-\beta_j(n)}, \tag{75b}$$

where $|I|$ and $|J|$ are the cardinalities of I and J. We refer to $\Phi(x)$ as the *multiplicative phase function* for the sum $g(n)$ in (64).

Thanks to positivity (70) the function $\Phi(x)$ is smooth and positive on (r_0, r_1). If we employ the convention $0^0 = 1$, which is natural in view of the limit $x^x = 1$ as $x \to +0$, then $\Phi(x)$ is continuous and positive at $x = r_0$ as well as at $x = r_1$ when $r_1 < +\infty$, even if some of the $l_i(x)$'s or $m_j(x)$'s vanish at one or both endpoints. When $r_1 = +\infty$, some calculations using balancedness condition (62) shows that

$$\Phi(x) = \left(\boldsymbol{\sigma^\lambda}/\boldsymbol{\tau^\mu}\right) \cdot \left(\boldsymbol{\sigma^\sigma}/\boldsymbol{\tau^\tau}\right)^x \cdot \{1 + O(1/x)\} \quad \text{as} \quad x \to +\infty, \tag{76}$$

where $\boldsymbol{\sigma^\sigma} := \prod_{i \in I} \sigma_i^{\sigma_i}$, $\boldsymbol{\sigma^\lambda} := \prod_{i \in I} \sigma_i^{\lambda_i}$ and so on; note that all of σ_i and τ_j are positive due to the admissibility condition (69) for the $r_1 = +\infty$ case. Thus it is natural to define

$$\Phi(+\infty) := \begin{cases} 0 & (\text{if } \boldsymbol{\sigma^\sigma} < \boldsymbol{\tau^\tau}), \\ \boldsymbol{\sigma^\lambda}/\boldsymbol{\tau^\mu} & (\text{if } \boldsymbol{\sigma^\sigma} = \boldsymbol{\tau^\tau}), \\ +\infty & (\text{if } \boldsymbol{\sigma^\sigma} > \boldsymbol{\tau^\tau}). \end{cases} \tag{77}$$

With this understanding we assume the *continuity at infinity*:

$$\boldsymbol{\sigma^\sigma} \leq \boldsymbol{\tau^\tau} \quad (\text{when } r_1 = +\infty). \tag{78}$$

Then $\Phi(x)$ is continuous on $[r_0, r_1]$ even when $r_1 = +\infty$ and it makes sense to define

$$\Phi_{\max} := \max_{r_0 \leq x \leq r_1} \Phi(x),$$

as a positive finite number. Therefore the function

$$\phi(x) := -\log \Phi(x) \tag{79}$$

is a real-valued, continuous function on $[r_0, r_1)$, smooth in (r_0, r_1); if $r_1 < +\infty$ then it is also continuous at $x = r_1$; otherwise, $\phi(x)$ is either continuous at $x = +\infty$ or tends to $+\infty$ as $x \to +\infty$. We refer to $\phi(x)$ as the *additive phase function* for the sum $g(n)$ in (64).

When $r_1 = +\infty$ we have to think of the (absolute) *convergence* of infinite series (64). If the strict inequality $\boldsymbol{\sigma^\sigma} < \boldsymbol{\tau^\tau}$ holds in (78) then it certainly converges. Otherwise, in order to guarantee its convergence, suppose that there is a constant $\sigma > 0$ such that for any $n \in \mathbb{N}$,

$$\mathrm{Re}\, \gamma(n) \leq -1 - \sigma \quad (\text{if } \boldsymbol{\sigma^\sigma} = \boldsymbol{\tau^\tau}), \tag{80}$$

where

$$\gamma(n) := \sum_{i \in I} \alpha_i(n) - \sum_{j \in J} \beta_j(n) + \frac{|J| - |I|}{2}. \tag{81}$$

18

Thanks to positivity (70) the function $u(x; n)$ is also smooth and nowhere vanishing on (r_0, r_1), but it may be singular at one or both ends of the interval when some of the $l_i(x)$'s or $m_j(x)$'s vanish there. To deal with this situation we say that $g(n)$ is *left-regular* if $I_0 \cup J_0 = \emptyset$; *right-regular* if $I_1 \cup J_1 = \emptyset$; and *regular* if $I_0 \cup J_0 \cup I_1 \cup J_1 = \emptyset$. If $g(n)$ is left-regular resp. right-regular with $r_1 < +\infty$, then $u(x; n)$ is continuous at $x = r_0$ resp. $x = r_1$. When $r_1 < +\infty$ the reflectional symmetry (65) exchanges left and right regularities to each other. We remark that if $r_1 = +\infty$ then right-regularity automatically follows from admissibility.

The maximum of $\Phi(x)$ or equivalently the minimum of $\phi(x)$ plays a leading role in our analysis, so it is important to think of the first and second derivatives of $\phi(x)$. Differentiations of (79) with balancedness condition (62) took into account yield

$$\phi'(x) = \log \prod_{j \in J} m_j(x)^{\tau_j} \prod_{i \in I} l_i(x)^{-\sigma_i}, \tag{82a}$$

$$\phi''(x) = \sum_{j \in J} \frac{\tau_j^2}{m_j(x)} - \sum_{i \in I} \frac{\sigma_i^2}{l_i(x)}. \tag{82b}$$

Denote by Max the set of all maximum points of $\Phi(x)$ on $[r_0, r_1]$. Suppose that $\Phi(x)$ attains its maximum $\Phi_{\max}$ only within (r_0, r_1), that is, $r_0, r_1 \notin$ Max. Moreover suppose that every maximum point is *nondegenerate* to the effect that

$$\text{Max} \Subset (r_0, r_1), \qquad \phi''(x_0) > 0 \quad \text{at any} \quad x_0 \in \text{Max}, \tag{83}$$

which is referred to as *properness* of the maximum. By formula (82a) any $x \in$ Max is a root of

$$\chi(x) := \prod_{j \in J} m_j(x)^{\tau_j} - \prod_{i \in I} l_i(x)^{\sigma_i} = 0, \qquad x \in (r_0, r_1), \tag{84}$$

which is called the *characteristic equation* for $g(n)$, while $\chi(x)$ is referred to as the *characteristic function* for $g(n)$. It is easy to see that equation (84) has only a finite number of roots, unless $\chi(x) \equiv 0$, so Max must be a finite set. Note that $\phi'(x)$ and $\chi(x)$ have the same sign.

Equation (84) can be used to determine the set Max explicitly. In applications to hypergeometric series, one usually puts $\sigma_i, \tau_j = \pm 1$ and $\lambda_i, \mu_j \in \mathbb{Z}$, thus (84) is equivalent to an algebraic equation with integer coefficients and hence any $x \in$ Max must be an algebraic number. In this case with $r_1 = +\infty$, since $\boldsymbol{\sigma}^{\boldsymbol{\sigma}} = \boldsymbol{\tau}^{\boldsymbol{\tau}} = \boldsymbol{\sigma}^{\boldsymbol{\lambda}} = \boldsymbol{\tau}^{\boldsymbol{\mu}} = 1$, the continuity at infinity (78) is trivially satisfied with $\Phi(+\infty) = 1$ in (77), thus condition $\text{Max} \Subset (r_0, +\infty)$ in (83) includes $\Phi_{\max} > 1$.

Theorem 5.2 *If balancedness* (62), *boundedness* (63), *admissibility* (69), *genericness* (74) and *properness* (83) *are all satisfied, with continuity at infinity* (78) *and convergence* (80) *being added when* $r_1 = +\infty$, *then the sum* $g(n)$ *in* (64) *admits an asymptotic representation*

$$g(n) = n^{\gamma(n)+\frac{1}{2}} \cdot \Phi_{\max}^n \cdot \{C(n) + \Omega(n)\}, \tag{85}$$

where $\gamma(n)$ *is defined in formula* (81) *while* $\Phi_{\max}$ *and* $C(n)$ *are defined by*

$$\Phi_{\max} := \Phi(x_0) \quad \text{for any} \quad x_0 \in \text{Max}; \qquad C(n) := \sqrt{2\pi} \sum_{x_0 \in \text{Max}} \frac{u(x_0; n)}{\sqrt{\phi''(x_0)}} \tag{86}$$

19

in terms of the notations in (75), whereas the error term $\Omega(n)$ is estimated as

$$|\Omega(n)| \le K\{n^{-\frac{1}{2}} + \lambda^{-n}(\delta_0(n)^{-1} + \delta_1(n)^{-1})\}, \qquad \forall n \ge N, \tag{87}$$

for some constants $K > 0$, $\lambda > 1$ and $N \in \mathbb{N}$, where $\delta_0(n)$ and $\delta_1(n)$ are defined in (74). This estimate is valid uniformly for all $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ satisfying conditions (63) and (74) along with (80) when $r_1 = +\infty$, in which case $I_1 = \emptyset$ and so $\delta_1(n) = 1$.

Things are simpler when Max consists of a single point $x_0 \in (r_0, r_1)$, in which case the main idea for proving Theorem 5.2 is to divide the sum (64) into five components:

$$g(n) = g_0(n) + h_0(n) + h(n) + h_1(n) + g_1(n),$$

with each component being a partial sum of (64) defined by

$$
\begin{aligned}
g_0(n) &:= \text{sum of } G(k;n) \text{ over} & \lceil r_0 n \rceil \le k \le \lceil (r_0 + \varepsilon)n \rceil - 1, & \quad \text{(left end)} \\
h_0(n) &:= \text{sum of } G(k;n) \text{ over} & \lceil (r_0 + \varepsilon)n \rceil \le k \le \lceil (x_0 - \varepsilon)n \rceil - 1, & \quad \text{(left side)} \\
h(n) &:= \text{sum of } G(k;n) \text{ over} & \lceil (x_0 - \varepsilon)n \rceil \le k \le \lceil (x_0 + \varepsilon)n \rceil - 1, & \quad \text{(top)} \\
h_1(n) &:= \text{sum of } G(k;n) \text{ over} & \lceil (x_0 + \varepsilon)n \rceil \le k \le \lceil (r_1 - \varepsilon)n \rceil - 1, & \quad \text{(right side)} \\
g_1(n) &:= \text{sum of } G(k;n) \text{ over} & \lceil (r_1 - \varepsilon)n \le k \le \lceil r_1 n \rceil - 1, & \quad \text{(right end)}
\end{aligned}
\tag{88}
$$

where if $r_1 = +\infty$ then the right-end component should be omitted. In order for the division (88) to make sense, the number ε must satisfy

$$0 < \varepsilon < \varepsilon_0 := \min\{(x_0 - r_0)/2, (r_1 - x_0)/2\}. \tag{89}$$

How to take $\varepsilon \in (0, \varepsilon_0)$ will be specified in the course of establishing Theorem 5.2.

We want to think of $h(n)$ as the principal part of $g(n)$, while other four components as remainders. Thus estimating the top component $h(n)$ is the central issue of this section, but treatment of both ends $g_0(n)$ and $g_1(n)$ is also far from trivial. For the sake of simplicity we shall deal with the case $|\mathrm{Max}| = 1$ only, but even when $|\mathrm{Max}| \ge 2$ things are essentially the same and it will be clear how to modify the arguments. The reflectional symmetry (65) reduces the discussion at the right end or right side to the discussion at the left counterpart. The top and side sums are regular, so we shall begin by estimating regular sums in §5.3.

In the present article we are working in the balanced cases, that is, under condition (62); it is an interesting problem to extend our method so as to cover non-balanced cases.

In the sequel we shall often utilize the following version of Stirling's formula: For any positive number $c > 0$ and any compact subset $A \Subset \mathbb{C}$ we have

$$\Gamma(xn + a) = (2\pi)^{1/2}\, x^{a - \frac{1}{2}}\, n^{a - \frac{1}{2}}\, x^{xn}\, (n/e)^{xn}\, \{1 + O(1/n)\} \quad \text{as} \quad n \to +\infty, \tag{90}$$

where Landau's symbol $O(1/n)$ is uniform with respect to $(x, a) \in \mathbb{R}_{\ge c} \times A$.

5.3   Regular Sums and Side Components

In this subsection we assume that $g(n)$ in (64) satisfies balancedness (62), boundedness (63) and admissibility (69), along with continuity at infinity (78) and convergence (80) if $r_1 = +\infty$, while properness (83) is not assumed and genericness (74) is irrelevant to regular sums.

Lemma 5.3 *If the sum $g(n)$ in (64) is regular then there exists an integer $N_0 \in \mathbb{N}$ and a constant $C_0 > 0$ such that $H(x; n)$ in formula (67) can be written*

$$H(x; n) = u(x; n) \cdot n^{\gamma(n)} \cdot \Phi(x)^n \cdot \{1 + e(x; n)\}, \tag{91a}$$

$$|e(x; n)| \leq C_0/n, \qquad \forall n \geq N_0, \ r_0 \leq \forall x \leq r_1. \tag{91b}$$

Proof. Since $g(n)$ is regular, that is, $I_0 \cup I_1 \cup J_0 \cup J_1 = \emptyset$, we have the uniform positivity (72) for all $i \in I$, $j \in J$ and $x \in [r_0, r_1]$. This together with boundedness (63) allows us to apply Stirling's formula (90) to all gamma factors $\Gamma(l_i(x)n + \alpha_i(n))$ and $\Gamma(m_j(x)n + \beta_j(n))$ of $H(x; n)$ in (67). Taking definitions (75) and (81) into account we use formula (90) to have

$$H(x; n) = u(x; n) \cdot n^{\gamma(n)} \cdot \Phi(x)^n \cdot \left\{ (n/e)^{\sum_{i \in I} l_i(x) - \sum_{j \in J} m_j(x)} \right\}^n \cdot \{1 + O(1/n)\},$$

where the $O(1/n)$ term is uniform with respect to $x \in [r_0, r_1]$ as well as to $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ satisfying condition (63). Then balancedness (68) yields the desired formula (91). $\qquad\square$

Proposition 5.4 *If the sum $g(n)$ in formula (64) is regular then it admits an estimate*

$$|g(n)| \leq C_1 \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot \Phi_{\max}^n, \qquad \forall n \geq N_0,$$

for a constant $C_1 > 0$ and an integer $N_0 \in \mathbb{N}$ which is the same as in Lemma 5.3.

Proof. From representation (91) we have

$$|H(x; n)| \leq (1 + C_0) \cdot |u(x; n)| \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n, \qquad r_0 \leq \forall x \leq r_1, \ \forall n \geq N_0. \tag{92}$$

First we consider the case $r_1 < +\infty$. Since $g(n)$ is regular and $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ are bounded by assumption (63), the definition (75b) implies that $u(x; n)$ is bounded for $(x, n) \in [r_0, r_1] \times \mathbb{Z}_{\geq N_0}$. Replacing the constant C_0 by a larger one if necessary, we have $|H(x; n)| \leq C_0 \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n$ for any $x \in [r_0, r_1]$ and $n \geq N_0$. Thus by definitions (64) and (67) we have for any $n \geq N_0$,

$$|g(n)| \leq \sum_{k=\lceil r_0 n \rceil}^{\lceil r_1 n \rceil - 1} |H(k/n; n)| \leq C_0 \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n \sum_{k=\lceil r_0 n \rceil}^{\lceil r_1 n \rceil - 1} 1$$

$$= C_0 \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n \cdot (\lceil r_1 n \rceil - \lceil r_0 n \rceil) \leq C_1 \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot \Phi_{\max}^n,$$

with the constant $C_1 := C_0(1 + r_1 - r_0)$.

We proceed to the case $r_1 = +\infty$ and $\boldsymbol{\sigma}^{\boldsymbol{\sigma}} = \boldsymbol{\tau}^{\boldsymbol{\tau}}$ in which condition (80) takes place. Since $g(n)$ is regular and $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ are bounded by (63), the definition (75b) implies that

$$u(x; n) = (2\pi)^{\frac{|I| - |J|}{2}} \prod_{i \in I} \sigma_i^{\alpha_i(n) - \frac{1}{2}} \prod_{j \in J} \tau_j^{\frac{1}{2} - \beta_j(n)} \cdot x^{\gamma(n)} \cdot \{1 + O(1/x)\} \qquad \text{as} \quad x \to +\infty,$$

uniformly for $n \in \mathbb{N}$. By condition (80) there exists a constant $C_2 > 0$ such that

$$|u(x; n)| \leq C_2 (2 + x)^{-1-\sigma}, \qquad \forall x \geq r_0, \ \forall n \geq N_0.$$

In view of definitions (64) and (67) this estimate together with formula (92) yields

$$|g(n)| \leq \sum_{k=\lceil r_0 n \rceil}^{\infty} |H(k/n; n)| \leq C_2 \left(1 + C_0\right) \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot \Phi_{\max}^n \sum_{k=\lceil r_0 n \rceil}^{\infty} \left(2 + \frac{k}{n}\right)^{-1-\sigma} \frac{1}{n}$$

$$\leq C_2 \left(1 + C_0\right) \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot \Phi_{\max}^n \int_{r_0}^{\infty} (1+x)^{-1-\sigma} \, dx = C_1 \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot \Phi_{\max}^n,$$

for any integer $n \geq N_0$, where $C_1 := C_2 \left(1 + C_0\right) \left(1 + r_0\right)^{-\sigma}/\sigma$.

The proof ends with the case where $r_1 = +\infty$ and $\boldsymbol{\sigma^\sigma < \tau^\tau}$. By Stirling's formula (90) and asymptotic representation (76) there exists a constant $C_3 > 0$ such that

$$|H(x; n)| \leq C_3 \cdot (xn)^{\operatorname{Re}\gamma(n)} \cdot \Phi(x)^n, \qquad \Phi(x) \leq C_3 \cdot \rho^x, \qquad ^\forall x \geq r_0, \; ^\forall n \geq N_0,$$

with $0 < \rho := \boldsymbol{\sigma^\sigma/\tau^\tau} < 1$. Take a number $r_2 > r_0$ so large that $d := C_3 \cdot \rho^{r_2/2} < \Phi_{\max}$ and let $g(n) = g_1(n) + g_2(n)$ be the decomposition according to the division $[r_0, +\infty) = [r_0, r_2) \cup [r_2, +\infty)$. Then an estimate for the $r_1 < +\infty$ case applies to $g_1(n)$, while one has $|H(x; n)| \leq C_3 \cdot d^n \cdot (xn)^c \cdot \rho^{xn/2}$ for $x \geq r_2$, where $c := \sup_{n \geq N_0} \operatorname{Re}\gamma(n)$, and hence

$$|g_2(n)| \leq \sum_{k=\lceil r_2 n \rceil}^{\infty} |H(k/n; n)| \leq C_3 \cdot d^n \sum_{k=\lceil r_2 n \rceil}^{\infty} k^c \cdot \rho^{k/2} \leq C_3 \cdot d^n \sum_{k=1}^{\infty} k^c \cdot \rho^{k/2} = C_4 \cdot d^n$$

for any $n \geq N_0$. It is clear from $0 < d < \Phi_{\max}$ that the proposition follows. $\qquad \square$

Proposition 5.4 can be used to estimate the side components $h_0(n)$ and $h_1(n)$ in (88).

