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On the Law of Free Subordinators

Octavio Arizmendi, Takahiro Hasebe and Noriyoshi Sakuma

Universität des Saarlandes, FR 6.1−Mathematik, 66123 Saarbrücken, Germany
E-mail address: arizmendi@math.uni-sb.de

Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: thasebe@math.kyoto-u.ac.jp

Department of Mathematics, Aichi University of Education
1 Hirosawa, Igaya-cho, Kariya-shi, 448-8542, Japan
E-mail address: sakuma@auecc.aichi-edu.ac.jp

Abstract. We study the freely infinitely divisible distributions that appear as the laws of free subordinators. This is the free analog of classically infinitely divisible distributions supported on $[0, \infty)$, called the free regular measures. We prove that the class of free regular measures is closed under the free multiplicative convolution, t^{th} boolean power for $0 \leq t \leq 1$, t^{th} free multiplicative power for $t \geq 1$ and weak convergence. In addition, we show that a symmetric distribution is freely infinitely divisible if and only if its square can be represented as the free multiplicative convolution of a free Poisson and a free regular measure. This gives two new explicit examples of distributions which are infinitely divisible with respect to both classical and free convolutions: $\chi^2(1)$ and $F(1, 1)$. Another consequence is that the free commutator operation preserves free infinite divisibility.

1. Introduction

A one dimensional subordinator $(X_t)_{t\geq 0}$ is a Lévy process whose increments are always nonnegative. The marginal distributions $(\mu_t)_{t\geq 0}$ of a subordinator $(X_t)_{t\geq 0}$ are infinitely divisible and their Lévy-Khintchine representations have *regular* forms for any $t \geq 0$:

$$\mathcal{C}^*_{\mu_t}(z) := \log\left(\int_{\mathbb{R}} e^{izx}\mu_t(dx)\right) = it\eta' z + t\int_{(0,\infty)} (e^{izx} - 1)\nu(dx), \tag{1.1}$$

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where the drift term η' satisfies $\eta' \geq 0$ and the Lévy measure ν satisfies $\int_{(0,\infty)}(1 \wedge x)\nu(dx) < \infty$ and $\nu((-\infty, 0]) = 0$. Poisson processes, positive stable processes and Gamma processes are typical examples. Subordinators have been broadly studied, see for example, Bertoin (1999) and Sato (1999). For applications in financial modeling, see Cont and Tankov (2004). A matrix valued extension has been considered in Barndorff-Nielsen and Perez-Abreu (2008).

A crucial property is that the class of infinitely divisible distributions with regular Lévy-Khintchine representations is closed under $*$-convolution powers, where $*$ denotes classical convolution. Namely, a $*$-infinitely divisible distribution μ has a regular Lévy-Khintchine representation if and only if $\mu_t = \mu^{*t}$ is concentrated on $[0, \infty)$ for all $t > 0$. See for details Theorem 24.11 in p.146 of the book by Sato (1999).

In free probability theory, free convolution or $\boxplus$-convolution was introduced by Voiculescu (1986) in order to describe the sum of free random variables. The main analytic tool for the study of free convolution is the so-called Voiculescu's R-transform or free cumulant transform, denoted here by $\mathcal{C}^{\boxplus}_{\mu}(z)$. The basic property of the free cumulant transform is that it linearizes free convolution:

$$\mathcal{C}^{\boxplus}_{\mu\boxplus\rho}(z) = \mathcal{C}^{\boxplus}_{\mu}(z) + \mathcal{C}^{\boxplus}_{\rho}(z).$$

Similarly to the classical case, one can define free Lévy processes and free infinite divisibility with respect to free convolution. One obtains the corresponding Lévy-Khintchine representation for the free cumulant transform. This representation is also given in terms of a characteristic triplet (η, a, ν) that satisfies the same properties as in the classical case. This produces a bijection Λ, first introduced by Bercovici and Pata (1999), between classically and freely infinitely divisible distributions.

In this context, we can also define the free counterpart of laws of subordinators, that is $\rho_t = \Lambda(\mu_t)$, where μ_t has the regular form (1.1). The free cumulant transforms (see Barndorff-Nielsen and Thorbjornsen (2006)) of the laws $(\rho_t)_{t\geq 0} = (\rho^{\boxplus t})_{t\geq 0}$ have the *free regular representations*

$$\mathcal{C}^{\boxplus}_{\rho_t}(z) = t\eta' z + t\int_{\mathbb{R}} \left(\frac{1}{1 - zx} - 1\right) \nu(dx), \quad z \in \mathbb{C}_-, \tag{1.2}$$

where (η', ν) is the pair of (1.1) with the same conditions: $\eta' \geq 0$, $\int_{(0,\infty)}(1 \wedge x)\nu(dx) < \infty$ and $\nu((-\infty, 0]) = 0$. It is readily seen that this class is closed under the convolution $\boxplus$.

Let us note here an important difference between classically and freely infinitely divisible distributions on the cone $[0, \infty)$. Any classically infinitely divisible distribution μ supported on $[0, \infty)$ satisfies that $\mu_t = \mu^{*t}$ is concentrated on $[0, \infty)$ for all time $t > 0$, and thus has a regular representation. However, there exists a freely infinitely divisible distribution μ on $[0, \infty)$ such that $\mu_t = \mu^{\boxplus t}$ is not supported on $[0, \infty)$ for all time $t > 0$. For example, the semicircle distribution with mean 2 and variance 1. If we construct a free Lévy process from this distribution, the laws μ_t for $t \geq 1$ concentrate on $[0, \infty)$ but do not for $0 < t < 1$, see Sakuma (2011) for more details. Thus, in this sense the correct counterpart of the class of $*$-infinitely divisible distributions supported on $[0, \infty)$ is the class of free regular measures.

The main purpose of this paper is to show strong closure properties of the class of free regular measures under different convolutions as well as several important consequences. More specifically, we prove that the class of free regular measures is closed not only under free additive convolution $\boxplus$ but also under free multiplicative convolution $\boxtimes$ and boolean convolution powers.

As a first important consequence, we characterize the laws of free subordinators in terms of free regularity. More precisely, $(Z_t)_{t\geq 0}$ is a free Lévy process such that the distribution of $Z_t - Z_s$ has non-negative support if and only if the law Z_1 is free regular.

As a second important consequence, if X and Y are two free independent random variables with free regular distributions, then $X^{1/2}YX^{1/2}$ also follows a free regular distribution, which is not true in the classical case. See Example 11.3 in Chapter 2 of the book by Steutel and Van Harn (2004).

Other results and the organization of this paper are as follows. First, we state the main theorems in Section 2. In Section 3 we review some basic theory of non-commutative probability. We recall free additive and multiplicative convolutions and the analytic tools to calculate them. We state basic results on free infinite divisibility such as Lévy-Khintchine representations and the Bercovici-Pata bijection Λ. Also, we explain boolean additive convolution and recall the boolean-to-free Bercovici-Pata bijection $\mathbb{B}$. Section 4 is devoted to the description of different characterizations of free regular measures. In Section 5 we derive, using the characterizations of Section 3, closure properties as explained in Theorem 2.1. In Section 6 we essentially prove Theorem 2.2 below, which in particular shows that the square of a symmetric freely infinitely divisible distribution is freely infinitely divisible. We partially show that, for selfadjoint operators, the free infinite divisibility is preserved under the free commutator operation. This fact is fully proved in Appendix with combinatorial techniques. Finally, in Section 7 we gather examples using results of previous sections and present open problems regarding these examples. At the end of paper, we give an appendix where combinatorial interpretation of Theorem 2.2 is discussed. It contributes to study free commutators.

2. Main results

Let $\mathcal{M}$ be the class of all Borel probability measures on the real line $\mathbb{R}$ and let $\mathcal{M}^+$ be the subclass of $\mathcal{M}$ consisting of probability measures with support on $\mathbb{R}_+ = [0, \infty)$. Also, for two probability measures $\mu, \nu \in \mathcal{M}$, we denote by $\mu * \nu$, $\mu \boxplus \nu$ and $\mu \uplus \nu$ the classical, free and boolean additive convolutions, respectively. When $\nu \in \mathcal{M}^+$ we denote by $\mu \boxtimes \nu$ the free multiplicative convolution. They will be defined precisely in Section 3.

Let I^* be the class of all classically infinitely divisible distributions and $I^{\boxplus}$ be the class of all freely infinitely divisible distributions. An important subclass of I^* is the class of infinitely divisible measures supported on $\mathbb{R}_+$, that is, $I^* \cap \mathcal{M}^+$. This class has regular Lévy-Khintchine representations.

