Cut-free and Analytic Sequent Calculus of First-Order Intuitionistic Epistemic Logic

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[Artemov and Protopopescu, 2016] gave an intuitionistic epistemic logic based on a verification reading of the intuitionistic knowledge in terms of Brouwer-Heyting-Kolmogorov interpretation. According to this interpretation, a proof of $A \supset B$ is a construction such that when a proof of A is given, a proof of B can be constructed. [Artemov and Protopopescu, 2016] proposed that a proof of a formula KA (read "it is known that A"), is the conclusive verification of the existence of a proof of A. Then $A \supset KA$ expresses that, when a proof of A is given, the conclusive verification of the existence of the proof of A can be constructed. Since a proof of A itself is the conclusive verification of the existence of a proof of A, they claim that $A \supset KA$ is valid. But $KA \supset A$ (usually called *factivity* or *reflection*) is not valid, since the verification does not always give a proof. They provided a Hilbert system of intuitionistic epistemic logic **IEL** as the intuitionistic propositional logic plus the axioms schemes $K(A \supset B) \supset KA \supset KB$, $A \supset KA$ and $\neg K\bot$. Moreover they gave **IEL** the following Kripke semantics. We say that $M = (W, \leq, R, V)$ is a *Kripke model* for **IEL** if $(W, \leq, V)$ is a Kripke model for intuitionistic propositional logic and R is a binary relation such that $R \subseteq \leq$, $\leq ; R \subseteq R$ and R satisfies the seriality. Then KA is true on a state w of M if and only if for any v, wRv implies A is true in v of M. [Artemov and Protopopescu, 2016] also proved that their Hilbert system is sound and complete.

The study of **IEL** also casts light on the study of the knowability paradox. The knowability paradox, also known as the Fitch-Church paradox, states that, if we claim the knowability principle: every truth is knowable ($A \supset \Diamond KA$), then we are forced to accept the omniscience principle: every truth is known ($A \supset KA$) [Fitch, 1963]. This paradox is commonly recognized as a threat to Dummett's semantic anti-realism. It is because the semantic anti-realists claim the knowability principle but they do not accept the omniscience principle. However, as Dummett admitted that he had taken some of intuitionistic basic features as a model for an anti-realist view [Dummett, 1978, p.164], it is reasonable to consider an intuitionistic logic as a basis. In this sense, if we employ BHK-interpretation of KA as above to accept the **IEL** in the study of the knowability paradox, $A \supset KA$ becomes valid and the knowability paradox is trivialized.

Proof-theoretical studies of **IEL** have been investigated. In Krupski and Yatmanov [2016], the sequent calculus of **IEL** has been given, though an inference rule corresponding to $KA \supset \neg\neg A$ in their system for **IEL** does not satisfy a desired syntactic property, i.e., the subformula property. In Protopopescu [2015], a Gödel-McKinsey-Tarski translation from the intuitionistic epistemic propositional logic to the bimodal expansion of the classical modal logic **S4** has been studied.

In this paper, we study the first-order expansion **QIEL** of intuitionistic epistemic logic of **IEL**. Artemov and Protopopescu mentioned that the notion of the intuitionistic knowledge capture both mathematical knowledge and empirical knowledge. When we consider the mathematical knowledge, quantifiers become inevitable. Moreover when we are concerned with the empirical knowledge, we recall that Hintikka had given arguments for first-order epistemic logic in Hintikka [2005]. He mentioned that if we want to deal with the locutions like "knows who," "knows when," "knows where," we can translate these expressions into a language with quantifiers. For example, about "who" we can have variables ranging over the human being, about "where" over the location in space. In this sense, our first-order expansion can provide a fundamental basis when we concern the intuitionistic mathematical and empirical knowledge.

We give the first-order expansion of **IEL** as **QIEL**. An expanded Kripke model $M = (W, \leq, R, D, I)$ is obtained by adding D and I into the Kripke model for **IEL**. Here D is a function which assigns a nonempty domain $D(w)$ to $w \in W$ such that, for any $w, v \in W$, if $w \leq v$ then $D(w) \subseteq D(v)$. Moreover I is an interpretation such that $I(c) \in D(w)$ for all $w \in W$ for any constant symbol c and $I(P, w) \subseteq D(w)^n$ for every $w \in W$ and every n-arity predicate P such that if $u \leq v$ then $I(P, u) \subseteq I(P, v)$ for all $u, v \in W$.