Lemma 5.5 *For any $0 < \varepsilon < \varepsilon_0$ there exist $N_1^\varepsilon \in \mathbb{N}$ and $C_1^\varepsilon > 0$ such that*

$$|h_0(n)| \leq C_1^\varepsilon \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot (\Phi_0^\varepsilon)^n, \quad |h_1(n)| \leq C_1^\varepsilon \cdot n^{\operatorname{Re}\gamma(n)+1} \cdot (\Phi_1^\varepsilon)^n, \quad ^\forall n \geq N_1^\varepsilon,$$

where $\Phi_0^\varepsilon := \max\limits_{r_0 + \varepsilon \leq x \leq x_0 - \varepsilon} \Phi(x)$ and $\Phi_1^\varepsilon := \max\limits_{x_0 + \varepsilon \leq x \leq r_1 - \varepsilon} \Phi(x)$.

Proof. We have only to apply Proposition 5.4 with r_0 and r_1 replaced by $r_0 + \varepsilon$ and $x_0 - \varepsilon$ to deduce the estimate for $h_0(n)$. In a similar manner we apply the proposition this time with r_0 and r_1 replaced by $x_0 + \varepsilon$ and $r_1 - \varepsilon$ to get the estimate for $h_1(n)$. $\qquad \square$

5.4 Top Component

We consider the top component $h(n)$ in (88). Recall the setting in §5.2 that $\operatorname{Max} = \{x_0\} \Subset (r_0, r_1)$, $\Phi_{\max} = \Phi(x_0) = e^{-\phi(x_0)}$, $\phi'(x_0) = 0$ and $\phi''(x_0) > 0$. Since the sum $h(n)$ is regular, Lemma 5.3 implies that $H(x; n)$ can be written as in (91a) with estimate (91b) now being

$$|e(x; n)| \leq C_0(\varepsilon)/n, \qquad ^\forall n \geq N_0(\varepsilon), \; x_0 - \varepsilon \leq {}^\forall x \leq x_0 + \varepsilon. \tag{93}$$

The local study of $H(x; n)$ near $x = x_0$ is best performed in terms of new variables

$$y := x - x_0 \quad \text{(shift)}; \qquad z := \sqrt{n}\, y \quad \text{(scale change)}.$$

Taylor expansions around $x = x_0$ show that $\phi(x)$ and $u(x; n)$ can be written

$$\phi(x) = \phi(x_0) + a\, y^2 + \eta(y), \qquad |\eta(y)| \le b\, |y|^3, \qquad |{}^\forall y\,| \le \varepsilon_1, \tag{94a}$$

$$u(x; n) = u(x_0; n) + v(y; n), \qquad |v(y; n)| \le c\, |y|, \qquad |{}^\forall y\,| \le \varepsilon_1, \tag{94b}$$

with $a := \frac{1}{2}\, \phi''(x_0) > 0$ and some positive constants b, c, $\varepsilon_1 > 0$. It is clear that a and b are independent of n. We can also take c and ε_1 uniformly in n because $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ are bounded by assumption (63). If we put

$$H_{\mathrm{a}}(x; n) := u(x_0; n) \cdot n^{\gamma(n)} \cdot \Phi(x)^n = u(x_0; n) \cdot n^{\gamma(n)} \cdot \Phi_{\max}^n \cdot e^{-n\{a\, y^2 + \eta(y)\}}, \tag{95a}$$

$$H_{\mathrm{b}}(x; n) := v(y; n) \cdot n^{\gamma(n)} \cdot \Phi(x)^n = n^{\gamma(n)} \cdot \Phi_{\max}^n \cdot v(y; n) \cdot e^{-n\{a\, y^2 + \eta(y)\}}, \tag{95b}$$

$$H_{\mathrm{c}}(x; n) := u(x; n) \cdot n^{\gamma(n)} \cdot \Phi(x)^n \cdot e(x; n), \tag{95c}$$

then formula (91a) yields $H(x; n) = H_{\mathrm{a}}(x; n) + H_{\mathrm{b}}(x; n) + H_{\mathrm{c}}(x; n)$, which in turn gives

$$h(n) = h_{\mathrm{a}}(n) + h_{\mathrm{b}}(n) + h_{\mathrm{c}}(n), \qquad h_\nu(n) := \sum_{k=l}^{m-1} H_\nu(k/n; n), \quad \nu = \mathrm{a, b, c},$$

where $l := \lceil (x_0 - \varepsilon)n \rceil$ and $m := \lceil (x_0 + \varepsilon)n \rceil$.

To estimate $h_{\mathrm{a}}(n)$ we use some a priori estimates, which will be collected in §5.6.

Lemma 5.6 *For any $0 < \varepsilon < \varepsilon_2 := \min\{\varepsilon_0, \frac{\varepsilon_1}{2}, \frac{a}{4b}\}$ and $n \ge N_1(\varepsilon) := \max\{2/\varepsilon, N_0(\varepsilon)\}$,*

$$h_{\mathrm{a}}(n) = \sqrt{\pi/a} \cdot u(x_0; n) \cdot n^{\gamma(n)+1/2} \cdot \Phi_{\max}^n \cdot \left\{ 1 + e_{\mathrm{a}}(n) \cdot n^{-1/2} \right\}, \tag{96a}$$

$$|e_{\mathrm{a}}(n)| \le M_5(a, b; \varepsilon) := 2M_3(a, b) + (5/a) \cdot (2\varepsilon)^{-3/2}, \tag{96b}$$

where $M_3(a, b)$ is defined in Lemma 5.17 and currently $a := \frac{1}{2}\phi''(x_0) > 0$.

Proof. Put $\psi(z; a) := e^{-a\, z^2 + \delta(z)}$ with $\delta(z) := -n \cdot \eta\left(n^{-1/2} z\right)$. Then (95a) and (94a) read

$$H_{\mathrm{a}}(x; n) = u(x_0; n) \cdot n^{\gamma(n)+1/2} \cdot \Phi_{\max}^n \cdot \psi(z; a) \cdot \frac{1}{\sqrt{n}}, \tag{97a}$$

$$|\delta(z)| \le \frac{b}{\sqrt{n}}\, |z|^3 \qquad (|{}^\forall z\,| \le \varepsilon_1 \sqrt{n}\,). \tag{97b}$$

Consider the sequence $\Delta : \xi_k := (k - x_0 n)/\sqrt{n}\ (k = l, \ldots, m)$. From the definitions of l and m,

$$-\varepsilon\sqrt{n} \le \xi_l < -\varepsilon\sqrt{n} + 1/\sqrt{n}, \qquad \varepsilon\sqrt{n} \le \xi_m < \varepsilon\sqrt{n} + 1/\sqrt{n}, \tag{98}$$

which together with $0 < \varepsilon < \varepsilon_2$ and $n \ge N_1(\varepsilon)$ implies inclusion $[\xi_l, \xi_m] \subset [-\varepsilon_1\sqrt{n}, \varepsilon_1\sqrt{n}\,]$, so the estimate (97b) is available for all $z \in [\xi_l, \xi_m]$. From formula (97a) we have

$$h_{\mathrm{a}}(n) = u(x_0; n) \cdot n^{\gamma(n)+\frac{1}{2}} \cdot \Phi_{\max}^n \cdot R(\psi, \Delta) \quad \text{with} \quad R(\psi; \Delta) := \sum_{k=l}^{m-1} \psi(\xi_k; a)\frac{1}{\sqrt{n}}, \tag{99}$$

where $R(\psi; \Delta)$ is the left Riemann sum of $\psi(z; a)$ for equipartition Δ of the interval $[\xi_l, \xi_m]$.

23

Let $\varphi(z; a) := e^{-az^2}$. Since $|\xi_k - z| \leq 1/\sqrt{n}$ for any $z \in [\xi_k, \xi_{k+1}]$, Lemma 5.17 yields

$$\left| R(\psi; \Delta) - \int_{\xi_l}^{\xi_m} \varphi(z; a)\, dz \right| = \left| \sum_{k=l}^{m-1} \int_{\xi_k}^{\xi_{k+1}} \{\psi(\xi_k; a) - \varphi(z; a)\}\, dz \right|$$

$$\leq \sum_{k=l}^{m-1} \int_{\xi_k}^{\xi_{k+1}} |\psi(\xi_k; a) - \varphi(z; a)|\, dz \leq \frac{M_3(a, b)}{\sqrt{n}} \sum_{k=l}^{m-1} \int_{\xi_k}^{\xi_{k+1}} \varphi(z; a/4)\, dz$$

$$= \frac{M_3(a, b)}{\sqrt{n}} \int_{\xi_l}^{\xi_m} \varphi(z; a/4)\, dz \leq \frac{M_3(a, b)}{\sqrt{n}} \int_{-\infty}^{\infty} \varphi(z; a/4)\, dz = 2M_3(a, b)\sqrt{\frac{\pi}{an}},$$

where estimate (119) is used in the second inequality. By the partition of Gaussian integral

$$\sqrt{\pi/a} = \int_{-\infty}^{\infty} \varphi(z; a)\, dz = \int_{-\infty}^{\xi_l} + \int_{\xi_l}^{\xi_m} + \int_{\xi_m}^{\infty} \varphi(z; a)\, dz,$$

and bounds $\xi_l \leq -\varepsilon\sqrt{n}/2$ and $\xi_m \geq \varepsilon\sqrt{n}$, which follow from (98) and $n \geq 2/\varepsilon$, we have

$$\left| R(\psi; \Delta) - \sqrt{\pi/a} \right| \leq \int_{-\infty}^{-\varepsilon\sqrt{n}/2} \varphi(z; a)\, dz + 2M_3(a, b)\sqrt{\frac{\pi}{an}} + \int_{\varepsilon\sqrt{n}}^{\infty} \varphi(z; a)\, dz$$

$$\leq 2M_3(a, b)\sqrt{\frac{\pi}{an}} + \frac{5\sqrt{\pi}}{2a^{3/2}\varepsilon^2 \cdot n} \leq \sqrt{\frac{\pi}{an}} M_5(a, b; \varepsilon), \tag{100}$$

with $M_5(a, b; \varepsilon) := 2M_3(a, b) + (5/a) \cdot (2\varepsilon)^{-3/2}$, where the estimate

$$\int_z^{\infty} \varphi(t; a)\, dt \leq \frac{1}{2}\sqrt{\frac{\pi}{a}}\, \varphi(z; a) \leq \frac{\sqrt{\pi}}{2a^{3/2}z^2}, \qquad {}^{\forall}z \geq 0,$$

and $\sqrt{n} \geq \sqrt{2/\varepsilon}$ are used in the second and third inequalities respectively. Upon writing $R(\psi; \Delta) = \sqrt{\pi/a}\,\{1 + e_2(n) \cdot n^{-1/2}\}$, formula (96) follows from (99) and (100). $\qquad \square$

Lemma 5.7 *For any $0 < \varepsilon < \varepsilon_2$ and $n \geq N_1(\varepsilon)$ we have*

$$|h_{\mathrm{b}}(n)| \leq c\, M_6(a) \cdot n^{\mathrm{Re}\,\gamma(n)} \cdot \Phi_{\max}^n, \tag{101}$$

where $M_6(a) := 2M_4(a/2)\sqrt{\pi/a} + 2/a$ with $M_4(a)$ defined in Lemma 5.18 and $a := \frac{1}{2}\phi''(x_0) > 0$. For any $0 < \varepsilon < \varepsilon_0$ there exists a constant $C_2(\varepsilon) > 0$ such that

$$|h_{\mathrm{c}}(n)| \leq C_2(\varepsilon) \cdot n^{\mathrm{Re}\,\gamma(n)} \cdot \Phi_{\max}^n, \qquad {}^{\forall}n \geq N_0(\varepsilon). \tag{102}$$

Proof. If $|y| \leq 2\varepsilon_2 \ (\leq \varepsilon_1)$ then estimate (94a) yields

$$a\,y^2 + \eta(y) \geq a\,y^2 - b\,|y|^3 = a\,y^2\left(1 - \tfrac{b}{a}\,|y|\right) \geq a\,y^2\left(1 - \tfrac{2b}{a}\,\varepsilon_2\right) \geq \tfrac{a}{2}\,y^2,$$

which together with estimate (94b) and definition (95b) gives

$$|H_{\mathrm{b}}(x; n)| \leq c \cdot n^{\mathrm{Re}\,\gamma(n)} \cdot \Phi_{\max}^n \cdot e^{-\frac{a}{2}ny^2}\,|y|, \qquad {}^{\forall}y\,| \leq 2\varepsilon_2,$$

$$= c \cdot n^{\mathrm{Re}\,\gamma(n)} \cdot \Phi_{\max}^n \cdot \varphi_1(z; a/2) \cdot \tfrac{1}{\sqrt{n}}, \qquad {}^{\forall}z\,| \leq 2\varepsilon_2\sqrt{n}, \tag{103}$$

24

where $\varphi_1(z; a) := |z| e^{-az^2}$. If $0 < \varepsilon < \varepsilon_2$ and $n \geq N_1(\varepsilon)$ then $[\xi_l, \xi_m] \subset [-2\varepsilon_2\sqrt{n}, 2\varepsilon_2\sqrt{n}]$ follows from (98), so estimate (103) is available for all $z \in [\xi_l, \xi_m]$, yielding

$$|h_{\mathrm{b}}(n)| \leq \sum_{k=l}^{m-1} |H_{\mathrm{b}}(k/n; n)| \leq c \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n \cdot R(\varphi_1; \Delta),$$

where the Riemann sum $R(\varphi_1; \Delta) := \sum_{k=l}^{m-1} \varphi_1(\xi_k; a/2) \cdot \frac{1}{\sqrt{n}}$ is estimated as

$$
\begin{aligned}
R(\varphi_1; \Delta) &\leq \sum_{k=l}^{m-1} \int_{\xi_k}^{\xi_{k+1}} |\varphi_1(\xi_k; a/2) - \varphi_1(z; a/2)| \, dz + \int_{\xi_l}^{\xi_m} \varphi_1(z; a/2) \, dz \\
&\leq M_4(a/2) \sum_{k=l}^{m-1} \int_{\xi_k}^{\xi_{k+1}} |\xi_k - z| \, \varphi(z; a/4) \, dz + \int_{-\infty}^{\infty} \varphi_1(z; a/2) \, dz \\
&\leq \frac{M_4(a/2)}{\sqrt{n}} \int_{-\infty}^{\infty} \varphi(z; a/4) \, dz + \int_{-\infty}^{\infty} \varphi_1(z; a/2) \, dz = 2 M_4(a/2) \sqrt{\frac{\pi}{an}} + \frac{2}{a} \leq M_6(a),
\end{aligned}
$$

where the second inequality is obtained by Lemma 5.18. Now (101) follows readily.

Since $\boldsymbol{\alpha}(n)$ and $\boldsymbol{\beta}(n)$ are bounded by (63), there exists a constant $C_1(\varepsilon) > 0$ such that $|u(x; n)| \leq C_1(\varepsilon)$ for any $n \in \mathbb{N}$ and $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$, which together with (93) yields

$$|H_{\mathrm{c}}(x; n)| \leq C_1(\varepsilon) \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n \cdot C_0(\varepsilon)/n, \quad {}^{\forall} n \geq N_0(\varepsilon), \ x_0 - \varepsilon \leq {}^{\forall} x \leq x_0 + \varepsilon.$$

Since $m - l = \lceil (x_0 + \varepsilon)n \rceil - \lceil (x_0 - \varepsilon)n \rceil \leq (2\varepsilon + 1)n$, we have for any $n \geq N_0(\varepsilon)$,

$$|h_{\mathrm{c}}(n)| \leq \sum_{k=l}^{m-1} |H(k/n; n)| \leq C_0(\varepsilon) \cdot C_1(\varepsilon) \cdot \frac{m-l}{n} \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n = C_2(\varepsilon) \cdot n^{\operatorname{Re}\gamma(n)} \cdot \Phi_{\max}^n,$$

where $C_2(\varepsilon) := (2\varepsilon + 1) \cdot C_0(\varepsilon) \cdot C_1(\varepsilon)$. This establishes estimate (102). $\qquad\square$

Proposition 5.8 *For any $0 < \varepsilon < \varepsilon_2$, there is a constant $M(\varepsilon) > 0$ such that*

$$h(n) = \sqrt{2\pi} \, \frac{u(x_0; n)}{\sqrt{\phi''(x_0)}} \cdot n^{\gamma(n)+\frac{1}{2}} \cdot \Phi_{\max}^n \cdot \left\{ 1 + \frac{e(n)}{\sqrt{n}} \right\}, \qquad |e(n)| \leq M(\varepsilon), \qquad (104)$$

for any $n \geq N_1(\varepsilon)$, where ε_2 and $N_1(\varepsilon)$ are given in Lemma 5.6.