Free regular measures are the free analogue of $I^* \cap \mathcal{M}^+$. More precisely, let $I_{r+}^{\boxplus} := \Lambda(I^* \cap \mathcal{M}^+)$, where $\Lambda : I^* \to I^{\boxplus}$ is the Bercovici-Pata bijection, which is defined in Section 3. This class $I_{r+}^{\boxplus}$ was first considered in Pérez-Abreu and Sakuma (2012) in connection to free multiplicative mixtures of the Wigner distribution. It is remarkable that $I_{r+}^{\boxplus} \subset I^{\boxplus} \cap \mathcal{M}^+$ but $I_{r+}^{\boxplus} \neq I^{\boxplus} \cap \mathcal{M}^+$; the Bercovici-Pata bijection can send measures with support larger than $\mathbb{R}_+$ to measures concentrated on $[0, \infty)$.

The main results are as follows. First, we will see that $I_{r+}^{\boxplus}$ describes the distributions of free Lévy processes (see Biane (1998)) with positive increments, that we will call *free subordinators*. For free Lévy processes, contrary to the classical, boolean and monotone cases, the positivity of the marginal distribution at time $t = 1$ does not imply the positivity of all increments.

Second, $I_{r+}^{\boxplus}$ behaves well with respect to various operations in non-commutative probability. More specifically, we are able to prove the following.

Theorem 2.1. *Let μ, ν be free regular measures and let σ be a freely infinitely divisible distribution. Then the following properties hold.*

(1) *$\mu \boxtimes \nu$ is free regular.*
(2) *$\mu^{\boxtimes t}$ is free regular for $t \geq 1$.*
(3) *$\mu^{\uplus t}$ is free regular for $0 \leq t \leq 1$.*
(4) *$\mu \boxtimes \sigma$ is freely infinitely divisible.*

Of particular interest is the fact that $I_{r+}^{\boxplus}$ is closed under free multiplicative convolution. It was proved by Belinschi and Nica (2008) that the boolean-to-free Bercovici-Pata bijection $\mathbb{B}$ is a homomorphism with respect to free multiplicative convolution. This suggested strongly that free infinite divisibility was preserved under free multiplicative convolution. Surprisingly, this is not true, even if we restrict to measures in $\mathcal{M}^+$. Therefore, $I_{r+}^{\boxplus}$ is a natural class to consider, since it solves this apparent flaw.

The final result shows that if a symmetric random variable X has a distribution in $I^{\boxplus}$, so does the square X^2. This result is quite surprising since it has no analog in the classical world. We describe this result precisely below. For $p \geq 0$, let μ^p denote the probability measure on $[0, \infty)$ induced by the map $x \mapsto |x|^p$.

Theorem 2.2. *Let μ be a symmetric measure and m be the free Poisson law with density $\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$.*

(1) *If μ is $\boxplus$-infinitely divisible, then there is a free regular measure σ such that $\mu^2 = m \boxtimes \sigma$. In particular, $\mu^2 \in I_{r+}^{\boxplus}$. Conversely, if σ is free regular, then $\mathrm{Sym}\left((m \boxtimes \sigma)^{1/2}\right)$ is $\boxplus$-infinitely divisible distribution, where $\mathrm{Sym}(\nu)$ is the symmetrization of $\nu \in \mathcal{M}^+$: $\mathrm{Sym}(\nu)(dx) := \frac{1}{2}\left(\nu(dx) + \nu(-dx)\right)$.*
(2) *If μ is a compound free Poisson with rate λ and jump distribution ν, then σ from (1) is also a compound free Poisson with rate λ and jump distribution ν^2.*

As a consequence we find two new explicit examples of measures which are infinitely divisible in both free and classical senses : $\chi^2(1)$ and $F(1,1)$. To the best of our knowledge, apart from these two examples, there are only three known measures with this property: the normal law, the Cauchy distribution and the free 1/2 stable law.

Secondly, we get as a byproduct that the free commutator of freely infinitely divisible measures is also infinitely divisible.

3. Preliminaries

3.1. *Analytic tools for free convolutions.* Following Voiculescu et al. (1992), we recall that a pair $(\mathcal{A}, \varphi)$ is called a *W^*-probability space* if $\mathcal{A}$ is a von Neumann algebra and φ is a normal faithful trace. A family of unital von Neumann subalgebras $\{\mathcal{A}_i\}_{i\in I} \subset \mathcal{A}$ is said to be *free* if $\varphi(a_1 \cdots a_n) = 0$ whenever $\varphi(a_j) = 0, a_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n$. A self-adjoint operator X *is said to be affiliated with* $\mathcal{A}$ if $f(X) \in \mathcal{A}$ for any bounded Borel function f on $\mathbb{R}$. In this case it is also said that X is a (non-commutative) *random variable.* Given a self-adjoint operator X affiliated with $\mathcal{A}$, the *distribution* of X is the unique measure μ_X in $\mathcal{M}$ satisfying

$$\varphi(f(X)) = \int_{\mathbb{R}} f(x)\mu_X(\mathrm{d}x)$$

for every Borel bounded function f on $\mathbb{R}$. If $\{\mathcal{A}_i\}_{i\in I}$ is a family of free unital von Neumann subalgebras and X_i is a random variable affiliated with $\mathcal{A}_i$ for each $i \in I$, then the *random variables* $\{X_i\}_{i\in I}$ are said to be *free.*

Let $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the upper and lower half-planes, respectively. The *Cauchy transform* of a probability measure μ on $\mathbb{R}$ is defined, for $z \in \mathbb{C}\backslash\mathbb{R}$, by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu\left(\mathrm{d}x\right).$$

It is well known that $G_\mu : \mathbb{C}_+ \to \mathbb{C}_-$ is analytic and that G_μ determines uniquely the measure μ. The *reciprocal Cauchy transform* is the function $F_\mu : \mathbb{C}_+ \to \mathbb{C}_+$ defined by $F_\mu\left(z\right) = 1/G_\mu(z)$. It was proved in Bercovici and Voiculescu (1993) that there are positive numbers α and M such that F_μ has a right compositional inverse F_μ^{-1} defined on the region

$$\Gamma_{\alpha,M} := \{z \in \mathbb{C}; |\Re(z)| < \alpha\Im(z),\ \Im(z) > M\}.$$

The *Voiculescu transform of* μ is defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z$$

on any region of the form $\Gamma_{\alpha,M}$ where F_μ^{-1} is defined, see Bercovici and Voiculescu (1993). The *free cumulant transform* is a variant of ϕ_μ defined as

$$\mathcal{C}_\mu^{\boxplus}(z) = z\phi_\mu\left(\frac{1}{z}\right) = zF_\mu^{-1}\left(\frac{1}{z}\right) - 1,$$

for $z \in D_\mu := \{z \in \mathbb{C}_- : z^{-1} \in \Gamma_{\alpha,M}\}$, see Barndorff-Nielsen and Thorbjornsen (2006).

The *free additive convolution* $\mu_1 \boxplus \mu_2$ of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined so that $\phi_{\mu_1\boxplus\mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$, or equivalently, $\mathcal{C}^{\boxplus}_{\mu_1\boxplus\mu_2}(z) = \mathcal{C}^{\boxplus}_{\mu_1}(z) + \mathcal{C}^{\boxplus}_{\mu_2}(z)$ for $z \in D_{\mu_1} \cap D_{\mu_2}$. The measure $\mu_1 \boxplus \mu_2$ is the distribution of the sum $X_1 + X_2$ of two free random variables X_1 and X_2 having distributions μ_1 and μ_2 respectively.

The *free multiplicative convolution* $\mu_1 \boxtimes \mu_2$ of probability measures $\mu_1, \mu_2 \in \mathcal{M}$, one of them in $\mathcal{M}^+$, say $\mu_1 \in \mathcal{M}^+$, is defined as the distribution of $\mu_{X_1^{1/2}X_2X_1^{1/2}}$ where $X_1 \geq 0, X_2$ are free, self-adjoint elements such that $\mu_{X_i} = \mu_i$. The element $X_1^{1/2}X_2X_1^{1/2}$ is self-adjoint and its distribution depends only on μ_1 and μ_2. The operation $\boxtimes$ on $\mathcal{M}^+$ is associative and commutative.

The next result was proved in Bercovici and Voiculescu (1993).

Proposition 3.1. *Let* $\mu \in \mathcal{M}^+$ *such that* $\mu(\{0\}) < 1$. *The function* $\Psi_\mu(z) = \int_0^\infty \frac{zx}{1-zx}\mu(\mathrm{d}x)$ *defined in* $\mathbb{C}\backslash\mathbb{R}_+$ *is univalent in the left-plane* $i\mathbb{C}_+$ *and* $\Psi_\mu(i\mathbb{C}_+)$ *is a region contained in the circle with diameter* $(\mu(\{0\})-1, 0)$. *Moreover,* $\Psi_\mu(i\mathbb{C}_+) \cap \mathbb{R} = (\mu(\{0\}) - 1, 0)$.