We also propose the sequent calculus for **QIEL**. The sequent calculus for **IEL** has been given by Krupski and Yatmanov [2016]. Their sequent calculus is obtained from the propositional part of Gentzen's sequent calculus **LJ** (with structural rules of weakening and contraction) for the intuitionistic logic plus the following two inference rules on the knowledge operator:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow A}{\Gamma_1, K\Gamma_2 \Rightarrow KA}\ (KI) \qquad \frac{\Gamma \Rightarrow K\bot}{\Gamma \Rightarrow F.}\ (U)$$

where a sequent $\Gamma \Rightarrow A$ (where Γ is a finite multiset of formulas) can be read as "if all of Γ hold then A holds." They established the cut-elimination theorem of the calculus, i.e., a derivable sequent in their system is derivable without any application of the following cut rule:

$$\frac{\Gamma \Rightarrow B \quad B, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta}\ (Cut),$$

where Δ contains at most one formula. It is remarked, however, that this system does not enjoy the subformula property. That is, in the rule of (U), we have a formula $K\bot$ which might not be a subformula of a formula in the lower sequent of the rule (U).

This talk gives a new cut-free and analytic sequent calculus $\mathscr{G}(\mathbf{QIEL})$ of the first-order intuitionistic epistemic logic, which is obtained from adding the following rule (K_{IEL}) into Gentzen's **LJ** with quantifiers:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, K\Gamma_2 \Rightarrow K\Delta}\ (K_{IEL})$$

where Δ contains at most one formula. This rule is equivalent to the rules from Krupski and Yatmanov [2016] in a propositional setting. Moreover it is easy to see that (K_{IEL}) satisfies the subformula property.

Let $\mathscr{G}^{-}(\mathbf{QIEL})$ be the system $\mathscr{G}(\mathbf{QIEL})$ without the cut rule. By the standard syntactic argument, we can establish the following fundamental proof-theoretic result.

Theorem 1 (Cut-Elimination) *If* $\mathscr{G}(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$ *then* $\mathscr{G}^{-}(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$.

Corollary 1 (Disjunction Property, Existence Property, Craig Interpolation Theorem) *As a corollary of cut-elimination theorem we have:*

1. *If* $\Rightarrow A \vee B$ *is derivable in* $\mathscr{G}(\mathbf{QIEL})$*, then either* $\Rightarrow A$ *or* $\Rightarrow B$ *is derivable in* $\mathscr{G}(\mathbf{QIEL})$.

2. *For any formula of the form $\exists xA$, if $\Rightarrow \exists xA$ is derivable in $\mathscr{G}(\mathbf{QIEL})$ then there exists a term t such that $\Rightarrow A(t/x)$ is derivable in $\mathscr{G}(\mathbf{QIEL})$.*

3. *If $\Rightarrow A \supset B$ is derivable in $\mathscr{G}(\mathbf{QIEL})$, then there exists a formula C such that $\Rightarrow A \supset C$ and $\Rightarrow C \supset B$ are also derivable in $\mathscr{G}(\mathbf{QIEL})$, and all free variables, predicate symbols and constant symbols of C are shared by both A and B.*

Given a sequent $\Gamma \Rightarrow \Delta$, Γ_* denotes the conjunction of all formulas in Γ ($\Gamma_* \equiv \top$ if Γ is empty) and Δ^* denotes the unique formula in Δ if Δ is non-empty; it denotes $\perp$ otherwise. We say that a sequent $\Gamma \Rightarrow \Delta$ is valid in a class $\mathbb{M}$ of models (denoted by $\mathbb{M} \models \Gamma \Rightarrow \Delta$), if $\Gamma_* \supset \Delta^*$ is satisfied in any states of any Kripke models.

Theorem 2 (Soundness of $\mathscr{G}(\mathbf{QIEL})$) *Let $\Gamma \Rightarrow \Delta$ be any sequent. If $\mathscr{G}(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$ then $\mathbb{M} \models \Gamma \Rightarrow \Delta$.*

With the method from Hermant [2005], we prove the following:

Theorem 3 (Completeness of $\mathscr{G}^-(\mathbf{QIEL})$) *Let $\Gamma \Rightarrow \Delta$ be a sequent. If $\mathbb{M} \models \Gamma \Rightarrow \Delta$ then $\mathscr{G}^-(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$.*

Corollary 2 *The following are all equivalent.*

1. $\mathbb{M} \models A$, 2. $\mathscr{G}(\mathbf{QIEL}) \vdash \Rightarrow A$, 3. $\mathscr{G}^-(\mathbf{QIEL}) \vdash \Rightarrow A$,

In particular, we can also prove the cut elimination theorems semantically by Theorem 2 and Theorem 3.

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