Proof. This readily follows from $h(n) = h_{\mathrm{a}}(n) + h_{\mathrm{b}}(n) + h_{\mathrm{c}}(n)$ and Lemmas 5.6 and 5.7. $\qquad\square$

5.5 Irregular Sums and End Components

We shall estimate the left-end component $g_0(n)$ in (88). When $r_1 < +\infty$ the estimate for the right-end component $g_1(n)$ follows from the left-end counterpart by reflectional symmetry (65). If we make the translation $k \mapsto l := k - \lceil r_0 n \rceil$ for convenience, we can write

$$
\begin{aligned}
\sigma_i k + \lambda_i n + \alpha_i(n) &= \sigma_i l + \bar{\lambda}_i n + \bar{\alpha}_i(n), & \bar{\lambda}_i &:= l_i(r_0) & (i \in I), \\
\tau_j k + \mu_j n + \beta_j(n) &= \tau_j l + \bar{\mu}_j n + \bar{\beta}_j(n), & \bar{\mu}_j &:= m_j(r_0) & (j \in J),
\end{aligned}
$$

where $\bar{\alpha}_i(n) := \alpha_i(n) + \sigma_i(\lceil r_0 n \rceil - r_0 n)$ and $\bar{\beta}_j(n) := \beta_j(n) + \tau_j(\lceil r_0 n \rceil - r_0 n)$. Note that $\bar{\alpha}_i(n)$ here is the same as $\alpha_i^{(0)}(n)$ in formula (73). Put $I_0^+ := I \setminus I_0$ and $J_0^+ := J \setminus J_0$, where the index sets I_0 and J_0 are defined in (71). Then $G(k; n)$ factors as

$$G(k; n) = G_0(l; n) \cdot G_0^+(l; n), \qquad l := k - \lceil r_0 n \rceil, \qquad (105\text{a})$$

$$G_0(l; n) := \frac{\prod_{i \in I_0} \Gamma(\sigma_i\, l + \bar{\alpha}_i(n))}{\prod_{j \in J_0} \Gamma(\tau_j\, l + \bar{\beta}_j(n))}, \quad G_0^+(l; n) := \frac{\prod_{i \in I_0^+} \Gamma(\sigma_i\, l + \bar{\lambda}_i\, n + \bar{\alpha}_i(n))}{\prod_{j \in J_0^+} \Gamma(\tau_j\, l + \bar{\mu}_j\, n + \bar{\beta}_j(n))}. \qquad (105\text{b})$$

From Lemma 5.1 one has $\sigma_i > 0$ for $i \in I_0$ and $\tau_j > 0$ for $j \in J_0$, whereas condition (68) at $x = r_0$ implies that $(\bar{\lambda}_i)_{i \in I_0^+}$ and $(\bar{\mu}_j)_{j \in J_0^+}$ are balanced to the effect that

$$\sum_{i \in I_0^+} \bar{\lambda}_i = \sum_{j \in J_0^+} \bar{\mu}_j. \qquad (106)$$

However, since $(\sigma_i)_{i \in I_0}$ and $(\tau_j)_{j \in J_0}$, resp. $(\sigma_i)_{i \in I_0^+}$ and $(\tau_j)_{j \in J_0^+}$, may not be balanced, we put

$$\rho_0 := \sum_{i \in I_0} \sigma_i - \sum_{j \in J_0} \tau_j, \qquad \rho_0^+ := \sum_{i \in I_0^+} \sigma_i - \sum_{j \in J_0^+} \tau_j, \qquad \rho_0 = -\rho_0^+, \qquad (107)$$

where the relation $\rho_0 = -\rho_0^+$ follows from the first condition of (62).

We begin by giving an asymptotic behavior of $G_0(l; n)$ as $l \to \infty$ in terms of

$$\Phi_0 := e^{-\rho_0} \prod_{i \in I_0} \sigma_i^{\sigma_i} \prod_{j \in J_0} \tau_j^{-\tau_j},$$

$$u_0(n) := (2\pi)^{\frac{|I_0| - |J_0|}{2}} \prod_{i \in I_0} \sigma_i^{\bar{\alpha}_i(n) - \frac{1}{2}} \prod_{j \in J_0} \tau_j^{\frac{1}{2} - \bar{\beta}_j(n)},$$

$$\gamma_0(n) := \sum_{i \in I_0} \bar{\alpha}_i(n) - \sum_{j \in J_0} \bar{\beta}_j(n) + \frac{|J_0| - |I_0|}{2}.$$

Note that Φ_0 is positive and $u_0(n)$ is nonzero due to the positivity of σ_i and τ_j for $i \in I_0$ and $j \in J_0$. We use the following general fact about the gamma function.

Lemma 5.9 *For any $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and any integer m such that $m \geq 1 + |\operatorname{Re} z|$,*

$$|\Gamma(z)| \leq \frac{2|\Gamma(z + m)|}{\operatorname{dist}(z, \mathbb{Z}_{\leq 0})}.$$

Proof. If $\operatorname{Re} z > 0$ we have $\operatorname{dist}(z, \mathbb{Z}_{\leq 0}) = |z|$ and the results follows readily. If $\operatorname{Re} z \leq 0$ then $\operatorname{Re}(z + m) \geq 1$ and so the sequence $|z|, |z + 1|, \cdots, |z + m - 1|$ contains $\operatorname{dist}(z, \mathbb{Z}_{\leq 0})$ as its minimum with the next smallest $\geq 1/2$ and all the rest ≥ 1, thus $|(z; m)| = |z||z + 1| \cdots |z + m - 1| \geq \operatorname{dist}(z, \mathbb{Z}_{\leq 0})/2$, hence $|\Gamma(z)| = |\Gamma(z + m)/(z; m)| \leq 2|\Gamma(z + m)|/\operatorname{dist}(z, \mathbb{Z}_{\leq 0})$. $\square$

Lemma 5.10 *There exists a constant $K_0 > 0$ such that*

$$|G_0(l; n)| \leq K_0 \cdot \delta_0(n)^{-1} \cdot (1 + l)^{|\operatorname{Re} \gamma_0(n)|} \cdot l^{\rho_0\, l} \cdot \Phi_0^l, \qquad \forall l \in \mathbb{Z}_{\geq 0}, \ \forall n \in \mathbb{N}, \qquad (108)$$

where for $l = 0$ the convention $l^{\rho_0\, l} = 1$ is employed.

Proof. Note that $G_0(l; n)$ in (105b) takes a finite value for every $l \geq \kappa := \max_{i \in I_0} (R+1)/\sigma_i$ and $n \in \mathbb{N}$, since (63) implies that $\sigma_i l + \operatorname{Re} \bar{\alpha}_i(n) \geq \sigma_i l + \operatorname{Re} \alpha_i(n) \geq \sigma_i l - R \geq 1$ for $i \in I_0$. By Stirling's formula (90) we have $G_0(l; n) = u_0(n) \cdot l^{\gamma_0(n) + \rho_0 l} \cdot \Phi_0^l \cdot \{1 + O(1/l)\}$ as $l \to +\infty$ uniformly with respect to $n \in \mathbb{N}$. Thus there exists a constant $M_0 > 0$ such that

$$|G_0(l; n)| \leq M_0 \cdot (1+l)^{|\operatorname{Re} \gamma_0(n)|} \cdot l^{\rho_0 l} \cdot \Phi_0^l, \qquad \forall l \geq \kappa, \ \forall n \in \mathbb{N}.$$

Take the smallest integer $m \geq \max_{i \in I_0} \{1 + \sigma_i(\kappa+1) + R\}$ and put

$$\bar{G}_0(l; n) := \frac{\prod_{i \in I_0} \Gamma(\sigma_i l + \bar{\alpha}_i(n) + m)}{\prod_{j \in J_0} \Gamma(\tau_j l + \bar{\beta}_j(n))}.$$

Since $1 + |\sigma_i l + \operatorname{Re} \bar{\alpha}_i(n)| \leq 1 + \sigma_i l + |\operatorname{Re} \bar{\alpha}_i(n)| \leq 1 + \sigma_i l + |\operatorname{Re} \alpha_i(n)| + \sigma_i \leq m$ for any $0 \leq l < \kappa$, $n \in \mathbb{N}$ and $i \in I_0$, Lemma 5.9 implies that for any $0 \leq l < \kappa$ and $n \in \mathbb{N}$,

$$|G_0(l; n)| \leq \frac{2^{|I_0|} \cdot |\bar{G}_0(l; n)|}{\prod_{i \in I_0} \operatorname{dist}(\sigma_i l + \bar{\alpha}_i(n), \mathbb{Z}_{\leq 0})} \leq \frac{2^{|I_0|} \cdot |\bar{G}_0(l; n)|}{\prod_{i \in I_0} \operatorname{dist}(\bar{\alpha}_i(n), \mathbb{Z}_{\leq 0} + |\sigma_i| \mathbb{Z}_{\leq 0})} \leq \frac{2^{|I_0|} \cdot |\bar{G}_0(l; n)|}{\delta_0(n)}.$$

In view of condition (74) there exists a constant $M_0' > 0$ such that

$$2^{|I_0|} \cdot |\bar{G}_0(l; n)| \leq M_0' \cdot (1+l)^{|\operatorname{Re} \gamma_0(n)|} \cdot l^{\rho_0 l} \cdot \Phi_0^l, \qquad 0 \leq \forall l < \kappa, \ \forall n \in \mathbb{N}.$$

Then by $1 \leq \delta_0(n)^{-1}$ the estimate (108) holds with the constant $K_0 := \max\{M_0, M_0'\}$. $\qquad\square$

We proceed to the investigation into $G_0^+(l; n)$ by writing

$$G_0^+(l; n) = H_0^+(l/n; n), \qquad H_0^+(x; n) := \frac{\prod_{i \in I_0^+} \Gamma(\bar{l}_i(x) n + \bar{\alpha}_i(n))}{\prod_{j \in J_0^+} \Gamma(\bar{m}_j(x) n + \bar{\beta}_j(n))}, \tag{109}$$

where $\bar{l}_i(x) := \sigma_i x + \bar{\lambda}_i$ and $\bar{m}_j(x) := \tau_j x + \bar{\mu}_j$, and then by putting

$$\Phi_0^+(x) := e^{-\rho_0^+ x} \prod_{i \in I_0^+} \bar{l}_i(x)^{\bar{l}_i(x)} \prod_{j \in J_0^+} \bar{m}_j(x)^{-\bar{m}_j(x)},$$

$$u_0^+(x; n) := (2\pi)^{\frac{|I_0^+| - |J_0^+|}{2}} \prod_{i \in I_0^+} \bar{l}_i(x)^{\bar{\alpha}_i(n) - \frac{1}{2}} \prod_{j \in J_0^+} \bar{m}_j(x)^{\frac{1}{2} - \bar{\beta}_j(n)},$$

$$\gamma_0^+(n) := \sum_{i \in I_0^+} \bar{\alpha}_i(n) - \sum_{j \in J_0^+} \bar{\beta}_j(n) + \frac{|J_0^+| - |I_0^+|}{2}.$$

Note that $\Phi_0^+(x)$ and $u_0^+(x; n)$ are well-defined continuous functions on $[0, \varepsilon]$ with $\Phi_0^+(x)$ being positive while $u_0^+(x; n)$ non-vanishing and uniformly bounded in $n \in \mathbb{N}$.

Lemma 5.11 *For any $0 < \varepsilon < \varepsilon_0$ there exist an integer $N_0(\varepsilon) \in \mathbb{N}$ and a constant $K_0^+(\varepsilon) > 0$ such that for any $n \geq N_0(\varepsilon)$ and $0 \leq x \leq \varepsilon$,*

$$|H_0^+(x; n)| \leq K_0^+(\varepsilon) \cdot n^{\operatorname{Re} \gamma_0^+(n) + \rho_0^+ x n} \cdot \Psi_0^+(\varepsilon)^n \qquad with \quad \Psi_0^+(\varepsilon) := \max_{0 \leq x \leq \varepsilon} \Phi_0^+(x). \tag{110}$$

Proof. From the definitions of I_0^+, J_0^+, $\bar{l}_i(x)$, $\bar{m}_j(n)$, there is a constant $c(\varepsilon) > 0$ such that

$$\bar{l}_i(x) > c(\varepsilon) \quad (i \in I_0^+), \qquad \bar{m}_j(x) > c(\varepsilon) \quad (j \in J_0^+), \qquad 0 \leq {}^\forall x \leq \varepsilon.$$

By condition (63), $H_0^+(x; n)$ takes a finite value for any $x \in [0, \varepsilon]$ and $n \geq N_0(\varepsilon) := (R+1)/c(\varepsilon)$ and Stirling's formula (90) implies that $H_0^+(x; n)$ admits an asymptotic formula

$$H_0^+(x; n) = u_0^+(x; n) \cdot n^{\gamma_0^+(n) + \rho_0^+ x\, n} \cdot \Phi_0^+(x)^n \cdot \{1 + O(1/n)\} \qquad \text{as} \quad n \to +\infty,$$

uniform in $x \in [0, \varepsilon]$, where one also uses the equality $\sum_{i \in I_0^+} \bar{l}_i(x) - \sum_{j \in J_0^+} \bar{m}_j(x) = \rho_0^+ x$, which is due to balancedness condition (106) and definition (107). From this estimate and the boundedness of $u_0^+(x; n)$ coming from (63) the assertion (110) follows readily. $\qquad\square$

Now we are able to give an estimate for the left-end component $g_0(n)$ in terms of

$$\varepsilon_3 := \begin{cases} +\infty & (\text{if } \rho_0 \geq 0), \\ e^{-1} \Phi_0^{-1/\rho_0} & (\text{if } \rho_0 < 0), \end{cases} \tag{111a}$$

$$\Psi_0(\varepsilon) := \begin{cases} 1 & (\text{if } \rho_0 > 0, \text{ or } \rho_0 = 0 \text{ with } \Phi_0 \leq 1), \\ (\varepsilon^{\rho_0} \Phi_0)^\varepsilon & (\text{if } \rho_0 < 0, \text{ or } \rho_0 = 0 \text{ with } \Phi_0 > 1), \end{cases} \tag{111b}$$

$$\Delta_0(\varepsilon) := \Psi_0(\varepsilon) \cdot \Psi_0^+(\varepsilon). \tag{111c}$$

Lemma 5.12 *For any $0 < \varepsilon < \varepsilon_4 := \min\{\varepsilon_0, \varepsilon_2, \varepsilon_3\}$ with ε_0, ε_2 and ε_3 defined in (89), Lemma 5.6 and (111a) respectively, there exist $N_0(\varepsilon) \in \mathbb{N}$ and $K_0(\varepsilon) > 0$ such that*

$$|g_0(n)| \leq K_0(\varepsilon) \cdot \delta_0(n)^{-1} \cdot n^{|\mathrm{Re}\,\gamma_0(n)| + \mathrm{Re}\,\gamma_0^+(n) + 1} \cdot \Delta_0(\varepsilon)^n, \qquad {}^\forall n \geq N_0(\varepsilon).$$

Proof. It follows from formulas (109), (110) and (107) that

$$|G_0^+(l; n)| = |H_0^+(l/n; n)| \leq K_0^+(\varepsilon) \cdot n^{\mathrm{Re}\,\gamma_0^+(n)} \cdot n^{-\rho_0 l} \cdot \Psi_0^+(\varepsilon)^n.$$

Multiplying this estimate by inequality (108), we have from formula (105a),

$$|G(k; n)| \leq K(\varepsilon) \cdot \delta_0(n)^{-1} \cdot n^{\mathrm{Re}\,\gamma_0^+(n)} \cdot \varphi(l; n) \cdot \Psi_0^+(\varepsilon)^n \cdot (1 + l)^{|\mathrm{Re}\,\gamma_0(n)|}, \tag{112}$$

for any $n \geq N_0(\varepsilon)$ and $0 \leq l := k - \lceil r_0 n \rceil < \varepsilon n$, where $K(\varepsilon) := K_0 \cdot K_0^+(\varepsilon)$ and

$$\varphi(t; n) := (t/n)^{\rho_0 t} \cdot \Phi_0^t \quad (t > 0) \quad \text{with} \quad \varphi(0; n) = \lim_{t \to +0} \varphi(t; n) = 1.$$

A bit of differential calculus shows the following:

(i) If either $\rho_0 > 0$ or $\rho_0 = 0$ with $\Phi_0 \leq 1$, then $\varphi(t; n)$ is non-increasing in $t \geq 0$ and hence $\varphi(t; n) \leq \varphi(0; n) = 1 = \Psi_0(\varepsilon)^n$ for any $t \geq 0$.

(ii) If either $\rho_0 < 0$ or $\rho_0 = 0$ with $\Phi_0 > 1$, then $\frac{d}{dt}\varphi(t; n) \geq 0$ in $0 \leq t \leq \varepsilon_3 n$ with equality only when $t = \varepsilon_3 n$, so that $\varphi(t; n) \leq \varphi(\varepsilon n; n) = (\varepsilon^{\rho_0} \Phi_0)^{\varepsilon n} = \Psi_0(\varepsilon)^n$ for any $0 \leq t \leq \varepsilon n$ ($< \varepsilon_3 n$), where ε_3 and $\Psi_0(\varepsilon)$ are defined in (111a) and (111b) respectively.