Let $\mu \in \mathcal{M}^+$ and $\chi_\mu : \Psi_\mu(i\mathbb{C}_+) \to i\mathbb{C}_+$ be the inverse function of Ψ_μ. The *S-transform* of μ is the function

$$S_\mu(z) = \chi(z)\frac{1+z}{z}. \tag{3.1}$$

The S-transform is an analytic tool for computing free multiplicative convolutions. When the measure μ is symmetric or has compact support and vanishing mean the inverse of Ψ is not unique, but there are two possible choices. One can still define an S-transform as in Equation (3.1) by choosing any of these inverses. The following was first shown by Voiculescu (1987) for measures in $\mathcal{M}^+$ with bounded support, and then extended to: measures in $\mathcal{M}^+$ with unbounded support by Bercovici and Voiculescu (1993); measures in $\mathcal{M}$ with compact support by Raj Rao and Speicher (2007); symmetric measures by Arizmendi and Pérez-Abreu (2009).

Proposition 3.2. *Let $\mu_1 \in \mathcal{M}^+$ and μ_2 a probability measure in $\mathcal{M}^+$ or symmetric, with $\mu_i \neq \delta_0$, $i = 1, 2$. Then $\mu_1 \boxtimes \mu_2 \neq \delta_0$ and*

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) S_{\mu_2}(z)$$

in the common domain containing $(-\varepsilon, 0)$ for small $\varepsilon > 0$. Moreover, $(\mu_1 \boxtimes \mu_2)(\{0\}) = \max\{\mu_1(\{0\}), \mu_2(\{0\})\}$.

Using this S-transform it was proved in Arizmendi and Pérez-Abreu (2009) that, for a $\mu \in \mathcal{M}^+$ and ν a symmetric probability measure, the following relation holds:

$$(\mu \boxtimes \nu)^2 = \mu \boxtimes \mu \boxtimes \nu^2 \tag{3.2}$$

where, for a measure μ, we denote by μ^2 the measure induced by the push-forward $t \to t^2$.

3.2. *Free infinite divisibility.*

Definition 3.3. Let μ be a probability measure in $\mathbb{R}$. We say that μ is **freely (or $\boxplus$- for short) infinitely divisible**, if for all n, there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus .... \boxplus \mu_n}_{n \text{ times}}. \tag{3.3}$$

We denote by $I^{\boxplus}$ the class of such measures.

For $\mu \in I^{\boxplus}$, a free convolution semigroup $(\mu^{\boxplus t})_{t\geq 0}$ can always be defined so that $\mathcal{C}^{\boxplus}_{\mu^{\boxplus t}}(z) = t\mathcal{C}^{\boxplus}_{\mu}(z)$.

Now, recall that a probability measure μ is classically infinitely divisible if and only if its classical cumulant transform $\mathcal{C}^*_\mu(u) := \log\left(\int_{\mathbb{R}} e^{iux}\mu(dx)\right)$ has the Lévy-Khintchine representation

$$\mathcal{C}^*_\mu(u) = i\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}} (e^{iut} - 1 - iut 1_{[-1,1]}(t))\nu(dt), \quad u \in \mathbb{R}, \tag{3.4}$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ν is a Lévy measure on $\mathbb{R}$, that is, $\int_{\mathbb{R}} \min(1, t^2)\nu(dt) < \infty$ and $\nu(\{0\}) = 0$. If this representation exists, the triplet (η, a, ν) is unique and is called the classical characteristic triplet of μ.

A $\boxplus$-infinitely divisible measure has a free analogue of the Lévy-Khintchine representation (see Barndorff-Nielsen and Thorbjornsen (2006)).

Proposition 3.4. *A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if there are $\eta \in \mathbb{R}$, $a \geq 0$ and a Lévy measure ν on $\mathbb{R}$ such that*

$$\mathcal{C}^{\boxplus}_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - zt} - 1 - tz1_{[-1,1]}(t)\right)\nu(dt), \quad z \in \mathbb{C}_-. \tag{3.5}$$

The triplet (η, a, ν) is unique and is called the free characteristic triplet of μ.

The expressions (3.4) and (3.5) give a natural bijection between I^* and $I^{\boxplus}$. This bijection was introduced by Bercovici and Pata (1999) in their studies of domains of attraction in free probability. Explicitly, this bijection is given as follows.

Definition 3.5. By the **Bercovici-Pata bijection** we mean the mapping $\Lambda : I^* \to I^{\boxplus}$ that sends the measure μ in I^* with classical characteristic triplet (η, a, ν) to the measure $\Lambda(\mu)$ in $I^{\boxplus}$ with free characteristic triplet (η, a, ν).

The map $\Lambda(\mu)$ is both a homomorphism in the sense that $\Lambda(\mu * \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$, and a homeomorphism with respect to weak convergence.

Another type of Lévy-Khintchine representation in terms of ϕ_μ is sometimes more useful than the free cumulant case: for $\mu \in I^{\boxplus}$, there exists a unique $\gamma_\mu \in \mathbb{R}$ and a finite non-negative measure τ_μ on $\mathbb{R}$ such that

$$\phi_\mu(z) = \gamma_\mu + \int_{\mathbb{R}} \frac{1 + xz}{z - x} \tau_\mu(dx).$$

Finally let us mention very well known $\boxplus$-infinitely divisible measures that we will use often in this paper. The first one is the standard Wigner semicircle law w with density

$$\frac{1}{2\pi}(4 - x^2)^{1/2} \mathrm{d}x, \quad -2 < x < 2.$$

The second is the Marchenko-Pastur law m, also known as free Poisson, with density

$$\frac{1}{2\pi} x^{-1/2}(4 - x)^{1/2} \mathrm{d}x, \quad 0 < x < 4.$$

3.3. *Boolean convolutions.* The additive boolean convolution $\mu \uplus \nu$ of probability measures on $\mathbb{R}$ was introduced in Speicher and Woroudi (1997). It is characterized by $K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z)$, where

$$K_\mu(z) = z - F_\mu(z),$$

which is called the energy function and is defined by Speicher and Woroudi (1997). Any probability measure is infinitely divisible with respect to the boolean convolution and a kind of Lévy-Khintchine representation is written as

$$K_\mu(z) = \gamma_\mu + \int_{\mathbb{R}} \frac{1 + xz}{z - x} \eta_\mu(dx),$$

where $\gamma_\mu \in \mathbb{R}$ and η_μ is a finite non-negative measure (see Speicher and Woroudi (1997)). A boolean convolution semigroup $(\mu^{\uplus t})_{t \geq 0}$ can always be defined for any probability measure $\mu \in \mathcal{M}$. Moreover, if $\mu \in \mathcal{M}^+$ then $\mu^{\uplus t} \in \mathcal{M}^+$ for all $t > 0$. The Bercovici-Pata bijection $\mathbb{B}$ from the boolean convolution to the free one can be defined in the same way as for Λ, by the relation $K_\mu = \phi_{\mathbb{B}(\mu)}$. The reader is referred to Bercovici and Pata (1999) for the definition of $\mathbb{B}$ in terms of domains of attraction.

Similarly to Λ, $\mathbb{B}$ is a homomorphism between $(\mathcal{M}, \uplus)$ and $(I^{\boxplus}, \boxplus)$, in the sense that $\mathbb{B}(\mu \uplus \nu) = \mathbb{B}(\mu) \boxplus \mathbb{B}(\nu)$. Also, $\mathbb{B}$ is a homeomorphism with respect to weak convergence.

4. Free regular measures

Let us consider a probability measure $\sigma \in I^{\boxplus}$ whose Lévy measure ν of (3.5) satisfies $\int_{\mathbb{R}_+} \min(1,t)\nu(dt) < \infty$. Then the Lévy-Khintchine representation reduces to

$$\mathcal{C}_\sigma^{\boxplus}(z) = \eta' z + \int_{\mathbb{R}} \left(\frac{1}{1-zt} - 1 \right) \nu(dt), \quad z \in \mathbb{C}_-, \tag{4.1}$$

where $\eta' \in \mathbb{R}$. The measure σ is said to be a **free regular infinitely divisible (or free regular, for short) distribution** if $\eta' \geq 0$ and $\nu((-\infty,0]) = 0$. The most typical example is some compound free Poisson distributions. If the drift term η' is zero and the Lévy measure ν is $\lambda\rho$ for some $\lambda > 0$ and a probability measure ρ on $\mathbb{R}$, then we call σ a **compound free Poisson distribution** with rate λ and jump distribution ρ. To clarify these parameters, we denote $\sigma = \pi(\lambda, \rho)$.

Remark 4.1. 1) The Marchenko-Pastur law m is a compound free Poisson with rate 1 and jump distribution δ_1.

2) The compound free Poisson $\pi(1,\nu)$ coincides with the free multiplication $m \boxtimes \nu$.

This section is devoted to clarify several characterizations of free regular measures, some of which can be inferred from results of Benaych-Georges (2010); Hasebe (2010); Pérez-Abreu and Sakuma (2012) and Sakuma (2011), as we recollect in the following theorem. The final characterization uses free Lévy processes which we will describe in details.

