Sequent Calculi for Multi-Agent Epistemic Logics for Distributed Knowledge

Ryo Murai
rmurai270@gmail.com

Katsuhiko Sano
v-sano@let.hokudai.ac.jp

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1 Introduction

"Distributed knowledge" is a notion developed in the community of multi-agent epistemic logic [1, 8]. In [1, p. 3], the notion is explained informally as follows:

> A group has distributed knowledge of a fact φ if the knowledge of φ is distributed among its members, so that by pooling their knowledge together the members of the group can deduce φ, even though it may be the case that no member of the group individually knows φ.

For example, a group consisting of a and b has distributed knowledge of a fact q, when a knows $p \to q$ and b knows p. Formally, "A group G has distributed knowledge of a fact φ." is written as "$D_G\varphi$", whose meaning is usually defined in a Kripke model. Let W be a possibly countable set of states, Agt be a finite set of agents, $(R_a)_{a\in\mathsf{Agt}}$ be a family of binary relations on W, indexed by agents, and V be a valuation function $\mathsf{Prop} \to \mathcal{P}(W)$, where Prop is a countable set of propositional variables. We call a tuple $M = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ a (Kripke) model. For a group $G \subseteq \mathsf{Agt}$, satisfaction of $D_G\varphi$ at a state w in a model M is defined as follows:

$$M, w \models D_G\varphi \iff \text{for all } v, \text{ if } (w, v) \in \bigcap_{a\in G} R_a \text{ then } M, v \models \varphi$$

It is clear from the definition that the operator $D_{\{a\}}$ behaves the same as K_a, a box-like operator for an agent a, usually defined in multi-agent epistemic logic. Therefore, we do not introduce K_a-like operator as a primitive one in this abstract.

The study of distributed knowledge so far is mainly model-theoretic [16, 13, 4, 15] and proof-theoretic study has been not so active. As far as we know, existing sequent calculi for logic with distributed knowledge are presented only in [6] and [5]. The former contains a natural G3-style formalization, in which each formula has a label and the latter contains Getzen-style and Kanger-style sequent calculus for logic with distributed knowledge operator which is simpler than the one we are interested in, in that the operator is not parameterized by group G.

We propose Gentzen-style sequent calculi (without label) for five kinds of multi-agent epistemic propositional logics with distributed knowledge operators, parameterized by groups, which are reasonable generalization of sequent calculi for basic modal logic and prove the cut elimination theorem for four of them. Using a method described in [7], Craig's interpolation theorem is also established for the four system, in which not only condition of propositional variables but also that of agents is taken into account. This is a new result for logic for distributed knowledge, as far as we know.

In the following, we briefly sketch our proof systems, and then state and comment on the theorems we have on the systems.

2 Proof Systems

We denote a finte set of agents by Agt. We call a nonempty subset of Agt "group" and denote it by G, H, etc. Let Prop be a countable set of propositional variables and Form be the set of formulas defined inductively by the following clauses ($\vee$ and $\neg$ are defined in the same way as the classical propositional logic):

$$\mathsf{Form} \ni \varphi ::= p \in \mathsf{Prop} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid D_G\varphi$$

First, we explain known Hilbert-style axiomatization for logics with D_G operator (for detail, refer to [1]). The following are axioms for the logics:

- (Taut) all instantiations of propositional tautology
- (Incl) $D_G\varphi \to D_H\varphi \ \ (G \subseteq H)$
- (K) $D_G(\varphi \to \psi) \to D_G\varphi \to D_G\psi$
- (T) $D_G\varphi \to \varphi$
- (4) $D_G\varphi \to D_G D_G\varphi$
- (5) $\neg D_G\varphi \to D_G\neg D_G\varphi$

An axiom system $\mathsf{H}(\mathbf{K}_D)$ ($\mathsf{H}(\mathbf{KT}_D)$,$\mathsf{H}(\mathbf{K4}_D)$, $\mathsf{H}(\mathbf{S4}_D)$, and $\mathsf{H}(\mathbf{S5}_D)$) is a collection of the inference rules of modus ponens ("from $\varphi \to \psi$ and φ infer ψ") and necessitation ("from φ infer $D_G\varphi$"), axioms (Taut) and (Incl) (common to all the five systems), and (an) axiom(s) (K) ((K) and (T); (K) and (4); (K), (T), and (4); and (K), (T), and (5), respectively).