28

In either case $0 < \varphi(t; n) \leq \Psi_0(\varepsilon)^n$ for any $0 \leq t < \varepsilon n$ and thus (112) and (111c) lead to

$$|G(k; n)| \leq K(\varepsilon) \cdot \delta_0(n)^{-1} \cdot n^{\operatorname{Re} \gamma_0^+(n)} \cdot \Delta_0(\varepsilon)^n \cdot (1 + l)^{|\operatorname{Re} \gamma_0(n)|}, \tag{113}$$

for any $n \geq N_0(\varepsilon)$ and $0 \leq l := k - \lceil r_0 n \rceil < \varepsilon n$. Since

$$\sum_{0 \leq l < \varepsilon n} (1 + l)^{|\operatorname{Re} \gamma_0(n)|} \leq \int_0^{\varepsilon n + 1} (1 + t)^{|\operatorname{Re} \gamma_0(n)|} \, dt \leq \frac{(\varepsilon n + 2)^{|\operatorname{Re} \gamma_0(n)| + 1}}{|\operatorname{Re} \gamma_0(n)| + 1} \leq \{(2 + \varepsilon) \cdot n\}^{|\operatorname{Re} \gamma_0(n)| + 1},$$

summing up (113) over the integers $0 \leq l \leq \lceil (r_0 + \varepsilon) n \rceil - \lceil r_0 n \rceil - 1 \ (< \varepsilon n)$ yields

$$|g_0(n)| \leq K(\varepsilon) \cdot \delta_0(n)^{-1} \cdot (2 + \varepsilon)^{|\operatorname{Re} \gamma_0(n)| + 1} \cdot n^{|\operatorname{Re} \gamma_0(n)| + \operatorname{Re} \gamma_0^+(n) + 1} \cdot \Delta_0(\varepsilon)^n, \quad {}^\forall n \geq N_0(\varepsilon).$$

Since $\gamma_0(n)$ is bounded by condition (63) we can take a constant $K_0(\varepsilon) \geq K(\varepsilon) \cdot (2 + \varepsilon)^{|\operatorname{Re} \gamma_0(n)| + 1}$ to establish the lemma. $\qquad \square$

Proposition 5.13 *For any $d > \Phi(r_0)$ there exists a positive constant $\varepsilon_5 \leq \varepsilon_4$ such that*

$$|g_0(n)| \leq M_0(d, \varepsilon) \cdot \delta_0(n)^{-1} \cdot d^n, \qquad {}^\forall n \geq N_0(\varepsilon), \ 0 < {}^\forall \varepsilon \leq \varepsilon_5, \tag{114}$$

for some $M_0(d, \varepsilon) > 0$ and $N_0(\varepsilon) \in \mathbb{N}$ independent of d, where $\Phi(x)$ is defined in (75a) and ε_4 is given in Lemma 5.12. When $r_1 < +\infty$, a similar statement can be made for the right-end component $g_1(n)$ in (88); for any $d > \Phi(r_1)$ there exists a sufficiently small $\varepsilon_6 > 0$ such that

$$|g_1(n)| \leq M_1(d, \varepsilon) \cdot \delta_0(n)^{-1} \cdot d^n, \qquad {}^\forall n \geq N_1(\varepsilon), \ 0 < {}^\forall \varepsilon \leq \varepsilon_6.$$

Proof. We show the assertion for the left-end component $g_0(n)$ only as the right-end counterpart follows by reflectional symmetry (65). Observe that $\Psi_0(\varepsilon) \to 1$, $\Psi_0^+(\varepsilon) \to \Phi(r_0)$ and so $\Delta_0(\varepsilon) \to \Phi(r_0)$ as $\varepsilon \to +0$. Thus given $d > \Phi(r_0)$ there is a constant $0 < \varepsilon_5 < \varepsilon_4$ such that $d > \Delta_0(\varepsilon)$ for any $0 < \varepsilon \leq \varepsilon_5$. Then Lemma 5.12 enables us to take a constant $M_0(d, \varepsilon)$ as in (114). $\qquad \square$

Proof of Theorem 5.2. As is mentioned at the end of §5.2 only the singleton case $\mathrm{M_{ax}} = \{x_0\}$ is treated for the sake of simplicity. We can take a number d so that $\max\{\Phi(r_0), \Phi(r_1)\} < d < \Phi_{\max}$, since $\Phi(x)$ attains its maximum only at the interior point $x_0 \in (r_0, r_1)$. For this d take the numbers ε_5 and ε_6 as in Proposition 5.13 and put $\varepsilon := \min\{\varepsilon_5, \varepsilon_6\}$. For this ε consider the numbers Φ_0^ε and Φ_1^ε in Lemma 5.5, both of which are strictly smaller than $\Phi_{\max}$. Take a number d_0 so that $\max\{d, \Phi_0^\varepsilon, \Phi_1^\varepsilon\} < d_0 < \Phi_{\max}$ and put $\lambda := \Phi_{\max}/d_0 > 1$. Then the estimates in Propositions 5.8 and 5.13 and Lemma 5.5 are put together into equation (88) to yield

$$g(n) = C(n) \cdot n^{\gamma(n) + \frac{1}{2}} \cdot \Phi_{\max}^n \cdot \{1 + \Omega(n)\},$$

where $C(n)$ is defined in (86) and $\Omega(n)$ admits the estimate (87). $\qquad \square$

Even without assuming properness (83) we have the following convenient proposition.

Proposition 5.14 *Suppose that the sum $g(n)$ in (64) satisfies balancedness (62), boundedness (63), admissibility (69) and genericness (74) along with continuity at infinity (78) and convergence (80) when $r_1 = +\infty$. For any $d > \Phi_{\max}$ there exist $K > 0$ and $N \in \mathbb{N}$ such that*

$$|g(n)| \leq K \cdot d^n \cdot \{\delta_0(n)^{-1} + \delta_1(n)^{-1}\}, \qquad {}^\forall n \geq N.$$

Proof. Divide $g(n)$ into three components; sums over $[r_0, r_0 + \varepsilon]$, $[r_0 + \varepsilon, r_1 - \varepsilon]$ and $[r_1 - \varepsilon, r_1]$. Take $\varepsilon > 0$ sufficiently small depending on how d is close to $\Phi_{\max}$. Apply Proposition 5.13 to the left and right components and then use Lemma 5.5 in the middle one. $\qquad \square$

5.6 A Priori Estimates

We present the a priori estimates used in §5.4. In what follows we often use the inequality

$$|e^x - 1| \le |x| \, e^{|x|} \qquad (x \in \mathbb{R}). \tag{115}$$

Given a positive constant a, we consider the function $\varphi(x; a) := e^{-ax^2}$.

Lemma 5.15 *If x, $y \in \mathbb{R}$ and $|y - x| \le 1$, then*

$$|\varphi(y; a) - \varphi(x; a)| \le M_1(a) \, |y - x| \, \varphi(x; a/2), \tag{116}$$

where $M_1(a) := a \sup_{x \in \mathbb{R}} (2|x| + 1) e^{-\frac{a}{2}(x^2 - 4|x| - 2)} < \infty$.

Proof. Put $h := y - x$. It then follows from inequality (115) that

$$\left| e^{ax^2 - a(x+h)^2} - 1 \right| = \left| e^{-ah(2x+h)} - 1 \right| \le a \, |h| |2x + h| \, e^{a|h||2x+h|}$$

$$\le a \, |h| \, (2|x| + |h|) \, e^{a|h|(2|x|+|h|)} \le a \, |h| \, (2|x| + 1) \, e^{a(2|x|+1)},$$

whenever $|h| \le 1$. Dividing both sides by e^{ax^2} we have

$$\left| e^{-a(x+h)^2} - e^{-ax^2} \right| \le a \, |h| \, (2|x| + 1) \, e^{-a(x^2 - 2|x| - 1)}$$

$$= a \, (2|x| + 1) \, e^{-\frac{a}{2}(x^2 - 4|x| - 2)} \cdot e^{-\frac{a}{2}x^2} \, |h| \le M_1(a) \, e^{-\frac{a}{2}x^2} \, |h|,$$

which proves the lemma. $\qquad\square$

Let $b > 0$, $m \ge 1$, $0 < \varepsilon \le \frac{a}{4b}$, and suppose that a function $\delta(x)$ admits an estimate

$$|\delta(x)| \le \frac{b}{m} |x|^3 \qquad (|^{\forall} x \,| \le \varepsilon m). \tag{117}$$

Lemma 5.16 *Under condition (117), the function $\psi(x; a) := e^{-ax^2 + \delta(x)}$ satisfies*

$$|\psi(x; a) - \varphi(x; a)| \le \frac{b \, M_2(a)}{m} \, \varphi(x; a/2) \qquad (|^{\forall} x \,| \le \varepsilon m), \tag{118}$$

where $M_2(a) := \sup_{x \in \mathbb{R}} |x|^3 \, e^{-\frac{a}{4} x^2} < \infty$.

Proof. For $|x| \le \varepsilon m$, we have

$$\begin{aligned}
\left| e^{-ax^2 + \delta(x)} - e^{-ax^2} \right| &= e^{-ax^2} \left| e^{\delta(x)} - 1 \right| \le |\delta(x)| \, e^{-ax^2 + |\delta(x)|} && \text{by (115)}, \\
&\le \frac{b}{m} |x|^3 \, e^{-ax^2 + \frac{b}{m}|x|^3} = \frac{b}{m} |x|^3 \, e^{-ax^2 \left(1 - \frac{b}{am}|x|\right)} && \text{by (117)}, \\
&\le \frac{b}{m} |x|^3 \, e^{-ax^2 \left(1 - \frac{b\varepsilon}{a}\right)} && \text{by } |x| \le \varepsilon m, \\
&\le \frac{b}{m} |x|^3 \, e^{-\frac{3a}{4} x^2} = \frac{b}{m} |x|^3 \, e^{-\frac{a}{4} x^2} \cdot e^{-\frac{a}{2} x^2} && \text{by } 0 < \varepsilon \le \frac{a}{4b}, \\
&\le \frac{b M_2(a)}{m} \, e^{-\frac{a}{2} x^2},
\end{aligned}$$

where the last inequality is by the definition of $M_2(a)$. $\qquad\square$

Lemma 5.17 *Under condition (117), if* $|x| \leq \varepsilon m$, $|y| \leq \varepsilon m$ *and* $|y - x| \leq 1/m$, *then*

$$|\psi(y; a) - \varphi(x; a)| \leq \frac{M_3(a, b)}{m} \varphi(x; a/4). \tag{119}$$

where $M_3(a, b) := M_1(a) + b M_2(a) + b M_1(a/2) M_2(a)$.

Proof. Putting $y = x + h$ with $|h| \leq 1/m$, we have

$$
\begin{aligned}
&|\psi(y; a) - \varphi(x; a)| \\
&\quad \leq |\psi(y; a) - \varphi(y; a)| + |\varphi(y; a) - \varphi(x; a)| && \text{by t.i.,} \\
&\quad \leq \frac{b M_2(a)}{m} \varphi(y; \tfrac{a}{2}) + M_1(a) |h| \varphi(x; \tfrac{a}{2}) && \text{by (118) and (116),} \\
&\quad \leq \frac{b M_2(a)}{m} \left\{ |\varphi(y; \tfrac{a}{2}) - \varphi(x; \tfrac{a}{2})| + \varphi(x; \tfrac{a}{2}) \right\} + \frac{M_1(a)}{m} \varphi(x; \tfrac{a}{2}) && \text{by t.i. and } |h| \leq \tfrac{1}{m}, \\
&\quad \leq \frac{b M_2(a)}{m} \left\{ M_1(\tfrac{a}{2}) |h| \varphi(x; \tfrac{a}{4}) + \varphi(x; \tfrac{a}{2}) \right\} + \frac{M_1(a)}{m} \varphi(x; \tfrac{a}{2}) && \text{by (116),} \\
&\quad \leq \frac{b M_2(a)}{m} \left\{ M_1(\tfrac{a}{2}) \varphi(x; \tfrac{a}{4}) + \varphi(x; \tfrac{a}{2}) \right\} + \frac{M_1(a)}{m} \varphi(x; \tfrac{a}{2}) && \text{by } |h| \leq \tfrac{1}{m} \leq 1, \\
&\quad \leq \frac{M_3(a, b)}{m} \varphi(x; \tfrac{a}{4}) && \text{by } \varphi(x; \tfrac{a}{2}) \leq \varphi(x; \tfrac{a}{4}),
\end{aligned}
$$

where t.i. refers to trigonometric inequality. $\square$

Lemma 5.18 *If* $x, y \in \mathbb{R}$ *and* $|y - x| \leq 1$, *then* $\varphi_1(x; a) := |x| e^{-ax^2}$ *satisfies*

$$|\varphi_1(y; a) - \varphi_1(x; a)| \leq M_4(a) |y - x| \varphi(x; a/4), \tag{120}$$

where $M_4(a) := 1 + M_1(a) \cdot \max_{x \in \mathbb{R}} (|x| + 1) e^{-\frac{a}{4} x^2} < \infty$.

Proof. Putting $y = x + h$ with $|h| < 1$, one has

$$
\begin{aligned}
|\varphi_1(x + h; a) - \varphi_1(x; a)| &= ||x + h| \varphi(x + h; a) - |x| \varphi(x; a)| \\
&\leq |x + h||\varphi(x + h; a) - \varphi(x; a)| + ||x + h| - |x|| \varphi(x; a) && \text{by t.i.,} \\
&\leq (|x| + 1) M_1(a) |h| \varphi(x; a/2) + |h| \varphi(x; a) && \text{by } |h| < 1, \text{ (116) and t.i.,} \\
&= M_1(a) \cdot (|x| + 1) e^{-\frac{a}{4} x^2} |h| \varphi(x; a/4) + |h| \varphi(x; a) \\
&\leq \{ 1 + M_1(a) \cdot (|x| + 1) e^{-\frac{a}{4} x^2} \} |h| \varphi(x; a/4) && \text{by } \varphi(x; a) \leq \varphi(x; a/4), \\
&\leq M_4(a) |h| \varphi(x; a/4).
\end{aligned}
$$

Thus estimate (120) has been proved. $\square$

6 Dominant Sequences

Recall that the hypergeometric series ${}_3g_2(\boldsymbol{a})$ is defined in (25) and the subset $\mathcal{S}(\mathbb{Z}) \subset \mathbb{Z}^5$ is defined in (31). In what follows we fix any positive numbers R, $\sigma > 0$ and let

$$\mathbb{A}(R, \sigma) := \{ \, \boldsymbol{a} = (a_0, a_1, a_2; b_1, b_2) \in \mathbb{C}^5 \, : \, \|\boldsymbol{a}\| \leq R, \ \operatorname{Re} s(\boldsymbol{a}) > \sigma \, \},$$

where $\| \cdot \|$ is the standard norm on $\mathbb{C}^5$. As an application of §5 we shall show the following.

Theorem 6.1 *If $\boldsymbol{p} = (p_0, p_1, p_2; q_1, q_2) \in \mathcal{S}(\mathbb{Z})$ is any vector satisfying either*

$$\text{(a)} \quad \Delta(\boldsymbol{p}) \leq 0 \qquad or \qquad \text{(b)} \quad 2q_1^2 - 2(p_1 + p_2)q_1 + p_1 p_2 \geq 0, \tag{121}$$

where $\Delta(\boldsymbol{p})$ is the polynomial in (33), then $|D(\boldsymbol{p})| > 1$ and there exists an asymptotic formula

$$t(\boldsymbol{a}) \cdot {}_3g_2(\boldsymbol{a} + n\boldsymbol{p}) = B(\boldsymbol{a}; \boldsymbol{p}) \cdot D(\boldsymbol{p})^n \cdot n^{-s(\boldsymbol{a}) - \frac{1}{2}} \left\{ 1 + O(n^{-\frac{1}{2}}) \right\} \qquad as \quad n \to +\infty,$$

uniformly valid with respect to $\boldsymbol{a} \in \mathbb{A}(R, \sigma)$, where $D(\boldsymbol{p})$ is defined in (32) and

$$t(\boldsymbol{a}) := \sin \pi (b_1 - a_0) \cdot \sin \pi (b_2 - a_0), \quad B(\boldsymbol{a}; \boldsymbol{p}) := \frac{\pi^{\frac{1}{2}} \cdot p_0^{a_0 - \frac{1}{2}} p_1^{a_1 - \frac{1}{2}} p_2^{a_2 - \frac{1}{2}} \cdot s_2(\boldsymbol{p})^{s(\boldsymbol{a}) - 1}}{2^{\frac{3}{2}} \prod_{i=0}^{2} \prod_{j=1}^{2} (q_j - p_i)^{b_j - a_i - \frac{1}{2}}}, \tag{122}$$

with $s_2(\boldsymbol{p}) := p_0 p_1 + p_1 p_2 + p_2 p_0 - q_1 q_2$ as in Theorem 4.3.

Remark 6.2 Conditions (30) and (121) are invariant under multiplication of $\boldsymbol{p}$ by any positive scalar. This homogeneity allows one to restrict $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}_1(\mathbb{R}) := \mathcal{S}(\mathbb{R}) \cap \{q_1 = 1\}$, which is a 3-dimensional solid tetrahedron. A numerical integration shows that the domain in $\mathcal{S}_1(\mathbb{R})$ bounded by inequalities (121) occupies some 43 % of the whole $\mathcal{S}_1(\mathbb{R})$ in volume basis. Thus we may say that about 43 % of the vectors in $\mathcal{S}(\mathbb{Z})$ satisfy condition (121).

By the definition of ${}_3g_2(\boldsymbol{a})$ one can write $g(n) := {}_3g_2(\boldsymbol{a} + n\boldsymbol{p}) = \sum_{k=0}^{\infty} \varphi(k; n)$ with

$$\varphi(k; n) := \frac{\Gamma(k + p_0 n + a_0) \Gamma(k - (q_1 - p_0)n + a_0 - b_1 + 1) \Gamma(k - (q_2 - p_0)n + a_0 - b_2 + 1)}{\Gamma(k + 1) \Gamma(k + (p_0 - p_1)n + a_0 - a_1 + 1) \Gamma(k + (p_0 - p_2)n + a_0 - a_2 + 1)}.$$

We remark that the current $g(n)$ corresponds to the sequence $g_0(n)$ in (26a), not to $g(n)$ in (26b). In general a gamma factor $\Gamma(\sigma k + \lambda n + \alpha)$ is said to be *positive* resp. *negative* on an interval of k, if $\sigma k + \lambda n$ is positive resp. negative whenever k lies in that interval. Since $\boldsymbol{p} \in \mathcal{S}(\mathbb{Z})$, all lower and an upper gamma factors of $\varphi(k; n)$ are positive in $k > 0$, while the remaining two upper factors changes their signs when k goes across $(q_1 - p_0)n$ or $(q_2 - p_0)n$. Thus it is natural to make a decomposition $g(n) = g_1(n) + g_2(n) + g_2(n)$ with

$$g_1(n) := \sum_{k=0}^{(q_1 - p_0)n - 1} \varphi(k; n), \quad g_2(n) := \sum_{k=(q_1 - p_0)n}^{(q_2 - p_0)n - 1} \varphi(k; n), \quad g_3(n) := \sum_{k=(q_2 - p_0)n}^{\infty} \varphi(k; n),$$

where if $q_1 = q_2$ then $g_2(n)$ should be null so we always assume $q_1 < q_2$ when discussing $g_2(n)$. It turns out that the first component $g_1(n)$ is the most dominant among the three, yielding the leading asymptotics for $g(n)$. The proof of Theorem 6.1 is completed at the end of §6.3.