Theorem 4.2. *The following conditions for $\mu \in \mathcal{M}$ are equivalent:*

(i) *μ is free regular.*
(ii) *$\mu \in \Lambda(\mathcal{M}^+ \cap I^*)$.*
(iii) *$\mu \in \mathbb{B}(\mathcal{M}^+)$.*
(iv) *$\mu^{\boxplus t} \in \mathcal{M}^+$ for any $t > 0$.*
(v) *μ is $\boxplus$-infinitely divisible, $\tau_\mu(-\infty,0) = 0$ and $\phi_\mu(-0) \geq 0$, where τ_μ is the measure appearing in the representation of the Voiculescu transform.*
(vi) *There exists a free subordinator X_t such that X_1 is distributed as μ.*

4.1. *Characterizations (ii)–(v).* The equivalence between (i) and (ii) is clear from the Lévy-Khintchine representation. However, we remark again that not all non-negative $\boxplus$-infinitely divisible distributions are free regular; a typical example of a measure in $I^{\boxplus} \cap \mathcal{M}^+$ but not in $I^{\boxplus}_{r+}$ is w_+, a semicircle distribution with mean 2 and variance 1.

In a similar fashion, one can prove the equivalence between (i) and (iii). This can be seen from the boolean Lévy-Khintchine representation of $\mu \in \mathcal{M}^+$ in terms of K_μ, see Proposition 2.5 of Hasebe (2010) for the details.

The equivalence between (i) and (iv) was proved by Benaych-Georges (2010) as the following lemma, see also Sakuma (2011).

Lemma 4.3. *A probability measure μ is in $I^{\boxplus}_{r+}$, if and only if $\mu^{\boxplus t} \in \mathcal{M}^+$ for all $t > 0$.*

The equivalence between (i) and (v) is proved as follows. For a measure ν we denote by $a(\nu)$ the *left extremity* of ν: $a(\nu) = \min\{x : x \in \text{supp } \nu\}$.

Proposition 4.4. *Let μ be a $\boxplus$-infinitely divisible distribution. Then μ is free regular if and only if $a(\tau_\mu) \geq 0$ and $\phi_\mu(-0) \geq 0$.*

Proof: Denote by $\mathbb{B}$ the Bercovici-Pata bijection from boolean to free convolutions: $z - F_\mu(z) = \phi_{\mathbb{B}(\mu)}(z)$. Let us denote by $z - F_\mu(z) = \gamma_\mu + \int_{\mathbb{R}} \frac{1+xz}{z-x}\eta_\mu(dx)$ the boolean Lévy–Khintchine representation. As proved in Proposition 2.5 of Hasebe (2010) supp $\mu \subset [0,\infty)$ if and only if supp $\eta_\mu \subset [0,\infty)$ and $F_\mu(-0) \le 0$. By definition, ν is free regular if and only if $\mathbb{B}^{-1}(\nu)$ is supported on $[0,\infty)$, yielding the conclusion. □

As we saw, $\mu \in I^{\boxplus} \cap \mathcal{M}^+$ does not imply $\mu \in I^{\boxplus}_{r+}$. However, if μ has a singularity at 0, such an implication is possible. We need a lemma to prove it.

Lemma 4.5. *Let μ be a $\boxplus$-infinitely divisible distribution with $a(\mu) > -\infty$. Then $a(\tau_\mu) \ge F_\mu(a(\mu) - 0)$.*

Proof: Since F_μ is strictly increasing in $(-\infty, a(\mu))$, one can define F_μ^{-1} in an open set of $\mathbb{C}$ containing $(-\infty, F_\mu(a(\mu)-0))$. This gives an analytic continuation of F_μ^{-1} to $\mathbb{C}\backslash[F_\mu(a(\mu)-0),\infty)$. Therefore, τ_μ is supported on $[F_\mu(a(\mu)-0),\infty)$. □

Theorem 4.6. *Let μ be a $\boxplus$-infinitely divisible measure supported on $[0,\infty)$ satisfying either of the following conditions: (i) $\mu(\{0\}) > 0$; (ii) $\mu(\{0\}) = 0$ and $\int_0^1 \frac{\mu(dx)}{x} = \infty$. Then μ is free regular.*

Proof: By assumption, $F_\mu(-0) = 0$. Lemma 4.5 implies that $a(\tau_\mu) \ge 0$. Taking the limit $z \nearrow 0$ in the identity $\phi_\mu(F_\mu(z)) = z - F_\mu(z)$, one concludes that $\phi_\mu(-0) = 0$. Therefore, μ is free regular from Proposition 4.4. □

4.2. *Free subordinators and free regular measures.* A particularly important family of real-valued processes with independent increments is that of Lévy processes; see Bertoin (2002); Sato (1999). Let us recall the definition of a Lévy process. A continuous-time process $\{X_t\}_{t\ge0}$ with values in $\mathbb{R}$ is called a Lévy process if

(1) Its sample paths are right-continuous and have left limits at every time point t.
(2) For all $0 \le t_1 < \cdots < t_n$, the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.
(3) For all $0 \le s \le t$, the increments $X_t - X_s$ and $X_{t-s} - X_0$ have the same distribution.
(4) For any $s \ge 0$, $X_{s+t} \to X_s$ in probability, as $t \to 0$, i.e. the distribution of $X_{s+t} - X_s$ converges weakly to δ_0, as $t \to 0$.

We assume that $X_0 = 0$. Now, if we denote by μ_t the distribution of X_t, then these measures satisfy the property

$$\mu_{s+t} = \mu_s * \mu_t \tag{4.2}$$

for any $s,t \ge 0$. The relation between infinitely divisible distributions and Lévy processes can be stated in the following proposition.

Proposition 4.7. *If $\{X_t\}_{t\ge0}$ is a Lévy process, then for each $t > 0$ the random variable X_t has an infinitely divisible distribution. Conversely, if μ is an infinitely divisible distribution then there is a Lévy process such that X_1 has distribution μ.*

A *subordinator* is a real-valued Lévy process with non-decreasing path. This class has been broadly studied; see Bertoin (2002); Cont and Tankov (2004); Sato (1999).

Proposition 4.8. *Let $\{X_t\}_{t\ge0}$ be a Lévy process. The process X_t is a subordinator if and only if the distribution of X_1 is supported on the positive real line.*

Now, following Biane (1998), we define a process with free additive increments to be a map $t \mapsto X_t$ from $\mathbb{R}_+$ to the set of self-adjoint elements affiliated to some W^*-probability space (A, φ) such that, for any $0 \leq t_1 < \cdots < t_n$, the elements $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are free. We also require the weak continuity of the distributions. However, we do not require an analog of property (1) of a classical Lévy process since there is no sample path in the W^*-probability setting.

To define a free (additive) Lévy process, we need stationarity. As Biane proposed, there are two natural classes which deserve to be called free Lévy processes, depending on whether we ask for time homogeneity of the distributions of the increments or of the transition probabilities. Here, we will use the former since in this case the distributions of a process form a semi-group for the free additive convolution.

Definition 4.9. A free additive Lévy process is a map $t \mapsto X_t$ from $\mathbb{R}_+$ to the set of self-adjoint elements affiliated to some W^*-probability space (A, φ), such that:

(1) For all $t_1 < \cdots < t_n$, the elements $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are free.
(2) For all $0 \leq s \leq t$ the increments $X_t - X_s$ and $X_{t-s} - X_0$ have the same distribution.
(3) For any $s \geq 0$ in, $X_{s+t} \to X_s$ in probability, as $t \to 0$, i.e. the distribution of $X_{s+t} - X_s$ converges weakly to δ_0, as $t \to 0$.

We also assume that $X_0 = 0$.

If we denote by μ_t the distribution of X_t, these measures satisfy the analog of (4.2):

$$\mu_{s+t} = \mu_s \boxplus \mu_t$$

for $s, t \geq 0$.

Definition 4.10. A free additive Lévy process X_t is called a *free subordinator* if for all $0 < s < t$ the increment $X_t - X_s$ is positive.

We state the analog of Proposition 4.8 which clarifies the role of free regular measures in terms of free Lévy processes: they correspond to free subordinators.

Proposition 4.11. *Let X_t be a free additive Lévy process. The process X_t is a free subordinator if and only if the distribution of X_1 is free regular.*

Proof: If X_t is a free subordinator, it is clear that the distribution μ_1 of X_1 is free regular since $X_t - X_0 = X_t$ is positive and then the distribution $\mu_t = \mu_1^{\boxplus t}$ is supported on $\mathbb{R}_+$. Lemma 4.3 allows us to conclude.

Conversely, suppose that the distribution $\mu = \mu_1$ of X_1 is free regular. We want to see that $X_t - X_s$ is positive. Since X_t is a free Lévy process it is stationary and then $X_t - X_s$ has the same distribution as X_{t-s}, which is $\mu^{\boxplus(t-s)}$ and then, by Lemma 4.3, supported on $\mathbb{R}_+$, i.e. X_{t-s} positive. □

5. Closure properties

The following property was proved by Belinschi and Nica (2008). For $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}^+$,

$$\mathbb{B}(\mu \boxtimes \nu) = \mathbb{B}(\mu) \boxtimes \mathbb{B}(\nu). \tag{5.1}$$

This suggested strongly that if μ and ν are $\boxplus$-infinitely divisible then $\mu \boxtimes \nu$ is also $\boxplus$-infinitely divisible. However, this is not true in general, even if both μ and

ν belong to $\mathcal{M}^+$ or $\mu = \nu$. The following counterexample was given by Sakuma (2011).