We now propose our sequent calculi for the logics for distributed knowledge. To the ordinary LK system [2, 3], we add some of the following rules for each logic:

$$\frac{\varphi_1, \cdots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \cdots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D_G)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D_G \Rightarrow)$$

$$\frac{\varphi_1, \cdots, \varphi_n, D_{G_1}\varphi_1, \cdots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \cdots, D_{G_n}\varphi_n \Rightarrow D_G\psi} \ (\Rightarrow {D_G}^{\mathbf{K4}_D})$$

$$\frac{D_{G_1}\varphi_1, \cdots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \cdots, D_{G_n}\varphi_n \Rightarrow D_G\psi} \ (\Rightarrow {D_G}^{\mathbf{S4}_D})$$

$$\frac{\varphi_1, \cdots, \varphi_n \Rightarrow \psi_1, \cdots, \psi_m, \chi \quad (\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq G)}{D_{G_1}\varphi_1, \cdots, D_{G_n}\varphi_n \Rightarrow D_{H_1}\psi_1, \cdots, D_{H_m}\psi_m, D_G\chi} \ (\Rightarrow {D_G}^{\mathbf{S5}_D})$$

A sequent calculus $\mathsf{G}(\mathbf{K}_D)$ ($\mathsf{G}(\mathbf{KT}_D)$,$\mathsf{G}(\mathbf{K4}_D)$, $\mathsf{G}(\mathbf{S4}_D)$, and $\mathsf{G}(\mathbf{S5}_D)$) is LK with the rule(s) (D_G) ((D_G) and $(D_G \Rightarrow)$; $(\Rightarrow {D_G}^{\mathbf{K4}_D})$; $(D_G \Rightarrow)$ and $(\Rightarrow {D_G}^{\mathbf{S4}_D})$; and $(D_G \Rightarrow)$ and $(\Rightarrow {D_G}^{\mathbf{S5}_D})$, respectively).

The idea underlying the rule (D_G) is similar to that of an inference rule called "$R12$" described in [12, section 4]. Our calculi $\mathsf{G}(\mathbf{KT}_D)$,$\mathsf{G}(\mathbf{K4}_D)$, $\mathsf{G}(\mathbf{S4}_D)$, and $\mathsf{G}(\mathbf{S5}_D)$ are constructed based on the known sequent calculus for $\mathbf{KT}, \mathbf{K4}, \mathbf{S4}$, and $\mathbf{S5}$ (surveyed in [11, 14]).

We note that for any logic $\mathbf{X}$ of the logics described above, $\mathsf{H}(\mathbf{X})$ and $\mathsf{G}(\mathbf{X})$ are equivalent in derivability of formulas, and hence that each system $\mathsf{G}(\mathbf{X})$ deserves its own name.

Theorem 1 (Equivalence between Hilbert-style and Gentzen-style axiomatization) *Let $\mathbf{X}$ be any of $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, $\mathbf{S4}_D$, and $\mathbf{S5}_D$. Then, the following hold.*

1. *If $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, then $\vdash_{\mathsf{G}(\mathbf{X})} \Rightarrow \varphi$*
2. *If $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma \to \bigvee \Delta$*

We have the cut elimination theorem for our sequent calculi, except for $\mathsf{G}(\mathbf{S5}_D)$.

Theorem 2 (Cut elimination) *Let $\mathbf{X}$ be any of $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, and $\mathbf{S4}_D$. Then, the following holds: If $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$, where $\mathsf{G}^-(\mathbf{X})$ denotes a system "$\mathsf{G}(\mathbf{X})$ minus cut rule".*

Flexibility of choice of groups occurring in the left side of the lower sequent in the rule (D_G) and the three $(\Rightarrow D_G)$-type rules is a key to the result. The reason why cut elimination theorem does not hold for $\mathsf{G}(\mathbf{S5}_D)$ is that the sequent calculus for basic $\mathbf{S5}$, on which $\mathsf{G}(\mathbf{S5}_D)$ is based, is not cut-free [9].

As an application of the cut elimination theorem, Craig's interpolation theorem can be derived, using a method described in [7]. (Application of the method to basic modal logic is also described in [10].)

Theorem 3 (Craig's interpolation theorem) *Let $\mathbf{X}$ be any of $\mathbf{K}_D$, $\mathbf{KT}_D$, $\mathbf{K4}_D$, and $\mathbf{S4}_D$. Given that $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \psi$, there exists a formula χ satisfying the following conditions:*

1. *$\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \chi$ and $\vdash_{\mathsf{G}(\mathbf{X})} \chi \Rightarrow \psi$.*

2. $V(\chi) \subseteq V(\varphi) \cap V(\psi)$*, where* $V(\rho)$ *denotes the set of propositional variables occuring in formula* ρ*.*

3. $A(\chi) \subseteq A(\varphi) \cap A(\psi)$*, where* $A(\rho)$ *denotes the set of agents occuring in formula* ρ*.*

We note that not only the condition for propositional variables but also the condition for agents can be satisfied.

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