6.1 First Component

For the first component $g_1(n)$, applying Euler's reflection formula for the gamma function to the two negative gamma factors in the numerator of $\varphi(k; n)$, we have

$$t(\boldsymbol{a}) \cdot g_1(n) = \pi^2 \cdot (-1)^{(q_1 + q_2)n} \cdot G_1(n) \quad \text{with} \quad G_1(n) := \sum_{k=0}^{L_1 n - 1} \frac{\Gamma(\sigma_1 k + \lambda_1 n + \alpha_1)}{\prod_{j=1}^{5} \Gamma(\tau_j k + \mu_j n + \beta_j)},$$

where $L_1 = q_1 - p_0$, $\sigma_1 = 1$, $\lambda_1 = p_0$, $\alpha_1 = a_0$ and

$$
\begin{array}{lllll}
\tau_1 = 1, & \tau_2 = 1, & \tau_3 = 1, & \tau_4 = -1, & \tau_5 = -1, \\
\mu_1 = 0, & \mu_2 = p_0 - p_1, & \mu_3 = p_0 - p_2, & \mu_4 = q_1 - p_0 = L_1, & \mu_5 = q_2 - p_0, \\
\beta_1 = 1, & \beta_2 = a_0 - a_1 + 1, & \beta_3 = a_0 - a_2 + 1, & \beta_4 = b_1 - a_0, & \beta_5 = b_2 - a_0.
\end{array}
$$

Under the assumption of Theorem 6.1 the sum $G_1(n)$ satisfies all conditions in Theorem 5.2. Indeed, balancedness (62) follows from $s(\boldsymbol{p}) = 0$; boundedness (63) is trivial because α_1 and β_j are independent of n; admissibility (69) is fulfilled with $r_0 = 0$ and $r_1 = L_1$ due to condition (30); genericness (74) is trivial since $I_0 \cup I_1 = \emptyset$ with $J_0 = \{1\}$ and $J_1 = \{4\}$ by inequalities in (30). To verify properness (83), notice that the characteristic equation (84) now reads

$$
\chi_1(x) = \frac{x(x + p_0 - p_1)(x + p_0 - p_2)}{(-x + q_1 - p_0)(-x + q_2 - p_0)} - (x + p_0) = 0.
$$

Thanks to $s(\boldsymbol{p}) = 0$ this equation reduces to a linear equation in x having the unique root

$$
x_0 = \frac{p_0(q_1 - p_0)(q_2 - p_0)}{p_1 p_2 - (q_1 - p_0)(q_2 - p_0)} = \frac{p_0(q_1 - p_0)(q_2 - p_0)}{s_2(\boldsymbol{p})},
$$

where $s(\boldsymbol{p}) = 0$ again leads to $s_2(\boldsymbol{p}) = p_1 p_2 - (q_1 - p_0)(q_2 - p_0)$, which together with (30) yields $s(\boldsymbol{p}) - p_0(q_2 - p_0) = (q_1 - p_1)(q_1 - p_2) > 0$ and hence $s_2(\boldsymbol{p}) > p_0(q_2 - p_0) > 0$, that is,

$$
0 < x_0 < L_1 = q_1 - p_0.
$$

If $\phi_1(x)$ is the additive phase function for $G_1(n)$ then it follows from (82b) and (30) that

$$
\phi_1''(x_0) = \frac{1}{x_0} + \frac{1}{x_0 + p_0 - p_1} + \frac{1}{x + p_0 - p_2} + \frac{1}{q_1 - p_0 - x_0} + \frac{1}{q_2 - p_0 - x_0} - \frac{1}{x_0 + p_0}
$$

$$
= \frac{s_2(\boldsymbol{p})^4}{p_0 p_1 p_2 \prod_{i=0}^{2} \prod_{j=1}^{2}(q_j - p_i)} > 0.
$$

Thus in the interval $0 < x < L_1$ the function $\phi_1(x)$ has only one local and hence global minimum at $x = x_0$, which is non-degenerate. Therefore properness (83) is satisfied with $\mathrm{Max} = \{x_0\}$ and hence Theorem 5.2 applies to the sum $G_1(n)$.

Lemma 6.3 *For any $\boldsymbol{p} \in \mathcal{S}(\mathbb{Z})$ we have $|D(\boldsymbol{p})| > 1$ and an asymptotic representation*

$$
t(\boldsymbol{a}) \cdot g_1(n) = B(\boldsymbol{a}; \boldsymbol{p}) \cdot D(\boldsymbol{p})^n \cdot n^{-s(\boldsymbol{a}) - \frac{1}{2}} \left\{ 1 + O(n^{-\frac{1}{2}}) \right\} \qquad as \quad n \to +\infty,
$$

uniform with respect to $\boldsymbol{a} \in \mathbb{A}(R, \sigma)$, where $D(\boldsymbol{p})$, $t(\boldsymbol{a})$ and $B(\boldsymbol{a}; \boldsymbol{p})$ are as in (32) and (122).

Proof. Substituting $x = x_0$ into formulas (75) and using $s(\boldsymbol{p}) = 0$ repeatedly, one has

$$
(\Phi_1)_{\mathrm{max}} = \Phi_1(x_0) = \frac{p_0^{p_0} p_1^{p_1} p_2^{p_2}}{\prod_{i=0}^{2} \prod_{j=1}^{2}(q_j - p_i)^{q_j - p_i}}, \qquad u_1(x_0) = \frac{p_0^{a_0 - 1} p_1^{a_1 - 1} p_2^{a_2 - 1} s_2(\boldsymbol{p})^{s(\boldsymbol{a}) + 1}}{(2\pi)^2 \prod_{i=0}^{2} \prod_{j=1}^{2}(q_j - p_i)^{b_j - a_i}},
$$

while $\gamma_1 := \gamma(n)$ in definition (81) now reads $\gamma_1 = -s(\boldsymbol{a}) - 1$. Since $\delta_0(n) = \delta_1(n) = 1$ in (87) by $I_0 \cup I_1 = \emptyset$, formula (85) in Theorem 5.2 implies that

$$
G_1(n) = C_1 \cdot (\Phi_1)_{\mathrm{max}}^n \cdot n^{\gamma_1 + \frac{1}{2}} \cdot \left\{ 1 + O(n^{-\frac{1}{2}}) \right\} \qquad as \quad n \to +\infty,
$$

33

where formula (86) allows one to calculate the constant $C_1 := C(n)$ as

$$C_1 = \sqrt{2\pi} \, \frac{u_1(x_0)}{\sqrt{\phi_1''(x_0)}} = \frac{p_0^{a_0-\frac{1}{2}} p_1^{a_1-\frac{1}{2}} p_2^{a_2-\frac{1}{2}} s_2(\boldsymbol{p})^{s(\boldsymbol{a})-1}}{(2\pi)^{\frac{3}{2}} \prod_{i=0}^{2} \prod_{j=1}^{2} (q_j - p_i)^{b_j-a_i-\frac{1}{2}}}.$$

In view of the relation between $G_1(n)$ and $g_1(n)$ the above asymptotic formula for $G_1(n)$ gives the one for $g_1(n)$. Finally $|D(\boldsymbol{p})| > 1$ follows from Lemma 6.4 below. $\qquad\square$

Lemma 6.4 *Under condition* (30) *one has* $|D(\boldsymbol{p})| = (\Phi_1)_{\max} > \Phi_1(0) > 1$.

Proof. First, $|D(\boldsymbol{p})| = (\Phi_1)_{\max}$ is obvious from the definition (32) of $D(\boldsymbol{p})$ and the expression for $(\Phi_1)_{\max}$, while $(\Phi_1)_{\max} > \Phi_1(0)$ is also clear from $\mathrm{Max} = \{x_0\}$. Regarding $\boldsymbol{p} = (p_0, p_1, p_2; q_1, q_2)$ as real variables subject to the linear relation $s(\boldsymbol{p}) = 0$ and ranging over the closure of the domain (30), we shall find the minimum of

$$\Phi_1(0) = \frac{p_0^{p_0}}{(p_0 - p_1)^{p_0-p_1}(p_0 - p_2)^{p_0-p_2}(q_1 - p_0)^{q_1-p_0}(q_2 - p_0)^{q_2-p_0}}.$$

For any fixed (p_0, p_1, p_2), due to the constraint $s(\boldsymbol{p}) = 0$, one can thought of $\Phi_1(0)$ as a function of single variable q_1 in the interval $p_0 \le q_1 \le p_1 + p_2$. Differentiation with respect to q_1 shows that $\Phi_1(0)$ attains its minimum (only) at the endpoints $q_1 = p_0, \; p_1 + p_2$, whose value is

$$\Psi(p_0, p_1, p_2) := \frac{p_0^{p_0}}{(p_0 - p_1)^{p_0-p_1}(p_0 - p_2)^{p_0-p_2}(p_1 + p_2 - p_0)^{p_1+p_2-p_0}}.$$

so $\Phi_1(0) > \Psi(p_0, p_1, p_2)$ for any $p_0 < q_1 < p_1+p_2$. With a fixed $p_0 > 0$ we think of $\Psi(p_0, p_1, p_2)$ as a function of (p_1, p_2) in the closed simplex $p_0 \le p_1+p_2$, $p_1 \le p_0$, $p_2 \le p_0$. It has a unique critical value $\Psi(p_0, 2p_0/3, 2p_0/3) = 3^{p_0} > 1$ in the interior of the simplex, while on its boundary one has $\Psi(p_0, \alpha, p_0) = \Psi(p_0, p_0, \alpha) = \Psi(p_0, \alpha, p_0 - \alpha) = p_0^{p_0} \alpha^{-\alpha}(p_0 - \alpha)^{\alpha-p_0} \ge 1$ for any $0 \le \alpha \le p_0$. Therefore we have $\Phi_1(0) > \Psi(p_0, p_1, p_2) \ge 1$ under condition (30). $\qquad\square$

6.2   Second Component

Taking the shift $k \mapsto k + (q_1 - p_0)n$ in $\varphi(k; n)$ (see (66)) and applying the reflection formula to the unique negative gamma factor in the numerator of $\varphi(k + (q_1 - p_0)n; n)$, one has

$$g_2(n) = \frac{\pi \cdot (-1)^{(q_2-q_1)n} G_2(n)}{\sin \pi(b_2 - a_0)} \quad \text{with} \quad G_2(n) := \sum_{k=0}^{L_2 n-1} (-1)^k \frac{\prod_{i=1}^{2} \Gamma(\sigma_i k + \lambda_i n + \alpha_i)}{\prod_{j=1}^{4} \Gamma(\tau_j k + \mu_j n + \beta_j)},$$

where $L_2 = q_2 - q_1 > 0$, $\sigma_1 = \sigma_2 = 1$, $\lambda_1 = q_1$, $\lambda_2 = 0$, $\alpha_1 = a_0$, $\alpha_2 = a_0 - b_1 + 1$ and

$$
\begin{array}{llll}
\tau_1 = 1, & \tau_2 = 1, & \tau_3 = 1, & \tau_4 = -1, \\
\mu_1 = q_1 - p_0, & \mu_2 = q_1 - p_1, & \mu_3 = q_1 - p_2, & \mu_4 = q_2 - q_1 = L_2, \\
\beta_1 = 1, & \beta_2 = a_0 - a_1 + 1, & \beta_3 = a_0 - a_2 + 1, & \beta_4 = b_2 - a_0.
\end{array}
$$

Rewriting $k \mapsto 2k$ or $k \mapsto 2k + 1$ according as k is even or odd, we have a decomposition $G_2(n) = G_{20}(n) - G_{21}(n) + H_2(n)$, where $G_{2\nu}(n)$ is given by

$$G_{2\nu}(n) := \sum_{k=0}^{\lceil \frac{L_2}{2}n\rceil-1} \frac{\prod_{i=1}^{2} \Gamma(2\sigma_i k + \lambda_i n + \alpha_i + \nu\,\sigma_i)}{\prod_{j=1}^{4} \Gamma(2\tau_j k + \mu_j n + \beta_j + \nu\,\tau_j)}, \qquad \nu = 0, 1,$$

34

while if L_2 or n is even then $H_2(n) := 0$; otherwise, i.e., if both of L_2 and n are odd then

$$H_2(n) := \frac{\prod_{i=1}^2 \Gamma((\sigma_i L_2 + \lambda_i)n + \alpha_i)}{\prod_{j=1}^4 \Gamma((\tau_j L_2 + \mu_j)n + \beta_j)}.$$

Obviously, $G_{20}(n)$ and $G_{21}(n)$ have the same multiplicative phase function, which we denote by $\Phi_2(x)$. Let $\phi_2(x) := -\log \Phi_2(x)$ be the associated additive phase function. In order to make the second component $g_2(n)$ weaker than the first one $g_1(n)$, we want to make $\phi_2'(x) \geq 0$ or equivalently $\chi_2(x) \geq 0$ for every $0 \leq x \leq L_2/2$, where $\chi_2(x)$ is the common characteristic function (84) for the sums $G_{20}(n)$ and $G_{21}(n)$, which is given by

$$\chi_2(x) = \frac{(2x + \mu_1)^2(2x + \mu_2)^2(2x + \mu_3)^2}{(-2x + L_2)^2} - (2x + \lambda_1)^2(2x + \lambda_2)^2.$$

The non-negativity of $\chi_2(x)$ in the interval $0 \leq x \leq L_2/2$ is equivalent to

$$\begin{aligned}
\chi(x; \boldsymbol{p}) &:= (x + \mu_1)(x + \mu_2)(x + \mu_3) + (x + \lambda_1)(x + \lambda_2)(x - L_2) \\
&= (x + q_1 - p_0)(x + q_1 - p_1)(x + q_1 - p_2) + x(x + q_1)(x + q_1 - q_2) \quad (123) \\
&\geq 0 \qquad \text{for any} \qquad 0 \leq x \leq L_2 = q_2 - q_1.
\end{aligned}$$

It is easy to see that $G_{20}(n)$ and $G_{21}(n)$ satisfy balancedness (62), boundedness (63) and admissibility (69) conditions, where $r_0 = 0$, $r_1 = L_2/2$ and $I_0 = \{2\}$, $I_1 = J_0 = \emptyset$, $J_1 = \{4\}$, while genericness (74) for $G_{2\nu}(n)$ becomes $b_1 - a_0 \notin \mathbb{Z}_{\geq \nu+1}$ for $\nu = 0, 1$.

Lemma 6.5 *Under the assumption of Lemma 6.3, if $\boldsymbol{p}$ satisfies the additional condition (123) then there exist positive constants $0 < d_2 < |D(\boldsymbol{p})|$, $C_2 > 0$ and $N_2 \in \mathbb{N}$ such that*

$$|t(\boldsymbol{a}) \cdot g_2(n)| \leq C_2 \cdot d_2^n, \qquad \forall n \geq N_2, \; \forall \boldsymbol{a} \in \mathbb{A}(R, \sigma).$$

Proof. Condition (123) implies that $\Phi_2(x)$ is decreasing everywhere in $0 \leq x \leq L_2/2$ and is strictly so near $x = 0$ since $\chi(0; \boldsymbol{p}) = (q_1 - p_0)(q_1 - p_1)(q_1 - p_2) > 0$ by condition (30). Hence $\Phi_2(x)$ attains its maximum (only) at the left end $x = 0$ of the interval, having the value

$$(\Phi_2)_{\max} = \Phi_2(0) = \frac{q_1^{q_1}}{(q_1 - p_0)^{q_1 - p_0}(q_1 - p_1)^{q_1 - p_1}(q_1 - p_2)^{q_1 - p_2}(q_2 - q_1)^{q_2 - q_1}} = \Phi_1(L_1).$$

whereas $(\Phi_2)_{\max} = \Phi_1(L_1) < (\Phi_1)_{\max} = |D(\boldsymbol{p})|$ follows from Lemma 6.4. Thus if d_2 is any number such that $(\Phi_2)_{\max} < d_2 < |D(\boldsymbol{p})|$, then Proposition 5.14 shows that

$$|G_{2\nu}(n)| \leq \frac{K_2 \cdot d_2^n}{\min\{1, \, \text{dist}(b_1 - a_0, \mathbb{Z}_{\geq \nu+1})\}} \leq \frac{K_2 \cdot d_2^n}{\delta(b_1 - a_0)}, \quad \forall n \geq N_2, \; \nu = 0, 1,$$

for some $K_2 > 0$ and $N_2 \in \mathbb{N}$, where $\delta(z) := \min\{1, \, \text{dist}(z, \mathbb{N})\}$ for $z \in \mathbb{C}$.

We have to take care of $H_2(n)$ when L_2 and n are both odd. Stirling's formula (90) yields

$$H_2(n) = \frac{1}{2\pi} \cdot \frac{\prod_{i=1}^2 (\sigma_i L_2 + \lambda_i)^{\alpha_i - \frac{1}{2}}}{\prod_{j=1}^4 (\tau_j L_2 + \mu_j)^{\beta_j - \frac{1}{2}}} \cdot \Phi_2(L_2/2)^n \cdot n^{\gamma_2} \cdot \{1 + O(1/n)\} \quad \text{as} \quad n \to +\infty,$$

where $\gamma_2 := \alpha_1 + \alpha_2 - \beta_1 - \beta_2 - \beta_3 - \beta_4 + 1$. Since $\Phi_2(L_2/2) < (\Phi_2)_{\max} < d_2$, upon retaking $K_2 > 0$ suitably, one has $|H_2(n)| \leq K_2 \cdot d_2^n \leq K_2 \cdot d_2^n / \delta(b_1 - a_0)$ for any $n \geq N_2$.

35

Then from the relation between $g_2(n)$ and $G_2(n) = G_{20}(n) - G_{21}(n) + H_2(n)$ one has

$$|t(\boldsymbol{a}) \cdot g_2(n)| \leq 3\pi K_2 \cdot M_2(\boldsymbol{a}) \cdot d_2^n \quad \text{with} \quad M_2(\boldsymbol{a}) := \frac{|\sin \pi(b_1 - a_0)|}{\delta(b_1 - a_0)}.$$

Since $M_2(\boldsymbol{a})$ is bounded for $\boldsymbol{a} \in \mathbb{A}(R, \sigma)$ the lemma follows (here σ is irrelevant). $\quad\square$

Lemma 6.5 tempts us to ask when condition (123) is satisfied.

Lemma 6.6 *For any* $\boldsymbol{p} \in \mathcal{S}(\mathbb{R})$ *condition* (121) *implies condition* (123).

Proof. We use the following general fact. Let $\chi(x)$ be a real cubic polynomial with positive leading coefficient and Δ be its discriminant. If $\Delta < 0$ then $\chi(x)$ has only one real root so that once $\chi(c_0) > 0$ for some $c_0 \in \mathbb{R}$ then $\chi(x) > 0$ for every $x \geq c_0$. Even if $\Delta = 0$, once $\chi(c_0) > 0$ then $\chi(x) \geq 0$ for every $x \geq c_0$ with possible equality $\chi(c_1) = 0$, $c_1 > c_0$, only if $\chi(x)$ attains a local minimum at $x = c_1$. Currently, $\chi(x; \boldsymbol{p})$ has discriminant $\Delta(\boldsymbol{p})$ in formula (33) and $\chi(0; \boldsymbol{p}) = (q_1 - p_0)(q_1 - p_1)(q_1 - p_2) > 0$ by condition (30). Thus if $\Delta(\boldsymbol{p}) \leq 0$ then $\chi(x; \boldsymbol{p}) \geq 0$ for every $x \geq 0$; this is just the case (a) in condition (121).