Proposition 5.1. *Let w_+ be the Wigner distribution with density*

$$w_{2,1}(x) = \frac{1}{2\pi}\sqrt{4-(x-2)^2}\cdot 1_{[0,4]}(x)dx.$$

Then $w_+ \boxtimes w_+$ is not $\boxplus$-infinitely divisible.

It is not a coincidence that in this counterexample w_+ is not free regular. Indeed, if either ν or μ is free regular the problem is fixed.

Proposition 5.2. *Let $\mu \in I_{r+}^{\boxplus}$ and $\nu \in I^{\boxplus}$, then $\mu \boxtimes \nu$ is freely infinitely divisible. Moreover if $\nu \in I_{r+}^{\boxplus}$ then $\mu \boxtimes \nu \in I_{r+}^{\boxplus}$.*

Proof: If $\mu \in I_{r+}^{\boxplus}$ then $\mu = \mathbb{B}(\mu_0)$ for some $\mu_0 \in \mathcal{M}^+$. Similarly if $\nu \in I^{\boxplus}$ then $\mu = \mathbb{B}(\nu_0)$ for some $\nu_0 \in \mathcal{M}$. Then $\mu_0 \boxtimes \nu_0$ is a well defined probability measure and Equation (5.1) gives $\mu \boxtimes \nu = \mathbb{B}(\mu_0 \boxtimes \nu_0) \in I^{\boxplus}$.

Now, if $\nu \in I_{r+}^{\boxplus}$ then $\nu_0 \in \mathcal{M}^+$ and then $\mu_0 \boxtimes \nu_0 \in \mathcal{M}^+$. Therefore $\mu \boxtimes \nu = \mathbb{B}(\mu_0 \boxtimes \nu_0) \in I_{r+}^{\boxplus}$ since $\mathbb{B}$ sends positive measures to free regular ones. □

Remark 5.3. As a consequence we answer a question in Pérez-Abreu and Sakuma (2012): If μ is free regular then $\mu \boxtimes \mu$ is also free regular.

Remark 5.4. The previous proposition raises a question on a relation between the free subordinators associated to ν, μ and $\nu \boxtimes \mu$. Let D_a be the dilation operator defined by $\int_{\mathbb{R}} f(x)(D_a\mu)(dx) = \int_{\mathbb{R}} f(ax)\mu(dx)$ for any bounded continuous function f and measure μ. Equivalently, if a random variable X follows a distribution μ, $D_a\mu$ is the distribution of aX. For $\mu, \nu \in I_{r+}^{\boxplus}$, the identity

$$D_{1/t}(\mu^{\boxplus t} \boxtimes \nu^{\boxplus t}) = (\mu \boxtimes \nu)^{\boxplus t},\ t \geq 0 \tag{5.2}$$

was essentially proved in Belinschi and Nica (2008, Proposition 3.5). This can be interpreted as follows in terms of processes. Let X_t and Y_t be free subordinators with $X_1 \sim \mu$ and $Y_1 \sim \nu$, which are free between them. Then the process $\frac{1}{t}X_t^{1/2}Y_tX_t^{1/2}$ is distributed as a free subordinator Z_t such that $Z_1 \sim \mu \boxtimes \nu$.

It is clear from Proposition 5.2 that if μ is in $I_{r+}^{\boxplus}$ then $\mu^{\boxtimes n}$ also belongs to $I_{r+}^{\boxplus}$, for $n \in \mathbb{N}$. Furthermore, this is also true for $t \geq 1$, $\mu^{\boxtimes t} \in I_{r+}^{\boxplus}$, when t is not necessarily an integer, as we state in following proposition.

Proposition 5.5. *If $\mu \in I_{r+}^{\boxplus}$, then for all $s \geq 1$, $\mu^{\boxtimes s} \in I_{r+}^{\boxplus}$.*

Proof: By Lemma 4.3, it is enough to see that $(\mu^{\boxtimes s})^{\boxplus t} \in \mathcal{M}^+$ for all $t > 0$. For this, we use the following identity, essentially proved in Belinschi and Nica (2008, Proposition 3.5):

$$D_{t^{s-1}}((\mu^{\boxtimes s})^{\boxplus t}) = (\mu^{\boxplus t})^{\boxtimes s}. \tag{5.3}$$

Now, since μ is free regular, $\mu^{\boxplus t} \in \mathcal{M}^+$ and then $(\mu^{\boxplus t})^{\boxtimes s} \in \mathcal{M}^+$. Therefore, the RHS of Equation (5.3) defines a probability measure with positive support and then $(\mu^{\boxtimes s})^{\boxplus t} \in \mathcal{M}^+$, as desired. □

Also, boolean powers less than one preserve free regularity.

Proposition 5.6. *If $\mu \in I_{r+}^{\boxplus}$, then $\mu^{\uplus t} \in I_{r+}^{\boxplus}$ for $0 \leq t \leq 1$.*

Proof: It is shown in Arizmendi and Hasebe (2011b) that if μ is $\boxplus$-infinitely divisible then, for $0 < t < 1$,

$$\mathbb{B}((\mu^{\boxplus(1-t)})^{\uplus t/(1-t)}) = \mu^{\uplus t}.$$

Since μ is free regular, $\mu^{\boxplus(1-t)}$ has a positive support and then, since the boolean convolution of measures in $\mathcal{M}^+$ stays in $\mathcal{M}^+$, we see that $(\mu^{\boxplus(1-t)})^{\uplus t/(1-t)} \in \mathcal{M}^+$. On the other hand $\mathbb{B}$ sends positive measures to free regular measures (see Theorem 4.2(iii)). □

Finally we show that $I^{\boxplus}_{r+}$ is closed under weak convergence.

Proposition 5.7. *Let $(\mu_n)_{n>0}$ be a sequence of measures in $I^{\boxplus}_{r+}$, weakly converging to a probability measure μ. Then μ is also in $I^{\boxplus}_{r+}$.*

Proof: For each $n \in \mathbb{N}$, we can write $\mu_n = \mathbb{B}(\nu_n)$ for some ν_n in $\mathcal{M}^+$ from Theorem 4.2. Since $\mathbb{B}$ is a homeomorphism, ν_n weakly converge to a probability measure $\nu \in \mathcal{M}^+$, and its holds that $\mu = \mathbb{B}(\nu)$. Hence $\mu \in I^{\boxplus}_{r+}$, as desired. □

6. Squares of random variables with symmetric distributions in $I^{\boxplus}$

We will prove Theorem 2.2 in this section. Given a probability measure μ, we recall that μ^p for $p \geq 0$ denotes the probability measure in $\mathcal{M}^+$ induced by the map $x \mapsto |x|^p$. For a measure λ on $\mathbb{R}$ we denote by $\mathrm{Sym}(\lambda)$ the symmetric measure $\frac{1}{2}(\lambda(dx) + \lambda(-dx))$.

We quote a result from Pérez-Abreu and Sakuma (2012, Theorem 12).

Theorem 6.1. *A symmetric probability measure μ is $\boxplus$-infinitely divisible if and only if there is a free regular distribution σ such that $\mathcal{C}^{\boxplus}_{\mu}(z) = \mathcal{C}^{\boxplus}_{\sigma}(z^2)$. Moreover, the free characteristic triplets $(0, a_\mu, \nu_\mu)$ of μ and $(\eta_\sigma, 0, \nu_\sigma)$ of σ are related as follows: $\nu_\mu = \mathrm{Sym}(\nu_\sigma^{1/2})$ (or equivalently $\nu_\sigma = \nu_\mu^2$), $a_\mu = \eta_\sigma$.*

The following proposition implies that the square of a symmetric measure which is $\boxplus$-infinitely divisible is also $\boxplus$-infinitely divisible. A similar result is proved for the rectangular free convolution of Benaych-Georges (2010).

Proposition 6.2. *Let μ be a $\boxplus$-infinitely divisible symmetric measure. Then there exists a free regular measure σ such that $\mu^2 = m \boxtimes \sigma$, that is, μ^2 is the compound free Poisson with rate 1 and jump distribution σ. Conversely, if σ is free regular, then $\mathrm{Sym}\left((m \boxtimes \sigma)^{1/2}\right)$ is $\boxplus$-infinitely divisible.*

Proof: We prove that the following are equivalent:
(a) $\mu^2 = m \boxtimes \sigma$,
(b) $\mathcal{C}^{\boxplus}_{\mu}(z) = \mathcal{C}^{\boxplus}_{\sigma}(z^2)$.
Indeed, if $\mu^2 = m \boxtimes \sigma$, then $S_{\mu^2}(z) = S_m(z)S_\sigma(z) = \frac{1}{1+z}S_\sigma(z)$. Combined with the relation $S_{\mu^2}(z) = \frac{z}{1+z}S_\mu(z)^2$, this implies $zS_\sigma(z) = (zS_\mu(z))^2$. Since the inverse of $zS_\lambda(z)$ is equal to $\mathcal{C}^{\boxplus}_{\lambda}$ for a probability measure λ, we conclude that $(\mathcal{C}^{\boxplus}_{\sigma})^{-1}(z) = ((\mathcal{C}^{\boxplus}_{\mu})^{-1}(z))^2$, which is equivalent to (b). Clearly the converse is also true. The desired result immediately follows from the above equivalence and Theorem 6.1. □

This completes the proof of Theorem 2.2(1). The result (2) for compound free Poissons is a consequence of Theorem 6.1.

Now the following result of Pérez-Abreu and Sakuma (2012, Theorem 22) follows as a consequence of Theorem 2.2.

Theorem 6.3. *Let $\sigma \in \mathcal{M}^+$ and w be the standard semicircle law. Then $\sigma \boxtimes \sigma \in I_{r+}^{\boxplus}$ if and only if $\mu = w \boxtimes \sigma \in I^{\boxplus}$.*

Remark 6.4. It is not true that the square of a symmetric infinitely divisible distribution in the classical sense is also infinitely divisible. For instance, if N_1 and N_2 are independent Poissons then $N_1 - N_2$ is also infinitely divisible and $(N_1 - N_2)^2$ is not infinitely divisible since it is supported on $\{0, 1, 4, 9, 25...\}$. (See Steutel and Van Harn (2004, pp. 51.))

There are two interesting consequences of Proposition 6.2. First, Proposition 6.2 allows us to identify some non trivial free regular measures which are in $I^* \cap I^{\boxplus}$: χ^2 and $F(1, 1)$. This will be explained in Example 7.1.

The second consequence is on the commutator of two free even elements, which was pointed out to us by Speicher.[1] See A.2 in the Appendix for the definition of even elements. In this case, an even element simply means that its distribution is symmetric.

Corollary 6.5. *Let a_1, a_2 be free, self-adjoint and even elements whose distributions μ_1, μ_2 are $\boxplus$-infinitely divisible. Then the distribution of the free commutator $\mu_1 \square \mu_2 := \mu_{i(a_1 a_2 - a_2 a_1)}$ is also $\boxplus$-infinitely divisible.*

Remark 6.6. If a_1, a_2 are free, even and self-adjoint, the distribution of the anti-commutator $\mu_{a_1 a_2 + a_2 a_1}$ is the same as $\mu_{i(a_1 a_2 - a_2 a_1)}$, as proved by Nica and Speicher (1998).

Proof: It was proved by Nica and Speicher (1998) that $\mu_1 \square \mu_2$ is also symmetric and satisfies

$$((\mu_1 \square \mu_2)^{\boxplus 1/2})^2 = \mu_1^2 \boxtimes \mu_2^2. \tag{6.1}$$

Since, for $i = 1, 2$, the distribution μ_i is symmetric and belongs to $I^{\boxplus}$, by Proposition 6.2, we have the representation $\mu_i^2 = m \boxtimes \sigma_i$, for some σ_i free regular. Then $((\mu_1 \square \mu_1)^{\boxplus 1/2})^2 = m \boxtimes \sigma$ with $\sigma = m \boxtimes \sigma_1 \boxtimes \sigma_2$. Now, by Theorem 5.2, σ is free regular and then $(\mu_1 \square \mu_2)^{\boxplus 1/2}$ is $\boxplus$-infinitely divisible. The desired result now follows. □

When we restrict μ_1 to the standard semicircle law, we obtain the analog of Theorem 6.3 for the free commutator.

Corollary 6.7. *Let σ be a symmetric measure and w be the standard semicircle law. Then $\sigma^2 \in I_{r+}^{\boxplus}$ if and only if $\mu = w \square \sigma \in I^{\boxplus}$.*

Proof: It is well known that the $w^2 = m$ and then we get from Equation (6.1) that $((w \square \sigma)^{\boxplus 1/2})^2 = m \boxtimes \sigma^2$. The result now follows from Proposition 6.2. □

Moreover, Nica and Speicher reduced the problem of calculating the cumulants of the free commutator to symmetric measures. A further analysis of this reduction in combination with Corollary 6.5 enables us to omit the assumption of evenness.

Theorem 6.8. *Let a_1 and a_2 be free and self-adjoint elements, and let $\mu_1 := \mu_{a_1}$ and $\mu_2 := \mu_{a_2}$ be $\boxplus$-infinitely divisible distributions. Then the distribution of the free commutator $\mu_1 \square \mu_2 := \mu_{i(a_1 a_2 - a_2 a_1)}$ is also $\boxplus$-infinitely divisible.*

The proof uses combinatorial tools and will be given in the Appendix.

[1] Free commutators have received less attention and not that much is known on this operation.

Remark 6.9 (Polynomials on free variables). So far we have proved that if a_1, a_2, a_3 are free even random variables whose distributions are $\boxplus$-infinitely divisible, then $i(a_ia_j - a_ja_i)$, $a_ia_j + a_ja_i$ and a_i^2 also have $\boxplus$-infinitely divisible distributions (for the free commutator, the assumption of evenness is not needed). Combining these results one can easily see that the following polynomials are also $\boxplus$-infinitely divisible: $a_1^2 + a_2^2 + a_2a_1 + a_1a_2$, $i(a_1a_2^2 - a_2a_1^2)$, $a_1^4 + a_2^4 - a_2^2a_1^2 - a_1^2a_2^2$, $a_1a_2^2a_1 + a_2a_1^2a_2 + a_1a_2a_1a_2 + a_2a_1a_2a_1$, $a_1a_2^2a_1 + a_2a_1^2a_2 - a_1a_2a_1a_2 - a_2a_1a_2a_1$, $a_1a_2a_3 + a_2a_1a_3 + a_3a_1a_2 + a_3a_2a_1$, etc. Therefore, it is natural to ask for which polynomials free infinite divisibility is preserved.

7. Examples, conjectures and future problems

In this section, we gather some examples related to our results. From these examples, we also present open problems.

As a first example we use Theorem 2.2 to identify measures in $I^* \cap I_{r+}^{\boxplus}$.

Example 7.1. The following are measures which are both classically and freely infinitely divisible.

(1) Let χ^2 be a chi-squared with 1 degree of freedom with density

$$f(x) := \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \ x > 0.$$

It is well known that χ^2 is infinitely divisible in the classical sense. It was proved in Belinschi et al. (2011) that a symmetric Gaussian Z is $\boxplus$-infinitely divisible. Hence, by Theorem 2.2, Z^2 is free regular. $Z^2 \sim \chi^2$ and then $\chi^2 \in I^* \cap I_{r+}^{\boxplus}$

(2) Let $F(1,n)$ be an F-distribution with density

$$f(x) := \frac{1}{B(1/2, n/2)} \frac{1}{(nx)^{1/2}} \left(1 + \frac{x}{n}\right)^{-(1+n)/2}, \quad x > 0.$$

$F(1,n)$ is classically infinitely divisible, as can be seen in Ismail and Kelker (1979). On the other hand $F(1,n)$ is the square of a t-student with n degrees of freedom $t(n)$. In particular $t(1)$ is the Cauchy distribution, hence by Theorem 2.2, $F(1,1)$ belongs to $I^* \cap I_{r+}^{\boxplus}$.

Remark 7.2. Numeric computations of free cumulants have shown that the chi-squared with 2 degrees of freedom is not freely infinitely divisible. However, the free infinite divisibility of t-student with n degrees of freedom is still an open question.

Next, we give some examples of free regular measure from known distributions in non-commutative probability.

Example 7.3. (1) Free one-sided stable distributions with non-negative drifts. These distributions are found by Biane in Appendix in Bercovici and Pata (1999).

(2) The square of a symmetric $\boxplus$-stable law. By Theorem 2.2 it is free regular, and moreover, by the results of Arizmendi and Pérez-Abreu (2009) we can identify the Lévy measure σ of Theorem 2.2 with a $\boxplus$-stable law. Indeed, any symmetric stable measure has the representation $w \boxtimes \nu_{\frac{1}{1+t}}$ and then by Equation (3.2) the square is $w^2 \boxtimes \nu_{\frac{1}{1+t}} \boxtimes \nu_{\frac{1}{1+t}} = m \boxtimes \nu_{\frac{1}{1+2t}}$.

(3) Free multiplicative, free additive and boolean powers of the free Poisson m. In particular, for $t \geq 1$ the free Bessel laws $m^{\boxtimes t} \boxtimes m^{\boxplus s}$ studied in Banica et al. (2011) are free regular.

(4) The free Meixner laws, which are introduced by Saitoh and Yoshida (2001) and Anshelevich (2003), whose Lévy measures are given by

$$\nu_{a,b,c}(dx) = c\frac{\sqrt{4b-(x-a)^2}}{\pi x^2}1_{a-2\sqrt{b}<x<a+2\sqrt{b}}(x)dx.$$

If $a - 2\sqrt{b} \geq 0$, then the Lévy measure is concentrated on $[0,\infty)$ and $\int_{\mathbb{R}} \min(1,|x|)\nu_{a,b,c}(dx) < \infty$. Thus, if the drift term is non-negative, then it will be free regular. This case includes the free gamma laws, which come from interpretation by orthogonal polynomials not the Bercovici-Pata bijection.