We proceed to the case (b) in (121). The derivative of $\chi(x; \boldsymbol{p})$ in x is given by

$$\chi'(x; \boldsymbol{p}) = 6x^2 + 4(2q_1 - q_2)x + (3q_1 - p_1 - p_2)(p_1 + p_2 - q_2) + 2q_1^2 - 2(p_1 + p_2)q_1 + p_1 p_2.$$

Note that $2q_1 - q_2$, $3q_1 - p_1 - p_2$, $p_1 + p_2 - q_2 > 0$ by condition (30). Having axis of symmetry $x = -(2q_1 - q_2)/3 < 0$, the quadratic function $\chi'(x; \boldsymbol{p})$ is increasing in $x \geq 0$ and hence

$$\chi'(x; \boldsymbol{p}) \geq \chi'(0; \boldsymbol{p}) = (3q_1 - p_1 - p_2)(p_1 + p_2 - q_2) + 2q_1^2 - 2(p_1 + p_2)q_1 + p_1 p_2$$
$$> 2q_1^2 - 2(p_1 + p_2)q_1 + p_1 p_2 \geq 0 \qquad \text{for any} \quad x \geq 0,$$

where the last inequality stems from (b) in condition (121). Thus $\chi(x; \boldsymbol{p}) \geq \chi(0; \boldsymbol{p}) > 0$ for any $x \geq 0$, so condition (123) is satisfied. $\quad\square$

The converse to the implication in Lemma 6.6 is also true, accordingly conditions (121) and (123) are equivalent for any $\boldsymbol{p} \in \mathcal{S}(\mathbb{R})$, but the proof of this fact is omitted as it is not needed in this article. In the situation of Lemma 6.5 we proceed to the third component.

6.3 Third Component

For the third component $g_3(n)$, taking the shift $k \mapsto k + (q_2 - p_0)n$ in $\varphi(k; n)$, one has

$$g_3(n) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{3} \Gamma(\sigma_i k + \lambda_i n + \alpha_i)}{\prod_{j=1}^{3} \Gamma(\tau_j k + \mu_j n + \beta_j)},$$

where $\sigma_1 = \sigma_2 = \sigma_3 = \tau_1 = \tau_2 = \tau_3 = 1$ and

$$\lambda_1 = q_2, \qquad \lambda_2 = q_2 - q_1, \quad \lambda_3 = 0, \qquad \alpha_1 = a_0, \quad \alpha_2 = a_0 - b_1 + 1, \quad \alpha_3 = a_0 - b_2 + 1,$$
$$\mu_1 = q_2 - p_0, \quad \mu_2 = q_2 - p_1, \quad \mu_3 = q_2 - p_2, \quad \beta_1 = 1, \quad \beta_2 = a_0 - a_1 + 1, \quad \beta_3 = a_0 - a_2 + 1.$$

It is easy to see that $g_3(n)$ satisfies balancedness (62), boundedness (63), admissibility (69) with $r_0 = 0$ and $r_1 = +\infty$. Notice that $I_0 = \{3\}$ if $q_1 < q_2$ and $I_0 = \{2, 3\}$ if $q_1 = q_2$, while $I_1 = J_0 = J_1 = \emptyset$. Genericness (74) becomes $b_2 - a_0 \notin \mathbb{N}$ if $q_1 < q_2$, and $b_1 - a_0$, $b_2 - a_0 \notin \mathbb{N}$ if $q_1 = q_2$. Continuity at infinity (78) is satisfied with $\boldsymbol{\sigma}^{\boldsymbol{\sigma}} = \boldsymbol{\tau}^{\boldsymbol{\tau}} = 1$; convergence condition (80) is equivalent to $\operatorname{Re} s(\boldsymbol{a}) \geq \sigma$. Under the assumption of Lemma 6.5 we have the following.

Lemma 6.7 *There exist positive constants $0 < d_3 < |D(\boldsymbol{p})|$, $C_3 > 0$ and $N_3 \in \mathbb{N}$ such that*

$$|t(\boldsymbol{a}) \cdot g_3(n)| \le C_3 \cdot d_3^n, \qquad {}^\forall n \ge N_3, \ {}^\forall \boldsymbol{a} \in \mathbb{A}(R, \sigma).$$

Proof. In view of $s(\boldsymbol{p}) = 0$ the characteristic function (84) for $g_3(n)$ is given by

$$
\begin{aligned}
\chi_3(x) &= (x + q_2 - p_0)(x + q_2 - p_1)(x + q_2 - p_2) - (x + q_2)(x + q_2 - q_1)x \\
&= s(\boldsymbol{p})\, x^2 + \{2s(\boldsymbol{p})\, q_2 + s_2(\boldsymbol{p})\}x + (q_2 - p_0)(q_2 - p_1)(q_2 - p_2) \\
&= s_2(\boldsymbol{p})\, x + (q_2 - p_0)(q_2 - p_1)(q_2 - p_2).
\end{aligned}
$$

Since $s_2(\boldsymbol{p}) > 0$ and $(q_2 - p_0)(q_2 - p_1)(q_2 - p_2) > 0$ from condition (30), one has $\chi_3(x) > 0$ and hence the additive phase function $\phi_3(x)$ satisfies $\phi_3'(x) > 0$ for any $x \ge 0$. Thus $\Phi_3(x) = e^{-\phi_3(x)}$ is strictly decreasing in $x \ge 0$ and attains its maximum (only) at $x = 0$, having the value

$$(\Phi_3)_{\max} = \Phi_3(0) = \frac{q_2^{q_2}(q_2 - q_1)^{q_2 - q_1}}{(q_2 - p_0)^{q_2 - p_0}(q_2 - p_1)^{q_2 - p_1}(q_2 - p_2)^{q_2 - p_2}} = \Phi_2(L_2/2),$$

whereas $(\Phi_3)_{\max} = \Phi_2(L_2/2) < (\Phi_2)_{\max} = \Phi_2(0) = \Phi_1(L_1) < (\Phi_1)_{\max} = |D(\boldsymbol{p})|$. Thus if d_3 is any number with $(\Phi_3)_{\max} < d_3 < |D(\boldsymbol{p})|$ then Proposition 5.14 implies that for any $n \ge N_3$,

$$|g_3(n)| \le \frac{K_3 \cdot d_3^n}{\delta(b_2 - a_0)} \quad \text{if} \quad q_1 < q_2; \qquad |g_3(n)| \le \frac{K_3 \cdot d_3^n}{\delta(b_1 - a_0) \cdot \delta(b_2 - a_0)} \quad \text{if} \quad q_1 = q_2,$$

where the function $\delta(z)$ is defined in the proof of Lemma 6.5. Since $\sin \pi(b_j - a_0)/\delta(b_j - a_0)$, $j = 1, 2$, are bounded for $\boldsymbol{a} \in \mathbb{A}(R, \sigma)$, the lemma follows immediately. $\qquad \square$

Theorem 6.1 is now an immediate consequence of Lemmas 6.3, 6.5, 6.6 and 6.7. Theorems 4.3 and 6.1 then imply that if the shift vector $\boldsymbol{p} \in \mathcal{S}(\mathbb{Z})$ satisfies condition (121) then $f(n)$ in (23a) and $g(n)$ in (26b) are recessive and dominant solutions to the recurrence relation (28) whose coefficients $q(n)$ and $r(n)$ are given by (23b) and (23c). Now it is almost ready to apply the general error estimate (29) to $X(n) = f(n)$ and $Y(n) = g(n)$, where a precise asymptotic formula for the ratio $R(n) = f(n+2)/g(n+2)$ is available from Theorems 4.3 and 6.1.

Theorem 6.1 is established under balancedness condition $s(\boldsymbol{p}) = 0$. We wonder if the discrete Laplace method in §5 could be extended so as to work even when this condition fails to hold.

7　Casoratian and Error Estimates

All that remain are to evaluate the initial term $\omega(0)$ of the Casoratian determinant

$$\omega(n) := f(n) \cdot g(n+1) - f(n+1) \cdot g(n),$$

and to incorporate the ensuing formula with the asymptotic representation for $R(n)$ to complete the proofs of Theorems 3.2 and 3.3. The first task is done in §7.1, while the second in §7.2.

7.1　Casoratian

In order to evaluate $\omega(0)$, following [10, formulas (7), (8) and (10)] we define

$$\sigma_0^{(0)}(\boldsymbol{a}) := \boldsymbol{a} = (a_0, a_1, a_2; b_1, b_2),$$

$$\sigma_1^{(0)}(\boldsymbol{a}) := (a_0 + 1 - b_1, a_1 + 1 - b_1, a_2 + 1 - b_1; 2 - b_1, b_2 + 1 - b_1),$$

$$\sigma_2^{(0)}(\boldsymbol{a}) := (a_0 + 1 - b_2, a_1 + 1 - b_2, a_2 + 1 - b_2; b_1 + 1 - b_2, 2 - b_2),$$

$$\sigma_0^{(\infty)}(\boldsymbol{a}) := (a_0, a_0 + 1 - b_1, a_0 + 1 - b_2; a_0 + 1 - a_1, a_0 + 1 - a_2),$$

$$\boldsymbol{a}^* := (1 - a_0, 1 - a_1, 1 - a_2; 2 - b_1, 2 - b_2).$$

Table 3: Five parameter involutions (including identity).

$$y_i^{(0)}(\boldsymbol{a}; z) := z^{1-b_i}{}_3f_2(\sigma_i^{(0)}(\boldsymbol{a}); z), \quad i = 0, 1, 2, \quad b_0 := 1,$$

$$y_0^{(\infty)}(\boldsymbol{a}; z) := e^{i\pi s(\boldsymbol{a})} z^{-a_0}{}_3f_2(\sigma_0^{(\infty)}(\boldsymbol{a}); 1/z),$$

where $\sigma_i^{(\nu)}$ are involutions on the parameters $\boldsymbol{a}$ as in Table 3, and put $y_i^{(\nu)}(\boldsymbol{a}) := y_i^{(\nu)}(\boldsymbol{a}; 1)$. Note that $y_0^{(0)}(\boldsymbol{a}) = {}_3f_2(\boldsymbol{a})$ and $y_0^{(\infty)}(\boldsymbol{a}) = e^{i\pi s(\boldsymbol{a})}{}_3g_2(\boldsymbol{a})$. Moreover let $\boldsymbol{1} := (1, 1, 1; 1, 1)$.

Lemma 7.1 *For any $\boldsymbol{a} \in \mathbb{C}^5$ with* $\operatorname{Re} s(\boldsymbol{a}) > 1$ *one has*

$$W(\boldsymbol{a}) := y_0^{(0)}(\boldsymbol{a}) \cdot y_0^{(\infty)}(\boldsymbol{a} + \boldsymbol{1}) - y_0^{(0)}(\boldsymbol{a} + \boldsymbol{1}) \cdot y_0^{(\infty)}(\boldsymbol{a})$$

$$= -e^{i\pi s(\boldsymbol{a})} \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_0 - b_1 + 1)\Gamma(a_0 - b_2 + 1)\Gamma(s(\boldsymbol{a}) - 1)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)\Gamma(b_2 - a_1)\Gamma(b_2 - a_2)}. \tag{124}$$

Proof. A careful inspection of Bailey [4, §10.3, formulas (3) and (5)] shows that

$$w(\boldsymbol{a}; z) := y_0^{(0)}(\boldsymbol{a}; z) \cdot y_1^{(0)}(\boldsymbol{a} + \boldsymbol{1}; z) - y_0^{(0)}(\boldsymbol{a} + \boldsymbol{1}; z) \cdot y_1^{(0)}(\boldsymbol{a}; z)$$

$$= \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_0 - b_1 + 1)\Gamma(a_1 - b_1 + 1)\Gamma(a_2 - b_1 + 1)}{\Gamma(b_1)\Gamma(1 - b_1)\Gamma(b_2 - a_0)\Gamma(b_2 - a_1)\Gamma(b_2 - a_2)}$$

$$\times z^{1-b_1-b_2}(1 - z)^{s(\boldsymbol{a})-1} \cdot y_2^{(0)}(\boldsymbol{a}^*; z), \qquad |z| < 1,$$

where $\boldsymbol{a}^*$ is defined in Table 3, while Okubo, Takano and Yoshida [21, Lemma 2] shows that

$$\lim_{z\uparrow 1}(1 - z)^{s(\boldsymbol{a})-1} \cdot y_2^{(0)}(\boldsymbol{a}^*; z) = \Gamma(s(\boldsymbol{a}) - 1), \qquad \operatorname{Re} s(\boldsymbol{a}) > 1.$$

It follows from these facts that $w(\boldsymbol{a}) := w(\boldsymbol{a}; 1)$ admits a representation

$$w(\boldsymbol{a}) = \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_0 - b_1 + 1)\Gamma(a_1 - b_1 + 1)\Gamma(a_2 - b_1 + 1)\Gamma(s(\boldsymbol{a}) - 1)}{\Gamma(b_1)\Gamma(1 - b_1)\Gamma(b_2 - a_0)\Gamma(b_2 - a_1)\Gamma(b_2 - a_2)}.$$

By the connection formula $y_0^{(\infty)}(\boldsymbol{a}) = C_0(\boldsymbol{a}) y_0^{(0)}(\boldsymbol{a}) + C_1(\boldsymbol{a}) y_1^{(0)}(\boldsymbol{a})$ in [10, formula (16)], where

$$C_0(\boldsymbol{a}) = \frac{e^{i\pi s(\boldsymbol{a})} \cdot \sin \pi a_1 \cdot \sin \pi a_2}{\sin \pi b_1 \cdot \sin \pi(b_2 - a_0)}, \qquad C_1(\boldsymbol{a}) = -\frac{e^{i\pi s(\boldsymbol{a})} \cdot \sin \pi(b_1 - a_1) \cdot \sin \pi(b_1 - a_2)}{\sin \pi b_1 \cdot \sin \pi(b_2 - a_0)},$$

and the periodicity $C_i(\boldsymbol{a} + \boldsymbol{1}) = C_i(\boldsymbol{a})$, $i = 0, 1$, we have $W(\boldsymbol{a}) = C_1(\boldsymbol{a}) w(\boldsymbol{a})$. This together with the reflection formula for the gamma function yields formula (124). $\square$

Theorem 7.2 *The initial value of the Casoratian $\omega(n)$ is given by*

$$\omega(0) = \frac{\pi^2 \cdot \rho(\boldsymbol{a}; \boldsymbol{k}) \cdot \Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(s(\boldsymbol{a}))}{t(\boldsymbol{a}) \prod_{i=0}^{2} \prod_{j=1}^{2} \Gamma(b_j - a_i + (l_j - k_i)_+)}, \tag{125}$$

where $\rho(\boldsymbol{a}; \boldsymbol{k}) \in \mathbb{Q}[\boldsymbol{a}]$ is the polynomial in (13) and $t(\boldsymbol{a}) := \sin \pi(b_1 - a_0) \cdot \sin \pi(b_2 - a_0)$.

Proof. From definitions (23a) and (26b) we find that

$$\omega(0) = f_0(0) \cdot g_1(0) - f_1(0) \cdot g_0(0) = {}_3f_2(\boldsymbol{a}) \cdot {}_3g_2(\boldsymbol{a}+\boldsymbol{k}) - {}_3f_2(\boldsymbol{a}+\boldsymbol{k}) \cdot {}_3g_2(\boldsymbol{a})$$

$$= y_0^{(0)}(\boldsymbol{a}) \, e^{-i\pi s(\boldsymbol{a}+\boldsymbol{k})} y_0^{(\infty)}(\boldsymbol{a}+\boldsymbol{k}) - y_0^{(0)}(\boldsymbol{a}+\boldsymbol{k}) \, e^{-i\pi s(\boldsymbol{a})} y_0^{(\infty)}(\boldsymbol{a})$$

$$= e^{-i\pi s(\boldsymbol{a})} \{ y_0^{(0)}(\boldsymbol{a}) \, y_0^{(\infty)}(\boldsymbol{a}+\boldsymbol{k}) - y_0^{(0)}(\boldsymbol{a}+\boldsymbol{k}) \, y_0^{(\infty)}(\boldsymbol{a}) \}$$

$$= e^{-i\pi s(\boldsymbol{a})} r(\boldsymbol{a}; \boldsymbol{k}) \{ y_0^{(0)}(\boldsymbol{a}) \, y_0^{(\infty)}(\boldsymbol{a}+\boldsymbol{1}) - y_0^{(0)}(\boldsymbol{a}+\boldsymbol{1}) \, y_0^{(\infty)}(\boldsymbol{a}) \} = e^{-i\pi s(\boldsymbol{a})} r(\boldsymbol{a}; \boldsymbol{k}) \, W(\boldsymbol{a}),$$

where the fourth equality follows from $s(\boldsymbol{k}) = 0$ and the fifth from the three-term relation

$$y_0^{(\nu)}(\boldsymbol{a}+\boldsymbol{k}) = r_1(\boldsymbol{a}; \boldsymbol{k}) \, y_0^{(\nu)}(\boldsymbol{a}) + r(\boldsymbol{a}; \boldsymbol{k}) \, y_0^{(\nu)}(\boldsymbol{a}+\boldsymbol{1}), \quad \nu = 0, \infty,$$

where $r_1(\boldsymbol{a}; \boldsymbol{k})$ and $r(\boldsymbol{a}; \boldsymbol{k})$ are the $(1,1)$ and $(1,2)$ entries of the connection matrix $A(\boldsymbol{a}; \boldsymbol{k})$ as in [10, formulas (33) and (34)]. Using formula (124) one has

$$\omega(0) = -r(\boldsymbol{a}; \boldsymbol{k}) \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(a_0 - b_1 + 1)\Gamma(a_0 - b_2 + 1)\Gamma(s(\boldsymbol{a}) - 1)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)\Gamma(b_2 - a_1)\Gamma(b_2 - a_2)}$$

$$= -\frac{\pi^2 \cdot r(\boldsymbol{a}; \boldsymbol{k}) \cdot \Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma(s(\boldsymbol{a}) - 1)}{t(\boldsymbol{a}) \prod_{i=0}^{2} \prod_{j=1}^{2} \Gamma(b_j - a_i)}$$

$$= \frac{\pi^2 \cdot \rho(\boldsymbol{a}; \boldsymbol{k}) \cdot \Gamma(a_0)\Gamma(a_1)\Gamma(a_2) \cdot \{s(\boldsymbol{a}) - 1\}\Gamma(s(\boldsymbol{a}) - 1)}{t(\boldsymbol{a}) \prod_{i=0}^{2} \prod_{j=1}^{2} (b_j - a_i; (l_j - k_i)_+) \cdot \prod_{i=0}^{2} \prod_{j=1}^{2} \Gamma(b_j - a_i)} = \text{RHS of (125)},$$

where the second equality follows from the reflection formula for the gamma function, the third from (13) and the final one from the recursion formula for the gamma function. $\square$