(5) The beta distribution $B(1-a,1+a)$ $(0<a<1)$ has the density

$$p_a(x) = \frac{\sin(\pi a)}{\pi a}x^{-a}(1-x)^a,\ 0<x<1.$$

$B(1-a,1+a)$ is $\boxplus$-infinitely divisible if and only if $\frac{1}{2} \leq a < 1$ as shown in Arizmendi and Hasebe (2011a). Moreover, $B(1-a,1+a)$ is free regular for $\frac{1}{2} \leq a < 1$ since $\int_0^1 \frac{p_a(x)}{x}dx = \infty$ (see Theorem 4.6). We note that $B(\frac{1}{2},\frac{3}{2})$ coincides with the Marchenko-Pastur law up to scaling.

Example 7.4. Let w be the standard semicircle law. Then w^2 and w^4 are both free regular. It is well known that $w^2 = m$, which is free regular. From Arizmendi et al. (2010), if $\mathbf{b}_s$ is the symmetric beta $(1/2,3/2)$ distribution, $\mathbf{b}_s$ is freely infinitely divisible and then, by Theorem 2.2, $(\mathbf{b}_s)^2$ is free regular.

The symmetric beta distribution $\mathbf{b}_s$ has density

$$\mathbf{b}_s(\mathrm{d}x) = \frac{1}{2\pi}\left|x\right|^{-1/2}(2-|x|)^{1/2}\mathrm{d}x, \quad |x|<2.$$

Clearly $m_{2n}(\mathbf{b}_s) = m_{4n}(w)$ and then $(\mathbf{b}_s)^2 = w^4$. Also since $w^4 = m^2 = (\mathbf{b}_s)^2$, w^4 is free regular.

Remark 7.5. It is not known if w^n is $\boxplus$-infinitely divisible for the other positive integers $n = 3,5,6,7,8,\cdots$. In classical case, any positive integer power of the standard Gaussian is $*$-infinitely divisible as shown in Bondesson (1992, Theorem 7.3.6).

One may ask if the example w_+ is an exception but the following example shows that there are a lot of measures in $I^{\boxplus} \cap \mathcal{M}^+$ which are not $I^{\boxplus}_{r+}$. We also mention here that a quarter-circle distribution is not $\boxplus$-infinitely divisible.

Example 7.6. (1) We present a method to construct freely infinitely divisible measures with positive support, but not free regular. Let $\mu \neq \delta_0$ be $\boxplus$-infinitely divisible with compact support, say $[-a,b]$. Then $\mu_1 := \mu \boxplus \delta_a$ has support $[0,b+a]$ and $\mu_2 := \mu \boxplus \delta_{-b}$ has support $[-(a+b),0]$. Both μ_1 and $\tilde{\mu}_2(dx) := \mu_2(-dx)$ are in $I^{\boxplus} \cap \mathcal{M}^+$, but either μ_1 or $\tilde{\mu}_2$ must not be free regular.

Indeed suppose that μ_1 is free regular, then $\mu_1 = \Lambda(\nu_1)$ for some $\nu_1 \in \mathcal{M}^+$ with unbounded positive support, say $[c,\infty)$. Now, recall that Λ is a homomorphism, so that $\mu_2 = \Lambda(\nu_1 * \delta_{-a-b})$ but since the support of ν is $[b,\infty)$ then the support of $\widetilde{\nu_1 * \delta_{-a-b}}$ is $(-\infty, a+b-c]$ and intersects $\mathbb{R}_-$, which means that $\tilde{\mu}_2$ is not free regular. In particular, if μ is symmetric, any shift of μ is not free regular. Easy explicit examples can also be obtained from μ a free regular measure, for instance from (3), (4) and (5) of Example 7.3.

(2) Let a_α be the monotone α-stable law characterized by $F_{a_\alpha}(z) = (z^\alpha + e^{i\alpha\pi})^{1/\alpha}$, where the powers z^α and $z^{1/\alpha}$ are respectively defined as $e^{\alpha \log z}$ and $e^{\frac{1}{\alpha}\log z}$ in $\mathbb{C}\backslash[0,\infty)$. The function log is not the principal value, but is defined so that $\mathrm{Im}(\log z) \in (0, 2\pi)$. If $\alpha \in [\frac{1}{2}, 1)$, this measure is $\boxplus$-infinitely divisible and supported on $[0,\infty)$ (see Arizmendi and Hasebe (2011a); Biane (1998)). However this measure is not free regular, since the Voiculescu transform $\phi_{a_\alpha}(z) = (z^\alpha - e^{i\alpha\pi})^{1/\alpha} - z$ is not analytic in $\mathbb{C}\backslash[0,\infty)$. In this case the support of the Lévy measure is $[-1,\infty)$.

(3) Let $\sigma > 0$. Suppose q be the quarter-circle distribution, that is, it has density

$$f_q(x) = \begin{cases} \frac{1}{\pi\sigma^2}\sqrt{4\sigma^2 - x^2} & (x \in [0, 2\sigma]), \\ 0 & (\text{otherwise}). \end{cases}$$

It is not freely infinitely divisible for any $\sigma > 0$. We can find it by the following proposition of free kurtosis.

Proposition 7.7. *If μ is freely infinitely divisible then the free kurtosis* $\mathrm{kurt}^{\boxplus}(\mu)$ *of μ is positive, that is,*

$$\mathrm{kurt}^{\boxplus}(\mu) = \frac{\widetilde{m_4}(\mu)}{(\widetilde{m_2}(\mu))^2} - 2 > 0,$$

where $\widetilde{m_2}(\mu)$, $\widetilde{m_4}(\mu)$ are 2nd and 4th moments around mean.

For more detail of free kurtosis, see p.171 in Arizmendi and Pérez-Abreu (2010). Here we can obtain moments of q as follows:

$$m_1(q) = \frac{8\sigma}{3\pi},\ m_2(q) = \sigma^2,\ m_3(q) = \frac{2^6\sigma^3}{15\pi},\ m_4(q) = 2\sigma^4.$$

Therefore,

$$\frac{\left(2 - \frac{2^{12}}{3^3\pi^4}\right)}{\left(1 - \frac{2^6}{3^2\pi^2}\right)^2} - 2 < 0$$

for any $\sigma > 0$. In fact, this amount is around -0.0233443.

Recall from Proposition 5.1 that $w_+ \boxtimes w_+$ is not freely infinitely divisible. Therefore, we have the following conjecture.

Conjecture 7.8. *If $\mu \in \mathcal{M}^+$ is $\boxplus$-infinitely divisible, then $\mu \boxtimes \mu$ is $\boxplus$-infinitely divisible if and only if μ is free regular.*

Example 7.9 (free commutators). (1) Let σ_s and σ_t be two symmetric free stable distributions of index s and t, respectively. Then by Corollary 6.5 the free commutator $\sigma_s \Box \sigma_t$ is $\boxplus$-infinitely divisible. For the case $t = s = 2$ (the Wigner semicircle distribution) the density of $w \Box w$ is given by Nica and Speicher (1998)

$$f(t) = \frac{\sqrt{3}}{2\pi \mid t \mid}\left(\frac{3t^2+1}{9h(t)} - h(t)\right),\ \mid t \mid \leq \sqrt{(11+5\sqrt{5})/2}, \tag{7.1}$$

where

$$h(t) = \sqrt[3]{\frac{18t^2+1}{27} + \sqrt{\frac{t^2(1+11t^2-t^4)}{27}}}.$$

(2) Let w be the standard semicircle law and let $\nu_{\frac{1}{1+2s}}$ be a positive free stable law, for some $s > 0$. If we denote $\hat{\nu}_{\frac{1}{1+2s}} = \mathrm{Sym}(\nu^{1/2}_{\frac{1}{1+2s}})$ then $\mu := w \Box \hat{\nu}_{\frac{1}{1+2s}}$ is a symmetric free stable distribution with index $\frac{2}{1+2s}$. Indeed, by Equation (6.1), μ satisfies

$$(\mu^{\boxplus 1/2})^2 = ((w \Box \hat{\nu}_{\frac{1}{1+2s}})^{\boxplus 1/2})^2 = w^2 \boxtimes \nu_{\frac{1}{1+2s}} = m \boxtimes \nu_{\frac{1}{1+2s}}.$$

From Equation (3.2) and results in Arizmendi and Pérez-Abreu (2009) we see that $m \boxtimes \nu_{\frac{1}{1+2s}} = (w \boxtimes \nu_{\frac{1}{1+s}})^2$. This means that $\mu^{\boxplus 1/2} = w \boxtimes \nu_{\frac{1}{1+s}}$ which is a symmetric free stable distribution with index $\frac{2}{1+2s}$. The case $s = 1/2$ was treated in Nica and Speicher (1998, Example 1.14).