7.2 Error Estimates

We are now in a position to establish our main results in §3.2 by means of the general estimate (29) upon putting $X(n) = f(n)$ and $Y(n) = g(n)$. In this subsection, unless otherwise mentioned explicitly, Landau's symbols $O(\cdot)$ are uniform in any compact subset of

$$\mathbb{A} := \{\boldsymbol{a} \in \mathbb{C}^5 : \operatorname{Re} s(\boldsymbol{a}) > 0\}.$$

Proof of Theorem 3.2. In the straight case in Definition 2.2 the sequences in (23a) and (26b) are given by $f(n) = {}_3f_2(\boldsymbol{a} + n\boldsymbol{k})$ and $g(n) = {}_3g_2(\boldsymbol{a} + n\boldsymbol{k})$ respectively. Under the assumption of Theorem 3.2 we can use Theorems 4.3 and 6.1 with $\boldsymbol{p}$ replaced by $\boldsymbol{k}$ to get

$$f(n) = {}_3f_2(\boldsymbol{a} + n\boldsymbol{k}) = \Gamma(s(\boldsymbol{a})) \cdot s_2(\boldsymbol{k})^{-s(\boldsymbol{a})} \cdot n^{-2s(\boldsymbol{a})} \cdot \{1 + O(1/n)\},$$

$$g(n) = {}_3g_2(\boldsymbol{a} + n\boldsymbol{k}) = \frac{B(\boldsymbol{a}; \boldsymbol{k})}{t(\boldsymbol{a})} \cdot D(\boldsymbol{k})^n \cdot n^{-s(\boldsymbol{a}) - \frac{1}{2}} \cdot \left\{ 1 + O(n^{-\frac{1}{2}}) \right\},$$

39

where $D(\boldsymbol{k})$, $t(\boldsymbol{a})$ and $B(\boldsymbol{a};\boldsymbol{k})$ are given by (32) and (122) with $\boldsymbol{p}$ replaced by $\boldsymbol{k}$, and hence

$$R(n) = \frac{f(n+2)}{g(n+2)} = \frac{t(\boldsymbol{a}) \cdot \Gamma(s(\boldsymbol{a})) \cdot s_2(\boldsymbol{k})^{-s(\boldsymbol{a})}}{B(\boldsymbol{a};\boldsymbol{k}) \cdot D(\boldsymbol{k})^2} \cdot D(\boldsymbol{k})^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\}.$$

Combining this formula with (125) in Theorem 7.2, we have

$$\omega(0)\,R(n) = \rho(\boldsymbol{a};\boldsymbol{k}) \cdot e_{\mathrm{s}}(\boldsymbol{a};\boldsymbol{k}) \cdot \gamma(\boldsymbol{a};\boldsymbol{k}) \cdot D(\boldsymbol{k})^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\},$$

where $e_{\mathrm{s}}(\boldsymbol{a};\boldsymbol{k})$ and $\gamma(\boldsymbol{a};\boldsymbol{k})$ are defined in (37) and (34). To cope with the error term in formula (29), we also need to care about how $R(n) \cdot Y(0)/X(0)$ depends on $\boldsymbol{a} \in \mathbb{A}$. Observe that

$$\frac{R(n) \cdot Y(0)}{X(0)} = \frac{R(n) \cdot {}_3g_2(\boldsymbol{a})}{{}_3f_2(\boldsymbol{a})} = \psi_1(\boldsymbol{a}) \cdot \psi_2(\boldsymbol{a}) \cdot D(\boldsymbol{k})^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\},$$

$$\text{with} \quad \psi_1(\boldsymbol{a}) := \frac{\Gamma(s(\boldsymbol{a})) \cdot s_2(\boldsymbol{k})^{-s(\boldsymbol{a})}}{B(\boldsymbol{a};\boldsymbol{k}) \cdot D(\boldsymbol{k})^2}, \quad \psi_2(\boldsymbol{a}) := \frac{t(\boldsymbol{a}) \cdot {}_3g_2(\boldsymbol{a})}{{}_3f_2(\boldsymbol{a})}.$$

It is obvious that $\psi_1(\boldsymbol{a})$ is holomorphic in $\mathbb{A}$. It is also easy to see that $\psi_2(\boldsymbol{a})$ is holomorphic in $\mathbb{A}_0 := \{\boldsymbol{a} \in \mathbb{A} : {}_3f_2(\boldsymbol{a}) \neq 0\}$. Indeed ${}_3g_2(\boldsymbol{a})$ has a pole when $a_0 - b_1 + 1 \in \mathbb{Z}_{\leq 0}$ or $a_0 - b_2 + 1 \in \mathbb{Z}_{\leq 0}$ but the pole is canceled by a zero of $t(\boldsymbol{a}) = \sin\pi(b_1 - a_0) \cdot \sin\pi(b_2 - a_0)$; similarly ${}_3g_2(\boldsymbol{a})$ has a pole when $a_0 \in \mathbb{Z}_{\leq 0}$ but it is canceled by a pole of ${}_3f_2(\boldsymbol{a})$. Now estimate (29) leads to asymptotic formula (36), in which Landau's symbol is uniform in any compact subset of $\mathbb{A}_0$. $\square$

Proof of Theorem 3.3. In the twisted case in Definition 2.2, if $n = 3m + i$, $m \in \mathbb{Z}_{\geq 0}$, $i = 0, 1, 2$, then the sequences $f(n)$ in (23a) and $g(n)$ in (26b) are given by

$$f(3m+i) = {}_3f_2(\boldsymbol{a} + \boldsymbol{j}_i + m\boldsymbol{p}), \qquad g(3m+i) = {}_3g_2(\boldsymbol{a} + \boldsymbol{j}_i + m\boldsymbol{p}),$$

where $\boldsymbol{j}_0 = \boldsymbol{0}$, $\boldsymbol{j}_1 = \boldsymbol{k}$, $\boldsymbol{j}_2 = \boldsymbol{l}$ and $\boldsymbol{p}$ is the shift vector in formula (20).

Observe that $\boldsymbol{p}$ belongs to $\mathcal{S}(\mathbb{Z})$ and satisfies condition (121), if and only if the seed vector $\boldsymbol{k} \in \mathbb{Z}^5$ fulfills condition (39). Indeed, since $p_0 = p_1 = p_2 = l_1 + l_2$, $q_1 = 3l_1$ and $q_2 = 3l_2$, the inequalities in (30) becomes $l_1 + l_2 < 3l_1 \leq 3l_2 < 2(l_1 + l_2)$, which is equivalent to $l_1 \leq l_2 < 2l_1$. Case (a) in condition (121) now reads $\Delta(\boldsymbol{p}) = -27(l_2^2 - 4l_1 l_2 + l_1^2)(l_2^2 + 2l_1 l_2 - 2l_1^2)(2l_2^2 - 2l_1 l_2 - l_1^2) \leq 0$, which together with $l_1 \leq l_2 < 2l_1$ yields $l_1 \leq l_2 \leq \tau l_1$ in condition (39), where $\tau = (1 + \sqrt{3})/2$. On the other hand, case (b) in condition (121) becomes $l_2^2 - 10l_1 l_2 + 7l_1^2 \geq 0$, that is, $l_2 \leq (5 - 3\sqrt{2})l_1$ or $l_2 \geq (5 + 3\sqrt{2})l_1$, but neither of which is possible when $l_1 \leq l_2 < 2l_1$.

Thus under the assumption of Theorem 3.3 one can apply Theorems 4.3 and 6.1 to the shift vector $\boldsymbol{p}$ in (20) with $\boldsymbol{a}$ replaced by $\boldsymbol{a} + \boldsymbol{j}_i$ to obtain

$$f(3m+i) = {}_3f_2(\boldsymbol{a} + \boldsymbol{j}_i + m\boldsymbol{p}) = \Gamma(s(\boldsymbol{a})) \cdot s_2(\boldsymbol{p})^{-s(\boldsymbol{a})} \cdot m^{-2s(\boldsymbol{a})} \cdot \{1 + O(1/m)\},$$

$$g(3m+i) = {}_3g_2(\boldsymbol{a} + \boldsymbol{j}_i + m\boldsymbol{p}) = \frac{B(\boldsymbol{a} + \boldsymbol{j}_i;\boldsymbol{p})}{t(\boldsymbol{a} + \boldsymbol{j}_i)} \cdot D(\boldsymbol{p})^m \cdot m^{-s(\boldsymbol{a})-\frac{1}{2}} \cdot \left\{1 + O(m^{-\frac{1}{2}})\right\},$$

where $s(\boldsymbol{j}_i) = 0$ is also used. Substituting the settings (20) and (21) into definitions (32) and (122) and taking $s(\boldsymbol{k}) = 0$ into account, one has $D(\boldsymbol{p}) = E(l_1, l_2)^3$ and

$$t(\boldsymbol{a} + \boldsymbol{j}_i) = (-1)^{i(l_1+l_2)} \cdot t(\boldsymbol{a}), \quad B(\boldsymbol{a} + \boldsymbol{j}_i;\boldsymbol{p}) = (-1)^{i(l_1+l_2)} \cdot B(\boldsymbol{a};\boldsymbol{p}) \cdot E(l_1, l_2)^i, \quad i = 0, 1, 2,$$

40

where $E(l_1, l_2)$ is defined in (41). These formulas and $s_2(\boldsymbol{p}) = 3(l_1^2 - l_1 l_2 + l_2^2)$ lead to

$$f(n) = 3^{s(\boldsymbol{a})} \cdot \Gamma(s(\boldsymbol{a})) \cdot (l_1^2 - l_1 l_2 + l_2^2)^{-s(\boldsymbol{a})} \cdot n^{-2s(\boldsymbol{a})} \cdot \{1 + O(1/n)\},$$

$$g(n) = 3^{s(\boldsymbol{a})+\frac{1}{2}} \cdot \frac{B(\boldsymbol{a};\boldsymbol{p})}{t(\boldsymbol{a})} \cdot E(l_1, l_2)^n \cdot n^{-s(\boldsymbol{a})-\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\},$$

so the ratio $R(n) = f(n+2)/g(n+2)$ is estimated as

$$R(n) = \frac{t(\boldsymbol{a}) \cdot \Gamma(s(\boldsymbol{a})) \cdot (l_1^2 - l_1 l_2 + l_2^2)^{-s(\boldsymbol{a})}}{3^{\frac{1}{2}} \cdot B(\boldsymbol{a};\boldsymbol{p}) \cdot E(l_1, l_2)^2} \cdot E(l_1, l_2)^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\}.$$

Substituting $\boldsymbol{p} = (l_1 + l_2, l_1 + l_2, l_1 + l_2; 3l_1, 3l_2)$ into definition (122) yields

$$B(\boldsymbol{a};\boldsymbol{p}) = \frac{\pi^{\frac{1}{2}}(l_1 + l_2)^{a_0+a_1+a_2-\frac{3}{2}} \cdot 3^{s(\boldsymbol{a})-1} \cdot (l_1^2 - l_1 l_2 + l_2^2)^{s(\boldsymbol{a})-1}}{2^{\frac{3}{2}} \cdot (2l_1 - l_2)^{2b_1-b_2+s(\boldsymbol{a})-\frac{3}{2}}(2l_2 - l_1)^{2b_2-b_1+s(\boldsymbol{a})-\frac{3}{2}}},$$

which is put together with formula (125) in Theorem 7.2 to give

$$\omega(0)\,R(n) = \rho(\boldsymbol{a};\boldsymbol{k}) \cdot e_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k}) \cdot \gamma(\boldsymbol{a};\boldsymbol{k}) \cdot E(l_1, l_2)^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\},$$

where $e_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k})$ and $\gamma(\boldsymbol{a};\boldsymbol{k})$ are given in (42) and (34). The treatment of $R(n) \cdot Y(0)/X(0)$ is similar to the one in the straight case and the estimate (29) leads to asymptotic formula (40), in which Landau's symbol is uniform in any compact subset of $\mathbb{A}_0$. $\qquad\square$

8   Back to Original Series and Specializations

Theorems 3.2 and 3.3 are stated in terms of the renormalized series ${}_3f_2(\boldsymbol{a})$. It is interesting to reformulate them in terms of the original series ${}_3F_2(1)$. Multiplying equations (36) and (40) by

$$\frac{\Gamma(b_1+l_1)\Gamma(b_2+l_2)}{\Gamma(a_0+k_0)\Gamma(a_1+k_1)\Gamma(a_2+k_2)} \cdot \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_1)\Gamma(b_2)} = \frac{(b_1;\,l_1)(b_2;\,l_2)}{(a_0;\,k_0)(a_1;\,k_1)(a_2;\,k_2)}$$

and using relation (9) between ${}_3f_2(\boldsymbol{a})$ and ${}_3F_2(\boldsymbol{a})$, we find that

$$\frac{{}_3F_2(\boldsymbol{a}+\boldsymbol{k})}{{}_3F_2(\boldsymbol{a})} - \underset{j=0}{\overset{n}{\mathbf{K}}} \frac{r^*(j)}{q^*(j)} = \frac{c_{\mathrm{s}}^*(\boldsymbol{a};\boldsymbol{k})}{{}_3F_2(\boldsymbol{a})^2} \cdot D(\boldsymbol{k})^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\}, \tag{126a}$$

$$\frac{{}_3F_2(\boldsymbol{a}+\boldsymbol{k})}{{}_3F_2(\boldsymbol{a})} - \underset{j=0}{\overset{n}{\mathbf{K}}} \frac{r^*(j)}{q^*(j)} = \frac{c_{\mathrm{t}}^*(\boldsymbol{a};\boldsymbol{k})}{{}_3F_2(\boldsymbol{a})^2} \cdot E(l_1, l_2)^{-n} \cdot n^{-s(\boldsymbol{a})+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\}, \tag{126b}$$

as $n \to +\infty$, where the quantities marked with an asterisk are defined by

$$q^*(0) := u(\boldsymbol{a})\textstyle\prod_{i=0}^{2}(a_i;\,k_i), \qquad q^*(n) := q(n), \quad n \geq 1,$$

$$r^*(0) := (b_1;\,l_1)(b_2;\,l_2), \qquad r^*(1) := v(\boldsymbol{a})\textstyle\prod_{i=0}^{2}(a_i;\,k_i), \qquad r^*(n) := r(n), \quad n \geq 2,$$

$$c_{\iota}^*(\boldsymbol{a};\boldsymbol{k}) := \rho(\boldsymbol{a};\boldsymbol{k}) \cdot e_{\iota}(\boldsymbol{a};\boldsymbol{k}) \cdot \gamma^*(\boldsymbol{a};\boldsymbol{k}), \qquad \iota = \mathrm{s}, \mathrm{t}, \tag{127}$$

41

with $\rho(\boldsymbol{a};\boldsymbol{k})$ and $e_\iota(\boldsymbol{a};\boldsymbol{k})$ unaltered while

$$\gamma^*(\boldsymbol{a};\boldsymbol{k}) := \frac{\Gamma(b_1+l_1)\Gamma(b_2+l_2)\Gamma(b_1)\Gamma(b_2)\Gamma^2(s(\boldsymbol{a}))}{\Gamma(a_0+k_0)\Gamma(a_1+k_1)\Gamma(a_2+k_2)\prod_{i=0}^2\prod_{j=1}^2\Gamma(b_j-a_i+(l_j-k_i)_+)}.$$

It follows from (9) that $\mathbb{A}_0^* := \{\boldsymbol{a}\in\mathbb{A} : b_1, b_2 \notin \mathbb{Z}_{\leq 0},\ {}_3F_2(\boldsymbol{a})\neq 0\} \subset \mathbb{A}_0$, where $\mathbb{A}$ and $\mathbb{A}_0$ are defined in §7.2, so Landau's symbols in (126) are uniform in any compact subset of $\mathbb{A}_0^*$.

Take an index $\lambda \in \{0,1,2\}$ such that $k_\lambda > 0$ and put $\{\lambda,\mu,\nu\} = \{0,1,2\}$. For any nonzero vector $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^5$ with $s(\boldsymbol{k}) = 0$ such an index λ always exists since $k_0 + k_1 + k_2 = l_1 + l_2 > 0$. In formulas (126) take the limit $a_\lambda \to 0$ and make the substitutions $a_i \mapsto a_i - k_i$, $b_j \mapsto b_j - l_j$ for $i = \mu,\nu$ and $j = 1, 2$. This procedure is referred to as the λ-th *specialization*. If this is well defined then ${}_3F_2(\boldsymbol{a}) \to 1$ as $a_\lambda \to 0$ so formulas (126a) and (126b) lead to

$$_3F_2\begin{pmatrix} k_\lambda, & a_\mu, & a_\nu \\ & b_1, & b_2 \end{pmatrix} - \mathbf{K}_{j=0}^n \frac{\hat{r}(j)}{\hat{q}(j)} = \hat{c}_{\mathrm{s}}(\boldsymbol{a};\boldsymbol{k}) \cdot D(\boldsymbol{k})^{-n} \cdot n^{-\hat{s}+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\}, \qquad (128\mathrm{a})$$

$$_3F_2\begin{pmatrix} k_\lambda, & a_\mu, & a_\nu \\ & b_1, & b_2 \end{pmatrix} - \mathbf{K}_{j=0}^n \frac{\hat{r}(j)}{\hat{q}(j)} = \hat{c}_{\mathrm{t}}(\boldsymbol{a};\boldsymbol{k}) \cdot E(l_1,l_2)^{-n} \cdot n^{-\hat{s}+\frac{1}{2}} \cdot \left\{1 + O(n^{-\frac{1}{2}})\right\}, \qquad (128\mathrm{b})$$

where $\hat{q}(n)$ and $\hat{r}(n)$ are derived from $q^*(n)$ and $r^*(n)$, while $\hat{c}_\iota(\boldsymbol{a};\boldsymbol{k}) := \hat{\rho}(\boldsymbol{a};\boldsymbol{k})\cdot\hat{e}_\iota(\boldsymbol{a};\boldsymbol{k})\cdot\hat{\gamma}(\boldsymbol{a};\boldsymbol{k})$, $\iota = \mathrm{s},\mathrm{t}$, are obtained from (127) through the specialization; in particular one has

$$\hat{\gamma}(\boldsymbol{a};\boldsymbol{k}) := \frac{\Gamma(b_1)\Gamma(b_2)\Gamma^2(\hat{s})}{\Gamma(k_\lambda)\Gamma(a_\mu)\Gamma(a_\nu)\prod\limits_{j=1,2}(b_j-l_j;\ (l_j-k_\lambda)_+)\cdot\prod\limits_{i=\mu,\nu}\prod\limits_{j=1,2}\Gamma(b_j-a_i+(k_i-l_j)_+)},$$

with $\hat{s} := b_1 + b_2 - a_\mu - a_\nu - k_\lambda$. Landau's symbols in (128) are uniform in compact subsets of

$$\hat{\mathbb{A}} := \{\,(a_\mu, a_\nu; b_1, b_2) \in \mathbb{C}^4 : \operatorname{Re}\hat{s} > 0,\ b_1, b_2 \notin \mathbb{Z}_{\leq 0}\,\}.$$

The specialization is indeed well defined. It follows from [10, Proposition 4.9] and Lemma 2.1 that for any $\boldsymbol{c} \in \mathbb{Q}^5$ the restriction $\rho(\boldsymbol{a}+\boldsymbol{c};\boldsymbol{k})\big|_{a_0=a_1=a_2=0}$ is a nonzero polynomial in $\mathbb{Q}[b_1, b_2]$ and hence $\rho(\boldsymbol{a}+\boldsymbol{c};\boldsymbol{k})\big|_{a_\lambda=0}$ is a nonzero polynomial in $\mathbb{Q}[a_\mu, a_\nu, b_1, b_2]$. This is also the case for $\sigma(\boldsymbol{k})$ and $\boldsymbol{l}$ in place of $\boldsymbol{k}$ in §2.2. Thus formula (19a) implies that the specialization for $q^*(0)$,

$$\hat{q}(0) := \lim_{a_\lambda \to 0} u(\boldsymbol{a})\prod_{i=0}^2 (a_i;\ k_i) \quad \text{followed by} \quad a_i \mapsto a_i - k_i,\ b_j \mapsto b_j - l_j,\quad i = \mu, \nu,\ j = 1, 2,$$

is well defined and the ensuing $\hat{q}(0)$ is a nontrivial rational function in $\mathbb{Q}(a_\mu, a_\nu, b_1, b_2)$. In a similar manner formula (19b) tells us that the specialization for $r^*(1)$, that is, $\hat{r}(1) \in \mathbb{Q}(a_\mu, a_\nu, b_1, b_2)$ is well defined and nontrivial. The specialization for $q^*(n)$ with $n \geq 1$ is also well defined, since $q^*(n) = q(n)$ is of the form ${}^{\sigma^i}u(\boldsymbol{a}+\boldsymbol{c})$, where $i \in \{0,1,2\}$ and $\boldsymbol{c}$ is a vector in $\mathbb{Z}_{\geq 0}^5$ whose λ-th upper component, say c_λ, is positive, in which case one can take $\lim_{a_\lambda\to 0}{}^{\sigma^i}u(\boldsymbol{a}+\boldsymbol{c})$ without trouble, because the critical factorial $(a_\lambda;\ k_\lambda) \to 0$ in the denominator of (19a) is now replaced by a safe one $(a_\lambda + c_\lambda;\ k_\lambda) \to (c_\lambda;\ k_\lambda) \neq 0$. The resulting $\hat{q}(n)$ is nontrivial in $\mathbb{Q}(a_\mu, a_\nu, b_1, b_2)$. A similar argument can be made for $\hat{r}(n)$ with $n \geq 2$. Thus the procedure of specialization is well defined over the rational function field $\mathbb{Q}(a_\mu, a_\nu, b_1, b_2)$.

$$q_0(n) := \frac{(3n + b_1)(3n + b_2) - (2n + a_0)(2n + a_2)}{(2n + a_0)(2n + a_1)} \qquad (n \geq 0),$$

$$q_1(n) := \frac{(3n + b_1 + 1)(3n + b_2 + 1) - (2n + a_0 + 1)(2n + a_1 + 1)}{(2n + a_1 + 1)(2n + a_2)} \qquad (n \geq 0),$$

$$q_2(n) := \frac{(3n + b_1 + 2)(3n + b_2 + 2) - (2n + a_1 + 2)(2n + a_2 + 1)}{(2n + a_0 + 1)(2n + a_2 + 1)} \qquad (n \geq 0),$$

$$r_0(n) := -\frac{(n + b_1 - a_2)(n + b_2 - a_2)}{(2n + a_0 - 1)(2n + a_2 - 1)} \qquad (n \geq 1),$$

$$r_1(n) := -\frac{(n + b_1 - a_0)(n + b_2 - a_0)}{(2n + a_0)(2n + a_1)} \qquad (n \geq 0),$$

$$r_2(n) := -\frac{(n + b_1 - a_1)(n + b_2 - a_1)}{(2n + a_1 + 1)(2n + a_2)} \qquad (n \geq 0).$$

Table 4: Partial denominators and numerators in Example 9.1 with $r_0(0) := 1$.

9 Some Examples

To illustrate Theorems 3.2 and 3.3 we present a couple of the simplest examples.

Example 9.1 The simplest example of twisted type is given by

$$\boldsymbol{k} = \begin{pmatrix} 1, & 1, & 0 \\ & 1, & 1 \end{pmatrix}, \qquad \boldsymbol{l} = \begin{pmatrix} 1, & 2, & 1 \\ & 2, & 2 \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} 2, & 2, & 2 \\ & 3, & 3 \end{pmatrix},$$

together with $\sigma(a_0, a_1, a_2; b_1, b_2) = (a_2, a_0, a_1; b_1, b_2)$. The recipe described in §2.1 readily yields $\rho(\boldsymbol{a}; \boldsymbol{l}) = b_1 b_2 - a_0 a_2$ and $\rho(\boldsymbol{a}; \boldsymbol{k}) = \rho(\boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k})) = 1$, so formula (19) yields

$$u(\boldsymbol{a}) = \frac{b_1 b_2 - a_0 a_2}{a_0 a_1}, \qquad v(\boldsymbol{a}) = -\frac{(b_1 - a_0)(b_2 - a_0)}{a_0 a_1}.$$

Thus the partial denominators and numerators of the continued fraction in Theorem 3.3 are given as in Table 4, so the continued fraction for ${}_3f_2(\boldsymbol{a} + \boldsymbol{k})/{}_3f_2(\boldsymbol{a})$ is well defined when

$$a_0, a_1, a_2 \notin \mathbb{Z}_{\leq 0}; \qquad b_j - a_i \notin \mathbb{Z}_{\leq 0}, \quad b_j - a_2 \notin \mathbb{Z}_{\leq -1}, \quad i = 0, 1, \ j = 1, 2.$$

In the error estimate (40) we have $E(l_1, l_2) = E(1, 1) = 4$ and

$$c_t(\boldsymbol{a}; \boldsymbol{k}) = \frac{\pi^{\frac{3}{2}} \cdot \Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma^2(s(\boldsymbol{a}))}{2^{a_0 + a_1 + a_2 + 1} \cdot 3^{s(\boldsymbol{a}) - \frac{1}{2}} \prod_{j=1}^{2} \prod_{i=0}^{2} \Gamma(b_j - a_i + \delta_{i2})}.$$

Passing to the continued fraction for ${}_3F_2(\boldsymbol{a} + \boldsymbol{k})/{}_3F_2(\boldsymbol{a})$, we have $q_0^*(0) = b_1 b_2 - a_0 a_2$, $r_0^*(0) = b_1 b_2$, $r_1^*(0) = -(b_1 - a_0)(b_2 - a_0)$ and $q_i^*(n) = q_i(n)$, $r_i^*(n) = r_i(n)$ for all other (i, n) in formula (126b). The 0-th specialization of (126b) then leads to $\hat{q}_0(0) = \hat{r}_0(0) = (b_1 - 1)(b_2 - 1)$, $\hat{r}_1(0) = -(b_1 - 1)(b_2 - 1)$ in formula (128b) while all other $\hat{q}_i(n)$ and $\hat{r}_i(n)$ are given as in Table 2, where circumflex "^" is deleted for the sake of simplicity. Clearly we can make $\hat{q}_0(0) = \hat{r}_0(0) = 1$, $\hat{r}_1(0) = -1$ up to equivalence of continued fractions and we have established Theorem 1.1.

Example 9.2 The next simplest example of twisted type is given by

$$\boldsymbol{k} = \begin{pmatrix} 2, & 0, & 0 \\ & 1, & 1 \end{pmatrix}, \qquad \boldsymbol{l} = \begin{pmatrix} 2, & 2, & 0 \\ & 2, & 2 \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} 2, & 2, & 2 \\ & 3, & 3 \end{pmatrix},$$

where $\boldsymbol{p}$ and σ are the same as in Example 9.1. In this case the recipe in §2.1 gives

$$\rho(\boldsymbol{a}; \boldsymbol{l}) = b_1 b_2 + a_0 a_1 - (a_2 - 1)(a_0 + a_1 + 1),$$
$$\rho(\boldsymbol{a}; \boldsymbol{k}) = b_1 b_2 - a_1 a_2 - (a_0 + 1)(b_1 + b_2 - a_1 - a_2),$$
$$\rho(\boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k})) = (b_1 + 1)(b_2 + 1) - (a_0 + 2)a_2 - (a_1 + 1)(b_1 + b_2 - a_0 - a_2).$$

With these data the formula (19) yields

$$u(\boldsymbol{a}) = \frac{(b_1 - a_1)(b_2 - a_1) \cdot \rho(\boldsymbol{a}; \boldsymbol{l})}{a_0(a_0 + 1) \cdot \rho(\boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k}))}, \qquad v(\boldsymbol{a}) = -\frac{(b_1 - a_2 + 1)(b_2 - a_2 + 1) \cdot \rho(\boldsymbol{a}; \boldsymbol{k})}{a_0(a_0 + 1) \cdot \rho(\boldsymbol{a} + \boldsymbol{k}; \sigma(\boldsymbol{k}))}.$$

In the error estimate (40) in Theorem 3.3 we have $E(l_1, l_2) = E(1, 1) = 4$ and

$$c_{\mathrm{t}}(\boldsymbol{a}; \boldsymbol{k}) = \frac{\pi^{\frac{3}{2}} \cdot \rho(\boldsymbol{a}; \boldsymbol{k}) \cdot \Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma^2(s(\boldsymbol{a}))}{2^{a_0 + a_1 + a_2 + 1} \cdot 3^{s(\boldsymbol{a}) - \frac{1}{2}} \cdot \prod_{j=1}^2 \Gamma(b_j - a_0) \cdot \prod_{i=1}^2 \prod_{j=1}^2 \Gamma(b_j - a_i + 1)}.$$

The 0-th specialization leads to a continued fraction expansion (128b) for $_3F_2(2, a_1, a_2; b_1, b_2)$.

Example 9.3 The simplest example of straight type is given by $\boldsymbol{k} = (2, 2, 2; 3, 3)$, $\boldsymbol{l} = 2\boldsymbol{k}$ and $\boldsymbol{p} = 3\boldsymbol{k}$. The recipe in §2.1 shows that $\rho(\boldsymbol{a}; \boldsymbol{k}) = a_0 a_1 a_2(b_1 + b_2 + 1) + b_1 b_2\{s(\boldsymbol{a}) - s_2(\boldsymbol{a})\}$ and $\rho(\boldsymbol{a}; 2\boldsymbol{k})$ is a polynomial of degree 10 (explicit formula is omitted). Formula (38) yields

$$u(\boldsymbol{a}) = \frac{\rho(\boldsymbol{a}; 2\boldsymbol{k})}{\rho(\boldsymbol{a} + \boldsymbol{k}; \boldsymbol{k}) \prod_{i=0}^2 a_i(a_i + 1)}, \qquad v(\boldsymbol{a}) = -\frac{\rho(\boldsymbol{a}; \boldsymbol{k}) \prod_{i=0}^2 \prod_{j=1}^2 (b_j - a_i + 1)}{\rho(\boldsymbol{a} + \boldsymbol{k}; \boldsymbol{k}) \prod_{i=0}^2 a_i(a_i + 1)}.$$

In the error estimate (36) in Theorem 3.2 we have $D(\boldsymbol{k}) = 2^6 = 64$ and

$$c_{\mathrm{s}}(\boldsymbol{a}; \boldsymbol{k}) = \frac{\pi^{\frac{3}{2}} \cdot \rho(\boldsymbol{a}; \boldsymbol{k}) \cdot \Gamma(a_0)\Gamma(a_1)\Gamma(a_2)\Gamma^2(s(\boldsymbol{a}))}{2^{a_0 + a_1 + a_2 + 9} \cdot 3^{2s(\boldsymbol{a}) - 1} \cdot \prod_{i=0}^2 \prod_{j=1}^2 \Gamma(b_j - a_i + 1)}.$$

References

[1] G.E. Andrews, R. Askey, and R. Roy. *Special Functions*. Cambridge Univ. Press, Cambridge, 1999.

[2] T. Aoki and M. Tanda. Parametric Stokes phenomena of the Gauss hypergeometric differential equation with a large parameter. *J. Math. Soc. Japan*, 68(3):1099–1132, 2016.

[3] M. Asakura, N. Otsubo, and T. Terasoma. An algebro-geometric study of special values of hypergeometric functions $_3F_2$. e-Print arXiv:1603.04558v2[math.NT].

[4] W.N. Bailey. *Generalized Hypergeometric Series*. Cambridge Univ. Press, Cambridge, 1935.

[5] W.N. Bailey. Contiguous hypergeometric functions of the type $_3F_2(1)$. *Proc. Glasgow Math. Assoc.*, 2:62–65, 1954-1956.

[6] B.C. Berndt, R.L. Lamphere, and B.M. Wilson. Chapter 12 of Ramanujan's second notebook: continued fractions. *Rocky Mountain J. Math.*, 15(2):235–310, 1985.

[7] J.M. Borwein, K.-K.S. Choi, and W. Pigulla. Continued fractions of tails of hypergeometric series. *Amer. Math. Monthly*, 112(6):493–501, 2005.

[8] M. Colman, A. Cuyt, and J. Van Deun. Validated computation of certain hypergeometric functions. *ACM Trans. Math. Softw.*, 38(2):Article 11, 20 pages, 2011.

[9] R.Y. Denis and S.N. Singh. Hypergeometric functions and continued fractions. *Far East J. Math. Sci.*, 2(3):385–400, 2000.

[10] A. Ebisu and K. Iwasaki. Three-term relations for $_3F_2(1)$. *J. Math. Anal. Appl.*, 463(2):593–610, 2018. https://doi.org/10.1016/j.jmaa.2018.03.034.

[11] J.L. Fields. Asymptotic expansions of a class of hypergeometric polynomials with respect to the order. Ⅲ. *J. Math. Anal. Appl.*, 12:593–601, 1965.

[12] E. Frank. A new class of continued fraction expansions for the ratios of hypergeometric functions. *Trans. Amer. Math. Soc.*, 81(2):453–476, 1956.

[13] C.F. Gauss. Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 +$ etc. In *Werke*, volume 3, pages 123–162. Göttingen, 1876.

[14] W. Gautschi. Computational aspects of three-term recurrence relations. *SIAM Rev.*, 9:24–82, 1967.

[15] A. Gil, J. Segura, and N.M. Temme. *Numerical Methods for Special Functions*. SIAM, Philadelphia, 2007.

[16] K. Iwasaki. Hypergeometric series with gamma product formula. *Indag. Math. (N.S.)*, 28(2):463–493, 2017.

[17] W.B. Jones and W. Thron. *Continued Fractions. Analytic Theory and Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1980.

[18] S. Farid Khwaja and A.B. Olde Daalhuis. Uniform asymptotic expansions for hypergeometric functions with large parameters IV. *Anal. Appl. (Singap.)*, 12(6):667–710, 2014.

[19] C. Krattenthaler and T. Rivoal. How can we escape Thomae's relations? *J. Math. Soc. Japan*, 58(1):183–210, 2006.

[20] Y.L. Luke. *The Special Functions and Their Approximations, Vol. I.* Academic Press, New York, London, 1969.

[21] K. Okubo, K. Takano, and S. Yoshida. A connection problem for the generalized hypergeometric function. *Funkcial. Ekvac.*, 31:483–495, 1988.

[22] R.B. Paris. Asymptotics of the Gauss hypergeometric function with large parameters. II. *J. Classical Analysis*, 3:1–15, 2013.

[23] S. Pincherle. Delle funzioni ipergeometriche e di varie questioni ad esse attienti. *Giorn. Mat. Battaglini*, 32:209–291, 1894.

[24] S. Ramanujan. *Notebooks*, volume 2. Tata Institute of Fundamental Research, Bombay, 1957.

[25] N.M. Temme. Large parameter cases of the Gauss hypergeometric function. *J. Comput. Appl. Math.*, 153(1-2):441–462, 2003.

[26] E.B. Van Vleck. On the convergence of the continued fraction of Gauss and other continued fractions. *Ann. Math. (2)*, 3:1–18, 1901.

[27] J.A. Wilson. Three-term contiguous relations and some new orthogonal polynomials. In E.B. Saff and R.S. Varga, editors, *Padé and Rational Approximations*, pages 227–232, New York, 1977. Academic Press.

[28] T. Yabu. Algebro-geometric study of special values at $x = 1$ of the generalized hypergeometric function $_3F_2$. Master's thesis, Hokkaido University, March 2017. in Japanese.

[29] L.-C. Zhang. Ramanujan's continued fractions for products of gamma functions. *J. Math. Anal. Appl.*, 174(1):22–52, 1993.