(3) Assume that b is a symmetric Bernoulli distribution $\frac{1}{2}(\delta_{-1} + \delta_1)$. Let μ, ν be symmetric distributions. Then the free commutator $\mu \Box \nu$ is 2-$\boxplus$-divisible, but when $\mu = \nu$ we can identify $(\mu \Box \mu)^{\boxplus 1/2}$. Indeed, by Eq. (6.1), $(\mu \Box \mu)^{\boxplus 1/2} = \sqrt{\mu^2 \boxtimes \mu^2}$. On the other hand, by Equation (3.2), $(\mu^2 \boxtimes b)^2 = \mu^2 \boxtimes \mu^2$. Hence $(\mu^2 \boxtimes b)^{\boxplus 2} = \mu \Box \mu$.

When $\mu = w$ a strange thing happens: $w^2 = m$, and $m \boxtimes b$ is a compound free Poisson with rate 1 and jump distribution b, see Remark 4.1. This implies that $w \Box w = m \boxplus \widetilde{m}$, where $\widetilde{m}$ is defined by $\widetilde{m}(B) = m(-B)$. It is a free symmetrization of the Poisson distribution (not to be confused with the symmetric beta of Example 7.4). As pointed out in Nica and Speicher (1998), this gives another derivation of the density of $w \Box w$ given in Equation (7.1).

(4) For the free Poisson with mean 1, the free commutator becomes $m \Box m = (m \boxtimes m \boxtimes b)^{\boxplus 2}$, the compound free Poisson with rate 2 and jump distribution $m \boxtimes b$. Indeed, if we define $\hat{m} := m \boxtimes b$, we have that $m \Box m = \hat{m} \Box \hat{m}$ since the even free cumulants of $\hat{m}$ are all one, the same as those of m, and since the free commutator of measures depends only on the even cumulants of the measures; see Nica and Speicher (1998, Theorem 1.2). By Equation (3.2) we have $\hat{m}^2 = m \boxtimes m$, and therefore by Equation (6.1), we have

$$((m \Box m)^{\boxplus 1/2})^2 = m \boxtimes m \boxtimes m \boxtimes m.$$

Again using Equation (3.2) we see that $m \boxtimes m \boxtimes m \boxtimes m = (m \boxtimes m \boxtimes b)^2$. The claim then follows.

Appendix A. Combinatorial approach

In this appendix we prove Theorem 6.8, using combinatorial tools. We also give a combinatorial proof of Theorem 2.2 which was proved with analytic tools. We decided not to include them in the main section of this article not only because they are more involved but also since, in principle, these proofs are only valid when the existence of moments is assumed. However, we believe that a reader who is more acquainted with the combinatorial approach may find them more illuminating.

A.1. *Free cumulants.* A measure μ has *all moments* if $m_k(\mu) = \int_{\mathbb{R}} t^k \mu(\mathrm{d}t) < \infty$, for each even integer $k \geq 1$. Probability measures with compact support have all moments.

The **free cumulants** (κ_n) were introduced by Voiculescu (1986) as an analogue of classical cumulants, and were developed more by Speicher (1994) in his combinatorial approach to free probability theory. We refer the reader to the book of

Nica and R. Speicher (2006) for a nice introduction to this combinatorial approach. Let μ be a probability measure with compact support, then the cumulants are the coefficients $\kappa_n = \kappa_n(\mu)$ in the series expansion

$$\mathcal{C}_\mu^{\boxplus}(z) = \sum_{n=1}^{\infty} \kappa_n(\mu) z^n.$$

For a sequence $(t_n)_{n\geq 1}$ and a partition $\pi = \{V_1, ..., V_r\} \in NC(n)$ we denote $t_\pi := t_{|V_1|} \cdots \kappa_{|V_r|}$.

The relation between the free cumulants and the moments is described by the lattice of non-crossing partitions $NC(n)$, namely,

$$m_n(\mu) = \sum_{\pi \in NC(n)} \kappa_\pi(\mu). \tag{A.1}$$

Since free cumulants are just the coefficients of the series expansion of $\mathcal{C}_\mu^{\boxplus}(z)$, they linearize free convolution:

$$\kappa_n(\mu_1 \boxplus \mu_2) = \kappa_n(\mu_1) + \kappa_n(\mu_2).$$

A compound free Poisson μ with rate λ and jump distribution ν can be characterized as

$$\kappa_n(\mu) = \lambda m_n(\nu).$$

In particular, if μ is of the form $m \boxtimes \sigma$ for a probability measure σ on $\mathbb{R}$, then $\kappa_n(m \boxtimes \sigma) = m_n(\sigma)$.

Compound free Poissons are $\boxplus$-infinitely divisible, and moreover, any $\boxplus$-infinitely divisible probability measure is a weak limit of compound free Poissons.

A.2. *Even elements.* When μ has all moments, being symmetric is equivalent to having vanishing odd moments, that is $m_{2k+1}(\mu) = \int_{\mathbb{R}} t^{2k+1} \mu(\mathrm{d}t) = 0$. On the other hand μ^2 has moments $m_k(\mu^2) = m_{2k}(\mu)$.

An element $x \in (\mathcal{A}, \varphi)$ is said to be **even** if the only non vanishing moments are even, i.e. $\varphi(x^{2k+1}) = 0$. Even elements correspond to symmetric distributions. It is clear by the moment-cumulant formula (A.1) that $x \in \mathcal{A}$ is even if and only if the only non-vanishing free cumulants are even. In this case we call $(\alpha_n := \kappa_{2n}(x))_{n\geq 1}$ the determining sequence of x.

The next proposition gives a formula for the cumulants of the square of an even element in terms of the cumulants of this element and can be found in Nica and R. Speicher (2006, Proposition 11.25)

Proposition A.1. *Let $x \in \mathcal{A}$ be an even element and let $(\alpha_n = \kappa_{2n}(x))_{n\geq 1}$ be the determining sequence of x. Then the cumulants of x^2 are given as follows:*

$$\kappa_n(x^2) = \sum_{\pi \in NC(n)} \alpha_\pi.$$

Now we are able to prove the main result of this section.

Proposition A.2. *Let μ be symmetric distribution with all moments. If μ is freely infinitely divisible, then μ^2 is a compound free Poisson $\pi(\lambda, \rho)$ with $\rho \in I_{r+}^{\boxplus}$. If moreover μ is itself a compound free Poisson $\pi(\lambda, \nu)$, then ρ is also a compound free Poisson $\rho = \pi(\lambda, \nu^2)$.*

Proof: Let x be an even element with distribution μ and suppose that μ is a symmetric compound free Poisson with rate λ and jump distribution ν and let $\rho = \pi(\lambda, \nu^2)$ be a compound free Poisson with rate λ and jump distribution ν^2. Then the determining sequence of x is

$$\alpha_n = \kappa_{2n}(x) = \lambda m_{2n}(\nu) = \lambda m_n(\nu^2) = \kappa_n(\rho).$$

By Proposition A.1 we have that

$$\kappa_n(x^2) = \sum_{\pi \in NC(n)} \alpha_\pi = \sum_{\pi \in NC(n)} \kappa_\pi(\rho) = m_n(\rho)$$

and hence the distribution μ of x^2 is a compound free Poisson with rate 1 and jump distribution ρ.

More generally if $\mu \in I^{\boxplus}$ is symmetric, then μ can be approximated by compound free Poissons which are symmetric, say $\mu = \lim_{n\to\infty} \mu_n$. By the previous case for each $n > 0$, $\mu_n^2 = m \boxtimes \nu_n$ for some ν_n compound free Poisson, which is free regular. Since $\mu_n^2 \to \mu^2$ and $\nu_n \to \nu$ for some ν, then $\mu = m \boxtimes \nu$. The measure ν is free regular since $I_{r+}^{\boxplus}$ is closed under the convergence in distribution. $\square$

Finally, we use similar arguments to prove Theorem 6.8 on free commutators.

Proof of Theorem 6.8: By an approximation similar to Proposition A.2, it is enough to consider μ_1 and μ_2 compound free Poissons. Let $\mu_1 \Box \mu_2$ be the free commutator and $\kappa_n(\mu_i) = \lambda_i m_n(\nu_i)$ the free cumulants of μ_i, for $i = 1, 2$. It is clear that $m_{2n}(\nu_i) = m_{2n}(\mathrm{Sym}(\nu_i))$ and $m_{2n+1}(\mathrm{Sym}(\nu_i)) = 0$. Now, by Theorem 1.2 in Nica and Speicher (1998), the free cumulants of $\mu_1 \Box \mu_2$ only depend on the even free cumulants of μ_1 and μ_2, and therefore we can change μ_i by the symmetric compound Poisson with Lévy measure $\mathrm{Sym}(\nu_i)$. Thus by Corollary 6.5 $\mu_1 \Box \mu_2$ is $\boxplus$-infinitely divisible as desired. $\square$

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